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FUNCTIONAL DETERMINANT ON PSEUDO-EINSTEIN 3-MANIFOLDS

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Given a three-dimensional pseudo-Einstein CR manifold $(M, T^{1,0}M, \theta)$, we establish an expression for the difference of determinants of the Paneitz type operators A_θ, related to the problem of prescribing the Q'-curvature, under the conformal change $\theta \mapsto e^w\theta$ with $w \in \mathcal{P}$ the space of pluriharmonic functions. This generalizes the expression of the functional determinant in four-dimensional Riemannian manifolds established in (*Proc. Amer. Math. Soc.* 113:3 (1991), 669–682). We also provide an existence result of maximizers for the scaling invariant functional determinant as in (*Ann. of Math.* (2) 142:1 (1995), 171–212).

1. Introduction and statement of the results

There has been extensive work on the study of spectral invariants of differential operators defined on a Riemannian manifold (M, g) and the relations to their conformal invariants; see for instance [Branson and Ørsted 1986; 1991a; 1991b]. As an example, if we consider the two-dimensional case with the pair of the Laplace operator $-\Delta_g$, and the associated invariant which is the scalar curvature R_g, we know that under conformal change of the metric $g \mapsto \tilde{g} = e^{2w}g$, one has the relation

$$R_{\tilde{g}}e^{2w} = -\Delta_g w + R_g.$$

It is also known that the spectrum of $-\Delta_g$ is discrete and can be written as $0 < \lambda_1 \le \lambda_2 \le \cdots$ and the corresponding zeta function is then defined by

$$\zeta_{-\Delta_g}(s) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s}.$$

This series converges uniformly for $s > 1$ and can be extended to a meromorphic function in $\mathbb{C}$ with 0 as a regular value. The determinant of the operator $-\Delta_g$ can then be written as

$$\det(-\Delta_g) = e^{-\zeta'_{-\Delta_g}(0)}.$$

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The celebrated Polyakov formula [1981a; 1981b] states that if $\tilde{g}=e^{2w}g$, then

$$\ln\left(\frac{\det(-\Delta_{\tilde{g}})}{\det(-\Delta_g)}\right)=\frac{-1}{12\pi}\int_M |\nabla w|^2+2Rw\,dv_g,$$

for metrics with the same volume. The scaling invariant functional determinant F_2 can then be written as

$$F_2(w)=\frac{-1}{12\pi}\left(\int_M |\nabla w|^2+2Rw\,dv_g-\left(\int_M R\,dv_g\right)\ln\left(\fint_M e^{2w}\,dv_g\right)\right).$$

Notice that the right-hand side is a familiar quantity. It is the Beckner–Onofri energy [Beckner 1993], and we know that

$$\int_{S^2} |\nabla w|^2+2Rw\,dv_g-\left(\int_M R\,dv_g\right)\ln\left(\fint_{S^2} e^{2w}\,dv_g\right)\geq 0.$$

This notion of determinant was extended to dimension four for conformally invariant operators, keeping in mind that the substitute of the pair $(-\Delta_g, R)$ in dimension two is the pair (P_g, Q_g) in dimension four, where P_g is the Paneitz operator and Q_g is the Riemannian Q-curvature [Branson and Ørsted 1991b; Esposito and Malchiodi 2019]. In addition, two new terms appear in the scaling invariant functional determinant expression. Indeed, if (M, g) is a 4-dimensional manifold and A_g is a nonnegative self-adjoint conformally covariant operator, then there exist β_1, β_2 and $\beta_3\in\mathbb{R}$ such that the scaling invariant functional determinant F_4 reads as

$$F_4(w)=\beta_1 I(w)+\beta_2 II(w)+\beta_3 III(w), \tag{1}$$

where

$$\begin{cases} I(w)=4\int_M w|W_g|^2\,dv_g-\left(\int_M |W_g|^2\,dv_g\right)\ln\left(\fint_M e^{4w}\,dv_g\right),\\ II(w)=\int_M wP_gw+4Q_gw\,dv_g-\left(\int_M Q_g\,dv_g\right)\ln\left(\fint_M e^{4w}\,dv_g\right),\\ III(w)=12\int_M(\Delta_g w+|\nabla w|^2)^2\,dv_g-4\int_M w\Delta_g R_g+R_g|\nabla w|^2\,dv_g.\end{cases}$$

In the case of the sphere S^4, we see that the second term II corresponds again to the four-dimensional Beckner–Onofri energy. The existence and uniqueness of maximizers of this expression was heavily investigated and we refer the reader to [Chang and Yang 1995; Gursky and Malchiodi 2012; Esposito and Malchiodi 2019; Okikiolu 2001] for an in-depth study of this functional in the Riemannian setting.

