Pacific Journal of Mathematics

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Volume 309 No. 2 December 2020

PACIFIC JOURNAL OF MATHEMATICS
Vol. 309, No. 2, 2020
https://doi.org/10.2140/pjm.2020.309.333

ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR SOME ELLIPTIC EQUATIONS IN EXTERIOR DOMAINS

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This paper is concerned with the asymptotic behavior of solutions of the problems

$$-\Delta u = e^u \text{ in } \mathbb{R}^2\backslash B, \quad \int_{\mathbb{R}^2\backslash B} e^{u(x)}\,dx < \infty, \tag{0-1}$$

where $B = \{x \in \mathbb{R}^2 : |x| < 1\}$ is the unit ball of $\mathbb{R}^2$, and

$$\Delta^2 u = e^u \text{ in } \mathbb{R}^4\backslash B, \quad \int_{\mathbb{R}^4\backslash B} e^{u(x)}\,dx < \infty, \tag{0-2}$$

where $B = \{x \in \mathbb{R}^4 : |x| < 1\}$ is the unit ball of $\mathbb{R}^4$. It is seen that the asymptotic behavior of solutions for (0-1) and (0-2) is equivalent to the asymptotic behavior of singular solutions of the related problems (via the transformation $v(y) = u(x)$, $y = x/|x|^2$):

$$-\Delta_y v = |y|^{-4}e^v \text{ in } B\backslash\{0\}, \quad \int_{B\backslash\{0\}} |y|^{-4}e^{v(y)}\,dy < \infty \tag{0-3}$$

and

$$\Delta_y^2 v = |y|^{-8}e^v \text{ in } B\backslash\{0\}, \quad \int_{B\backslash\{0\}} |y|^{-8}e^{v(y)}\,dy < \infty, \tag{0-4}$$

respectively. We obtain the exact asymptotic behavior of solutions of (0-1) and (0-2) as $|x| \to \infty$. Meanwhile, we find that the singular solutions of the related problems (0-3) and (0-4) in $B\backslash\{0\}$ are asymptotic radial solutions and obtain the corresponding asymptotic behavior as $|y| \to 0$.

The research of Guo is supported by NSFC (No. 11171092, 11571093), the research of Liu is supported by NSFC (No. 11971147) and CPSF (No. 2019M662475).
MSC2020: 35B45, 35J40.
Keywords: asymptotic behavior of solutions, semilinear biharmonic problems, exterior domains, singular solutions.

1. Introduction

In this paper, we study the asymptotic behavior of solutions for the following problems:

$$-\Delta u = e^u \text{ in } \mathbb{R}^2 \backslash B, \quad \int_{\mathbb{R}^2 \backslash B} e^{u(x)}\, dx < \infty, \tag{1-1}$$

where $B = \{x \in \mathbb{R}^2 : |x| < 1\}$ is the unit ball of $\mathbb{R}^2$, and

$$\Delta^2 u = e^u \text{ in } \mathbb{R}^4 \backslash B, \quad \int_{\mathbb{R}^4 \backslash B} e^{u(x)}\, dx < \infty, \tag{1-2}$$

where $B = \{x \in \mathbb{R}^4 : |x| < 1\}$ is the unit ball in $\mathbb{R}^4$.

The equations in (1-1) and (1-2) have roots in conformal geometry. Let (M, g) be a complete Riemannian manifold. Associated to g, there are tensors such as the full curvature tensor R_g, the Ricci curvature tensor Ric_g and the scalar curvature S_g. The Laplace operator Δ_g is a well-known elliptic operator on M associated with the metric g. In dimension 4, the equation in (1-2) is closely related to the Q-curvature problem. The Q-curvature is similar to the scalar curvature in dimension 2. See [Chang and Yang 1995; 1997; Graham et al. 1992; Lin 1998; Martinazzi 2009; Xu 2006].

The structure of solutions of (1-1) and (1-2) in $\mathbb{R}^2$ and $\mathbb{R}^4$ respectively has been studied in [Chen and Li 1991; Lin 1998; Martinazzi 2009; Wei and Xu 1999; Wei and Ye 2008; Xu 2006]. For a solution $u \in C^4(\mathbb{R}^4)$ of the equation in (1-2), an important fact $-\Delta u \geq 0$ in $\mathbb{R}^4$ can be obtained. Using the moving-plane or moving-sphere arguments, Lin [1998] and Xu [2006] classified the solutions and obtained the asymptotic behavior of solutions as $|x| \to \infty$. Moreover, the singular solutions of the equation

$$-\Delta u = e^u \text{ in } B \backslash \{0\}, \quad \int_{B \backslash \{0\}} e^{u(x)}\, dx < \infty, \tag{1-3}$$

where B is the unit ball in $\mathbb{R}^2$, have also been studied in [Chou and Wan 1994] via the theory of complex variables. More precisely, Chou and Wan [1994] showed that the singular solutions of (1-3) are asymptotic radial solutions and obtain the asymptotic behavior of solutions as $|x| \to 0$. By the transformation

$$v(y) = u(x), \quad y = \frac{x}{|x|^2},$$

we see that the problems (1-1) and (1-2) are equivalent to the problems

$$-\Delta_y v = |y|^{-4} e^v \text{ in } B \backslash \{0\}, \quad \int_{B \backslash \{0\}} |y|^{-4} e^{v(y)}\, dy < \infty \tag{1-4}$$

and

$$\Delta_y^2 v = |y|^{-8} e^v \text{ in } B\backslash\{0\}, \quad \int_{B\backslash\{0\}} |y|^{-8} e^{v(y)}\, dy < \infty, \tag{1-5}$$

respectively. The asymptotic behavior of solutions for (1-1) and (1-2) is equivalent to the asymptotic behavior of singular solutions for (1-4) and (1-5). We will obtain the exact asymptotic behavior of solutions of (1-1) and (1-2) as $|x| \to \infty$. Moreover, we will show that the singular solutions of (1-4) and (1-5) are asymptotic radial solutions and obtain the asymptotic behavior of the singular solutions as $|y| \to 0$ by using the theory of PDEs. We find that the study of (1-2) is more complicated than that of (1-1). To obtain the result similar to that of (1-1), we need to put an extra assumption on the solution to avoid the appearance of an extra fundamental solution of the operator Δ^2. Our main results of this paper are the following theorems:

Theorem 1.1. *Assume that $u \in C^2(\mathbb{R}^2\backslash B)$ is a solution of* (1-1). *Then*

$$\frac{u(x)}{\ln|x|} \to \alpha \quad as \quad |x| \to \infty, \tag{1-6}$$

where $\alpha < -2$.

