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LOCAL NORMAL FORMS FOR MULTIPLICITY FREE $U(n)$ ACTIONS ON COADJOINT ORBITS

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Actions of $U(n)$ on $U(n+1)$ coadjoint orbits via embeddings of $U(n)$ into $U(n+1)$ are an important family of examples of multiplicity free spaces. They are related to Gelfand–Zeitlin completely integrable systems and multiplicity free branching rules in representation theory. This paper computes the Hamiltonian local normal forms of all such actions, at arbitrary points, in arbitrary $U(n+1)$ coadjoint orbits. The results are described using combinatorics of interlacing patterns; gadgets that describe the associated Kirwan polytopes.

1. Introduction

A Hamiltonian action of a compact connected Lie group K on compact symplectic manifold (M, ω) with an equivariant moment map is a *multiplicity free space* if the ring of K-invariant functions $C^\infty(M)^K$ is a commutative Poisson subalgebra [Guillemin and Sternberg 1984a]. The moment map of a multiplicity free space identifies the orbit space, M/K, with a convex polytope called the *Kirwan polytope* [1984]. Compact multiplicity free spaces are classified by their Kirwan polytope and the principal isotropy subgroup of the action [Knop 2011]. The local classification of multiplicity free spaces (in a neighbourhood of an orbit) is a crucial step in the proof of the classification theorem for compact multiplicity free spaces. It is equivalent to the classification of smooth affine spherical varieties for $G = K^{\mathbb{C}}$. Smooth affine spherical varieties are classified by their weight monoids [Losev 2009].

One particularly concrete family of examples of multiplicity free spaces is provided by the action of a unitary group, $U(n)$, on a coadjoint orbit of the unitary group $U(n+1)$ via an embedding of $U(n)$ into $U(n+1)$ (Section 3A). The Kirwan polytopes of these spaces can be described as the set of points $(\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ that satisfy the so-called *interlacing inequalities*,

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_n \geq \lambda_{n+1}, \tag{1}$$

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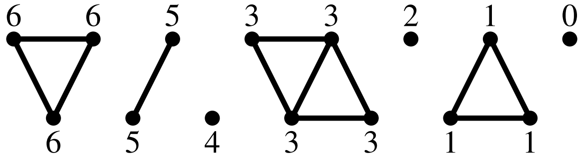
where $\lambda_1, \dots, \lambda_{n+1} \in \mathbb{R}$ are fixed parameters determined by the coadjoint orbit. The main result of this paper (Theorem 3.3) is the computation of the local classifying data of these spaces at arbitrary points in arbitrary $U(n+1)$ orbits. This result has two interesting features. First, the classifying data are described in terms of combinatorial gadgets called *interlacing patterns* that encode the combinatorics of the Kirwan polytope (see Section 3B). An example of an interlacing pattern is illustrated below. It corresponds to certain points in $U(8)$ coadjoint orbits diffeomorphic to $U(8)/U(2) \times U(1) \times U(2) \times U(1) \times U(1) \times U(1)$.



The second interesting feature is the proof (given in Section 4). Rather than using the classification of smooth affine spherical varieties, the classifying data are computed directly by elementary means. Following several standard reductions, the main step in this proof is the explicit computation of the isotropy representations (Section 4A). It is shown that they are certain products of standard representations and trivial representations of factors of the isotropy subgroup, which has a block diagonal form. The block diagonal factors of the isotropy subgroup that act by standard representations correspond to "parallelogram shapes" that appear in the interlacing pattern. For example, the isotropy subgroup corresponding to the interlacing pattern above is $U(1) \times U(1) \times 1 \times U(2) \times U(1)$ and the isotropy representation is $\{0\} \oplus \mathbb{C} \oplus \{0\} \oplus \mathbb{C}^2 \oplus \{0\}$ (see Example 5). The computation of this representation relies on the relationship between the combinatorics of interlacing patterns and divisibility properties of characteristic polynomials of certain Hermitian matrices.

Motivation for this work is provided by the *Gelfand–Zeitlin*[1] commutative completely integrable systems [Guillemin and Sternberg 1983]. Although Gelfand–Zeitlin systems have been studied extensively in recent years (see, e.g., [Alekseev et al. 2018; Bouloc et al. 2018; Cho et al. 2020; Lane 2020]), very little is known about their local normal forms as integrable systems near singular fibers (see Example 6). An ongoing program aims to use the results of this paper to prove topological and symplectic local normal forms for Gelfand–Zeitlin systems. The multiplicity free spaces studied in this paper, as well as the associated Gelfand–Zeitlin systems, have analogues for orthogonal groups and orthogonal coadjoint orbits. The local models of those multiplicity free spaces can be computed in a similar fashion.

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[1] Also spelled Gelfand–Cetlin and Gelfand–Tsetlin.

2. Hamiltonian group actions and local normal forms

This section fixes conventions, notation, and recalls the statement of the Marle–Guillemin–Sternberg local normal form. Standard references are [Audin 2004; Guillemin and Sternberg 1984c] modulo conventions.

2A. ***Hamiltonian group actions.*** Let K be a connected Lie group. Denote its Lie algebra by $\mathfrak{k}$, the dual vector space by $\mathfrak{k}^*$, and the dual pairing by $\langle\cdot,\cdot\rangle$. Let Ad and Ad* denote the adjoint and coadjoint actions respectively, i.e., $\langle \mathrm{Ad}_k^*\xi, X\rangle = \langle \xi, \mathrm{Ad}_{k^{-1}} X\rangle$ for $k \in K$, $\xi \in \mathfrak{k}^*$, and $X \in \mathfrak{k}$. Given a left action of K on a manifold M, the fundamental vector field of $X \in \mathfrak{k}$ is

$$\underline{X}_p = \frac{d}{dt}\bigg|_{t=0} \exp(tX)\cdot p, \qquad p \in M.$$

Let (M,ω) be a symplectic manifold. A left action of K on M is *Hamiltonian* if there exists an equivariant map $\Phi : M \to \mathfrak{k}^*$ such that

$$\iota_{\underline{X}}\omega = d\langle \Phi, X\rangle.$$

A map Φ with this property is called a *moment map*. The tuple (M,ω,Φ) is a *Hamiltonian K-manifold*. Hamiltonian K-manifolds (M,ω,Φ) and (M',ω',Φ') are *isomorphic* if there exists a K-equivariant, symplectic diffeomorphism $\varphi : (M,\omega) \to (M',\omega')$ such that $\Phi' \circ \varphi = \Phi$.

Example 1 (coadjoint orbits). Let $\mathcal{O} \subset \mathfrak{k}^*$ be an orbit of the coadjoint action of K. Given $\xi \in \mathcal{O}$, the tangent space $T_\xi\mathcal{O} \subset \mathfrak{k}^*$ is the set of elements of the form $\mathrm{ad}_X^*\xi$, $X \in \mathfrak{k}$. The *Kostant–Kirillov–Souriau* symplectic form ω_{KKS} on $\mathcal{O}$ is defined pointwise by the formula

$$(\omega_{\mathrm{KKS}})_\xi(\mathrm{ad}_X^*\xi, \mathrm{ad}_Y^*\xi) = \langle \xi, [X,Y]\rangle.$$

The inclusion map $\iota : \mathcal{O} \to \mathfrak{k}^*$ is a moment map for the coadjoint action of K on $(\mathcal{O}, \omega_{\mathrm{KKS}})$. △

Example 2 (homomorphisms). Let (M,ω,Φ) be a Hamiltonian K-manifold, H be a Lie group, and $\varphi : H \to K$ be a Lie group homomorphism. Let $(d\varphi)^* : \mathfrak{k}^* \to \mathfrak{h}^*$ denote the linear map dual to $d\varphi : \mathfrak{h} \to \mathfrak{k}$. Then the action of H on M defined via the action of K and the homomorphism φ is Hamiltonian and $(d\varphi)^* \circ \Phi$ is a moment map. △

Let $U(n)$ denote the group of $n \times n$ unitary matrices, with Lie algebra $\mathfrak{u}(n)$, and let $\mathcal{H}_n$ denote the set of $n \times n$ Hermitian matrices, $X = X^\dagger$, where $X \mapsto X^\dagger$ denotes conjugate transpose. Fix the isomorphism

$$\mathcal{H}_n \to \mathfrak{u}(n)^*, \quad X \mapsto \left(A \mapsto \frac{1}{\sqrt{-1}}\mathrm{Tr}(XA)\right). \tag{2}$$

It is equivariant with respect to the action of $U(n)$ on $\mathcal{H}_n$ by conjugation, $k\cdot X = kXk^\dagger$.

