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Huneke and Wiegand conjectured that, if M is a finitely generated, nonfree, torsion-free module with rank over a one-dimensional Cohen–Macaulay local ring R, then the tensor product of M with its algebraic dual has torsion. This conjecture, if R is Gorenstein, is a special case of a celebrated conjecture of Auslander and Reiten on the vanishing of self-extensions that stems from the representation theory of finite-dimensional algebras.

If R is a one-dimensional Cohen–Macaulay ring such that $R = S/(f)$ for some local ring $(S, \mathfrak{n})$, and a non-zero-divisor $f \in \mathfrak{n}^2$ on S, we make use of Hochster's theta invariant and prove that such R-modules M which have finite projective dimension over S satisfy the proposed torsion conclusion of the conjecture. Along the way we give several applications of our argument pertaining to torsion properties of tensor products of modules.

1. Introduction

This paper concerns commutative Noetherian local rings $(R, \mathfrak{m}, k)$ and finitely generated R-modules.

The aim of this paper is to study the torsion-freeness property of tensor products of modules, a subtle topic which stems from the beautiful work of Auslander [1961].

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Our focus is on the torsion of tensor products of the form $M \otimes_R M^*$ over one-dimensional Cohen–Macaulay local rings R, where M^* denotes $\operatorname{Hom}_R(M, R)$. In particular, we are concerned with the following long-standing conjecture of Huneke and Wiegand.

Conjecture 1.1 [Huneke and Wiegand 1994, page 473]. *Let R be a one-dimensional local ring and let M be a finitely generated, nonfree, torsion-free R-module. If M has rank (e.g., if R is a domain), then $M \otimes_R M^*$ has torsion.*

Recall that a finitely generated R-module M is said to have *rank* if there is a nonnegative integer r such that $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus r}$ for all associated primes $\mathfrak{p}$ of R; see [Bruns and Herzog 1993, 1.4.3].

Conjecture 1.1 stems from the seminal works of Auslander [1961], and Huneke and Wiegand [1994]. The conjecture is true over hypersurface rings [Huneke and Wiegand 1994, 3.7], but it is very much open in general, even for ideals over complete intersection domains of codimension two. It is worth noting that Conjecture 1.1 is a special version of the celebrated conjecture of Auslander and Reiten [1975] on the vanishing of Ext when the ring in question is a one-dimensional Gorenstein domain; see [Celikbas and Wiegand 2015, 8.6] for details.

There is strong evidence that Conjecture 1.1 should be true over complete intersections; see [Celikbas and Wiegand 2015; Huneke et al. 2019]. Moreover, there are various examples supporting the conjecture over rings that are not necessarily complete intersections. For example, it is proved in [Huneke et al. 2019, 3.6] that Conjecture 1.1 is true over Cohen–Macaulay rings with minimal multiplicity, e.g., over local Arf rings [Lipman 1971]. For some further examples, we refer to [Celikbas et al. 2019a] and point out the following:

Example 1.2. Let R be a one-dimensional, reduced, nonregular, local ring.

(i) If R is complete, and has prime characteristic p and perfect residue field, then it follows ${}^{\varphi^n}R \otimes_R ({}^{\varphi^n}R)^*$ has torsion for all $n \gg 0$. Here $\varphi^n : R \to R$ is the n-th iterate of the Frobenius endomorphism given by $r \mapsto r^{p^n}$, and ${}^{\varphi^n}R$ denotes R with the R-action given by $r \cdot s = r^{p^n} s$ for all $r, s \in R$; see [Celikbas et al. 2019a, 2.15; Miller 2003, 2.1.3 and 2.2.12].

(ii) If R is a Gorenstein domain and I is an Ulrich ideal of R which is not principal, then I is a self-dual R-module, i.e., $I \cong I^*$, and so $I \otimes_R I^*$ has torsion. In particular, if $R = \mathbb{C}[\![t^4, t^5, t^6]\!]$ and $I = (t^4, t^6)$, then $I \otimes_R I^*$ has torsion; see Example 4.17 and Proposition 4.18.

The purpose of this paper is to prove Theorem 1.3 and give some observations about Conjecture 1.1; see Theorem 3.2 for a higher dimensional version of the next result. The tool we employ to prove Theorem 1.3 is the Hochster's θ invariant, which was initially defined by Hochster [1981] to study the direct summand conjecture; it was further developed by Dao [2008; 2013], and more recently by Buchweitz and van Straten [2012], and Walker et al. [Moore et al. 2011; Walker 2017]. The invariant $\theta^R(M, N)$ for R-modules M and N is defined as $\operatorname{length}_R(\operatorname{Tor}^R_{2n+2}(M, N)) - \operatorname{length}_R(\operatorname{Tor}^R_{2n+1}(M, N))$ for $n \gg 0$; see §2.13 for the details.

Theorem 1.3. *Let R be a one-dimensional Cohen–Macaulay local ring, where $R = S/(f)$ for some local ring $(S, \mathfrak{n})$ and a non-zero-divisor $f \in \mathfrak{n}$ on S. Let M and N be nonzero finitely generated R-modules, and assume the following conditions hold*:

(i) $\operatorname{pd}_S(M) < \infty$ *or* $\operatorname{pd}_S(N) < \infty$ (*e.g., S is regular*).

(ii) $\operatorname{length}_R(\operatorname{Tor}^R_i(M, N)) < \infty$ *for all* $i \gg 0$ (*e.g., R is reduced*).

(iii) $\theta^R(M, N) = 0$.

If $M \otimes_R N$ is torsion-free, then $\operatorname{Tor}^R_i(M, N) = 0$ for all $i \geq 1$, and M and N are torsion-free.

To the best of our knowledge, Theorem 1.3 is new, even if S is a ramified regular ring; see [Celikbas et al. 2015a, 3.6] and Section 3. Next is a corollary of Theorem 1.3 concerning Conjecture 1.1; see Corollaries 4.6 and 4.8.

Corollary 1.4. *Let R be a one-dimensional Cohen–Macaulay ring such that, for some local ring $(S, \mathfrak{n})$ and a non-zero-divisor $f \in \mathfrak{n}^2$ on S, we have $R = S/(f)$. Assume M is a finitely generated R-module that has rank.*

If M is a nonfree torsion-free R-module and $\operatorname{pd}_S(M) < \infty$, then $M \otimes_R M^*$ has torsion. In particular, if $M = \operatorname{coker}(\alpha)$, where (α, β) is a reduced matrix factorization of f over S (i.e., a matrix factorization of f with entries in $\mathfrak{n}$), then $M \otimes_R M^*$ has torsion.

As mentioned previously, if S is regular, Corollary 1.4 follows from a result of Huneke and Wiegand [1994, 3.7]. In this case, as is well-known, maximal Cohen–Macaulay R-modules with no free summands occur as reduced matrix factorizations of f over S; see [Eisenbud 1980]. Similarly, if S is *G-regular* (i.e., when there are no nonfree totally reflexive S-modules), Takahashi [2008] proved that there is a one-to-one correspondence between reduced matrix factorizations of f and totally reflexive R-modules without free summands. Note that, if the ring R is as in Corollary 1.4, reduced matrix factorizations of f exist due to a result of Herzog, Ulrich, and Backelin; see [Herzog et al. 1991, 1.2 and 2.2], and also [Avramov 1998, 5.1.3, Avramov et al. 1997, 3.1, Yoshino 1990, Chapter 8].

In Sections 2 and 3 we collect some preliminary results and give a proof of Theorem 1.3, respectively. Sections 4 and 5 are devoted to several applications of Theorem 1.3 pertaining to torsion properties of tensor products of modules. As Theorem 1.3 relies upon the vanishing of theta invariant, in Appendix A we point out by an example that $\theta^R(M, N)$ can vanish nontrivially: in Example A.3, we record an example of a one-dimensional reduced hypersurface ring R, and finitely generated R-modules M and N such that $\theta^R(M, N) = 0$, but neither M nor N has rank, or equivalently, neither M nor N has zero class in the reduced Grothendieck group $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$. Moreover, in Appendix B, building on an argument of Huneke and Wiegand [1994, 4.7], we recall how to obtain examples of nonfree, torsion-free R-modules M with rank such that $M \otimes_R M$ is torsion-free over certain one-dimensional local rings R; see §B.1.

2. Preliminaries

In this section we recall definitions and collect some basic facts that will be used throughout the paper. We have, by definition, $\operatorname{depth}(0) = \infty$ and $\operatorname{pd}(0) = -\infty$. Moreover, ΩM denotes the syzygy of a given finitely generated R-module M.

2.1 *Torsion submodule*. Let R be a local ring and let M be a finitely generated R-module. The *torsion submodule* $\mathsf{T}_R M$ of M is the kernel of the natural map $M \to \mathrm{Q}(R) \otimes_R M$, where $\mathrm{Q}(R)$ is the total quotient ring of R. Hence there is an exact sequence of R-modules:

$$0 \to \mathsf{T}_R M \to M \to \perp_R M \to 0. \tag{2.1.1}$$

M is said to have *torsion* (or be, *torsion-free*) if $\mathsf{T}_R M \neq 0$ (respectively, $\mathsf{T}_R M = 0$). Note, M is torsion, i.e., $\mathsf{T}_R M = M$, if and only if $M_{\mathfrak{p}} = 0$ for each associated prime $\mathfrak{p}$ of R. Note also that $M = 0$ if and only if M is both torsion and torsion-free.

2.2. Let R be a local ring. If $0 \to X \to Y \to Z \to 0$ is a short exact sequence of finitely generated R-modules, then it follows that the sequence

$$0 \to \mathsf{T}_R X \to \mathsf{T}_R Y \to \mathsf{T}_R Z$$

is exact. In particular, if X and Z are torsion-free, then so is Y.

The next fact will be used several times throughout, for example, for Corollary 4.6.

2.3. Let R be a local ring and let M be a finitely generated R-module. Set $M^* = \operatorname{Hom}(M, R)$, the algebraic dual of M. If $M^* = 0$, then there is an $x \in R$, which is a non-zero-divisor on R, such that $xM = 0$; see [Bruns and Herzog 1993, 1.2.3(b)]. In other words, $M^* = 0$ if and only if M is a torsion R-module. In particular, if $M \neq 0$ and M is torsion-free, then $M \otimes_R M^* \neq 0$.

The following argument is from [Huneke and Wiegand 1994]; we will invoke it in the proofs of Theorem 3.2, Remark 3.5, Proposition 4.4, and Corollary 5.12.

2.4 [Huneke and Wiegand 1994, 1.1]. Let R be a local ring, and let M and N be nonzero finitely generated R-modules. Assume $M \otimes_R N$ is torsion-free. Set $U = \perp_R M$ and $V = \mathsf{T}_R M$. Then:

(i) $M \otimes_R N \cong U \otimes_R N$.

(ii) If $\operatorname{Tor}_1^R(U, N) = 0$, then M is torsion-free, i.e., $M = U$.

(iii) If $\operatorname{Tor}_i^R(U, N) = 0$ for all $i \geq 1$, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

To establish part (i), we tensor the exact sequence $0 \to V \to M \to U \to 0$ by N, and obtain the following exact sequence of R-modules:

$$\operatorname{Tor}_1^R(U, N) \to V \otimes_R N \xrightarrow{\alpha} M \otimes_R N \xrightarrow{\beta} U \otimes_R N \to 0.$$

As $V \otimes_R N$ is torsion, we see that the image of α is also torsion. Hence, since $M \otimes_R N$ is torsion-free, it follows that $\alpha = 0$. Therefore, β is an isomorphism and part (i) follows.

Now assume $\operatorname{Tor}_1^R(U, N) = 0$. Then α is both zero and injective so that $V \otimes_R N = 0$. This implies, as $N \neq 0$, that $V = 0$, i.e., M is torsion-free, i.e., $U = M$. This proves part (ii). Notice, part (iii) is a consequence of part (ii).

2.5 ***Gorenstein and complete intersection dimensions*** [Auslander and Bridger 1969; Avramov et al. 1997]. Let R be a local ring and let M be a finitely generated R-module.

M is said to be *totally reflexive* provided that the natural map $M \to M^{**}$ is bijective and $\operatorname{Ext}_R^i(M, R) = 0 = \operatorname{Ext}_R^i(M^*, R)$ for all $i \geq 1$. The infimum of n for which there exists an exact sequence $0 \to X_n \to \cdots \to X_0 \to M \to 0$ such that each X_i is totally reflexive is called the *Gorenstein dimension* of M. If M has Gorenstein dimension n, we write $\text{G-dim}_R(M) = n$. Therefore, M is totally reflexive if and only if $\text{G-dim}_R(M) \leq 0$, where it follows by convention that $\text{G-dim}_R(0) = -\infty$.

A diagram of local ring maps $R \to R' \twoheadleftarrow S$ is called a *quasi-deformation* if $R \to R'$ is flat and the kernel of the surjection $R' \twoheadleftarrow S$ is generated by a regular sequence on S. The *complete intersection dimension* of M is defined as follows:

$$\text{CI-dim}_R(M) = \inf\{\operatorname{pd}_S(M \otimes_R R') - \operatorname{pd}_S(R') : R \to R' \twoheadleftarrow S \text{ is a quasi-deformation}\}.$$

The following inequalities hold in general:

(2.5.1) $$\text{G-dim}_R(M) \leq \text{CI-dim}_R(M) \leq \operatorname{pd}_R(M).$$

Moreover, if any of the dimensions in (2.5.1) is finite, then it is equal to those to its left.

2.6 ***Complexity*** [Avramov 1989]. Let R be a local ring and let M be a finitely generated R-module. The *complexity* $\operatorname{cx}_R(M)$ of M is the smallest nonnegative integer r such that there exists a real number A with $\beta_n(M) \leq A \cdot n^{r-1}$ for all $n \gg 0$. Here $\beta_n(M)$ is the n-th Betti number of M. It follows that $\operatorname{cx}_R(M) = 0$ if and only if $\operatorname{pd}_R(M) < \infty$, and $\operatorname{cx}_R(M) \leq 1$ if and only if M has bounded Betti numbers.

Next we collect certain properties of complexity and complete intersection dimension. Prior to that we recall the definition of the transpose:

2.7 ***Auslander transpose*** [Auslander and Bridger 1969]. Let R be a local ring and let M be a finitely generated R-module with a projective presentation

$$P_1 \xrightarrow{f} P_0 \to M \to 0.$$

The *transpose* $\operatorname{Tr} M$ of M is the cokernel of $f^* = \operatorname{Hom}_R(f, R)$, and so is given by the following exact sequence:

$$0 \to M^* \to P_0^* \to P_1^* \to \operatorname{Tr} M \to 0. \tag{2.7.1}$$

In particular, up to projectives, $\operatorname{Tr} M$ is uniquely defined and $\operatorname{Tr}\operatorname{Tr} M \cong M$.

2.8. Let R be a local ring and let M and N be finitely generated R-modules such that $M \neq 0$.

(i) If $\text{CI-dim}_R(M) < \infty$, then it follows that $\operatorname{cx}_R(M) \leq \mathsf{embdim}(R) - \operatorname{depth}(R)$; see [Avramov et al. 1997, 5.6].

(ii) If $\text{CI-dim}_R(M) < \infty$, then it follows that $\text{CI-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{CI-dim}_R(M)$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$; see [Avramov et al. 1997, 1.6].

(iii) If $\text{CI-dim}_R(M) < \infty$, then it follows that $\text{CI-dim}_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M)$, which also equals $\sup\{i : \operatorname{Ext}^i_R(M, R) \neq 0\}$; see [Avramov et al. 1997, 1.4].

(iv) Assume $\text{CI-dim}_R(M) < \infty$. If f is a non-zero-divisor on R and $fM = 0$, then it follows that $\text{CI-dim}_{R/fR}(M) < \infty$. Also, if f is a non-zero-divisor on both R and M, then it follows that $\text{CI-dim}_{R/fR}(M/fM) = \text{CI-dim}_R(M)$; see [Avramov et al. 1997, 1.12.2–3].

(v) If $R \to R'$ is a flat local map of local rings and $\text{CI-dim}_{R'}(M \otimes_R R') < \infty$, then it follows that $\text{CI-dim}_R(M) = \text{CI-dim}_{R'}(M \otimes_R R')$; see [Avramov et al. 1997, 1.11].

(vi) If $\text{CI-dim}_R(M) = 0$, then it follows that $\text{CI-dim}(M^*) = \text{CI-dim}_R(\operatorname{Tr} M) = 0$, and also $\operatorname{cx}(M) = \operatorname{cx}(M^*) < \infty$. Moreover, $\text{CI-dim}_R(M) = 0$ if and only if $\text{CI-dim}_R(\operatorname{Tr} M) = 0$; see [Bergh and Jorgensen 2011, 3.5; 2014, 3.2; Celikbas et al. 2015b, 3.2(i)].