Theorem 1.2. *Assume that $u \in C^4(\mathbb{R}^4\backslash B)$ is a solution of* (1-2) *and*

$$u(x) = o(|x|^2) \quad as \quad |x| \to \infty. \tag{1-7}$$

Then

$$\frac{u(x)}{\ln|x|} \to \alpha \quad as \quad |x| \to \infty, \tag{1-8}$$

$$-|x|^2 \Delta u(x) \to \frac{1}{2|\mathbb{S}^3|} \int_{\mathbb{R}^4\backslash B} e^{u(y)}\, dy + \kappa \quad as \quad |x| \to \infty, \tag{1-9}$$

where $\alpha < -4$, $|\mathbb{S}^3|$ is the surface area of the unit sphere, κ is a constant.

Remark 1.3. We will see from the proof that the conclusions of Theorems 1.1 and 1.2 are still true if we assume $u \in C^2(\mathbb{R}^2\backslash \bar{B})$ and $u \in C^4(\mathbb{R}^4\backslash \bar{B})$ respectively or $u \in C^2(\mathbb{R}^2\backslash \overline{B_R(0)})$ and $u \in C^4(\mathbb{R}^4\backslash \overline{B_R(0)})$ respectively for some $R > 1$, where and in the following, $B_R(0) = \{x \in \mathbb{R}^2 : |x| < R\}$ or $B_R(0) = \{x \in \mathbb{R}^4 : |x| < R\}$. Our assumptions in Theorems 1.1 and 1.2 are only for convenience of using some expressions in our calculations.

As an application of Theorem 1.2, we can consider the following problem:

$$\Delta^2 v = e^v \text{ in } B\backslash\{0\}, \quad \int_{B\backslash\{0\}} e^{v(y)}\, dy < \infty, \tag{1-10}$$

where $B = \{y \in \mathbb{R}^4 : |y| < 1\}$ and obtain the asymptotic radial symmetry result for (1-10) in the punctured ball.

Theorem 1.4. *Assume that $v \in C^4(B\backslash\{0\})$ is a singular solution of the problem* (1-10) *with*

$$v(y) = o(|y|^{-2}) \quad \textit{as} \quad |y| \to 0.$$

Then

$$\frac{v(y)}{\ln|y|} \to \gamma \quad \textit{as} \quad |y| \to 0,$$

where $\gamma > -4$.

Similar results in $\mathbb{R}^4$ are well-known in [Lin 1998; Xu 2006]. Liouville theorem for harmonic functions plays the key role in obtaining these results in $\mathbb{R}^4$. However, the corresponding Liouville theorem does not hold in $\mathbb{R}^4 \setminus B$ and the methods in [Lin 1998; Xu 2006] cannot be used here. Moreover, we cannot show $-\Delta u \geq 0$ in $\mathbb{R}^4\backslash B$ for a solution $u \in C^4(\mathbb{R}^4\backslash B)$ of (1-2). To this end, we need to overcome some technical difficulties here and use some new idea to obtain the corresponding results in $\mathbb{R}^4\backslash B$.

The organization of the paper is the following: In Section 2, we give some qualitative properties of solutions for (1-2). The main results will be obtained in Section 3. In the Appendix, we present some estimates used in Section 3.

2. Preliminaries

In this section, we study the qualitative properties of solutions for (1-2). This is crucial to the proof of Theorem 1.2.

Let $u \in C^4(\mathbb{R}^4\backslash B)$ be a solution of problem (1-2) and

$$v(y) = u(x), \qquad y = \frac{x}{|x|^2}.$$

Then $v \in C^4(B\backslash\{0\})$ satisfies the problem

$$\Delta_y^2 v = |y|^{-8} e^v \text{ in } B\backslash\{0\}, \quad \int_{B\backslash\{0\}} |y|^{-8} e^{v(y)}\, dy < \infty, \tag{2-1}$$

where $B \subset \mathbb{R}^4$ is the unit ball. Moreover,

$$v(y) = o(|y|^{-2}) \quad \text{as} \quad |y| \to 0. \tag{2-2}$$

It is easy to see that 0 is a nonremovable singular point of v. Using the fact that

$$\int_{B\backslash\{0\}} |y|^{-8} e^{v(y)}\, dy = \int_{\mathbb{R}^4\backslash B} e^u\, dx,$$

we have

$$\infty > \int_{B\backslash\{0\}} |y|^{-8} e^{v(y)}\, dy = |\mathbb{S}^3| \int_0^1 \rho^{-5} \overline{e^v}\, d\rho \geq |\mathbb{S}^3| \int_0^1 \rho^{-5} e^{\bar{v}}\, d\rho, \tag{2-3}$$

where $\rho = |y|$, $|\mathbb{S}^3|$ is the surface area of the unit sphere and

$$\bar{v}(\rho) := \frac{1}{|\mathbb{S}^3|} \int_{\mathbb{S}^3} v(\rho, \theta)\, d\theta \quad \text{for all } \rho \in (0, 1).$$

In the following, we first consider the asymptotic behavior of $\bar{v}(\rho)$ as ρ tends to 0.

Lemma 2.1. *Let* $v \in C^4(B\backslash\{0\})$ *be a solution of* (2-1) *satisfying* (2-2). *Then*

$$\frac{\bar{v}(\rho)}{\ln \rho} \to \beta \quad as \quad \rho \to 0, \tag{2-4}$$

where $\beta > 4$.

Proof. Note that $\bar{v}(\rho)$ satisfies the problem

$$\Delta^2 \bar{v} = \rho^{-8} \overline{e^{v}} \text{ in } (0, 1), \qquad \int_0^1 \rho^{-5} \overline{e^{v}}(\rho)\, d\rho < \infty.$$

By (2-2), we find

$$\bar{v}(\rho) = o(\rho^{-2}). \tag{2-5}$$

Step 1: We claim that if $\lim_{\rho \to 0} \rho^3 \bar{v}'(\rho)$ exists, then it must be 0, i.e.,

$$\lim_{\rho \to 0} \rho^3 \bar{v}'(\rho) = 0. \tag{2-6}$$

On the contrary, there is $M \neq 0$ (M maybe $\pm\infty$) such that $\lim_{\rho \to 0} \rho^3 \bar{v}'(\rho) = M$. We consider two cases:

(i) $M > 0$,

(ii) $M < 0$.

For the case (i), we have that there exist $M_0 > 0$ and $\rho_0 > 0$ such that

$$\bar{v}'(\rho) \geq M_0 \rho^{-3} \text{ for all } \rho \in (0, \rho_0). \tag{2-7}$$

Integrating (2-7) on (ρ, ρ_0), we obtain

$$\bar{v}(\rho) \leq -\tfrac{1}{2} M_0 \rho^{-2} + \bar{v}(\rho_0) + \tfrac{1}{2} M_0 \rho_0^{-2} \text{ for all } \rho \in (0, \rho_0), \tag{2-8}$$

which is a contradiction with (2-5).

For the case (ii), we have that there exist $\rho_0 > 0$ and $M_0 < 0$ such that

$$\bar{v}'(\rho) \leq M_0 \rho^{-3} \text{ for all } \rho \in (0, \rho_0). \tag{2-9}$$

By integrating (2-9) on (ρ, ρ_0), we see

$$\bar{v}(\rho) \geq -\tfrac{1}{2} M_0 \rho^{-2} + \bar{v}(\rho_0) + \tfrac{1}{2} M_0 \rho_0^{-2} \text{ for all } \rho \in (0, \rho_0).$$

This also contradicts (2-5). Thus, our claim (2-6) holds.

