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JIAN-HONG WANG AND YU ZHENG

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ELLIPTIC GRADIENT ESTIMATES FOR A PARABOLIC EQUATION WITH V-LAPLACIAN AND APPLICATIONS

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In this paper, we establish a local elliptic gradient estimate for positive bounded solutions to a parabolic equation concerning the V-Laplacian

$$\big(\Delta_V - \partial_t - q(x,t)\big)u(x,t) = F(u(x,t))$$

on an n-dimensional complete Riemannian manifold with the Bakry–Émery Ricci curvature Ric_V bounded below, which is weaker than the m-Bakry–Émery Ricci curvature Ric_V^m bounded below considered by Chen and Zhao (2018). As applications, we obtain the local elliptic gradient estimates for the cases that $F(u) = au\ln u$ and au^γ. Moreover, we prove parabolic Liouville theorems for the solutions satisfying some growth restriction near infinity and study the problem about conformal deformation of the scalar curvature. In the end, we also derive a global Bernstein-type gradient estimate for the above equation with $F(u) = 0$.

1. Introduction and main results

In this paper, we will study local and global elliptic gradient estimates for positive smooth bounded solutions $u(x,t)$ to a parabolic equation

$$\big(\Delta_V - \partial_t - q(x,t)\big)u(x,t) = F(u(x,t)) \tag{1.1}$$

on an n-dimensional complete Riemannian manifold (M^n, g), where $q(x,t)$ is a function which is C^2 in the x-variable and C^1 in the t-variable, and $F(u)$ is a C^2 function of u.

The Equation (1.1) is an important extension of the Schrödinger equation. The V-Laplacian is defined by

$$\Delta_V := \Delta - V\cdot\nabla,$$

where V is a smooth vector field.

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As in [Chen et al. 2012], we define the m-Bakry–Émery Ricci curvature

$$\mathrm{Ric}_V^m := \mathrm{Ric} + \tfrac{1}{2}\mathcal{L}_V g - \frac{1}{m} V^* \otimes V^*$$

for any number $m \geq 0$, where Ric is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative in the direction of V, and V^* is the metric-dual of V. When $m = 0$, it means that $V \equiv 0$ and Ric_V^m returns to the usual Ricci tensor. The (∞)-Bakry–Émery Ricci curvature is

$$\mathrm{Ric}_V := \mathrm{Ric} + \tfrac{1}{2}\mathcal{L}_V g.$$

It is easy to see that $\mathrm{Ric}_V^m \geq c$ implies $\mathrm{Ric}_V \geq c$, but not vice versa.

If $\mathrm{Ric}_V = \lambda g$ for some real constant λ, then (M^n, g) is a Ricci soliton, which is a natural extension of Einstein metric. A Ricci soliton is called shrinking, steady or expanding, if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. In particular, when $V = \nabla f$ for some function $f \in C^\infty(M)$, since $\mathcal{L}_{\nabla f} g = 2\,\mathrm{Hess}\, f$ (Hess is the Hessian with respect to the metric g), a Ricci soliton becomes a gradient Ricci soliton. The gradient Ricci soliton plays an important role in the formation of singularities of the Ricci flow, and has been studied by many authors; see [Cao 2010; Hamilton 1995] for nice surveys.

Relating to the V-Laplacian, we have, for $\mathrm{Ric}_V^m (0 < m < \infty)$, the following Bochner formula:

$$\begin{aligned} \tfrac{1}{2}\Delta_V |\nabla u|^2 &= |\nabla\nabla u|^2 + \langle \nabla u, \nabla \Delta_V u\rangle + \mathrm{Ric}_V^m(\nabla u, \nabla u) + \frac{1}{m}|\langle V, \nabla u\rangle|^2 \\ &\geq \frac{(\Delta_V u)^2}{m+n} + \langle \nabla u, \nabla \Delta_V u\rangle + \mathrm{Ric}_V^m(\nabla u, \nabla u). \end{aligned} \tag{1.2}$$

When $m = \infty$, we have

$$\tfrac{1}{2}\Delta_V |\nabla u|^2 = |\nabla\nabla u|^2 + \langle \nabla u, \nabla \Delta_V u\rangle + \mathrm{Ric}_V(\nabla u, \nabla u). \tag{1.3}$$