Now let us move to the CR setting. We consider a 3-dimensional CR manifold $(M, T^{1,0}M, J, \theta)$ and we recall that the substitute for the pair (P_g, Q_g) is (P_θ, Q_θ), where P_θ is the CR Paneitz operator and Q_θ is the CR Q-curvature [Fefferman and Hirachi 2003; Gover and Graham 2005]. The problem with this pair is that the total Q-curvature is always zero. In fact in pseudo-Einstein manifolds the Q-curvature vanishes identically. Hence, we do not have a decent substitute for the CR Beckner–Onofri inequality. Fortunately, if we restrict our study to pseudo-Einstein manifolds and variations in the space of pluriharmonic functions $\mathcal{P}$, then we have a better substitute for the pair (P_g, Q_g), namely (P'_θ, Q'_θ). These quantities were first introduced on odd dimensional spheres in [Branson et al. 2013] and then on pseudo-Einstein manifolds in [Case et al. 2016; Case and Yang 2013; Hirachi 2014]. In particular one has a Beckner–Onofri type inequality involving the operator P'_θ acting on pluriharmonic functions as proved in [Branson et al. 2013]. We also recall that the total Q'-curvature corresponds to a geometric invariant, namely the Burns–Epstein invariant $\mu(M)$ [Burns and Epstein 1988; Chêng and Lee 1990].

One is tempted to see what the spectral invariants of the operator P' are or the restriction of P' to the space $\mathcal{P}$ of pluri-harmonic functions and link them to geometric quantities such as the total Q'-curvature.

We recall that the quantity Q'_θ changes as follows: if $\tilde{\theta} = e^w\theta$ with $w \in \mathcal{P}$, then

$$P'_\theta w + Q'_\theta = Q'_{\tilde{\theta}} e^{2w} + \tfrac{1}{2} P_\theta(w^2), \tag{2}$$

which we can write as

$$P'_\theta w + Q'_\theta = Q'_{\tilde{\theta}} e^{2w} \mod \mathcal{P}^\perp.$$

We let $\tau_\theta : L^2 \to \mathcal{P}$ be the orthogonal projection on $\mathcal{P}$ with respect to the inner product induced by θ and set $A_\theta = \tau_\theta P'_\theta \tau_\theta$. Then equation (2) can be rewritten as

$$A_\theta w + \tau_\theta(Q'_\theta) = \tau_\theta(Q'_{\tilde{\theta}} e^{2w}).$$

Prescribing the quantity $\bar{Q}'_\theta = \tau_\theta(Q'_\theta)$ was thoroughly investigated in [Maalaoui 2019b; Case et al. 2016; Ho 2019] mainly because of the property that

$$\int_M \bar{Q}'_\theta \, dv_\theta = \int_M Q'_\theta \, dv_\theta = -\frac{\mu(M)}{16\pi^2}.$$

We recall that in [Maalaoui 2019a], we proved that the dual of the Beckner–Onofri inequality, namely the logarithmic Hardy–Littlewood–Sobolev inequality, can be linked to the regularized zeta function of the operator A_θ evaluated at one. This was proved in the Riemannian setting in [Morpurgo 1996; 2002; Okikiolu 2008].

In this paper, we will generalize the expression (1) by studying the determinant of the operator A_θ. In all that follows we assume that $(M, T^{1,0}M, J, \theta)$ is an embeddable pseudo-Einstein manifold such that P'_θ is nonnegative and has trivial kernel. First we show that:

Theorem 1.1 (conformal index). *Let ζ_{A_θ} be the spectral zeta function of the operator A_θ. Then $\zeta_{A_\theta}(0)$ is a conformal invariant in $\mathcal{P}$. Moreover,*

$$\zeta_{A_\theta}(0) = \frac{-1}{24\pi^2} \int_M Q'_\theta \, dv_\theta - 1.$$

In order to compute the determinant of the operator A_θ we introduce the quantities $\widetilde{A}_1(w)$, $\widetilde{A}_2(w)$ and $\widetilde{A}_3(w)$ defined by

$$\begin{cases} \widetilde{A}_1(w) := \int_M wA_\theta w + 2Q'_\theta w\, d\nu - \frac{1}{c_1}\ln\bigl(⨍_M e^{2w}\, d\nu\bigr), \\ \widetilde{A}_2(w) := 2\int_M R\bigl(\Delta_b w + \frac{1}{2}|\nabla_b w|^2\bigr) - \bigl(\Delta_b w + \frac{1}{2}|\nabla_b w|^2\bigr)^2\, d\nu, \\ \widetilde{A}_3(w) := 2\int_M w_{,0}R - \frac{1}{3}w_{,0}|\nabla_b w|^2 - w_{,0}\Delta_b w\, d\nu. \end{cases} \tag{3}$$

One can also write $\widetilde{A}_2(w)$ as

$$\widetilde{A}_2(w) = 2\int_M R\left(\frac{\Delta_b e^{\frac{1}{2}w}}{2e^{\frac{1}{2}w}}\right) - \left(\frac{\Delta_b e^{\frac{1}{2}w}}{2e^{\frac{1}{2}w}}\right)^2 d\nu.$$