Step 2: We claim that there is a negative constant M satisfying

$$\lim_{\rho\to 0}\rho^3(\Delta\bar{v})'(\rho)=M. \tag{2-10}$$

Since $\bar{v}(\rho)$ satisfies the equation

$$(\rho^3(\Delta\bar{v})'(\rho))'=\rho^{-5}\overline{e^{v}} \text{ for all } \rho\in(0,1). \tag{2-11}$$

Then $f(\rho):=\rho^3(\Delta\bar{v})'(\rho)$ is an increasing function and hence $\lim_{\rho\to 0} f(\rho)=M<\infty$ exists and M maybe $-\infty$. For $\epsilon>0$ sufficiently small, by integrating (2-11) on $(\epsilon,1)$, we get

$$(\Delta\bar{v})'(1)-\epsilon^3(\Delta\bar{v})'(\epsilon)=\int_\epsilon^1\rho^{-5}\overline{e^{v}}(\rho)\,d\rho. \tag{2-12}$$

Since $\int_0^1\rho^{-5}\overline{e^{v}}(\rho)\,d\rho<\infty$, we easily see that $M>-\infty$.

We next show that $M<0$. On the contrary, we have

$$(\Delta\bar{v})'(\rho)=\left(M+\int_0^\rho t^{-5}\overline{e^{v}}(t)\,dt\right)\rho^{-3}>0 \text{ for all } \rho\in(0,1). \tag{2-13}$$

Thus $\lim_{\rho\to 0}\Delta\bar{v}(\rho)=\tilde{M}_1<\infty$ exists and $\tilde{M}_1$ maybe $-\infty$. We now consider three cases here:

(a) $\tilde{M}_1>0$,

(b) $\tilde{M}_1=0$,

(c) $\tilde{M}_1<0$.

For the case (a), we have that there exist $\rho_1>0$ and $0<M_1\le\frac{1}{2}\tilde{M}_1$ such that

$$\Delta\bar{v}(\rho)\ge M_1 \text{ for all } \rho\in(0,\rho_1).$$

Hence,

$$(\rho^3\bar{v}'(\rho))'\ge M_1\rho^3>0 \text{ for all } \rho\in(0,\rho_1)$$

and

$$\lim_{\rho\to 0}\rho^3\bar{v}'(\rho) \text{ exists.}$$

By Step 1, we find

$$\lim_{\rho\to 0}\rho^3\bar{v}'(\rho)=0.$$

On the other hand, since $\tilde{M}_1<\infty$, we see that there exist $\rho_2>0$ and $M_2\ge 2\tilde{M}_1$ such that

$$(\rho^3\bar{v}'(\rho))'\le M_2\rho^3 \text{ for all } \rho\in(0,\rho_2]. \tag{2-14}$$

Integrating (2-14) on $(0, \rho)$, we infer

$$\bar{v}'(\rho) \le \tfrac{1}{4} M_2 \rho \text{ for all } \rho \in (0, \rho_2].$$

Thus

$$\bar{v}(\rho) \ge C > -\infty \text{ for all } \rho \in (0, \rho_2]. \tag{2-15}$$

This contradicts the fact that

$$e^C \int_0^1 \rho^{-5}\, d\rho \le \int_0^1 \rho^{-5} e^{\bar{v}(\rho)}\, d\rho \le \int_0^1 \rho^{-5} \overline{e^{v}}(\rho)\, d\rho < \infty.$$

For the case (b), we see that there exists $\rho_3 > 0$ such that

$$\Delta \bar{v}(\rho) \ge 0 \text{ for all } \rho \in (0, \rho_3).$$

By Step 1, we see

$$\lim_{\rho \to 0} \rho^3 \bar{v}'(\rho) = 0.$$

Similarly, there are $\rho_4 > 0$ and $M_3 > 0$ satisfying

$$(\rho^3 \bar{v}'(\rho))' \le M_3 \rho^3 \text{ for all } \rho \in (0, \rho_4]. \tag{2-16}$$

We can also derive a contradiction from (2-16) as in the proof of the case (a).

For the case (c), we see that there exist $\rho_5 > 0$ and $-\infty < \frac{1}{2}\tilde{M}_1 < M_4 < 0$ such that

$$(\rho^3 \bar{v}'(\rho))' \le M_4 \rho^3 < 0 \text{ for all } \rho \in (0, \rho_5]. \tag{2-17}$$

By Step 1, we get

$$\lim_{\rho \to 0} \rho^3 \bar{v}'(\rho) = 0.$$

Integrating (2-17) on $(0, \rho)$, we obtain

$$\bar{v}'(\rho) \le \tfrac{1}{4} M_4 \rho \text{ for all } \rho \in (0, \rho_5]$$

and

$$\bar{v}(\rho_5) - \bar{v}(\rho) \le \tfrac{1}{8} M_4 (\rho_5^2 - \rho^2) \text{ for all } \rho \in (0, \rho_5],$$

which implies

$$\bar{v}(\rho) \ge C > -\infty \text{ for all } \rho \in (0, \rho_5].$$

This is a contradiction with (2-3).

Step 3: We prove (2-4).

In view of (2-10) and (2-11), we deduce that

$$(\Delta \bar{v})'(\rho) = \left(M + \int_0^\rho s^{-5} \overline{e^{v}}(s)\, ds \right) \rho^{-3} = (M + \eta(\rho)) \rho^{-3} \text{ for } \rho \text{ near } 0, \tag{2-18}$$

where $\eta(\rho)=\int_0^\rho s^{-5}\overline{e^{v}}(s)\,ds$. Since $M<0$, we see that $\lim_{\rho\to 0}\Delta\bar v(\rho)=\gamma$ exists. As in Step 2, we infer

(2-19) $$\lim_{\rho\to 0}\rho^3\bar v'(\rho)=0.$$

Integrating (2-18) on $[\rho,\rho_*]$, we obtain

$$\Delta\bar v(\rho_*)-\Delta\bar v(\rho)=-\tfrac12 M(\rho_*^{-2}-\rho^{-2})+\int_\rho^{\rho_*}\eta(s)s^{-3}\,ds \text{ for } \rho\in(0,\rho_*),$$

where $\rho_*>0$ is sufficiently small. Then

$$\Delta\bar v(\rho)=\Delta\bar v(\rho_*)+\tfrac12 M\rho_*^{-2}-\tfrac12 M\rho^{-2}-\int_\rho^{\rho_*}\eta(s)s^{-3}\,ds \text{ for } \rho\in(0,\rho_*)$$

and

(2-20) $$(\rho^3\bar v'(\rho))' = \big[\Delta\bar v(\rho_*)+\tfrac12 M\rho_*^{-2}\big]\rho^3-\tfrac12 M\rho-\rho^3\int_\rho^{\rho_*}\eta(s)s^{-3}\,ds \text{ for } \rho\in(0,\rho_*).$$

