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GLOBAL REGULARITY OF THE NAVIER–STOKES EQUATIONS ON 3D PERIODIC THIN DOMAIN WITH LARGE DATA

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We consider the Navier–Stokes equations on a 3D periodic thin domain $T_\epsilon = (0, l_1) \times (0, l_2) \times (0, \epsilon)$. We show that there exists an absolute (large) constant C such that for any $C^* > 0$ which can be arbitrarily large, there exists an $\epsilon_0 > 0$ such that the Navier–Stokes equations are globally well-posed for a class of large initial data satisfying

$$\|\partial_h u_0\|_{L^2(T_\epsilon)} \leqslant \frac{C^*}{\epsilon^{\frac{1}{2}} |\ln \epsilon|^{\frac{3}{2}}}, \quad \|\partial_3 u_0\|_{L^2(T_\epsilon)} \leqslant \frac{1}{C\epsilon^{\frac{1}{2}}},$$

where $\partial_h = (\partial_1, \partial_2)$ and $0 < \epsilon \leqslant \epsilon_0$. This improves the result of Kukavica and Ziane (*Journal of Differential Equations* 234:(2) (2007), 485–506), where the initial data u_0 is required to satisfy

$$\|\nabla u_0\|_{L^2(T_\epsilon)} \leqslant \frac{1}{C\epsilon^{\frac{1}{2}} |\ln \epsilon|^{\frac{3}{2}}}.$$

1. Introduction

The Navier–Stokes equations describe the time evolution of solutions of mathematical models of viscous incompressible fluids. The research of solutions has attracted many experts. To our knowledge, in the whole space case, Leray [1934] proved that if the divergence-free initial data u_0 belongs to L^2, there exists a weak solution $u(t)$ which is defined for all $t \geqslant 0$ and satisfies a global energy inequality. Hopf [1951] extended the result to the bounded domain case. Furthermore, if the initial data possesses certain regularity, say $u_0 \in H^1(\Omega)$, where Ω is a smooth bounded or periodic domain, then the Leray solution is smooth and unique at least for some short time interval; see [Temam 1984].

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In this paper, we consider the Navier–Stokes equations of the incompressible fluid flow on a periodic domain T_ϵ,

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1-1}$$

where u and p denote the velocity and the pressure, respectively, and T_ϵ is a 3D periodic thin domain, $T_\epsilon = T^2 \times T^1_\epsilon$, $T^2 = (0, l_1) \times (0, l_2)$, $0 < l_1, l_2 < \infty$, $T^1_\epsilon = (0, \epsilon)$, $0 < \epsilon < 1$. We assume that the initial data satisfies $u_0 \in H^1_{\text{per}}(T_\epsilon)$ with $\int_{T_\epsilon} u_0 = 0$. As we have mentioned above, there exists a local smooth solution. However, we don't know whether the solution can be global. In fact, in the 3D case, there is a global solution provided the initial data is sufficiently small; see [Fujita and Kato 1964]. It is unknown for the global existence in the large initial data case.

Our goal in this paper is to find how large the initial data can be to ensure the global existence of strong solutions on thin periodic domain. Hale and Raugel [1992a; 1992b] studied reaction diffusion equations and damped wave equations on thin domain. Raugel and Sell [1993; 1994] further studied the existence of strong solutions of the Navier–Stokes equations on thin domain. In particular, in [Raugel and Sell 1993], they proved that, in the periodic boundary condition case, the global existence holds with initial data in a *large set* of $H^1(T_\epsilon)$. Subsequent works concerning various boundary conditions complemented and extended their result; see [Temam and Ziane 1996; Montgomery-Smith 1999; Iftimie 1999; Iftimie and Raugel 2001; Kukavica and Ziane 2006; 2007; Hou et al. 2008; Kukavica et al. 2013; 2014]. It is worth mentioning that Temam and Ziane [1996] proved that in the case with Dirichlet boundary condition, global existence holds if the initial data satisfies

$$\|\nabla u_0\|_{L^2(T_\epsilon)} \leqslant \frac{\nu}{C\epsilon^{\frac{1}{2}}}, \tag{1-2}$$

where ν denotes the viscosity. It would be very interesting to understand how far we can go in the periodic case.

However, the periodic case is quite different with the Dirichlet boundary condition case. In the case of the periodic boundary condition, there is no Poincaré inequality in the vertical direction. For this reason, in the periodic case, the global regularity is still unclear under (1-2). Montgomery and Smith [1999] proved the global existence of solutions if

$$\|\nabla u_0\|_{L^2(T_\epsilon)} \leq \frac{\nu}{C(l_1, l_2)},$$

which was later on improved by Kukavica and Ziane [2006] to

$$\|\nabla u_0\|_{L^2(T_\epsilon)} \leq \frac{\nu}{C(l_1, l_2)\epsilon^{\frac{1}{6}}}.$$

Then after a year, Kukavica and Ziane [2007] improved their result to

$$\|\nabla u_0\|_{L^2(T_\epsilon)} \le \frac{\nu}{C(l_1,l_2)\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}},$$

where C is a sufficiently large constant.

In this paper, we prove that the global existence holds if the initial data satisfies

$$\|\partial_h u_0\|_{L^2(T_\epsilon)} \leqslant \frac{C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}, \qquad \|\partial_3 u_0\|_{L^2(T_\epsilon)} \leqslant \frac{1}{C\epsilon^{\frac{1}{2}}},$$

where $\partial_h = (\partial_1, \partial_2)$, C^* is an arbitrarily large constant and C is a sufficiently large constant. Here without loss of generality, we have taken the viscosity to be 1. We emphasize that the vertical derivative of the velocity $\partial_3 u$ has reached the desired result with the power of $-\frac{1}{2}$ of the exponent of ϵ. This is due to the observation that the Poincaré inequality for $\partial_3 u$ in the vertical direction holds since the average of $\partial_3 u$ in the vertical direction is automatically 0 for the periodic boundary condition. More precisely, it holds that

$$\frac{1}{\epsilon}\int_0^\epsilon \partial_3 u \,\mathrm{d}x_3 = \frac{1}{\epsilon}\big[u(x_1,x_2,\epsilon) - u(x_1,x_2,0)\big] = 0,$$

since u is periodic in vertical direction. To deal with the horizontal derivative $\partial_h u$, we use the same method as [Kukavica and Ziane 2007]. However, our result allows C^* to be arbitrarily large which is required to be sufficiently small in [Kukavica and Ziane 2007]. The key improvement lies in that in the estimate of $\|u_3\|_{L^\alpha(T_\epsilon)}$, we take $\alpha = 3 + 2|\ln\epsilon|/\ln|\ln\epsilon|$ instead of $\alpha = 3 + |\ln\epsilon|$ to gain more room for C^*.

Before we state our main result, we recall our hypothesis and introduce our notations. We assume that u satisfies the periodic boundary conditions

$$\begin{cases} u(x + l_i e_i, t) = u(x,t), & i = 1,2, \\ u(x + \epsilon e_3, t) = u(x,t), \end{cases}$$

where $\{e_1, e_2, e_3\}$ is the natural basis in $\mathbb{R}^3$. In addition, we require that the initial data $u(x,0) = u_0(x)$ satisfies

$$\int_{T_\epsilon} u_0(x)\,\mathrm{d}x = 0. \tag{1-3}$$

It then follows that any solution of (1-1) with the initial data $u_0(x)$ will also satisfy $\int_{T_\epsilon} u(x,t)\,\mathrm{d}x = 0$ for all $t > 0$. Let $L^p(T_\epsilon) \equiv L^p(T_\epsilon, \mathbb{R}^3)$ be the space of L^p vector functions u with the usual norm

$$\|u\|_{L^p(T_\epsilon)} = \left(\int_{T_\epsilon} |u|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

Let $H^m_{\mathrm{per}}(T_\epsilon) \equiv H^m_{\mathrm{per}}(T_\epsilon, \mathbb{R}^3)$ denote the closure in $H^m(T_\epsilon, \mathbb{R}^3)$ of those smooth functions that are periodic in space, i.e., $u(x+l_i e_i) = u(x)$, $i = 1, 2, 3$, where $l_3 = \epsilon$. Throughout this paper, the symbol C denotes a sufficiently large constant, which depends only on l_1 and l_2. Its value may change from one inequality to another. On the other hand, the constant $C_0, C_1, \dots$, which depend on l_1 and l_2, are fixed.

We are ready to state the main result in this paper.

Theorem 1.1. *Consider the Navier–Stokes equations (1-1) with the initial data $u_0 \in H^1_{\mathrm{per}}(T_\epsilon)$ which satisfies (1-3). For any given arbitrarily large constant C^*, there exists an $\epsilon_0 = \epsilon_0(C^*) \in (0, 1)$ such that for every $\epsilon \in (0, \epsilon_0]$, assuming that u_0 satisfies*

$$\begin{cases} \|\partial_h u_0\|_{L^2(T_\epsilon)} \leqslant \dfrac{C^*}{\epsilon^{\frac{1}{2}} |\ln \epsilon|^{\frac{3}{2}}}, \\ \|\partial_3 u_0\|_{L^2(T_\epsilon)} \leqslant \dfrac{1}{C_0 \epsilon^{\frac{1}{2}}}, \end{cases}$$

where $\partial_h = (\partial_1, \partial_2)$ and $C_0 > 0$ is a sufficiently large constant which depends only on l_1 and l_2, then (1-1) has a unique global solution u that belongs to $C([0, \infty), H^1_{\mathrm{per}}(T_\epsilon))$.

The following result is a key step in the proof of Theorem 1.1. We emphasize that this theorem is given by Kukavica and Ziane [2007]. However, their proof seems incomplete for us and needs some modifications. For completeness, we will present the details in Section 3.

Theorem 1.2. *Let $3 \leqslant \alpha \leqslant \widetilde{C} |\ln \epsilon|$ be arbitrary, where $\widetilde{C}$ is a large constant. Assume that the initial data $u_0 = (u_{01}, u_{02}, u_{03}) \in H^1_{\mathrm{per}}(T_\epsilon)$ satisfies*

$$\|\nabla u_{0k}\|_{L^2(T_\epsilon)} \leqslant \frac{1}{C \epsilon^{\frac{1}{2}} \alpha (\alpha + |\ln \epsilon|)^{\frac{1}{2}}}, \qquad k = 1, 2$$

and

$$\|u_{03}\|_{L^\alpha(T_\epsilon)} \leqslant \frac{1}{C \epsilon^{\frac{\alpha-3}{\alpha}}}.$$

Then (1-1) has a unique global solution u. Moreover,

$$\|\nabla u_k(\cdot, t)\|_{L^2(T_\epsilon)} \leqslant \frac{C}{\epsilon^{\frac{1}{2}} \alpha (\alpha + |\ln \epsilon|)^{\frac{1}{2}}}, \qquad k = 1, 2$$

and

$$\|u_3(\cdot, t)\|_{L^\alpha(T_\epsilon)} \leqslant \frac{C}{\epsilon^{\frac{\alpha-3}{\alpha}}}$$

for all $t > 0$, where $C > 0$ is a constant which depends only on l_1 and l_2.

The remaining part of this paper is organized as follows. Section 2 focuses on the Sobolev imbedding theorems for thin domain. Section 3 is devoted to proving Theorem 1.2. In Section 4, we finish the proof of Theorem 1.1 by dividing the whole time into three time intervals and using Theorem 1.2 in the third time interval to get the global regularity.

2. Preliminaries

In this section, we will introduce the average operator M and give the Sobolev imbedding theorems for thin domain. In addition, we will give an inequality about the L^α norm of u_3, which will play an important role in proving the main result.

For any $u \in L^1(T_\epsilon)$, as in [Kukavica and Ziane 2006; 2007; Raugel and Sell 1993; Temam and Ziane 1996], the average operator M is defined by

$$(Mu)(x_1,x_2) = \frac{1}{\epsilon}\int_0^\epsilon u(x_1,x_2,x_3)\,\mathrm{d}x_3.$$

We also define the operator N by (see [Kukavica and Ziane 2006; 2007; Raugel and Sell 1993; Temam and Ziane 1996])

$$Nu(x_1,x_2,x_3) = u(x_1,x_2,x_3) - (Mu)(x_1,x_2).$$

It is clear that Mu is independent of x_3 and $MNu = 0$. In addition, we also have

$$\|u\|^2_{L^2(T_\epsilon)} = \|Mu\|^2_{L^2(T_\epsilon)} + \|Nu\|^2_{L^2(T_\epsilon)}.$$

In the following lemma, we will recall the Sobolev imbedding theorems for thin domain which will be frequently used in the proof of the main result. The following estimates can be found in [Kukavica and Ziane 2006; 2007; Temam and Ziane 1996].

Lemma 2.1. *Assume $u \in H^1_{\mathrm{per}}(T_\epsilon)$.*

(i) *We have*

$$\|Nu\|_{L^2(T_\epsilon)} \leqslant C\epsilon\|\partial_3 u\|_{L^2(T_\epsilon)} \quad \textit{and} \quad \|Nu\|_{L^6(T_\epsilon)} \leqslant C\|\nabla u\|_{L^2(T_\epsilon)}.$$

For all $a \in [2,6]$, we have

$$\|Nu\|_{L^a(T_\epsilon)} \leqslant C\|u\|^{\frac{6-a}{2a}}_{L^2(T_\epsilon)}\|\nabla u\|^{\frac{3a-6}{2a}}_{L^2(T_\epsilon)}.$$

Moreover,

$$\|Nu\|_{L^a(T_\epsilon)} \leqslant C\epsilon^{\frac{6-a}{2a}}\|\nabla u\|_{L^2(T_\epsilon)}.$$

Here C depends only on l_1 and l_2.

(ii) *For all $a \in [2,\infty)$, we have*

$$\|Mu\|_{L^a(T_\epsilon)} \leqslant \frac{Ca^{\frac12}}{\epsilon^{\frac{a-2}{2a}}}\|u\|^{\frac{2}{a}}_{L^2(T_\epsilon)}\|\nabla u\|^{\frac{a-2}{a}}_{L^2(T_\epsilon)} + \frac{C}{\epsilon^{\frac{a-2}{2a}}}\|u\|_{L^2(T_\epsilon)}.$$

Moreover, if $\int_{T_\epsilon} u\,\mathrm{d}x = 0$, *then*

$$\|Mu\|_{L^a(T_\epsilon)} \leqslant \frac{Ca^{\frac{1}{2}}}{\epsilon^{\frac{a-2}{2a}}} \|u\|_{L^2(T_\epsilon)}^{\frac{2}{a}} \|\nabla u\|_{L^2(T_\epsilon)}^{\frac{a-2}{a}} \leqslant \frac{Ca^{\frac{1}{2}}}{\epsilon^{\frac{a-2}{2a}}} \|\nabla u\|_{L^2(T_\epsilon)}.$$

Here C *depends only on* l_1 *and* l_2.