The formula (1.2) looks like the classical Bochner formula on an $(m+n)$-dimensional manifold with Ricci tensor, therefore many geometric results for the Laplacian on n-dimensional manifolds with Ric bounded below can be possibly extended to the V-Laplacian on $(m+n)$-dimensional manifolds with Ric_V^m bounded below, such as the mean curvature comparison theorem, the volume comparison theorem, etc. However, for Ric_V, due to lack of the term $\frac{1}{m}|\langle V, \nabla u\rangle|^2$, there seems essential obstacles to obtaining some important conclusions when Ric_V is only bounded below.

To the best of our knowledge, the gradient estimate technique was originated by S.-T. Yau [1975] in the 1970s, who first proved a gradient estimate for the harmonic function on manifolds. In the 1980s, this technique was developed by Li and Yau [1986] for the heat equation on manifolds, and yielded a parabolic gradient estimate (sometimes called Li–Yau's gradient estimate). More precisely,

Theorem A [Li and Yau 1986]. *Let M be a complete manifold with dimension $n \geq 2$, $\operatorname{Ric}(M) \geq -k,\ k \geq 0$. Suppose u is any positive solution to the heat equation in $B(x_0, R) \times [t_0-T, t_0]$. Then*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{c_n}{R^2} + \frac{c_n}{T} + c_n k$$

in $B\big(x_0, \frac{R}{2}\big) \times \big[t_0 - \frac{T}{2}, t_0\big]$. Here c_n depends only on n.

Li and Yau [1986] also proved a parabolic gradient estimate for the Schrödinger equation

$$\big(\Delta - \partial_t - q(x, t)\big)u(x, t) = 0,$$

which can be seen as the special case of (1.1) (see [Li and Yau 1986, Theorem 1.2]).

In the 1990s, R. Hamilton [1993] proved a global elliptic gradient estimate (sometimes called Hamilton's gradient estimate) for the heat equation on closed manifolds.

Theorem B [Hamilton 1993]. *Let M be an n-dimensional closed manifold with $\operatorname{Ric} \geq -K$ for nonnegative constant K, and let u be a positive solution of the heat equation*

$$\frac{\partial u}{\partial t} = \Delta u$$

with $u \leq A$ for all time. Then

$$t|\nabla u|^2 \leq (1 + 2Kt)u^2 \ln \frac{A}{u}.$$

Hamilton's gradient estimate requires that the equation be defined on closed manifolds. Later, Souplet and Zhang [2006] proved a local elliptic gradient estimate (sometimes called Souplet–Zhang's gradient estimate) for the heat equation on noncompact manifolds by inserting a necessary logarithmic correction term.

Theorem C [Souplet and Zhang 2006]. *Let M be a Riemannian manifold with dimension $n \geq 2$ and $\operatorname{Ric} \geq -k,\ k \geq 0$. Assume u is any positive solution to the heat equation in $Q_{R,T} = B_{x_0}(R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$ with $u \leq M$. Then there exists a dimensional constant c such that*

$$|\nabla \ln u| \leq c\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right)\left(1 + \ln \frac{M}{u}\right)$$

in $Q_{R/2, T/2}$.

Moreover, if M has nonnegative Ricci curvature and u is any positive solution of the heat equation on $M \times (0, \infty)$, then there exist dimensional constants c_1, c_2 such that

$$|\nabla \ln u| \leq c_1 \frac{1}{\sqrt{t}}\left(c_2 + \ln \frac{u(x, 2t)}{u(x, t)}\right)$$

for all $x \in M$ and $t > 0$.

Apart from the above theorems, Li–Yau's, Hamilton's and Souplet–Zhang's gradient estimates have been generalized to other linear and nonlinear equations on Riemannian manifolds, see, e.g., [Brighton 2013; Chow and Hamilton 1997; Chen and Qiu 2016; Cao and Zhang 2011; Huang and Ma 2016; Li and Xu 2011; Li 1991; 2012; 2015; Ma 2006; Ruan 2007; Wu 2015; Yang 2008; Zhu 2016].