Example 3 (representations). Identify $\mathbb{C}^n \cong M_{n\times 1}(\mathbb{C})$. The standard symplectic form on $\mathbb{C}^n$ is

$$\omega_{\mathrm{std}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2\sqrt{-1}}(\boldsymbol{x}^\dagger \boldsymbol{y} - \boldsymbol{y}^\dagger \boldsymbol{x}), \quad \boldsymbol{x}, \boldsymbol{y} \in M_{n\times 1}(\mathbb{C}). \tag{3}$$

The action of $U(n)$ on $\mathbb{C}^n$ by the standard representation is Hamiltonian with moment map

$$\Phi(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{x}\boldsymbol{x}^\dagger. \tag{4}$$

More generally, suppose that V is a real vector space equipped with a linear symplectic form ω_V. Let $\rho : K \to Sp(V, \omega_V)$ be a representation of K on V by symplectic transformations. Then the action of K on (V, ω_V) defined by ρ is Hamiltonian with moment map Φ_V defined by the condition

$$\frac{1}{2}\omega_V(d\rho(X)\boldsymbol{v}, \boldsymbol{v}) = \langle \Phi_V(\boldsymbol{v}), X\rangle \quad \text{for all } \boldsymbol{v} \in V. \tag{5}$$

△

Example 4 (isotropy representations). Let (M, ω, Φ) be a Hamiltonian K-manifold. Given $p \in M$, let $K \cdot p$ denote the orbit of the action of K through p and let $K_p \leq K$ denote the *isotropy subgroup*; the subgroup of elements that fix p. Let $K_{\Phi(p)}$ denote the isotropy subgroup of $\Phi(p)$. Then $K_p \leq K_{\Phi(p)}$. The *symplectic slice at $p \in M$* is the vector space

$$W_p = T_p(K\cdot p)^\omega / (T_p(K\cdot p) \cap T_p(K\cdot p)^\omega),$$

where $T_p(K\cdot p)^\omega$ denotes the subspace of elements $X \in T_pM$ such that $\omega_p(X, Y) = 0$ for all $Y \in T_p(K\cdot p)$. The restriction of ω_p to $T_p(K\cdot p)^\omega$ descends to a symplectic form on W_p denoted $\overline{\omega}_p$. The linearization of the action of K_p, a.k.a. the *isotropy representation*, preserves the subspaces $T_p(K\cdot p)^\omega$ and $T_p(K\cdot p) \cap T_p(K\cdot p)^\omega$, so it descends to an action of K_p on $(W_p, \overline{\omega}_p)$ by symplectic transformations. Thus $(W_p, \overline{\omega}_p, \Phi_W)$ is a Hamiltonian K_p-manifold, where Φ_W is defined as in Example 3. △

2B. ***Marle–Guillemin–Sternberg local normal forms.*** Given a connected Lie group K, *Marle–Guillemin–Sternberg data* (MGS data) is a tuple (ξ, L, W, ω_W) where $\xi \in \mathfrak{k}^*$, L is a Lie subgroup of K_ξ, and (W, ω_W) is a symplectic vector space equipped with a representation of L by symplectic transformations.

Given MGS data (ξ, L, W, ω_W), [Guillemin and Sternberg 1984c; Marle 1985] construct a Hamiltonian K-manifold, denoted $M(\xi, L, W, \omega_W)$, with the following properties. Let $\mathfrak{m} = \mathfrak{k}_\xi/\mathfrak{l}$ and identify $\mathfrak{m}^*$ with a L-invariant complement of $\mathfrak{l}^*$ in $\mathfrak{k}_\xi^*$.

As a manifold, $M(\xi, L, W, \omega_W)$ is the total space of the vector bundle

$$K \times_L (\mathfrak{m}^* \times W) \to K/L \tag{6}$$

associated to the principal bundle $L \to K \to K/L$ and the representation $\mathfrak{m}^* \times W$. The symplectic structure on $M(\xi, L, W, \omega_W)$ is determined by the data (ξ, L, W, ω_W) (see [Guillemin and Sternberg 1984b; 1984c; Marle 1985] for more details). With respect to this diffeomorphic description of $M(\xi, L, W, \omega_W)$, the Hamiltonian action of K and the corresponding moment map are

$$k' \cdot [k, \eta, w] = [k'k, \eta, w], \qquad \Phi([k, \eta, w]) = \mathrm{Ad}^*_k(\eta + \Phi_W(w) + \xi). \tag{7}$$

Let (M, ω, Φ) be a Hamiltonian K-manifold. The *Marle–Guillemin–Sternberg data of a point* $p \in M$ is $(\Phi(p), K_p, W_p, \overline{\omega}_p)$, where K_p is the isotropy subgroup of p and $(W_p, \overline{\omega}_p)$ is the symplectic slice at p equipped with the isotropy representation of K_p as described in Example 4.

Theorem 2.1 (Marle–Guillemin–Sternberg local normal forms [Guillemin and Sternberg 1984b; Marle 1985]). *Let (M, ω, Φ) be a Hamiltonian K-manifold. For all $p \in M$ there exists K-invariant neighbourhoods $U \subset M$ of the orbit $K \cdot p$ and $U' \subset M(\Phi(p), K_p, W_p, \overline{\omega}_p)$ of the orbit $K \cdot [e, 0, 0]$ and an isomorphism of Hamiltonian K-manifolds $\varphi : U \to U'$ such that $\varphi(p) = [e, 0, 0]$.*

Hamiltonian K-manifolds (M, ω, Φ) and (M', ω', Φ') are *equivalent* if there exists an automorphism ψ of K, a symplectomorphism $F : (M, \omega) \to (M', \omega')$, and an Ad^*_K-fixed element $\xi \in \mathfrak{k}^*$ such that

(1) $\psi(k) \cdot F(m) = F(k \cdot m)$, and

(2) $\Phi + \xi = (d\psi)^* \circ \Phi' \circ F$.

Marle–Guillemin–Sternberg data (ξ, L, W, ω_W) and $(\xi', L', W', \omega_{W'})$ for K are *equivalent* if the corresponding model spaces are equivalent as Hamiltonian K-manifolds. For instance, if p and p' are in the same K-orbit, then the MGS data of p and p' are equivalent.

3. Statement of the main theorem

The following notation will be useful in the remainder of the paper. Given a sequence of real numbers $\underline{\tau} = (\tau_1, \dots, \tau_n)$, let $[\underline{\tau}]$ denote the set of elements in $\underline{\tau}$. Let $\underline{\tau}_i$ denote the i-th element of $[\underline{\tau}]$ in decreasing order. Let $m(\underline{\tau})$ denote the size of $[\underline{\tau}]$. Let $n_\tau(\underline{\tau})$ denote the number of times τ occurs in $\underline{\tau}$. Let $n_i(\underline{\tau})$ denote the number of times $\underline{\tau}_i$ occurs in $\underline{\tau}$.