(iii) *Assume* $\int_{T_\epsilon} u\,\mathrm{d}x = 0$. *Then for* $a \in [2,6]$, *we have*

$$\|u\|_{L^a(T_\epsilon)} \leqslant \frac{Ca^{\frac{1}{2}}}{\epsilon^{\frac{a-2}{2a}}} \|u\|_{L^2(T_\epsilon)}^{\frac{6-a}{2a}} \|\nabla u\|_{L^2(T_\epsilon)}^{\frac{3a-6}{2a}} \leqslant \frac{Ca^{\frac{1}{2}}}{\epsilon^{\frac{a-2}{2a}}} \|\nabla u\|_{L^2(T_\epsilon)},$$

where C *depends only on* l_1 *and* l_2.

One can find the proof of the above lemma in [Kukavica and Ziane 2006; 2007; Temam and Ziane 1996]. It should be pointed out that to get the last two inequalities in Lemma 2.1, we used the following Poincaré inequality on the periodic domain $T_\epsilon = (0, l_1) \times (0, l_2) \times (0, \epsilon)$:

$$\|u\|_{L^2(T_\epsilon)} \le C\|\nabla u\|_{L^2(T_\epsilon)}, \tag{2-1}$$

where C depends on l_1 and l_2. Inequality (2-1) is valid under the assumption $\int_{T_\epsilon} u\,\mathrm{d}x = 0$. To prove this, we first see that $u = Mu + Nu$. This means that the integral average of Nu on the vertical direction and Mu on the horizontal direction are 0, respectively, i.e.,

$$\frac{1}{\epsilon}\int_0^\epsilon Nu\,\mathrm{d}x_3 = 0, \qquad \frac{1}{|T^2|}\int_{T^2} Mu\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0.$$

Using the Poincaré inequality for Nu on the vertical direction and Mu on T^2, respectively, we get

$$\begin{aligned} &\|Nu\|_{L^2(T_\epsilon)} \le C\epsilon\|\partial_3 u\|_{L^2(T_\epsilon)}, \qquad \text{(see Lemma 2.1(i))}\\ &\|Mu\|_{L^2(T^2)} \le C\|\partial_h Mu\|_{L^2(T^2)} \Rightarrow \|Mu\|_{L^2(T_\epsilon)} \le C\|\partial_h u\|_{L^2(T_\epsilon)}. \end{aligned}$$

Hence,

$$\begin{aligned} \|u\|_{L^2(T_\epsilon)} &\le \|Nu\|_{L^2(T_\epsilon)} + \|Mu\|_{L^2(T_\epsilon)}\\ &\le C\epsilon\|\partial_3 u\|_{L^2(T_\epsilon)} + C\|\partial_h u\|_{L^2(T_\epsilon)} \le C\|\nabla u\|_{L^2(T_\epsilon)}. \end{aligned}$$

Next, we will give an estimate concerning the L^α norm of u_3 which has appeared in [Kukavica and Ziane 2006, Lemma 3] for $\alpha = 6$ and in [Kukavica et al. 2013, Lemma 4.2] for general α in the two-dimensional case. Below is the three-dimensional case. We remark that this has been proven for $\alpha = 6$ in [Kukavica and Ziane 2006, Lemma 4]. For completeness, we will present a proof below which seems even simpler.

Lemma 2.2. *Consider u_3, the third component of the velocity, which is defined on T_ϵ. Let $\alpha \in [2,\infty)$ be arbitrary. Assume that $u_3 \in H^1_{\text{per}}(T_\epsilon) \cap L^\alpha(T_\epsilon)$ satisfies $\nabla(|u_3|^{\frac{\alpha}{2}}) \in L^2_{\text{per}}(T_\epsilon)$ and $\int_{T_\epsilon} u_3 \, \mathrm{d}x = 0$. Then*

$$\big\| |u_3|^{\frac{\alpha}{2}} \big\|^2_{L^2(T_\epsilon)} \leqslant C \big\| \nabla(|u_3|^{\frac{\alpha}{2}}) \big\|^2_{L^2(T_\epsilon)}, \tag{2-2}$$

where C depends only on l_1, l_2 and α.

Remark 2.3. Lemma 2.2 will be used to prove the main result Theorem 1.1. We don't need to add the assumption $\int_{T_\epsilon} u_3 \, \mathrm{d}x = 0$ which appears in Lemma 2.2 to Theorem 1.1. Actually, in Theorem 1.1, we have made an assumption to the initial data $u_0 = (u_{01}, u_{02}, u_{03})$, that is,

$$\int_{T_\epsilon} u_0(x) \, \mathrm{d}x = 0;$$

see (1-3). Under this assumption, we can see that any solution $u = (u_1, u_2, u_3)$ of Navier–Stokes equations with this initial data will satisfy

$$\int_{T_\epsilon} u(x,t) \, \mathrm{d}x = 0$$

for all $t > 0$. Hence when we use Lemma 2.2 to prove Theorem 1.1, we don't need to make extra assumptions.

Proof. Since the size of T_ϵ is not order one, we make a transform to map T_ϵ onto $\widetilde{\Omega} = (0, l_1) \times (0, l_2) \times (0, 1)$. The transform is defined by

$$u_3(x_1, x_2, x_3) = u_3(y_1, y_2, \epsilon y_3) = v(y_1, y_2, y_3), \tag{2-3}$$

where $x = (x_1, x_2, x_3) \in T_\epsilon$, $y = (y_1, y_2, y_3) \in \widetilde{\Omega}$ and $x_i = y_i$, $i = 1, 2$; $x_3 = \epsilon y_3$. Then we know that v is defined on $\widetilde{\Omega}$ whose size is order one. Let $\tilde{u}(x) = |u_3|^{\frac{\alpha}{2}}(x)$ and $\tilde{v}(y) = |v|^{\frac{\alpha}{2}}(y)$. Since $\int_{T_\epsilon} u_3 \, \mathrm{d}x = 0$, it is obvious $\int_{\widetilde{\Omega}} v(y) \, \mathrm{d}y = 0$. By a similar argument as that of Lemma 3 in [Kukavica and Ziane 2006], we have

$$\|\tilde{v}\|^2_{L^2(\widetilde{\Omega})} \leqslant C \|\nabla \tilde{v}\|^2_{L^2(\widetilde{\Omega})}, \tag{2-4}$$

where C depends only on l_1, l_2 and α. Moreover, we can conclude from (2-3) that

$$\|\tilde{v}\|^2_{L^2(\widetilde{\Omega})} = \frac{1}{\epsilon} \|\tilde{u}\|^2_{L^2(T_\epsilon)}, \tag{2-5}$$

and

$$\|\nabla \tilde{v}\|^2_{L^2(\widetilde{\Omega})} \leqslant \frac{1}{\epsilon} \|\nabla \tilde{u}\|^2_{L^2(T_\epsilon)}. \tag{2-6}$$

It then follows from (2-4)–(2-6) that

$$\|\tilde{u}\|^2_{L^2(T_\epsilon)} \leqslant C\|\nabla\tilde{u}\|^2_{L^2(T_\epsilon)},$$

where C depends only on l_1, l_2 and α. Thus we complete the proof of (2-2). □

3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. We follow the idea of [Kukavica and Ziane 2007]. However, from our point of view, compared with the proof in [Kukavica and Ziane 2007], two places need to be modified when we estimate K_3 coming from the estimate of $\|u_3\|_{L^\alpha}$. We will show the details in the following proof.

Proof. Since the initial data $u_0 \in H^1_{\text{per}}(T_\epsilon)$, we know that the solution of (1-1) is smooth and unique on an initial time interval $(0, T_{\max})$, where $T_{\max} > 0$ depends on u_0. Take t_1, $0 < t_1 < T_{\max}$ and suppose $t \in [0, t_1]$. By (1-1), the componentwise Navier–Stokes equations become

$$\partial_t u_k - \Delta u_k + \sum_{j=1}^{3} u_j \partial_j u_k + \partial_k p = 0, \tag{3-1}$$

where $k = 1, 2, 3$.

Consider the Navier–Stokes equations (3-1) for $k = 1, 2$. We multiply the equations with $-\Delta u_k$ respectively and integrate over $T_\epsilon \times [0, t]$, and sum. Let $u_h = (u_1, u_2)$. It then follows that

$$\begin{aligned} \|\nabla u_h(t)\|^2_{L^2_x} - \|\nabla u_{0h}\|^2_{L^2_x} + \|\Delta u_h\|^2_{L^2_t L^2_x} &= \sum_{j=1}^{2} \iint u_j \partial_j u_h \Delta_h u_h + \sum_{j=1}^{2} \iint u_j \partial_j u_h \partial_{33} u_h \\ &\quad + \iint u_3 \partial_3 u_h \Delta u_h + \iint \partial_h p \Delta u_h \\ &= J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{3-2}$$

where $\Delta_h = \partial_{11} + \partial_{22}$, $\partial_h = (\partial_1, \partial_2)$ and we abbreviate

$$\|\cdot\|_{L^s_t L^r_x} = \|\cdot\|_{L^s((0,t), L^r(T_\epsilon))}.$$

We remark that above and in the sequel, all unmarked double integrals are understood to be over $T_\epsilon \times [0, t]$ and all unmarked single integrals are understood to be over T_ϵ.

For the term J_1, using integration by parts together with the fact $\nabla\cdot u=0$, we get

$$\begin{aligned}
J_1 &= -\sum_{i,j=1}^{2}\iint \partial_i u_j \partial_j u_h \partial_i u_h - \sum_{i,j=1}^{2}\iint u_j \partial_j \partial_i u_h \partial_i u_h \\
&= -\sum_{i,j=1}^{2}\iint \partial_i u_j \partial_j u_h \partial_i u_h + \frac{1}{2}\sum_{i,j=1}^{2}\iint \partial_j u_j \partial_i u_h \partial_i u_h \\
&= -\sum_{i,j=1}^{2}\iint \partial_i u_j \partial_j u_h \partial_i u_h - \frac{1}{2}\sum_{i=1}^{2}\iint \partial_i u_h \partial_i u_h \partial_3 u_3 \\
&= \frac{1}{2}\sum_{i=1}^{2}\iint \partial_i u_h \partial_i u_h \partial_3 u_3 + \iint \partial_1 u_2 \partial_2 u_1 \partial_3 u_3 - \iint \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 \\
&= J_{11} + J_{12} + J_{13}.
\end{aligned}$$

We next estimate J_{11}, J_{12}, J_{13}. Define

$$J(t) = \|\nabla u_h\|_{L_t^\infty L_x^2} + \|\nabla^2 u_h\|_{L_t^2 L_x^2},$$

where $\nabla^2 = (\partial_{ij})$, $i, j = 1,2,3$. Then we have the following useful estimate:

$$\|\partial_3 u_k\|_{L_t^2 L_x^a} \leqslant C\epsilon^{\frac{6-a}{2a}} J(t_1), \qquad a\in[2,6],\ k=1,2,3. \tag{3-3}$$

Since $\int_0^\epsilon \partial_3 u_k \, dx_3 = 0$ for $k = 1, 2, 3$, by using Lemma 2.1(i), we have

$$\|\partial_3 u_k\|_{L_t^2 L_x^a} \leqslant C\epsilon^{\frac{6-a}{2a}} \|\nabla\partial_3 u_k\|_{L_t^2 L_x^2} \leqslant C\epsilon^{\frac{6-a}{2a}} J(t_1), \qquad k=1,2.$$

By using the divergence-free condition, we get

$$\|\partial_3 u_3\|_{L_t^2 L_x^a} \leqslant C\epsilon^{\frac{6-a}{2a}} \|\nabla(\partial_1 u_1 + \partial_2 u_2)\|_{L_t^2 L_x^2} \leqslant C\epsilon^{\frac{6-a}{2a}} J(t_1).$$

Thus we finish the proof of the inequality (3-3). For the term J_{11}, we decompose it into three parts:

$$\begin{aligned}
J_{11} &= \frac{1}{2}\sum_{i=1}^{2}\iint M(\partial_i u_h)M(\partial_i u_h)\partial_3 u_3 + \sum_{i=1}^{2}\iint M(\partial_i u_h)N(\partial_i u_h)\partial_3 u_3 \\
&\qquad + \frac{1}{2}\sum_{i=1}^{2}\iint N(\partial_i u_h)N(\partial_i u_h)\partial_3 u_3 \\
&= J_{111} + J_{112} + J_{113}.
\end{aligned} \tag{3-4}$$

Regarding J_{111}, due to the fact that $M(\partial_i u_h)$ is independent of x_3, we have $\partial_3 M(\partial_i u_h) = 0$, thus

$$J_{111} = -\frac{1}{2}\sum_{i=1}^{2}\iint \partial_3 M(\partial_i u_h) M(\partial_i u_h) u_3 - \frac{1}{2}\sum_{i=1}^{2}\iint M(\partial_i u_h)\partial_3 M(\partial_i u_h) u_3 = 0. \tag{3-5}$$

Regarding J_{113}, we have

$$J_{113} \leqslant C \sum_{i=1}^{2} \|N(\partial_i u_h)\|_{L_t^4 L_x^3} \|N(\partial_i u_h)\|_{L_t^4 L_x^3} \|\partial_3 u_3\|_{L_t^2 L_x^3}. \tag{3-6}$$

Since $\int_0^\epsilon N(\partial_i u_h)\,\mathrm{d}x_3 = 0$, we have

$$\begin{aligned} \|N(\partial_i u_h)\|_{L_t^4 L_x^3} &\leqslant C \big\| \|\partial_i u_h\|_{L_x^2}^{\frac{1}{2}} \|\nabla \partial_i u_h\|_{L_x^2}^{\frac{1}{2}} \big\|_{L_t^4} \\ &\leqslant C \|\partial_i u_h\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|\nabla \partial_i u_h\|_{L_t^2 L_x^2}^{\frac{1}{2}} \\ &\leqslant C(\|\partial_i u_h\|_{L_t^\infty L_x^2} + \|\nabla \partial_i u_h\|_{L_t^2 L_x^2}) \leqslant C J(t_1). \end{aligned} \tag{3-7}$$

