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In one of the fundamental results of Arakelov's arithmetic intersection theory, Faltings and Hriljac (independently) proved the Hodge index theorem for arithmetic surfaces by relating the intersection pairing to the negative of the Néron–Tate height pairing. More recently, this has been generalized to higher dimensions by Moriwaki and by Yuan and Zhang. We extend these results to projective varieties over transcendence degree one function fields. The new challenge is dealing with nonconstant but numerically trivial line bundles coming from the constant field via Chow's K/k-trace functor. As an application of the Hodge index theorem, we also prove a rigidity theorem for the set of canonical height zero points of polarized algebraic dynamical systems over function fields. For function fields over finite fields, this gives a rigidity theorem for preperiodic points, generalizing previous work of Mimar, of Baker and DeMarco, and of Yuan and Zhang.

1. Introduction

The Hodge index theorem states classically that the divisor intersection pairing on an algebraic surface has signature $+1, -1, \ldots, -1$. The corresponding result for line bundles on arithmetic surfaces, i.e., relative curves over the ring of integers of a number field, was proven independently by Faltings [1984] and Hriljac [1985], and is a fundamental result in Arakelov theory. More recently, Moriwaki [1996] extended this to higher-dimensional arithmetic varieties, and Yuan and Zhang [2017] proved a Hodge index theorem for adelic metrized line bundles over $\overline{\mathbb{Q}}$.

In their work, Yuan and Zhang also conjectured that a similar result should hold over function fields. Here we prove their conjecture. Our theorem statement differs slightly from their conjecture, however, so as to be stated more directly and to avoid reliance on a noncanonical isogeny between Chow's function field K/k-trace and K/k-image.

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Let k be an algebraically closed field of arbitrary characteristic, and let $K = k(B)$ be the function field of B, a smooth, projective curve over k. Let $\pi : X \to \mathrm{Spec}(K)$ be a geometrically normal, geometrically integral, projective variety of dimension $n \geq 1$. We will consider the group $\widehat{\mathrm{Pic}}(X)$ of adelic metrized line bundles on X in the sense of [Zhang 1995]; definitions will be recalled in Section 2.1. Since an adelic metric can be specified, for example, by a line bundle on a model $\mathcal{X} \to B$ of X, this setting also covers relative varieties fibered over B, in the same way that Yuan and Zhang's work over number fields encompasses Arakelov's setting of arithmetic varieties over the spectrum of the ring of integers of a number field.

Chow's K/k-trace functor $\mathrm{Tr}_{K/k}$ identifies the part of the Picard variety of X which is defined over k, and the line bundles in $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))$ can all be given adelic metrics in a well-defined canonical way using isotrivial models over B. This construction is detailed in Section 2.5. Let $\mathrm{Pic}^\tau(X)$ be the numerically trivial subgroup of $\mathrm{Pic}(X)$. We prove the following result, with more detailed versions stated in Section 3:

Theorem 1.1. *Let $M, N \in \mathrm{Pic}^\tau(X)$, and let $L_1, \dots L_{n-1} \in \mathrm{Pic}(X)$ be ample. There exist canonical metrics on M and N so that*

$$\langle M, N\rangle_{L_1,\dots,L_{n-1}} := \overline{M} \cdot \overline{N} \cdot \bar{L}_1 \cdots \bar{L}_{n-1}$$

is a well-defined bilinear pairing, independent of the choice of the metrics on $L_1, \dots, L_{n-1}$. This extends to a symmetric $\mathbb{R}$-bilinear form on $\mathrm{Pic}^\tau(X) \otimes_{\mathbb{Z}} \mathbb{R}$ which is negative semidefinite with kernel

$$\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X)) \otimes_{\mathbb{Z}} \mathbb{R}.$$

If one removes the function field trace (so that the kernel is trivial), this is the same result that Yuan and Zhang prove for number fields. It is straightforward to see that $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X)) \otimes_{\mathbb{Z}} \mathbb{R}$ is in the kernel. Thus, the main new difficulty is showing that numerically trivial adelic metrized line bundles which are nonconstant must all come from isotrivial subgroups of the Picard group of X. In essence, all arguments of the proof must commute with the K/k-trace functor.

1.1. ***Arithmetic dynamics.*** Again let X be a projective variety over a function field K. A polarized dynamical system (f, L, q) is an endomorphism $f : X \to X$ along with an ample line bundle $L \in \mathrm{Pic}(X)$ such that $f^*L \cong L^{\otimes q}$ for some $q > 1$. The set of preperiodic points of f is defined as

$$\mathrm{Prep}(f) := \{x \in X(\overline{K}) \mid x \text{ has a finite forward orbit under } f\}.$$

Call and Silverman [1993] show that such a polarized endomorphism defines a canonical Weil height $\widehat{h}_f$. Here we show that L can be given an *admissible* metric $\bar{L}_f$ so that the height $h_{\bar{L}_f}$ defined by $\bar{L}_f$ via arithmetic intersections agrees with $\widehat{h}_f$ on $X(\overline{K})$. The advantage to our definition is that $h_{\bar{L}_f}$ defines not only heights of points, but heights of subvarieties of X as well. By applying the Hodge index theorem to compare the canonical heights defined by two different polarized dynamical systems, we prove the following rigidity theorem:

Theorem 1.2. *Let X be a projective variety defined over a transcendence degree one function field K over any base k, and let (f, L, q) and (g, M, r) be two polarized dynamical systems on X. If the points with height zero under $h_{\bar{L}_f}$ and the points with height zero under $h_{\overline{M}_g}$ agree on a Zariski dense subset of $X(\overline{K})$, then they are identical.*

This is stated more generally in Section 4. When k is the algebraic closure of a finite field, the Northcott property implies that the points with canonical height zero under f are exactly the preperiodic points $\mathrm{Prep}(f)$, giving an immediate corollary.

Corollary 1.3. *In the same setting as Theorem 1.2 but with the additional hypothesis that $k = \overline{\mathbb{F}}_q$, if $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is Zariski dense in $X(\overline{K})$, then $\mathrm{Prep}(f) = \mathrm{Prep}(g)$.*

This was conjectured by Yuan and Zhang, and they prove a similar result over number fields.

Over general function fields the corollary does not hold, as not all canonical height zero points are preperiodic. The proofs differ as well, as while it is clear that the set $\mathrm{Prep}(f)$ does not depend on the choice of polarization L, this must be proven for canonical height zero points, and then the heights compared in a more indirect way. Even so, some limited things can be said.

Corollary 1.4. *Let K be the function field of a smooth projective curve over any field k, and let f and g be two rational functions $\mathbb{P}^1_K \to \mathbb{P}^1_K$ which are not isotrivial. If $\mathrm{Prep}(f)$ and $\mathrm{Prep}(g)$ intersect on an infinite subset of $\mathbb{P}^1(\overline{K})$, they are equal.*

This is a direct consequence of our theorem and a theorem of Baker [2009]. Chatzidakis and Hrushovski [2008a; 2008b] prove results comparing preperiodic points and height zero points using a model-theoretic nonisotriviality condition in a much more general setting, but it is difficult to combine that result into a useful rigidity statement. This is discussed further in Section 5.

1.2. ***Outline of paper and sketch of methods.*** Definitions and basic properties of adelic metrized line bundles and Chow's K/k-image and trace are recalled in Section 2. Additionally, this section includes technical lemmas, such as the existence of flat metrics, which will be needed throughout the paper.

Our main Hodge index theorem, a classification of numerically trivial line bundles, and an $\mathbb{R}$-linear variant (Theorems 3.1, 3.2, and 3.3) are fully stated and proven in Section 3. We begin with the case of X being a curve. Decomposing adelic metrized line bundles into flat and vertical pieces, and addressing intersections of the vertical parts using the local Hodge index theorem of [Yuan and Zhang 2017, Theorem 2.1], we reduce to the case of flat metrics. Then, following the methods of Faltings [1984] and Hriljac [1985], we relate the intersection pairing to the Néron–Tate height pairing on the Jacobian variety of X, and complete the result for curves using properties of heights on the Jacobian.

Next we prove the inequality part of Theorem 3.1 by induction on the dimension of X, using a Bertini-type theorem of Seidenberg [1950] to find sections which cut out nice subvarieties of X. Along the way we prove a Cauchy–Schwarz inequality for this intersection pairing. Theorem 3.2 and the equality part of Theorem 3.1 are then also proved by induction, where we again decompose into flat and vertical metrics and must show that the K/k-trace and image functors behave nicely when restricted to a subvariety. This is more difficult than the inequality, however. For the inequality, we write each metrized line bundle as a limit of model metrics, and prove the result for model metrics, thus getting the same inequality on their limit. We can write the same limit in the equality case, but we cannot assume that the same equality hypothesis holds for the model metrics, and must argue by other means. Finally, Theorem 3.3 is easily deduced from Theorem 3.1 and its proof.

Section 4 proves the application of our result to polarized algebraic dynamical systems. We first describe and prove the existence of admissible metrics for a given polarized algebraic dynamical system, which generalize flat metrics, and give rise to canonical heights defined by intersections. This transforms the rigidity statement into a statement comparing two different admissible adelic metrized line bundles, which is proved using the Hodge index theorem.

Section 5 gives corollaries of the main results proven here, and discusses what can still be said about preperiodic points over larger fields without the Northcott property.

2. Preliminaries

Here we introduce the definitions, basic properties, and lemmas which will be needed throughout the paper. The core theory used in this paper is built on local intersection theory as developed by Gubler [1998; 2007b], Chambert-Loir [2006], Chambert-Loir and Thuillier [2009], and Zhang [1995]. More generally, one can find an introduction to Arakelov theory in [Moriwaki 2014; Lang 1988; Soulé 1992].

2.1. ***Metrized line bundles over local fields.*** Let K be a complete non-Archimedean field with nontrivial absolute value $|\cdot|$. Denote the valuation ring of K by

$$K^\circ := \{a \in K : |a| \le 1\},$$

and its maximal ideal by

$$K^{\circ\circ} := \{a \in K : |a| < 1\},$$

so that $\widetilde{K} := K^\circ/K^{\circ\circ}$ is the residue field.

Let X be a variety over K and denote by X^{an} its Berkovich analytification [1990]. For $x \in X^{\mathrm{an}}$, write $K(x)$ for the residue field of x. A line bundle L on X has an analytification, denoted L^{an}, as a line bundle on X^{an}.

Definition 2.1. A *continuous metric* $\|\cdot\|$ on L consists of a $K(x)$-metric $\|\cdot\|_x$ on $L^{\mathrm{an}}(x)$ for every $x \in X^{\mathrm{an}}$, where this collection of metrics is continuous in the sense that for every rational section s of L, the map $X^{\mathrm{an}} \to \mathbb{R}$ defined by $x \mapsto \|s(x)\|_x$ is continuous away from the poles of s. We call L with a continuous metric a *metrized line bundle* and denote this by $\bar{L} = (L, \|\cdot\|)$. For a fixed line bundle L, limits of metrics are taken with respect to the topology induced by the supremum norm.