(vii) If $\text{CI-dim}_R(M) < \infty$ and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, then it follows $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq \text{CI-dim}_R(M) + 1$; see [Avramov and Buchweitz 2000, 4.9]. Hence, if $\text{CI-dim}_R(M) < \infty$, then $\operatorname{Tor}_i^R(M, N)$ is torsion for all $i \gg 0$ if and only if $\operatorname{Tor}_i^R(M, N)$ is torsion for all $i \geq 1$; see §2.1.

(viii) If $\text{CI-dim}_R(M) < \infty$ and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, then the *depth formula* for M and N holds, i.e., $\operatorname{depth}(M) + \operatorname{depth}(N) = \operatorname{depth}(R) + \operatorname{depth}(M \otimes_R N)$; see [Araya and Yoshino 1998, 2.5].

2.9 [Avramov and Martsinkovsky 2002, 3.1]. Let R be a local ring and let N be a finitely generated R-module such that $\text{G-dim}_R(N) < \infty$. Then there is an exact sequence of finitely generated R-modules $0 \to L \to Z \to N \to 0$, where $\text{G-dim}_R(Z) = 0$ and $\text{pd}_R(L) < \infty$.

The next result is due to Sather-Wagstaff [2004]; here we record the module case, but in fact his result holds for homologically finite complexes.

2.10 [Sather-Wagstaff 2004, 3.6]. Let R be a local ring and let

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

be a short exact sequence of finitely generated R-modules. Assume i, j, s are integers with $\{i, j, s\} = \{1, 2, 3\}$. If $\text{pd}_R(X_i) < \infty$ and $\text{CI-dim}_R(X_j) < \infty$, then $\text{CI-dim}_R(X_s) < \infty$.

We use §2.11 in the proofs of Theorem 3.2, Corollary 4.1, and Proposition 4.4.

2.11. Let R be a one-dimensional Cohen–Macaulay local ring, and let N be a finitely generated R-module such that $\text{CI-dim}_R(N) < \infty$. Then there is an exact sequence of finitely generated R-modules

$$0 \to F \to Z \to N \to 0, \tag{2.11.1}$$

where F is free, $\text{CI-dim}_R(Z) = 0$, and $\text{cx}_R(Z) = \text{cx}_R(N)$.

To see this, first note that we have $\text{G-dim}_R(N) < \infty$ since $\text{CI-dim}_R(N) < \infty$; see (2.5.1). Hence a short exact sequence as in (2.11.1) exists by §2.9, where $\text{G-dim}_R(Z) = 0$ and $\text{pd}_R(F) < \infty$. Now §2.10 implies that $\text{CI-dim}_R(Z) < \infty$. Consequently, we deduce that $\text{CI-dim}_R(Z) = \text{G-dim}_R(Z) = 0$; see (2.5.1). Note, by §2.8(iii), we have that $\text{depth}_R(Z) = 1$. Thus, the depth lemma applied to (2.11.1) yields $\text{depth}_R(F) = 1$. So, we conclude that F is free by the Auslander–Buchsbaum formula. Finally, notice that as F is free, by tensoring (2.11.1) with k, we obtain that $\beta_i^R(Z) = \beta_i^R(N)$ for each $i \geq 2$. This yields the equality $\text{cx}_R(Z) = \text{cx}_R(N)$; see §2.6.

We use the following exact sequence in §2.13, and also in the proofs of Theorem 3.2 and Proposition 5.9.

2.12 [Rotman 1979, 11.65]. Let $(S, \mathfrak{n})$ be a local ring and let $R = S/(f)$ for some non-zero-divisor $f \in \mathfrak{n}$ on S. If M and N are finitely generated R-modules, then we have the change of rings long exact sequence of Tors:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\text{Tor}_i^R(M,N) & \to & \text{Tor}_{i+1}^S(M,N) & \to & \text{Tor}_{i+1}^R(M,N) & \to & \\
\vdots & & \vdots & & \vdots & & \\
\text{Tor}_0^R(M,N) & \to & \text{Tor}_1^S(M,N) & \to & \text{Tor}_1^R(M,N) & \to & 0
\end{array}$$

In the following, we recall the definition of a version of Hochster's θ pairing [1981], developed by Dao [2007]. This pairing can be defined in a more general setting, but the definition recorded here suffices for our argument; see [Dao 2008; 2013] for more details.

2.13 θ ***pairing*** [Dao 2007; Hochster 1981]. Let M and N be finitely generated R-modules. Assume $R \to R' \twoheadleftarrow S$ is a codimension one quasi-deformation with zero-dimensional closed fibre, i.e., we have a diagram of local ring maps such that $R \to R'$ is flat, $R' \cong S/(f)$ for some non-zero-divisor f on S, and $\dim(R'/\mathfrak{m}R') = 0$. We set $(-)' = - \otimes_R R'$ and assume the following conditions hold:

(a) $\text{CI-dim}_S(N') < \infty$ and $\text{Tor}_i^S(M', N') = 0$ for all $i \gg 0$ (e.g., $\text{pd}_S(N') < \infty$).

(b) $\text{length}_R(\text{Tor}_i^R(M, N)) < \infty$ for all $i \gg 0$ (e.g., R is an isolated singularity).

It follows that

$$\text{CI-dim}_{R'}(N') < \infty \quad \text{and} \quad \text{CI-dim}_R(N) = \text{CI-dim}_{R'}(N');$$

see §2.8(iv, v). Note we have, by (a) and §2.8(vii), that $\text{Tor}_i^S(M', N') = 0$ for all $i > \text{CI-dim}_S(N')$. Therefore §2.12 yields the following isomorphisms:

$$\text{Tor}_i^{R'}(M', N') \cong \text{Tor}_{i+2}^{R'}(M', N') \quad \text{for all } i > \text{CI-dim}_{R'}(N'). \tag{2.13.1}$$

For a nonmaximal prime ideal $\mathfrak{p}$, we have

$$\text{Tor}_i^R(M, N)_{\mathfrak{p}} = 0 \quad \text{for all } i > \text{CI-dim}_R(N);$$

see §2.8(ii, vii). Thus $\text{length}(\text{Tor}_i^R(M, N)) < \infty$ for all $i > \text{CI-dim}_R(N)$, and hence

$$\text{length}_{R'}(\text{Tor}_i^{R'}(M', N')) < \infty \quad \text{for all } i > \text{CI-dim}_{R'}(N'). \tag{2.13.2}$$

Let $\ell = \text{length}_{R'}(R'/\mathfrak{m}R')$. Then, by (2.13.1) and (2.13.2), we see that the difference

$$\begin{aligned}&\text{length}_R(\text{Tor}_{2n+2}^R(M, N)) - \text{length}_R(\text{Tor}_{2n+1}^R(M, N)) \\ &\qquad = \frac{1}{\ell} \cdot \big(\text{length}_{R'}(\text{Tor}_{2n+2}^{R'}(M', N')) - \text{length}_{R'}(\text{Tor}_{2n+1}^{R'}(M', N'))\big)\end{aligned}$$

is independent of n if $2n > \text{CI-dim}_{R'}(N') - 1$. One defines the theta pairing over R as

$$\theta^R(M, N) = \text{length}_R(\text{Tor}_{2n+2}^R(M, N)) - \text{length}_R(\text{Tor}_{2n+1}^R(M, N)), \tag{2.13.3}$$

where n is an integer such that $2n > \text{CI-dim}_R(N) - 1$.

It follows θ^R is additive on short exact sequence of finitely generated R-modules, whenever it is well-defined on each pair of modules in question; see, for example, [Dao 2007, 4.3(2)].

2.14. Let M and N be finitely generated R-modules. Assume the following conditions hold:

(a) $$\text{CI-dim}_R(N) < \infty \quad \text{and} \quad \text{cx}_R(N) = 1.$$

(b) $$\text{length}_R(\text{Tor}_i^R(M, N)) < \infty \quad \text{for all } i \gg 0.$$

Then we can choose a codimension one quasi-deformation of the form $R \to R' \overset{\alpha}{\twoheadleftarrow} S$, where $\text{pd}_S(N') < \infty$; see [Avramov and Buchweitz 2000, 4.1.3]. Localizing at some $\mathfrak{p} \in \text{Min}_{R'}(R'/\mathfrak{m}R')$, set $\mathfrak{q} = \alpha^{-1}(\mathfrak{p})$: we see that $R \to R'_{\mathfrak{p}} \overset{\alpha}{\twoheadleftarrow} S_{\mathfrak{q}}$ is a codimension one quasi-deformation with $\text{pd}_{S_{\mathfrak{q}}}(N \otimes_R R'_{\mathfrak{p}}) < \infty$; see the proof of [Sather-Wagstaff 2004, 2.11]. Therefore, replacing the original quasi-deformation with the aforementioned one, we may assume $\dim(R'/\mathfrak{m}R') = 0$.

So it follows from (2.13.3) that $\theta^R(M, N)$ is well-defined, as long as n is an integer such that $2n > \text{CI-dim}_R(N) - 1$.

Next we record two more preliminary results prior to moving to the next section; both of these results are used in the next section for our proof of Theorem 3.2.

2.15. Let R be a local ring, and let L and M be finitely generated R-modules. If M is maximal Cohen–Macaulay and $\text{pd}_R(L) < \infty$, then $\text{Tor}_i^R(L, M) = 0$ for all $i \geq 1$; see, for example, [Celikbas 2011, 3.8].

2.16. Let R be a local ring and let M be a finitely generated R-module. Assume M has a finite free resolution, i.e., there is an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where each F_i is a finitely generated free R-module. The *Euler number* of M is defined as $\chi(M) = \sum (-1)^i \text{ rank } F_i$.

It is known that $\chi(M)$ is independent of the choice of the finite free resolution of M. Moreover, it follows that $\chi(M) = 0$ if and only if there is a non-zero-divisor f on R such that $fM = 0$; see [Matsumura 1986, 19.8 and page 158].

3. Proof of the main result

In this section we prove the main result of this paper; see Theorem 3.2. Our motivation comes from the following result, which is recorded for the one-dimensional case:

3.1 [Celikbas et al. 2015a, 3.6]. Let R be a one-dimensional local ring with $\widehat{R} = S/(f)$ for some unramified regular local ring S, and a non-zero-divisor $f \in \mathfrak{n}$ on S. Let M and N be finitely generated R-modules. If $\text{length}_R(\text{Tor}_i^R(M, N)) < \infty$ for all $i \gg 0$, $\theta^R(M, N) = 0$, and $M \otimes_R N$ is torsion-free, then $\text{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, and M and N are both torsion-free.

A consequence of our argument gives an extension of §3.1 and establishes the vanishing of $\text{Tor}_i^R(M, N)$ when S is an arbitrary two-dimensional Cohen–Macaulay local ring, and $\mathrm{pd}_S(N) < \infty$ or $\mathrm{pd}_S(M) < \infty$; see Theorem 3.2. As is clear, since we do not work over hypersurface rings, our method of proof is different from that employed to prove §3.1. Among other things, one of the properties that is not available to us under our setup is that, when R is a hypersurface, every torsion-free module can be embedded in a free R-module; see [Huneke and Wiegand 1994, 1.5]. Also, over a ring R as in §3.1, for a pair of finitely generated R-modules (M, N), if $\theta^R(M, N)$ is defined and vanishes, then the pair (M, N) is Tor-rigid, i.e., if $\mathrm{Tor}_n^R(M, N) = 0$ for some $n \geq 0$, then $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq n$ [Dao 2013, 2.8]; this Tor-rigidity result depends on the fact that S is an unramified regular ring. Thus the properties that play an important role in the proof of §3.1 do not apply directly under our setup.

The following is our main result; although we are interested in the one dimensional case (due to Conjecture 1.1), our argument works over Cohen–Macaulay local rings of arbitrary positive dimension if the modules considered have sufficiently large depth; see also Remark 3.5.

Theorem 3.2. *Let R be a Cohen–Macaulay local ring, and let M and N be finitely generated R modules. Assume* $\dim(R) = d \geq 1$ *and the following conditions hold*:

(i) $\text{CI-dim}_R(N) < \infty$ *and* $\mathrm{cx}_R(N) = 1$.

(ii) $\mathrm{length}_R(\mathrm{Tor}_i^R(M, N)) < \infty$ *for all* $i \gg 0$.

(iii) $\mathrm{depth}_R(M) \geq d-1$ *and* $\mathrm{depth}_R(N) \geq d-1$.

(iv) *If* $d = 1$, *assume further* $\theta^R(M, N) = 0$.

If $M \otimes_R N$ is (nonzero) maximal Cohen–Macaulay, then $\mathrm{Tor}_i^R(M, N) = 0$ *for all* $i \geq 1$, *and M and N are both maximal Cohen–Macaulay.*

Proof. It suffices to prove the vanishing of $\mathrm{Tor}_i^R(M, N)$ for all $i \geq 1$; see §2.8(viii).

First we assume $d \geq 2$, and choose a non-zero-divisor x on R, M, N, and $M \otimes_R N$. Set $T = R/xR$, $A = M/xM$, and $B = N/xN$. Notice

$$\text{CI-dim}_T(B) = \text{CI-dim}_R(N) < \infty \quad \text{and} \quad \mathrm{cx}_T(B) = \mathrm{cx}_R(N) = 1;$$

see §2.8(iv). We have the following exact sequence:

$$0 \to M \xrightarrow{x} M \to A \to 0. \tag{3.2.1}$$

Tensoring (3.2.1) with N, we obtain the following long exact sequence for all $i \geq 0$:

$$\begin{aligned}(3.2.2) \quad \cdots \to \mathrm{Tor}_{i+1}^R(M,N) \xrightarrow{x} \mathrm{Tor}_{i+1}^R(M,N) \to \mathrm{Tor}_{i+1}^R(A,N) \to \mathrm{Tor}_i^R(M,N) \to \\ \cdots \to \mathrm{Tor}_1^R(A,N) \to M \otimes_R N \xrightarrow{x} M \otimes_R N \to A \otimes_R N \to 0.\end{aligned}$$

As x is a non-zero-divisor on R and N, and $xA = 0$, we have that

$$\mathrm{Tor}_i^T(A, B) \cong \mathrm{Tor}_i^R(A, N) \quad \text{for all } i \geq 0.$$

It follows from (3.2.2) that $\mathrm{length}_T(\mathrm{Tor}_i^T(A,B))<\infty$ for all $i\gg 0$. Moreover, $\theta^R(A,N)$ is well-defined and we see that $\theta^R(M,N)=\theta^R(M,N)+\theta^R(A,N)$, by additivity applied to (3.2.1). Therefore, $0=\theta^R(A,N)=\theta^T(A,B)$. It follows from (3.2.2) that $(M\otimes_R N)/x(M\otimes_R N)\cong A\otimes_R N\cong A\otimes_T B$. This implies that $A\otimes_T B$ is maximal Cohen–Macaulay over T. Moreover, $\mathrm{depth}_T(A)\geq \mathrm{depth}(T)-1$ and $\mathrm{depth}_T(B)\geq \mathrm{depth}(T)-1$. Consequently, we may use induction to go all the way down to dimension one. More precisely, we may replace the pair (M,N) over the ring R with the pair (A,B) over the ring T, and we may assume $\dim(T)=1$: this case yields the vanishing of $\mathrm{Tor}_i^T(A,B)$ for all $i\geq 1$, and in view of Nakayama's lemma, we conclude by (3.2.2) that $\mathrm{Tor}_i^R(M,N)=0$ for all $i\geq 1$, as claimed.

We now proceed by assuming $d=1$. Set $U=\perp_R M$ and $V=\top_R M$. Note that U is a nonzero maximal Cohen–Macaulay R-module. Choose a quasi-deformation $R\to R'\twoheadleftarrow S$ such that $\dim(R'/\mathfrak{m}R')=0$, $R'\cong S/(f)$ for some local ring $(S,\mathfrak{n})$ and a non-zero-divisor $f\in\mathfrak{n}$ on S such that $\mathrm{pd}_S(N\otimes_R R')<\infty$. So we may assume $R=R'=S/(f)$, where $\mathrm{pd}_S(N)<\infty$. As $\dim(R)=1$, we know that $\mathrm{length}_R(V)<\infty$. Thus $\theta^R(V,N)$ is well-defined; see §2.13.

Next we record two claims; we use these claims to prove the vanishing of $\mathrm{Tor}_i^R(M,N)$ for all $i\geq 1$, and defer the proofs of the claims until the end.