By using the inequality (3-3) with $a = 3$, we have

$$\|\partial_3 u_3\|_{L_t^2 L_x^3} \leqslant C \epsilon^{\frac{1}{2}} J(t_1). \tag{3-8}$$

It then follows from (3-6)–(3-8) that

$$J_{113} \leqslant C \epsilon^{\frac{1}{2}} J(t_1)^3. \tag{3-9}$$

Regarding J_{112}, we have

$$J_{112} \leqslant C \sum_{i=1}^{2} \|M(\partial_i u_h)\|_{L_t^4 L_x^4} \|N(\partial_i u_h)\|_{L_t^4 L_x^3} \|\partial_3 u_3\|_{L_t^2 L_x^{\frac{12}{5}}}. \tag{3-10}$$

Since $\int_{T_\epsilon} \partial_i u_h \,\mathrm{d}x = 0$, we can see from Lemma 2.1(ii) with $a = 4$ that

$$\|M(\partial_i u_h)\|_{L_x^4} \leqslant C \epsilon^{-\frac{1}{4}} \|\partial_i u_h\|_{L_x^2}^{\frac{1}{2}} \|\nabla \partial_i u_h\|_{L_x^2}^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned} \|M(\partial_i u_h)\|_{L_t^4 L_x^4} &\leqslant C \epsilon^{-\frac{1}{4}} \|\partial_i u_h\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|\nabla \partial_i u_h\|_{L_t^2 L_x^2}^{\frac{1}{2}} \\ &\leqslant C \epsilon^{-\frac{1}{4}} (\|\partial_i u_h\|_{L_t^\infty L_x^2} + \|\nabla \partial_i u_h\|_{L_t^2 L_x^2}) \leqslant C \epsilon^{-\frac{1}{4}} J(t_1). \end{aligned} \tag{3-11}$$

By using the inequality (3-3) with $a = \frac{12}{5}$, we have

$$\|\partial_3 u_3\|_{L_t^2 L_x^{\frac{12}{5}}} \leqslant C\epsilon^{\frac{3}{4}} J(t_1). \tag{3-12}$$

It then follows from (3-7), (3-10)–(3-12) that

$$J_{112} \leqslant C\epsilon^{\frac{1}{2}} J(t_1)^3. \tag{3-13}$$

Based on (3-4), (3-5), (3-9) and (3-13), we have

$$J_{11} \leqslant C\epsilon^{\frac{1}{2}} J(t_1)^3.$$

The terms J_{12} and J_{13} are estimated in the same way as J_{11}. Therefore, we obtain

$$J_1 \leqslant C\epsilon^{\frac{1}{2}} J(t_1)^3. \tag{3-14}$$

For the term J_2, using integration by parts together with $\nabla \cdot u = 0$, we have

$$\begin{aligned} J_2 &= \sum_{j=1}^{2} \iint u_j \partial_j u_h \partial_{33} u_h = -\sum_{j=1}^{2} \iint \partial_3 u_j \partial_j u_h \partial_3 u_h - \sum_{j=1}^{2} \iint u_j \partial_j \partial_3 u_h \partial_3 u_h \\ &= -\sum_{j=1}^{2} \iint \partial_3 u_j \partial_j u_h \partial_3 u_h + \frac{1}{2} \sum_{j=1}^{2} \iint \partial_j u_j \partial_3 u_h \partial_3 u_h = J_{21} + J_{22}. \end{aligned}$$

Regarding J_{21}, we have

$$J_{21} \leqslant \sum_{j=1}^{2} \|\partial_3 u_j\|_{L_t^2 L_x^4} \|\partial_j u_h\|_{L_t^\infty L_x^2} \|\partial_3 u_h\|_{L_t^2 L_x^4} \leqslant C\epsilon^{\frac{1}{2}} J(t_1)^3,$$

where we have used (3-3) with $a = 4$. The same estimate holds for J_{22}. Therefore,

$$J_2 \leqslant C\epsilon^{\frac{1}{2}} J(t_1)^3. \tag{3-15}$$

For the term J_3, define

$$K(t) = \left(\big\| |u_3|^{\frac{\alpha}{2}} \big\|_{L_t^\infty L_x^2} + \big\| \nabla(|u_3|^{\frac{\alpha}{2}}) \big\|_{L_t^2 L_x^2}\right)^{\frac{2}{\alpha}}, \quad t \in [0, T_{\max}).$$

Then we get

$$J_3 \leqslant \|u_3\|_{L_t^\infty L_x^\alpha} \|\partial_3 u_h\|_{L_t^2 L_x^{\frac{2\alpha}{\alpha-2}}} \|\Delta u_h\|_{L_t^2 L_x^2}.$$

Since $3 \leqslant \alpha \leqslant \widetilde{C} |\ln \epsilon|$ implies that $2 < \frac{2\alpha}{\alpha-2} \leqslant 6$, it follows from (3-3) that

$$\|\partial_3 u_h\|_{L_t^2 L_x^{\frac{2\alpha}{\alpha-2}}} \leqslant C\epsilon^{\frac{\alpha-3}{\alpha}} J(t_1).$$

Thus,

$$J_3 \leqslant C\epsilon^{\frac{\alpha-3}{\alpha}} K(t_1) J(t_1)^2. \tag{3-16}$$

For the term J_4 which includes Δp, we need to take the divergence of (1-1) and obtain that

$$-\Delta p = \nabla\cdot(u\cdot\nabla u) = \sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i.$$

Then we have

$$\begin{aligned} J_4 &= -\iint \Delta p \partial_h u_h = \sum_{i,j=1}^{3} \iint \partial_i u_j \partial_j u_i \partial_h u_h \\ &= \sum_{i,j=1}^{2} \iint \partial_i u_j \partial_j u_i \partial_h u_h + 2\sum_{j=1}^{2} \iint \partial_3 u_j \partial_j u_3 \partial_h u_h \\ &= -\sum_{i,j=1}^{2} \iint \partial_i u_j \partial_j u_i \partial_3 u_3 + 2\sum_{j=1}^{2} \iint \partial_3 u_j \partial_j u_3 \partial_h u_h \\ &= J_{41} + J_{42}. \end{aligned}$$

The term J_{41} can be estimated in a similar way to J_{11}, giving

$$J_{41} \leqslant C\epsilon^{\frac{1}{2}} J(t_1)^3. \tag{3-17}$$

Regarding J_{42}, using integration by parts, we have

$$\begin{aligned} J_{42} &= -2\sum_{j=1}^{2} \iint \partial_j \partial_3 u_j u_3 \partial_h u_h - 2\sum_{j=1}^{2} \iint \partial_3 u_j u_3 \partial_j \partial_h u_h \\ &= 2\sum_{j=1}^{2} \iint \partial_j \partial_3 u_j u_3 \partial_3 u_3 - 2\sum_{j=1}^{2} \iint \partial_3 u_j u_3 \partial_j \partial_h u_h \\ &= J_{421} + J_{422}. \end{aligned}$$

Estimate J_{421} and J_{422} to obtain

$$J_{421} \leqslant C\|\partial_j \partial_3 u_j\|_{L_t^2 L_x^2} \|u_3\|_{L_t^\infty L_x^\alpha} \|\partial_3 u_3\|_{L_t^2 L_x^{\frac{2\alpha}{\alpha-2}}} \leqslant C\epsilon^{\frac{\alpha-3}{\alpha}} K(t_1) J(t_1)^2$$

$$J_{422} \leqslant C\sum_{j=1}^{2} \|\partial_3 u_j\|_{L_t^2 L_x^{\frac{2\alpha}{\alpha-2}}} \|u_3\|_{L_t^\infty L_x^\alpha} \|\partial_j \partial_h u_h\|_{L_t^2 L_x^2} \leqslant C\epsilon^{\frac{\alpha-3}{\alpha}} K(t_1) J(t_1)^2.$$

Thus we conclude

$$J_{42} \leqslant C\epsilon^{\frac{\alpha-3}{\alpha}} K(t_1) J(t_1)^2. \tag{3-18}$$

It then follows from (3-17) and (3-18) that

$$J_4 \leqslant C\epsilon^{\frac{1}{2}} J(t_1)^3 + C\epsilon^{\frac{\alpha-3}{\alpha}} K(t_1) J(t_1)^2. \tag{3-19}$$

Therefore, by (3-2), (3-14)–(3-16) and (3-19), we obtain the final estimate about $J(t)$:

$$J(t)^2 \leqslant C\epsilon^{\frac{1}{2}} J(t)^3 + C\epsilon^{\frac{\alpha-3}{\alpha}} K(t)J(t)^2 + J(0)^2, \tag{3-20}$$

where we have used the second derivative estimate

$$\|\nabla^2 u_h\|_{L_t^2 L_x^2} \leqslant C\|\Delta u_h\|_{L_t^2 L_x^2}$$

together with the fact

$$J(0) = \|\nabla u_{0h}\|_{L_x^2}.$$

The next objective is to estimate $K(t)$. Consider the Navier–Stokes equations (3-1) for $k = 3$. We multiply it with $|u_3|^{\alpha-1}\operatorname{sgn} u_3$ and integrate over $T_\epsilon \times [0, t]$. There holds

$$\iint \partial_t u_3 |u_3|^{\alpha-1}\operatorname{sgn} u_3 + \iint u\cdot\nabla u_3 |u_3|^{\alpha-1}\operatorname{sgn} u_3 - \iint \Delta u_3 |u_3|^{\alpha-1}\operatorname{sgn} u_3 \\ = -\iint \partial_3 p |u_3|^{\alpha-1}\operatorname{sgn} u_3.$$

After a short calculation, we have

$$\frac{1}{\alpha}\|u_3(t)\|_{L_x^\alpha}^\alpha + \frac{4(\alpha-1)}{\alpha^2}\|\nabla(|u_3|^{\frac{\alpha}{2}})\|_{L_t^2 L_x^2}^2 \\ = -\iint \partial_3 p |u_3|^{\alpha-1}\operatorname{sgn} u_3 + \frac{1}{\alpha}\|u_{03}\|_{L_x^\alpha}^\alpha. \tag{3-21}$$

It remains to estimate $-\iint \partial_3 p|u_3|^{\alpha-1}\operatorname{sgn} u_3$. From (1-1), we know that

$$p = (-\Delta)^{-1}\nabla\cdot\nabla\cdot(u\otimes u) = \sum_{i,j=1}^{3} R_i R_j(u_i u_j) = \sum_{i,j=1}^{3} R_{i,j}(u_i u_j),$$

where R_1, R_2, R_3 are the Riesz transforms. Since ∂_3 can commute with the Riesz transforms, we have

$$\begin{aligned}\partial_3 p &= \partial_3 \sum_{i,j=1}^{3} R_{i,j}(u_i u_j) = 2\sum_{i,j=1}^{3} R_{i,j}(\partial_3 u_i u_j)\\ &= 2\sum_{i=1}^{3}\sum_{j=1}^{2} R_{i,j}(\partial_3 u_i N u_j) + 2\sum_{i=1}^{3}\sum_{j=1}^{2} R_{i,j}(\partial_3 u_i M u_j) + 2\sum_{i=1}^{3} R_{i,3}(\partial_3 u_i u_3)\\ &= q_1 + q_2 + q_3.\end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
-\iint \partial_3 p |u_3|^{\alpha-1} \operatorname{sgn} u_3
&= -\iint (q_1+q_2+q_3) u_3 |u_3|^{\alpha-2} \\
&= -\iint q_1 |u_3|^{\frac{\alpha-2}{2}} N(|u_3|^{\frac{\alpha}{2}}) \operatorname{sgn} u_3 - \iint q_1 |u_3|^{\frac{\alpha-2}{2}} M(|u_3|^{\frac{\alpha}{2}}) \operatorname{sgn} u_3 \\
&\qquad - \iint q_2 u_3 |u_3|^{\alpha-2} - \iint q_3 |u_3|^{\frac{\alpha-2}{2}} N(|u_3|^{\frac{\alpha}{2}}) \operatorname{sgn} u_3 \\
&\qquad - \iint q_3 |u_3|^{\frac{\alpha-2}{2}} M(|u_3|^{\frac{\alpha}{2}}) \operatorname{sgn} u_3 \\
&= K_1+K_2+K_3+K_4+K_5.
\end{aligned}
$$

For the term K_1,

$$
\begin{aligned}
K_1 &\leqslant \|q_1\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} \big\| |u_3|^{\frac{\alpha-2}{2}} \big\|_{L_t^\infty L_x^{\frac{2\alpha}{\alpha-2}}} \big\| N(|u_3|^{\frac{\alpha}{2}}) \big\|_{L_t^2 L_x^6} \\
&\leqslant C \|q_1\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} \big\| |u_3|^{\frac{\alpha}{2}} \big\|^{\frac{\alpha-2}{\alpha}}_{L_t^\infty L_x^2} \big\| \nabla(|u_3|^{\frac{\alpha}{2}}) \big\|_{L_t^2 L_x^2} \leqslant C \|q_1\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} K(t_1)^{\alpha-1},
\end{aligned}
$$

where we have used Lemma 2.1(i) for $\big\| N(|u_3|^{\frac{\alpha}{2}}) \big\|_{L_t^2 L_x^6}$. Regarding $\|q_1\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}}$, we have

$$
\begin{aligned}
\text{(3-22)} \qquad \|q_1\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} &\leqslant C \sum_{i=1}^{3} \sum_{j=1}^{2} \|R_{i,j}\|_{L_x^{\frac{3\alpha}{\alpha+3}}} \|\partial_3 u_i N u_j\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} \\
&\leqslant C \sum_{i=1}^{3} \sum_{j=1}^{2} \|R_{i,j}\|_{L_x^{\frac{3\alpha}{\alpha+3}}} \|\partial_3 u_i\|_{L_t^2 L_x^{\frac{6\alpha}{\alpha+6}}} \|N u_j\|_{L_t^\infty L_x^6}.
\end{aligned}
$$