3A. ***Multiplicity free $U(n)$ actions on $U(n+1)$ coadjoint orbits.*** Given a non-increasing sequence of real numbers $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n+1})$, let $\mathcal{O}_\Lambda$ denote the set of matrices in $\mathcal{H}_{n+1}$ with eigenvalues $\lambda_1, \dots, \lambda_{n+1}$. Then $\mathcal{O}_\Lambda$ is the orbit of

$$\Lambda := \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n+1} \end{pmatrix} \tag{8}$$

under the action of $U(n+1)$ by conjugation and the map $k \mapsto k\lambda k^\dagger$ descends to a $U(n+1)$-equivariant diffeomorphism

$$U(n+1)/U(n_1(\underline{\lambda})) \times \cdots \times U(n_{m(\underline{\lambda})}(\underline{\lambda})) \to \mathcal{O}_\Lambda. \tag{9}$$

The map (2) defines a $U(n)$-equivariant diffeomorphism of $\mathcal{O}_\Lambda$ with a coadjoint orbit of $U(n+1)$. Let ω_Λ denote the symplectic form on $\mathcal{O}_\Lambda$ defined by this identification and the Kostant–Kirillov–Souriau symplectic form defined in Example 1. For all $p \in \mathcal{O}_\Lambda$,

$$(\omega_\Lambda)_p([X, p], [Y, p]) = \frac{1}{\sqrt{-1}} \operatorname{Tr}(p[X, Y]) \qquad \text{for all } X, Y \in \mathfrak{u}(n+1). \tag{10}$$

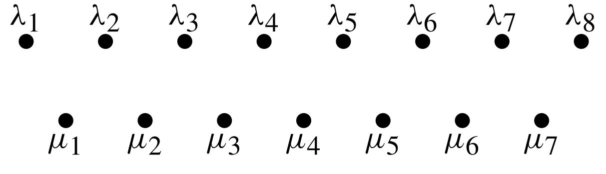
With respect to (2), $(\mathcal{O}_\Lambda, \omega_\Lambda, \iota : \mathcal{O}_\Lambda \to \mathcal{H}_{n+1})$ is a Hamiltonian $U(n+1)$-manifold, where ι denotes inclusion. Let $K = U(n)$ and let $\varphi : K \to U(n+1)$ be an embedding of K as a Lie subgroup of $U(n+1)$. With respect to the identification (2), $(d\varphi)^*$ is a linear projection $\mathcal{H}_{n+1} \to \mathcal{H}_n$. By Example 2, $(\mathcal{O}_\Lambda, \omega_\Lambda, \Phi)$ is a Hamiltonian K-manifold with moment map

$$\Phi = (d\varphi)^* \circ \iota : \mathcal{O}_\Lambda \to \mathcal{H}_n. \tag{11}$$

It is well-known that $(\mathcal{O}_\Lambda, \omega_\Lambda, \Phi)$ are multiplicity free spaces for all possible choices of $\underline{\lambda}$ and φ (this follows from Lemma 4.1 below).

3B. ***Interlacing patterns.*** Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n+1})$ and $\underline{\mu} = (\mu_1, \dots, \mu_n)$ be non-increasing sequences of numbers that satisfy the interlacing inequalities (1). The inequalities (1) are represented by attaching labels to a fixed set of $2n+1$ vertices arranged on a triangular grid as illustrated by the following example.

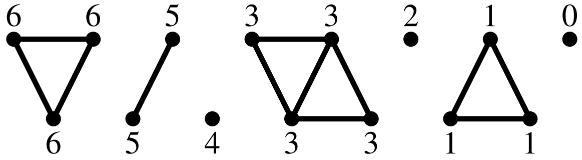
(12)



If a vertex labelled x appears to the left of a vertex labelled y, then $x \geq y$. The labels on the top row correspond to $\underline{\lambda}$ and the labels on the bottom row correspond to $\underline{\mu}$.

The (labelled) *interlacing pattern* of a pair of sequences $(\underline{\lambda}, \underline{\mu})$ that satisfy (1) is the labelled undirected plane graph obtained by adding straight edges to the diagram above according to the following rule: two vertices are connected by an edge if and only if they are nearest neighbours and their labels are equal. For example, the following is the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ where $\underline{\lambda} = (6, 6, 5, 3, 3, 2, 1, 0)$ and $\underline{\mu} = (6, 5, 4, 3, 3, 1, 1)$.

(13)



Three types of connected components can occur in interlacing patterns: $\triangledown$-*shapes*, $\triangle$-*shapes*, and ▱-*shapes*. In the example (13): the components labelled 6, 2, and 0 are $\triangledown$-shapes, the components labelled 4 and 1 are $\triangle$-shapes, and the components labelled 5 and 3 are ▱-shapes. By convention, an isolated vertex on the top row is a $\triangledown$-shape and an isolated vertex on the bottom row is a $\triangle$-shape.

If $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n+1})$ is fixed, then the set of pairs $(\underline{\lambda}, \underline{\mu})$ that satisfy (1) (equivalently, the set of labelled interlacing patterns whose labels on the top row are given by $\underline{\lambda}$) is in bijection with elements of the polytope

$$\Delta_{\underline{\lambda}} := \{\underline{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n \mid (\underline{\lambda}, \underline{\mu}) \text{ satisfies (1)}\}.$$

Given $(\mathcal{O}_\Lambda, \omega_\Lambda, \Phi)$ as in the previous section, a point $p \in \mathcal{O}_\Lambda$ determines a pair $(\underline{\lambda}, \underline{\mu})$ that satisfies (1), where $\underline{\mu} = (\mu_1, \dots, \mu_n)$ denotes the eigenvalues of $\Phi(p)$ arranged in nonincreasing order. Thus, every $p \in \mathcal{O}_\Lambda$ has an associated labelled interlacing pattern. As observed in [Guillemin and Sternberg 1983], the polytope $\Delta_{\underline{\lambda}}$ defined above is the Kirwan polytope of $(\mathcal{O}_\Lambda, \omega_\Lambda, \Phi)$, i.e.,

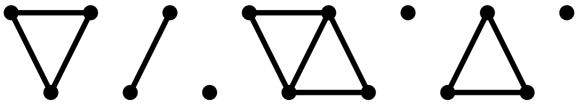
$$\Delta_{\underline{\lambda}} = \{(\mu_1, \dots, \mu_n) \in \mathbb{R}^n \mid \mu_1 \geq \cdots \geq \mu_n,\ \exists p \in \mathcal{O}_\Lambda \text{ with eigenvalues } \mu_1, \dots, \mu_n\}.$$

The notation $\substack{\lambda\in[\underline{\lambda}] \\ \triangledown\text{-shape}}$ denotes the set of all $\lambda \in [\underline{\lambda}]$ such that the connected component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled by λ is a $\triangledown$-shape. Similar notation is used for other sets. For example, any pair $(\underline{\lambda}, \underline{\mu})$ satisfying (1) satisfies the identity

$$\sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^{n} \mu_i = \sum_{\substack{\lambda\in[\underline{\lambda}] \\ \triangledown\text{-shape}}} \lambda - \sum_{\substack{\mu\in[\underline{\mu}] \\ \triangle\text{-shape}}} \mu. \tag{14}$$

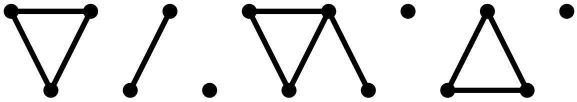
Remark 3.1. An *unlabelled interlacing pattern* is an undirected plane graph that can be obtained from a labelled interlacing pattern by erasing the labels. In other words, the edges in an unlabelled interlacing pattern must correspond to a configuration of equalities and strict inequalities that is allowed by (1). For instance, the following is an unlabelled interlacing pattern.

(15)



On the other hand, the following is not an unlabelled interlacing pattern.