An important example of a continuous metric is a *model metric*: Let $\mathcal{X}$ be a model of X over K°, i.e., a projective, flat, finitely presented, integral scheme over $\operatorname{Spec} K^\circ$ whose generic fiber $\mathcal{X}_K$ is isomorphic to X, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ whose generic fiber $\mathcal{L}_K$ is isomorphic to L. Then we can define a continuous metric on L by specifying that for any trivialization $\mathcal{L}_{\mathcal{U}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{U}}$ on an open set $\mathcal{U} \subset \mathcal{X}$ given by a rational section ℓ, we have $\|\ell(x)\|_x = 1$ for any x reducing to $\mathcal{U}_{\widetilde{K}}$ in the reduction $\widetilde{X}$ over $\widetilde{K}$.

We now define some important properties and notation.

Definition 2.2. Let $\bar{L} = (L, \|\cdot\|)$ and $\overline{M}$ be metrized line bundles on X.

(1) A model metric is *nef* if it is given by a relatively nef line bundle on the corresponding model.

(2) Call both $\bar{L}$ and $\|\cdot\|$ *nef* if $\|\cdot\|$ is equal to a limit of nef model metrics.

(3) $\bar{L}$ is *arithmetically positive* if it is nef and L is ample.

(4) $\bar{L}$ is *integrable* if it can be written as $\bar{L} = \bar{L}_1 - \bar{L}_2$ with $\bar{L}_1$ and $\bar{L}_2$ nef.

(5) $\overline{M}$ is $\bar{L}$*-bounded* if there exists a positive integer m such that $m\bar{L} + \overline{M}$ and $m\bar{L} - \overline{M}$ are both nef.

(6) $\bar{L}$ is *vertical* if it is integrable and $L \cong \mathcal{O}_X$.

(7) $\bar{L}$ is *constant* if it is isometric to the pull-back of a metrized line bundle on $\operatorname{Spec} K$.

(8) $\widehat{\operatorname{Pic}}(X)$ is defined to be the group of isometry classes of integrable metrized line bundles.

Remark. When we say a line bundle is relatively ample or nef, we always mean with respect to the structure morphism; here $\mathcal{X} \to \operatorname{Spec} K^\circ$. A concise discussion of relative amplitude and nefness can be found in [Lazarsfeld 2004, Chapter 1.7].

We also have a local intersection theory for metrized line bundles on X, due to [Gubler 1998; 2007a], and to [Zhang 1995] when K has a discrete valuation. Let Z

be a d-dimensional cycle on X, let $\bar{L}_0, \dots, \bar{L}_d$ be integrable metrized line bundles on X, and $\ell_0, \dots, \ell_d$ rational sections of each, respectively, such that

$$\left(\bigcap_i |\mathrm{div}(\ell_i)|\right) \cap |Z| = \varnothing,$$

where $|Z|$ means the underlying topological space of the cycle Z. Then Z has a local height $\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z]$ with the following properties:

(1) The local height is linear in $\widehat{\mathrm{div}}(\ell_i)$ and Z.

(2) For fixed sections, it is continuous with respect to the metrics.

(3) When $\bar{L}_i$ has a model metric given by $\mathcal{L}_i$ on a common model $\mathcal{X}$, the height is given by classical intersections:

$$\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z] = \mathrm{div}_{\mathcal{X}}(\ell_0) \cdots \mathrm{div}_{\mathcal{X}}(\ell_d) \cdot [\mathcal{Z}],$$

where $\mathcal{Z}$ is the Zariski closure of Z in $\mathcal{X}$.

(4) If the support of $\mathrm{div}(\ell_0)$ contains no component of Z, there is a measure $c_1(\bar{L}_1) \cdots c_1(\bar{L}_d)\delta_Z$ on X^{an} due to [Chambert-Loir 2006] which allows us to compute $\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z]$ inductively as

$$\widehat{\mathrm{div}}(\ell_1) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [\mathrm{div}(\ell_0) \cdot Z] - \int_{X^{\mathrm{an}}} \log\|\ell_0(x)\|_x c_1(\bar{L}_1) \cdots c_1(\bar{L}_d)\, \delta_Z.$$

This notation is meant to suggest that $c_1(\bar{L}_i)$ should be thought of as the arithmetic version of the classical Chern form $c_1(L_i)$.

(5) If $\ell_0|_{Z_j}$ is constant and $c_1(L_1) \cdots c_1(L_d) \cdot [Z_j] = 0$ for every irreducible component Z_j of Z, then this pairing does not depend on the choice of sections, so we may simply write

$$\bar{L}_0 \cdots \bar{L}_d \cdot Z = \widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z].$$

When $Z = X$, we typically omit Z in all of the above notation.

By definition, every integrable metric can be written as a limit of model metrics (with respect to the supremum norm). Properties (3) and (4) above guarantee that intersections of integrable metrized line bundles are equal to the corresponding limits of intersections of models which approximate them.

2.2. ***Adelic metrized line bundles.*** We now move to the global theory, which is built from the theory of metrized line bundles over each localization, discussing first models and then adelic metrized line bundles. We return to the setting of the main theorems of this paper, where k is any algebraically closed field, B is a smooth projective curve over k, $K = k(B)$ is its function field, and $\pi : X \to \mathrm{Spec}(K)$ is a geometrically normal, geometrically integral, projective variety.

Let $\mathcal{X}$ be a model for X, meaning that $\mathcal{X} \to B$ is geometrically integral, projective, and flat, and the generic fiber $\mathcal{X}_K$ is isomorphic to X. Given a geometrically integral subvariety $\mathcal{Y}$ of dimension $d+1$ in $\mathcal{X}$ and line bundles $\mathcal{L}_0, \dots, \mathcal{L}_d$ on $\mathcal{X}$ each with a respective section $\ell_0, \dots, \ell_d$ such that their common support has empty intersection with $\mathcal{Y}_K$, the arithmetic intersection pairing on $\mathrm{Pic}(\mathcal{X})$ is defined locally as

$$\mathcal{L}_0 \cdots \mathcal{L}_d \cdot \mathcal{Y} := \widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [\mathcal{Y}] := \sum_{\nu} \big(\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [\mathcal{Y}]\big)_{\nu},$$

where ν ranges over the closed points (places) of B, and

$$\big(\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [\mathcal{Y}]\big)_{\nu}$$

means the local intersection number after base-change to the complete field K_ν. As the notation suggests, this does not depend on the choice of sections. Again we typically drop $\mathcal{Y}$ in the notation if $\mathcal{Y} = \mathcal{X}$, and when $\mathcal{X}$ is one-dimensional, we call $\widehat{\deg}(\mathcal{L}_0) := \mathcal{L}_0 \cdot \mathcal{X}$ the arithmetic degree of $\mathcal{L}_0$.

Remark. This arithmetic intersection theory for $\mathcal{X} \to B$ is equal to the classical intersection theory given by viewing $\mathcal{X}$ as a variety over the field k, but is written using the fibration so as to align notationally with Arakelov's arithmetic intersection theory [1974; 1975]. In the function field setting there are no Archimedean places to consider, as B is projective.

Given a line bundle L on X we call a line bundle $\mathcal{L}$ on $\mathcal{X}$ a model for L provided that $\mathcal{L}_K \cong L$. For each place ν of B, completing with respect to ν induces a model over K_ν° and a model metric $\|\cdot\|_{\mathcal{L},\nu}$ of $L_{K_\nu}^{\mathrm{an}}$ on $X_\nu^{\mathrm{an}} := X_{K_\nu}^{\mathrm{an}}$.

Definition 2.3. The collection $\|\cdot\|_{\mathcal{L},\mathbb{A}} = \{\|\cdot\|_{\mathcal{L},\nu}\}_\nu$ of continuous metrics for every place ν of B given by $(\mathcal{X}, \mathcal{L})$ is called a *model adelic metric* on L. More generally, an *adelic metric* $\|\cdot\|_{\mathbb{A}}$ on L is a collection of continuous metrics $\|\cdot\|_\nu$ of $L_{K_\nu}^{\mathrm{an}}$ on X_ν^{an} for every place ν, which agrees with some model adelic metric at all but finitely many places. A line bundle on X with an adelic metric is called an *adelic metrized line bundle*, and is denoted $\bar{L} = (L, \|\cdot\|_{\mathbb{A}})$. When the context is clear we will frequently drop adelic and simply write *metrized line bundle*. For a fixed line bundle L, limits of adelic metrics are taken with respect to the topology induced by $\max_\nu \|\cdot\|_{\mathrm{sup}}$, the maximum of the supremum norm on each fiber. Such a limit does not require fixing a single model $\mathcal{X}$.

We extend our local definitions of properties of metrized line bundles to the global case.

Definition 2.4. Let $\bar{L}$ be an adelic metrized line bundle.

(1) $\bar{L}$ is *nef* if it is equal to a limit of model metrics induced by nef line bundles on models of X.

(2) $\bar{L}$ is *integrable* if it can be written as $\bar{L} = \bar{L}_1 - \bar{L}_2$, where each $\bar{L}_i$ is nef.

(3) $\bar{L}$ is *arithmetically positive* if L is ample and $\bar{L} - \pi^*\overline{N}$ is nef for some adelic metrized line bundle $\overline{N}$ on Spec K with $\widehat{\deg}(\overline{N}) > 0$.

(4) $\overline{M}$ is $\bar{L}$*-bounded* if there exists a positive integer m such that $m\bar{L} + \overline{M}$ and $m\bar{L} - \overline{M}$ are both nef.

(5) $\bar{L}$ is *vertical* if it is integrable and $L \cong \mathcal{O}_X$

(6) $\bar{L}$ is *constant* if it is isometric to the pull-back of a metrized line bundle on Spec K.

(7) $\widehat{\text{Pic}}(X)$ is defined to be the group of isometry classes of integrable metrized line bundles.

Remark. In the definition of arithmetically positive, we have thus far only defined the arithmetic degree in the model case, but every adelic metrized line bundle in Spec K has a model metric, so we may use that definition. The definition is also made more general in the following material.

Remark. The definition of arithmetically positive is equivalent to requiring that L is ample and for every $\overline{N} \in \widehat{\text{Pic}}(K)$, there exists some positive integer m such that $m\bar{L} - \pi^*\overline{N}$ is nef. This means, in particular, that all of $\pi^*\widehat{\text{Pic}}(K)$ is $\bar{L}$-bounded for arithmetically positive $\bar{L}$.

Remark. To avoid confusion, note that the preceding definitions are not equivalent to requiring that the local property of the same name holds at every fiber. In fact, since relative amplitude (resp. nefness) holds if and only if the restriction to every fiber is ample (resp. nef), if a property holds in the global setting then the corresponding property holds locally at every place, but the converse is false. For example, if $\bar{L}_v$ is nef on X_v for every place, each $\bar{L}_v$ can be written as a limit of nef models $\mathcal{L}_{v,i}$ on $\mathcal{X}_{v,i}$, but it may not be possible to assemble these into global models $\mathcal{L}_i$ on models $\mathcal{X}_i$ of X.