As we know, the Riesz transforms $R_{i,j}$ $(i, j = 1, 2, 3)$ are bounded on $L^p(T_\epsilon)$ for $1 < p < \infty$. Furthermore, the bound is given by (see [Grafakos 2004, p. 362])

$$
\text{(3-23)} \qquad \|R_{i,j}\|_{L^p} \leqslant C \max\Big(p, \frac{1}{p-1}\Big),
$$

where C is independent of p. Here, $\frac{3}{2} \leqslant \frac{3\alpha}{\alpha+3} < 3$ when $3 \leqslant \alpha \leqslant \widetilde{C}|\ln \epsilon|$. Thus, $\|R_{i,j}\|_{L_x^{\frac{3\alpha}{\alpha+3}}} \leqslant C$ for $i, j = 1, 2, 3$. Since $2 \leqslant \frac{6\alpha}{\alpha+6} < 6$ when $3 \leqslant \alpha \leqslant \widetilde{C}|\ln \epsilon|$, we can see from (3-3) that

$$
\text{(3-24)} \qquad \|\partial_3 u_i\|_{L_t^2 L_x^{\frac{6\alpha}{\alpha+6}}} \leqslant C \epsilon^{\frac{3}{\alpha}} J(t_1).
$$

Also, by using Lemma 2.1(i) we have

$$\sum_{j=1}^{2}\|Nu_j\|_{L_t^\infty L_x^6} \leqslant \sum_{j=1}^{2}\|\nabla u_j\|_{L_t^\infty L_x^2} \leqslant CJ(t_1). \tag{3-25}$$

Thus, by (3-22)–(3-25), we get

$$\|q_1\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} \leqslant C\epsilon^{\frac{3}{\alpha}} J(t_1)^2.$$

Therefore, we obtain the estimate of K_1,

$$K_1 \leqslant C\epsilon^{\frac{3}{\alpha}} J(t_1)^2 K(t_1)^{\alpha-1}. \tag{3-26}$$

For the term K_2, we have

$$K_2 \leqslant \|q_1\|_{L_t^2 L_x^{r_1}} \big\| |u_3|^{\frac{\alpha-2}{2}} \big\|_{L_t^\infty L_x^{\frac{2\alpha}{\alpha-2}}} \big\| M(|u_3|^{\frac{\alpha}{2}}) \big\|_{L_t^2 L_x^b}, \tag{3-27}$$

where r_1 and b satisfy $\frac{1}{r_1}+\frac{\alpha-2}{2\alpha}+\frac{1}{b}=1$. Let $b \geqslant 2\alpha$, then we have $r_1 \in \big(\frac{2\alpha}{\alpha+2}, \frac{2\alpha}{\alpha+1}\big]$. Now we estimate the three terms on the right-hand side of (3-27). Regarding $\|q_1\|_{L_t^2 L_x^{r_1}}$, we have

$$\|q_1\|_{L_t^2 L_x^{r_1}} \leqslant C \sum_{i=1}^{3}\sum_{j=1}^{2} \|\partial_3 u_i\|_{L_t^2 L_x^{2r_1}} \|Nu_j\|_{L_t^\infty L_x^{2r_1}},$$

Because of the fact that $2r_1 \in \big(\frac{4\alpha}{\alpha+2}, \frac{4\alpha}{\alpha+1}\big] \subset \big[\frac{12}{5}, 4\big)$ when $3 \leqslant \alpha \leqslant C|\ln\epsilon|$, we conclude from (3-3) that

$$\|\partial_3 u_i\|_{L_t^2 L_x^{2r_1}} \leqslant C\epsilon^{\frac{3-r_1}{2r_1}} J(t_1).$$

Also, by using Lemma 2.1(i) we have

$$\sum_{j=1}^{2}\|Nu_j\|_{L_t^\infty L_x^{2r_1}} \leqslant C\epsilon^{\frac{3-r_1}{2r_1}} \sum_{j=1}^{2}\|\nabla u_j\|_{L_t^\infty L_x^2} \leqslant C\epsilon^{\frac{3-r_1}{2r_1}} J(t_1).$$

Thus, we have

$$\|q_1\|_{L_t^2 L_x^{r_1}} \leqslant C\epsilon^{\frac{3-r_1}{r_1}} J(t_1)^2. \tag{3-28}$$

Regarding $\big\| |u_3|^{\frac{\alpha-2}{2}} \big\|_{L_t^\infty L_x^{\frac{2\alpha}{\alpha-2}}}$, we have

$$\big\| |u_3|^{\frac{\alpha-2}{2}} \big\|_{L_t^\infty L_x^{\frac{2\alpha}{\alpha-2}}} = \big\| |u_3|^{\frac{\alpha}{2}} \big\|_{L_t^\infty L_x^2}^{\frac{\alpha-2}{\alpha}} \leqslant CK(t_1)^{\frac{\alpha-2}{2}}. \tag{3-29}$$

Regarding $\left\|M(|u_3|^{\frac{\alpha}{2}})\right\|_{L_t^2 L_x^b}$, by using Lemma 2.1(ii) we have

$$\begin{aligned}
&\left\|M(|u_3|^{\frac{\alpha}{2}})\right\|_{L_t^2 L_x^b} \\
&\leqslant C b^{\frac{1}{2}} \epsilon^{-\frac{b-2}{2b}} \left\| \left\| |u_3|^{\frac{\alpha}{2}} \right\|_{L_x^2}^{\frac{2}{b}} \left\| \nabla(|u_3|^{\frac{\alpha}{2}}) \right\|_{L_x^2}^{\frac{b-2}{b}} \right\|_{L_t^2} + C\epsilon^{-\frac{b-2}{2b}} \left\| |u_3|^{\frac{\alpha}{2}} \right\|_{L_t^2 L_x^2} \\
&\leqslant C b^{\frac{1}{2}} \epsilon^{-\frac{b-2}{2b}} \left\| \left\| |u_3|^{\frac{\alpha}{2}} \right\|_{L_x^2}^{\frac{2}{b}} \right\|_{L_t^b} \left\| \left\| \nabla(|u_3|^{\frac{\alpha}{2}}) \right\|_{L_x^2}^{\frac{b-2}{b}} \right\|_{L_t^{\frac{2b}{b-2}}} + C\epsilon^{-\frac{b-2}{2b}} \left\| |u_3|^{\frac{\alpha}{2}} \right\|_{L_t^2 L_x^2} \\
&\leqslant C b^{\frac{1}{2}} \epsilon^{-\frac{b-2}{2b}} \left\| |u_3|^{\frac{\alpha}{2}} \right\|_{L_t^2 L_x^2}^{\frac{2}{b}} \left\| \nabla(|u_3|^{\frac{\alpha}{2}}) \right\|_{L_t^2 L_x^2}^{\frac{b-2}{b}} + C\epsilon^{-\frac{b-2}{2b}} \left\| |u_3|^{\frac{\alpha}{2}} \right\|_{L_t^2 L_x^2}.
\end{aligned}$$

Meanwhile, by using Lemma 2.2, we know that

$$\left\|M(|u_3|^{\frac{\alpha}{2}})\right\|_{L_t^2 L_x^b} \leqslant C b^{\frac{1}{2}} \epsilon^{-\frac{b-2}{2b}} \left\| \nabla(|u_3|^{\frac{\alpha}{2}}) \right\|_{L_t^2 L_x^2} \leqslant C b^{\frac{1}{2}} \epsilon^{-\frac{b-2}{2b}} K(t_1)^{\frac{\alpha}{2}}. \tag{3-30}$$

Therefore, it follows from (3-27)–(3-30) that

$$K_2 \leqslant C b^{\frac{1}{2}} \epsilon^{\frac{3}{\alpha}-\frac{2}{b}} J(t_1)^2 K(t_1)^{\alpha-1}. \tag{3-31}$$

For the term K_3, we first rewrite $q_2 = 2\sum_{i=1}^{3}\sum_{j=1}^{2} R_{i,j}(\partial_3 u_i M u_j)$. Since Mu_j is independent of x_3, we have $\partial_3 u_i M u_j = \partial_3(Mu_i + Nu_i) M u_j = \partial_3(Nu_i M u_j)$. Let

$$\widetilde{q_2} = 2\sum_{i=1}^{3}\sum_{j=1}^{2} R_{i,j}(Nu_i M u_j).$$

Then we have $q_2 = \partial_3 \widetilde{q_2}$ as the derivative can commute with the Riesz transforms. Thus, we obtain the following result

$$\begin{aligned}
K_3 &= -\iint \partial_3 \widetilde{q_2} u_3 |u_3|^{\alpha-2} = \iint \widetilde{q_2} \partial_3 u_3 |u_3|^{\alpha-2} + \iint \widetilde{q_2} u_3 \partial_3(|u_3|^{\alpha-2}) \\
&= (\alpha-1) \iint \widetilde{q_2} \partial_3 u_3 |u_3|^{\alpha-2} = \frac{2(\alpha-1)}{\alpha} \iint \widetilde{q_2} \partial_3(|u_3|^{\frac{\alpha}{2}}) |u_3|^{\frac{\alpha-2}{2}} \operatorname{sgn} u_3.
\end{aligned}$$

According to the above result, we have

$$\begin{aligned}
K_3 &\leqslant \frac{2(\alpha-1)}{\alpha} \|\widetilde{q_2}\|_{L_t^2 L_x^\alpha} \left\| \partial_3(|u_3|^{\frac{\alpha}{2}}) \right\|_{L_t^2 L_x^2} \left\| |u_3|^{\frac{\alpha-2}{2}} \right\|_{L_t^\infty L_x^{\frac{2\alpha}{\alpha-2}}} \\
&\leqslant C \|\widetilde{q_2}\|_{L_t^2 L_x^\alpha} \left\| |u_3|^{\frac{\alpha}{2}} \right\|_{L_t^\infty L_x^2}^{\frac{\alpha-2}{\alpha}} K(t_1)^{\frac{\alpha}{2}} \\
&\leqslant C \|\widetilde{q_2}\|_{L_t^2 L_x^\alpha} K(t_1)^{\alpha-1}.
\end{aligned}$$

It remains to estimate $\|\widetilde{q_2}\|_{L^2_t L^\alpha_x}$.

$$\begin{aligned}\|\widetilde{q_2}\|_{L^\alpha_x} &\leqslant C\sum_{i=1}^{3}\sum_{j=1}^{2}\|R_{i,j}Nu_iMu_j\|_{L^\alpha_x} \leqslant C\sum_{i=1}^{3}\sum_{j=1}^{2}\|R_{i,j}\|_{L^\alpha_x}\|Nu_iMu_j\|_{L^\alpha_x}\\ &\leqslant C\alpha\sum_{i=1}^{3}\sum_{j=1}^{2}\|Nu_iMu_j\|_{L^\alpha_x}.\end{aligned}$$

Here, compared with [Kukavica and Ziane 2007], we modified the estimate of $\|\widetilde{q_2}\|_{L^\alpha_x}$ by adding the L^α norm of Riesz transforms given by (3-23). The reason is that we will take α to be very large, roughly like $|\ln\epsilon|$, when proving the main result. Hence

$$\|\widetilde{q_2}\|_{L^2_tL^\alpha_x}\leqslant C\alpha\sum_{i=1}^{3}\sum_{j=1}^{2}\|Nu_iMu_j\|_{L^2_tL^\alpha_x}\leqslant C\alpha\sum_{i=1}^{3}\sum_{j=1}^{2}\|Nu_i\|_{L^2_tL^{r_2}_x}\|Mu_j\|_{L^\infty_tL^b_x},$$

where $b\geqslant 2\alpha$ and $r_2=\frac{b\alpha}{b-\alpha}\in(\alpha,2\alpha]$. By using Lemma 2.1(ii) we have

$$\|Mu_j\|_{L^\infty_tL^b_x}\leqslant Cb^{\frac{1}{2}}\epsilon^{-\frac{b-2}{2b}}\|\nabla u_j\|_{L^\infty_tL^2_x}\leqslant Cb^{\frac{1}{2}}\epsilon^{-\frac{b-2}{2b}}J(t_1).$$

One expects to bound $\|Nu_i\|_{L^2_tL^{r_2}_x}$ by $\|\nabla^2 u_h\|_{L^2_tL^2_x}$ and thus by $J(t)$. In [Kukavica and Ziane 2007], the authors considered two cases: $2\le r_2\le 6$ and $6\le r_2<\infty$. When $2\le r_2\le 6$, by Lemma 2.1(i),

$$\|Nu_i\|_{L^2_tL^{r_2}_x}\le C\epsilon^{\frac{6-r_2}{2r_2}}\sum_{k=1}^{3}\|\partial_k(Nu_i)\|_{L^2_tL^2_x}.$$

For the case $i=3, k=1,2$, they used the Poincaré inequality to get

$$\|\partial_k(Nu_3)\|_{L^2_tL^{r_2}_x}\le C\epsilon\|\partial_3\partial_k u_3\|_{L^2_tL^2_x}\le C\epsilon\|\partial_k\partial_h u_h\|_{L^2_tL^2_x}.$$

When $6\le r_2<\infty$, they first used the Gagliardo–Nirenberg inequality to get

$$\|Nu_i\|_{L^2_tL^{r_2}_x}\le C\sum_{k=1}^{3}\|\partial_k(Nu_i)\|_{L^2_tL^{\widetilde{r_2}}_x},\quad \widetilde{r_2}=\frac{3r_2}{r_2+3}.$$

Then by Lemma 2.1(i),

$$\|\partial_k(Nu_i)\|_{L^2_tL^{\widetilde{r_2}}_x}\le C\epsilon^{\frac{6-\widetilde{r_2}}{2\widetilde{r_2}}}\|\nabla\partial_k u_i\|_{L^2_tL^2_x}=C\epsilon^{\frac{6+r_2}{2r_2}}\|\nabla\partial_k u_i\|_{L^2_tL^2_x},\tag{3-32}$$

However, it seems that $\|\nabla\partial_k u_i\|_{L^2_tL^2_x}$ can't be controlled by $J(t)$ when $i=3$, $k=1,2$ because $J(t)$ doesn't contain the L^2 norm of $\nabla^2 u_3$.