Claim 1. $$\theta^R(V,N)=0.$$

Assuming Claim 1 is true, consider the following short exact sequence of R-modules:

$$0\to V\to M\to U\to 0. \tag{3.2.3}$$

Recall that we have $\mathrm{length}_R(V)<\infty$ and $\mathrm{length}(\mathrm{Tor}_i^R(M,N))<\infty$ for all $i\gg 0$. Hence it follows from (3.2.3) that:

$$\mathrm{length}_R(\mathrm{Tor}_i^R(U,N))<\infty \quad \text{for all } i\gg 0. \tag{3.2.4}$$

In particular, $\theta^R(U,N)$ is well-defined; see §2.13. Moreover, by the additivity of θ pairing applied to the short exact sequence in (3.2.3), we see that:

$$\theta^R(M,N)=\theta^R(U,N)+\theta^R(V,N). \tag{3.2.5}$$

We know, by the hypothesis, that $\theta^R(M,N)=0$ and, by Claim 1, that $\theta^R(V,N)=0$. Therefore, it follows from (3.2.5) that:

$$\theta^R(U,N)=0. \tag{3.2.6}$$

As $\dim(R)=1$ and $\text{CI-dim}_R(N)<\infty$, it follows from §2.11 that there is a short exact sequence of finitely generated R-modules

$$0\to F\to Z\to N\to 0, \tag{3.2.7}$$

where F is free, $\text{CI-dim}_R(Z) = 0$, and $\text{cx}_R(Z) = \text{cx}_R(N) = 1$.

Claim 2. $U \otimes_R Z$ is a torsion-free R-module, $\text{length}_R(\text{Tor}_i^R(U, Z)) < \infty$ for each $i \geq 1$, and $\text{Tor}_i^R(U, Z) \cong \text{Tor}_i^R(U, N)$ for all $i \geq 1$.

Assuming Claim 2 is true, note that (3.2.7) implies $\text{pd}_S(Z) < \infty$ since $\text{pd}_S(N)$ and $\text{pd}_S(F)$ are finite. As $\text{length}_R(\text{Tor}_i^R(U, Z)) < \infty$ for all $i \geq 1$, we see from §2.13 that $\theta^R(U, Z)$ is well-defined and the following holds:

$$\theta^R(U, Z) = \text{length}_R(\text{Tor}_{2n+2}^R(U, Z)) - \text{length}_R(\text{Tor}_{2n+1}^R(U, Z)) \tag{3.2.8}$$

for each integer n with $2n > \text{CI-dim}_R(Z) - 1$, i.e., for each $n \geq 0$ (because $\text{CI-dim}_R(Z) = 0$).

We know, by Claim 2, that $\text{Tor}_i^R(U, Z) \cong \text{Tor}_i^R(U, N)$ for all $i \geq 1$. Hence it follows that $\theta^R(U, N) = \theta^R(U, Z)$. Thus (3.2.6) and (3.2.8) yield the following equalities of lengths:

$$\text{length}_R(\text{Tor}_{2i+2}^R(U, Z)) = \text{length}_R(\text{Tor}_{2i+1}^R(U, Z)) \quad \text{for all } i \geq 0. \tag{3.2.9}$$

Notice $0 = \text{CI-dim}_R(Z) = \text{depth}(R) - \text{depth}_R(Z)$, i.e., $\text{depth}_R(Z) = 1$; see §2.8(iii). Hence $\text{depth}_S(Z) = 1$. Since $\dim(S) = 2$ and $\text{pd}_S(Z) < \infty$, we conclude, by the Auslander–Buchsbaum formula, that $\text{pd}_S(Z) = 1$. Now we consider the following exact sequence that follows from §2.12 applied for the pair (U, Z):

$$\text{Tor}_2^S(U, Z) \to \text{Tor}_2^R(U, Z) \to U \otimes_R Z \to \text{Tor}_1^S(U, Z) \to \text{Tor}_1^R(U, Z) \to 0. \tag{3.2.10}$$

As $\text{pd}_S(Z) = 1$, we have $\text{Tor}_2^S(U, Z) = 0$. So, by (3.2.10), we see that $\text{Tor}_2^R(U, Z)$ embeds in $U \otimes_R Z$. Moreover, we know by Claim 2 that $\text{length}_R(\text{Tor}_2^R(U, Z)) < \infty$ and $U \otimes_R Z$ is a torsion-free R-module. So we conclude from (3.2.10) that

$$\text{Tor}_2^R(U, Z) = 0. \tag{3.2.11}$$

Consequently, (3.2.9) and (3.2.11) yield

$$\text{Tor}_1^R(U, Z) = 0 = \text{Tor}_2^R(U, Z). \tag{3.2.12}$$

On the other hand, as $\text{pd}_S(Z) = 1$, we can use §2.12 once more, this time for the pair (U, Z), and obtain

$$\text{Tor}_i^R(U, Z) \cong \text{Tor}_{i+2}^R(U, Z) \quad \text{for all } i \geq 1. \tag{3.2.13}$$

Therefore, we see from (3.2.12) and (3.2.13) that $\text{Tor}_i^R(U, Z) = 0$ for all $i \geq 1$. Thus Claim 2 yields the vanishing of $\text{Tor}_i^R(U, N)$ for all $i \geq 1$. Finally we can now invoke §2.4(iii) and deduce that $\text{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, as required.

Now we establish the claims and complete the proof of the theorem.

Proof of Claim 1. To prove the claim, we follow the argument of [Celikbas and Dao 2014, 4.6]. Note that $\Omega_R N$ is a maximal Cohen–Macaulay R-module. So

$\operatorname{depth}_R(\Omega_R N) = 1$, which equals $\operatorname{depth}_S(\Omega_R N)$. Note also that there is a short exact sequence $0 \to \Omega_R N \to T \to N \to 0$, where T is a finitely generated free R-module. This exact sequence, since both $\operatorname{pd}_S(N)$ and $\operatorname{pd}_S(T)$ are finite, implies that $\operatorname{pd}_S(\Omega_R N) < \infty$. Hence, since $\operatorname{depth}(S) = 2$, we conclude from the Auslander-Buchsbaum formula that $\operatorname{pd}_S(\Omega_R N) = 1$. It now follows from §2.12 that there is an exact sequence:

$$(3.2.14)\quad 0 \to \operatorname{Tor}_2^R(\Omega_R N, k) \to \Omega_R N \otimes_R k \to \operatorname{Tor}_1^S(\Omega_R N, k) \to \operatorname{Tor}_1^R(\Omega_R N, k) \to 0.$$

Taking the alternating sum of lengths of modules in (3.2.14), we obtain

$$(3.2.15)\qquad \theta^R(k, \Omega_R N) = \beta_2^R(\Omega_R N) - \beta_1^R(\Omega_R N) = \beta_0^S(\Omega_R N) - \beta_1^S(\Omega_R N),$$

where $\beta_i^R(\Omega_R N)$ denotes the i-th Betti number of $\Omega_R N$.

As $\Omega_R N$ has a finite free resolution over S, we can now apply § 2.16 for the module $\Omega_R N$ over the ring S: the Euler number of $\Omega_R N$ over S, which is $\beta_0^S(\Omega_R N) - \beta_1^S(\Omega_R N)$, vanishes since $f \cdot \Omega_R N = 0$. So, by (3.2.15), we have $\theta^R(k, \Omega_R N) = 0$. As $\theta^R(k, N) = -\theta^R(k, \Omega_R N)$, we see $\theta^R(k, N) = 0$. Moreover, since V has a finite filtration by copies of k, it follows that $\theta^R(V, N)$ vanishes. This justifies Claim 1. □

Proof of Claim 2. Notice, as F is free, tensoring (3.2.7) with U over R, we see that there are isomorphisms

$$(3.2.16)\qquad \operatorname{Tor}_i^R(U, Z) \cong \operatorname{Tor}_i^R(U, N) \quad \text{for each } i \geq 2,$$

and there is an exact sequence of the form

$$(3.2.17)\quad 0 \to \operatorname{Tor}_1^R(U, Z) \to \operatorname{Tor}_1^R(U, N) \xrightarrow{\gamma} U \otimes_R F \to U \otimes_R Z \to U \otimes_R N \to 0.$$

Let $\mathfrak{p}$ be a minimal prime ideal of R. Then it follows from (3.2.4) that

$$\operatorname{Tor}_i^R(U, N)_{\mathfrak{p}} = 0 \quad \text{for all } i \gg 0,$$

and hence (3.2.16) shows that $\operatorname{Tor}_i^R(U, Z)_{\mathfrak{p}} = 0$ for all $i \gg 0$. So it follows from §2.8(vii) that $\operatorname{Tor}_i^R(U, Z)_{\mathfrak{p}} = 0$ for all $i \geq \operatorname{CI-dim}_{R_{\mathfrak{p}}}(Z_{\mathfrak{p}}) + 1$. Also, by §2.8(ii), we know $\operatorname{CI-dim}_{R_{\mathfrak{p}}}(Z_{\mathfrak{p}}) \leq \operatorname{CI-dim}_R(Z) = 0$. So we see that $\operatorname{Tor}_i^R(U, Z)_{\mathfrak{p}} = 0$ for all $i \geq 1$. As $\mathfrak{p}$ is an arbitrary minimal prime ideal of R, this argument shows that $\operatorname{length}_R(\operatorname{Tor}_i^R(U, Z)) < \infty$ for all $i \geq 1$, as claimed.

Note that it follows from (3.2.16) that $\operatorname{length}_R(\operatorname{Tor}_i^R(U, N)) < \infty$ for all $i \geq 2$. In particular, $\operatorname{Tor}_i^R(U, N)$ is torsion for all $i \geq 2$. However, this forces $\operatorname{Tor}_i^R(U, N)$ to be torsion for each $i \geq 1$; see §2.8(vii). Thus the image of the map γ in (3.2.17) is torsion. On the other hand, $U \otimes_R F$, being a finite direct sum of copies of U, is torsion-free. So $\gamma = 0$, and it follows from (3.2.17) that $\operatorname{Tor}_i^R(U, Z) \cong \operatorname{Tor}_i^R(U, N)$, for $i = 1$ as well. Hence, by (3.2.16), we establish that $\mathrm{Tor}_i^R(U, Z) \cong \mathrm{Tor}_i^R(U, N)$ for all $i \geq 1$.

In light of the fact that $\gamma = 0$, the following exact sequence is induced from (3.2.17):

$$0 \to U \otimes_R F \to U \otimes_R Z \to U \otimes_R N \to 0. \tag{3.2.18}$$

As $M \otimes_R N$ and $U \otimes_R F$ are torsion-free and $U \otimes_R N \cong M \otimes_R N$, we conclude from (3.2.18) that $U \otimes_R Z$ is torsion-fee; see §2.2 and §2.4(i). This completes the proof of Claim 2. □

We finish this section by recording some remarks concerning Theorem 3.2.

Remark 3.3. It is worth mentioning that the conclusion of Theorem 3.2 is not necessarily true if the ring in question is Artinian. For example, if $R = \mathbb{C}[\![x]\!]/(x^2)$ and $M = N = R/(x)$, then R is an Artinian hypersurface (so that each R-module has finite complete intersection dimension and $\theta^R(-, -)$ is well-defined), $\mathrm{cx}_R(N) = 1$, $\mathrm{Tor}_i^R(M, N) \cong N \neq 0$ for all $i \geq 0$, and $\theta^R(M, N) = 0$.

In Section 4 we refer to the next fact to prove Proposition 4.4 and Remark 4.7.

Remark 3.4. Let R be a local ring and let N be a nonzero finitely generated R-module. Assume $\text{CI-dim}_R(N) = 0$ and $\mathrm{cx}_R(N) = 1$. Then it follows that $\beta_i^R(N) = \beta_{i+1}^R(N)$ and $\Omega_R^i(N) \cong \Omega_R^{i+2}(N)$, for all $i \geq 0$; see [Avramov et al. 1997, 7.3].

Note that, in Theorem 3.2, we have $\text{CI-dim}_R(\Omega_R N) = 0$ and $\mathrm{cx}_R(\Omega_R N) = 1$. Therefore Remark 3.4 implies $\beta_2^R(\Omega_R N) = \beta_1^R(\Omega_R N)$, and so $\theta^R(k, \Omega_R N) = 0$; see (3.2.15) in the proof of Theorem 3.2. This gives an alternative way of establishing the vanishing of $\theta^R(k, \Omega_R N)$ without appealing to the property of the Euler number recorded in §2.16.

Next we consider Theorem 3.2 for the case where $\mathrm{cx}_R(N) = 0$, i.e., $\mathrm{pd}_R(N) < \infty$. In this case we can obtain the vanishing of $\mathrm{Tor}_i^R(M, N)$ without any depth assumption on M.

Remark 3.5. Let R be a d-dimensional Cohen–Macaulay local ring, and let M and N be finitely generated R-modules. Assume $\mathrm{pd}_R(N) < \infty$ and $\mathrm{depth}_R(N) \geq d - 1$. If $M \otimes_R N$ is (nonzero) maximal Cohen–Macaulay, then N is free and M is maximal Cohen–Macaulay.

To establish this, we may assume $\mathrm{pd}_R(N) \neq 0$, as otherwise N would be free. In particular, we may assume $d \geq 1$. Note, by the Auslander–Buchsbaum formula and the hypothesis, we have that $\mathrm{pd}_R(N) = 1$. Set $U = \perp_R M$. Then, since U is a torsion-free R-module, [Celikbas and Takahashi 2019, 2.7] implies that $\mathrm{Tor}_i^R(U, N) = 0$ for all $i \geq 1$. Hence § 2.4(iii) gives the vanishing of $\mathrm{Tor}_i^R(M, N)$ for all $i \geq 1$. As $\mathrm{pd}_R(N) < \infty$, the depth formula for M and N over R holds; see § 2.8(viii). This shows, since $M \otimes_R N$ is maximal Cohen–Macaulay, that both M and N

are maximal Cohen–Macaulay R-modules. Consequently, N is free due to the Auslander–Buchsbaum formula.

4. Some corollaries of the main result

In this section we proceed to give various corollaries of Theorem 3.2 concerning the torsion in tensor products of modules, especially those of the form $M \otimes_R M^*$ over one dimensional local rings. In particular, we give a proof of Corollary 1.4; see Corollaries 4.6 and 4.8. Along the way we extend results of Huneke and Wiegand [1994], and Auslander [1961] on the reflexivity of tensor products of modules which justify a higher dimensional version of Conjecture 1.1 over normal domains; see Proposition 4.12 and Conjecture 4.13.

We denote by $\mathsf{G}(R)$ the *Grothendieck group* of finitely generated R-modules, i.e., the quotient of the free abelian group of all isomorphism classes of finitely generated R-modules by the subgroup generated by the relations coming from short exact sequences of finitely generated R-modules. We write $[M]$ for the class of a finitely generated R-module M in $\mathsf{G}(R)$ and denote by $\overline{\mathsf{G}}(R)$ the group $\mathsf{G}(R)/\mathbb{Z}\cdot[R]$, the reduced Grothendieck group of R. We set $\overline{\mathsf{G}}(R)_{\mathbb{Q}} = (\mathsf{G}(R)/\mathbb{Z}\cdot[R]) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The next corollary corroborates [Celikbas 2011, 1.2], which examines the vanishing of Tor for modules of complexity at most one over complete intersection rings.

Corollary 4.1. *Let R be a one-dimensional local ring, and let M and N be nonzero finitely generated R-modules. Assume* $\text{CI-dim}_R(N) < \infty$, $\text{cx}_R(N) \leq 1$, *and $N_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime ideal $\mathfrak{p}$ of R.*

(i) *If* $\text{pd}_R(N) < \infty$ *and $M \otimes_R N$ is torsion-free, then M is torsion-free and N is free.*

(ii) *Assume* $\text{pd}_R(N) = \infty$, *or equivalently*, $\text{cx}_R(N) = 1$, *and* $[M] = 0$ *in* $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$. *Then*:

  (a) *If $M \otimes_R N$ is torsion-free, then so are M and N, and* $\text{Tor}_i^R(M, N) = 0$ *for all $i \geq 1$.*

  (b) *If* $\text{Tor}_n^R(M, N) = 0$ *for some $n \geq 1$, then* $\text{Tor}_i^R(M, N) = 0$ *for all $i \geq n$, i.e., the pair (M, N) is Tor-rigid.*

Proof. We may assume R is Cohen–Macaulay; as otherwise N would be free and all the claims follow. In particular, part (i) is a special case of Remark 3.5. Hence we assume $\text{cx}_R(N) = 1$ and $[M] = 0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$, and proceed to prove part (ii).

Let X be a finitely generated R-module. As $N_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime ideal $\mathfrak{p}$ of R, we have $\text{length}_R(\text{Tor}_i^R(N, X)) < \infty$ for all $i \geq 1$. In particular, $\theta^R(N, X)$ is well-defined; see §2.14. This yields, since θ is additive on short exact sequence of finitely generated R-modules, a linear map $\theta^R(N, -) : \mathsf{G}(R) \to \mathbb{Z}$.

Moreover, as $\theta^R(N, R) = 0$, this map induces a map $\theta^R(N, -) : \overline{\mathsf{G}}(R)_{\mathbb{Q}} \to \mathbb{Q}$. Hence, since $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$, we have that $\theta^R(M, N) = 0$. Now, if $M \otimes_R N$ is torsion-free, then it follows from Theorem 3.2 that M and N are torsion-free, and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. This establishes part (ii)(a).