Global intersections are defined similarly to the model case, except with the local metrics given explicitly by the adelic metric instead of induced by a model. Given a d-dimensional integral subvariety Z of X and integrable adelic metrized line bundles $\bar{L}_0, \ldots, \bar{L}_d$ with respective sections $\ell_0, \ldots, \ell_d$ with empty common intersection with Z, their intersection is

$$\begin{aligned}\bar{L}_0 \cdots \bar{L}_d \cdot Z &:= \widehat{\text{div}}(\ell_0) \cdots \widehat{\text{div}}(\ell_d) \cdot [Z] \\ &= \sum_v \widehat{\text{div}}(\ell_0|_{X_v}) \cdots \widehat{\text{div}}(\ell_d|_{X_v}) \cdot [Z|_{X_v}],\end{aligned}$$

where again this is independent of the choice of sections. Summing the local induction formula at each place produces a global induction formula: letting ℓ_0 be a rational section of $\bar{L}_0$ whose support does not contain Z,

$$\bar{L}_0 \cdots \bar{L}_d \cdot Z \\ = \bar{L}_1 \cdots \bar{L}_d \cdot (Z \cdot \operatorname{div}(\ell_0)) - \sum_\nu \int_{X_\nu^{\mathrm{an}}} \log\|\ell_0(x)\|_\nu c_1(\bar{L}_1, \nu) \cdots c_1(\bar{L}_d, \nu)\, \delta_Z|_{X_\nu}.$$

As before, we drop Z when $Z = X$, and when X is zero-dimensional, we call $\widehat{\deg}(\bar{L}_0) := \bar{L}_0 \cdot X$ the arithmetic degree of $\bar{L}_0$.

As in the local case, we can always compute intersections of adelic metrized line bundles by approximating them with model metrics and computing the limit of the corresponding arithmetic intersections of the models.

Definition 2.5. An adelic metrized line bundle $\overline{M}$ on X of dimension n is called *numerically trivial* if for any $\bar{L}_1, \dots, \bar{L}_n \in \widehat{\operatorname{Pic}}(X)$,

$$\overline{M} \cdot \bar{L}_1 \cdots \bar{L}_n = 0.$$

Call two adelic metrized line bundles *numerically equivalent* if their difference is numerically trivial.

2.3. ***Flat metrics.*** Adelic metrized line bundles with flat metrics form an especially nice class of adelic metrized line bundles. We will often be able to split a metrized line bundle into a bundle with a flat metric plus a vertical bundle, and then work with each of these separately, as flatness will tell us that these have trivial intersection.

Definition 2.6. Let X be a projective variety over a complete field K, and let $\bar{L}$ be a metrized line bundle on X. Then $\bar{L}$ is *flat* if for any morphism $f : C \to X$ of a projective curve over K into X, we have $c_1(f^*\bar{L}) = 0$ on the Berkovich analytification C^{an}. If now X is a projective variety over a global field and $\bar{L}$ an adelic metrized line bundle on X, call $\bar{L}$ flat provided it is flat at every place.

Note that if $\bar{L}$ is flat, L must be numerically trivial, as

$$\deg(L|_C) = \int_{C^{\mathrm{an}}} c_1(\bar{L}|_C) = 0.$$

Lemma 2.7. *Let L be a numerically trivial line bundle on a projective, normal variety X over a function field K. Then L has a flat metric, which is unique up to constant multiple.*

Remark. When X is a curve, this lemma has a much simpler proof using linear algebra; see for example [Hriljac 1985, Theorem 1.3]. If $\mathcal{X} \to B$ is a model for X and $\mathcal{X}_\nu$ is geometrically normal (for example, every place ν of good reduction), then the flat metric on L at ν is induced by the model metric corresponding to the closure in $\mathcal{X}$ of a divisor on X in the class of L.

To prove the lemma in general, the following related notion will be useful. We will show that it is equivalent to flatness for abelian varieties.

Definition 2.8. Let $\bar{L}$ be a metrized line bundle on an abelian variety A such that L is algebraically trivial. We call $\bar{L}$ *admissible* if $[2]^*\bar{L} \cong 2\bar{L}$.

Proof of Lemma 2.7. First, suppose X is an abelian variety. Then L is algebraically trivial, and we have an isomorphism $\phi : [2]^*L \cong 2L$. Take any metric $\|\cdot\|_1$ on L. Then Tate's limiting argument, as in [Zhang 1995, Theorem 2.2], shows that

$$\|\cdot\|_n := \phi^*[2]^*\|\cdot\|_{n-1}^{\frac{1}{2}}$$

converges to an admissible metric $\|\cdot\|_0$ on L, and that further this is the unique admissible metric on L up to constant multiples.

Let $C \to X$ be a smooth projective curve in X. After a translation and extension of K, we can fix a point $x_0 \in C(K)$ which maps to $0 \in X$. By the universal property of the Jacobian, $C \to X$ factors through the Jacobian map $C \to \mathrm{Jac}(C)$ taking $x_0 \to 0$, and the pullback of $(L, \|\cdot\|_0)$ to $\mathrm{Jac}(C)$ is also admissible. Then by [Gubler 2007b, Remark 3.14], $c_1(L, \|\cdot\|_0) = 0$, and hence L has a flat metric. By taking the tensor product of this metric with the inverse of any other flat metric on L, uniqueness up to constant multiple is reduced to showing that $\|1\|$ is constant for any flat metric on $\mathcal{O}_X$. Any two points on X are connected by a curve; let D be its normalization. Then $\|1\|$ is constant by the local Hodge index theorem [Yuan and Zhang 2017, Theorem 2.1] in dimension one at each place.

Now let X be an arbitrary projective, normal variety, choose a point $x_0 \in X(K)$ (extending K if necessary) and recall the Albanese map $i : X \to \mathrm{Alb}(X)$ taking x_0 to 0. Since L is numerically trivial, we may replace it by a multiple and assume it is algebraically trivial. Then L corresponds to a K point ξ of $\mathrm{Pic}^0_{\mathrm{red},X} = \mathrm{Alb}(X)^\vee$. By definition, L is (isomorphic to) the Poincaré bundle P on $\mathrm{Alb}(X) \times \mathrm{Alb}(X)^\vee$ restricted to $\mathrm{Alb}(X) \times \{\xi\}$, then pulled back through

$$i \times \mathrm{id} : X \times \mathrm{Alb}(X)^\vee \to \mathrm{Alb}(X) \times \mathrm{Alb}(X)^\vee.$$

$P|_{\mathrm{Alb}(X)\times\{\xi\}}$ is algebraically trivial, and hence has a flat metric. But the pullback of a flat metric is also flat, so this defines a flat metric for L. □

The reason we care about flat metrics is shown by Lemma 2.9 and Corollary 2.10:

Lemma 2.9. *Let K be a complete non-Archimedean field, and $X \to \operatorname{Spec} K$ a geometrically connected, geometrically normal, projective variety of dimension n, with a flat metrized line bundle $\overline{M}$. Then given any integrable metrized line bundles $\bar{L}_1, \dots, \bar{L}_{n-1}$ on X,*

$$c_1(\overline{M})c_1(\bar{L}_1)\cdots c_1(\bar{L}_{n-1}) = 0.$$

Proof. We show that

$$\int_{X^{\mathrm{an}}} \log\|\ell_n(x)\|_x c_1(\overline{M}) \cdot c_1(\bar{L}_1) \cdots c_1(\bar{L}_{n-1}) = 0$$

for every section ℓ_n of any metrized line bundle $\bar{L}_n$. Proceed by induction on n.

Since any integral metrized line bundle can be written as a difference of arithmetically positive metrized line bundles and the measure is additive with respect to the metrized line bundles, we may assume that $\bar{L}_{n-1}$ is arithmetically positive without loss of generality. Further, by approximation, it suffices to treat the case where $\bar{L}_{n-1}$ is a model metric, induced by some ample line bundle $\mathcal{L}$ on a model $\mathcal{X}$ for X. By Seidenberg's Bertini theorem [Seidenberg 1950, Theorem 7'], $\mathcal{L}$ has a section s which cuts out a horizontal, geometrically integral, normal subvariety $\mathcal{Y}$. After base changing to a finite extension K' of K, we may assume that this subvariety is geometrically normal. Since this extension merely scales the intersection number by $[K':K]$ it has no effect on the proof of this lemma. Let Y be the generic fiber of $\mathcal{Y}$, and let Z be $\mathrm{div}(\ell_n)$ restricted to Y.

We compute an intersection product in two different ways. First,

$$\begin{aligned}\overline{M}\cdot\bar{L}_1\cdots\bar{L}_n &= \overline{M}|_Y\cdot\bar{L}_1|_Y\cdots\bar{L}_{n-2}|_Y\cdot\bar{L}_n|_Y\\ &= \overline{M}|_Z\cdot\bar{L}_1|_Z\cdots\bar{L}_{n-2}|_Z-\int_{X^{\mathrm{an}}}\log\|\ell_n(x)\|_x c_1(\overline{M})c_1(\bar{L}_1)\cdots c_1(\bar{L}_{n-2})\,\delta_Y\\ &= \overline{M}|_Z\cdot\bar{L}_1|_Z\cdots\bar{L}_{n-2}|_Z,\end{aligned}$$

where the first equality follows from $\mathcal{Y}$ being horizontal, the second from the induction formula for local intersection numbers, and the third from the induction hypothesis. We now compute this in a different order:

$$\begin{aligned}&\overline{M}\cdot\bar{L}_1\cdots\bar{L}_n\\ &\quad= \overline{M}\cdot\bar{L}_1\cdots\bar{L}_{n-1}\cdot(\mathrm{div}(\ell_n))-\int_{X^{\mathrm{an}}}\log\|\ell_n(x)\|_x c_1(\overline{M})c_1(\bar{L}_1)\cdots c_1(\bar{L}_{n-1})\\ &\quad= \overline{M}|_Z\cdot\bar{L}_1|_Z\cdots\bar{L}_{n-2}|_Z-\int_{X^{\mathrm{an}}}\log\|\ell_n(x)\|_x c_1(\overline{M})c_1(\bar{L}_1)\cdots c_1(\bar{L}_{n-1}),\end{aligned}$$

where now the first inequality follows from the induction formula, and the second from $\mathcal{Y}$ being horizontal. Comparing the two equalities completes the proof. □

Corollary 2.10. *Let $\overline{M}$ be flat, $\overline{N}$ be vertical, and $\bar{L}_1,\dots,\bar{L}_{n-1}$ be any integrable adelic metrized line bundles. Then*

$$\overline{M}\cdot\overline{N}\cdot\bar{L}_1\cdots\bar{L}_{n-1}=0.$$

Proof. Since $\overline{N}$ is vertical, $N=\mathcal{O}_X$. Compute this intersection using the induction formula with the section $s=1$ of $\mathcal{O}_X$:

$$\begin{aligned}&\overline{M}\cdot\overline{N}\cdot\bar{L}_1\cdots\bar{L}_{n-1}\\ &\qquad= \overline{M}\cdot\bar{L}_1\cdots\bar{L}_{n-1}\cdot(\mathrm{div}(s))-\int_{X^{\mathrm{an}}}\log\|1\|_x c_1(\overline{M})\cdot c_1(\bar{L}_1)\cdots c_1(\bar{L}_{n-1}).\end{aligned}$$

The first term is zero since $\mathrm{div}(s)$ is empty, and the integral is zero by Lemma 2.9. □

2.4. ***Heights of points and subvarieties.*** An important application of the intersection theory of adelic metrized line bundles is to define height functions.

Definition 2.11. Let $\overline{N} \in \widehat{\mathrm{Pic}}(X)$. We define the *height* of a point $x \in X(\overline{K})$ by

$$h_{\overline{N}}(x) := \frac{1}{[K(x):K]} \overline{N} \cdot \tilde{x},$$

where $\tilde{x}$ is the image of x in X via $X_{K(x)} \to X_K = X$.