To modify this, we will use the idea of anisotropic interpolations. Obviously

$$\|Nu_i\|_{L_x^{r_2}} = \big\| \|Nu_i\|_{L_{x_3}^{r_2}} \big\|_{L_{x_h}^{r_2}},$$

where we abbreviate $\|\cdot\|_{L_{x_3}^p} = \|\cdot\|_{L^p((0,\epsilon))}$. In the sequel, we will also abbreviate $\|\cdot\|_{L_{x_h}^q} = \|\cdot\|_{L^q(T^2)}$. Interpolating through the vertical direction, we have

$$\|Nu_i\|_{L_{x_3}^{r_2}} \leqslant C\|Nu_i\|_{L_{x_3}^2}^{\frac{1}{2}+\frac{1}{r_2}} \|\partial_3 Nu_i\|_{L_{x_3}^2}^{\frac{1}{2}-\frac{1}{r_2}} \leqslant C\epsilon^{\frac{1}{2}+\frac{1}{r_2}} \|\partial_3 Nu_i\|_{L_{x_3}^2}.$$

This implies that

$$\|Nu_i\|_{L_x^{r_2}} \leqslant C\epsilon^{\frac{1}{2}+\frac{1}{r_2}} \big\| \|\partial_3 Nu_i\|_{L_{x_3}^2} \big\|_{L_{x_h}^{r_2}} \leqslant C\epsilon^{\frac{1}{2}+\frac{1}{r_2}} \big\| \|\partial_3 Nu_i\|_{L_{x_h}^{r_2}} \big\|_{L_{x_3}^2}.$$

Interpolating through the horizontal direction, we obtain that

$$\|\partial_3 Nu_i\|_{L_{x_h}^{r_2}} \leqslant C r_2^{\frac{1}{2}} \|\partial_3 Nu_i\|_{L_{x_h}^2}^{\frac{2}{r_2}} \|\partial_h\partial_3 Nu_i\|_{L_{x_h}^2}^{1-\frac{2}{r_2}} + C\|\partial_3 Nu_i\|_{L_{x_h}^2}.$$

As a result, we have

$$\begin{aligned}
\big\| \|\partial_3 Nu_i\|_{L_{x_h}^{r_2}} \big\|_{L_{x_3}^2} &\leqslant C r_2^{\frac{1}{2}} \Big\| \|\partial_3 Nu_i\|_{L_{x_h}^2}^{\frac{2}{r_2}} \|\partial_h\partial_3 Nu_i\|_{L_{x_h}^2}^{1-\frac{2}{r_2}} \Big\|_{L_{x_3}^2} + C\|\partial_3 Nu_i\|_{L_x^2} \\
&\leqslant C r_2^{\frac{1}{2}} \|\partial_3 Nu_i\|_{L_x^2}^{\frac{2}{r_2}} \|\partial_h\partial_3 Nu_i\|_{L_x^2}^{1-\frac{2}{r_2}} + C\|\partial_3 Nu_i\|_{L_x^2} \\
&\leqslant C r_2^{\frac{1}{2}} \epsilon^{\frac{2}{r_2}} \|\partial_{33} Nu_i\|_{L_x^2}^{\frac{2}{r_2}} \|\partial_h\partial_3 Nu_i\|_{L_x^2}^{1-\frac{2}{r_2}} + C\epsilon\|\partial_{33} Nu_i\|_{L_x^2} \\
&\leqslant C r_2^{\frac{1}{2}} \epsilon^{\frac{2}{r_2}} \|\nabla\partial_3 u_i\|_{L_x^2}.
\end{aligned}$$

Therefore, we get the estimate

$$\|Nu_i\|_{L_x^{r_2}} \leqslant C r_2^{\frac{1}{2}} \epsilon^{\frac{1}{2}+\frac{3}{r_2}} \|\nabla\partial_3 u_i\|_{L_x^2},$$

which yields

$$\|Nu_i\|_{L_t^2 L_x^{r_2}} \leqslant C r_2^{\frac{1}{2}} \epsilon^{\frac{1}{2}+\frac{3}{r_2}} \|\nabla\partial_3 u_i\|_{L_t^2 L_x^2} \leqslant C r_2^{\frac{1}{2}} \epsilon^{\frac{1}{2}+\frac{3}{r_2}} J(t_1) \leqslant C b^{\frac{1}{2}} \epsilon^{\frac{1}{2}+\frac{3}{\alpha}-\frac{3}{b}} J(t_1).$$

Thus we have

$$\|\widetilde{q_2}\|_{L_t^2 L_x^{\alpha}} \leqslant C\alpha b \epsilon^{\frac{3}{\alpha}-\frac{2}{b}} J(t_1)^2.$$

It follows that

$$K_3 \leqslant C\alpha b \epsilon^{\frac{3}{\alpha}-\frac{2}{b}} J(t_1)^2 K(t_1)^{\alpha-1}. \tag{3-33}$$

For the term K_4, we have

$$\begin{aligned} K_4 &\leqslant \|q_3\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} \big\| |u_3|^{\frac{\alpha-2}{2}} \big\|_{L_t^\infty L_x^{\frac{2\alpha}{\alpha-2}}} \big\| N(|u_3|^{\frac{\alpha}{2}}) \big\|_{L_t^2 L_x^6} \\ &\leqslant C \|q_3\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} \big\| |u_3|^{\frac{\alpha-2}{2}} \big\|_{L_t^\infty L_x^2}^{\frac{\alpha-2}{\alpha}} \big\| \nabla(|u_3|^{\frac{\alpha}{2}}) \big\|_{L_t^2 L_x^2} \\ &\leqslant C \|q_3\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} K(t_1)^{\alpha-1}. \end{aligned}$$

Next we estimate $\|q_3\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}}$,

$$\|q_3\|_{L_t^2 L_x^{\frac{3\alpha}{\alpha+3}}} \leqslant C \sum_{i=1}^{3} \|\partial_3 u_i\|_{L_t^2 L_x^3} \|u_3\|_{L_t^\infty L_x^\alpha} \leqslant C \epsilon^{\frac{1}{2}} J(t_1) K(t_1),$$

where we have used (3-3) with $a = 3$. According to this estimate, we have

$$K_4 \leqslant C \epsilon^{\frac{1}{2}} J(t_1) K(t_1)^\alpha. \tag{3-34}$$

For the term K_5, using a similar method as for K_2, we have

$$K_5 \leqslant \|q_3\|_{L_t^2 L_x^{r_1}} \big\| |u_3|^{\frac{\alpha-2}{2}} \big\|_{L_t^\infty L_x^{\frac{2\alpha}{\alpha-2}}} \big\| M(|u_3|^{\frac{\alpha}{2}}) \big\|_{L_t^2 L_x^b},$$

where r_1 and b satisfy $\frac{1}{r_1} + \frac{\alpha-2}{2\alpha} + \frac{1}{b} = 1$. According to (3-29) and (3-30), we also have

$$K_5 \leqslant C b^{\frac{1}{2}} \epsilon^{-\frac{b-2}{2b}} \|q_3\|_{L_t^2 L_x^{r_1}} K(t_1)^{\alpha-1}. \tag{3-35}$$

It remains to estimate $\|q_3\|_{L_t^2 L_x^{r_1}}$:

$$\|q_3\|_{L_t^2 L_x^{r_1}} \leqslant C \sum_{i=1}^{3} \|\partial_3 u_i\|_{L_t^2 L_x^{r_3}} \|u_3\|_{L_t^\infty L_x^\alpha} \leqslant C \sum_{i=1}^{3} \|\partial_3 u_i\|_{L_t^2 L_x^{r_3}} K(t_1),$$

where $\frac{1}{r_3} + \frac{1}{\alpha} = \frac{1}{r_1}$. Since r_3 satisfies $\frac{1}{r_3} + \frac{1}{b} = \frac{1}{2}$, we get that $2 < r_3 \leqslant 3$ when $3 \leqslant \alpha \leqslant \widetilde{C} |\ln \epsilon|$ and $b \geqslant 2\alpha$. Thus we can see from (3-3) that

$$\|\partial_3 u_i\|_{L_t^2 L_x^{r_3}} \leqslant C \epsilon^{\frac{6-r_3}{2r_3}} J(t_1).$$

Hence $\|q_3\|_{L_t^2 L_x^{r_1}} \leqslant C \epsilon^{\frac{6-r_3}{2r_3}} J(t_1) K(t_1)$. Then (3-35) yields that

$$K_5 \leqslant C b^{\frac{1}{2}} \epsilon^{\frac{1}{2} - \frac{2}{b}} J(t_1) K(t_1)^\alpha. \tag{3-36}$$

Finally, by summarizing (3-21), (3-26), (3-31), (3-33), (3-34) and (3-36), we have

$$K(t)^\alpha \leqslant C\alpha^2 b\epsilon^{\frac{3}{\alpha}-\frac{2}{b}} J(t)^2 K(t)^{\alpha-1} + C\alpha b^{\frac{1}{2}}\epsilon^{\frac{1}{2}-\frac{2}{b}} J(t)K(t)^\alpha + K(0)^\alpha$$

for all $t \in (0, T_{\max})$. Letting $b = 2\alpha + |\ln\epsilon|$ and C_1 be a sufficiently large constant, we get the following estimate:

$$K(t)^\alpha \leqslant C_1\alpha^2(\alpha+|\ln\epsilon|)\epsilon^{\frac{3}{\alpha}} J(t)^2 K(t)^{\alpha-1} + C_1\alpha(\alpha+|\ln\epsilon|)^{\frac{1}{2}}\epsilon^{\frac{1}{2}} J(t)K(t)^\alpha + K(0)^\alpha.$$

Meanwhile, by (3-20), we have

$$J(t)^2 \leqslant C_1\epsilon^{\frac{1}{2}} J(t)^3 + C_1\epsilon^{\frac{\alpha-3}{\alpha}} K(t)J(t)^2 + J(0)^2. \tag{3-37}$$

Assume that the initial data u_0 satisfies

$$J(0) = \|\nabla u_{0h}\|_{L^2(T_\epsilon)} \leqslant \frac{1}{C\epsilon^{\frac{1}{2}}\alpha(\alpha+|\ln\epsilon|)^{\frac{1}{2}}},$$

$$K(0) = \|u_{03}\|_{L^\alpha(T_\epsilon)} \leqslant \frac{1}{C\epsilon^{\frac{\alpha-3}{\alpha}}}.$$

We claim that

$$J(t) \leqslant \frac{2}{C\epsilon^{\frac{1}{2}}\alpha(\alpha+|\ln\epsilon|)^{\frac{1}{2}}}, \tag{3-38}$$

$$K(t) \leqslant \frac{2}{C\epsilon^{\frac{\alpha-3}{\alpha}}}, \tag{3-39}$$

for all $t \in (0, T_{\max})$ provided C is sufficiently large. This fact implies that $T_{\max} = \infty$. Our claim can be established by contradiction. Suppose that the claim is not true, then there exists a time $t^* \in (0, T_{\max})$ such that (3-38) and (3-39) hold for all $t \in [0, t^*]$ and

$$J(t^*) = \frac{2}{C\epsilon^{\frac{1}{2}}\alpha(\alpha+|\ln\epsilon|)^{\frac{1}{2}}}, \tag{3-40}$$

or

$$K(t^*) = \frac{2}{C\epsilon^{\frac{\alpha-3}{\alpha}}}. \tag{3-41}$$

Using (3-37) with $t = t^*$, we get

$$J(t^*)^2 \leqslant J(t^*)^2\left(\frac{2C_1\epsilon^{\frac{1}{2}}}{C\epsilon^{\frac{1}{2}}\alpha(\alpha+|\ln\epsilon|)^{\frac{1}{2}}} + C_1\epsilon^{\frac{\alpha-3}{\alpha}}\frac{2}{C\epsilon^{\frac{\alpha-3}{\alpha}}}\right) + \frac{1}{C^2\epsilon\alpha^2(\alpha+|\ln\epsilon|)}.$$

Choose C be large enough such that

$$\frac{2C_1\epsilon^{\frac{1}{2}}}{C\epsilon^{\frac{1}{2}}\alpha(\alpha+|\ln\epsilon|)^{\frac{1}{2}}} + C_1\epsilon^{\frac{\alpha-3}{\alpha}}\frac{2}{C\epsilon^{\frac{\alpha-3}{\alpha}}} < \frac{3}{4}.$$

Then we get

$$J(t^*)^2 < \frac{4}{C^2\epsilon\alpha^2(\alpha+|\ln\epsilon|)},$$

which contradicts (3-40). Similarly we can also prove

$$K(t^*) < \frac{2}{C\epsilon^{\frac{\alpha-3}{\alpha}}},$$

which contradicts (3-41) provided C is sufficiently large. Therefore we establish our claim and finish the proof of Theorem 1.2. □

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Our proof will be divided into three steps.

First, we consider the solution on a very small time interval $[0, t_0]$. We will prove that $\|\partial_3 u\|_{L^2(T_\epsilon)}$ decay very fast and $\|\partial_h u\|_{L^2(T_\epsilon)}$ should not increase quickly after a very short time. Furthermore, at the time t_0, we have

$$\begin{cases} \|\partial_h u(t_0)\|_{L^2(T_\epsilon)} \leqslant \dfrac{2C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}, \\ \|\partial_3 u(t_0)\|_{L^2(T_\epsilon)} \leqslant \dfrac{2C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}. \end{cases}$$

This implies that

$$\|\nabla u(t_0)\|_{L^2(T_\epsilon)} \leqslant \frac{4C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}.$$

Second, we regard t_0 as the initial time and $\|\nabla u(t_0)\|_{L^2(T_\epsilon)}$ as the initial data. Consider the solution on a small time interval $[t_0, t_1]$. We will prove that at the time t_1, the data will satisfy the condition of Theorem 1.2.

Finally, we regard t_1 as the initial time and apply Theorem 1.2 directly to get a solution on the time interval $[t_1, \infty)$.

After the above three steps, we will obtain a solution on $[0, \infty)$. Now let us expatiate the details of the proof.

Proof.

Step 1: Solution on $[0, t_0]$.