(16)



If $\underline{\mu}$ and $\underline{\mu}'$ are contained in the relative interior of the same face of $\Delta_{\underline{\lambda}}$, then the unlabelled interlacing patterns of $(\underline{\lambda}, \underline{\mu})$ and $(\underline{\lambda}, \underline{\mu}')$ are the same. Thus the set of unlabelled interlacing patterns obtained by erasing labels from labelled interlacing patterns of pairs $(\underline{\lambda}, \underline{\mu})$, $\underline{\lambda}$ fixed, is in natural bijection with the set of faces of $\Delta_{\underline{\lambda}}$. The partial order on faces of $\Delta_{\underline{\lambda}}$ corresponds to an obvious partial order on the set of all such unlabelled interlacing patterns. Thus, they encode $\Delta_{\underline{\lambda}}$ as an abstract polytope. It is also straightforward to read the local moment cone of a point $\underline{\mu} \in \Delta_{\underline{\lambda}}$ from the unlabelled interlacing pattern of $(\underline{\lambda}, \underline{\mu})$. The intersection of this local moment cone with the standard lattice in $\mathbb{R}^n$ is the weight monoid of the corresponding smooth affine spherical variety that appears in the classification of [Knop 2011].

Remark 3.2. The interlacing patterns described here occur as rows in larger diagrams, also called interlacing patterns, that describe points and faces of Gelfand–Zeitlin polytopes as well as fibers of Gelfand–Zeitlin systems (see, e.g., [An et al. 2018; Cho et al. 2020; Pabiniak 2014; Bouloc et al. 2018]). Some authors use an equivalent combinatorial gadget called *ladder diagrams* and introduce terminology such as W-blocks, M-blocks, and N-blocks that is equivalent to the notions of *▽-shapes*, *△-shapes*, and *▱-shapes* used here.

3C. ***Statement of the main theorem.*** Let $K = U(n)$ and let $(\underline{\lambda}, \underline{\mu})$ be a pair of nonincreasing sequences $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n+1})$ and $\underline{\mu} = (\mu_1, \dots, \mu_n)$ that satisfy the interlacing inequalities (1). Let $\mathrm{M} := \operatorname{diag}(\mu_1, \dots, \mu_n)$. The stabilizer subgroup K_{M} for the conjugation action of K is a block diagonal subgroup isomorphic to $U(n_1(\underline{\mu})) \times \cdots \times U(n_{m(\underline{\mu})}(\underline{\mu}))$. Define

$$W_{(\underline{\lambda},\underline{\mu})} := \bigoplus_{\substack{\mu \in [\underline{\mu}] \\ ▱\text{-shape}}} \mathbb{C}^{n_\mu(\underline{\mu})}, \tag{17}$$

and the block-diagonal subgroup

$$L_{(\underline{\lambda},\underline{\mu})} := L_1 \times \cdots \times L_{m(\underline{\mu})} \leq U(n_1(\underline{\mu})) \times \cdots \times U(n_{m(\underline{\mu})}(\underline{\mu})) = K_{\mathrm{M}}, \tag{18}$$

where

$$L_i = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & k \end{array} \right) \mid k \in U(n_i(\underline{\mu}) - 1) \right\} \leq U(n_i(\underline{\mu})) \tag{19}$$

if the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\underline{\mu}_i$ is a ⏢-shape, and $L_i = U(n_i(\underline{\mu}))$ otherwise. Equip $W_{(\underline{\lambda},\underline{\mu})}$ with the representation of $L_{(\underline{\lambda},\underline{\mu})}$ where the factor L_i acts by the standard representation on the corresponding factor $\mathbb{C}^{n_i(\underline{\mu})}$ if the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\underline{\mu}_i$ is a ▱-shape, and it acts trivially otherwise.

Example 5. Consider the interlacing pattern in equation (13). It follows that $\mathrm{M} = \mathrm{diag}(6, 5, 4, 3, 3, 1, 1)$,

$$L_{(\underline{\lambda},\underline{\mu})} = \left\{ \left(\begin{array}{c|c|c|c|c} k_6 & & & & \\ \hline & k_5 & & & \\ \hline & & 1 & & \\ \hline & & & k_3 & \\ \hline & & & & \begin{matrix} 1 & \\ & k_1 \end{matrix} \end{array} \right) \;\middle|\; k_6, k_5, k_1 \in U(1),\ k_3 \in U(2) \right\} \tag{20}$$

$$W_{(\underline{\lambda},\underline{\mu})} = \{0\} \oplus \mathbb{C} \oplus \{0\} \oplus \mathbb{C}^2 \oplus \{0\}.$$

The representation of $L_{(\underline{\lambda},\underline{\mu})}$ on $W_{(\underline{\lambda},\underline{\mu})}$ is $(k_6, k_5, k_3, k_1) \cdot (z_5, z_3) = (k_5 z_5, k_3 z_3)$. △

For $\mu \in \underline{\mu}$, define $r_\mu \geq 0$ such that

$$r_\mu^2 = -\Bigg(\prod_{\substack{\lambda \in [\underline{\lambda}] \\ \text{▽-shape}}} (\mu - \lambda) \Bigg) \Bigg(\prod_{\substack{\tau \in [\underline{\mu}] \\ \text{⏢-shape} \\ \tau \neq \mu}} \frac{1}{(\mu - \tau)} \Bigg) \tag{21}$$

if the connected component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled μ is a ⏢-shape, and $r_\mu = 0$ otherwise. If the connected component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled μ is a ⏢-shape, then $r_\mu^2 > 0$.

Provided that the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\mu = \underline{\mu}_i$ is not a ⏢-shape, define

$$C_i := C_\mu := \sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^{n} \mu_i - \mu + \sum_{\substack{\tau \in [\underline{\mu}] \\ \text{⏢-shape}}} \frac{r_\tau^2}{\mu - \tau}. \tag{22}$$

Finally, define a linear symplectic form on $W_{(\underline{\lambda},\underline{\mu})}$ by the formula

$$\omega_{(\underline{\lambda},\underline{\mu})}(\boldsymbol{u},\boldsymbol{w}) := \frac{1}{\sqrt{-1}} \sum_{\substack{\mu\in[\underline{\mu}] \\ \text{▱-shape}}} \frac{-\boldsymbol{u}_\mu^\dagger \boldsymbol{w}_\mu + \boldsymbol{w}_\mu^\dagger \boldsymbol{u}_\mu}{C_\mu}, \tag{23}$$

for all $\boldsymbol{u}, \boldsymbol{w} \in W_{(\underline{\lambda},\underline{\mu})}$, where $\boldsymbol{u}_\mu$ denotes the projection of $\boldsymbol{u}$ to the factor $\mathbb{C}^{n_\mu(\underline{\mu})}$.

Theorem 3.3. *Let* $K = U(n)$ *and let* $(\mathcal{O}_\Lambda, \omega_\Lambda, \Phi)$ *be the Hamiltonian* K*-manifold associated to a nonincreasing sequence* $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n+1})$ *and an embedding* $\varphi : K \to U(n+1)$ *as in Section 3A. Then, the Marle–Guillemin–Sternberg local normal form data of* $p \in \mathcal{O}_\Lambda$ *is equivalent to*

$$(\mathrm{M}, L_{(\underline{\lambda},\underline{\mu})}, W_{(\underline{\lambda},\underline{\mu})}, \omega_{(\underline{\lambda},\underline{\mu})}), \tag{24}$$

where $(\underline{\lambda}, \underline{\mu})$ *is determined by* p *as in Section 3B and* M, $L_{(\underline{\lambda},\underline{\mu})}$, $W_{(\underline{\lambda},\underline{\mu})}$, *and* $\omega_{(\underline{\lambda},\underline{\mu})}$ *are as defined above.*

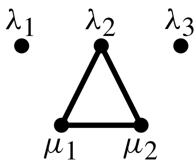
The proof of Theorem 3.3, given in Section 4, describes an explicit linear isomorphism between the isotropy representation at p and the symplectic representation $(W_{(\underline{\lambda},\underline{\mu})}, \omega_{(\underline{\lambda},\underline{\mu})})$.