Integrating (2-20) on $(0,\rho]$ and using (2-19), we have

(2-21) $$\bar v'(\rho)=\tfrac14\big[\Delta\bar v(\rho_*)+\tfrac12 M\rho_*^{-2}\big]\rho-\tfrac14 M\rho^{-1} -\rho^{-3}\int_0^\rho t^3\int_t^{\rho_*}\eta(s)s^{-3}\,ds\,dt \text{ for } \rho\in(0,\rho_*).$$

Integrating (2-21) on $[\rho,\rho_*]$, we deduce

(2-22) $$\bar v(\rho)=\bar v(\rho_*)-\tfrac18\big[\Delta\bar v(\rho_*)+\tfrac12 M\rho_*^{-2}\big](\rho_*^2-\rho^2)+\tfrac14 M(\ln\rho_*-\ln\rho) +\int_\rho^{\rho_*}\xi^{-3}\int_0^\xi t^3\int_t^{\rho_*}\eta(s)s^{-3}\,ds\,dt\,d\xi \text{ for } \rho\in(0,\rho_*).$$

Note that

$$\int_\rho^{\rho_*}\xi^{-3}\int_0^\xi t^3\int_t^{\rho_*}\eta(s)s^{-3}\,ds\,dt\,d\xi=o_\rho(1)\ln\rho+O(1) \text{ for } \rho \text{ near } 0.$$

Then

(2-23) $$\frac{\bar v(\rho)}{\ln\rho}\to\beta \quad\text{as}\quad \rho\to 0,$$

where $\beta=-\frac14 M$. Since $\int_0^1\rho^{-5}e^{\bar v(\rho)}\,d\rho<\infty$, we easily see that $\beta>4$ and this completes the proof of this lemma. □

Next, we need the following key lemma. Similar results are well-known from [Lin 1998; Wei and Xu 1999; Xu 2006] for solutions of the equation of (1-2) in $\mathbb{R}^4$ by using the fact that $-\Delta u\ge 0$ in $\mathbb{R}^4$. However, we cannot obtain such “nice” property for solutions of (1-2) in $\mathbb{R}^4 \backslash B$. To do so, we will use some new arguments here, which are interesting themselves.

Lemma 2.2. *Let $u \in C^4(\mathbb{R}^4 \backslash B)$ be a solution of* (1-2) *and* (1-7) *hold. Then, there is a constant C such that*

$$u(x) \leq C \quad \textit{for} \quad x \in \mathbb{R}^4 \backslash B. \tag{2-24}$$

Moreover,

$$\Delta u(x) \to 0 \quad \textit{as} \quad |x| \to \infty. \tag{2-25}$$

Proof. We divide the proof into several steps.

Step 1: We first show that

$$\overline{\lim}_{|x|\to\infty} \Delta u(x) \leq 0. \tag{2-26}$$

Suppose $\overline{\lim}_{|x|\to\infty} \Delta u(x) > 0$. Then, there is a sequence $\{x_k\} \subset \mathbb{R}^4 \backslash B$ with $|x_k| \to \infty$ as $k \to \infty$ and $\epsilon > 0$ independent of k, such that

$$\Delta u(x_k) \geq \epsilon > 0 \quad \text{for} \quad k \geq 1.$$

Let $w = -\Delta u$. Then

$$\Delta u + w = 0 \text{ in } \mathbb{R}^4 \backslash B, \quad \Delta w + e^u = 0 \text{ in } \mathbb{R}^4 \backslash B.$$

Define

$$\bar{u}_k(r) = \frac{1}{|\partial B_r(x_k)|} \int_{\partial B_r(x_k)} u(x) d\sigma, \quad 0 \leq r \leq \tfrac{1}{2}|x_k|.$$

Using Jensen's inequality, we have

$$\Delta \bar{u}_k + \bar{w}_k = 0 \text{ for } r \in \big[0, \tfrac{1}{2}|x_k|\big], \quad \Delta \bar{w}_k + e^{\bar{u}_k} \leq 0 \text{ for } r \in \big[0, \tfrac{1}{2}|x_k|\big]. \tag{2-27}$$

Since $r^3 \bar{w}_k'(r) < 0$, we find that

$$\bar{w}_k(r) \leq \bar{w}_k(0) \leq -\epsilon.$$

By (2-27), we have

$$(r^3 \bar{u}_k')' \geq \epsilon r^3,$$

which implies

$$\bar{u}_k'(r) \geq \tfrac{1}{4} \epsilon r.$$

Integrating both the sides, we deduce

$$\bar{u}_k(r) \geq \bar{u}_k(0) + \tfrac{1}{8} \epsilon r^2 \text{ for all } r \in \big(0, \tfrac{1}{2}|x_k|\big].$$

Note that $\bar{u}_k(0) = u(x_k) = o(|x_k|^2)$ for k sufficiently large, we find

$$\bar{u}_k\big(\tfrac{1}{2}|x_k|\big) \geq \big(\tfrac{1}{8}\epsilon + o(1)\big)\big(\tfrac{1}{2}|x_k|\big)^2,$$

which contradicts the fact $u(x) = o(|x|^2)$ as $|x| \to \infty$.

Step 2: We show

$$\underline{\lim}_{|x|\to\infty} \Delta u(x) \geq 0.$$

On the contrary, there exist $\epsilon > 0$ and a sequence $\{x_k\} \subset \mathbb{R}^4 \setminus B$ with $|x_k| \to \infty$ as $k \to \infty$ such that

$$\Delta u(x_k) \leq -\epsilon \quad \text{for} \quad k \geq 1.$$

Setting $v_k(y) = u(x)$, $y = x - x_k$, we see that

$$\Delta_y^2 v_k = e^{v_k}, \quad \Delta_y v_k(0) = \Delta_x u(x_k) \leq -\epsilon.$$

Let

$$z_k(y) = \frac{\Delta v_k(y)}{\Delta v_k(0)}, \quad \bar{z}_k(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} z_k(y)\, d\sigma \text{ for } r \in \left[0, \tfrac{1}{2}|x_k|\right].$$

Then, $z_k(0) = 1$ and

$$\Delta \bar{z}_k = \frac{\overline{e^{v_k}}}{\Delta v_k(0)}.$$

Integrating on $(0, r)$ yields

$$r^3 \bar{z}_k'(r) = \frac{1}{\Delta v_k(0)|\mathbb{S}^3|} \int_{B_r(0)} e^{v_k(y)}\, dy < 0. \tag{2-28}$$

For any fixed $R > 0$, we have

$$\int_{B_R(0)} e^{v_k(y)} dy \leq \int_{B_{|x_k|/2}(x_k)} e^{u(y)}\, dy \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore, it follows from (2-28) that

$$\bar{z}_k'(r) \to 0 \text{ uniformly for } r \in (0, R] \text{ as } k \to \infty,$$

which implies

$$\bar{z}_k(r) \to 1 \text{ uniformly for } r \in [0, R] \text{ as } k \to \infty. \tag{2-29}$$