Remark. The heights produced by this definition are Weil heights, which can be defined without intersection theory [Bombieri and Gubler 2006; Call and Silverman 1993], but we use the above definition as it generalizes to define heights of subvarieties.

Definition 2.12. Let $d = \dim Y$. The *height* of Y with respect to $\overline{N}$ is defined to be

$$h_{\overline{N}}(Y) := \frac{(\overline{N}|_Y)^{d+1}}{(d+1)(N|_Y)^d}$$

and the *essential minimum* of Y with respect to $\overline{N}$ is

$$\lambda_1(Y, \overline{N}) := \sup_{\substack{U \subset Y \\ \text{open}}} \Big(\inf_{x \in U(\overline{K})} h_{\overline{N}|_Y}(x) \Big).$$

By the successive minima of Zhang [1995, Theorem 1.1], and proven in the function field setting by Gubler [2007a, Theorem 4.1], we can state the following.

Proposition 2.13. *When $\overline{N}$ is nef*,

$$\lambda_1(Y, \overline{N}) \geq h_{\overline{N}}(Y) \geq 0.$$

2.5. ***Abelian varieties and Chow's K/k-trace and image.*** Proofs of the existence and properties of the trace and image can be found in [Lang 1983] and [Conrad 2006]. Let A be an abelian variety defined over K. The K/k-image $(\mathrm{Im}_{K/k}(A), \lambda)$ consists of an abelian variety $\mathrm{Im}_{K/k}(A)$ over k and a surjective morphism

$$\lambda : A \to \mathrm{Im}_{K/k}(A)_K$$

with the following universal property: If V is an abelian variety defined over k, and $\phi : A \to V_K$ is a morphism, then ϕ factors through λ. Provided the fields K and k are clear, we will often drop the K/k subscript and just write $\mathrm{Im}(A)$.

The K/k-trace is $(\mathrm{Tr}_{K/k}(A), \tau)$ where $\mathrm{Tr}_{K/k}(A)$ is an abelian variety over k, and

$$\tau : \mathrm{Tr}_{K/k}(A)_K \to A$$

is universal among all morphisms from k-abelian varieties to A. Again we will often drop the K/k when the fields are unambiguous. The image can be thought of as the largest quotient of A that can be defined over k and the trace as the largest abelian subvariety that can be defined over k. This heuristic is literally true in characteristic zero, but in positive characteristic the trace map may have an infinitesimal kernel; see [Conrad 2006, Section 6].

These constructions are dual to each other in the sense that

$$\mathrm{Tr}(A^\vee) = \mathrm{Im}(A)^\vee,$$

and the image and trace are isogenous via the composition $\lambda \circ \tau$ (descended to the k-varieties).

Given a morphism of abelian varieties $f : A \to B$, we get morphisms $f_{\mathrm{Tr}} : \mathrm{Tr}(A) \to \mathrm{Tr}(B)$ and $f_{\mathrm{Im}} : \mathrm{Im}(A) \to \mathrm{Im}(B)$ commuting with τ and λ.

Now suppose X is a geometrically normal projective variety over K of dimension n, and assume that K is large enough so that $X(K)$ is nonempty. We write $\mathbf{Pic}_X$ for the Picard scheme of X, representing the Picard functor on X. This scheme exists (i.e., the Picard functor is representable), and its reduced neutral component, denoted $\mathbf{Pic}^0_{\mathrm{red},X}$, is an abelian variety [Kleiman 2005; Grothendieck 1962, Lecture 236]. Note that we do require the reduction, as $\mathbf{Pic}^0_X$ may fail to be reduced in positive characteristic. Write $\mathrm{Pic}(X)$ and $\mathrm{Pic}^0(X)$ for the abelian groups of K points of $\mathbf{Pic}_X$ and $\mathbf{Pic}^0_{\mathrm{red},X}$, respectively.

We can then define Alb_X, called the Albanese variety of X, to be the abelian variety dual to $\mathbf{Pic}^0_{\mathrm{red},X}$. Choosing a point $x_0 \in X(K)$ fixes an Albanese morphism

$$\iota : X \to \mathrm{Alb}_X$$

taking x_0 to 0, and then (Alb_X, ι) uniquely satisfies the Albanese universal property: any morphism from X to an abelian variety taking x_0 to zero must factor through ι [Wittenberg 2008].

We now have the language to differentiate between metrized line bundles defined over the constant field k and those which are not. Define a group homomorphism

$$\widehat{\tau}_{K/k} : \mathrm{Tr}_{K/k}(\mathbf{Pic}^0_{\mathrm{red},X})(k) \to \widehat{\mathrm{Pic}}(X)$$

as follows. First, by the duality of the K/k-trace and image,

$$\mathrm{Tr}_{K/k}(\mathbf{Pic}^0_{\mathrm{red},X})(k) = \mathrm{Pic}^0(\mathrm{Im}_{K/k}(\mathrm{Alb}_X)).$$

Then we can map

$$\mathrm{Pic}^0(\mathrm{Im}_{K/k}(\mathrm{Alb}_X)) \hookrightarrow \mathrm{Pic}(\mathrm{Im}_{K/k}(\mathrm{Alb}_X)) \to \mathrm{Pic}(\mathrm{Im}_{K/k}(\mathrm{Alb}_X) \times_k B),$$

where the map on the right is the pullback of projection onto the first factor. Since $\mathrm{Im}_{K/k}(\mathrm{Alb}_X)$ is defined over k, the fibered product $\mathrm{Im}_{K/k}(\mathrm{Alb}_X) \times_k B$ is a model for $\mathrm{Im}_{K/k}(\mathrm{Alb}_X) \times_k K$, and thus we get a map

$$\mathrm{Pic}(\mathrm{Im}_{K/k}(\mathrm{Alb}_X) \times_k B) \to \widehat{\mathrm{Pic}}(\mathrm{Im}_{K/k}(\mathrm{Alb}_X)_K)$$

given by taking model metrics. Finally, X maps to $\mathrm{Im}_{K/k}(\mathrm{Alb}_X)_K$ via the Albanese map followed by the image map, and pulling this back gives

$$\widehat{\mathrm{Pic}}(\mathrm{Im}_{K/k}(\mathrm{Alb}_X)_K) \to \widehat{\mathrm{Pic}}(X).$$

We can thus define $\widehat{\tau}_{K/k}$ as the composition of the above maps. While it took several steps to formally define $\widehat{\tau}_{K/k}$, it is very natural; if we define

$$\phi : \widehat{\mathrm{Pic}}(X) \to \mathrm{Pic}(X)$$

by forgetting the metric, then

$$\phi \circ \widehat{\tau}_{K/k} = \tau_{K/k}$$

is the K/k-trace morphism (on field-valued points), and the image of this composition lands in $\mathrm{Pic}^0(X)$. To simplify notation, we write $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))$ to mean the image of $\widehat{\tau}_{K/k}$ in $\widehat{\mathrm{Pic}}(X)$. By construction $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))$ is flat and numerically trivial, as on every fiber X_ν this group restricts to $\mathrm{Tr}_{K/k}(\mathbf{Pic}^0_{\mathrm{red},X})_{K_\nu}(k)$, which is algebraically trivial, and so in particular $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))$ has zero intersection with every vertical metrized line bundle.

3. Proof of Hodge index theorem

3.1. ***Statement of results.*** Let k be any algebraically closed field, let B be a smooth projective curve over that field, and let $K = k(B)$ be the corresponding function field. Let X be a geometrically normal projective variety over K of dimension n, and assume that K is large enough so that $X(K)$ is nonempty. Then choosing a point $x \in X(K)$ we may fix an Albanese morphism $\iota : X \to \mathrm{Alb}_X$. We impose these conditions on X as well as this choice of Albanese morphism throughout the rest of the paper.

We can now state our main theorem:

Theorem 3.1 (arithmetic Hodge index theorem for function fields). *Let $\overline{M}$ be an integrable adelic $\mathbb{Q}$-line bundle on X and $\bar{L}_1, \ldots, \bar{L}_{n-1}$ nef adelic $\mathbb{Q}$-line bundles on X. Suppose if $n \geq 2$ that $M \cdot L_1 \ldots L_{n-1} = 0$ and each L_i is big, or that* $\deg M = 0$ *if $n = 1$. Then*

$$\overline{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} \leq 0.$$

Further, if every $\bar{L}_i$ is arithmetically positive, and $\overline{M}$ is $\bar{L}_i$-bounded for every i, then

$$\overline{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} = 0$$

if and only if

$$\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}.$$

When $n = 1$ so that X is a curve,

$$\overline{M}^2 = -2h_{\mathrm{NT}}(M),$$

where h_{NT} is the Néron–Tate height on the Jacobian of X.

Remark. When k is the algebraic closure of a finite field, $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$ is zero, since all elements are torsion.

Remark. For the "if" direction of the equality, note that all of

$$\pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$$

is $\bar{L}$ bounded for any arithmetically positive $\bar{L}$. This follows from the remark after Definition 2.4 for $\pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$, and from the fact that $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$ is numerically trivial by construction.

Call a metrized line bundle $\overline{M}$ on X numerically trivial if

$$\overline{M} \cdot \bar{L}_1 \cdots \bar{L}_n = 0$$

for every choice of metrized line bundles $\bar{L}_1, \ldots, \bar{L}_n$. The classical Hodge index theorem says that the only divisors on a surface with zero self-intersection are the numerically trivial divisors. We show that that is nearly, but not quite the case here:

Theorem 3.2. *The following are equivalent for* $\overline{M} \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$:

(1) $\overline{M}$ *is numerically trivial.*

(2) *The height* $h_{\overline{M}}$ *is identically zero on* $X(\bar{K})$.

(3) $\overline{M} \in \pi^*\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$, *where* $\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}}$ *is defined to be the elements of* $\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$ *with arithmetic degree zero.*

Define $\mathrm{Pic}^\tau(X)$ to be the group of isomorphism classes of numerically trivial line bundles on X. We define a pairing on $\mathrm{Pic}^\tau(X)$ to give an $\mathbb{R}$-linear version of Theorem 3.1. Let $M, N \in \mathrm{Pic}^\tau(X)_{\mathbb{R}}$, and let $L_1, \ldots, L_{n-1} \in \mathrm{Pic}(X)_{\mathbb{Q}}$ be nef. Then define a pairing by

$$\langle M, N\rangle_{L_1,\ldots,L_{n-1}} := \overline{M} \cdot \overline{N} \cdot \bar{L}_1 \cdots \bar{L}_{n-1},$$

using any choice of flat metrics on M and N, and any choice of metrics on L_i. By Lemma 5.19 of [Yuan and Zhang 2017], (proven as a simple consequence of Lemma 2.9 here) this pairing does not depend on the choice of metric.

Theorem 3.3. *For any* $M \in \mathrm{Pic}^\tau(X)_{\mathbb{R}}$ *and nef* $L_1, \ldots, L_{n-1} \in \mathrm{Pic}(X)_{\mathbb{Q}}$,

$$\langle M, M\rangle_{L_1,\ldots,L_{n-1}} \leq 0.$$

Further, if every L_i *is ample, then equality holds if and only if* $M \in \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{R}}$. *When* X *is a curve,*

$$\langle \cdot, \cdot \rangle = -2\langle \cdot, \cdot \rangle_{\mathrm{NT}},$$

where $\langle \cdot, \cdot \rangle_{\mathrm{NT}}$ *is the Néron–Tate height pairing on the Jacobian of* X.