Remark 3.4. It is straightforward to check that as $L_{(\underline{\lambda},\underline{\mu})}$-representations,

$$\mathfrak{m}^* \cong \bigoplus_{\substack{\mu\in[\underline{\mu}] \\ \text{⏢-shape}}} (\mathbb{R} \times \mathbb{C}^{n_\mu(\underline{\mu})-1}), \tag{25}$$

where if the component of the interlacing pattern labelled $\underline{\mu}_i$ is a ⏢-shape, then the factor $L_i \cong U(n_i(\underline{\mu}) - 1)$ acts on the corresponding factor $\mathbb{R} \times \mathbb{C}^{n_i(\underline{\mu})-1}$ as the product of the trivial representation and the standard representation. Otherwise the factor L_i acts trivially. The moment map of the local normal form $M(\mathrm{M}, L_{(\underline{\lambda},\underline{\mu})}, W_{(\underline{\lambda},\underline{\mu})}, \omega_{(\underline{\lambda},\underline{\mu})})$ is easily computed by combining Example 3 and (7).

Example 6. Let $\lambda_1 > \lambda_2 > \lambda_3$ and let $p \in \mathcal{O}_\Lambda$ such that the eigenvalues of $\Phi(p)$ are $\mu_1 = \mu_2 = \lambda_2$. The interlacing pattern of p is



It follows from Theorem 3.3 that the orbit through p is a Lagrangian $U(2)/U(1) \cong S^3$ and a neighbourhood of this orbit is isomorphic to a neighbourhood of the zero section in T^*S^3, equipped with the Hamiltonian action of $U(2)$ by cotangent lift of the action of $U(2)$ on S^3. This particular example was derived by Alamiddine

[2009], who used it to show that the Gelfand–Zeitlin systems on regular $U(3)$ coadjoint orbits are isomorphic, in a neighbourhood of this Lagrangian S^3 fiber, to an integrable system for the normalized geodesic flow on T^*S^3 for the round metric on S^3. △

4. Proof of Theorem 3.3

Let $K = U(n)$ and fix an arbitrary nonincreasing sequence $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n+1})$. Several standard reductions are in order.

First, any two embeddings $K \to U(n+1)$ endow $\mathcal{O}_\Lambda$ with equivalent Hamiltonian K-manifold structures: the restricted coadjoint actions differ by the coadjoint action of an element $g \in U(n+1)$. Thus, it is sufficient to compute the MGS data with respect to the embedding

$$\varphi : K \to U(n+1), \quad k \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & k \end{array}\right). \tag{26}$$

With respect to (2),

$$(d\varphi)^* : \mathcal{H}_{n+1} \to \mathcal{H}_n, \quad (d\varphi)^*(X) = X^{(n)}, \tag{27}$$

where $X^{(n)}$ is the bottom right principal $n \times n$ submatrix of X. Thus $\Phi(X) = X^{(n)}$.

Second, it is sufficient to compute the MGS data for points of the form

$$p = \left(\begin{array}{c|c} c & \mathbf{z}^\dagger \\ \hline \mathbf{z} & \mathrm{M} \end{array}\right) = \left(\begin{array}{c|ccccc} c & \bar{z}_1 & \bar{z}_2 & \cdots & \bar{z}_{n-1} & \bar{z}_n \\ \hline z_1 & \mu_1 & & & & \\ z_2 & & \mu_2 & & & \\ \vdots & & & \ddots & & \\ z_{n-1} & & & & \mu_{n-1} & \\ z_n & & & & & \mu_n \end{array}\right), \tag{28}$$

$$z_i \in \mathbb{C} \quad \text{and} \quad c = \sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^{n} \mu_i,$$

where $\mu_1 \geq \cdots \geq \mu_n$. Indeed, every point in $\mathcal{O}_\Lambda$ can be brought to this form by the action $U(n)$, so its MGS data is equivalent to the MGS data of a point of this form. Note that $p \in \Phi^{-1}(\mathrm{M})$ if and only if p is of the form (28).

Before giving the final reduction, recall from [Guillemin and Sternberg 1983] that the condition $p \in \mathcal{O}_\Lambda$, for p of the form (28), is equivalent to the following equality of characteristic polynomials,

$$\prod_{i=1}^{n+1} (x - \lambda_i) = (x - c) \prod_{i=1}^{n} (x - \mu_i) - \sum_{i=1}^{n} |z_i|^2 \prod_{\substack{j=1 \\ i \neq j}}^{n} (x - \mu_j). \tag{29}$$

Rewrite p in block form

$$p = \left(\begin{array}{c|c|c|c|c} c & z_1^\dagger & z_2^\dagger & \cdots & z_m^\dagger \\ \hline z_1 & \underline{\mu}_1 I_{n_1(\underline{\mu})} & & & \\ \hline z_2 & & \underline{\mu}_2 I_{n_2(\underline{\mu})} & & \\ \hline \vdots & & & \ddots & \\ \hline z_m & & & & \underline{\mu}_m I_{n_m(\underline{\mu})} \end{array}\right), \qquad z_i \in M_{n_i(\underline{\mu})\times 1}(\mathbb{C}). \tag{30}$$

where $m = m(\underline{\mu})$. If $\mu = \underline{\mu}_i$, let $z_\mu = z_i$ denote the corresponding block. Then (29) becomes

$$\begin{aligned} &\prod_{\lambda\in[\underline{\lambda}]} (x-\lambda)^{n_\lambda(\underline{\lambda})} \\ &\quad = (x-c)\prod_{\mu\in[\underline{\mu}]} (x-\mu)^{n_\mu(\underline{\mu})} - \sum_{\mu\in[\underline{\mu}]} \|z_\mu\|^2 (x-\mu)^{n_\mu(\underline{\mu})-1} \prod_{\substack{\tau\in[\underline{\mu}]\\ \tau\neq\mu}} (x-\tau)^{n_\tau(\underline{\tau})}. \end{aligned} \tag{31}$$

The following lemma is well-known. Its proof is left as an exercise using the fact that $p \in \mathcal{O}_\Lambda$ if and only if p satisfies (31).

Lemma 4.1. *Let p be of the form* (30). *Then $p \in \mathcal{O}_\Lambda$ if and only if for all $\mu \in \underline{\mu}$, $\|z_\mu\|^2 = r_\mu^2$. Moreover, the action of K_{M} on $\Phi^{-1}(\mathrm{M})$ is transitive.*

The final reduction concerns the isotropy subgroup. Given $(\underline{\lambda}, \underline{\mu})$, define $\tilde{p} \in \mathcal{O}_\Lambda$ of the form (30) such that for all $\mu \in [\underline{\mu}]$,

$$z_\mu = \begin{pmatrix} r_\mu \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{32}$$

By construction, $K_{\tilde{p}} = L_{(\underline{\lambda},\underline{\mu})}$. The MGS data of every other point $p \in \Phi^{-1}(\mathrm{M})$ is equivalent to that of $\tilde{p}$ by Lemma 4.1.

Remark 4.2. Many of the facts mentioned in this section are also useful for studying Gelfand–Zeitlin systems [Guillemin and Sternberg 1983; Cho et al. 2020].

4A. ***The isotropy representation.*** Continuing from the previous section, this section computes the isotropy representations at the points $\tilde{p} \in \Phi^{-1}(\mathrm{M})$ as described in (30), (32) and Lemma 4.1.