We now give the main theorems, a local elliptic (Souplet–Zhang's) gradient estimate for positive smooth solutions to (1.1), which is based on the arguments of Souplet and Zhang [2006] for the heat equation, Brighton [2013] for the f-harmonic function and J.-Y. Wu [2015] for the f-heat equation. It is important that our gradient estimate does not depend on any assumption on V.

Theorem 1.1. *Let (M^n, g) be an n-dimensional complete Riemannian manifold, and let $B_{x_0}(R)$ be a geodesic ball of radius R around x_0 and $R \geq 2$. Assume* $\mathrm{Ric}_V \geq -k$ *in $B_{x_0}(R)$ for some constant $k \geq 0$. Let u be a positive solution of* (1.1) *in $Q_{R,T} = B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$ with $u \leq M$ for some positive constant M, where $t_0 \in \mathbb{R}$ and $T > 0$. Then there exists a dimensional constant $C(n)$ such that*

$$|\nabla \ln u| \leq C(n)\left(\sqrt{\frac{1+|\delta|}{R}} + \frac{1}{\sqrt{t-t_0+T}} + \sqrt{k+\lambda_1+\lambda_2+\lambda_3+\lambda_4}\right)\left(1+\ln\frac{M}{u}\right) \tag{1.4}$$

in $Q_{R/2,T}$ with $t \neq t_0 - T$. Here

$$\lambda_1 = -\min\left\{0, \min_{Q_{R,T}} \frac{F(u)}{u}\right\}, \quad \lambda_2 = -\min\left\{0, \min_{Q_{R,T}}\left(F'(u) - \frac{F(u)}{u}\right)\right\},$$

$$\lambda_3 = 2\max_{Q_{R,T}}\{q^-\} \quad (q^- = \max\{-q, 0\} \text{ is the negative part of } q),$$

$$\lambda_4 = \max_{Q_{R,T}}\left|\nabla\sqrt{|q|}\right|,$$

which are nonnegative constants, and $\delta = \max_{\{x \mid d(x,x_0)=1\}} \Delta_V r(x)$.

Remark 1.2. Theorem 1.1 describes local elliptic gradient estimate under only Ric_V bounded below, whose assumption on Ric_V is obviously weaker than the assumption on $\mathrm{Ric}_V^m (m < \infty)$ which was considered by Chen and Zhao [2018].

On one hand, we apply Theorem 1.1 to analyze the existence of solutions to the special case of (1.1). Moreover, we study the problem about conformal deformation of the scalar curvature on complete noncompact manifolds; see Corollary 2.7 in Section 2.

Theorem 1.3. *Let (M^n, g) be an n-dimensional complete manifold with* $\mathrm{Ric}_V \geq 0$. *Consider the equation*

$$\big(\Delta_V - \partial_t - q(x)\big)u(x,t) = au^\gamma \tag{1.5}$$

for some constants $a \geq 0$ *and* $\gamma > 1$. *Suppose that* $q(x) \neq 0$ *and*

$$q^- = o(R^{-1}), \quad \left|\nabla\sqrt{|q|}\right| = o(R^{-1}) \quad \text{as } R \to \infty.$$

Then there does not exist any positive ancient solution (*that is*, *a solution defined in all space and negative time*) *to* (1.5) *such that* $u(x,t) = o(r(x)^{1/2} + |t|^{1/2})$ *near infinity. In particular*, *if* $V \equiv 0$, *we only assume* $u(x,t) = o(r(x) + |t|^{1/2})$ *near infinity.*

Remark 1.4. If $q(x)$ is a positive constant, it naturally satisfies the growth conditions of $q(x)$ in Theorem 1.3. There also exist many nontrivial functions $q(x)$ satisfying these growth conditions, such as $q(x) = e^{-x}$ in R^1.

On the other hand, we apply Theorem 1.1 to prove the parabolic Liouville theorem for the V-heat equation under certain growth conditions of solutions. This result is similar to the cases of the heat equation and the f-heat equation, obtained by Souplet and Zhang [2006] and Wu [2015], respectively.