Then we have the following:

Theorem 1.2. *There exists c_2 and $c_3 \in \mathbb{R}$ such that for all $w \in \mathcal{P}$, we have*

$$\ln\left(\frac{\det(A_\theta)}{\det(A_{e^w\theta})}\right) = -\frac{1}{24\pi^2}\widetilde{A}_1(w) + c_2\widetilde{A}_2(w) - c_3\widetilde{A}_3(w). \tag{4}$$

Notice that the expression (4) is not scaling invariant, because for $\tilde{\theta} = c^2\theta$, with c a positive constant, we have

$$\det(A_{\tilde{\theta}}) = c^{-4\zeta_\theta(0)}\det(A_\theta).$$

So we fix the volume V of (M, θ) and define the scaling invariant functional

$$S_{A_\theta} = \left(\frac{\mathrm{Vol}(\theta)}{V}\right)^{\zeta_{A_\theta}(0)}\det(A).$$

Now we can define the scaling invariant functional $F : W_H^{2,2}(M) \cap \mathcal{P} \to \mathbb{R}$ by

$$F(w) = \ln(S_{A_\theta}) - \ln(S_{A_{e^w\theta}}),$$

where $W_H^{2,2}(M)$ is Folland–Stein space. Then one can write the following expression of F,

$$F(w) = c_1 II(w) + c_2 III(w) + c_3 IV(w),$$

where

$$\begin{cases} II(w) = \int_M wA_\theta w + 2Q'_\theta w\, d\nu - \int_M Q'_\theta\, d\nu \ln\bigl(⨍ e^{2w} d\nu\bigr). \\ III(w) = \widetilde{A}_2(w). \\ IV(w) = -\widetilde{A}_3(w). \end{cases} \tag{5}$$

Notice that the functional II is the CR Beckner–Onofri functional studied first in [Branson et al. 2013] on the standard sphere S^3. In particular one has on the standard sphere

$$II(w) \geq 0.$$

This functional was also investigated in [Case et al. 2016] and its critical points correspond to the pseudo-Einstein structures with constant $\overline{Q'}$-curvature. The functional III is also similar to the Riemannian one defined in [Chang and Yang 1995] and its critical points are pseudo-Einstein contact forms $\tilde{\theta}$ satisfying

$$\tilde{\tau}(\widetilde{\Delta_b}\widetilde{R}) = 0.$$

The functional IV is a bit different, in fact if we let $\mathcal{H}$ defined by

$$\mathcal{H}(w) = R_{,0} - \tfrac{1}{3}|\nabla_b w|^2_{,0} - \tfrac{2}{3}\operatorname{div}_b(w_{,0}\nabla_b w) + \Delta_b w_{,0} - (\Delta_b w)_{,0},$$

then the critical points of IV satisfy

$$\tilde{\tau}(e^{-2w}\mathcal{H}(w)) = 0.$$

We set

$$a = \frac{\int_M Q'_\theta \, d\nu}{16\pi^2}.$$

Since the coefficients c_2 and c_3 are still unknown for the operator A_θ and the corresponding functional S_{A_θ}, we will be considering them as parameters in our setting. Then we show the following for the functional F:

Theorem 1.3. *Assume that $c_2 > 0$ and $c_3 \geq 0$. Then there exists a constant μ depending on θ, such that if*

$$c_3 < \mu\left(\sqrt{25c_2^2 + \frac{1}{3\pi^2}c_2(1-a)} - 5c_2\right), \tag{6}$$

then F has a maximizer $w_\infty \in W_H^{2,2}(M) \cap \mathcal{P}$ under the constraint $\int_M e^{2w}\, d\nu = 1$. Moreover, this maximizer satisfies the Euler–Lagrange equation

$$\tau_{\theta_\infty}\left[-\frac{1}{24\pi^2}\widetilde{Q'_\theta} + c_2\widetilde{\Delta_b}\widetilde{R} + c_3 e^{-2w}\mathcal{H}(w)\right] = cte,$$

where the tilde refers to quantities computed using the contact form $\theta_\infty = e^{w_\infty}\theta$.

Notice the condition (6) implies in particular that $\int_M Q'_\theta d\nu < 16\pi^2$. Hence, as a consequence, we have that $(M, T^{1,0}M, \theta)$ is not equivalent to the standard sphere. We point out that in order to verify the sharpness of (6) one needs to check specific examples which is not as easy as in the Riemannian case, since we are dealing with CR pluriharmonic functions and we lack explicit examples of manifolds where one can have an explicit expression of the spectrum of the P'-operators.