To prove part (ii)(b), we proceed by assuming $\operatorname{Tor}_n^R(M, N) = 0$ for some $n \geq 1$. Then we have $\operatorname{Tor}_1^R(M, \Omega_R^{n-1}(N)) = 0$ and $\text{CI-dim}_R(\Omega_R^{n-1}(N)) < \infty$; see §2.10. Hence we use the exact sequence that follows from §2.11 for the module $\Omega_R^{n-1}(N)$:

$$0 \to F \to Z \to \Omega_R^{n-1}(N) \to 0. \tag{4.1.1}$$

Here F is free, $\text{CI-dim}_R(Z) = 0$, and $\operatorname{cx}_R(Z) = \operatorname{cx}_R(\Omega_R^{n-1}(N)) = 1$. Note that, by (2.5.1), we have that $\text{G-dim}_R(Z) = 0$, i.e., Z is totally reflexive and hence Z is torsion-free. Moreover, $\operatorname{Tor}_1^R(M, Z)$ vanishes since

$$\operatorname{Tor}_1^R(M, Z) \hookrightarrow \operatorname{Tor}_1^R(M, \Omega_R^{n-1}(N)).$$

Applying $- \otimes_R Z$ to the syzygy exact sequence $0 \to \Omega_R(M) \to R^{\oplus v} \to M \to 0$, where v is a positive integer, we see that $\Omega_R(M) \otimes_R Z$ is contained in $Z^{\oplus v}$. So, $\Omega_R(M) \otimes_R Z$ is torsion-free. As $[M] = 0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$, it follows from the syzygy sequence that $[\Omega_R(M)] = 0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$. Furthermore, by (4.1.1), we know $Z_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime ideal $\mathfrak{p}$ of R. Hence, by using part (ii)(a) for the pair $(\Omega_R(M), Z)$, we conclude that $\operatorname{Tor}_i^R(\Omega_R(M), Z) = 0$ for all $i \geq 1$. This implies the vanishing of $\operatorname{Tor}_i^R(M, Z)$ for all $i \geq 2$. Consequently, we deduce $\operatorname{Tor}_i^R(M, Z) = 0$ for all $i \geq 1$ since we already know that $\operatorname{Tor}_1^R(M, Z)$ vanishes. This implies that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq n+1$, as claimed. □

The next two remarks are concerned with Corollary 4.1.

Remark 4.2. If R is a one-dimensional local ring and M is a finitely generated R-module (not necessarily torsion-free) which has rank, then it follows that $[M] = 0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$; see [Celikbas and Dao 2011, 2.5; Huneke and Wiegand 1994, 1.3]. In view of this fact and Corollary 4.1, we have the following result:

If R is a one-dimensional local ring, and M and N are nonzero finitely generated R-modules such that $\text{CI-dim}_R(N) < \infty$, $\operatorname{cx}_R(N) = 1$, $N_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime $\mathfrak{p}$ of R, M has rank and $M \otimes_R N$ is torsion-free, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, and both M and N are torsion-free.

Remark 4.3. Let R be a one-dimensional complete intersection domain, and let M and N be nonzero finitely generated R-modules.

It is not known whether R-modules are Tor-rigid, in general. Moreover, if $M \otimes_R N$ is torsion-free, it is also not known whether M or N must be torsion-free; see [Celikbas and Wiegand 2015, 2.10]. In fact, Tor-rigidity yields an affirmative answer to the latter query: if $M \otimes_R N$ is torsion-free and N is Tor-rigid, then it follows $\operatorname{Ext}^1_R(\operatorname{Tr} M, N) = 0$ and hence $\operatorname{Ext}^1_R(\operatorname{Tr} M, R) = 0$, i.e., M is torsion-free; see, for example, [Auslander and Bridger 1969, 2.8; Celikbas et al. 2019b, 3.4].

Corollary 4.1 gives a partial affirmative answer to the aforementioned open problems. It points out that modules of complexity at most one, i.e., modules of bounded Betti numbers, are Tor-rigid over R. Moreover, it shows that, if $M \otimes_R N$ is torsion-free, and M or N has complexity at most one, then both M and N are torsion-free (note that, for a one-dimensional local domain R, one has $\overline{\mathsf{G}}(R)_{\mathbb{Q}} = 0$; see, for example, [Celikbas and Dao 2011, 2.5]).

It is known that the conclusion of Corollary 4.1 may fail in case $[M] \neq 0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$. For example, if $R = k[\![x, y]\!]/(xy)$, $M = R/(x)$, and $N = R/(x^2)$, then $M_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ are both free over $R_{\mathfrak{p}}$ for each associated prime ideal $\mathfrak{p}$ of R (as R is reduced) and $M \otimes_R N$ is torsion-free, but $\operatorname{Tor}^R_{2i-1}(M, N) \neq 0 = \operatorname{Tor}^R_{2i}(M, N)$ for all $i \geq 1$; see [Huneke and Wiegand 1997, page 164] and also §A.1. This example also illustrates the following:

Proposition 4.4. *Let R be a one-dimensional local ring, and let M and N be finitely generated R-modules. Assume* $\operatorname{CI-dim}_R(N) < \infty$ *and* $\operatorname{cx}_R(N) = 1$. *Assume further $M_{\mathfrak{p}}$ or $N_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime $\mathfrak{p}$ of R (e.g., R is reduced). If $M \otimes_R N$ is torsion-free, then* $\operatorname{Tor}^R_{2i}(\perp_R M, N) = 0$ *for all $i \geq 1$.*

Proof. Note that we may assume $M \neq 0 \neq N$. We may further assume R is Cohen–Macaulay, as otherwise M or N would be free and the claim would follow. Moreover, if $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime ideal $\mathfrak{p}$ of R, then so is $\perp_R M$. Therefore, we may replace M with $\perp_R M$, and assume M is a nonzero torsion-free R-module; see §2.4(i).

There is a short exact sequence of R-modules of the form

$$0 \to F \to Z \to N \to 0, \tag{4.4.1}$$

where F is free, $\operatorname{CI-dim}_R(Z) = 0$, and $\operatorname{cx}_R(Z) = \operatorname{cx}_R(N)$; see §2.11. By tensoring (4.4.1) with M over R, we obtain an exact sequence,

$$\operatorname{Tor}^R_1(M, N) \xrightarrow{\nu} M \otimes_R F \to M \otimes_R Z \to M \otimes_R N \to 0. \tag{4.4.2}$$

As $\operatorname{Tor}^R_1(M, N)$ is torsion and M is torsion-free, we conclude ν is the zero map and that $M \otimes_R Z$ is torsion-free; see §2.2.

Next we consider a pushforward sequence of Z, i.e., a short exact sequence of R-modules as

$$0 \to Z \to G \to Z_1 \to 0, \tag{4.4.3}$$

where G is free, $\operatorname{CI-dim}_R(Z_1) = 0$, and also $\operatorname{cx}_R(Z_1) = \operatorname{cx}_R(Z) = \operatorname{cx}_R(N) = 1$; see §2.8(iii), §2.10, and §B.3. Notice it follows from Remark 3.4 that $Z_1 \cong \Omega^2_R(Z_1)$.

Therefore, as $Z \cong \Omega_R(Z_1)$, we conclude that $\Omega_R(Z) \cong \Omega^2_R(Z_1) \cong Z_1$. This yields the short exact sequence of R-modules

$$0 \to Z \to G \to \Omega_R(Z) \to 0. \tag{4.4.4}$$

Tensoring (4.4.4) with M over R, we obtain an injection

$$\operatorname{Tor}_1^R(\Omega_R(Z), M) \hookrightarrow M \otimes_R Z.$$

As $M \otimes_R Z$ is torsion-free and $\operatorname{Tor}_1^R(\Omega_R(Z), M)$ is torsion, we see $\operatorname{Tor}_1^R(\Omega_R(Z), M)$ vanishes, i.e., $0 = \operatorname{Tor}_1^R(\Omega_R(Z), M) \cong \operatorname{Tor}_2^R(Z, M)$. This forces $\operatorname{Tor}_{2i}^R(Z, M) = 0$ for all $i \geq 1$ since $Z \cong \Omega^2_R(Z)$. This completes the proof of the proposition: due to (4.4.1), we have that $\operatorname{Tor}_{2i}^R(Z, M) \cong \operatorname{Tor}_{2i}^R(N, M)$ for all $i \geq 1$. □

Our next observation may be of independent interest: the first part examines the complete intersection dimension of a torsion-free module with its algebraic dual over one-dimensional local rings without any depth assumption on their tensor products. The second part of Lemma 4.5 is our first step to establish consequences of Theorem 3.2 concerning Conjecture 1.1 — it is used in the proof of Corollary 4.6.

Lemma 4.5. *Let R be a one-dimensional local ring, and let M be a finitely generated R-module.*

(i) *Assume M is torsion-free. Then*

$$\text{CI-dim}_R(M) < \infty \quad \textit{if and only if} \quad \text{CI-dim}_R(M^*) < \infty.$$

(ii) *Assume* $\text{CI-dim}_R(M) < \infty$, *and $M \otimes_R M^*$ is a nonzero torsion-free R-module. If* $\operatorname{Tor}_i^R(M, M^*) = 0$ *for all $i \gg 0$, then M is free.*

Proof. (i) If $\text{CI-dim}_R(M) < \infty$, then, since $\operatorname{depth}_R(M) = 1$, it follows from §2.8(iii) that $\text{CI-dim}_R(M) = 0$; this implies, in view of §2.8(vi), that $\text{CI-dim}_R(M^*) = 0$. Hence it suffices to assume $\text{CI-dim}_R(M^*) < \infty$ and show that $\text{CI-dim}_R(M) < \infty$.

Assume $\text{CI-dim}_R(M^*) < \infty$. Then (2.7.1) and §2.10 show that

$$\text{CI-dim}_R(\operatorname{Tr} M) < \infty.$$

Let $\mathfrak{p}$ be an associated prime ideal of R. Then, by (2.5.1), we have

$$\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \text{CI-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}),$$

and it follows from §2.8(iii) that $\text{CI-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \operatorname{depth}(R_\mathfrak{p}) = 0$. Hence $M_\mathfrak{p}$ is totally reflexive over $R_\mathfrak{p}$. Therefore, it follows that $\operatorname{Ext}^1_R(\operatorname{Tr} M, R)_\mathfrak{p} = 0$. This shows, as $\mathfrak{p}$ is an arbitrary associated prime ideal of R, that $\operatorname{Ext}^1_R(\operatorname{Tr} M, R)$ is a torsion R-module. Now, since $\operatorname{Ext}^1_R(\operatorname{Tr} M, R) \hookrightarrow M$ and as M is torsion-free, it follows that $\operatorname{Ext}^1_R(\operatorname{Tr} M, R) = 0$; see [Auslander and Bridger 1969, 2.8]. Consequently §2.8(iii) shows that $\text{CI-dim}_R(\operatorname{Tr} M) = 0$. Note, up to free summands, we have that $\operatorname{Tr}\operatorname{Tr} M \cong M$. Thus §2.8(vi) implies that $\text{CI-dim}_R(\operatorname{Tr}\operatorname{Tr} M) = 0$, i.e., $\text{CI-dim}_R(M) = 0$, as required.

(ii) Notice, since $\operatorname{Tor}_i^R(M, M^*) = 0$ for all $i \gg 0$, we have that $\operatorname{Tor}_i^R(M, \operatorname{Tr} M) = 0$ for all $i \gg 0$; see (2.7.1). Hence it suffices to prove that $\text{CI-dim}_R(M) = 0$: in this case, §2.8(vii) implies $\operatorname{Tor}_1^R(M, \operatorname{Tr} M) = 0$ so that M is free; by, for example, [Yoshino 1990, 3.9]. Consequently, as $\text{CI-dim}_R(M) \leq \operatorname{depth}(R)$, we may assume $\operatorname{depth}(R) \neq 0$, i.e., we may assume R is a one-dimensional Cohen–Macaulay local ring.

Let $\mathfrak{p}$ be a minimal prime ideal of R. Then $\text{CI-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$; see §2.8(iii). Moreover, as $\operatorname{Tor}_i^R(M, M^*)_\mathfrak{p} = 0$ for all $i \gg 0$, we conclude from § 2.8(vii) that $\operatorname{Tor}_i^R(M, M^*)_\mathfrak{p} = 0$ for all $i \geq \text{CI-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) + 1$. This implies that each $\operatorname{Tor}_i^R(M, M^*)$ has finite length for $i \geq 1$. Thus it follows from [Celikbas et al. 2015b, 3.6] that $\operatorname{Tor}_i^R(M, M^*) = 0$ for all $i \geq 1$. Since $M \otimes_R M^*$ is nonzero and torsion-free, the depth formula implies $\operatorname{depth}_R(M) = 1$; see §2.8(viii). Finally, we conclude by §2.8(iii) that $\text{CI-dim}_R(M) = 0$, as claimed. □

If $R = S/(f)$, where $(S, \mathfrak{n})$ is a two-dimensional regular local ring and $0 \neq f \in \mathfrak{n}$, it follows from a result of Huneke and Wiegand [1994, 3.7] that $M \otimes_R M^*$ has torsion for each nonfree, torsion-free finitely generated R-module M with rank. In particular, Conjecture 1.1 is true over hypersurface rings. In Corollary 4.6, we generalize this fact and show that it carries over to Cohen–Macaulay rings (not necessarily hypersurcases) under mild conditions; see also Corollaries 4.8 and 4.10 for related results.

In Corollary 4.6, we assume $R = S/(f)$, where $(S, \mathfrak{n})$ is a two-dimensional Cohen–Macaulay local ring, and $f \in \mathfrak{n}$ is a non-zero-divisor on S. We assume further M is a finitely generated R-module such that $\operatorname{pd}_S(M) < \infty$. Then, by using the quasi-deformation $R \xrightarrow{=} R \twoheadleftarrow S$, we have that $\text{CI-dim}_R(M) < \infty$; see §2.5. Moreover, [Avramov 1989, 3.2(3)] shows that $\operatorname{cx}_R(X) \leq \operatorname{cx}_S(X) + 1$ for each finitely generated R-module X, in particular, $\operatorname{cx}_R(M) \leq 1$. In Corollary 4.6, we also consider the case where $\operatorname{pd}_S(M^*) < \infty$, i.e., $\operatorname{pd}_S(\operatorname{Hom}_R(M, R)) < \infty$. It is worth noting that, in general, a module can have finite projective dimension, even though its algebraic dual has infinite projective dimension; see, for example, [Huneke and Jorgensen 2003, 2.3].

Corollary 4.6. *Let R be a one-dimensional Cohen–Macaulay ring such that, for some local ring $(S, \mathfrak{n})$ and a non-zero-divisor $f \in \mathfrak{n}$ on S, we have $R = S/(f)$. Let M be a finitely generated R-module. Assume the following hold*:

(a) $\operatorname{pd}_S(M) < \infty$ *or* $\operatorname{pd}_S(M^*) < \infty$.

(b) *$M \otimes_R M^*$ is a torsion-free R-module.*

Then M is free provided that at least one of the following conditions hold:

(i) *$M \otimes_R M^*$ is not zero, and M has rank over R.*

(ii) *M is a torsion-free module that has rank over R.*

(iii) *$M \otimes_R M^*$ is not zero,*

$$\operatorname{length}_R(\operatorname{Tor}_i^R(M, M^*)) < \infty \quad \textit{for all } i \gg 0, \quad \textit{and} \quad \theta^R(M, M^*) = 0.$$

(iv) *M is torsion-free,*

$$\operatorname{length}_R(\operatorname{Tor}_i^R(M, M^*)) < \infty \quad \textit{for all } i \gg 0, \quad \textit{and} \quad \theta^R(M, M^*) = 0.$$

(v) *M is torsion-free, and*

$$\operatorname{length}_R(\operatorname{Tor}_n^R(M, M^*)) = \operatorname{length}_R(\operatorname{Tor}_{n+q}^R(M, M^*)) < \infty$$

for an even integer $n \geq 2$ and an odd integer $q \geq 1$.

Proof. We may assume $M \neq 0$. Recall, if $M \neq 0$ and M is torsion-free, then $M \otimes_R M^* \neq 0$; see §2.3. This implies, for each part, we have that $M \otimes_R M^*$ is a nonzero torsion-free R-module. Furthermore, it shows that part (ii) follows from part (i), and part (iv) follows from part (iii).