On the other hand, we see that, for $r \in \left[R, \tfrac{1}{2}|x_k|\right]$,

$$-\bar{z}_k'(r) \leq \frac{r^{-3}}{|\Delta v_k(0)||\mathbb{S}^3|} \int_{B_{|x_k|/2}(0)} e^{v_k(y)} dy. \tag{2-30}$$

Integrating both sides on $[R, r]$, we find

$$0 \leq \bar{z}_k(R) - \bar{z}_k(r) \leq \frac{1}{2R^2|\Delta v_k(0)||\mathbb{S}^3|} \int_{B_{|x_k|/2}(0)} e^{v_k(y)}\, dy \to 0 \tag{2-31}$$

uniformly on $r \in \big[R, \tfrac{1}{2}|x_k|\big]$ as $k \to \infty$. By (2-29) and (2-31), we deduce

$$\bar{z}_k(r) \to 1 \text{ uniformly on } r \in \big[0, \tfrac{1}{2}|x_k|\big] \text{ as } k \to \infty. \tag{2-32}$$

Hence, for k sufficiently large, we have

$$\Delta \bar{v}_k(r) \le \tfrac{1}{2}\Delta v_k(0) < -\tfrac{1}{2}\epsilon \quad \text{for} \quad r \in \big[0, \tfrac{1}{2}|x_k|\big].$$

Using the similar arguments as in Step 1, we infer

$$\bar{v}_k\big(\tfrac{1}{2}|x_k|\big) - \bar{v}_k(0) < -\tfrac{1}{16}M\big(\tfrac{1}{2}|x_k|\big)^2$$

and

$$\bar{u}_k\big(\tfrac{1}{2}|x_k|\big) \le -\big(\tfrac{1}{16}M + o(1)\big)\big(\tfrac{1}{2}|x_k|\big)^2.$$

This contradicts the fact $u(x) = o(|x|^2)$ as $|x| \to \infty$.

Combining Steps 1 and 2, we can obtain

$$\lim_{|x|\to\infty} \Delta u(x) = 0. \tag{2-33}$$

Step 3: we show (2-24). On the contrary, there is a sequence $\{x_k\} \subset \mathbb{R}^4 \backslash B$ with $|x_k| \to \infty$ as $k \to \infty$ such that $u(x_k) \to \infty$ as $k \to \infty$. Setting $v_k(y) = u(x)$, $y = x - x_k$, we see from (2-33) that, for k sufficiently large,

$$\Delta_y v_k(y) \ge -\vartheta \text{ for all } y \in B_{|x_k|/2}(0),$$

where ϑ is a positive constant. Thus

$$\Delta \bar{v}_k(r) \ge -\vartheta \text{ for all } r \in \big(0, \tfrac{1}{2}|x_k|\big].$$

Then, for k sufficiently large,

$$\bar{v}_k(r) \ge \bar{v}_k(0) - \tfrac{1}{8}\vartheta r^2 \text{ for all } r \in \big(0, \tfrac{1}{2}|x_k|\big]$$

and

$$e^{\bar{v}_k(r)} \ge e^{u(x_k)} e^{-\vartheta r^2/8} \ge M e^{-\vartheta r^2/8} \text{ for all } r \in \big(0, \tfrac{1}{2}|x_k|\big], \tag{2-34}$$

for some $M > 0$ suitably large. Note that $u(x_k) \to \infty$ as $k \to \infty$. From (2-34), we have

$$\int_{B_{|x_k|/2}(x_k)} e^{u(y)}\, dy \ge M|\mathbb{S}^3| \int_0^2 r^3 e^{-\vartheta r^2/8}\, dr > 0,$$

which is a contradiction with

$$\int_{B_{|x_k|/2}(x_k)} e^{u(y)}\, dy \to 0 \quad \text{as} \quad k \to \infty. \qquad \square$$

3. Proof of the main results

In this section, we present the proof of Theorems 1.1 and 1.2. The proof of Theorem 1.1 is simple by using the result in [Chou and Wan 1994]. We mainly concentrate our attention to the proof of Theorem 1.2.

Proof of Theorem 1.1. Let $u \in C^2(\mathbb{R}^2\backslash B)$ be a solution to (1-1). Using the transformation

$$v(y) = u(x), \quad y = \frac{x}{|x|^2},$$

we see that $v \in C^2(B\backslash\{0\})$ satisfies the problem

$$-\Delta_y v = |y|^{-4} e^v \text{ in } B\backslash\{0\}, \quad \int_{B\backslash\{0\}} |y|^{-4} e^{v(y)}\,dy < \infty. \tag{3-1}$$

It is clear that 0 is a nonremovable singular point of v. To obtain the asymptotic behavior of $u(x)$ as $|x| \to \infty$, we only need to obtain the asymptotic behavior of $v(y)$ as $|y| \to 0$.

Let $w(y) = v(y) - 4\ln|y|$. We find that $w(y)$ satisfies the problem

$$-\Delta w = e^w \text{ in } B\backslash\{0\}, \quad \int_{B\backslash\{0\}} e^{w(y)}\,dy < \infty. \tag{3-2}$$

It follows from [Chou and Wan 1994, Theorem 5] that

$$\frac{w(y)}{\ln|y|} \to \beta_0 \quad \text{as} \quad |y| \to 0, \tag{3-3}$$

where $\beta_0 > -2$, which implies

$$\frac{v(y)}{\ln|y|} \to \beta \quad \text{as} \quad |y| \to 0, \tag{3-4}$$

where $\beta = \beta_0 + 4 > 2$. Therefore, (1-6) can be obtained from (3-4) and the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Define

$$w(x) = \frac{1}{4|\mathbb{S}^3|} \int_{\mathbb{R}^4\backslash B} \ln\Big(\frac{|x-y|}{|y|}\Big) e^{u(y)}\,dy \tag{3-5}$$

and

$$\bar{w}(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} w(y)\,d\sigma, \quad r > 1. \tag{3-6}$$

Then, we have

$$\Delta^2 w(x) = -e^{u(x)} \text{ and } \Delta^2(u+w)(x) = 0 \text{ for } x \in \mathbb{R}^4\backslash B.$$

Note that u is upper bounded, as in [Lin 1998], we can deduce

$$\Delta w = \frac{1}{2|\mathbb{S}^3|}\int_{\mathbb{R}^4\setminus B}\frac{1}{|x-y|^2}e^{u(y)}\,dy \tag{3-7}$$

and

$$\Delta w(x)\to 0 \quad \text{as} \quad |x|\to\infty.$$

Set $\psi = u+w$. Then $\Delta^2\psi = 0$ in $\mathbb{R}^4\setminus B$. Let $k(t,\theta)=\psi(r,\theta)$, $\bar{k}(t)=\bar{\psi}(r)$, $t=\ln r$, $r=|x|$, $r>1$. Then $\Delta^2\bar{\psi}(r)=0$, $r>1$. By Lemmas 2.1, A.1, we know

$$\frac{\bar{k}(t)}{t}\to\alpha_0-\beta \quad \text{as} \quad t\to\infty,$$

where

$$\alpha_0=\frac{1}{4|\mathbb{S}^3|}\int_{\mathbb{R}^4\setminus B}e^{u(y)}\,dy.$$