These results are proven over the next three subsections, with the bulk of the work going into proving Theorem 3.1, with Theorems 3.2 and 3.3 following as corollaries.

3.2. ***Curves.*** We begin when X is a curve. Here we can work directly in $\widehat{\mathrm{Pic}}(X)$ as opposed to $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$. Then the theorem discusses the self-intersection $\overline{M}^2$ when $\deg M = 0$. By Lemma 2.7, M has a flat metric $\overline{M}_0 = (M, \|\cdot\|_0)$.

Let $\overline{N}$ be the vertical line bundle defined by

$$\overline{M} = \overline{M}_0 + \overline{N}.$$

Since $\overline{M}_0$ is flat, $\overline{M}_0 \cdot \overline{N} = 0$ so that

$$\overline{M}^2 = \overline{M}_0^2 + \overline{N}^2 = \overline{M}_0^2 + \sum_{\nu} \overline{N}_{\nu}^2,$$

where $\overline{N}_\nu$ is the restriction of $\overline{N}$ to $X_\nu := X \otimes_K K_\nu$ for each place ν of K (i.e., each closed point of B). Now $\overline{N}_\nu^2 \leq 0$ with equality if and only if $\overline{N}_\nu$ is constant by the local Hodge index theorem [Yuan and Zhang 2017, Theorem 2.1]. Hence,

$$\sum_{\nu} \overline{N}_{\nu}^2 \leq 0,$$

with equality if and only if $\overline{N} \in \pi^* \widehat{\mathrm{Pic}}(K)$.

Next, we consider $\overline{M}_0^2$. Since M has degree zero, it corresponds naturally to a K-point on the Jacobian, Jac_X, of X. Given any two points $P, Q \in \mathrm{Jac}_X(K)$, let L_P and L_Q be the corresponding algebraically trivial line bundles on X. These each have a flat metric, $\bar{L}_P$ and $\bar{L}_Q$, respectively, unique up to constant metric, and thus we get a well-defined symmetric bilinear pairing

$$(P, Q) \mapsto -\bar{L}_P \cdot \bar{L}_Q$$

on $\mathrm{Jac}_X(K)$, as the intersection does not depend on the choice of flat metric. As is noted in [Faltings 1984] and [Hriljac 1985] in the arithmetic setting, this pairing is exactly the Néron–Tate height pairing. Then the Shioda–Tate theorem [Shioda 1999, Theorem 7] states that this pairing descends to a positive definite pairing on $\mathrm{Jac}_X(K)_{\mathbb{Q}} / \mathrm{Tr}_{K/k}(\mathrm{Jac}_X)(k)$, and

$$\overline{M}^2 = -2h_{\mathrm{NT}}(M).$$

Since $\widehat{\tau}_{K/k}$ produces elements of $\widehat{\mathrm{Pic}}(X)$ with flat metrics, our pairing on

$$\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))$$

matches that considered by Shioda, and this completes the proof of Theorem 3.1 in dimension one. Since Shioda's pairing extends $\mathbb{R}$-linearly, this also proves Theorem 3.3 in dimension one.

We now turn to Theorem 3.2 in dimension one. (1) $\Rightarrow$ (2), as heights are defined using intersections. In particular, fix any model $\mathcal{X} \to B$ for X, and for $x \in X(\overline{K})$ let $\bar{L}_x$ be the model metric corresponding to the Zariski closure of $\tilde{x}$ in $\mathcal{X}$. Then for $\overline{M} \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ the height $h_{\overline{M}}(x)$ is just $\overline{M} \cdot \bar{L}_x / [K(x) : K] = 0$.

Now suppose $h_{\overline{M}}$ is identically zero on $X(\overline{K})$. Then $\deg M = 0$ as otherwise M or $-M$ is ample and $\overline{M}$ defines an unbounded Weil height. Suppose $\bar{L}$ is a model metric induced by a very ample line bundle $\mathcal{L}$ on a model $\mathcal{X} \to B$. Then, extending K if necessary, by Seidenberg's Bertini theorem [Seidenberg 1950], $\mathcal{L}$ has a section which cuts out a normal, irreducible horizontal subvariety of $\mathcal{X}$. Since $\mathcal{X}$ is a surface, this is just the closure of a point $x_0 \in X(\overline{K})$ on the generic fiber. Then

$$\overline{M} \cdot \bar{L} = \overline{M}|_{x_0} = h_{\overline{M}}(x_0) = 0.$$

Since $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ is generated by linear combinations of very ample line bundles and $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ consists of limits of such, this proves (2) ⇒ (1).

If $\overline{M}$ is numerically trivial, then $\overline{M} \in \widehat{\mathrm{Pic}}(K) + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$, as necessarily $\overline{M}^2 = 0$. If $\overline{M}_1 \in \widehat{\mathrm{Pic}}(K)$, then $h_{\overline{M}_1}$ is constant, with value equal to $\widehat{\deg}(\overline{M}_1)$. Thus (2) ⇒ (3).

Finally, if $\overline{M} \in \pi^*\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$ then $\overline{M}$ is numerically trivial, as $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$ is numerically trivial on every fiber by construction, and the intersection of $\overline{N} \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$ with $\bar{L} \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ is the arithmetic degree of $\overline{N}$ times the degree of L. Then (3) ⇒ (1), completing the proof.

3.3. ***Inequality and Cauchy–Schwarz.*** We will now prove the inequality part of Theorem 3.1 by induction on $n = \dim X$, and get a version of the Cauchy–Schwarz inequality as a corollary. As in [Yuan and Zhang 2017, Section 3.3, Assumption (2)], we may assume that each $\bar{L}_i$ is arithmetically positive (instead of just big) by a limiting argument. Thus we may assume L_i is ample.

Since $\overline{M}$ and each $\bar{L}_i$ can be approximated by model metrics, it suffices to prove

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} \leq 0,$$

under the assumption that $\mathcal{M}$ and every $\mathcal{L}_i$ are line bundles on a model $\mathcal{X}$ for X, that $\mathcal{L}_i$ is ample with respect to k, and that the intersection $\mathcal{M}_K \cdot (\mathcal{L}_1)_K \cdots (\mathcal{L}_{n-1})_K$ on the generic fiber is zero.

Replacing $\mathcal{L}_1$ by a positive tensor power if necessary, we may assume it is very ample. Then by a Bertini-type result of Seidenberg [1950, Theorem 7'], a generic section of $\mathcal{L}_1$ cuts out an integral normal subvariety $\mathcal{Y}$ of $\mathcal{X}$, and we may further stipulate that $\mathcal{Y}$ is horizontal. Then

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = \mathcal{M}|_{\mathcal{Y}}^2 \cdot \mathcal{L}_1|_{\mathcal{Y}} \cdots \mathcal{L}_{n-2}|_{\mathcal{Y}}.$$

This reduces the problem to a lower dimension, but we require that $\mathcal{Y}_K$ have a K point to conclude the result by induction. This is certainly true if we replace K with a finite extension K', or equivalently replace B with a finite cover. Since intersection numbers simply scale by $[K' : K]$ and the subgroup $\widehat{\mathrm{Pic}}(K) + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))$ is equal to $\widehat{\mathrm{Pic}}(K') + \mathrm{Tr}_{K'/k}(\mathrm{Pic}^0(X_{K'}))$ intersected with $\widehat{\mathrm{Pic}}(X)$, such a base change is permissible.

Given $M \in \mathrm{Pic}^{\tau}(X)_{\mathbb{R}}$, we can write it as an $\mathbb{R}$-linear combination of numerically trivial line bundles on X, and each has a flat metric by Lemma 2.7. Then the inequality of Theorem 3.1 immediately implies the inequality of Theorem 3.3 when every L_i is big. If L_i is merely nef, choose any ample line bundle A and $\epsilon > 0$, and then $L_{i_\epsilon} := L_i + \epsilon A$ is big. Thus the inequality holds with $L_{i,\epsilon}$ replacing L_i, and taking the limit as $\epsilon \to 0$, it holds in general.

As a corollary, we have the following Cauchy–Schwarz inequality:

Corollary 3.4. *Let $\overline{M}$ and $\overline{N}$ be two integral adelic line bundles on X, and let $\bar{L}_1, \ldots, \bar{L}_{n-1}$ be nef adelic line bundles on X such that*

$$M \cdot L_1 \cdots L_{n-1} = N \cdot L_1 \cdots L_{n-1} = 0.$$

Then

$$(\overline{M} \cdot \overline{N} \cdot \bar{L}_1 \cdots \bar{L}_{n-1})^2 \le (\overline{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1})(\overline{N}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1}).$$

Proof. This follows from the inequality part of the Hodge index theorem proven above, and from the standard proof of the Cauchy–Schwarz inequality using the (negative semidefinite) inner product

$$\langle M, N\rangle_{\bar{L}_1,\ldots,\bar{L}_{n-1}} := \overline{M} \cdot \overline{N} \cdot \bar{L}_1 \cdots \bar{L}_{n-1}. \qquad \square$$

3.4. *Equality.* We now proceed to the equality part of Theorem 3.1. To prove the "if" direction, suppose $\overline{M} \in \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$. Then $\overline{M}$ is numerically trivial, as it is numerically trivial on every fiber by construction. If $\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$, then $\overline{M}^2$ consists of self-intersections of whole fibers, which are equal to zero.

To prove "only if," suppose that each $\bar{L}_i$ is arithmetically positive, that $\overline{M}$ is $\bar{L}_i$-bounded for all i, and that

$$\overline{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} = 0.$$

Note that as a consequence of the Cauchy–Schwarz inequality above, the set of metrized line bundles $\overline{M}$ satisfying these properties forms a group via tensor products.

By [Yuan and Zhang 2017, Lemma 3.7] (this requires that $\bar{L}_i$ is arithmetically positive), M is numerically trivial on X. Thus it has a flat metric; let $\overline{M}_0 = (M, \|\cdot\|)$ be flat. Then, similar to the curve case, $\overline{N} := \overline{M} - \overline{M}_0$ is vertical, and

$$\overline{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} = \overline{M}_0^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} + \overline{N}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1}.$$

The inequality part of the Hodge index theorem guarantees that both terms on the right are zero, and then by the local Hodge index theorem at every place occurring in $\overline{N}$, we have $\overline{N} \in \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$. Hence we are reduced to proving the statement in the flat metric case $\overline{M} = \overline{M}_0$.

We again replace $\bar{L}_1$ by a positive multiple to assume that L_1 is very ample, then apply Seidenberg's Bertini theorem to conclude that $(L_1)_{\overline{K}}$ has a section s which cuts out an integral, normal subvariety Y. Such Y is defined over some finite extension of K'/K, and thus after a base change from K to K', we may assume that L_1 has a section which cuts out a geometrically integral and geometrically normal subvariety Y. As in the proof of the inequality, this finite extension merely scales the intersection numbers by a positive factor. We thus continue writing K, assuming it has been made large enough, to avoid excessive additional notation.

Lemma 3.5. *If $\overline{M}$ is flat, and Y is a geometrically normal subvariety of X, then*

$$\overline{M}|_Y^2 \cdot \bar{L}_2|_Y \cdots \bar{L}_{n-1}|_Y = 0.$$

Proof. By the induction formula of Chambert-Loir [2006], recalled in Section 2.1,

$$\overline{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} = \overline{M}|_Y^2 \cdot \bar{L}_2|_Y \cdots \bar{L}_{n-1}|_Y - \sum_v \int_{X_v^{\mathrm{an}}} \log\|s\|_v c_1(\overline{M})^2 c_1(\bar{L}_2) \cdots c_1(\bar{L}_{n-1}).$$

Since $\overline{M}$ is flat, all the integrals are zero. □

Thus we may assume

$$\overline{M}|_Y \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(Y))_{\mathbb{Q}}$$

by induction.