Theorem 1.5. *Let* (M^n, g) *be an n-dimensional complete Riemannian manifold with* $\mathrm{Ric}_V \geq 0$. *Let* $u(x,t)$ *be an eternal solution* (*that is*, *a solution defined in all space and time*) *to*

$$(\Delta_V - \partial_t)u = 0. \tag{1.6}$$

Then the following conclusions hold.

(i) *If* $u(x,t) = e^{o(r^{1/2}(x)+|t|^{1/2})}$ *near infinity and* $u > 0$, *then u is a constant.*

(ii) *If* $u(x,t) = o(r^{1/2}(x) + |t|^{1/2})$ *near infinity*, *then u is a constant.*

Remark 1.6. The growth condition of u is necessary. For example, let $u = e^{x+2t}$, $V = \nabla f$, $f = -x$ in $\mathbb{R}^1$. Then u is a nonconstant positive eternal solution to (1.6). Any complete shrinking or steady Ricci solitons satisfy $\mathrm{Ric}_V \geq 0$, hence Theorem 1.5 also holds on shrinking or steady Ricci solitons.

In the end, we derive a global Bernstein-type gradient estimate for positive bounded solution to (1.1) with $F(u) = 0$ on complete Riemannian manifolds with Ric_V bounded below, which is inspired by the works of Kotschwar [2007] for the heat equation and Wu [2015] for the f-heat equation.

Theorem 1.7. *Let* (M^n, g) *be an n-dimensional complete Riemannian manifold with* $\mathrm{Ric}_V \geq -k$ *for some constant* $k \geq 0$. *Let u be a solution to*

$$\big(\Delta_V - \partial_t - q(x,t)\big)u(x,t) = 0 \tag{1.7}$$

in $M^n \times [0, T]$ *with* $0 < T < \infty$. *Suppose that*

$$0 < u \leq M, \quad q^-(x,t) \leq \alpha \quad \text{and} \quad \left|\nabla\sqrt{|q|}\right| \leq \beta,$$

where M, α, β are positive constants. Then there exists an absolute constant C such that

$$t|\nabla u|^2 \le CM^2\big(1+(k+\alpha+\beta)T\big)$$

in $M^n \times [0, T]$.

The rest of this paper is organized as follows. In Section 2, we give a useful lemma and a cut-off function to prove Theorem 1.1 via the maximum principle and V-Laplacian comparison theorem. As applications of Theorem 1.1, we prove Theorems 1.3 and 1.5. Moreover, we apply Theorem 1.3 to discuss Yamabe type problems and obtain Corollary 2.7. In Section 3, we prove Theorem 1.7 by using another local elliptic gradient estimate for (1.7).

2. Local elliptic gradient estimate

In this section, we first follow the techniques of [Souplet and Zhang 2006; Brighton 2013; Wu 2015] to prove Theorem 1.1. Notice that $0 < u \le M$ is a solution of (1.1). Define a smooth function

$$f = \ln \frac{u}{M} \quad \text{in } Q_{R,T}.$$

Obviously, $f \le 0$. By (1.1), we have

$$(\Delta_V - \partial_t)f + |\nabla f|^2 - q(x,t) = A(f), \quad \text{where } A(f) = \frac{F(Me^f)}{Me^f} = \frac{F(u)}{u}. \tag{2.1}$$

Let

$$g := |\nabla \ln(1-f)|^2 = \frac{|\nabla f|^2}{(1-f)^2},$$

we have the following lemma.

Lemma 2.1. *Let (M^n, g) be an n-dimensional complete Riemannian manifold with* $\mathrm{Ric}_V \ge -k$ *for some constant $k \ge 0$. Then g satisfies*

$$(\Delta_V - \partial_t)g \ge \frac{2f}{1-f}\langle \nabla f, \nabla g\rangle + 2(1-f)g^2 - 2(k+\lambda_1+\lambda_2+\lambda_3)g - 2\lambda_4^2, \tag{2.2}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the same as in Theorem 1.1.