Define

$$z(t,\theta)=k(t,\theta)-\bar{k}(t).$$

Then

$$z_t^{(4)}-4z_{tt}+2\Delta_\theta z_{tt}+\Delta_\theta^2 z=0, \quad (t,\theta)\in(0,\infty)\times\mathbb{S}^3. \tag{3-8}$$

Let

$$z(t,\theta)=\sum_{i=1}^{\infty}\sum_{j=1}^{m_i} z_i^j(t)Q_i^j(\theta),$$

where $Q_i^j(\theta)$ is an eigenfunction corresponding to the eigenvalue σ_i of the problem

$$\Delta_\theta^2 Q=\sigma Q, \quad \theta\in\mathbb{S}^3.$$

It is known from [Guo et al. 2015] that $\sigma_i=\lambda_i^2$, $\lambda_i=i(2+i)$, $m_i=(1+i)^2$ is the multiplicity of σ_i and Q_i^j also satisfies

$$-\Delta_\theta Q_i^j=\lambda_i Q_i^j, \quad \theta\in\mathbb{S}^3.$$

Hence for each (i,j) with $i=1,2,\dots$, $j=1,2,\dots,m_i$,

$$(z_i^j)^{(4)}(t)-2(\lambda_i+2)(z_i^j)_{tt}(t)+\lambda_i^2(z_i^j)(t)=0. \tag{3-9}$$

The characteristic equation of (3-9) is

$$\tau^{(4)}-2(2+\lambda_i)\tau^2+\lambda_i^2=0$$

and the corresponding characteristic roots are given by

$$\tau_1^{(i)}=\sqrt{2+\lambda_i+2\sqrt{1+\lambda_i}}, \quad \tau_2^{(i)}=-\sqrt{2+\lambda_i+2\sqrt{1+\lambda_i}},$$
$$\tau_3^{(i)}=\sqrt{2+\lambda_i-2\sqrt{1+\lambda_i}}, \quad \tau_4^{(i)}=-\sqrt{2+\lambda_i-2\sqrt{1+\lambda_i}}.$$

Moreover,

$$\tau_2^{(i)} < \tau_4^{(i)} < 0 < \tau_3^{(i)} < \tau_1^{(i)}.$$

By the standard ODE theory, we see that there is $T \gg 1$ such that for $t > T$

$$z_i^j(t) = B_1 e^{\tau_1^{(i)} t} + B_2 e^{\tau_2^{(i)} t} + B_3 e^{\tau_3^{(i)} t} + B_4 e^{\tau_4^{(i)} t}.$$

Since $|z(t,\theta)| \le Ct$, we have $B_1 = B_3 = 0$. Thus

$$z_i^j(t) = B_2 e^{\tau_2^{(i)} t} + B_4 e^{\tau_4^{(i)} t}$$

and

$$B_2 = O(T) e^{-\tau_2^{(i)} T}, \quad B_4 = O(T) e^{-\tau_4^{(i)} T}.$$

Thus

$$z_i^j(t) = O(T) e^{\tau_2^{(i)}(t-T)} + O(T) e^{\tau_4^{(i)}(t-T)}.$$

Let $Z^2(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} [z_i^j(t)]^2$. Note that $\tau_2^{(i)} < \tau_4^{(i)} < 0$, then

$$Z^2(t) \le CT \sum_{i=1}^{\infty} m_i \big(e^{2\tau_2^{(i)}(t-T)} + e^{2\tau_4^{(i)}(t-T)}\big) \le CT \sum_{i=1}^{\infty} m_i e^{2\tau_4^{(i)}(t-T)} \le CT e^{2\tau_4^{(1)}(t-T)},$$

where C is a positive constant independent of t. Here we have used the fact that for $t > T_* := 10T$,

$$\lim_{i\to\infty} \frac{m_{i+1}}{m_i} e^{2(\tau_4^{(i+1)} - \tau_4^{(i)})(t-T)} \le e^{-2(t-T)} \le \tfrac{1}{2}.$$

Note that $\|Q_i^j\|_{L^2(\mathbb{S}^3)} = 1$ for each (i, j). Hence

$$\|z\|_{L^2(\mathbb{S}^3)} \le C e^{\tau_4^{(1)}(t-T)} \le C e^{\tau_4^{(1)} t}.$$

By the interior L^∞-estimate of (3-8) in $(t-1, t+1) \times \mathbb{S}^3$, we obtain

$$|z(t,\theta)| \le C \|z\|_{L^2((t-1,t+1)\times\mathbb{S}^3)} \le C e^{\tau_4^{(1)} t}, \tag{3-10}$$

which implies

$$\max_{\theta\in\mathbb{S}^3} |z(t,\theta)| \le C e^{\tau_4^{(1)} t} \quad \text{for} \quad t \in (T_*, \infty).$$

Since $\tau_4^{(1)} = -1$, we find

$$u(x) = -w(x) + \bar{\psi}(|x|) + O(|x|^{-1}).$$

By Lemma A.1, we see

$$\frac{w(x)}{\ln|x|} \to \alpha_0 \quad \text{as} \quad |x| \to \infty.$$

Therefore

(3-11)
$$\lim_{|x|\to\infty} \frac{u(x)}{\ln|x|} = -\beta, \quad \beta > 4.$$

Next we show

(3-12)
$$-|x|^2 \Delta u(x) \to \frac{1}{2|\mathbb{S}^3|} \int_{\mathbb{R}^4\setminus B} e^{u(y)}\, dy + \kappa \quad \text{as} \quad |x| \to \infty,$$

where κ is a constant.