Lemma 4.3. *Let $p \in \Phi^{-1}(\mathrm{M})$ and let c, z be defined as in* (30)*. The subspace $T_p(K\cdot p)^\omega$ consists of all matrices of the form*

$$\left(\begin{array}{c|c} 0 & (c-\mathrm{M})\boldsymbol{x}^\dagger + z^\dagger X^\dagger \\ \hline (c-\mathrm{M})\boldsymbol{x} + Xz & 0 \end{array}\right), \qquad X \in \mathfrak{k},\ \boldsymbol{x} \in M_{n\times 1}(\mathbb{C}) \tag{33}$$

such that

$$\begin{aligned} 0 &= \boldsymbol{x}^{\dagger}\boldsymbol{z} + \boldsymbol{z}^{\dagger}\boldsymbol{x}, \\ 0 &= \boldsymbol{x}\boldsymbol{z}^{\dagger} + \boldsymbol{z}\boldsymbol{x}^{\dagger} + [X, \mathrm{M}]. \end{aligned} \tag{34}$$

The subspace $T_p(K \cdot p) \cap T_p(K \cdot p)^{\omega}$ *consists of all matrices of the form*

$$\left(\begin{array}{c|c} 0 & \boldsymbol{z}^{\dagger}Y^{\dagger} \\ \hline Y\boldsymbol{z} & 0 \end{array}\right), \quad Y \in \mathfrak{k}_{\mathrm{M}}. \tag{35}$$

Proof. Denote

$$\eta := \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & Y \end{array}\right), \quad \xi := \left(\begin{array}{c|c} x_0 & -\boldsymbol{x}^{\dagger} \\ \hline \boldsymbol{x} & X \end{array}\right), \quad X, Y \in \mathfrak{k},\ x_0 \in \sqrt{-1}\mathbb{R},\ \boldsymbol{x} \in M_{n\times 1}(\mathbb{C}).$$

The tangent space $T_p\mathcal{O}_{\Lambda}$ consists of elements of the form $[\xi, p]$. Since diagonal elements of $\mathfrak{u}(n+1)$ act trivially, set $x_0 = 0$. Then elements of $T_p\mathcal{O}_{\Lambda}$ have block form

$$[\xi, p] = \left(\begin{array}{c|c} -\boldsymbol{x}^{\dagger}\boldsymbol{z} - \boldsymbol{z}^{\dagger}\boldsymbol{x} & (c-\mathrm{M})\boldsymbol{x}^{\dagger} + \boldsymbol{z}^{\dagger}X^{\dagger} \\ \hline (c-\mathrm{M})\boldsymbol{x} + X\boldsymbol{z} & \boldsymbol{x}\boldsymbol{z}^{\dagger} + \boldsymbol{z}\boldsymbol{x}^{\dagger} + [X, \mathrm{M}] \end{array}\right), \quad X \in \mathfrak{k},\ \boldsymbol{x} \in M_{n\times 1}(\mathbb{C}).$$

Elements of $T_p(K \cdot p)$ have block form

$$[\eta, p] = \left(\begin{array}{c|c} 0 & \boldsymbol{z}^{\dagger}Y^{\dagger} \\ \hline Y\boldsymbol{z} & [Y, \mathrm{M}] \end{array}\right), \quad Y \in \mathfrak{k}.$$

Recall,

$$T_p(K \cdot p)^{\omega} = \bigl\{[\xi, p] \in T_p\mathcal{O}_{\Lambda} \mid (\omega_{\Lambda})_p([\xi, p], [\eta, p]) = 0 \text{ for all } Y \in \mathfrak{k}\bigr\}.$$

By (10),

$$\begin{aligned} \sqrt{-1}(\omega_{\Lambda})_p([\xi, p], [\eta, p]) &= \operatorname{Tr}(p[\xi, \eta]) \\ &= -\operatorname{Tr}(\boldsymbol{z}^{\dagger}Y\boldsymbol{x}) - \operatorname{Tr}(\boldsymbol{z}\boldsymbol{x}^{\dagger}Y) + \operatorname{Tr}(\mathrm{M}[X, Y]) \\ &= \operatorname{Tr}(([\mathrm{M}, X] - \boldsymbol{x}\boldsymbol{z}^{\dagger} - \boldsymbol{z}\boldsymbol{x}^{\dagger})Y). \end{aligned}$$

Let $\sqrt{-1}E_{i,i}$, $E_{i,j} - E_{j,i}$, and $\sqrt{-1}(E_{i,j} + E_{j,i})$ be standard basis elements for $\mathfrak{k}$ (where $E_{i,j}$ denotes the matrix whose i, j-entry is 1 and all other entries are 0). Plugging these elements in for Y yields a system of equations,

$$\begin{aligned} 0 &= x_i\bar{z}_i + z_i\bar{x}_i && \text{for all } i, \\ 0 &= (\mu_j - \mu_i)(X_{j,i} + X_{i,j}) - (x_j\bar{z}_i + z_j\bar{x}_i - x_i\bar{z}_j - z_i\bar{x}_j) && \text{for all } i \neq j, \\ 0 &= (\mu_j - \mu_i)(X_{j,i} - X_{i,j}) - (x_j\bar{z}_i + z_j\bar{x}_i + x_i\bar{z}_j + z_i\bar{x}_j) && \text{for all } i \neq j, \end{aligned} \tag{36}$$

(where $X_{i,j}$ denotes the i, j entry of X) which in turn is equivalent to the system of equations

$$\begin{aligned} 0 &= x_i\bar{z}_i + z_i\bar{x}_i && \text{for all } i, \\ 0 &= (\mu_j - \mu_i)X_{j,i} - (x_j\bar{z}_i + z_j\bar{x}_i) && \text{for all } i \neq j. \end{aligned} \tag{37}$$

This system of equations is equivalent to the system of matrix equations (34). It follows from (34) that the block diagonal parts of $[\xi, p] \in T_p(K \cdot p)^{\omega}$ are zero, so $[\xi, p]$ has the form (33) subject to the equations (34). By properties of equivariant moment maps, $T_p(K \cdot p) \cap T_p(K \cdot p)^{\omega} = T_p(K_{\mathrm{M}} \cdot p)$ [Guillemin and Sternberg 1984c]. Elements of $T_p(K_{\mathrm{M}} \cdot p)$ have block form of (35), which completes the proof. □

Equations (34) dictate the form of the vectors $(c - \mathrm{M})\boldsymbol{x} + X\boldsymbol{z}$, as the next two lemmas demonstrate.

Lemma 4.4. *Let $p \in \Phi^{-1}(\mathrm{M})$ and let $\boldsymbol{z}$ be defined as in* (30). *Let $X \in \mathfrak{k}$ and $\boldsymbol{x} \in M_{n \times 1}(\mathbb{C})$ such that*

$$0 = \boldsymbol{x}\boldsymbol{z}^{\dagger} + \boldsymbol{z}\boldsymbol{x}^{\dagger} + [X, \mathrm{M}]. \tag{38}$$

If the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled μ is not a △-shape, then

$$(X\boldsymbol{z})_{\mu} = \Bigg(\sum_{\substack{\tau \in [\underline{\mu}] \\ \triangle\text{-shape}}} \frac{r_{\tau}^2}{\mu - \tau} \Bigg) \boldsymbol{x}_{\mu}.$$

Proof. Let $\mu \neq \nu$ distinct elements of $\underline{\mu}$. Let $X_{\mu,\nu}$, $\boldsymbol{x}_{\mu}$, $\boldsymbol{z}_{\mu}$, etc. denote the corresponding blocks of X, $\boldsymbol{x}$, and $\boldsymbol{z}$. By (38), the μ, ν block of X is given by the formula

$$X_{\mu,\nu} = \frac{1}{\mu - \nu}(\boldsymbol{x}_{\mu}\boldsymbol{z}_{\nu}^{\dagger} + \boldsymbol{z}_{\mu}\boldsymbol{x}_{\nu}^{\dagger}) \qquad \text{for all } \mu \neq \nu.$$

By Lemma 4.1, if the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled μ is not a △-shape, then $\boldsymbol{z}_{\mu} = 0$. Thus

$$\begin{aligned}(X\boldsymbol{z})_{\mu} &= \sum_{\substack{\tau \in [\underline{\mu}] \\ \tau \neq \mu}} X_{\mu,\tau}\boldsymbol{z}_{\tau} = \sum_{\substack{\tau \in [\underline{\mu}] \\ \tau \neq \mu}} \frac{1}{\mu - \tau}\boldsymbol{x}_{\mu}\boldsymbol{z}_{\tau}^{\dagger}\boldsymbol{z}_{\tau} \\ &= \Bigg(\sum_{\substack{\tau \in [\underline{\mu}] \\ \triangle\text{-shape}}} \frac{\|\boldsymbol{z}_{\tau}\|^2}{\mu - \tau} \Bigg) \boldsymbol{x}_{\mu} = \Bigg(\sum_{\substack{\tau \in [\underline{\mu}] \\ \triangle\text{-shape}}} \frac{r_{\tau}^2}{\mu - \tau} \Bigg) \boldsymbol{x}_{\mu}. \end{aligned}$$

□

Recall the definition of C_{μ} from (22).