Write $\overline{M}|_Y = \overline{M}' + \pi^*\overline{M}_1$, with $\overline{M}' \in \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(Y))_{\mathbb{Q}}$ and $\overline{M}_1 \in \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$. Then define $\overline{M}_2 = \overline{M} - \overline{\pi}^*\overline{M}_1$. Since M is numerically trivial, replacing $\overline{M}$ by a positive integer multiple if necessary, we may further assume M is algebraically trivial, and then that π^*M_1, $M_2 \in \mathrm{Pic}^0(X)$.

As noted earlier, if we drop the metric structure the map $\widehat{\tau}$ is simply the K/k-trace map on field-valued points. The following lemma then proves that $M_2|_Y = M'$ lifts via the pullback of $Y \hookrightarrow X$ to an element of $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$.

Lemma 3.6. *Let $f : A \to B$ be a morphism of abelian varieties defined over K. In the commutative diagram*

$$\begin{array}{ccc} \mathrm{Tr}(A)(k)_{\mathbb{Q}} & \xrightarrow{\tau_A} & A(K)_{\mathbb{Q}} \\ \downarrow{\scriptstyle f_{\mathrm{Tr}}} & & \downarrow{\scriptstyle f} \\ \mathrm{Tr}(B)(k)_{\mathbb{Q}} & \xrightarrow{\tau_B} & B(K)_{\mathbb{Q}} \end{array}$$

$(f \circ \tau_A)(\mathrm{Tr}(A)(k)_{\mathbb{Q}})$ *is equal to* $f(A(K)_{\mathbb{Q}}) \cap \tau_B(\mathrm{Tr}(B)(k)_{\mathbb{Q}})$.

Proof. To shorten notation, we will drop writing the map τ_A and consider $\mathrm{Tr}(A)(k)$ directly as a subgroup of $A(K)$ (and similarly for B). First reduce to the case where f is surjective: let B' be the image of f, an abelian subvariety of B. By Poincaré reducibility, B is isogenous to $B' \times B''$, for some abelian variety B''. Then $\mathrm{Tr}(B)$ is isogenous to $\mathrm{Tr}(B') \times \mathrm{Tr}(B'')$, and the intersection of $\mathrm{Tr}(B')(k) \times \mathrm{Tr}(B'')(k)$ with $B'(K)$ is just $\mathrm{Tr}(B')(k)$.

Now assume f is surjective. By [Conrad 2006, Theorem 6.4], $\mathrm{Tr}(A)_K$ is isogenous to an abelian subvariety $A' \subset A$ such that $\mathrm{Tr}(A) \cong \mathrm{Tr}(A')$ and $\mathrm{Tr}(A/A') = 0$. Similarly, B has an abelian subvariety B' with the same properties. Composing with these isogenies, we get a surjection

$$\mathrm{Tr}(A)_K \times (A/A') \twoheadrightarrow \mathrm{Tr}(B)_K \times (B/B')$$

where the map on the first component is f descended to the traces. Now consider the map $A/A' \to \mathrm{Tr}(B)_K$ obtained from the above map composed with projection onto the first component. This map must factor through $\mathrm{Im}(A/A')_K$, which is trivial as $\mathrm{Im}(A/A')$ is isogenous to $\mathrm{Tr}(A/A') = 0$. Thus $\mathrm{Tr}(A)_K \to \mathrm{Tr}(B)_K$ is surjective, and we get a surjection $\mathrm{Tr}(A)(k) \twoheadrightarrow \mathrm{Tr}(B)(k)$, proving the lemma. □

Hence we may lift $\overline{M}_2|_Y$ to an element $\overline{M}'_2 \in \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$, and we must have

$$\overline{M}_2 - \overline{M}'_2 \in \ker\bigl(\widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{Pic}}(Y)\bigr).$$

Since $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ has finite kernel [Kleiman 2005, Remark 9.5.8], replacing M with a positive integer multiple, we may assume $M_2 - M'_2 = \mathcal{O}_X$ and thus $\overline{M}_2 - \overline{M}'_2$ is vertical. Additionally, by the Cauchy–Schwarz inequality, Corollary 3.4,

$$(\overline{M}_2 - \overline{M}'_2)^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} = (\overline{M} - \pi^*\overline{M}_1 - \overline{M}'_2)^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} = 0,$$

so that by the local Hodge index theorem the metric must be constant at each place and $\overline{M}_2 - \overline{M}'_2 \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$. Note that the local Hodge index theorem requires that $\overline{M}_2 - \overline{M}'_2$ be $\bar{L}_i$-bounded, but this holds, as $\overline{M}$ is $\bar{L}_i$-bounded by hypothesis, and all of $\pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$ is $\bar{L}_i$-bounded as well. This means that

$$\overline{M} = (\pi^*\overline{M}_1 + \overline{M}_2 - \overline{M}'_2) + \overline{M}'_2 \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}.$$

This proves that when $\overline{M}$ is $\bar{L}_i$-bounded and $\bar{L}_i$ is arithmetically positive for all i, then

$$\overline{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} = 0$$

if and only if $\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$, which completes the proof of Theorem 3.1.

The inequality part of Theorem 3.3 is implied immediately by the inequality of Theorem 3.1 provided each L_i is big. To accomplish this, chose an ample line bundle A on X, and define $L_{i,\epsilon} := L_i + \epsilon A$ for $\epsilon > 0$. Then extending these to nef metrics $\bar{L}_{i,\epsilon}$, we have

$$\langle M, M\rangle_{L_{1,\epsilon},\dots,L_{n-1,\epsilon}} = \overline{M}^2 \cdot \bar{L}_{1,\epsilon} \cdots \bar{L}_{n-1,\epsilon} \leq 0,$$

and the result follows letting $\epsilon \to 0$.

To prove the equality of Theorem 3.3, again split $M \in \mathrm{Pic}^\tau(X)_{\mathbb{R}}$ into an $\mathbb{R}$-linear combination of numerically trivial line bundles. Using the inequality, the equality can be proven for each of these individually. L_1 is ample, so Lemma 3.5 applies, and then by the induction hypothesis $M|_Y \in \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(Y))_{\mathbb{R}}$. Then by Lemma 3.6 we conclude $M \in \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{R}}$.

Finally, we prove Theorem 3.2. If $\overline{M}$ is numerically trivial it is flat by Theorem 3.1, and then its restriction to any geometrically normal subvariety is also numerically trivial by the proof of Lemma 3.5. Thus (1) $\Rightarrow$ (2) follows from the dimension one case, as we can compute the height of a point on any curve passing through that point.

To show (2) $\Rightarrow$ (3), assume $h_{\overline{M}}$ is trivial on $X(\overline{K})$, and chose a curve $C \subset X$. Since the height is trivial on all of C, we have $\overline{M}|_C \in \pi^*\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(C))_{\mathbb{Q}}$, as was proven earlier for curves. Then by the induction argument above,

$$\overline{M} \in \pi^*\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}.$$

We have established previously that $\pi^*\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$ is numerically trivial, so (3) $\Rightarrow$ (1).

4. Algebraic dynamical systems

As before, K is the function field of a smooth projective curve B over an algebraically closed field k, and let X be a projective variety over K. Suppose (X, f, L) and (X, g, M) are two polarized dynamical systems on X, so that f and g are endomorphisms of X, and L and M are ample line bundles such that $f^*L \cong L^q$ and $g^*M \cong M^r$ for some $q, r > 1$.

Remark. If X is not normal, we may replace X by its normalization $\psi : X' \to X$, replace f by the normalization $f' : X' \to X'$ of $f \circ \psi$, and replace L by $L' = \psi^*L$ to get a new polarized algebraic dynamical system (X', f', L') with $\mathrm{Prep}(f') = \psi^{-1}\,\mathrm{Prep}(f)$, and similarly for (X, g, M). By first replacing K with an extension if necessary, we may further assume that the normalization is geometrically normal. Hence from here on out we assume without loss of generality that X is geometrically normal.

Our main goal in this section is to prove a comparison theorem for the points with dynamical height 0 under f and g, with an important corollary comparing the preperiodic points of f and g when k is the algebraic closure of a finite field. We begin with general properties of polarized algebraic dynamical systems, then define the particular arithmetic dynamical heights involved before stating the theorem.

4.1. ***An f^*-splitting of the Néron–Severi sequence.*** We first show that the projection from $\mathrm{Pic}(X)$ onto the Néron–Severi group has a unique f^* equivariant section.

The pullback f^* preserves the exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0,$$

defining the Néron–Severi group $\mathrm{NS}(X)$, and the Néron–Severi theorem [SGA 6 1971, Exposé XII, Théorème 5.1, p. 650] tells us that $\mathrm{NS}(X)$ is a finitely generated $\mathbb{Z}$-module. For arbitrary k, the $\mathbb{Z}$-module $\mathrm{Pic}^0(X)$ need not be finitely generated, but by the Lang–Néron theorem [1959],

$$\mathrm{Pic}^0(X) \,\big/\, \mathrm{Tr}_{K/k}\,\mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(X) \,\big/\, \mathrm{Pic}^0(\mathrm{Im}_{K/k}(\mathrm{Alb}(X)))$$

is a finitely generated $\mathbb{Z}$-module. To shorten our notation, define

$$\mathrm{Pic}^0_{\mathrm{tr}}(X) := \mathrm{Pic}^0(X) \,\big/\, \mathrm{Tr}_{K/k}\,\mathrm{Pic}^0(X),$$

$$\mathrm{Pic}_{\mathrm{tr}}(X) := \mathrm{Pic}(X) \,\big/\, \mathrm{Tr}_{K/k}\,\mathrm{Pic}^0(X),$$

so that we have an exact sequence of finite-dimensional $\mathbb{C}$-vector spaces

$$0 \to \mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{C}} \to \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{C}} \to \mathrm{NS}(X)_{\mathbb{C}} \to 0,$$

which is also an exact sequence of f^*-modules.

Lemma 4.1. *The operator f^* is semisimple on $\mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{C}}$ with eigenvalues of absolute value $q^{1/2}$, and is semisimple on $\mathrm{NS}(X)$ with eigenvalues of absolute value q.*

Proof. As usual, let $n = \dim X$. By the classical Hodge index theorem [SGA 6 1971, Exposé XIII, Corollaire 7.4], we can decompose $\mathrm{NS}(X)_{\mathbb{R}}$ as

$$\mathrm{NS}(X)_{\mathbb{R}} := \mathbb{R}L \oplus P(X), \qquad P(X) := \{\xi \in \mathrm{NS}(X)_{\mathbb{R}} : \xi \cdot L^{n-1} = 0\},$$

and define a negative definite pairing on $P(X)$ by

$$\langle \xi_1, \xi_2 \rangle := \xi_1 \cdot \xi_2 \cdot L^{n-2}.$$

The projection formula for intersection numbers applied to L^n gives us $\deg f = q^n$, and then applied to this pairing, we have

$$\langle f^*\xi_1, f^*\xi_2 \rangle = q^2 \langle \xi_1, \xi_2 \rangle.$$

Hence $\frac{1}{q} f^*$ is orthogonal with respect to this pairing, and $\frac{1}{q} f^*$ is diagonalizable on $\mathrm{NS}(X)_{\mathbb{C}}$ with eigenvalues all of absolute value 1.