Thanks to (3-7), (3-11), we deduce

(3-13)
$$|x|^2 \Delta w(x) \to \frac{1}{2|\mathbb{S}^3|} \int_{\mathbb{R}^4\setminus B} e^{u(y)}\, dy \quad \text{as} \quad |x| \to \infty.$$

Let $h(x) = \Delta(u + w)(x)$,

$$\bar{h}(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} h(y)\, d\sigma.$$

Since $\lim_{|x|\to\infty} \Delta(u + w)(x) = 0$. Then, we see that $\lim_{r\to\infty} \bar{h}(r) = 0$,

$$\Delta \bar{h}(r) = 0 \text{ for all } r \in (1, \infty).$$

Then, there is a constant c such that

(3-14)
$$\bar{h}'(r) \equiv c r^{-3} \text{ for all } r \in (1, \infty).$$

By integrating (3-14) in (r, ∞), we obtain

(3-15)
$$\bar{h}(r) = -\kappa r^{-2}, \quad \text{where} \quad \kappa = \tfrac{1}{2} c.$$

To obtain (3-12), we only need to show

(3-16)
$$|x|^2 (h(x) - \bar{h}(|x|)) \to 0 \quad \text{as} \quad |x| \to \infty.$$

Let

$$z(t, \theta) = h(r, \theta) - \bar{h}(r), \quad t = \ln r.$$

Then $z(t, \theta)$ satisfies

(3-17)
$$z_{tt}(t, \theta) + 2 z_t(t, \theta) + \Delta_\theta z(t, \theta) = 0, \quad (t, \theta) \in (0, \infty) \times \mathbb{S}^3.$$

Set

$$z(t, \theta) = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} z_i^j(t) Q_i^j(\theta),$$

where $Q_i^j(\theta)$ is an eigenfunction corresponding to the eigenvalue λ_i of the problem

$$-\Delta_\theta Q = \lambda Q, \quad \theta \in \mathbb{S}^3.$$

It is well-known that $\lambda_i = i(2+i)$ and $m_i = (1+i)^2$ is the multiplicity of λ_i. Then $z_i^j(t)$ satisfies

$$(z_i^j)''(t) + 2(z_i^j)'(t) - \lambda_i z_i^j(t) = 0, \quad t \in (0,\infty). \tag{3-18}$$

The characteristic equation of (3-18) is

$$\tau^2 + 2\tau - \lambda_i = 0,$$

whose characteristic roots are given by

$$\tau_1^{(i)} = -1 - \sqrt{1+\lambda_i} < 0 \quad \text{and} \quad \tau_2^{(i)} = -1 + \sqrt{1+\lambda_i} > 0, \quad i = 1, 2, \dots$$

Therefore, for $T \gg 1$ and $t > T$, we see that

$$z_i^j(t) = Ae^{\tau_1^{(i)}t} + Be^{\tau_2^{(i)}t}, \text{ where } A, B \text{ are generic constants.}$$

Using the fact that $h(x)$ is bounded and hence $z(t,\theta)$ is bounded, we deduce that $B = 0$ and

$$z_i^j(t) = Ae^{\tau_1^{(i)}t} \quad \text{with} \quad A = O(1)e^{-\tau_1^{(i)}T}.$$

Hence, for $j = 1, 2, \dots, m_i$, we have

$$z_i^j(t) = O(1)e^{\tau_1^{(i)}(t-T)} \text{ for all } t > T.$$

Since

$$\lim_{i\to\infty} \frac{m_{i+1}}{m_i} e^{2(\tau_1^{(i+1)} - \tau_1^{(i)})(t-T)} \le e^{-2(t-T)} < \tfrac{1}{2}.$$

Thus, for $t > 10T$, we obtain

$$\|z\|^2_{L^2(\mathbb{S}^3)} \le C \sum_{i=1}^{\infty} m_i e^{2\tau_1^{(i)}(t-T)} \le Ce^{2\tau_1^{(1)}t}, \tag{3-19}$$

where $C > 0$ is independent of t.

For any fixed $(t,\theta) \in (T_* + 1, \infty) \times \mathbb{S}^3$, by the interior L^∞-estimate of (3-17) in $(t-1, t+1) \times \mathbb{S}^3$, we obtain from (3-19) that

$$|z(t,\theta)| \le C\|z\|_{L^2((t-1,t+1)\times\mathbb{S}^3)} \le Ce^{\tau_1^{(1)}t}, \tag{3-20}$$

where $C > 0$ is independent of t. Thus

$$\max_{\theta\in\mathbb{S}^3} |z(t,\theta)| \le Ce^{-3t} \text{ for } t \in (T_*, \infty). \tag{3-21}$$

Therefore, for $|x| \ge 2e^{T_*}$, we have

$$|x|^2 |h(x) - \bar{h}(|x|)| \le C|x|^{-1},$$

where $C > 0$ is independent of $|x|$. Then (3-12) holds and the proof of the theorem is complete. □

Now we are in the position to give the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $w(y) = v(y) + 8\ln|y|$. Then $w(y)$ satisfies the problem

$$\Delta^2 w = |y|^{-8} e^w \text{ in } B\backslash\{0\}, \qquad \int_{B\backslash\{0\}} |y|^{-8} e^{w(y)}\, dy < \infty.$$

It is known from the proof of Theorem 1.2 and Remark 1.3 that

$$\frac{w(y)}{\ln|y|} \to \beta \quad \text{as} \quad |y| \to 0$$

with $\beta > 4$. Then we have

$$\frac{v(y)}{\ln|y|} \to \gamma \quad \text{as} \quad |y| \to 0$$

with $\gamma = \beta - 8 > -4$. This completes the proof of the theorem. □

Appendix: Some estimates

In this section, we shall present some estimates used in the proof of Theorem 1.2. The proof is similar to that in [Lin 1998]. For the reader's convenience, we give the proof.

Let u be a solution of (1-2). Define

$$\alpha_0 = \frac{1}{4|\mathbb{S}^3|} \int_{\mathbb{R}^4\backslash B} e^{u(y)}\, dy, \tag{A-1}$$

$$w(x) = \frac{1}{4|\mathbb{S}^3|} \int_{\mathbb{R}^4\backslash B} \ln\Bigl(\frac{|x-y|}{|y|}\Bigr) e^{u(y)}\, dy, \tag{A-2}$$

$$\bar{w}(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} w(y)\, d\sigma \text{ for } r > 1. \tag{A-3}$$

Lemma A.1. *Let u be a solution of* (1-2) *and $\bar{v}(r)$ is defined in* (A-3). *Then*

$$\frac{w(x)}{\ln|x|} \to \alpha_0 \quad \textit{as} \quad |x| \to \infty, \tag{A-4}$$

$$\frac{\bar{w}(r)}{\ln r} \to \alpha_0 \quad \textit{as} \quad r \to \infty. \tag{A-5}$$

Proof. We only need to show (A-4). We first show that

$$w(x) \le \alpha_0 \ln|x| + C, \text{ where } C \text{ is a positive constant.} \tag{A-6}$$

For $|x| \ge 4$, we split $\mathbb{R}^4 \setminus B = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \bigl\{y \in \mathbb{R}^4\backslash B : |y - x| \le \frac{1}{2}|x|\bigr\}$, $\Omega_2 = \bigl\{y \in \mathbb{R}^4\backslash B : |y - x| > \frac{1}{2}|x|\bigr\}$. For $y \in \Omega_1$, we have

$$|y| \ge |x| - |x - y| \ge \tfrac{1}{2}|x| \ge |x - y|.$$

Then $\ln(|x-y|/|y|) \le 0$. Note that $|x-y| \le |x|+|y| \le |x||y|$ for $|x|, |y| \ge 2$. Since $\frac{3}{2} \ge |x-y|/|x| \ge \frac{1}{2}$ for $|x| \ge 4$, $|y| \le 2$. Thus, we find

$$\ln|x-y| \le \ln|x| + C.$$

Then

$$\begin{aligned} w(x) &\le \frac{1}{4|\mathbb{S}^3|}\int_{\Omega_2} \ln\Big(\frac{|x-y|}{|y|}\Big) e^{u(y)}\,dy \\ &\le \frac{1}{4|\mathbb{S}^3|}\ln|x|\int_{\Omega_2} e^{u(y)}\,dy + \frac{1}{4|\mathbb{S}^3|}\int_{\Omega_2\cap\{|y|\le 2\}} (C-\ln|y|)e^{u(y)}\,dy \\ &\le \alpha_0 \ln|x| + C. \end{aligned}$$