Our first goal is to estimate $\|\partial_3 u\|_{L^2(T_\epsilon)}$. Applying ∂_3 to (1-1), we obtain a new equation

$$\partial_t\partial_3 u - \Delta\partial_3 u + \partial_3(u\cdot\nabla u) + \nabla\partial_3 p = 0. \tag{4-1}$$

Take the L^2 inner product with $\partial_3 u$ in (4-1) to get

$$\text{(4-2)}\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 u\|_{L^2(T_\epsilon)}^2 + \|\nabla\partial_3 u\|_{L^2(T_\epsilon)}^2 = -\int_{T_\epsilon} \partial_3 u\cdot\nabla u\partial_3 u\,\mathrm{d}x = -\int_{T_\epsilon} \partial_3 u\cdot\nabla Mu\partial_3 u\,\mathrm{d}x - \int_{T_\epsilon} \partial_3 u\cdot\nabla Nu\partial_3 u\,\mathrm{d}x = I_1 + I_2.$$

For the term I_1, we note that Mu is independent of x_3, thus we rewrite it as

$$\text{(4-3)}\quad I_1 = -\int_{T_\epsilon} \partial_3 u_h\cdot\partial_h Mu\partial_3 u\,\mathrm{d}x = -\int_{T^2}\int_0^\epsilon \partial_3 u_h\cdot\partial_h Mu\partial_3 u\,\mathrm{d}x_3\,\mathrm{d}x_h,$$

where $\partial_3 u_h = (\partial_3 u_1, \partial_3 u_2)$, $\partial_h = (\partial_1, \partial_2)$ and $\mathrm{d}x_h = \mathrm{d}x_1\,\mathrm{d}x_2$. Using Hölder's inequality to the vertical direction, we get that

$$\text{(4-4)}\quad \int_0^\epsilon \partial_3 u_h\cdot\partial_h Mu\partial_3 u\,\mathrm{d}x_3 \leqslant \|\partial_3 u\|_{L^2_{x_3}}\|\partial_h Mu\|_{L^\infty_{x_3}}\|\partial_3 u\|_{L^2_{x_3}},$$

where $\|\partial_h Mu\|_{L^\infty_{x_3}} = |\partial_h Mu|$ since $\partial_h Mu$ is independent of x_3. Then by applying Hölder's inequality to the horizontal direction, we see from (4-3) and (4-4) that

$$I_1 \leqslant \int_{T^2} \|\partial_3 u\|_{L^2_{x_3}}\|\partial_h Mu\|_{L^\infty_{x_3}}\|\partial_3 u\|_{L^2_{x_3}}\,\mathrm{d}x_h \leqslant \big\|\|\partial_3 u\|_{L^2_{x_3}}\big\|_{L^4_{x_h}}^2\big\||\partial_h Mu|\big\|_{L^2_{x_h}} \leqslant \epsilon^{-\frac{1}{2}}\big\|\|\partial_3 u\|_{L^4_{x_h}}\big\|_{L^2_{x_3}}^2\|\partial_h Mu\|_{L^2(T_\epsilon)}.$$

Interpolating through the horizontal direction, we have

$$\text{(4-5)}\quad \|\partial_3 u\|_{L^4_{x_h}} \leqslant C\|\partial_3 u\|_{L^2_{x_h}}^{\frac{1}{2}}\|\partial_h\partial_3 u\|_{L^2_{x_h}}^{\frac{1}{2}} + C\|\partial_3 u\|_{L^2_{x_h}}.$$

It then follows that

$$I_1 \leqslant C\epsilon^{-\frac{1}{2}}\big(\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_h\partial_3 u\|_{L^2(T_\epsilon)} + \|\partial_3 u\|_{L^2(T_\epsilon)}^2\big)\|\partial_h Mu\|_{L^2(T_\epsilon)}.$$

Since $\int_0^\epsilon \partial_3 u\,\mathrm{d}x_3 = 0$, by Lemma 2.1(i) we have that

$$\text{(4-6)}\quad \|\partial_3 u\|_{L^2(T_\epsilon)} \leqslant C\epsilon\|\partial_{33} u\|_{L^2(T_\epsilon)}.$$

Hence

$$\text{(4-7)}\quad I_1 \leqslant C\epsilon^{\frac{1}{2}}(\|\partial_{33} u\|_{L^2(T_\epsilon)}\|\partial_h\partial_3 u\|_{L^2(T_\epsilon)} + \|\partial_{33} u\|_{L^2(T_\epsilon)}^2)\|\partial_h u\|_{L^2(T_\epsilon)} \leqslant C\epsilon^{\frac{1}{2}}\|\partial_h u\|_{L^2(T_\epsilon)}\|\nabla\partial_3 u\|_{L^2(T_\epsilon)}^2.$$

For the term I_2, by using Hölder's inequality to the vertical direction, we have

$$\text{(4-8)}\quad I_2 = -\int_{T^2}\int_0^\epsilon \partial_3 u\cdot\nabla Nu\partial_3 u\,\mathrm{d}x_3\,\mathrm{d}x_h \leqslant \int_{T^2} \|\partial_3 u\|_{L^2_{x_3}}\|\nabla Nu\|_{L^\infty_{x_3}}\|\partial_3 u\|_{L^2_{x_3}}\,\mathrm{d}x_h.$$

Regarding $\|\nabla Nu\|_{L^\infty_{x_3}}$, by interpolating through the vertical direction, we have

$$\|\nabla Nu\|_{L^\infty_{x_3}} \leqslant C\|\nabla Nu\|^{\frac{1}{2}}_{L^2_{x_3}}\|\nabla\partial_3 Nu\|^{\frac{1}{2}}_{L^2_{x_3}}. \tag{4-9}$$

Then applying Hölder's inequality to the horizontal direction, we see from (4-8) and (4-9) that

$$\begin{aligned} I_2 &\leqslant C\big\|\|\partial_3 u\|_{L^2_{x_3}}\big\|^2_{L^4_{x_h}}\big\|\|\nabla Nu\|^{\frac{1}{2}}_{L^2_{x_3}}\big\|_{L^4_{x_h}}\big\|\|\nabla\partial_3 Nu\|^{\frac{1}{2}}_{L^2_{x_3}}\big\|_{L^4_{x_h}} \\ &\leqslant C\big\|\|\partial_3 u\|_{L^4_{x_h}}\big\|^2_{L^2_{x_3}}\|\nabla Nu\|^{\frac{1}{2}}_{L^2(T_\epsilon)}\|\nabla\partial_3 Nu\|^{\frac{1}{2}}_{L^2(T_\epsilon)}. \end{aligned}$$

To deal with $\|\partial_3 u\|_{L^4_{x_h}}$, we use the same method as (4-5). Therefore

$$\begin{aligned} I_2 &\leqslant C\big(\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_h\partial_3 u\|_{L^2(T_\epsilon)}+\|\partial_3 u\|^2_{L^2(T_\epsilon)}\big)\|\nabla Nu\|^{\frac{1}{2}}_{L^2(T_\epsilon)}\|\nabla\partial_3 Nu\|^{\frac{1}{2}}_{L^2(T_\epsilon)} \\ &\leqslant C\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_h\partial_3 u\|_{L^2(T_\epsilon)}\|\nabla Nu\|^{\frac{1}{2}}_{L^2(T_\epsilon)}\|\nabla\partial_3 u\|^{\frac{1}{2}}_{L^2(T_\epsilon)} \\ &\qquad +C\|\partial_3 u\|^2_{L^2(T_\epsilon)}\|\nabla Nu\|^{\frac{1}{2}}_{L^2(T_\epsilon)}\|\nabla\partial_3 u\|^{\frac{1}{2}}_{L^2(T_\epsilon)} \\ &= I_{21}+I_{22}. \end{aligned}$$

Regarding I_{21}, we have

$$\begin{aligned} I_{21} &\leqslant C\|\partial_3 u\|_{L^2(T_\epsilon)}\|\nabla Nu\|^{\frac{1}{2}}_{L^2(T_\epsilon)}\|\nabla\partial_3 u\|^{\frac{3}{2}}_{L^2(T_\epsilon)} \\ &\leqslant C\epsilon^{\frac{1}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\nabla\partial_3 Nu\|^{\frac{1}{2}}_{L^2(T_\epsilon)}\|\nabla\partial_3 u\|^{\frac{3}{2}}_{L^2(T_\epsilon)} \\ &\leqslant C\epsilon^{\frac{1}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\nabla\partial_3 u\|^2_{L^2(T_\epsilon)}, \end{aligned}$$

where we have used

$$\|\nabla Nu\|_{L^2(T_\epsilon)} \leqslant C\epsilon\|\nabla\partial_3 u\|_{L^2(T_\epsilon)} \tag{4-10}$$

because of $\int_0^\epsilon \nabla Nu\,\mathrm{d}x_3 = 0$. Regarding I_{22}, by (4-6) and (4-10), we have

$$\begin{aligned} I_{22} &\leqslant C\epsilon^{\frac{3}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_{33} u\|_{L^2(T_\epsilon)}\|\nabla\partial_3 Nu\|^{\frac{1}{2}}_{L^2(T_\epsilon)}\|\nabla\partial_3 u\|^{\frac{1}{2}}_{L^2(T_\epsilon)} \\ &\leqslant C\epsilon^{\frac{3}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\nabla\partial_3 u\|^2_{L^2(T_\epsilon)}. \end{aligned}$$

Therefore, we get the following estimate of I_2:

$$I_2 \leqslant C\epsilon^{\frac{1}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\nabla\partial_3 u\|^2_{L^2(T_\epsilon)}. \tag{4-11}$$

Summarizing (4-2), (4-7) and (4-11), we obtain the estimate of $\|\partial_3 u\|_{L^2(T_\epsilon)}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 u\|^2_{L^2(T_\epsilon)} + (2 - C\epsilon^{\frac{1}{2}}\|\partial_h u\|_{L^2(T_\epsilon)} - C\epsilon^{\frac{1}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)})\|\nabla\partial_3 u\|^2_{L^2(T_\epsilon)} \leqslant 0.$$

If

$$C\epsilon^{\frac{1}{2}}\|\partial_h u\|_{L^2(T_\epsilon)} + C\epsilon^{\frac{1}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)} < 1 \tag{4-12}$$

for all $t \in [0, t_0]$, where t_0 is given by (4-18), then we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 u\|^2_{L^2(T_\epsilon)} + \|\nabla\partial_3 u\|^2_{L^2(T_\epsilon)} \leqslant 0. \tag{4-13}$$

Integrating from 0 to t, we have

$$\|\partial_3 u\|_{L^\infty_t L^2_x} \leqslant \|\partial_3 u_0\|_{L^2(T_\epsilon)}, \tag{4-14}$$

and

$$\|\nabla\partial_3 u\|_{L^2_t L^2_x} \leqslant \|\partial_3 u_0\|_{L^2(T_\epsilon)}. \tag{4-15}$$

In addition, from (4-6), we get

$$\|\nabla\partial_3 u\|^2_{L^2(T_\epsilon)} \geqslant C^{-1}\epsilon^{-2}\|\partial_3 u\|^2_{L^2(T_\epsilon)}.$$

Hence (4-13) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_3 u\|^2_{L^2(T_\epsilon)} + C^{-1}\epsilon^{-2}\|\partial_3 u\|^2_{L^2(T_\epsilon)} \leqslant 0.$$

This implies that

$$\|\partial_3 u\|^2_{L^2(T_\epsilon)} \leqslant \mathrm{e}^{-C^{-1}\epsilon^{-2}t}\|\partial_3 u_0\|^2_{L^2(T_\epsilon)}. \tag{4-16}$$

Assume that the initial data satisfies

$$\|\partial_3 u_0\|_{L^2(T_\epsilon)} \leqslant \frac{1}{C_0\epsilon^{\frac{1}{2}}}. \tag{4-17}$$

Let

$$t_0 = 3C_2\epsilon^2 \ln|\ln\epsilon|, \tag{4-18}$$

where C_2 is the constant C on the right-hand side of (4-16). Then when $t \geqslant t_0$, we have

$$\|\partial_3 u\|_{L^2(T_\epsilon)} \leqslant \frac{2C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}. \tag{4-19}$$

Next, we want to estimate $\|\partial_h u\|_{L^2(T_\epsilon)}$. Similarly, applying ∂_h to (1-1), we get

$$\partial_t\partial_h u - \Delta\partial_h u + \partial_h(u\cdot\nabla u) + \nabla\partial_h p = 0. \tag{4-20}$$

Taking the L^2 inner product with $\partial_h u$ in (4-20), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_h u\|^2_{L^2(T_\epsilon)} + \|\nabla\partial_h u\|^2_{L^2(T_\epsilon)} = -\int_{T_\epsilon} \partial_h u\cdot\nabla u\,\partial_h u\,\mathrm{d}x = -\int_{T_\epsilon} \partial_h u_3\,\partial_3 u\,\partial_h u\,\mathrm{d}x - \int_{T_\epsilon} \partial_h u_{\bar h}\cdot\partial_{\bar h} u\,\partial_h u\,\mathrm{d}x = I_3 + I_4, \tag{4-21}$$

where $u_{\bar h} = (u_1, u_2)$ and $\partial_{\bar h} = (\partial_1, \partial_2)$. For the term I_3, we rewrite it as

$$I_3 = -\int_{T_\epsilon} \partial_h M u_3\,\partial_3 u\,\partial_h M u\,\mathrm{d}x - \int_{T_\epsilon} \partial_h M u_3\,\partial_3 u\,\partial_h N u\,\mathrm{d}x - \int_{T_\epsilon} \partial_h N u_3\,\partial_3 u\,\partial_h M u\,\mathrm{d}x - \int_{T_\epsilon} \partial_h N u_3\,\partial_3 u\,\partial_h N u\,\mathrm{d}x = I_{31} + I_{32} + I_{33} + I_{34}.$$

Regarding I_{31}, we have

$$I_{31} \leqslant \|\partial_h M u_3\|_{L^4(T_\epsilon)}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_h M u\|_{L^4(T_\epsilon)} \leqslant \|\partial_h M u\|^2_{L^4(T_\epsilon)}\|\partial_3 u\|_{L^2(T_\epsilon)}.$$

Since u satisfies the periodic boundary condition, we know that

$$\int_{T_\epsilon} \partial_h u\,\mathrm{d}x = 0.$$

Hence by Lemma 2.1(ii) with $a = 4$, we have

$$\|\partial_h M u\|_{L^4(T_\epsilon)} \leqslant C\epsilon^{-\frac{1}{4}}\|\partial_h u\|^{\frac{1}{2}}_{L^2(T_\epsilon)}\|\nabla\partial_h u\|^{\frac{1}{2}}_{L^2(T_\epsilon)}.$$

Therefore

$$\begin{aligned} I_{31} &\leqslant C\epsilon^{-\frac{1}{2}}\|\partial_h u\|_{L^2(T_\epsilon)}\|\nabla\partial_h u\|_{L^2(T_\epsilon)}\|\partial_3 u\|_{L^2(T_\epsilon)} \\ &\leqslant C\epsilon^{-1}\|\partial_h u\|^2_{L^2(T_\epsilon)}\|\partial_3 u\|^2_{L^2(T_\epsilon)} + \frac{1}{8}\|\nabla\partial_h u\|^2_{L^2(T_\epsilon)} \\ &\leqslant C\epsilon\|\partial_h u\|^2_{L^2(T_\epsilon)}\|\partial_{33} u\|^2_{L^2(T_\epsilon)} + \frac{1}{8}\|\nabla\partial_h u\|^2_{L^2(T_\epsilon)}, \end{aligned} \tag{4-22}$$

where we have used (4-6). Regarding I_{32}, we have

$$I_{32} \leqslant \|\partial_h M u\|_{L^3(T_\epsilon)}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_h N u\|_{L^6(T_\epsilon)}.$$