Note, as $\operatorname{pd}_S(M) < \infty$ or $\operatorname{pd}_S(M^*) < \infty$, we have $\operatorname{CI-dim}_R(M) < \infty$ or $\operatorname{CI-dim}_R(M^*) < \infty$; see §2.5. Hence it suffices to show $\operatorname{Tor}_i^R(M, M^*) = 0$ for all $i \gg 0$, and M is torsion-free: in that case we can use Lemma 4.5: the first part of the lemma implies $\operatorname{CI-dim}_R(M) < \infty$, and hence the second part shows that M is free.

If $\operatorname{pd}_R(M^*) < \infty$, then M^* is free by the Auslander–Buchsbaum formula. This implies, since $M \otimes_R M^*$ is a nonzero torsion-free R-module, that M is torsion-free. So M must be free. Similarly, if $\operatorname{pd}_R(M) < \infty$, then Remark 3.5 shows that M is free. Moreover, we know that $\operatorname{cx}_R(M) \leq 1$, or $\operatorname{cx}_R(M^*) \leq 1$; see [Avramov 1989, 3.2(3)]. Consequently, we may assume $\operatorname{CI-dim}_R(M) < \infty$ and $\operatorname{cx}_R(M) = 1$, or $\operatorname{CI-dim}_R(M^*) < \infty$ and $\operatorname{cx}_R(M^*) = 1$.

(i) Assume M has rank over R. Then M^* also has rank, which equals the rank of M. In particular, both M and M^* are free when localized at each associated prime ideal of R. Thus Remark 4.2 yields the vanishing of $\operatorname{Tor}_i^R(M, M^*)$ for all $i \geq 1$ and the fact that M is torsion-free.

(iii) Assume $\operatorname{length}_R(\operatorname{Tor}_i^R(M, M^*)) < \infty$ for all $i \gg 0$, and $\theta^R(M, M^*) = 0$. In that case Theorem 3.2 yields the vanishing of $\operatorname{Tor}_i^R(M, M^*)$ for all $i \geq 1$, and that M is torsion-free.

(v) Recall that, by Lemma 4.5(i), we know $\operatorname{CI-dim}_R(M) < \infty$. Let $\mathfrak{p} \in \operatorname{Supp}_R(M)$ be a minimal prime ideal of R. Then it follows

$$\operatorname{CI-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0 \quad \text{and} \quad \operatorname{Tor}_n^R(M, M^*)_{\mathfrak{p}} = \operatorname{Tor}_{n+q}^R(M, M^*)_{\mathfrak{p}} = 0;$$

see §2.8(ii). Therefore, by [Bergh 2008, 3.2], we see $\operatorname{Tor}_i^R(M, M^*)_{\mathfrak{p}} = 0$ for all $i \geq 1$. This implies that $\operatorname{length}_R(\operatorname{Tor}_i^R(M, M^*)) < \infty$ for each $i \geq 1$. We also know that $\operatorname{Tor}_i^R(M, M^*) \cong \operatorname{Tor}_{i+2}^R(M, M^*)$ for each $i \geq 1$; see Remark 3.4. So the hypothesis $\operatorname{length}_R(\operatorname{Tor}_n^R(M, M^*)) = \operatorname{length}_R(\operatorname{Tor}_{n+q}^R(M, M^*))$ implies that

$$\operatorname{length}_R(\operatorname{Tor}_i^R(M, M^*)) = \operatorname{length}_R(\operatorname{Tor}_{i+1}^R(M, M^*)) \quad \text{for each } i \geq 1.$$

Hence $\theta^R(M, M^*) = 0$ and the result follows from part (iv). □

If M and N are finitely generated modules over a local ring R such that M is totally reflexive, then it follows from [Avramov and Buchweitz 2000, 4.4.7] that

$$\widehat{\operatorname{Tor}}_i^R(M, N) \cong \widehat{\operatorname{Ext}}_R^{-i-1}(M^*, N) \quad \text{for all } i \in \mathbb{Z}.$$

Here $\widehat{\operatorname{Tor}}$ and $\widehat{\operatorname{Ext}}$ denote the Tate Tor and Ext, respectively; see, for example, [Avramov and Buchweitz 2000] for details.

In the following, for Gorenstein rings, we provide an alternative proof of Corollary 4.6(iv) that does not appeal to Theorem 3.2.

Remark 4.7. Let R be a one-dimensional Cohen–Macaulay ring such that, for some local ring $(S, \mathfrak{n})$ and $f \in \mathfrak{n}$ a non-zero-divisor on S, we have $R = S/(f)$. Let M be a nonzero finitely generated torsion-free R-module such that $\operatorname{pd}_S(M) < \infty$ or $\operatorname{pd}_S(M^*) < \infty$.

If $\operatorname{pd}_R(M) < \infty$, then M is free by the Auslander–Buchsbaum formula. So we may assume $\operatorname{pd}_R(M) = \infty$. Then it follows that $\operatorname{CI-dim}_R(M) = 0$ and $\operatorname{cx}_R(M) = 1$; see Lemma 4.5(i). Then we have

$$\widehat{\operatorname{Tor}}_i^R(M, N) \cong \widehat{\operatorname{Tor}}_{i+2}^R(M, N) \quad \text{and} \quad \widehat{\operatorname{Ext}}_R^i(M, N) \cong \widehat{\operatorname{Ext}}_R^{i+2}(M, N) \quad \text{for all } i \in \mathbb{Z};$$

see Remark 3.4. Therefore, as M is totally reflexive, we have the following isomorphisms for all $i \geq 1$:

$$\operatorname{Tor}_{2i-1}^R(M^*, M) \cong \widehat{\operatorname{Tor}}_{2i-1}^R(M^*, M) \cong \widehat{\operatorname{Ext}}_R^{-2i}(M, M) \cong \widehat{\operatorname{Ext}}_R^{2i}(M, M) \cong \operatorname{Ext}_R^{2i}(M, M),$$

$$\operatorname{Tor}_{2i}^R(M^*, M) \cong \widehat{\operatorname{Tor}}_{2i}^R(M^*, M) \cong \widehat{\operatorname{Ext}}_R^{-2i-1}(M, M) \cong \widehat{\operatorname{Ext}}_R^{2i-1}(M, M) \cong \operatorname{Ext}_R^{2i-1}(M, M).$$

In particular, if $\operatorname{Tor}_{2i-1}^R(M^*, M) = 0$ for some $i \geq 1$, then M is free; see [Avramov and Buchweitz 2000, 4.2].

Now assume R is Gorenstein and $\operatorname{length}_R(\operatorname{Tor}_i^R(M, M^*)) < \infty$ for all $i \gg 0$. Then it follows $\operatorname{length}_R(\operatorname{Tor}_i^R(M, M^*)) < \infty$ for all $i \geq 1$; see §2.8(ii, viii). In particular, we have that $\operatorname{length}_R(\operatorname{Ext}_R^1(M, M)) < \infty$.

Now assume $M \otimes_R M^*$ is torsion-free. Then [Huneke and Jorgensen 2003, 5.9] implies $\operatorname{Ext}_R^1(M, M) = 0$. Therefore, $M \otimes_R M^*$ is torsion-free if and only if $\operatorname{Tor}_{2i}^R(M^*, M) = 0$ for all $i \geq 1$. Consequently, if $\theta^R(M, M^*) = 0$ and $M \otimes_R M^*$ is torsion-free, then $\operatorname{Tor}_j^R(M, M^*) = 0$ for all $j \geq 1$, and hence M is free, by, for example, Lemma 4.5(ii).

If $(S, \mathfrak{n})$ is a Cohen–Macaulay local ring and $f \in \mathfrak{n}^2$ is a non-zero-divisor on S, then f has a reduced matrix factorization (φ, ψ) over S. In this case, $\operatorname{coker}(\varphi)$ is a nonfree, maximal Cohen–Macaulay module over $S/(f)$ which has projective dimension one over S; see [Herzog et al. 1991, 1.2 and 2.2].

A local ring S is called *G-regular* [Takahashi 2008] if each totally reflexive S-module is free. It is known that each regular ring, as well as each Golod ring that is not a hypersurface, is G-regular. In particular, every non-Gorenstein Cohen–Macaulay local ring with minimal multiplicity is G-regular; see [Takahashi 2008, 5.1]. Note that, if $R = S/(f)$, where $(S, \mathfrak{n})$ is a G-regular ring, and $f \in \mathfrak{n}^2$ is a non-zero-divisor on S, then R is not G-regular; see [Takahashi 2008, 4.6].

The following, advertised in Corollary 1.4, follows from Corollary 4.6 and [Takahashi 2008, 2.13].

Corollary 4.8. *Let $R = S/(f)$ be a one-dimensional Cohen–Macaulay ring, where $(S, \mathfrak{n})$ is a local ring and $f \in \mathfrak{n}^2$ is a non-zero-divisor on S. Assume M is a finitely generated R-module that has rank. Then $M \otimes_R M^*$ has torsion if at least one of the following holds*:

(i) $M = \operatorname{coker}(\varphi)$, *where (φ, ψ) is a reduced matrix factorization of f.*

(ii) *S is G-regular and M is a nonfree totally reflexive R-module.*

Proof. (i) We know M, the cokernel of φ, is a nonfree, torsion-free module over R. Since $\operatorname{pd}_S(M) < \infty$ and M has rank over R, it follows from Corollary 4.6(ii) that $M \otimes_R M^*$ has torsion.

(ii) As M is a totally reflexive R-module, it follows that $\text{G-dim}_S(M) < \infty$; see, for example, [Takahashi 2008, 1.5(3)(ii)]. Hence, since S is G-regular, we conclude that $\operatorname{pd}_S(M) < \infty$, and so the claim follows from Corollary 4.6(ii). □

Here is an example for which we can employ Corollary 4.8(i); note the ring in question is a complete intersection, but is not a hypersurface; see also [Celikbas 2011, 4.17; Huneke and Wiegand 1994, 3.7].

Example 4.9. Let $R = S/(f)$, where $S = \mathbb{C}[\![x, y, z]\!]/(xz - y^2)$ and $f = x^3 - z^2$. Then it follows that $R \cong \mathbb{C}[\![t^4, t^5, t^6]\!]$ is a one-dimensional local domain. Moreover,

$$(\varphi, \psi) = \left(\begin{pmatrix} -z & x \\ x^2 & -z \end{pmatrix}, \begin{pmatrix} z & x \\ x^2 & z \end{pmatrix} \right)$$

is a reduced matrix factorization of f over S. Therefore, by Corollary 4.8(i), the tensor product $M \otimes_R M^*$ has torsion, where M is the finitely generated R-module given by the exact sequence $0 \to S^{\oplus 2} \xrightarrow{\varphi} S^{\oplus 2} \to M \to 0$.

Next is a reformulation of Corollary 4.6(i, ii); it shows that Conjecture 1.1 is true for modules that have finite complete intersection dimension and bounded Betti numbers. We separate this result for the convenience of the reader as it is stated slightly different to Corollary 4.6.

Corollary 4.10. *Let R be a one-dimensional local ring, and let M be a non-free finitely generated R-module that has rank (e.g., R is a domain). Assume* $\text{CI-dim}_R(M) < \infty$ *and* $\text{cx}_R(M) \leq 1$ *(e.g., R is a hypersurface ring). If $M \otimes_R M^*$ is not zero, then $M \otimes_R M^*$ has torsion. In particular, if M is torsion-free, then $M \otimes_R M^*$ has torsion.*

Proof. Note that, since M is not free but has rank, R is a Cohen–Macaulay ring. Note also that, if M is torsion-free, then $M \otimes_R M^* \neq 0$; see §2.3. Hence it suffices to assume $M \otimes_R M^*$ is not zero and prove that $M \otimes_R M^*$ has torsion.

Suppose $M \otimes_R M^*$ is a nonzero torsion-free R-module, and seek a contradiction. It follows from Remark 3.5 that $\text{pd}_R(M) = \infty$, i.e., $\text{cx}_R(M) = 1$. Then we may choose a codimension one quasi-deformation $R \to R' \twoheadleftarrow S$ with zero-dimensional closed fibre such that $\text{pd}_S(M \otimes_R R') < \infty$; see §2.14. Thus $R' \cong S/(f)$ for some local ring $(S, \mathfrak{n})$, and a non-zero-divisor $f \in \mathfrak{n}$ on S. Moreover, it follows that R' is a one-dimensional Cohen–Macaulay ring, $M' = M \otimes_R R'$ has rank over R', and $M' \otimes_{R'} (M')^* \neq 0$. Now Corollary 4.6(i), applied to the module M' over R', shows that M' is free over R', which implies M is free over R, i.e., a contradiction. Hence, if $M \otimes_R M^*$ is not zero, then $M \otimes_R M^*$ must have torsion. □

Further remarks related to Conjecture 1.1. Huneke and Wiegand, motivated by a theorem of Auslander [1961, 3.3], proved that, if R is a local domain satisfying Serre's condition (S_2) such that $R_{\mathfrak{p}}$ is a hypersurface for each height-one prime ideal $\mathfrak{p}$ of R, and M is a finitely generated torsion-free R-module such that $M \otimes_R M^*$ is reflexive, then M is free; see [Huneke and Wiegand 1994, 5.2]. In this subsection we slightly strengthen this result, and show that it holds for R-modules M (not necessarily torsion-free) such that $M \otimes_R M^*$ is nonzero and reflexive; see Proposition 4.12. We also discuss a higher dimensional version of Conjecture 1.1; see Conjecture 4.13 and also Proposition 4.14.

We proceed with a lemma:

Lemma 4.11. *Let R be a local ring, and let M be a finitely generated R-module such that $M^* \neq 0$. If $\text{Ext}^1_R(\text{Tr}\, M, M^*) = \text{Ext}^2_R(\text{Tr}\, M, M^*) = 0$, then M is free.*

Proof. There is an exact sequence $0 \to M^* \to F \to G \to \text{Tr}\, M \to 0$, where F and G are finitely generated free R-modules; see (2.7.1). This yields the following exact sequences:

$$(4.11.1) \qquad 0 \to M^* \to F \to L \to 0 \quad \text{and} \quad 0 \to L \to G \to \text{Tr}\, M \to 0.$$

As $0 = \text{Ext}^2_R(\text{Tr}\, M, M^*) \cong \text{Ext}^1_R(L, M^*)$, the exact sequence $0 \to M^* \to F \to L \to 0$ splits; this implies M^* is free. Since $\text{Ext}^1_R(\text{Tr}\, M, M^*) = \text{Ext}^2_R(\text{Tr}\, M, M^*) = 0$, we conclude that $\operatorname{Ext}^1_R(\operatorname{Tr} M, R) = \operatorname{Ext}^2_R(\operatorname{Tr} M, R) = 0$. Therefore, the natural map $M \to M^{**}$ is bijective, i.e., M is reflexive. Note, as M^* is free, so is M^{**}. Consequently, we deduce that M is free. □

Note that, if R is a local ring and M is a finitely generated R-module such that M has positive rank, then $\operatorname{Supp}_R(M) = \operatorname{Spec}(R)$: to see this, notice, given $\mathfrak{q} \in \operatorname{Spec}(R)$, there is a minimal prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \subseteq \mathfrak{q}$. As $M_{\mathfrak{p}} \neq 0$, we conclude that $M_{\mathfrak{q}} \neq 0$. In particular, if R is a local ring and M is a finitely generated R-module such that M has rank r and $M^* \neq 0$, then $r \geq 1$ and M^* has rank r, so that $\operatorname{Supp}_R(M) = \operatorname{Supp}_R(M^*) = \operatorname{Spec}(R)$; see §2.3. We make use of this observation in the next results.

Proposition 4.12. *Let R be a local ring satisfying Serre's condition (S_2) and let M be a finitely generated R-module such that $M^* \neq 0$ and $M \otimes_R M^*$ is reflexive. Then M is free if one of the following conditions holds*:

(i) *$M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ of R of height at most one.*

(ii) *M has rank, and $R_{\mathfrak{p}}$ is a hypersurface ring for each height-one prime ideal $\mathfrak{p}$ of R.*

Proof. For part (i), we may assume $\dim(R) \geq 2$. Consider the following exact sequence which follows from [Auslander and Bridger 1969, 2.6(a)]:

$$(4.12.1)\quad 0 \to \operatorname{Ext}^1_R(\operatorname{Tr} M, M^*) \to M \otimes_R M^* \xrightarrow{\phi} \operatorname{Hom}_R(M^*, M^*) \to \operatorname{Ext}^2_R(\operatorname{Tr} M, M^*) \to 0.$$

It follows by part (i) that $\operatorname{Ext}^1_R(\operatorname{Tr} M, M^*)_{\mathfrak{p}} = 0 = \operatorname{Ext}^2_R(\operatorname{Tr} M, M^*)_{\mathfrak{p}}$. Hence the map $\phi_{\mathfrak{p}}$ is an isomorphism for each prime ideal $\mathfrak{p}$ of R of height at most one. Notice $\operatorname{Hom}_R(M^*, M^*)$ is a torsion-free R-module since M^* is torsion-free. This implies ϕ is an isomorphism; see, for example, [Celikbas and Wiegand 2015, page 446]. Therefore, $\operatorname{Ext}^1_R(\operatorname{Tr} M, M^*) = \operatorname{Ext}^2_R(\operatorname{Tr} M, M^*) = 0$, and hence M is free by Lemma 4.11.