On $\mathrm{Pic}^0(X)_{\mathbb{R}}$ we can define a pairing as follows: for $\xi_1, \xi_2 \in \mathrm{Pic}^0(X)_{\mathbb{R}}$, let $\overline{\xi}_1$ and $\overline{\xi}_2$ be flat metrized extensions, and let $\bar{L}$ be any integrable adelic line bundle extending L. Then define

$$\langle \xi_1, \xi_2 \rangle := \overline{\xi}_1 \cdot \overline{\xi}_2 \cdot \bar{L}^{n-1}.$$

It follows from Corollary 2.10 that this pairing does not depend on the choice of metrics. Since $\mathrm{Tr}_{K/k} \mathrm{Pic}^0(X)$ is numerically trivial, this pairing descends to $\mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{R}}$, and by Theorem 3.1, it is negative definite on this quotient.

Again applying the projection formula,

$$(f^*\overline{\xi}_1) \cdot (f^*\overline{\xi}_2) \cdot (f^*\bar{L})^{n-1} = q^n(\overline{\xi}_1 \cdot \overline{\xi}_2 \cdot \bar{L}^{n-1}),$$

since each $f^*\overline{\xi}_i$ is still flat. We may also replace $f^*\bar{L}$ by $\bar{L}^q$ because the pairing is independent of the choice of metric on L, and have

$$\langle f^*\xi_1, f^*\xi_2 \rangle = q\langle \xi_1, \xi_2 \rangle.$$

Hence, $q^{-\frac{1}{2}} f^*$ is orthogonal on $\mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{R}}$ with respect to the negative of this pairing, making it diagonalizable with eigenvalues of absolute value 1 as a transformation on $\mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{C}}$. □

By the theorem,

$$0 \to \mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{C}} \to \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{C}} \to \mathrm{NS}(X)_{\mathbb{C}} \to 0$$

has a unique splitting as f^*-modules by a section

$$\ell_f : \mathrm{NS}(X)_{\mathbb{C}} \to \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{C}}.$$

Let $P, Q \in \mathbb{Q}[T]$ be the minimal polynomials of f^* on $\mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{Q}}$ and $\mathrm{NS}(X)_{\mathbb{Q}}$ respectively. Because the eigenvalues of f^* are different on $\mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{Q}}$ and $\mathrm{NS}(X)_{\mathbb{Q}}$, we see that P and Q are coprime, and $R := PQ$ is the minimal polynomial of f^* on $\mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}$. Define

$$\mathrm{Pic}_{\mathrm{tr},f}(X)_{\mathbb{Q}} := \ker Q(f^*)|_{\mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}}$$

and then this splitting can be given over $\mathbb{Q}$ as

$$\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Pic}_{\mathrm{tr},f}(X)_{\mathbb{Q}} \hookrightarrow \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}.$$

4.2. ***Admissible metrics.*** Adding to the notation above, define

$$\widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)_{\mathbb{Q}} := \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} / \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}.$$

Theorem 4.2. *The projection* $\widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)_{\mathbb{Q}} \to \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}$ *has a unique section* $M \mapsto \overline{M}_f$ *as* f^**-modules, satisfying*:

(1) *If* $M \in \mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{Q}}$ *then* $\overline{M}_f$ *is flat.*

(2) *If* $M \in \mathrm{Pic}_{\mathrm{tr},f}(X)_{\mathbb{Q}}$ *is ample then* $\overline{M}_f$ *is nef.*

Adelic metrized line bundles of the form $\overline{M}_f$ *are called* f*-admissible.*

Remark. Since $\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}} \subset \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ is flat and numerically trivial, and the underlying line bundles in $\mathrm{Pic}(X)$ are also numerically trivial, the notions of ampleness, nefness, and flatness are all well defined modulo the trace. While $\overline{M}_f$ represents a coset of the trace in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ instead of a single metrized line bundle, all coset representatives will produce the same height functions and intersection numbers, by Theorem 3.2.

Proof. Define $\widehat{\mathrm{Pic}}(X)'$ to be the group of adelic line bundles on X with continuous (but not necessarily integrable) metrics, and $\widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)' := \widehat{\mathrm{Pic}}(X)'/\,\mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))$. This contains $\widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)$. We will show that if the projection $\widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)'_{\mathbb{Q}} \to \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}$ has a unique section, then properties (1) and (2) of the theorem hold for this section. Since $\mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{Q}}$ and the ample classes in $\mathrm{Pic}_{\mathrm{tr},f}(X)_{\mathbb{Q}}$ generate $\mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}$, the section does in fact produce integrable metrics, proving the theorem.

The kernel of the projection $\widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)'_{\mathbb{Q}} \to \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}$ is

$$D(X) = \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} \bigoplus_v C(X_v^{\mathrm{an}}),$$

where $C(X_v^{\mathrm{an}})$ is the ring of continuous $\mathbb{R}$-valued functions on X_v^{an}, via the association $\|\cdot\|_v \to -\log\|1\|_v$. Recall that $R = PQ$ was defined to be the minimal polynomial of f^* on $\mathrm{Pic}(X)_{\mathbb{Q}}$ and now consider the action of $R(f^*)$ on $D(X)$.

Lemma 4.3. *$R(f^*)$ is invertible on $D(X)$.*

Proof. The pullback f^* acts as the identity on $\widehat{\mathrm{Pic}}(K)$, hence $R(f^*)$ acts as $R(1)$, and this is not zero because the roots of R all have absolute value q or $q^{1/2}$. So it suffices to show that $R(f^*)$ is invertible on $C(X)_{\mathbb{C}} := \left(\bigoplus_v C(X_v^{\mathrm{an}})\right) \otimes_{\mathbb{R}} \mathbb{C}$. Factor R over $\mathbb{C}$ as

$$R(T) = a \prod_i \left(1 - \frac{T}{\lambda_i}\right),$$

where $a \neq 0$, and by Lemma 4.1, $|\lambda_i|$ is either $q^{1/2}$ or q. $R(f^*)$ is invertible provided each term $1 - f^*/\lambda_i$ is, and each term has inverse

$$\left(1 - \frac{f^*}{\lambda_i}\right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{f^*}{\lambda_i}\right)^k,$$

provided this series converges absolutely with respect to the operator norm, which is defined with respect to the supremum norm $\|\cdot\|_{\sup}$ on $C(X_v^{\mathrm{an}})_{\mathbb{C}}$ for every place v. The pullback f^* does not change the supremum norm, so the operator norm of f^* is 1, and

$$\left\| \left(\frac{f^*}{\lambda_i}\right)^k \right\| = \frac{1}{|\lambda_i|^k} \leq q^{-\frac{k}{2}},$$

so the series converges absolutely. □

Corollary 4.4. *The exact sequence*

$$0 \to D(X) \to \widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)'_{\mathbb{Q}} \to \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}} \to 0$$

has a unique f^*-equivariant splitting.

Proof. Define

$$E(X) := \ker\big(R(f^*) : \widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)'_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)'_{\mathbb{Q}}\big).$$

Since $R(f^*)$ kills all of $\mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}$, this gives an f^*-invariant decomposition

$$\widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)'_{\mathbb{Q}} = D(X) \bigoplus E(X)$$

such that the projection onto $\mathrm{Pic}_{\mathrm{tr}}(X)$ gives an isomorphism $E(X) \xrightarrow{\sim} \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}$, whose inverse is the desired splitting.

We can write this down even more explicitly. For $M \in \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}}$, let $\overline{M}$ be any choice of metric in $\widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)'_{\mathbb{Q}}$. Then define

$$\overline{M}_f := \overline{M} - R(f^*)|_{D(X)}^{-1} R(f^*)\overline{M}. \qquad \square$$

It now remains to show that this splitting satisfies (1) and (2). To start, suppose M is in $\mathrm{Pic}^0_{\mathrm{tr}}(X)_{\mathbb{Q}}$. After extending K if necessary, we can find a preperiodic point $x_0 \in X(K)$ (in fact, by [Fakhruddin 2003], $\mathrm{Prep}(f)$ is dense in $X(\overline{K})$), and by replacing f with an iterate we may assume that x_0 is a fixed point. Let $i : X \to \mathrm{Alb}(X)$ be the Albanese map taking $x_0 \mapsto 0$, then f^* and i^* induce the following commutative diagram, where $f' := (f^*)^\vee$:

$$\begin{array}{ccc}
\mathrm{Pic}^0_{\mathrm{tr}}(\mathrm{Alb}(X)) & \underset{\sim}{\overset{i^*}{\rightleftarrows}} & \mathrm{Pic}^0_{\mathrm{tr}}(X) \\
{\scriptstyle (f')^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathrm{Pic}^0_{\mathrm{tr}}(\mathrm{Alb}(X)) & \underset{\sim}{\overset{i^*}{\rightleftarrows}} & \mathrm{Pic}^0_{\mathrm{tr}}(X) \\
{\scriptstyle M \mapsto \overline{M}_{f'}}\downarrow & & \downarrow{\scriptstyle M \mapsto \overline{M}_f} \\
\widehat{\mathrm{Pic}}_{\mathrm{tr}}(\mathrm{Alb}(X))' & \xrightarrow{i^*} & \widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)'
\end{array}$$

Because this commutes, it suffices to show (1) for abelian varieties, as i^* takes $M_{f'}$ to $\overline{M}_f$, and the pullback of a flat metric is also flat. Now $[2]^*M = 2M$, and since $[2]$ commutes with f',

$$[2]^*\overline{M}_{f'} = 2\overline{M}_{f'},$$

so that as in the proof of Lemma 2.7, we have that $\overline{M}_{f'}$, and hence also $\overline{M}_f$ is flat.

Finally, we show that (2) also holds. This is proven when K is a number field in [Yuan and Zhang 2017, Theorem 4.9], however the proof works identically in our geometric setting, as it only relies on the fact that $\mathrm{Pic}_f(X)_{\mathbb{Q}}$ (here $\mathrm{Pic}_{\mathrm{tr},f}(X)_{\mathbb{Q}}$) is a finite-dimensional $\mathbb{Q}$-vector space on which the operator $q^{-1}f^*$ has eigenvalues with absolute value one. □

Thus, we have an f^*-equivariant linear map

$$\widehat{\ell}_f : \mathrm{NS}(X)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)_{\mathbb{Q}}$$

given by the composition of the section developed in Theorem 4.2 and the map just preceding it. Importantly, we can think of this as a map into $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ which is well defined up to a numerically trivial factor, and thus sufficient to specify heights and intersections. Given $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$, we will write $\overline{M}_f$ to mean any lift of the image of M under $\mathrm{Pic}(X)_{\mathbb{Q}} \to \mathrm{Pic}_{\mathrm{tr}}(X)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}_{\mathrm{tr}}(X)_{\mathbb{Q}}$ to $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.

4.3. ***Rigidity of height zero points and preperiodic points.*** Heights given by f-admissible metrized line bundles have particularly nice properties and correspond to the dynamical canonical heights defined by Call and Silverman [1993].