2. Heat coefficients and conformal invariance

Let $(M, T^{1,0}M, \theta)$ be a pseudo-Einstein 3-manifold and P'_θ its P'-operator defined by

$$P'_\theta f = 4\Delta_b^2 f - 8\operatorname{Im}\big(\nabla^1(A_{11}\nabla^1 f)\big) - 4\operatorname{Re}\big(\nabla^1(R\nabla_1 f)\big). \tag{7}$$

Denote by $\tau : L^2(M) \to \mathcal{P}$ the orthogonal projection on the space of pluriharmonic functions with respect to the L^2-inner product induced by θ. We consider the operator $A_\theta = \tau P'_\theta \tau$ and for the conformal change $\tilde{\theta} = e^w \theta$, with $w \in \mathcal{P}$, we let

$$A_{\tilde{\theta}} = \tau_{\tilde{\theta}}(e^{-2w} A_\theta),$$

where $\tau_{\tilde{\theta}}$ is the orthogonal projection with respect to the L^2-inner product induced by $\tilde{\theta}$.

In order to evaluate and manipulate the spectral invariants, we need to study the expression of the heat kernel of the operator A_θ. Unfortunately, this operator is not elliptic or subelliptic (as an operator on $C^\infty(M)$), and does not have an invertible principal symbol in the sense of $\Psi_H(M)$-calculus (see [Ponge 2007]). In fact A_θ can be seen as a Toeplitz operator, and one might adopt the approach introduced in [Boutet de Monvel and Guillemin 1981] in order to study it. But instead, we will modify the operator in order to be able to use the classical computations done for the heat kernel.

Consider the operator $\mathcal{L} = A_\theta + \tau^\perp L \tau^\perp$, where L is chosen so that $\mathcal{L}$ has an invertible principal symbol in $\Psi^4_H(M)$. Notice that $\tau\mathcal{L} = \mathcal{L}\tau = A_\theta$. Based on [Ponge 2007], if $\mathcal{K}$ is the heat kernel of $\mathcal{L}$ one has the following expansion near zero:

$$\mathcal{K}(t, x, x) \sim \sum_{j=0}^{\infty} \tilde{a}_j(x) t^{\frac{1}{4}(j-4)} + \ln(t) \sum_{j=1}^{\infty} t^j \tilde{b}_j(x).$$

Since $e^{-t\mathcal{L}} = e^{-tA_\theta}\tau + e^{-tL}\tau^\perp$, we have that the kernel K of e^{-tA_θ} which is the restriction to $\mathcal{P}$ of $\mathcal{K}$, reads as

$$K(t, x, x) \sim \sum_{j=0}^{\infty} t^{\frac{1}{4}(j-4)} a_j(x) + \ln(t) \sum_{j=1}^{\infty} t^j b_j(x), \tag{8}$$

and this will be the main expansion that we will be using for the rest of the paper.

Now we want to define the infinitesimal variation of a quantity under a conformal change. Fix $w \in \mathcal{P}$ and for a given quantity F_θ depending on θ denote $\delta F_\theta := \frac{d}{dr}_{|r=0} F_{e^{rw}\theta}$. Next, we define the zeta function of A_θ by

$$\zeta_{A_\theta}(s) := \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s},$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ is the spectrum of the operator $A_\theta : W^{2,2}(M) \cap \mathcal{P} \to \mathcal{P}$. In what follows $\mathrm{TR}[A]$ is to be understood as the trace of the operator A in $\mathcal{P}$. Then we have the following proposition.

Proposition 2.1. *With the notations above, we have*

$$\zeta_{A_\theta}(0) = \int_M a_4(x)dx - 1.$$

Moreover,

$$\delta\zeta_{A_\theta}(0) = 0 \quad \textit{and} \quad \delta\zeta'_{A_\theta}(0) = 2\int_M w\Big(a_4(x) - \frac{1}{V}\Big)\,d\nu,$$

where $V = \int_M d\nu_\theta$ is the volume of M.

Proof. Most of the computations in this part are relatively standard and they can be found in [Branson and Ørsted 1986; 1991a; 1991b] in the Riemannian setting. First we use the Mellin transform and (8) to write

$$\begin{aligned}
\zeta_{A_\theta}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(\mathrm{TR}[e^{-tA_\theta}]-1)\,dt\\
&= \frac{1}{\Gamma(s)}\bigg(-\frac{1}{s}+\int_0^1 t^{s-1}\sum_{j=0}^{N} t^{\frac{1}{4}(j-4)}\int_M a_j(x)\,d\nu\,dt\\
&\qquad +\int_M t^{s-1}O(t^{\frac{1}{4}(N+1-4)})\,dt+\sum_{j=1}^{N}\int_0^1 t^{j+s-1}\ln(t)\int_M b_j\,d\nu\,dt\\
&\qquad\qquad +\int_0^1 t^{s-1}O(t^{N+1}\ln(t))\,dt+\int_1^\infty t^{s-1}\sum_{j=1}^{\infty}e^{-\lambda_j t}\,dt\bigg)\\
&= \frac{1}{\Gamma(s)}\bigg(\frac{-1}{s}+\sum_{j=0}^{N}\frac{1}{s+\frac{1}{4}(j-4)}\int_M a_j(x)\,d\nu+\int_0^1 t^{s-1}O(t^{\frac{1}{4}(N+1-4)})\,dt\\
&\qquad +\sum_{j=1}^{N}\frac{1}{(s+j)^2}\int_M b_j\,d\nu+\int_0^1 t^{s-1}O(t^{N+1}\ln(t))\,dt\int_1^\infty t^{s-1}\sum_{j=1}^{\infty}e^{-\lambda_j t}\,dt\bigg).
\end{aligned}$$