For part (ii), let $\mathfrak{p}$ be a height-one prime ideal of R. Then $M_{\mathfrak{p}}$ has rank over $R_{\mathfrak{p}}$ and $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M^*_{\mathfrak{p}}$ is a nonzero torsion-free $R_{\mathfrak{p}}$ module. Hence it follows from [Huneke and Wiegand 1994, 3.1] that $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$; see also Corollary 4.10. Now part (i) shows that M is free. □

In passing we consider a higher dimensional version of Conjecture 1.1.

Conjecture 4.13. *Let R be a local ring satisfying (S_2) and let M be a finitely generated R-module. If M has rank and $M \otimes_R M^*$ is a nonzero reflexive R-module, then M is free.*

It is known that Conjecture 1.1 can be stated over local rings of arbitrary dimension under extra assumptions; see, for example, [Celikbas and Wiegand 2015, 8.6]. However, we could not find a suitable reference that proves, if Conjecture 1.1

is true, then so is Conjecture 4.13. Next we use Proposition 4.12(i) and give an argument to point out this fact.

Proposition 4.14. *If Conjecture 1.1 is true, then Conjecture 4.13 is also true.*

Proof. Assume Conjecture 1.1 is true. Let R be a d-dimensional local ring satisfying (S_2), and let M be a finitely generated R-module such that M has rank and $M \otimes_R M^*$ is a nonzero reflexive R-module. To show M must be free, we proceed by induction on d.

If $d = 0$, then M is free since M has rank. Hence assume $d = 1$. Then R is a one-dimensional Cohen–Macaulay ring and $M \otimes_R M^*$ is a nonzero torsion-free R-module. Set $U = \perp_R M$. Then U is a nonzero torsion-free R-module that has rank. Moreover, we have that $M \otimes_R M^* \cong U \otimes_R M^*$; see § 2.3. By dualizing the short exact sequence (2.1.1), we obtain the following exact sequence: $0 \to U^* \to M^* \to (\mathsf{T}_R M)^*$. As $(\mathsf{T}_R M)^* = 0$, we see that $M^* \cong U^*$. Thus we have

$$M \otimes_R M^* \cong U \otimes_R M^* \cong U \otimes_R U^*.$$

In particular, $U \otimes_R U^*$ is torsion-free. As Conjecture 1.1 is true, we conclude that U is a free R-module. This forces M to be free; see [Huneke and Wiegand 1994, 1.1].

Next assume $d \geq 2$ and let $\mathfrak{p}$ be a height-one prime ideal of R. Then $R_{\mathfrak{p}}$ is a local ring satisfying (S_2), and $M_{\mathfrak{p}}$ is a finitely generated R-module such that $M_{\mathfrak{p}}$ has rank over $R_{\mathfrak{p}}$ and $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (M_{\mathfrak{p}})^*$ is a nonzero reflexive $R_{\mathfrak{p}}$-module. Hence the induction hypothesis shows that $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$. Consequently Proposition 4.12(i) shows that M is free. □

The fact that maximal Cohen–Macaulay modules are reflexive over Gorenstein rings shows that, if Conjecture 4.13 is true, then so is Conjecture 1.1 over (one-dimensional) Gorenstein rings. However, in general, over Cohen–Macaulay rings (not necessarily Gorenstein), we do not know whether or not this implication is true.

Our next aim is to establish Example 1.2(ii) from the introduction, and hence obtain a new class of ideals that satisfy the torsion conclusion of Conjecture 1.1.

Let R be a Cohen–Macaulay local ring, and let I be an $\mathfrak{m}$-primary ideal of R containing a parameter ideal $\mathfrak{q}$ of R as a reduction. Then I is said to be an *Ulrich ideal* provided that $I^2 = \mathfrak{q}I$ and I/I^2 is a free R/I-module; see [Goto et al. 2014, 1.1]. We refer the reader to [Goto et al. 2014] for details about Ulrich ideals; here we record a few observations about them related to our argument. For our purpose, we only consider Ulrich ideals that are not parameter ideals.

Remark 4.15. If R is a Gorenstein ring and I is an Ulrich ideal of R, then I has bounded Betti numbers, i.e., $\operatorname{cx}_R(I) \leq 1$; see [Goto et al. 2014, 7.4].

Corollary 4.16. *Let R be a one-dimensional complete intersection domain, and let I be an Ulrich ideal of R. Then R/I is a Tor-rigid R-module. Moreover, if M is a finitely generated R-module that has torsion, then $M \otimes_R I$ has torsion.*

Proof. This follows from Corollary 4.1(ii) and Remark 4.15 (note that $\overline{\mathsf{G}}(R)_{\mathbb{Q}} = 0$; see [Celikbas and Dao 2011, 2.5]). □

Example 4.17. Let $R = \mathbb{C}[[t^4, t^5, t^6]] \cong \mathbb{C}[[x, y, z]]/(xz - y^2, x^3 - z^2)$ and let $I = (t^4, t^6)$. Then R is a one-dimensional complete intersection domain, and I is an Ulrich ideal of R; see [Goto et al. 2014, 6.3]. Hence, R/I is Tor-rigid, and $I \otimes_R M$ has torsion for each finitely generated R-module M that has torsion; see Corollary 4.16. This fact can fail if M does not have torsion. For example, letting J be the ideal (t^4, t^5) of R, we have that $I \otimes_R J$ is torsion-free, i.e., $\mathrm{Tor}_2^R(R/I, R/J) = 0$; see [Huneke and Wiegand 1994, 4.3]. Hence, since R/I is Tor-rigid, we conclude that $\mathrm{Tor}_i^R(R/I, R/J) = 0$ for all $i \geq 2$, i.e., $\mathrm{Tor}_i^R(I, J) = 0$ for all $i \geq 1$.

Notice, Remark 4.15, together with Corollary 4.10, establishes Example 1.2(ii) over complete intersection rings. In fact, this result is true over Gorenstein rings that are not necessarily complete intersections. This fact can be shown as follows:

Proposition 4.18. *Let $(R, \mathfrak{m})$ be a one-dimensional Gorenstein local ring. If I is a nonprincipal Ulrich ideal of R, then it follows that $I \cong I^*$, and $I \otimes_R I^*$ has torsion.*

Proof. Note that, since $\mathfrak{q} \subsetneq I$, we conclude from [Goto et al. 2014, 2.6(b)] that I is generated by two elements. Moreover, since $I^2 = \mathfrak{q}I$, there is an exact sequence $0 \to \mathfrak{q}/I^2 \to I/I^2 \to I/\mathfrak{q} \to 0$, where $I/I^2 \cong (R/I)^{\oplus 2}$, $\mathfrak{q}/I^2 \cong R/I$, and $I/\mathfrak{q} \cong R/I$. Thus the multiplicity of I is equal to $2 \cdot \mathrm{length}_R(R/I) = \mathrm{length}_R(I/I^2)$. Hence [Ooishi 1996, 2.3] implies that I is a self-dual R-module, i.e., $I \cong I^*$. This yields the isomorphism $I \otimes_R I \cong I^* \otimes_R I$.

We now follow the idea discussed in the paragraph preceding [Huneke and Wiegand 1994, 4.4] and observe that $I \otimes_R I$ has torsion; this implies that $I \otimes_R I^*$ has torsion, as claimed. We see, by applying $- \otimes_R I$ to the exact sequence $0 \to I \to R \to R/I \to 0$, that there is an exact sequence

$$0 \to \mathrm{Tor}_1^R(R/I, I) \to I \otimes_R I \xrightarrow{\alpha} I \xrightarrow{\beta} I \otimes_R R/I \to 0. \tag{4.18.1}$$

Note that I contains a non-zero-divisor on R (as it contains a parameter ideal). Hence $\mathrm{Tor}_1^R(R/I, I)$ is a torsion R-module. Now suppose $I \otimes_R I$ is torsion-free. Then it follows from (4.18.1) that $\mathrm{Tor}_1^R(R/I, I) = 0$, α is injective, and $I \otimes_R I \cong \ker(\beta) \cong I^2$. In particular, we have that $\mu_R(I \otimes_R I) = \mu_R(I^2)$, where $\mu_R(-)$ denotes the number of elements in a minimal generating set. It follows from Nakayama's lemma that $\mu_R(I \otimes_R I) = \mu_R(I)^2$. Therefore we obtain $\mu_R(I)^2 = \mu_R(I^2)$, which forces I to be principal since $\mu_R(I^2) \leq \mu_R(I)(\mu_R(I) + 1)/2$. Hence, since I is not principal, we conclude that $I \otimes_R I$ has torsion. □

5. On tensor products of totally reflexive modules

Huneke and Wiegand proved that tensor products of two nonfree modules over a local domain — that is a quotient of a regular ring by a nonzero element — cannot be maximal Cohen–Macaulay. In fact, this result is true over such rings that are not domains as long as one of the modules in question has rank; see [Huneke and Wiegand 1994, 3.1].

The main purpose of this section is to prove a consequence of Theorem 3.2 that is somewhat of a different nature. Namely, we would like to show that tensor products of two nonfree totally reflexive modules over a Cohen–Macaulay local domain – that is a quotient of a G-regular ring by a non-zero-divisor – cannot be totally reflexive; see Proposition 5.7. Recall that R is called G-regular [Takahashi 2008] if there are no nonfree totally reflexive R-modules. Since each regular ring is G-regular, and each totally reflexive module is maximal Cohen–Macaulay over Cohen–Macaulay rings, our conclusion may be considered as a G-hypersurface version of the result of Huneke and Wiegand [1994, 3.1] mentioned above.

We start by giving a few examples which illustrate the fact that, in general, tensor products of nonfree totally reflexive modules may or may not be totally reflexive.

Example 5.1. Let $R = \mathbb{C}[\![x, y]\!]/(xy)$ and $M = R/(x)$. Then R is a Gorenstein ring and $M \otimes_R M \cong M$ is totally reflexive.

Recall, over a local ring $(R, \mathfrak{m})$, an element $0 \neq x \in \mathfrak{m}$ is said to be an *exact zero-divisor* [Henriques and Şega 2011] if there exists $y \in R$ such that $(0 :_R x) = (y)$ and $(0 :_R y) = (x)$.

Example 5.2. Let $R = \mathbb{C}[\![x, y, z, w]\!]/(x^2, xy, y^2, z^2, w^2)$. Note that R is not Gorenstein, and z and w are exact zero-divisors on R. Set $M = R/(z)$ and $N = R/(w)$. Then M and N are both totally reflexive R-modules. Moreover, $M \otimes_R N$ is a totally reflexive R-module since

$$\begin{aligned}\text{G-dim}_R(M \otimes_R N) = \text{G-dim}_R(R/(z, w)) &= \text{G-dim}_{R/zR}(R/(z, w))\\ &= \text{G-dim}_{R/zR}((R/zR)/w(R/zR)) = 0.\end{aligned}$$

Here the second equality follows from [Soto 2000, Corollary on page 53], while the last one is due to the fact that w is an exact zero-divisor on $R/(z)$.

In the next example, we observe that over local rings with $\mathfrak{m}^3 = 0$, the tensor product of two totally reflexive modules, given by a pair of exact zero-divisors, is not totally reflexive.

Example 5.3. Let $(R, \mathfrak{m})$ be a local ring such that $\mathfrak{m}^3 = 0$ and R is not Gorenstein; e.g., $R = \mathbb{C}[\![x, y, z]\!]/(x^2, y^2, z^2, yz)$. Assume $\{x, y\}$ is a pair of distinct exact zero-divisors. Let $M = R/(x)$ and $N = R/(y)$, and consider the following short exact sequence of R-modules:

$$0 \to (x, y)/(x) \to R/(x) \to R/(x, y) \to 0. \tag{5.3.1}$$

Notice $(x, y)/(x) \cong R/(x :_R y)$, and $y \cdot \mathfrak{m}^2 = 0$ so that $y\mathfrak{m} \subseteq (0 :_R y) = (x)$, i.e., $(x :_R y) = \mathfrak{m}$. So, if $\text{G-dim}_R(M \otimes_R N) < \infty$, then (5.3.1) shows

$$\text{G-dim}_R(R/(x :_R y)) = \text{G-dim}_R(R/\mathfrak{m}) < \infty,$$

i.e., R is Gorenstein; see [Christensen 2000, 1.2.9 and 1.4.9]. Hence it follows $\text{G-dim}_R(M \otimes_R N) = \infty$.

It also seems worth noting, even if $M \otimes_R N$ has finite Gorenstein dimension, M or N may not have finite Gorenstein dimension. We record such an example next.

Example 5.4. Let $R = \mathbb{C}[\![x, y, z]\!]/(x^2, xy, y^2)$, $M = R/(xz)$, and let $N = R/(z)$. Then R is not Gorenstein. Moreover, it follows that $M \otimes_R N \cong N$ so that

$$\text{pd}_R(M \otimes_R N) = \text{pd}_R(N) = 1 < \infty$$

since z is a non-zero-divisor on R. We proceed to show that $\text{G-dim}_R(M) = \infty$.

Note that we have the following isomorphisms:

$$(xz) \cong (x) \cong R/(x, y). \tag{5.4.1}$$

The first isomorphism in (5.4.1) between the ideals of R holds since z is a non-zero-divisor on R, while the second one is due to the fact that $(0 :_R x) = (x, y)$. Therefore it follows from (5.4.1) that there is a short exact sequence of R-modules

$$0 \to R/(x, y) \to R \to M \to 0. \tag{5.4.2}$$

Set $T = R/(x, y)$. Then it follows from (5.4.2) that $\text{G-dim}_R(M) < \infty$ if and only if $\text{G-dim}_R(T) < \infty$; see [Christensen 2000, 1.2.9]. Hence it suffices to observe that $\text{G-dim}_R(T) = \infty$.

As z is a non-zero-divisor on R and T, we have that

$$\text{G-dim}_R(T) = \text{G-dim}_{R/(z)}(T/zT);$$

see [Christensen 2000, 1.4.5]. Note also $R/(z) \cong \mathbb{C}[\![x, y]\!]/(x^2, xy, y^2)$ is a non-Gorenstein local ring. Therefore, since T/zT is isomorphic to the residue field of the ring $R/(z)$, we conclude that $\text{G-dim}_{R/(z)}(T/zT) = \infty$; see [Christensen 2000, 1.4.9]. Consequently we deduce $\text{G-dim}_R(M) = \infty$, as claimed.

The next observation is known; see, for example, the proof of [Miller 1998, 1.1]. Recall that, if $R = S/(f)$,where $(S, \mathfrak{n})$ is a local ring and $f \in \mathfrak{n}$ is a non-zero-divisor on S, then it follows that $\text{cx}_R(M) \leq \text{cx}_S(M) + 1$ for each finitely generated R-module M; see [Avramov 1989, 3.2.3].

Remark 5.5. Let $R = S/(f)$, where $(S, \mathfrak{n})$ is a local ring and $f \in \mathfrak{n}$ is a non-zero-divisor on S. Let M and N be finitely generated R-modules such that

$$\operatorname{pd}_S(M \otimes_R N) < \infty.$$

If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, then it follows that $\operatorname{pd}_R(M) < \infty$ or $\operatorname{pd}_R(N) < \infty$.

To see this, let P and Q be minimal free resolutions of M and N, respectively, over R. Then, as $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, we see that $P \otimes_R Q$ is a minimal free resolution of $M \otimes_R N$. Moreover, for each $n \geq 0$, it follows that

$$\beta_n^R(M \otimes_R N) = \operatorname{rank}(P \otimes_R Q)_n = \sum_{i+j=n} \operatorname{rank}(P_i \otimes_R Q_j) = \sum_{i=0}^{n} \beta_i^R(M)\beta_{n-i}^R(N) \tag{5.5.1}$$

Now, if P and Q are infinite resolutions, then (5.5.1) shows that $\beta_n^R(M \otimes_R N) \geq n+1$ for each $n \geq 0$. However, since $\operatorname{pd}_S(M \otimes_R N) < \infty$, we have that $\operatorname{cx}_R(M \otimes_R N) \leq 1$, i.e., there is a real number A such that $\beta_n^R(M \otimes_R N) \leq A$ for each $n \geq 0$; see §2.6. So, P or Q must a finite complex, i.e., $\operatorname{pd}_R(M) < \infty$ or $\operatorname{pd}_R(N) < \infty$. Furthermore, if $\operatorname{pd}_R(N) < \infty$, it follows from (5.5.1) that $\beta_i^R(M)$ is bounded by a real number for each $i \geq 0$, i.e., $\operatorname{cx}_R(M) \leq 1$.