Proposition 4.5. *Let $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$. Then*:

(1) *If $f^*M = M^\lambda$ for some $\lambda \in \mathbb{Q}$, then $f^*\overline{M}_f = \overline{M}_f^\lambda$ in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$, and*

$$h_{\overline{M}_f}(f(\cdot)) = \lambda h_{\overline{M}_f}(\cdot).$$

(2) *For $x \in \mathrm{Prep}(f)$, $\overline{M}_f|_x$ is trivial on $\widehat{\mathrm{Pic}}(x)_{\mathbb{Q}}$, and in particular $h_{\overline{M}_f}$ is zero on* $\mathrm{Prep}(f)$.

Further, if M is ample and $f^*M = \lambda M$ for some $\lambda > 1$ (in particular, if $M = L$), then

(3) $h_{\overline{M}_f}(x) \geq 0$ *for all $x \in X(K)$, and*

(4) *if k is finite, $h_{\overline{M}_f}(x) = 0$ if and only if $x \in \mathrm{Prep}(f)$.*

Call and Silverman [1993] establish that our height agrees with the dynamical canonical height $\widehat{h}_f$, and then the above properties all follow from well-known properties of dynamical heights proven in [loc. cit.].

We can now state and prove our main theorem of this section.

Theorem 4.6. *Let (f, L) and (g, M) be two polarized algebraic dynamical systems on X. Define $Z_f := \{x \in X(\overline{K}) | h_{\overline{L}_f}(x) = 0\}$ to be the set of height zero points with respect to $\overline{L}_f$, and Z_g the set of height zero points with respect to $\overline{M}_g$, and let Z be the Zariski closure of $Z_f \cap Z_g$ in X. Then*

$$Z_f \cap Z(\overline{K}) = Z_g \cap Z(\overline{K}).$$

When k is finite, $Z_f = \mathrm{Prep}(f)$ and $Z_g = \mathrm{Prep}(g)$, so Corollary 1.3 stated in the introduction follows as an immediate consequence. If k is not finite, it is still true that $Z_f \supseteq \mathrm{Prep}(f)$, but there may be height zero points with infinite forward orbit. See Section 5 for further discussion.

Proof. We begin by proving a simpler lemma, justifying the notation that Z_f does not depend on the polarization L.

Lemma 4.7. *Let $f : X \to X$, and let L and M be two ample line bundles which polarize f. Then*

$$\{x \in X(\bar{K}) \mid h_{\bar{L}_f}(x) = 0\} \text{ is equal to } \{x \in X(\bar{K}) \mid h_{\overline{M}_f}(x) = 0\},$$

and we unambiguously call both sets Z_f.

Proof. Since L is ample, there exists a constant $c > 0$ such that $cL - M$ is also ample. Then by Proposition 4.5, the canonical heights $h_{\overline{M}_f}$ and $h_{\overline{cL}_f} = ch_{\bar{L}_f}$ are related by

$$0 \le h_{\overline{M}_f}(x) \le ch_{\bar{L}_f}(x)$$

for all $x \in X(\bar{K})$. Thus

$$\{x \in X(\bar{K}) | h_{\bar{L}_f}(x) = 0\} \subseteq \{x \in X(\bar{K}) | h_{\overline{M}_f}(x) = 0\}.$$

By symmetry, we also have containment in the other direction. □

We now prove the theorem.

Let Y be the normalization of an irreducible component of Z, assume K is replaced by a finite extension if necessary so that Y is geometrically normal, and say $\dim Y = d$. Let ξ be the image of L in $\mathrm{NS}(X)$. Then ξ has two different lifts $\widehat{\ell}_f(\xi)$ and $\widehat{\ell}_g(\xi)$ to $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} / \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}$, and we can pick representatives $\bar{L}_f$ and $\bar{L}_g$ in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$. By Theorem 4.2, $\bar{L}_f$ and $\bar{L}_g$ are both nef, and are f- and g-admissible, respectively. Since L, L_g, and L_f are all in the same numerical equivalence class in $\mathrm{Pic}(X)_{\mathbb{Q}}$, all are ample.

Their sum $\overline{N} := \bar{L}_f + \bar{L}_g$ is also nef, and defines a height function $h_{\overline{N}}$, which does not depend on the choice of representatives of cosets modulo the trace.

By Lemma 4.7 and the premise that $Z_f \cap Z_g \cap Z(\bar{K})$ is dense, Y has a dense set of points which have height zero under $h_{\overline{N}}$. By the successive minima (see Proposition 2.13),

$$\lambda_1(Y, \overline{N}) = h_{\overline{N}}(Y) = 0.$$

Rewriting the height of Y in terms of intersections,

$$0 = (\bar{L}_f|_Y + \bar{L}_g|_Y)^{d+1} = \sum_{i=0}^{d+1} \binom{d+1}{i} (\bar{L}_f|_Y)^i \cdot (\bar{L}_g|_Y)^{d+1-i}.$$

Since both $\bar{L}_f$ and $\bar{L}_g$ are nef, every term in the sum on the right is nonnegative, hence all must be zero. Then

$$(\bar{L}_f|_Y - \bar{L}_g|_Y)^2 \cdot (\bar{L}_f|_Y + \bar{L}_g|_Y)^{d-1} = 0,$$

as well. Because $L_f - L_g$ is zero in the Néron–Severi group, and thus numerically trivial, we also have

$$(L_f|_Y - L_g|_Y)\cdot(L_f|_Y + L_g|_Y)^{d-1} = 0.$$

Additionally, $(\bar{L}_f - \bar{L}_g)$ is clearly $(\bar{L}_f + \bar{L}_g)$-bounded, and we are nearly in the right setting to apply Theorem 3.1, except that $(\bar{L}_f + \bar{L}_g)$ is nef, but not necessarily arithmetically positive.

To fix this, we simply adjust the metric by a small positive factor: let $\bar{C} \in \widehat{\mathrm{Pic}}(K)$ with $\widehat{\deg}(\bar{C}) > 0$. Replace the pair $(\bar{L}_f - \bar{L}_g, \bar{L}_f + \bar{L}_g)$ by $(\bar{L}_f - \bar{L}_g, \bar{L}_f + \bar{L}_g + \overline{\pi}^* C)$. Since $L_f - L_g$ is numerically trivial, the metric on $\bar{L}_f - \bar{L}_g$ is flat, so adding $\overline{\pi}^* C$, which is vertical, does not change the intersection number. All the conditions of the theorem are now satisfied, so that the theorem tells us

$$(\bar{L}_f - \bar{L}_g) \in \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \mathrm{Tr}_{K/k}(\mathrm{Pic}^0(X))_{\mathbb{Q}}.$$

We therefore conclude by Theorem 3.2 that $h_{\bar{L}_f} - h_{\bar{L}_g}$ is a constant height function on Y. Since these two heights both take value zero on a dense set in Z, they must be equal on Y. Thus these heights define the same sets of height zero points, and then by Lemma 4.7, Z_f and Z_g agree on Y, and hence on all of Z. □

5. Related results and further questions

5.1. ***Rigidity of preperiodic points over global function fields.*** We first summarize some basic consequences of Theorem 4.6 when K is a global function field, particularly in the case when $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is dense in X.

Lemma 5.1. *Let K be a global function field, and let f and g be two polarized algebraic dynamical systems on a projective variety X. Then the following are equivalent*:

(1) $\mathrm{Prep}(f) = \mathrm{Prep}(g)$.

(2) $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is dense in X.*

(3) $\mathrm{Prep}(f) \subset \mathrm{Prep}(g)$.

(4) $g(\mathrm{Prep}(f)) \subset \mathrm{Prep}(f)$.

Proof. The equivalence of (1) and (2) is an immediate consequence of Theorem 4.6 and the fact that over a global function field, all dynamical height zero points are preperiodic. Clearly (1) implies (4). By Fakhruddin [2003], $\mathrm{Prep}(f)$ is always dense in X, hence (3) implies (2). We now show (4) implies (3).

Stratify $\mathrm{Prep}(f)$ by degree, writing

$$\mathrm{Prep}(f) = \bigcup_{d\geq 0} \mathrm{Prep}(f, d),$$

where

$$\mathrm{Prep}(f, d) := \{x \in \mathrm{Prep}(f) \mid [K(x) : K] \leq d\}.$$

Since each $\mathrm{Prep}(f, d)$ has height zero and bounded degree, it is finite. Now (4) says that g fixes $\mathrm{Prep}(f)$, but since g is defined over K, it fixes each $\mathrm{Prep}(f, d)$ as well. Thus every point of $\mathrm{Prep}(f)$ has finite forward orbit under g. □

This lemma suggests two related questions which we do not answer here.

(1) When is $\mathrm{Prep}(f)$ equal to $\mathrm{Prep}(g)$?

(2) If $\mathrm{Prep}(f) = \mathrm{Prep}(g)$, how closely related must f and g be?

In the case of $f : \mathbb{P}^1 \to \mathbb{P}^1$, Mimar [2013] gives a variety of partial answers to these questions, with the general implication being that if f and g have the same preperiodic points, their Julia sets must also be very similar. But this is likely very difficult in dimension greater than one.

5.2. ***Preperiodic points over larger function fields.*** Theorem 3.1 and most of the proof of Theorem 4.6 hold over all transcendence degree one function fields, not just global function fields. But because the Northcott principal fails when k is not a finite field or the algebraic closure of a finite field, we cannot equate height zero points with preperiodic points over arbitrary function fields, and thus Theorem 4.6 is a statement about height zero points and not preperiodic points. In this broader setting, however, some things can still be said.

Baker [2009] proves the following theorem, first proven by Benedetto [2005] in the case of polynomials.

Theorem 5.2. *Let $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a rational function of degree ≥ 2, and suppose that f is not isotrivial, in the sense that there exists no finite extension K' of K and Möbius transformation $M \in \mathrm{PGL}_2(K')$ such that*

$$f' := M^{-1} \circ f \circ M$$

is defined over k. Then

$$\mathrm{Prep}(f) = Z_f.$$

Thus Theorem 4.6 proven here immediately implies Corollary 1.4.

In higher-dimension isotriviality is less straightforward to classify. When A is an abelian variety, its K/k-trace classifies how isotrivial it is, and then the Lang–Néron theorem provides a Northcott-like result for the Néron–Tate canonical height (the dynamical height induced by $[n]$): height zero points fall into only finitely many cosets of $\mathrm{Tr}_{K/k}(A)(k) \hookrightarrow A(K)$.

There is no notion of a trace for general varieties, however, and $\iota^{-1}\,\mathrm{Tr}_{K/k}(\mathrm{Alb}(X))$ is not a sufficient substitute, as $\mathrm{Alb}(X)$ will often be trivial. Chatzidakis and Hrushovski [2008a; 2008b] instead use model theory, and a variant of isotriviality called *constructible descent* to k. Their theorem generalizes both Baker's result and the Lang–Néron theorem.

Theorem 5.3. *Let K be any function field and let k be its field of constants. Let $f : X \to X$ be an algebraic dynamical system defined over K, and assume f does not constructibly descend to k. Then for every point $x \in X(\overline{K})$ with dynamical height zero there exists a proper Zariski closed subset $Y_x \subsetneq X$ such that the orbit of x is contained in Y_x.*

The author is optimistic that the methods of arithmetic heights and rigidity theorem of this paper, combined with model-theoretic treatment of isotriviality will yield stronger dynamics results over general function fields in the future.

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