Proof. Let $h = \ln(1-f)$, i.e., $g = |\nabla h|^2$. By the Bochner formula (1.3) and $\mathrm{Ric}_V \ge -k$, we have

$$\begin{aligned} \Delta_V g &= 2\big(|\nabla\nabla h|^2 + \langle \nabla h, \nabla \Delta_V h\rangle + \mathrm{Ric}_V(\nabla h, \nabla h)\big) \\ &\ge 2\big(\langle \nabla h, \nabla \Delta_V h\rangle - k|\nabla h|^2\big). \end{aligned} \tag{2.3}$$

Since $\nabla h = -\frac{\nabla f}{1-f}$ and

$$\Delta_V h = \Delta h - \langle V, \nabla h\rangle = -\frac{(1-f)\Delta f + |\nabla f|^2}{(1-f)^2} + \left\langle V, \frac{\nabla f}{1-f}\right\rangle$$
$$= -\frac{\Delta_V f}{1-f} - \frac{|\nabla f|^2}{(1-f)^2}.$$

By a direct computation, we obtain

$$\langle \nabla h, \nabla \Delta_V h\rangle = \frac{2\nabla\nabla f(\nabla f, \nabla f)}{(1-f)^3} + \frac{2|\nabla f|^4}{(1-f)^4} + \frac{\langle \nabla f, \nabla \Delta_V f\rangle}{(1-f)^2} + \frac{|\nabla f|^2 \Delta_V f}{(1-f)^3}.$$

Hence, (2.3) becomes

$$\Delta_V g \geq \frac{4\nabla\nabla f(\nabla f, \nabla f)}{(1-f)^3} + \frac{4|\nabla f|^4}{(1-f)^4} + \frac{2\langle \nabla f, \nabla \Delta_V f\rangle}{(1-f)^2} + \frac{2|\nabla f|^2 \Delta_V f}{(1-f)^3} - 2k\frac{|\nabla f|^2}{(1-f)^2}. \tag{2.4}$$

By using (2.1), we obtain

$$\begin{aligned}\partial_t g &= \frac{2\langle \nabla f, \nabla f_t\rangle}{(1-f)^2} + \frac{2|\nabla f|^2 f_t}{(1-f)^3}\\ &= \frac{2\langle \nabla f, \nabla \Delta_V f\rangle}{(1-f)^2} + \frac{4\nabla\nabla f(\nabla f, \nabla f)}{(1-f)^2} + \frac{2|\nabla f|^2 \Delta_V f}{(1-f)^3} + \frac{2|\nabla f|^4}{(1-f)^3}\\ &\quad - \frac{2\langle \nabla f, \nabla (q+A)\rangle}{(1-f)^2} - \frac{2(q+A)|\nabla f|^2}{(1-f)^3}.\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$\begin{aligned}(\Delta_V - \partial_t) g \geq{}& \frac{4\nabla\nabla f(\nabla f, \nabla f)}{(1-f)^3} + \frac{4|\nabla f|^4}{(1-f)^4} - \frac{4\nabla\nabla f(\nabla f, \nabla f)}{(1-f)^2} - \frac{2|\nabla f|^4}{(1-f)^3}\\ &- 2k\frac{|\nabla f|^2}{(1-f)^2} + \frac{2\langle \nabla f, \nabla (q+A)\rangle}{(1-f)^2} + \frac{2(q+A)|\nabla f|^2}{(1-f)^3}.\end{aligned} \tag{2.6}$$

Since $g = \frac{|\nabla f|^2}{(1-f)^2}$, then

$$\langle \nabla g, \nabla f\rangle = \frac{2\nabla\nabla f(\nabla f, \nabla f)}{(1-f)^2} + \frac{2|\nabla f|^4}{(1-f)^3},$$

which implies

$$\begin{aligned}0 ={}& -2\langle \nabla g, \nabla f\rangle + \frac{4\nabla\nabla f(\nabla f, \nabla f)}{(1-f)^2} + \frac{4|\nabla f|^4}{(1-f)^3}\\ &+ \frac{1}{1-f}\left(2\langle \nabla g, \nabla f\rangle - \frac{4|\nabla f|^4}{(1-f)^3}\right) - \frac{4\nabla\nabla f(\nabla f, \nabla f)}{(1-f)^3}.\end{aligned} \tag{2.7}$$