Next is a corollary of Theorem 3.2 and Remark 5.5.

Corollary 5.6. *Let $R = S/(f)$ be a Cohen–Macaulay ring, where $(S, \mathfrak{n})$ is a local ring and $f \in \mathfrak{n}$ is a non-zero-divisor on S. Let M and N be finitely generated R-modules such that*:

(a) $\operatorname{pd}_S(N) < \infty$ *and* $\operatorname{pd}_S(M \otimes_R N) < \infty$.

(b) *M, N, and $M \otimes_R N$ are maximal Cohen–Macaulay R-modules.*

Then M or N is free provided that at least one of the following holds:

(i) *M has rank and $N_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime ideal $\mathfrak{p}$ of R.*

(ii) $\dim(R) \geq 2$ *and* $\operatorname{length}_R(\operatorname{Tor}_i^R(M, N)) < \infty$ *for all $i \gg 0$.*

Proof. Note, by Remark 5.5, it suffices to prove $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. Note also that $\operatorname{CI-dim}_R(N) < \infty$ and $\operatorname{cx}_R(N) \leq 1$.

For part (i), since M has rank, we may assume $\dim(R) \geq 1$. First, consider the case where $\dim(R) = 1$. Then, if $\operatorname{pd}_R(N) < \infty$, Remark 3.5 shows that N is free. Hence we assume $\operatorname{cx}_R(N) = 1$. In that case Remark 4.2 yields $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

Next assume $\dim(R) \geq 2$. Localizing at a nonmaximal prime ideal $\mathfrak{p}$ of R, we see that the hypotheses are preserved. Hence, by the induction hypothesis, we have that $M_{\mathfrak{p}}$ or $N_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$. In particular, $\operatorname{length}_R(\operatorname{Tor}_i^R(M, N)) < \infty$ for all

$i \geq 1$. Thus, to prove part (i), it suffices to prove part (ii). The fact that part (ii) is an immediate consequence of Theorem 3.2 completes the proof. □

Here is the main result of this section: recall, if $R = S/(f)$, where $(S, \mathfrak{n})$ is a G-regular local ring and $f \in \mathfrak{n}^2$ is a non-zero-divisor on S, then R is not G-regular; see [Takahashi 2008, 4.6].

Proposition 5.7. *Let $R = S/(f)$, where $(S, \mathfrak{n})$ is a Cohen–Macaulay G-regular local ring and $f \in \mathfrak{n}^2$ is a non-zero-divisor on S. Let M and N be finitely generated, nonfree, and totally reflexive R-modules.*

If M has rank and $N_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for each associated prime $\mathfrak{p}$ of R, then $M \otimes_R N$ is not totally reflexive. In particular, if R is a domain, then $M \otimes_R N$ is not totally reflexive.

Proof. If $M \otimes_R N$ is totally reflexive, then M, N, and $M \otimes_R N$ have finite projective dimension over S; see, for example, [Takahashi 2008, 1.5(4)]; in that case Corollary 5.6 implies that M or N is free. Therefore, $M \otimes_R N$ is not totally reflexive. □

Example 5.8. Let $R = S/(x^2+y^2+z^2+w^2)$, where $S = \mathbb{C}[\![x, y, z, w]\!]/(xy, yz, zw)$. Note that R is reduced, $\dim(S) = 2$, and $\{x + y + z, y + z + w\}$ is an S-regular sequence. Moreover, it follows that

$$S/(x+y+z, y+z+w) \cong \mathbb{C}[\![z, w]\!]/(-zw - w^2, -z^2 - zw, zw)$$

is an Artinian ring with radical square zero. Hence, S is G-regular but R is not G-regular; see [Takahashi 2008, 4.2 and 4.6]. Therefore, if M and N are nonfree totally reflexive R-modules, either of which has rank, then Proposition 5.7 shows that $M \otimes_R N$ is not totally reflexive.

We finish this section by proving a result similar to Theorem 3.2: our aim is to show that, in case the module M in Theorem 3.2 is maximal Cohen–Macaulay, then one can prove the vanishing of Tor under weaker assumptions, for example, regardless of the depth of N. Subsequently, we give an application of our result concerning tensor products of totally reflexive modules over hypersurfaces; see Corollary 5.12.

Note, by §2.13, $\theta^R(M, N)$ is well-defined under the hypotheses of Proposition 5.9.

Proposition 5.9. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 1$ such that $R = S/(f)$ for some local ring $(S, \mathfrak{n})$ and a non-zero-divisor $f \in \mathfrak{n}$ on S. Let M be a (finitely generated) maximal Cohen–Macaulay R-module, and let N be a finitely generated R-module. Assume the following conditions hold*:

(i) $\text{CI-dim}_S(N) < \infty$ *and* $\text{Tor}_i^S(M, N) = 0$ *for all* $i \gg 0$ (*e.g.*, $\text{pd}_S(N) < \infty$).

(ii) $\text{length}_R(\text{Tor}_i^R(M, N)) < \infty$ *for all* $i \gg 0$.

(iii) $\theta^R(M, N) = 0$.

If $M \otimes_R N$ *is torsion-free, then* $\operatorname{Tor}_i^R(M, N) = 0$ *for all* $i \geq 1$.

Proof. We start by noting $\text{CI-dim}_R(N) < \infty$; see §2.8(iv). Hence $\text{G-dim}_R(N) < \infty$ so that there is an exact sequence of finitely generated R-modules

$$0 \to L \to Z \to N \to 0, \tag{5.9.1}$$

where Z is a totally reflexive R-module and $\operatorname{pd}_R(L) < \infty$; see (2.5.1) and §2.9.

As $\operatorname{pd}_R(L) < \infty$, it follows that $\operatorname{pd}_S(L) < \infty$ [Rotman 1979, 9.32]. Hence, by (5.9.1), we have $\text{CI-dim}_S(Z) < \infty$ as $\text{CI-dim}_S(N) < \infty$; see § 2.10. Thus $\text{CI-dim}_S(Z) = 1$ and $\text{CI-dim}_R(Z) = 0$; see §2.8(iii). Moreover, since $\operatorname{pd}_S(L) < \infty$ and $\operatorname{Tor}_i^S(M, N) = 0$ for all $i \gg 0$, it follows from (5.9.1) that $\operatorname{Tor}_i^S(M, Z) = 0$ for all $i \gg 0$. Consequently $\operatorname{Tor}_i^S(M, Z) = 0$ for all $i \geq 2$; see §2.8(vii). Note also, as $\operatorname{length}_R(\operatorname{Tor}_i^R(M, N)) < \infty$ for all $i \gg 0$, we have that $\operatorname{Tor}_i^R(M, N)$ is torsion for all $i \geq 1$; see §2.8(vii).

<u>Claim</u>. $M \otimes_R Z$ is torsion-free, and $\operatorname{Tor}_i^R(M, Z) \cong \operatorname{Tor}_i^R(M, N)$ for all $i \geq 1$.

We first show that the claim is sufficient to complete the proof. Note, by the claim, we have that $\operatorname{length}_R(\operatorname{Tor}_i^R(M, Z)) < \infty$ for all $i \gg 0$. Hence the fact $\text{CI-dim}_R(Z) = 0$ forces $\operatorname{length}_R(\operatorname{Tor}_i^R(M, Z)) < \infty$ for all $i \geq 1$; see §2.8(ii, vii). In particular, as $\operatorname{Tor}_i^S(M, Z) = 0$ for all $i \geq 2$ and $\text{CI-dim}_S(Z) < \infty$, we conclude that $\theta^R(M, Z)$ is well-defined; see § 2.13. Hence it follows by the claim $\theta^R(M, Z) = \theta^R(M, N)$ so that $\theta^R(M, Z) = 0$.

As $M \otimes_R Z$ is torsion-free, $\operatorname{Tor}_2^S(M, Z) = 0$ and $\operatorname{Tor}_2^R(M, N)$ is torsion, we use §2.12 for the pair (M, Z), and deduce that $\operatorname{Tor}_2^R(M, Z)$ vanishes. So, in view of (2.13.3), we have $\theta^R(M, Z) = \operatorname{length}_R(\operatorname{Tor}_2^R(M, Z)) - \operatorname{length}_R(\operatorname{Tor}_1^R(M, Z))$ and hence $\operatorname{Tor}_1^R(M, Z) = 0$. Now, as $\operatorname{Tor}_i^R(M, Z) \cong \operatorname{Tor}_{i+2}^R(M, Z)$ for all $i \geq 1$, we see that $\operatorname{Tor}_i^R(M, Z) = 0$ for all $i \geq 1$. Therefore, it remains to justify the claim.

To prove the claim, we will first show $M \otimes_R L$ is torsion-free, or equivalently, $M \otimes_R L$ satisfies (S_1), i.e., $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p} \otimes_{R_\mathfrak{p}} L_\mathfrak{p}) \geq \min\{1, \dim(R_\mathfrak{p})\}$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$. Let $\mathfrak{p} \in \operatorname{Supp}_R(M \otimes_R L)$ and assume $\dim(R_\mathfrak{p}) \geq 1$ (recall $\operatorname{depth}(0) = \infty$). Since $\operatorname{Tor}_i^R(M, L) = 0$ for all $i \geq 1$, the equality

$$\operatorname{depth}_{R_\mathfrak{p}}(L_\mathfrak{p}) + \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{depth}(R_\mathfrak{p}) + \operatorname{depth}_{R_\mathfrak{p}}(L_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p})$$

holds; see § 2.8(viii) and § 2.15. So $\operatorname{depth}_{R_\mathfrak{p}}(L_\mathfrak{p}) = \operatorname{depth}_{R_\mathfrak{p}}(L_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p})$. Notice (5.9.1) localized at $\mathfrak{p}$ shows that $L_\mathfrak{p} \hookrightarrow Z_\mathfrak{p} \neq 0$. Since $Z_\mathfrak{p}$ is a torsion-free module over $R_\mathfrak{p}$, we have that $\operatorname{depth}_{R_\mathfrak{p}}(L_\mathfrak{p}) \geq 1$. Consequently this shows $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p} \otimes_{R_\mathfrak{p}} L_\mathfrak{p}) \geq 1$, and hence $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p} \otimes_{R_\mathfrak{p}} L_\mathfrak{p}) \geq \min\{1, \dim(R_\mathfrak{p})\}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. In particular, $M \otimes_R L$ is torsion-free.

Now, as $M \otimes_R L$ is torsion-free and $\operatorname{Tor}_1^R(M, N)$ is torsion, by tensoring (5.9.1) with M, we obtain the exact sequence

$$0 \to M \otimes_R L \to M \otimes_R Z \to M \otimes_R N \to 0.$$

This implies $M \otimes_R Z$ is torsion-free; see §2.2. Moreover, as $\operatorname{Tor}_i^R(M, L) = 0$ for all $i \geq 1$, in view of (5.9.1), we conclude that $\operatorname{Tor}_i^R(M, Z) \cong \operatorname{Tor}_i^R(M, N)$ for all $i \geq 1$. This proves the claim and completes the proof. □

We use the next observation to prove Corollary 5.11:

Remark 5.10. Let R be a local ring, M a finitely generated reflexive R-module, and let x be a non-zero-divisor on R. Then M/xM is a torsionless R/xR-module. In particular, M/xM is a torsion-free module over R/xR. One can show this as follows:

Note that, since M is torsionless, there is a short exact sequence

$$0 \to M \to F \to C \to 0,$$

where F is free and $\operatorname{Ext}_R^1(C, R) = 0$; see, for example, §B.3. Dualizing this short exact sequence, we have the following commutative diagram, where λ denotes the natural map:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \cong \downarrow \lambda_M & & \cong \downarrow \lambda_F & & \downarrow \lambda_C & & \\
0 & \longrightarrow & M^{**} & \longrightarrow & F^{**} & \longrightarrow & C^{**} & \longrightarrow & \operatorname{Ext}_R^1(M^*, R)
\end{array}$$

This shows λ_C is injective, i.e., C is torsionless. So this implies that

$$\operatorname{Tor}_1^R(C, R/xR) = 0.$$

Hence the top row yields an injection $M/xM \hookrightarrow F/xF$, as claimed.

Corollary 5.11. *Let R be a Cohen–Macaulay ring of dimension $d \geq 2$ such that $R = S/(f)$ for some local ring $(S, \mathfrak{n})$ and a non-zero-divisor $f \in \mathfrak{n}$ on S. Let M and N be finitely generated R-modules, and assume the following hold*:

(i) $\operatorname{pd}_S(N) < \infty$ *and* $\operatorname{depth}_R(N) \geq 1$.

(ii) *M is a maximal Cohen–Macaulay R-module.*

(iii) $\operatorname{length}_R(\operatorname{Tor}_i^R(M, N)) < \infty$ *for all* $i \gg 0$.

If $M \otimes_R N$ is reflexive, then $\operatorname{Tor}_i^R(M, N) = 0$ *for all* $i \geq 1$.

Proof. Let $x \in \mathfrak{m}$ be a non-zero-divisor on R, M, and N. For any R-module X we let $\overline{X}$ denote X/xX. Then it follows that $\overline{M}$ is a maximal Cohen–Macaulay

$\bar{R}$-module and $\overline{M \otimes_R N} \cong \overline{M} \otimes_{\bar{R}} \overline{N}$, where $\overline{M} \otimes_{\bar{R}} \overline{N}$ is a torsion-free $\bar{R}$-module by Remark 5.10. Now consider the following short exact sequence of R-modules:

$$0 \to N \xrightarrow{x} N \to \overline{N} \to 0. \tag{5.11.1}$$

We see from (iii) and (5.11.1) that $\text{length}_R(\text{Tor}_i^R(M, \overline{N})) < \infty$ for all $i \gg 0$. Hence $\theta^R(M, \overline{N})$ is well-defined, and the additivity of the θ-pairing applied to the short exact sequence (5.11.1) yields $\theta^R(M, N) = \theta^R(M, N) + \theta^R(M, \overline{N})$, i.e., $\theta^R(M, \overline{N}) = 0$; see §2.13.

Write $x = y + (f)$ for some $y \in \mathfrak{n}$. Then $\{y, f\}$ is an S-regular sequence. Hence we can write $\bar{R} = T/(f)$, where $T = S/(y)$ and f is a non-zero-divisor on T contained in the maximal ideal of T.

Notice $\text{Tor}_i^{\bar{R}}(\overline{M}, \overline{N}) \cong \text{Tor}_i^R(M, \overline{N})$ for all $i \geq 0$; this implies $\theta^{\bar{R}}(\overline{M}, \overline{N})$ is well-defined, and hence $\theta^{\bar{R}}(\overline{M}, \overline{N}) = \theta^R(M, \overline{N}) = 0$. Moreover, since y is a non-zero-divisor on S and N, it follows that

$$\text{pd}_T(\overline{N}) = \text{pd}_T(N/xN) = \text{pd}_{S/(y)}(N/yN) = \text{pd}_S(N) < \infty.$$

So we have $\text{pd}_T(\overline{N}) < \infty$, $\text{length}_{\bar{R}}(\text{Tor}_i^{\bar{R}}(\overline{M}, \overline{N})) < \infty$ for all $i \gg 0$, and also $\theta^{\bar{R}}(\overline{M}, \overline{N}) = 0$. Hence we use Proposition 5.9 with the pair $(\overline{M}, \overline{N})$ over the ring $\bar{R} = T/(f)$, and conclude that $\text{Tor}_i^{\bar{R}}(\overline{M}, \overline{N}) = 0$ for all $i \geq 1$. Consequently, by using (5.11.1) and Nakayama's lemma, we see that $\text{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. □

The following result corroborates with the Tor vanishing conclusion of [Huneke and Wiegand 1994, 2.7].

Corollary 5.12. *Let R be a local hypersurface ring, i.e., $R = S/(f)$ for some regular local ring $(S, \mathfrak{n})$ and $0 \neq f \in \mathfrak{n}$. Let M be a nonfree maximal Cohen–Macaulay R-module, and let N be a finitely generated R-module such that $\text{pd}_R(N) = \infty$. Assume $\dim(R) \geq 2$, and $R_\mathfrak{p}$ is a domain for each $\mathfrak{p} \in \text{Spec}(R) - \{\mathfrak{m}\}$. Then $M \otimes_R N$ is not reflexive.*

Proof. We assume $M \otimes_R N$ is reflexive and seek a contradiction. Note that it follows $M \otimes_R \perp_R N$ is reflexive; see §2.4(i).

Pick a prime ideal $\mathfrak{p} \in \text{Supp}_R(M \otimes_R \perp_R N) - \{\mathfrak{m}\}$. Then $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} (\perp_R N)_\mathfrak{p}$ is a reflexive $R_\mathfrak{p}$-module over the hypersurface domain $R_\mathfrak{p}$. So [Huneke and Wiegand 1994, 2.7] implies that $\text{Tor}_i^R(M, \perp_R N)_\mathfrak{p} = 0$ for all $i \geq 1$. This shows that $\text{length}_R(\text{Tor}_i^R(M, \perp_R N)) < \infty$ for all $i \geq 1$.