Lemma 4.5. *Let p, X, and $\boldsymbol{x}$ as in Lemma 4.4 such that* (38) *holds. Assume that the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled μ is not a △-shape. Then, $C_{\mu} = 0$ if and only if the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled μ is a ▽-shape.*

Proof. First, note that it is sufficient to prove

$$\prod_{\substack{\lambda\in[\underline{\lambda}]\\ \triangledown\text{-shape}}}(x-\lambda)=(x-c)\prod_{\substack{\mu\in[\underline{\mu}]\\ \vartriangle\text{-shape}}}(x-\mu)-\sum_{\substack{\mu\in[\underline{\mu}]\\ \vartriangle\text{-shape}}} r_\mu^2\prod_{\substack{\tau\in[\underline{\mu}]\\ \vartriangle\text{-shape}\\ \tau\neq\mu}}(x-\tau). \tag{39}$$

Indeed, since the component of the interlacing pattern labelled μ is not a $\vartriangle$-shape, plugging in $x=\mu$ yields

$$\prod_{\substack{\lambda\in[\underline{\lambda}]\\ \triangledown\text{-shape}}}(\mu-\lambda)=\Bigg(\mu-c-\sum_{\substack{\tau\in[\underline{\mu}]\\ \vartriangle\text{-shape}}}\frac{r_\tau^2}{\mu-\tau}\Bigg)\prod_{\substack{\tau\in[\underline{\mu}]\\ \vartriangle\text{-shape}}}(\mu-\tau)=-C_\mu\prod_{\substack{\tau\in[\underline{\mu}]\\ \vartriangle\text{-shape}}}(\mu-\tau) \tag{40}$$

and the factor

$$\prod_{\substack{\tau\in[\underline{\mu}]\\ \vartriangle\text{-shape}}}(\mu-\tau) \tag{41}$$

is nonzero.

Second, applying Lemma 4.1 ($r_\mu=0$ when the component labelled μ is not a $\vartriangle$-shape) and rearranging, observe that

$$\begin{aligned}
&(x-c)\prod_{\mu\in[\underline{\mu}]}(x-\mu)^{n_\mu([\underline{\mu}])}-\sum_{\mu\in[\underline{\mu}]} r_\mu^2(x-\mu)^{n_\mu([\underline{\mu}])-1}\prod_{\substack{\tau\in[\underline{\mu}]\\ \tau\neq\mu}}(x-\tau)^{n_\tau([\underline{\mu}])} \\
&=(x-c)\prod_{\substack{\mu\in[\underline{\mu}]\\ \vartriangle\text{-shape}}}(x-\mu)^{n_\mu([\underline{\mu}])}\prod_{\substack{\tau\in[\underline{\mu}]\\ \triangledown,\,\text{▱-shape}}}(x-\tau)^{n_\tau([\underline{\mu}])} \\
&\quad-\sum_{\substack{\mu\in[\underline{\mu}]\\ \vartriangle\text{-shape}}} r_\mu^2(x-\mu)^{n_\mu([\underline{\mu}])-1}\prod_{\substack{\tau\in[\underline{\mu}]\\ \vartriangle\text{-shape}\\ \tau\neq\mu}}(x-\tau)^{n_\tau([\underline{\mu}])}\prod_{\substack{\tau\in[\underline{\mu}]\\ \triangledown,\,\text{▱-shape}}}(x-\tau)^{n_\tau([\underline{\mu}])} \\
&=\Bigg((x-c)\prod_{\substack{\mu\in[\underline{\mu}]\\ \vartriangle\text{-shape}}}(x-\mu)-\sum_{\substack{\mu\in[\underline{\mu}]\\ \vartriangle\text{-shape}}} r_\mu^2\prod_{\substack{\tau\in[\underline{\mu}]\\ \vartriangle\text{-shape}\\ \tau\neq\mu}}(x-\tau)\Bigg) \\
&\qquad\cdot\prod_{\substack{\tau\in[\underline{\mu}]\\ \vartriangle\text{-shape}}}(x-\tau)^{n_\tau([\underline{\mu}])-1}\prod_{\substack{\tau\in[\underline{\mu}]\\ \triangledown,\,\text{▱-shape}}}(x-\tau)^{n_\tau([\underline{\mu}])}.
\end{aligned} \tag{42}$$

Then (39) follows by combining (42) and (31), which completes the proof. □

For $p \in \Phi^{-1}(\mathrm{M})$, let $V_p \subset \mathbb{C}^n$ denote the image of injective linear map

$$T : T_p(K \cdot p)^{\omega} \to \mathbb{C}^n,$$

$$\left(\begin{array}{c|c} 0 & (c-\mathrm{M})\boldsymbol{x}^{\dagger} + \boldsymbol{z}^{\dagger} X^{\dagger} \\ \hline (c-\mathrm{M})\boldsymbol{x} + X\boldsymbol{z} & 0 \end{array}\right) \mapsto (c-\mathrm{M})\boldsymbol{x} + X\boldsymbol{z} \tag{43}$$

and let $U_p \subset V_p$ denote the image of $T_p(K \cdot p) \cap T_p(K \cdot p)^{\omega}$. Specialize to the case of $\tilde{p}$ and recall that $K_{\tilde{p}} = L_{(\underline{\lambda},\underline{\mu})}$. The map T is $K_{\tilde{p}}$-equivariant with respect to the action of $K_{\tilde{p}}$ on $\mathbb{C}^n$ as a block-diagonal subgroup of $K = U(n)$ acting by the standard representation. Decompose $\mathbb{C}^n = \bigoplus_{i=1}^{m} \mathbb{C}^{n_i(\underline{\mu})}$, $m = m(\underline{\mu})$. The subspaces $V_{\tilde{p}}$ and $U_{\tilde{p}}$ have the forms $\bigoplus_{i=1}^{m} V_i$ and $\bigoplus_{i=1}^{m} U_i$, respectively, for some subspaces $U_i \subset V_i \subset \mathbb{C}^{n_i(\underline{\mu})}$. The map T descends to an isomorphism of $K_{\tilde{p}}$-representations,

$$W_{\tilde{p}} = T_{\tilde{p}}(K \cdot \tilde{p})^{\omega} / (T_{\tilde{p}}(K \cdot \tilde{p}) \cap T_{\tilde{p}}(K \cdot \tilde{p})^{\omega}) \cong \bigoplus_{i=1}^{m} V_i / U_i. \tag{44}$$

The representation of $K_{\tilde{p}} = L_1 \times \cdots \times L_m$ on the right is given in each component by the inclusion $L_i \subset U(n_i(\underline{\mu}))$ and the standard representation of $U(n_i(\underline{\mu}))$ on $\mathbb{C}^{n_i(\underline{\mu})}$. This representation of L_i preserves the subspaces $U_i \subset V_i$ so it induces a representation on V_i / U_i.

Recall that if the component of the interlacing pattern labelled $\underline{\mu}_i$ is a ▱-shape, then $L_i = U(n_i(\underline{\mu}))$.

Proposition 4.6. *For all $i = 1, \dots, m$, $m = m(\underline{\mu})$, there is an isomorphism of L_i representations*

$$V_i / U_i \cong \begin{cases} \mathbb{C}^{n_i(\underline{\mu})} & \textit{if the component of the interlacing pattern} \\ & \qquad \textit{of } (\underline{\lambda}, \underline{\mu}) \textit{ labelled } \underline{\mu}_i \textit{ is a ▱-shape}, \\ \{0\} & \textit{otherwise}, \end{cases}$$

where $\mathbb{C}^{n_i(\underline{\mu})}$ denotes the standard representation of $U(n_i(\underline{\mu}))$.