Since, Γ has a simple pole at $s = 0$ with residue 1, we see that by taking $s \to 0$, there are only two terms that survive, leading to

$$\zeta_{A_\theta}(0) = \int_M a_4(x)\,d\nu - 1.$$

Next we move to the study of the variation of ζ_{A_θ}. Let $f \in C^\infty(M)$ and $v \in \mathcal{P}$. Then

$$\int_M \tau_{rw}(f)v\,d\nu_{rw} = \int_M f v e^{2rw}\,d\nu.$$

Differentiating with respect to r and evaluating at 0 yields

$$\int_M (\delta\tau(f) + 2w\tau(f) - 2wF)v\,d\nu = 0.$$

Hence,

$$\delta\tau(f) = 2\tau(wf - w\tau(f)).$$

If we let M_w be the multiplication by w, then

$$\delta\tau = 2(\tau M_w - \tau M_w \tau).$$

In particular, if $f \in \mathcal{P}$, then $\delta\tau(f) = 0$.

Next we want to evaluate δA_θ. Recall that $A_{e^{rw}\theta} = \tau_{e^{rw}\theta} e^{-2rw} A_\theta$. Therefore,

$$\delta A_\theta = \delta\tau A_\theta - 2\tau M_w A_\theta = -2\tau M_w A_\theta.$$

Thus,

$$\begin{aligned}\delta\,\mathrm{TR}[e^{-tA_\theta}] &= -t\,\mathrm{TR}[\delta A_\theta e^{-tA_\theta}] \\ &= 2t\,\mathrm{TR}[\tau M_w A_\theta e^{-tA_\theta}] \\ &= 2t\,\mathrm{TR}[M_w A_\theta e^{-tA_\theta}].\end{aligned}$$

The last equality follows from $\mathrm{TR}[AB] = \mathrm{TR}[BA]$ and $e^{-tA_\theta}\tau = e^{-tA_\theta}$. But

$$-tM_w A_\theta e^{-tA_\theta} = \frac{d}{d\varepsilon}_{|\varepsilon=0} M_w e^{-t(1+\varepsilon)A_\theta}.$$

Using the expansion (8), we have

$$K(t(1+\varepsilon), x, x) \sim \sum_{j=0}^{\infty} (1+\varepsilon)^{\frac{1}{4}(j-4)} t^{\frac{1}{4}(j-4)} a_j(x) + H,$$

where H is the logarithmic part. Hence, comparing the terms in the expansion after integration, we get

$$\delta \int_M a_j\,dv = \frac{4-j}{2} \int_M w a_j\,dv.$$

In particular, we have $\delta \int_M a_4\,dv = 0$.

Similarly,

$$\Gamma(s)\zeta_{A_\theta}(s) = \Gamma(s)\big(\zeta_{A_\theta}(0) + s\zeta'_{A_\theta}(0) + O(s^2)\big).$$

Hence, since $\delta\zeta_{A_\theta}(0) = 0$, and $s\Gamma(s) \sim 0$ when $s \to 0$, we have

$$\delta\zeta'_{A_\theta}(0) = [\Gamma(s)\delta\zeta_{A_\theta}(s)]_{s=0}.$$

But,

$$\Gamma(s)\delta\zeta_{A_\theta}(s) = \int_0^\infty 2t^s \operatorname{TR}[wA_\theta e^{-tA_\theta}]\,dt \tag{9}$$
$$= -\int_0^\infty 2t^s \frac{d}{dt} \operatorname{TR}[we^{-tA_\theta}]\,dt$$
$$= \int_0^\infty 2st^{s-1} \operatorname{TR}\Big[w\Big(e^{-tA_\theta} - \frac{1}{V}\Big)\Big]\,dt.$$

Using again the expansion (8) and a similar computation as in the previous case yields

$$\delta\zeta'_{A_\theta}(0) = 2\int_M w\Big(a_4 - \frac{1}{V}\Big)\,dv. \qquad \square$$

Proposition 2.2. *There exists $c \neq 0$ such that*

$$\zeta_{A_\theta}(0) = c\int_M Q'_\theta\,dv - 1.$$

Moreover $c = -\dfrac{1}{24\pi^2}$.