From (2.7), we know that

$$\frac{4\nabla\nabla f(\nabla f,\nabla f)}{(1-f)^3}+\frac{4|\nabla f|^4}{(1-f)^4}-\frac{4\nabla\nabla f(\nabla f,\nabla f)}{(1-f)^2}-\frac{2|\nabla f|^4}{(1-f)^3} = \frac{2f}{1-f}\langle\nabla g,\nabla f\rangle+\frac{2|\nabla f|^4}{(1-f)^3}.$$

Using the above equality, (2.6) becomes

$$(\Delta_V-\partial_t)g \ge \frac{2f}{1-f}\langle\nabla f,\nabla g\rangle+2(1-f)g^2-2\Big(k-A'(f)-\frac{A}{1-f}\Big)g - \frac{2}{1-f}|\nabla q|\sqrt{g}+\frac{2q}{1-f}g. \tag{2.8}$$

Since $0<\frac{1}{1-f}\le 1$ and

$$2|\nabla q|\sqrt{g}\le 2|q|g+\frac{|\nabla q|^2}{2|q|}=2|q|g+2\big|\nabla\sqrt{|q|}\big|^2. \tag{2.9}$$

Noticing that the inequality (2.9) is trivial when $q=0$. Hence,

$$-\frac{2|\nabla q|\sqrt{g}}{1-f}\ge-\frac{2|q|g}{1-f}-\frac{2\big|\nabla\sqrt{|q|}\big|^2}{1-f}\ge-\frac{2|q|g}{1-f}-2\big|\nabla\sqrt{|q|}\big|^2.$$

Using this inequality, (2.8) can be rewritten as

$$(\Delta_V-\partial_t)g \ge \frac{2f}{1-f}\langle\nabla f,\nabla g\rangle+2(1-f)g^2-2\bigg(k-A'(f)-\frac{A}{1-f}+\frac{2q^-}{1-f}\bigg)g - 2\big|\nabla\sqrt{|q|}\big|^2.$$

By the definitions of $\lambda_1,\lambda_2,\lambda_3,\lambda_4$, we have

$$-\frac{A}{1-f}\le\frac{A^-}{1-f}\le A^-=-\min\{0,\frac{F(u)}{u}\}\le\lambda_1,$$
$$-A'(f)=-\Big(F'(u)-\frac{F(u)}{u}\Big)\le\lambda_2,\qquad \frac{2q^-}{1-f}\le\lambda_3,\qquad \big|\nabla\sqrt{|q|}\big|\le\lambda_4.$$

Hence, (2.2) immediately follows. □

In order to prove Theorem 1.1, we introduce a useful cut-off function which originated with Li and Yau [1986] (see also [Bailesteanu et al. 2010; Souplet and Zhang 2006]).

Lemma 2.2. *Fix $t_0\in\mathbb{R}$ and $T>0$. For any given $\tau\in(t_0-T,t_0]$, there exists a smooth function $\bar\eta:[0,\infty)\times[t_0-T,t_0]\to\mathbb{R}$ satisfying following propositions*:

(1) *$0\le\bar\eta(r,t)\le 1$ in $[0,R]\times[t_0-T,t_0]$, and it is supported in a subset of $[0,R]\times[t_0-T,t_0]$.*

(2) $\bar{\eta}(r,t)=1$ *and* $\partial_r\bar{\eta}(r,t)=0$ *in* $\left[0,\frac{R}{2}\right]\times[\tau,t_0]$ *and* $\left[0,\frac{R}{2}\right]\times[t_0-T,t_0]$, *respectively.*

(3) $|\partial_t\bar{\eta}|\le C\bar{\eta}^{1/2}/(\tau-t_0+T)$ *in* $[0,\infty)\times[t_0-T,t_0]$ *for some constant* $C>0$, *and* $\bar{\eta}(r,t_0-T)=0$ *for all* $r\in[0,\infty)$.