Proof. In general,

$$U_i = \{(Y\boldsymbol{z})_i \mid Y \in \mathfrak{k}_{\mathrm{M}}\} = \{Y_{i,i}\boldsymbol{z}_i \mid Y_{i,i} \in \mathfrak{u}(n_i(\underline{\mu}))\}.$$

If the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\underline{\mu}_i$ is a ⏢-shape, then, by Lemma 4.1, $\boldsymbol{z}_i \neq 0$, so $U_i = \mathbb{C}^{n_i(\underline{\mu})}$ and $V_i / U_i \cong \{0\}$. If the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\underline{\mu}_i$ is not a ⏢-shape, then, $\boldsymbol{z}_i = 0$, so $U_i = \{0\}$.

It remains to determine the subspace V_i when the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\underline{\mu}_i$ is not a ⏢-shape. In this case, it follows by Lemma 4.4 that the block

$$((c-\mathrm{M})\boldsymbol{x} + X\boldsymbol{z})_i = (c-\mathrm{M})\boldsymbol{x}_i + (X\boldsymbol{z})_i = C_i \boldsymbol{x}_i,$$

where $C_i = C_{\underline{\mu}_i}$ as defined in (22). By Lemma 4.3,

$$\begin{aligned}(45)\qquad V_i &= \{((c-\mathrm{M})\boldsymbol{x} + X\boldsymbol{z})_i \mid X \in \mathfrak{k}, \boldsymbol{x} \in \mathbb{C}^n, \boldsymbol{x}\boldsymbol{z}^\dagger + \boldsymbol{z}\boldsymbol{x}^\dagger + [X, \mathrm{M}]\} \\ &= \{C_i \boldsymbol{x}_i \mid \boldsymbol{x}_i \in \mathbb{C}^{n_i(\underline{\mu})}\}.\end{aligned}$$

By Lemma 4.5, $C_i = 0$ if and only if the component of the interlacing pattern of $(\underline{\lambda}, \underline{\mu})$ labelled $\underline{\mu}_i$ is a ▱-shape. This completes the proof. ☐

Thus $\bigoplus_{i=1}^m V_i/U_i$ is isomorphic to the $L_{(\underline{\lambda},\underline{\mu})}$-representation $W_{(\underline{\lambda},\underline{\mu})}$.

Proposition 4.7. *The linear symplectic structure on* $W_{(\underline{\lambda},\underline{\mu})}$ *defined via the symplectic form* $\overline{\omega}_{\tilde{p}}$ *and the isomorphism* (44) *equals the linear symplectic form* $\omega_{(\underline{\lambda},\underline{\mu})}$ *defined in* (23).

Proof. Denote

$$\eta := \left(\begin{array}{c|c} 0 & -\boldsymbol{y}^\dagger \\ \hline \boldsymbol{y} & Y \end{array}\right), \quad \xi := \left(\begin{array}{c|c} 0 & -\boldsymbol{x}^\dagger \\ \hline \boldsymbol{x} & X \end{array}\right), \qquad X, Y \in \mathfrak{k},\ \boldsymbol{x}, \boldsymbol{y} \in M_{n\times 1}(\mathbb{C}).$$

Then, using Lemma 4.4,

$$\begin{aligned}(46)\quad &\sqrt{-1}(\omega_\Lambda)_{\tilde{p}}([\xi, \tilde{p}], [\eta, \tilde{p}]) \\ &= \mathrm{Tr}(\tilde{p}[\xi, \eta]) = \mathrm{Tr}([\tilde{p}, \xi]\eta) \\ &= -\mathrm{Tr}\left(\left(\begin{array}{c|c} 0 & (c-\mathrm{M})\boldsymbol{x}^\dagger + \boldsymbol{z}^\dagger X^\dagger \\ \hline (c-\mathrm{M})\boldsymbol{x} + X\boldsymbol{z} & 0 \end{array}\right)\left(\begin{array}{c|c} 0 & -\boldsymbol{y}^\dagger \\ \hline \boldsymbol{y} & Y \end{array}\right)\right) \\ &= -((c-\mathrm{M})\boldsymbol{x}^\dagger + \boldsymbol{z}^\dagger X^\dagger)\boldsymbol{y} + \mathrm{Tr}(((c-\mathrm{M})\boldsymbol{x} + X\boldsymbol{z})\boldsymbol{y}^\dagger) \\ &= -((c-\mathrm{M})\boldsymbol{x}^\dagger + \boldsymbol{z}^\dagger X^\dagger)\boldsymbol{y} + \mathrm{Tr}(\boldsymbol{y}^\dagger((c-\mathrm{M})\boldsymbol{x} + X\boldsymbol{z})) \\ &= -(c-\mathrm{M})(\boldsymbol{x}^\dagger\boldsymbol{y} - \boldsymbol{y}^\dagger\boldsymbol{x}) - \boldsymbol{z}^\dagger X^\dagger \boldsymbol{y} + \boldsymbol{y}^\dagger X\boldsymbol{z} \\ &= -(c-\mathrm{M})(\boldsymbol{x}^\dagger\boldsymbol{y} - \boldsymbol{y}^\dagger\boldsymbol{x}) - (X\boldsymbol{z})^\dagger \boldsymbol{y} + \boldsymbol{y}^\dagger X\boldsymbol{z} \\ &= -(c-\mathrm{M})(\boldsymbol{x}^\dagger\boldsymbol{y} - \boldsymbol{y}^\dagger\boldsymbol{x}) + \sum_{i=1}^m \Bigg(\sum_{\substack{\text{⏢-shape}\\ j\neq i}} \frac{r_j^2}{\mu_i - \mu_j}\Bigg)(-\boldsymbol{x}_i^\dagger \boldsymbol{y}_i + \boldsymbol{y}_i^\dagger \boldsymbol{x}_i).\end{aligned}$$

Viewing $[\xi, \tilde{p}]$ and $[\eta, \tilde{p}]$ as representatives of vectors in the isotropy representation,

$$\begin{aligned}(47)\quad (\overline{\omega}_\lambda)_{\tilde{p}}([\xi, \tilde{p}], [\eta, \tilde{p}]) &= \frac{1}{\sqrt{-1}} \sum_{\substack{\text{▱-shape}\\ i=1}}^m \Bigg(c - \mu_i + \sum_{\substack{\text{⏢-shape}\\ j\neq i}} \frac{r_j^2}{\mu_i - \mu_j}\Bigg)(-\boldsymbol{x}_i^\dagger \boldsymbol{y}_i + \boldsymbol{y}_i^\dagger \boldsymbol{x}_i) \\ &= \frac{1}{\sqrt{-1}} \sum_{\substack{\text{▱-shape}\\ i=1}}^m C_i(-\boldsymbol{x}_i^\dagger \boldsymbol{y}_i + \boldsymbol{y}_i^\dagger \boldsymbol{x}_i).\end{aligned}$$

Applying the isomorphism $T : W_{\tilde{p}} \to W_{(\underline{\lambda},\underline{\mu})}$, $[\xi, p] \mapsto \boldsymbol{u} = (C_i \boldsymbol{x}_i)_i$, $[\eta, p] \mapsto \boldsymbol{v} = (C_i \boldsymbol{y}_i)_i$ yields

$$\omega_{(\underline{\lambda},\underline{\mu})}(\boldsymbol{u}, \boldsymbol{w}) = \frac{1}{\sqrt{-1}} \sum_{\substack{\square\text{-shape} \\ i=1}}^{m} \frac{-\boldsymbol{u}_i^{\dagger} \boldsymbol{w}_i + \boldsymbol{w}_i^{\dagger} \boldsymbol{u}_i}{C_i}. \qquad \square$$

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