Proof. First we notice that a_4 is a pseudo-Hermitian invariant of order -2, that is

$$a_{4,e^r\theta} = e^{-2r}a_{4,\theta}$$

for all $r \in \mathbb{R}$. So from [Hirachi 2014], we have the existence of $c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$ such that

$$a_4 = c_1 Q'_\theta + c_2\Delta_b R + c_3 R_{,0} + c_4 R^2 + c_5 Q_\theta,$$

where $Q'_\theta = 2\Delta_b R - 4|A|^2 + R^2$ and $Q_\theta = -\frac{2}{3}\Delta_b R + 2\operatorname{Im}(A_{11,\bar{1}\bar{1}})$. Since we are in a pseudo-Einstein manifold and $w \in \mathcal{P}$ we can assume that $Q_\theta = 0$. So after integration, we have

$$\int_M a_4\,dv = c_1\int_M Q'_\theta\,dv + c_4\int_M R^2\,dv.$$

Since $\int_M a_4\,dv$ is invariant under the conformal change $e^w\theta$, it is easy to see that $c_4 = 0$. Hence,

$$\int_M a_4\,dv = c_1\int_M Q'_\theta\,dv.$$

Next we want to calculate c_1 (compare to [Stanton 1989], where the invariant k_2 is always 0). We take the case of the sphere S^3. Based on the computations in [Branson et al. 2013], we have

$$\zeta_{A_\theta}(s) = 2\sum_{j=1}^\infty \frac{j+1}{(j(j+1))^s} = 2\sum_{j=2}^\infty \frac{1}{j^{2s-1}}\left(\frac{1}{1-\frac{1}{j}}\right)^s.$$

Using the expansion of

$$\left(1-\frac{1}{j}\right)^{-s}=1+\frac{s}{j}+\frac{s(s+1)}{2j^2}+sO\left(\frac{1}{j^3}\right),$$

we see that

$$\zeta_{A_\theta}(s)=2\left(\zeta_R(2s-1)-1+s(\zeta_R(2s)-1)-\frac{s(s+1)}{2}(\zeta_R(2s+1)-1)\right)+sH(s),$$

with $H(s)$ holomorphic near $s=0$ and ζ_R the classical Riemann Zeta function. Now we recall that ζ_R is regular at $s=0$ and $s=-1$ but has a simple pole at $s=1$ with residue equal to 1. Hence

$$\zeta_{A_\theta}(0)=2\left(-\frac{1}{12}-1+\frac{1}{4}\right)=-\frac{5}{3}\neq 0.$$

Knowing that $\int_{S^3} Q'_\theta\,dv=16\pi^2$, we have

$$16\pi^2c-1=-\frac{5}{3}.$$

Thus,

$$c=-\frac{1}{24\pi^2}.\qquad\square$$

3. The expression for the determinant

Recall that in the previous section, we found that $a_4=c_1Q'+c_2\Delta_bR+c_3R_{,0}$. In particular,

$$\begin{aligned}\delta\zeta'_{A_\theta}(0)&=\int_M 2w\left(a_4(x)-\frac{1}{V}\right)dv\\&=c_1\int_M 2w\left(Q'_\theta-\frac{1}{c_1V}\right)dv+c_2\int_M 2R\Delta_bw\,dv-c_3\int_M 2w_{,0}R\,dv\\&=c_1A_1+c_2A_2+c_3A_3.\end{aligned}$$

We will calculate the change of each term under conformal change of θ. The easiest term to handle is the first one. Indeed, recall that if $\tilde\theta=e^w\theta$ then

$$\begin{aligned}\widetilde{Q'_\theta}e^{2w}&=P'_\theta w+Q'_\theta \mod \mathcal{P}^\perp,\\\widetilde{R}&=[R-|\nabla_bw|^2-2\Delta_bw]e^{-w},\\\widetilde{\Delta}_bf&=e^{-w}[\Delta_bf+\nabla_bf\cdot\nabla_bw].\end{aligned}$$

So if $\hat\theta_u=e^{uw}\theta$, we have

$$\int_M 2w\left[\widehat{Q'_\theta}-\frac{1}{c_1}\frac{1}{\widehat{V}}\right]d\hat v=\int_M 2uwP'_\theta w+2Q'w-\frac{1}{c_1}\frac{2we^{2uw}}{\int_M e^{2uw}\,dv}\,dv.$$

Integrating u in [0, 1] yields

$$\widetilde{A}_1(w) = \int_M wA_\theta w + Q'_\theta w\, dv - \frac{1}{c_1} \ln\left(\fint_M e^{2w}\, dv \right).$$

For the second term, we have

$$\begin{aligned}\int_M \widehat{R}\hat{\Delta}_b w\, d\hat{v} &= \int_M \left[R - u^2|\nabla_b w|^2 - 2u\Delta_b w\right]\left[\Delta_b w + u|\nabla_b w|^2\right] dv \\ &= \int_M R\Delta_b w - u^2|\nabla_b w|^2\Delta_b w - 2u(\Delta_b w)^2 + Ru|\nabla_b w|^2 \\ &\qquad\qquad - u^3|\nabla_b w|^4 - 2u^2|\nabla_b w|^2\Delta_b w\, dv.\end{aligned}$$