Hence by Lemma 2.1(ii) with $a = 3$ and Lemma 2.1(i) with $a = 6$, we have

$$\|\partial_h M u\|_{L^3(T_\epsilon)} \leqslant C\epsilon^{-\frac{1}{6}}\|\partial_h u\|^{\frac{2}{3}}_{L^2(T_\epsilon)}\|\nabla\partial_h u\|^{\frac{1}{3}}_{L^2(T_\epsilon)},$$

and

$$\|\partial_h N u\|_{L^6(T_\epsilon)} \leqslant C\|\nabla\partial_h u\|_{L^2(T_\epsilon)}.$$

Thus

$$I_{32} \leqslant C\epsilon^{-\frac{1}{6}} \|\partial_h u\|^{\frac{2}{3}}_{L^2(T_\epsilon)} \|\partial_3 u\|_{L^2(T_\epsilon)} \|\nabla \partial_h u\|^{\frac{4}{3}}_{L^2(T_\epsilon)}$$
$$\leqslant C\epsilon^{-\frac{1}{2}} \|\partial_h u\|^2_{L^2(T_\epsilon)} \|\partial_3 u\|^3_{L^2(T_\epsilon)} + \frac{1}{8} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)}$$
$$\leqslant C\epsilon^{\frac{3}{2}} \|\partial_3 u\|_{L^2(T_\epsilon)} \|\partial_{33} u\|^2_{L^2(T_\epsilon)} \|\partial_h u\|^2_{L^2(T_\epsilon)} + \frac{1}{8} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)}. \tag{4-23}$$

The estimate of I_{33} is as same as I_{32}, i.e.,

$$I_{33} \leqslant C\epsilon^{\frac{3}{2}} \|\partial_3 u\|_{L^2(T_\epsilon)} \|\partial_{33} u\|^2_{L^2(T_\epsilon)} \|\partial_h u\|^2_{L^2(T_\epsilon)} + \frac{1}{8} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)}. \tag{4-24}$$

Regarding I_{34}, we have

$$I_{34} \leqslant \|\partial_h N u\|_{L^4(T_\epsilon)} \|\partial_3 u\|_{L^2(T_\epsilon)} \|\partial_h N u\|_{L^4(T_\epsilon)}.$$

By using Lemma 2.1(i) with $a = 4$, we have

$$\|\partial_h N u\|_{L^4(T_\epsilon)} \leqslant C\epsilon^{\frac{1}{4}} \|\nabla \partial_h u\|_{L^2(T_\epsilon)}.$$

Thus, by (4-14), we obtain that

$$I_{34} \leqslant C\epsilon^{\frac{1}{2}} \|\partial_3 u\|_{L^2(T_\epsilon)} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)} \leqslant C\epsilon^{\frac{1}{2}} \|\partial_3 u_0\|_{L^2(T_\epsilon)} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)}. \tag{4-25}$$

Consequently, summarizing (4-22)–(4-25), we get the estimate of I_3,

$$I_3 \leqslant C(\epsilon \|\partial_{33} u\|^2_{L^2(T_\epsilon)} + \epsilon^{\frac{3}{2}} \|\partial_3 u\|_{L^2(T_\epsilon)} \|\partial_{33} u\|^2_{L^2(T_\epsilon)}) \|\partial_h u\|^2_{L^2(T_\epsilon)}$$
$$+ C\epsilon^{\frac{1}{2}} \|\partial_3 u_0\|_{L^2(T_\epsilon)} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)} + \frac{3}{8} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)}. \tag{4-26}$$

For the term I_4, we rewrite it as

$$I_4 = -\int_{T_\epsilon} \partial_h u_{\bar{h}} \cdot \partial_{\bar{h}} u \partial_h u \, \mathrm{d}x$$
$$= -\int_{T_\epsilon} \partial_h u_{\bar{h}} \cdot \partial_{\bar{h}} M u \partial_h u \, \mathrm{d}x - \int_{T_\epsilon} \partial_h u_{\bar{h}} \cdot \partial_{\bar{h}} N u \partial_h u \, \mathrm{d}x = I_{41} + I_{42}.$$

Regarding I_{41}, by using Hölder's inequality to the vertical direction and the horizontal direction respectively, we get that

$$I_{41} = -\int_{T^2} \int_0^\epsilon \partial_h u_{\bar{h}} \cdot \partial_{\bar{h}} M u \partial_h u \, \mathrm{d}x_3 \, \mathrm{d}x_h$$
$$\leqslant \int_{T^2} \|\partial_h u\|_{L^2_{x_3}} \|\partial_h M u\|_{L^\infty_{x_3}} \|\partial_h u\|_{L^2_{x_3}} \, \mathrm{d}x_h$$
$$\leqslant \big\| \|\partial_h u\|_{L^2_{x_3}} \big\|^2_{L^4_{x_h}} \|\partial_h M u\|_{L^2_{x_h}} \leqslant \epsilon^{-\frac{1}{2}} \big\| \|\partial_h u\|_{L^4_{x_h}} \big\|^2_{L^2_{x_3}} \|\partial_h M u\|_{L^2(T_\epsilon)}.$$

Interpolating through the horizontal direction together with $\int_{T^2} \partial_h u \, \mathrm{d}x_h = 0$, we have

$$\|\partial_h u\|_{L^4_{x_h}} \leqslant C \|\partial_h u\|^{\frac{1}{2}}_{L^2_{x_h}} \|\partial_h \partial_h u\|^{\frac{1}{2}}_{L^2_{x_h}}. \tag{4-27}$$

Thus

$$\begin{aligned} I_{41} &\leqslant C\epsilon^{-\frac{1}{2}} (\|\partial_h u\|_{L^2(T_\epsilon)} \|\partial_h \partial_h u\|_{L^2(T_\epsilon)}) \|\partial_h u\|_{L^2(T_\epsilon)} \\ &\leqslant C\epsilon^{-1} \|\partial_h u\|^4_{L^2(T_\epsilon)} + \frac{1}{8} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)}. \end{aligned} \tag{4-28}$$

Regarding I_{42}, by using Hölder's inequality to the vertical direction, we have

$$\begin{aligned} I_{42} &= -\int_{T^2} \int_0^\epsilon \partial_h u_{\bar{h}} \cdot \partial_{\bar{h}} N u \partial_h u \, \mathrm{d}x_3 \, \mathrm{d}x_h \\ &\leqslant \int_{T^2} \|\partial_h u\|_{L^2_{x_3}} \|\partial_h N u\|_{L^\infty_{x_3}} \|\partial_h u\|_{L^2_{x_3}} \, \mathrm{d}x_h. \end{aligned}$$

Interpolating through the vertical direction, we have

$$\|\partial_h N u\|_{L^\infty_{x_3}} \leqslant C \|\partial_h N u\|^{\frac{1}{2}}_{L^2_{x_3}} \|\partial_3 \partial_h N u\|^{\frac{1}{2}}_{L^2_{x_3}}.$$

Then by using Hölder's inequality to the horizontal direction, we get

$$\begin{aligned} I_{42} &\leqslant C \big\| \|\partial_h u\|_{L^2_{x_3}} \big\|^2_{L^4_{x_h}} \big\| \|\partial_h N u\|^{\frac{1}{2}}_{L^2_{x_3}} \big\|_{L^4_{x_h}} \big\| \|\partial_h \partial_3 N u\|^{\frac{1}{2}}_{L^2_{x_3}} \big\|_{L^4_{x_h}} \\ &\leqslant C \big\| \|\partial_h u\|_{L^4_{x_h}} \big\|^2_{L^2_{x_3}} \|\partial_h N u\|^{\frac{1}{2}}_{L^2(T_\epsilon)} \|\partial_h \partial_3 N u\|^{\frac{1}{2}}_{L^2(T_\epsilon)}. \end{aligned}$$

By (4-27) and Lemma 2.1(i) with $a = 2$, we have

$$\begin{aligned} I_{42} &\leqslant C \|\partial_h u\|_{L^2(T_\epsilon)} \|\partial_h \partial_h u\|_{L^2(T_\epsilon)} \|\partial_h N u\|^{\frac{1}{2}}_{L^2(T_\epsilon)} \|\partial_h \partial_3 N u\|^{\frac{1}{2}}_{L^2(T_\epsilon)} \\ &\leqslant C\epsilon^{\frac{1}{2}} \|\partial_h u\|_{L^2(T_\epsilon)} \|\partial_h \partial_h u\|_{L^2(T_\epsilon)} \|\partial_h \partial_3 N u\|_{L^2(T_\epsilon)} \\ &\leqslant C\epsilon^{\frac{1}{2}} \|\partial_h u\|_{L^2(T_\epsilon)} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)}. \end{aligned} \tag{4-29}$$

Consequently, summarizing (4-28) and (4-29), we get the estimate of I_4,

$$\begin{aligned} I_4 \leqslant C\epsilon^{-1} \|\partial_h u\|^2_{L^2(T_\epsilon)} \|\partial_h u\|^2_{L^2(T_\epsilon)} + C\epsilon^{\frac{1}{2}} \|\partial_h u\|_{L^2(T_\epsilon)} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)} \\ + \frac{1}{8} \|\nabla \partial_h u\|^2_{L^2(T_\epsilon)}. \end{aligned} \tag{4-30}$$

Finally, combining (4-21), (4-26) and (4-30), we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_h u\|^2_{L^2(T_\epsilon)} + \Big(\frac{3}{2} - C\epsilon^{\frac{1}{2}}\|\partial_3 u_0\|_{L^2(T_\epsilon)} - C\epsilon^{\frac{1}{2}}\|\partial_h u\|_{L^2(T_\epsilon)}\Big)\|\nabla\partial_h u\|^2_{L^2(T_\epsilon)} \leqslant C(\epsilon\|\partial_{33}u\|^2_{L^2(T_\epsilon)} + \epsilon^{\frac{3}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_{33}u\|^2_{L^2(T_\epsilon)} + \epsilon^{-1}\|\partial_h u\|^2_{L^2(T_\epsilon)})\|\partial_h u\|^2_{L^2(T_\epsilon)}.$$

Assuming that the initial data satisfies

$$\|\partial_3 u_0\|_{L^2(T_\epsilon)} \leqslant \frac{1}{C_0\epsilon^{\frac{1}{2}}}, \tag{4-31}$$

we have

$$C\epsilon^{\frac{1}{2}}\|\partial_3 u_0\|_{L^2(T_\epsilon)} \leqslant \frac{C}{C_0} < \frac{1}{4}$$

provided C_0 is sufficiently large. If

$$C\epsilon^{\frac{1}{2}}\|\partial_h u\|_{L^2(T_\epsilon)} < \frac{1}{4} \tag{4-32}$$

for all $t \in [0, t_0]$, where t_0 is given by (4-18), then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_h u\|^2_{L^2(T_\epsilon)} + \|\nabla\partial_h u\|^2_{L^2(T_\epsilon)} \leqslant C(\epsilon\|\partial_{33}u\|^2_{L^2(T_\epsilon)} + \epsilon^{\frac{3}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_{33}u\|^2_{L^2(T_\epsilon)} + \epsilon^{-1}\|\partial_h u\|^2_{L^2(T_\epsilon)})\|\partial_h u\|^2_{L^2(T_\epsilon)}.$$

Using Gronwall's inequality, we get that

$$\|\partial_h u\|^2_{L^2(T_\epsilon)} + \int_0^t \|\nabla\partial_h u\|^2_{L^2(T_\epsilon)}\,\mathrm{d}s \leqslant \mathrm{e}^{G(t)}\|\partial_h u_0\|^2_{L^2(T_\epsilon)},$$

where

$$G(t) = \int_0^t g(s)\,\mathrm{d}s$$

and

$$g(t) = C(\epsilon\|\partial_{33}u\|^2_{L^2(T_\epsilon)} + \epsilon^{\frac{3}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_{33}u\|^2_{L^2(T_\epsilon)} + \epsilon^{-1}\|\partial_h u\|^2_{L^2(T_\epsilon)}).$$

Our next goal is to show $G(t)$ can be very small when $t \in [0, t_0]$, where t_0 is given by (4-18). Then we will obtain

$$\|\partial_h u(t)\|_{L^2(T_\epsilon)} \leqslant 2\|\partial_h u_0\|_{L^2(T_\epsilon)}. \tag{4-33}$$

We write $G(t)$ as $G_1(t)+G_2(t)+G_3(t)$, where

$$G_1(t)=\int_0^t C\epsilon\|\partial_{33}u\|^2_{L^2(T_\epsilon)}\,\mathrm{d}s,$$
$$G_2(t)=\int_0^t C\epsilon^{\frac{3}{2}}\|\partial_3 u\|_{L^2(T_\epsilon)}\|\partial_{33}u\|^2_{L^2(T_\epsilon)}\,\mathrm{d}s,$$
$$G_3(t)=\int_0^t C\epsilon^{-1}\|\partial_h u\|^2_{L^2(T_\epsilon)}\,\mathrm{d}s.$$

For the term $G_1(t)$, we conclude from (4-15) and (4-31) that

$$G_1(t)\leqslant C\epsilon\|\partial_3 u_0\|^2_{L^2(T_\epsilon)}\leqslant \frac{C}{C_0^2}.$$

For the term $G_2(t)$, by (4-14), (4-15) and (4-31), we have

$$G_2(t)\leqslant C\epsilon^{\frac{3}{2}}\|\partial_3 u\|_{L_t^\infty L_x^2}\|\partial_{33}u\|^2_{L_t^2L_x^2}\leqslant C\epsilon^{\frac{3}{2}}\|\partial_3 u_0\|^3_{L^2(T_\epsilon)}\leqslant \frac{C}{C_0^3}.$$

For the term $G_3(t)$, by (4-33), we have

$$G_3(t)\leqslant C\epsilon^{-1}\|\partial_h u\|^2_{L_t^\infty L_x^2}t_0\leqslant C\epsilon^{-1}\|\partial_h u_0\|^2_{L^2(T_\epsilon)}t_0.$$

Assume that the initial data satisfies

$$\|\partial_h u_0\|_{L^2(T_\epsilon)}\leqslant \frac{C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}. \tag{4-34}$$

Then

$$G_3(t)\leqslant \frac{3C_1C(C^*)^2\ln|\ln\epsilon|}{|\ln\epsilon|^3}\to 0\quad \text{as } \epsilon\to 0.$$

Based on the above analysis, we can take $C_0>1$ and $0<\epsilon_1<1$ such that for every $\epsilon\in(0,\epsilon_1)$, there holds $G(t)<1$ for all $t\in[0,t_0]$. Then (4-33) holds for all $t\in[0,t_0]$. By (4-14), (4-17), (4-33) and (4-34), we can take $\epsilon_2\in(0,1)$ such that for every $\epsilon\in(0,\epsilon_2)$, conditions (4-12) and (4-32) hold for all $t\in[0,t_0]$. Therefore, we completed the a prior estimate. Additionally, by (4-19) and (4-33), we get that at t_0, there hold

$$\|\partial_3 u(t_0)\|_{L^2(T_\epsilon)}\leqslant \frac{2C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}$$

and

$$\|\partial_h u(t_0)\|_{L^2(T_\epsilon)}\leqslant \frac{2C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}.$$

Step 2: Solution on $[t_0, t_1]$.