(4) $-(C_\epsilon/R)\bar{\eta}^\epsilon\le\partial_r\bar{\eta}\le 0$ *and* $|\partial_r^2\bar{\eta}|\le C_\epsilon\bar{\eta}^\epsilon/R^2$ *in* $[0,\infty)\times[t_0-T,t_0]$ *for every* $\epsilon\in(0,1)$ *with some constant* C_ϵ *depending on* ϵ.

Now, we apply Lemmas 2.1 and 2.2 to prove Theorem 1.1 via the maximum principle and the V-Laplacian comparison theorem [Wu 2018, Theorem 2.1] in a local space-time supported set.

Proof of Theorem 1.1. Fix any number $\tau\in(t_0-T,t_0]$, we will show that (1.4) holds at (x,τ) for all $x\in B_{x_0}\left(\frac{R}{2}\right)$. The assertion of theorem will immediately follows due to τ is arbitrary.

Choose a cut-off function $\bar{\eta}(r,t)$ satisfying the propositions of Lemma 2.2. Let $\eta: M\times[t_0-T,t_0]\to\mathbb{R}$ such that $\eta(x,t)=\bar{\eta}(r(x),t)$, where $r(x)=d(x,x_0)$. It is easy to see that $\eta(x,t)$ is supported in $Q_{R,T}$. Our aim is to calculate $(\Delta_V-\partial_t)(\eta g)$ and estimate each term at a space-time point where ηg attains its maximum.

From Lemma 2.1, we conclude

$$
\begin{aligned}
(2.10)\quad (\Delta_V-\partial_t)(\eta g)&-\left(\frac{2f}{1-f}\nabla f+2\frac{\nabla\eta}{\eta}\right)\nabla(\eta g)\\
&\ge 2(1-f)\eta g^2-\left(\frac{2f}{1-f}\langle\nabla f,\nabla\eta\rangle\right)g-2\frac{|\nabla\eta|^2}{\eta}g+(\Delta_V\eta)g-\eta_t g\\
&\qquad-2(k+\lambda_1+\lambda_2+\lambda_3)\eta g-2\lambda_4^2\eta.
\end{aligned}
$$

Assume

$$(\eta g)(x_1,t_1)=\max_{B_{x_0}(R)\times[t_0-T,\tau]}(\eta g).$$

We may assume $(\eta g)(x_1,t_1)>0$, otherwise, $g(x,\tau)\le 0$ and (1.4) naturally holds at (x,τ) whenever $d(x,x_0)<\frac{R}{2}$. Notice that $t_1\ne t_0-T$ due to $(\eta g)(x_1,t_1)>0$. We may also assume that $\eta(x,t)$ is smooth at (x_1,t_1) by the standard Calabi argument [1958]. Using the maximum principle, at (x_1,t_1), we have

$$\Delta_V(\eta g)\le 0,\quad (\eta g)_t\ge 0\quad\text{and}\quad\nabla(\eta g)=0.$$

Hence, (2.10) can be simplified as

$$
\begin{aligned}
(2.11)\quad 2(1-f)\eta g^2\le\left(\frac{2f}{1-f}\langle\nabla f,\nabla\eta\rangle+2\frac{|\nabla\eta|^2}{\eta}\right)g-(\Delta_V\eta)g+\eta_t g&\\
+2(k+\lambda_1+\lambda_2+\lambda_3)\eta g+2\lambda_4^2\eta.&
\end{aligned}
$$

at (x_1,t_1). In the following, we will estimate each term on the right hand side of (2.11) and obtain the desired gradient estimate in Theorem 1.1. We will get it by two steps.

Case I. Assume $x_1 \notin B_{x_0}(1)$. Since $\mathrm{Ric}_V \geq -k$ and $d(x_1, x_0) \geq 1$ in $B_{x_0}(R)$, $R \geq 2$, by the V-Laplacian comparison theorem, we have

$$\Delta_V r(x_1) \leq \delta + k(R-1),$$

where $\delta = \max_{\{x|d(x,x_0)=1\}} \Delta_V r(x)$.