In particular after integrating over u between 0 and 1, we get

$$\begin{aligned}\widetilde{A}_2(w) &= 2\int_M R\Delta_b w - |\nabla_b w|^2\Delta_b w - (\Delta_b w)^2 + \frac{1}{2}R|\nabla_b w|^2 - \frac{1}{4}|\nabla_b w|^4\, dv \\ &= 2\int_M R\Delta_b w - \left(\Delta_b w + \frac{1}{2}|\nabla_b w|^2\right)^2 + \frac{R}{2}|\nabla_b w|^2\, dv.\end{aligned}$$

Next we compute

$$\int_M \widehat{T} w \widehat{R} d\hat{v} = \int_M \left[w_{,0}R - u^2 w_{,0}|\nabla_b w|^2 - 2uw_{,0}\Delta_b w\right] dv,$$

where T is the characteristic vector field of θ and we are adopting the notation $Tf = f_{,0}$. Integrating as above yields

$$\widetilde{A}_3(w) = 2\int_M w_{,0}R - \frac{1}{3}w_{,0}|\nabla_b w|^2 - w_{.0}\Delta_b w\, dv.$$

Therefore, one has

$$\zeta'_{\widetilde{A}_\theta}(0) - \zeta'_{A_\theta}(0) = c_1\widetilde{A}_1(w) + c_2\widetilde{A}_2(w) - c_3\widetilde{A}_3(w)$$

or equivalently

$$\ln\left(\frac{\det(A_\theta)}{\det(A_{\tilde{\theta}})}\right) = c_1\widetilde{A}_1(w) + c_2\widetilde{A}_2(w) - c_3\widetilde{A}_3(w). \qquad \square$$

4. Scaling invariant functional

We focus now on the study of the functional F defined by

$$F(w) = c_1 II(w) + c_2 III(w) + c_3 IV(w),$$

where $c_1 = -1/(24\pi^2)$. For the sake of notation, we will keep using c_1 instead of its numerical value. We will also be using constants C_k depending on M and θ.

We recall first that there exists a constant C such that

$$\frac{1}{16\pi^2}\int_M wA_\theta w+2Q'w\,dv-\ln\left(\fint e^{2w}\,dv\right)\geq C. \tag{10}$$

In fact this follows from the CR Beckner–Onofri inequality proved in [Branson et al. 2013] and also treated in [Case and Yang 2013]. Since the functional F is scaling invariant (that is, $F(w+c)=F(w)$), we can assume without loss of generality that $\bar{w}=\int_M w\,dv=0$. Also, we recall that $a_4=c_1Q'_\theta+c_2\Delta_bR+c_3R_{,0}$. Therefore,

$$\int_M a_4w\,dv=\int_M c_1Q'_\theta w+c_2R\Delta_bw-c_3Rw_{,0}\,dv.$$

Hence, we can write F as

$$\begin{aligned}F(w)=2\int_M a_4w\,dv+c_1\left(\int_M wA_\theta w\,dv-\int_M Q'_\theta\,dv\ln\left(\fint e^{2w}\right)\right)\\ -2c_2\int_M\left(\Delta_bw+\frac{1}{2}|\nabla_bw|^2\right)^2dv+c_2\int_M R|\nabla_bw|^2\,dv\\ +2c_3\int_M w_{,0}\left(\frac{1}{3}|\nabla_bw|^2+\Delta_bw\right)dv.\end{aligned}$$

Using (10), we have

$$\int_M Q'_\theta\,dv\ln\left(\fint e^{2w}\,dv\right)\leq\frac{\int_M Q'_\theta\,dv}{16\pi^2}\left[\int_M wA_\theta w+2Q'_\theta w\,dv\right]-C.$$

Therefore, for

$$a=\frac{\int_M Q'_\theta\,dv}{16\pi^2}$$

and since $c_1<0$, we have

$$\begin{aligned}&c_1\left(\int_M wA_\theta w\,dv-\int_M Q'_\theta\,dv\ln\left(\fint e^{2w}\right)\right)\\ &\quad\leq c_1\left[(1-a)\int_M wA_\theta w\,dv+2a\int Q'_\theta w\,dv\right]+C_1\\ &\quad\leq 4c_1(1-a)\int_M(\Delta_bw)^2\,dv+C_2\int_M|\nabla_bw|^2\,dv+2ac_1\int Q'_\theta w\,dv+C_3,\end{aligned} \tag{11}$$

where in the second line we used the expression (7). On the other hand, for the mixed term of $III(w)$, we have for every $\alpha>0$,

$$2\int_M\Delta_bw|\nabla_bw|^2\,dv\leq\alpha\int_M(\Delta_bw)^2+\frac{1}{\alpha}\int_M|\nabla_bw|^4\,dv. \tag{12}$$