We consider the solution from t_0. At this time, $\|\nabla u(t_0)\|_{L^2(T_\epsilon)}$ satisfies

$$\|\nabla u(t_0)\|_{L^2(T_\epsilon)} \leqslant \frac{4C^*}{\epsilon^{\frac{1}{2}}|\ln \epsilon|^{\frac{3}{2}}}.$$

In what follows we will estimate $\|\nabla u(t)\|_{L^2(T_\epsilon)}$ for $t \in [t_0, t_0 + T]$, where T will be given by (4-36). We emphasize that in [Kukavica and Ziane 2007], the authors proved the case when C^* is sufficient small. In our case, C^* can be arbitrarily large. Below, we will show that it can be proved by using the same method as [Kukavica and Ziane 2007].

Take the L^2 inner product with $-\Delta u$ in (1-1) to obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{L^2(T_\epsilon)} + \|\Delta u\|^2_{L^2(T_\epsilon)}
&= \int_{T_\epsilon} u\cdot\nabla u \Delta u \,\mathrm{d}x = -\int_{T_\epsilon} \nabla u\cdot\nabla u\nabla u\,\mathrm{d}x \\
&= -\int_{T_\epsilon} \nabla Mu\cdot\nabla Mu\nabla u\,\mathrm{d}x - \int_{T_\epsilon} \nabla Mu\cdot\nabla Nu\nabla u\,\mathrm{d}x \\
&\qquad - \int_{T_\epsilon} \nabla Nu\cdot\nabla Mu\nabla u\,\mathrm{d}x - \int_{T_\epsilon} \nabla Nu\cdot\nabla Nu\nabla u\,\mathrm{d}x \\
&= L_1 + L_2 + L_3 + L_4.
\end{aligned}$$

For L_1, by using Hölder's inequality and Lemma 2.1(ii), we have

$$\begin{aligned}
L_1 &\leqslant \|\nabla Mu\|^2_{L^4(T_\epsilon)}\|\nabla u\|_{L^2(T_\epsilon)} \leqslant C\epsilon^{-\frac{1}{2}}\|\nabla u\|^2_{L^2(T_\epsilon)}\|\Delta u\|_{L^2(T_\epsilon)} \\
&\leqslant C\epsilon^{-1}\|\nabla u\|^4_{L^2(T_\epsilon)} + \frac{1}{4}\|\Delta u\|^2_{L^2(T_\epsilon)}.
\end{aligned}$$

For L_2, we have

$$\begin{aligned}
L_2 &\leqslant \|\nabla Mu\|_{L^3(T_\epsilon)}\|\nabla Nu\|_{L^6(T_\epsilon)}\|\nabla u\|_{L^2(T_\epsilon)} \\
&\leqslant C\epsilon^{-\frac{1}{6}}\|\nabla u\|^{\frac{2}{3}}_{L^2(T_\epsilon)}\|\Delta u\|^{\frac{1}{3}}_{L^2(T_\epsilon)}\|\Delta u\|_{L^2(T_\epsilon)}\|\nabla u\|_{L^2(T_\epsilon)} \\
&\leqslant C\epsilon^{-\frac{1}{6}}\|\nabla u\|^{\frac{5}{3}}_{L^2(T_\epsilon)}\|\Delta u\|^{\frac{4}{3}}_{L^2(T_\epsilon)} \leqslant C\epsilon^{-\frac{1}{2}}\|\nabla u\|^5_{L^2(T_\epsilon)} + \frac{1}{4}\|\Delta u\|^2_{L^2(T_\epsilon)}.
\end{aligned}$$

In the same way, we see that

$$L_3 \leqslant C\epsilon^{-\frac{1}{2}}\|\nabla u\|^5_{L^2(T_\epsilon)} + \frac{1}{4}\|\Delta u\|^2_{L^2(T_\epsilon)}.$$

For L_4, by using Hölder's inequality and Lemma 2.1(i), we obtain

$$\begin{aligned}
L_4 &\leqslant \|\nabla Nu\|_{L^3(T_\epsilon)}\|\nabla Nu\|_{L^6(T_\epsilon)}\|\nabla u\|_{L^2(T_\epsilon)}\\
&\leqslant C\|\nabla u\|_{L^2(T_\epsilon)}^{\frac{1}{2}}\|\Delta u\|_{L^2(T_\epsilon)}^{\frac{1}{2}}\|\Delta u\|_{L^2(T_\epsilon)}\|\nabla u\|_{L^2(T_\epsilon)}\\
&\leqslant C\|\nabla u\|_{L^2(T_\epsilon)}^{\frac{3}{2}}\|\Delta u\|_{L^2(T_\epsilon)}^{\frac{3}{2}} \leqslant C\|\nabla u\|_{L^2(T_\epsilon)}^{6} + \frac{1}{4}\|\Delta u\|_{L^2(T_\epsilon)}^{2}.
\end{aligned}$$

As a result, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2(T_\epsilon)}^2 + \|\Delta u\|_{L^2(T_\epsilon)}^2 \leqslant C\epsilon^{-1}\|\nabla u\|_{L^2(T_\epsilon)}^4 + C\|\nabla u\|_{L^2(T_\epsilon)}^6, \tag{4-35}$$

where we have used

$$\begin{aligned}
C\epsilon^{-\frac{1}{2}}\|\nabla u\|_{L^2(T_\epsilon)}^5 &= C\epsilon^{-\frac{1}{2}}\|\nabla u\|_{L^2(T_\epsilon)}^2\|\nabla u\|_{L^2(T_\epsilon)}^3\\
&\leqslant C\epsilon^{-1}\|\nabla u\|_{L^2(T_\epsilon)}^4 + C\|\nabla u\|_{L^2(T_\epsilon)}^6.
\end{aligned}$$

Applying Gronwall's inequality to (4-35), we get

$$\|\nabla u(t)\|_{L^2(T_\epsilon)}^2 + \int_{t_0}^{t}\|\Delta u\|_{L^2(T_\epsilon)}^2\,\mathrm{d}s \leqslant \mathrm{e}^{H(t)}\|\nabla u(t_0)\|_{L^2(T_\epsilon)}^2, \quad t \in (t_0, t_0+T].$$

where

$$H(t) = \int_{t_0}^{t} C_3\epsilon^{-1}\|\nabla u\|_{L^2(T_\epsilon)}^2 + C_3\|\nabla u\|_{L^2(T_\epsilon)}^4\,\mathrm{d}s,$$

C_3 is the constant C on the right-hand side of (4-35) and

$$T = \min\left\{\frac{\epsilon^2|\ln\epsilon|^3}{128C_3(C^*)^2}, \frac{\epsilon^2|\ln\epsilon|^6}{2\times 64^2 C_3(C^*)^4}\right\}.$$

Take $\epsilon_3 \in (0,1)$ such that for every $\epsilon \in (0,\epsilon_3)$, we have

$$64(C^*)^2 \leqslant |\ln\epsilon|^3.$$

Then we get

$$T = \frac{\epsilon^2|\ln\epsilon|^3}{128C_3(C^*)^2}, \tag{4-36}$$

and $H(t) \leqslant 1$ for $t \in (t_0, t_0+T]$. Consequently, there hold

$$\|\nabla u(t)\|_{L^2(T_\epsilon)} \leqslant 2\|\nabla u(t_0)\|_{L^2(T_\epsilon)} \leqslant \frac{8C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}, \quad t \in (t_0, t_0+T],$$

and

$$\int_{t_0}^{t_0+T}\|\Delta u\|_{L^2(T_\epsilon)}^2\,\mathrm{d}s \leqslant 4\|\nabla u(t_0)\|_{L^2(T_\epsilon)}^2 \leqslant \frac{64(C^*)^2}{\epsilon|\ln\epsilon|^3}.$$

Hence, there exists $t_1 \in (t_0, t_0 + T)$ such that

$$\|\Delta u(t_1)\|_{L^2(T_\epsilon)}^2 \leqslant \frac{C(C^*)^4}{\epsilon^3|\ln\epsilon|^6}$$

and

$$\|\nabla u(t_1)\|_{L^2(T_\epsilon)} \leqslant \frac{8C^*}{\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}. \tag{4-37}$$

Now, let us turn to $\|u(t_1)\|_{L^\alpha(T_\epsilon)} \leqslant \|Nu(t_1)\|_{L^\alpha(T_\epsilon)} + \|Mu(t_1)\|_{L^\alpha(T_\epsilon)}$. For $\|Nu(t_1)\|_{L^\alpha(T_\epsilon)}$, we have

$$\begin{aligned}\|Nu(t_1)\|_{L^\alpha(T_\epsilon)} &\leqslant C\|Nu(t_1)\|_{L^2(T_\epsilon)}^{\frac{1}{4}+\frac{3}{2\alpha}}\|\Delta Nu(t_1)\|_{L^2(T_\epsilon)}^{\frac{3}{4}-\frac{3}{2\alpha}}\\ &\leqslant C\epsilon^{\frac{1}{4}+\frac{3}{2\alpha}}\|\nabla u(t_1)\|_{L^2(T_\epsilon)}^{\frac{1}{4}+\frac{3}{2\alpha}}\|\Delta u(t_1)\|_{L^2(T_\epsilon)}^{\frac{3}{4}-\frac{3}{2\alpha}}\\ &\leqslant C(C^*)^{\frac{7}{4}-\frac{3}{2\alpha}}\epsilon^{\frac{3-\alpha}{\alpha}}|\ln\epsilon|^{-\frac{21}{8}+\frac{9}{4\alpha}}\\ &\leqslant C_4(C^*)^{\frac{7}{4}}\epsilon^{\frac{3-\alpha}{\alpha}}|\ln\epsilon|^{-\frac{15}{8}},\end{aligned}$$

since $3 \leqslant \alpha \leqslant \widetilde{C}|\ln\epsilon|$. Take $\epsilon_4 \in (0,1)$ such that for every $\epsilon \in (0, \epsilon_4)$, we have

$$|\ln\epsilon|^{\frac{15}{8}} \geqslant CC_4(C^*)^{\frac{7}{4}}.$$

Then we get

$$\|Nu(t_1)\|_{L^\alpha(T_\epsilon)} \leqslant \frac{1}{C\epsilon^{\frac{\alpha-3}{\alpha}}}. \tag{4-38}$$

For $\|Mu(t_1)\|_{L^\alpha(T_\epsilon)}$, we have

$$\|Mu(t_1)\|_{L^\alpha(T_\epsilon)} \leqslant \frac{C\alpha^{\frac{1}{2}}}{\epsilon^{\frac{\alpha-2}{2\alpha}}}\|\nabla u(t_1)\|_{L^2(T_\epsilon)} \leqslant \frac{C_5C^*\alpha^{\frac{1}{2}}}{\epsilon^{\frac{\alpha-1}{\alpha}}|\ln\epsilon|^{\frac{3}{2}}} = \frac{1}{\epsilon^{\frac{\alpha-3}{\alpha}}}\frac{C_5C^*\alpha^{\frac{1}{2}}}{\epsilon^{\frac{2}{\alpha}}|\ln\epsilon|^{\frac{3}{2}}}.$$

Fix

$$\alpha = 3 + \frac{2|\ln\epsilon|}{\ln|\ln\epsilon|}, \tag{4-39}$$

then we have

$$\frac{C_5C^*\alpha^{\frac{1}{2}}}{\epsilon^{\frac{2}{\alpha}}|\ln\epsilon|^{\frac{3}{2}}} \leqslant \frac{2C_5C^*}{(\ln|\ln\epsilon|)^{\frac{1}{2}}}.$$

Take $\epsilon_5 \in (0,1)$ such that for every $\epsilon \in (0, \epsilon_5)$, we have

$$(\ln|\ln\epsilon|)^{\frac{1}{2}} \geqslant 2CC_5C^*.$$

Therefore we have

$$\|Mu(t_1)\|_{L^\alpha(T_\epsilon)} \leqslant \frac{1}{C\epsilon^{\frac{\alpha-3}{\alpha}}}. \tag{4-40}$$

Moreover, for the fixed α (4-39), we know that

$$\frac{1}{C\epsilon^{\frac{1}{2}}\alpha(\alpha+|\ln\epsilon|)^{\frac{1}{2}}} \geqslant \frac{\ln|\ln\epsilon|}{C\epsilon^{\frac{1}{2}}|\ln\epsilon|^{\frac{3}{2}}}.$$

Take $\epsilon_6 \in (0,1)$ such that for every $\epsilon \in (0,\epsilon_6)$, we have

$$\ln|\ln\epsilon| \geqslant 8CC^*.$$

Then by (4-37), we see that

$$\|\nabla u(t_1)\|_{L^2(T_\epsilon)} \leqslant \frac{1}{C\epsilon^{\frac{1}{2}}\alpha(\alpha+|\ln\epsilon|)^{\frac{1}{2}}}. \tag{4-41}$$

Step 3: Solution on $[t_1,\infty)$.

We regard t_1 as the initial time. It follows from (4-38), (4-40) and (4-41) that the data at t_1 satisfies the condition of Theorem 1.2. Then the solution on $[t_1,\infty)$ can be proved by a direct use of Theorem 1.2. Thus, taking

$$\epsilon_0 = \min\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6\},$$

we finish the proof of Theorem 1.1. □

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