Now, as $\text{pd}_S(\perp_R N) < \infty$ and $\text{depth}_R(\perp_R N) \geq 1$, from Corollary 5.11 we conclude that $\text{Tor}_i^R(M, \perp_R N) = 0$ for all $i \geq 1$. Consequently, by §2.4(iii), we obtain the vanishing of $\text{Tor}_i^R(M, N)$ for all $i \geq 1$. This forces M or N to have finite projective dimension; see [Huneke and Wiegand 1997, 1.9]. Consequently $M \otimes_R N$ cannot be reflexive. □

The conclusion of Corollary 5.12 can fail over arbitrary hypersurfaces that are not locally domains on the punctured spectrum.

Example 5.13. Let $R = \mathbb{C}[\![x, y, z]\!]/(xy)$, $M = R/(x)$ and let $N = R/(x^2)$. Then R is a two-dimensional hypersurface, M is a nonfree maximal Cohen–Macaulay R-module and $\mathrm{pd}_R(N) = \infty$. Note that, $M \otimes_R N$, being isomorphic to M, is reflexive, and $R_{\mathfrak{p}}$ is not a domain for $\mathfrak{p} = (x, y) \in \mathrm{Spec}(R)$.

It is worth noting that totally reflexive modules over a ring as in Corollary 5.12 cannot be defined by exact zero-divisors.

Remark 5.14. Let $(R, \mathfrak{m})$ be a local ring such that $\mathrm{depth}(R) \geq 2$ and $R_{\mathfrak{p}}$ is a domain for each $\mathfrak{p} \in \mathrm{Spec}(R) - \{\mathfrak{m}\}$. Then R does not admit exact zero divisors.

To see this, suppose $x \in \mathfrak{m}$ is an exact zero divisor on R, and seek a contradiction. It follows from the definition that there is $0 \neq y \in \mathfrak{m}$ such that $(0 :_R x) = (y)$ and $(0 :_R y) = (x)$; i.e., the following is the minimal free resolution of R/xR over R:

$$\cdots \to R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to R/xR \to 0.$$

First assume that there is a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R) - \{\mathfrak{m}\}$ such that $x, y \in \mathfrak{p}$. Then, by localizing the minimal free resolution of R/xR at $\mathfrak{p}$, we obtain the minimal free resolution of $(R/xR)_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$:

$$\cdots \to R_{\mathfrak{p}} \xrightarrow{x} R_{\mathfrak{p}} \xrightarrow{y} R_{\mathfrak{p}} \xrightarrow{x} R_{\mathfrak{p}} \xrightarrow{y} R_{\mathfrak{p}} \xrightarrow{x} R_{\mathfrak{p}} \to R_{\mathfrak{p}}/xR_{\mathfrak{p}} \to 0.$$

In particular, we see that (x, y) is also pair of exact zero divisors on $R_{\mathfrak{p}}$. However, this is not possible since $R_{\mathfrak{p}}$ is a domain and local domains cannot admit exact zero divisors.

Now let I be the ideal of R generated by x and y. Suppose $\mathfrak{m}$ is minimal prime over I. Then $2 \leq \mathrm{depth}(R) \leq \dim(R) = \mathrm{height}_R(\mathfrak{m}) \leq 2$, i.e., R is Cohen–Macaulay of dimension two, and that $\mathrm{height}_R(I) = 2$. This implies that $\{x, y\}$ is a regular sequence on R, which is not possible. Therefore $\mathfrak{m}$ is not minimal over I. Consequently, there is a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R) - \{\mathfrak{m}\}$ such that $x, y \in \mathfrak{p}$; this gives a contradiction by the previous argument.

Appendix A: On the vanishing of the theta invariant

Recall that, if R is a one-dimensional reduced hypersurface ring, then $\theta^R(M, N)$ is defined and vanishes for all finitely generated R-modules M and N, either of which has rank; see Remark 4.2. Since Theorem 3.2 relies upon the vanishing of θ pairing, we would like to find out whether θ can vanish nontrivially. More precisely, we would like to find out whether there is a one-dimensional reduced hypersurface ring R, and modules M and N over R — neither of which has rank — such that $\theta^R(M, N) = 0$. We were unable to find an example (or a result) from the literature that addresses our query. The aim of this section is to record such an example suggested to us by Hailong Dao; see Example A.3. First, in §A.1, we will record a related fact that was shown to us by Mark Walker: over one-dimensional reduced local rings R, a finitely generated R-module M has rank if and only if its class is zero in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$. A similar result that makes use of θ pairing is established in [Dao 2013, 3.3] for hypersurface rings.

A.1. Let R be a one-dimensional Cohen–Macaulay local ring, and let M be a finitely generated R-module.

(i) There exists a rational number r such that $\operatorname{length}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = r \cdot \operatorname{length}(R_{\mathfrak{p}})$ for each associated prime ideal $\mathfrak{p}$ of R if and only if $[M]=0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$.

(ii) Assume R is reduced. Then M has rank if and only if $[M]=0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$.

Proof. (i) Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ be the set of all minimal (associated) prime ideals of R. Note that $\mathsf{G}(k) = \mathbb{Z} \cdot [k]$ and $\mathsf{G}(R_{\mathfrak{p}_j}) = \mathbb{Z} \cdot [k(\mathfrak{p}_j)]$, where $k(\mathfrak{p}_j)$ is the residue field of $R_{\mathfrak{p}_j}$, for all $j = 1, \dots, n$.

There is a right exact sequence of the form

(A.1.1)
$$\mathsf{G}(k) \xrightarrow{\alpha} \mathsf{G}(R) \xrightarrow{\beta} \bigoplus_{j=1}^{n} \mathsf{G}(R_{\mathfrak{p}_j}) \to 0.$$

Here α is the natural inclusion with

$$\alpha([k]) = [k] \quad \text{and} \quad \beta([M]) = \big(\operatorname{length}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})[k(\mathfrak{p}_j)]\big)_j.$$

In (A.1.1), by identifying $\mathsf{G}(k)$ with $\mathbb{Z}$, and $\bigoplus_{j=1}^{n} \mathsf{G}(R_{\mathfrak{p}_j})$ with $\mathbb{Z}^{\oplus n}$, we obtain another right exact sequence of the form

(A.1.2)
$$\mathbb{Z} \xrightarrow{\alpha} G(R) \xrightarrow{\beta} \mathbb{Z}^{\oplus n} \to 0,$$

where $\alpha(1) = [k]$ and $\beta([M]) = \big(\operatorname{length}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})\big)_j$. Applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$ to (A.1.2), we see there is a right exact sequence of the form

(A.1.3)
$$\mathbb{Q} \xrightarrow{\alpha \otimes 1} \mathsf{G}(R)_{\mathbb{Q}} \xrightarrow{\beta \otimes 1} \mathbb{Q}^{\oplus n} \to 0.$$

Here $\alpha \otimes 1(1) = [k]$, which is zero in $\mathsf{G}(R)_{\mathbb{Q}}$. Hence $\alpha \otimes 1$ is the zero map so that $\beta \otimes 1$ is an isomorphism.

Consequently $[M]=0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$ if and only if $[M] = r \cdot [R]$ for some rational number r if and only if $\beta \otimes 1([M]) = r \cdot \beta \otimes 1([R])$ if and only if

$$\operatorname{length}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) = r \cdot \operatorname{length}(R_{\mathfrak{p}_j}) \quad \text{for all } j = 1, \dots, n.$$

(ii) If M has rank, then $[M]=0$ in $\overline{G}(R)_{\mathbb{Q}}$; see Remark 4.2. To see the converse, let $\mathfrak{p}$ be an associated prime ideal of R. Then, by (i), we have

$$\text{length}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = r \cdot \text{length}(R_{\mathfrak{p}})$$

for some rational number r. Since $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus n}$ for some positive integer n, we see $n=r$ and hence M has rank r. □

The next example shows that the conclusion of § A.1(ii) can fail if R is not reduced. It also shows that Conjecture 1.1 would fail if the module M in question does not have rank, even if $[M]=0$ in $\overline{G}(R)_{\mathbb{Q}}$.

Example A.2. Let $R=\mathbb{C}[\![x, y]\!]/(x^2)$ and let $M=R/(x)$. Then $M\cong M^*$, and so M is torsion-free. The exact sequence $0\to M\to R\to M\to 0$ implies that $[M]=0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$. Moreover, M does not have rank. Note also that $\operatorname{Tor}_i^R(M, M^*)\cong M$ for all $i\geq 0$, and hence $\text{length}(\operatorname{Tor}_i^R(M, M^*))=\infty$ for all $i\geq 0$.

Here is an example we seek on the vanishing of θ invariant.

Example A.3. Let

$$R=\mathbb{C}[\![x, y]\!]/(xy(x-y)), \quad M=R/(x) \quad \text{and,} \quad N=M\oplus R/(y)\oplus R/(y).$$

Then R is a one-dimensional reduced hypersurface ring, and M and N are nonfree, finitely generated, torsion-free R-modules.

The minimal free resolution of M is given by

$$F = \ \cdots \xrightarrow{(x-y)y} R \xrightarrow{x} R \xrightarrow{(x-y)y} R \xrightarrow{x} R \longrightarrow 0\ .$$

Thus $\operatorname{Tor}_1^R(M, M)\cong k[\![y]\!]/(y^2)$ and $\operatorname{Tor}_2^R(M, M)=0$ so that $\theta^R(M, M)=-2$. Similarly one can check $\operatorname{Tor}_1^R(R/(y), R/(y))\cong k[\![x]\!]/(x^2)$ and $\operatorname{Tor}_2^R(R/(y), R/(y))=0$. So it follows $\theta^R(R/(y), R/(y))=-2$. Tensoring F with $R/(y)$, we see

$$\operatorname{Tor}_2^R(M, R/(y))\cong k \quad \text{and} \quad \operatorname{Tor}_1^R(M, R/(y))=0.$$

This yields $\theta^R(M, R/(y))=1$.

Therefore we have

$$\theta^R(N, N)=-6 \quad \text{and} \quad \theta^R(M, N)=\theta^R(M, M)+2\theta^R(M, R/(y))=0.$$

Note that, since $\theta^R(M, M)\neq 0$ and $\theta^R(N, N)\neq 0$, neither M nor N has rank.

Remark A.4. It seems interesting that, contrary to Example A.3, over certain reduced hypersurface rings, $\theta(M, N)$ can vanish only when M and N have rank. For example, if $R=\mathbb{C}[\![x, y]\!]/(xy)$, and M and N are finitely generated R-modules, then one can check that $\theta^R(M, N)$ vanishes if and only if M and N both have rank. Note, by § A.1, one concludes for this particular hypersurface ring R, and

R-modules M and N that, $\theta^R(M, N) = 0$ if and only if $[M] = [N] = 0$ in $\overline{\mathsf{G}}(R)_{\mathbb{Q}}$ if and only if M and N both have rank.

Appendix B: Some examples of torsion-free tensor products

In this section we recall that Conjecture 1.1 may fail if one considers the tensor product $M \otimes_R M$ instead of $M \otimes_R M^*$. Huneke and Wiegand showed that, if R is a one-dimensional local domain that is not Gorenstein, then there exists a finitely generated torsion-free module R-module M such that M is not free and $M \otimes_R M$ is torsion-free; see the proof of [Huneke and Wiegand 1994, 4.7]. However their argument seems to not yield an explicit example of such a module M. Building on the proof of Huneke and Wiegand, we will point out how to construct nonfree torsion-free R-modules M with rank such that $M \otimes_R M$ is torsion-free over certain one-dimensional local rings R.

B.1. Let R be a one-dimensional Cohen–Macaulay local ring with canonical module ω. Set $M = \operatorname{Tr} \Omega \operatorname{Tr} \Omega\omega$. If R is generically Gorenstein but not Gorenstein, then M is a finitely generated, nonfree, torsion-free R-module with rank such that $M \otimes_R M$ is torsion-free.

Proof. It follows from [Auslander and Bridger 1969, 2.21] that there is an exact sequence of the form

$$0 \to F \to M \oplus G \to \omega \to 0, \tag{B.1.1}$$

where F and G are finitely generated free R-modules. In particular, M and M^* are torsion-free modules such that M has rank and M^* is nonzero. As syzygy modules are torsionless, we have $\operatorname{Ext}^1_R(M, R) = 0$. It follows that $M \otimes_R \omega$ is torsion-free, and the sequence (B.1.1) does not split; see [Avramov et al. 2005, B.4; Araya et al. 2018, 2.5]. Now tensoring (B.1.1) with M, we see that $M \otimes_R M$ is torsion-free; see §2.2. □

The observation in §B.1 raises the following question; an affirmative answer to this question yields a counterexample to Conjecture 1.1.

Question B.2. Is there a one-dimensional, generically Gorenstein, Cohen–Macaulay local ring R with a canonical module $\omega \ncong R$ such that $(\operatorname{Tr} \Omega \operatorname{Tr} \Omega\omega)^* \cong \operatorname{Tr} \Omega \operatorname{Tr} \Omega\omega$?

Modules yielding torsion-free tensor products as in §B.1 can also be obtained without appealing to the short exact sequence involving the transpose. Such a module can be realized as the pushforward of the first syzygy of the canonical module of the ring R. We observe this below by including a few additional details to the argument of [Huneke and Wiegand 1994, 4.7].

B.3 [Huneke and Wiegand 1994, 4.7]. Let M be a finitely generated R-module, and let $\pi : F \to M^*$ be a minimal free presentation of M^*. Denote $\mu : M \to F^*$ by the composition of the natural map $\delta_M : M \to M^{**}$ and $\pi^* : M^{**} \to F^*$. Then μ^* is surjective, and the cokernel of μ, denoted by $\mathrm{PF}(M)$, is called the *pushforward* of M (pushforward is unique up to free summands; see, for example, [Celikbas 2011, page 174]).

Now assume M is torsion-free and $\operatorname{Ext}^1_R(M, R) \neq 0$. Take a minimal generating set $\alpha_1, \dots, \alpha_t$ of $\operatorname{Ext}^1_R(M, R)$. Then each α_i represents a short exact sequence of the form $0 \to R \to N_i \to M \to 0$. Let $\alpha : 0 \to R^{\oplus t} \to N \to M \to 0$ be a pullback of the short exact sequence $\bigoplus_{i=1}^t \alpha_i : 0 \to R^{\oplus t} \to \bigoplus_{i=1}^t N_i \to M^{\oplus t} \to 0$ by the diagonal map $\Delta : M \to M^{\oplus t}$. Then $\alpha = (\alpha_1, \dots, \alpha_t) \in \operatorname{Ext}^1_R(M, R^{\oplus t}) \cong \operatorname{Ext}^1_R(M, R)^{\oplus t}$. Next consider the induced exact sequence

$$0 \to M^* \to N^* \to (R^{\oplus t})^* \xrightarrow{\alpha} \operatorname{Ext}^1_R(M, R) \to \operatorname{Ext}^1_R(N, R) \to 0.$$

Since the map $(R^{\oplus t})^* \xrightarrow{\alpha} \operatorname{Ext}^1_R(M, R)$ is surjective, we see that $\operatorname{Ext}^1_R(N, R) = 0$. Thus, in the following pullback diagram, W, being a direct sum of $R^{\oplus s}$ and $R^{\oplus t}$, is free. So the vanishing of $\operatorname{Ext}^1_R(N, R)$ shows that $N = \mathrm{PF}(\Omega M)$.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & R^{\oplus t} & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & R^{\oplus t} & \longrightarrow & W & \longrightarrow & R^{\oplus s} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \Omega M & = & \Omega M & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now, if R is as in §B.1 and $M = \omega$, via the argument there, $N \otimes_R N$ is torsion-free.

In the next example we record a nonfree, torsion-free module N over a one-dimensional local domain R, where $N \otimes_R N$ is torsion-free, but $N \otimes_R N^*$ has torsion.

Example B.4. Let $R = \mathbb{C}[\![t^3, t^4, t^5]\!] \cong \mathbb{C}[\![x, y, z]\!]/(y^2 - xz, x^3 - yz, x^2y - z^2)$. Then R is a one-dimensional local domain which is not Gorenstein. Let N be the R-module given by the following exact sequence:

$$R^{\oplus 3} \xrightarrow{\begin{bmatrix} -y & x & z \\ x^2 & -z & -xy \\ -z & y & x^2 \end{bmatrix}} R^{\oplus 3} \longrightarrow N \longrightarrow 0.$$

One can check, for example, by using [Macaulay2 1993], that both N and $N \otimes_R N$ are torsion-free R-modules. Moreover, it follows that $N \otimes_R N^*$ has torsion; see [Huneke et al. 2019, 3.6].

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