Next, we let λ denote the best constant appearing in the estimate

$$\|f_{,0}\|_{L^2}\leq\lambda\|\Delta_bf\|_{L^2}.$$

Then we have

$$
\begin{aligned}
(13)\quad 2\int_M w_{,0}\Big(\frac{1}{3}|\nabla_b w|^2+\Delta_b w\Big)\,dv &\le 2\Big(\lambda\|\Delta_b w\|_{L^2}\Big\|\frac{1}{3}|\nabla_b w|^2+\Delta_b w\Big\|_{L^2}\Big)\\
&\le 2\lambda\Big(\|\Delta_b w\|_{L^2}^2+\frac{1}{3}\|\Delta_b w\|_{L^2}\big\||\nabla_b w|^2\big\|_{L^2}\Big)\\
&\le \lambda\bigg(\Big(2+\frac{\alpha}{3}\Big)\|\Delta_b w\|_{L^2}^2+\frac{1}{3\alpha}\int_M |\nabla_b w|^4\,dv\bigg).
\end{aligned}
$$

Hence, combining (11), (12) and (13) and assuming that $c_2>0$ and $c_3\ge 0$, we get

$$
\begin{aligned}
(14)\quad & c_1 II(w)+c_2 III(w)+c_3 IV(w)\\
&\le \Big(4c_1(1-a)+c_2(\alpha-2)+c_3\Big(2+\frac{\alpha}{3}\Big)\lambda\Big)\int_M (\Delta_b w)^2\,dv\\
&\quad +\Big(c_2\Big(\frac{1}{\alpha}-\frac{1}{2}\Big)+c_3\frac{\lambda}{3\alpha}\Big)\int_M |\nabla_b w|^4\,dv+C_4\int_M |\nabla_b w|^2\,dv\\
&\quad +2\int_M a_4 w\,dv+2ac_1\int_M Q'_\theta w\,dv+C_5.
\end{aligned}
$$

Now we need to choose α in a way that the coefficients of $\int_M(\Delta_b w)^2\,dv$ and $\int_M|\nabla_b w|^4\,dv$ are both negative. For this to happen, one needs that

$$
\begin{cases}
4c_1(1-a)-2c_2+2\lambda c_3<0,\\
\alpha<\dfrac{2c_2-2\lambda c_3-4c_1(1-a)}{c_2+\frac{1}{3}\lambda c_3},\\
\dfrac{1}{\alpha}<\dfrac{c_2}{2c_2+\frac{2}{3}\lambda c_3}.
\end{cases}
$$

Hence, we need

$$
\frac{2c_2+\frac{2}{3}\lambda c_3}{c_2}<\frac{2c_2-2\lambda c_3-4c_1(1-a)}{c_2+\frac{1}{3}\lambda c_3}.
$$

This is possible if condition (6) is satisfied for $\mu=3/(2\lambda)$. This yields in particular that if $w_k\in W_H^{2,2}(M)\cap\mathcal{P}$ is a maximizing sequence for F, then there exists $C>0$ such that

$$
\int_M(\Delta_b w_k)^2\,dv+\int_M|\nabla_b w_k|^4\,dv\le C.
$$

Indeed, we have

$$
-\varepsilon<F(0)\le F(w_k).
$$

Therefore, there exists $c > 0$ such that

$$-\varepsilon \le -c\left(\int_M (\Delta_b w_k)^2\,dv + \int_M |\nabla_b w_k|^4\,dv\right) + C_4\int_M |\nabla_b w_k|^2\,dv + 2\int_M a_4 w_k\,dv + 2ac_1\int_M Q'_\theta w_k\,dv + C_5. \tag{15}$$

Thus

$$\int_M (\Delta_b w_k)^2\,dv+\int_M |\nabla_b w_k|^4\,dv \le C_6\left(\int_M |\nabla_b w_k|^4\,dv\right)^{\frac{1}{2}} + C_7\left(\int_M |\nabla_b w_k|^4\,dv\right)^{\frac{1}{4}} + C_8,$$

where here we used the fact that $\bar{w} = 0$ and

$$\int_M f w\,dv \le \|f\|_{L^2}\|w\|_{L^2} \le \|f\|_{L^2}\|\nabla_b w\|_{L^2} \le \|f\|_{L^2} V^{\frac{1}{4}}\|\nabla_b w\|_{L^4},$$

in order to bound the terms of $\int_M a_4 w_k\,dv$ and $\int_M Q'_\theta w_k\,dv$ and Hölder's inequality to bound the term $\int_M |\nabla_b w_k|^2\,dv$.

Hence, $(w_k)_k$ is bounded in $W^{2,2}_H(M)$ and has a weakly convergent subsequence that converges to w_∞. Passing to the $\limsup$ in $F(w_k)$, we get that the weak limit w_∞ is in fact a maximizer of F.

Finally, based on the remark below (5), we see that the critical points of F under the constraint $\int_M e^{2w} dv = 1$ satisfy the equation

$$\tilde{\tau}\big[c_1\widetilde{Q'_\theta} + c_2\widetilde{\Delta_b}\widetilde{R} + c_3 e^{-2w}\mathcal{H}\big] = cte.$$

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