Next, we claim that for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that for $|x| > R$,

$$w(x) \ge \big(\alpha_0 - \tfrac{1}{2}\varepsilon\big)\ln|x| + \frac{1}{4|\mathbb{S}^3|}\int_{B_1(x)} \ln(|x-y|)e^{u(y)}\,dy. \tag{A-7}$$

We decompose $\mathbb{R}^4\backslash B$ into A_1, A_2 and A_3, where $A_1 = \{y : 1 < |y| \le R_0\}$, $A_2 = \big\{y : |y-x| \le \frac{1}{2}|x|, |y| \ge R_0\big\}$, $A_3 = \{y : |y-x| > \frac{1}{2}|x|, |y| \ge R_0\}$. For any $\varepsilon > 0$, choosing R_0 large, and taking $|x|$ sufficiently large, we have

$$\begin{aligned} &\frac{1}{4|\mathbb{S}^3|}\int_{A_1} \ln\Big(\frac{|x-y|}{|y|}\Big)e^{u(y)}\,dy - \frac{1}{4|\mathbb{S}^3|}\ln|x|\int_{\mathbb{R}^4\backslash B} e^{u(y)}\,dy \\ &\quad \ge \frac{1}{4|\mathbb{S}^3|}\ln|x|\int_{A_1} \ln\Big(\frac{|x-y|}{|x||y|}\Big)e^{u(y)}\,dy - \frac{1}{4|\mathbb{S}^3|}\ln|x|\int_{A_1^c} e^{u(y)}\,dy \ge -\tfrac{1}{4}\varepsilon\ln|x|. \end{aligned}$$

Then

$$\frac{1}{4|\mathbb{S}^3|}\int_{A_1} \ln\Big(\frac{|x-y|}{|y|}\Big)e^{u(y)}\,dy \ge \big(\alpha_0 - \tfrac{1}{4}\varepsilon\big)\ln|x|.$$

Since $|y| < 2|x|$ in A_2, we find

$$\begin{aligned} &\frac{1}{4|\mathbb{S}^3|}\int_{A_2} \ln\Big(\frac{|x-y|}{|y|}\Big)e^{u(y)}\,dy \\ &\qquad\qquad \ge \frac{1}{4|\mathbb{S}^3|}\int_{B_1(x)} \ln(|x-y|)e^{u(y)}\,dy - \frac{1}{4|\mathbb{S}^3|}\ln(2|x|)\int_{A_2} e^{u(y)}\,dy. \end{aligned}$$

For A_3, if $|y| \le 2|x|$, $|x-y| \ge \frac{1}{2}|x| \ge \frac{1}{4}|y|$. If $|y| \ge 2|x|$, $|x-y| \ge |y|-|x| \ge \frac{1}{2}|y|$. Then $|x-y|/|y| \ge \frac{1}{4}$. Hence

$$\frac{1}{4|\mathbb{S}^3|}\int_{A_3} \ln\Big(\frac{|x-y|}{|y|}\Big)e^{u(y)}\,dy \ge -\frac{1}{4|\mathbb{S}^3|}\ln 4\int_{A_3} e^{u(y)}\,dy.$$

Therefore, our claim follows.

Let δ_0 small, R_0 sufficiently large such that

$$\int_{B_4(x)} e^{u(y)}\,dy \le \delta_0, \quad |x| \ge R_0. \tag{A-8}$$

Set h be the solution of

$$\Delta^2 h = e^u \text{ in } B_4(x), \quad h = \Delta h = 0 \text{ on } \partial B_4(x).$$

By [Lin 1998, Lemma 2.3], we find, for small $\delta_0 > 0$,

$$\int_{B_4(x)} e^{2|h(y)|}\, dy \le \sigma, \tag{A-9}$$

where $\sigma > 0$ is independent of x.

Let $\varphi = u - h$. Then

$$\Delta^2 \varphi = 0 \text{ in } B_4(x), \quad \Delta\varphi = \Delta u, \varphi = u \text{ on } \partial B_4(x).$$

Setting $\phi(y) = -\Delta\varphi(y)$, then

$$\Delta\phi = 0 \text{ in } B_4(x), \quad \phi = -\Delta u \text{ on } \partial B_4(x).$$

By Lemma 2.2, we see that Δu is bounded. Thus

$$|\phi(y)| \le c_0, \quad y \in \overline{B_2(x)}.$$

Note that

$$\Delta\varphi = -\phi \text{ in } B_4(x), \quad \varphi = u \text{ on } \partial B_4(x).$$

By the elliptic estimates, we have

$$\sup_{B_1(x)} \varphi \le C(\|\varphi^+\|_{L^1(B_2(x))} + \|\phi\|_{L^q(B_2(x))}), \quad q > 2.$$

Since $\varphi = u - h$, then $\varphi^+ \le u^+ + |h|$. Thus

$$\int_{B_2(x)} \varphi^+\, dy \le C \int_{B_2(x)} e^{\varphi^+/2}\, dy \le C\left(\int_{B_2(x)} e^{u^+(y)}\, dy\right)^{1/2}\left(\int_{B_2(x)} e^{|h(y)|}\, dy\right)^{1/2}.$$

Note $e^{u^+} \le 1 + e^u$, we find that $\sup_{B_1(x)} \varphi(x) \le C$, $u \le C + |h(y)|$, $y \in B_1(x)$. Then

$$\int_{B_1(x)} e^{2u(y)}\, dy \le C \int_{B_1(x)} e^{2|h(y)|}\, dy \le C.$$

Thus

$$\left|\int_{B_1(x)} \ln(|x-y|) e^{u(y)}\, dy\right| \le \left(\int_{B_1(x)} \ln^2|x-y|\, dy\right)^{1/2}\left(\int_{B_1(x)} e^{2u(y)}\, dy\right)^{1/2} \le C.$$

By (A-7), we deduce, for $|x|$ large enough,

$$w(x) \ge (\alpha_0 - \varepsilon) \ln|x|. \tag{A-10}$$

In view of (A-6),(A-10), we can obtain (A-4). □

Acknowledgements

The authors are very grateful to the referee for many valuable suggestions and comments, which greatly improved this paper.

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Received April 30, 2020. Revised August 26, 2020.

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PACIFIC JOURNAL OF MATHEMATICS

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 309 No. 2 December 2020

0030-8730(202012)309:2;1-H