Below the Young's inequality and Lemma 2.2 will be repeatedly used in the following estimate. Let c be a constant depending only on n whose value may change from line to line. Then we have the following inequalities:

$$\begin{aligned}\frac{2f}{1-f}\langle\nabla f, \nabla\eta\rangle g &\leq 2|f||\nabla\eta|g^{3/2}\\ &= 2\big[\eta(1-f)g^2\big]^{3/4}\cdot\frac{|f||\nabla\eta|}{\big[\eta(1-f)\big]^{3/4}}\\ &\leq \eta(1-f)g^2 + c\frac{f^4|\nabla\eta|^4}{(1-f)^3\eta^3}\\ &\leq \eta(1-f)g^2 + c\frac{f^4}{R^4(1-f)^3}.\end{aligned}\tag{2.12}$$

$$2\frac{|\nabla\eta|^2}{\eta}g \leq \tfrac{1}{8}\eta g^2 + 8\frac{|\nabla\eta|^4}{\eta^3} \leq \tfrac{1}{8}\eta g^2 + \frac{c}{R^4}.\tag{2.13}$$

$$\begin{aligned}-(\Delta_V\eta)g &= -\big(\partial_r^2\bar\eta + \partial_r\bar\eta\Delta_V r\big)g\\ &\leq \Big(|\partial_r^2\bar\eta| + \big(|\delta| + k(R-1)\big)|\partial_r\bar\eta|\Big)g\\ &= \eta^{1/2}g\frac{|\partial_r^2\bar\eta|}{\bar\eta^{1/2}} + \big(|\delta| + k(R-1)\big)\eta^{1/2}g\frac{|\partial_r\bar\eta|}{\bar\eta^{1/2}}\\ &\leq \tfrac{1}{8}\eta g^2 + c\bigg(\frac{|\partial_r^2\bar\eta|^2}{\bar\eta} + \delta^2\frac{|\partial_r\bar\eta|^2}{\bar\eta} + k^2(R-1)^2\frac{|\partial_r\bar\eta|^2}{\bar\eta}\bigg)\\ &\leq \tfrac{1}{8}\eta g^2 + \frac{c}{R^4} + \frac{c\delta^2}{R^2} + ck^2\frac{(R-1)^2}{R^2}\\ &\leq \tfrac{1}{8}\eta g^2 + \frac{c}{R^4} + \frac{c\delta^2}{R^2} + ck^2.\end{aligned}\tag{2.14}$$

$$|\eta_t|g = \eta^{1/2}g\frac{|\bar\eta_t|}{\bar\eta^{1/2}} \leq \tfrac{1}{8}\eta g^2 + 8\frac{|\bar\eta_t|^2}{\bar\eta} \leq \tfrac{1}{8}\eta g^2 + \frac{c}{(\tau-t_0+T)^2}.\tag{2.15}$$

$$2(k+\lambda_1+\lambda_2+\lambda_3)\eta g \leq \tfrac{1}{8}\eta g^2 + 8(k+\lambda_1+\lambda_2+\lambda_3)^2.\tag{2.16}$$

$$2\lambda_4^2\eta \leq 2\lambda_4^2.\tag{2.17}$$

Substituting (2.12)–(2.17) into the right hand side of (2.11), at (x_1, t_1), we have

$$2\eta(1-f)g^2 \le \eta(1-f)g^2+\tfrac{1}{2}\eta g^2+c\frac{f^4}{R^4(1-f)^3}+\frac{c}{R^4}+\frac{c\delta^2}{R^2}+ck^2+\frac{c}{(\tau-t_0+T)^2} +8(k+\lambda_1+\lambda_2+\lambda_3)^2+2\lambda_4^2.$$

Since $1-f \ge 1$, the above estimate implies

$$\begin{aligned}(\eta g^2)(x_1,t_1) &\le \frac{1}{1-f}\Big(\tfrac{1}{2}\eta g^2+c\frac{f^4}{R^4(1-f)^3}+\frac{c}{R^4}+\frac{c\delta^2}{R^2}+ck^2+\frac{c}{(\tau-t_0+T)^2} \\ &\qquad +8(k+\lambda_1+\lambda_2+\lambda_3)^2+2\lambda_4^2\Big)\\ &\le \tfrac{1}{2}\eta g^2+c\frac{f^4}{R^4(1-f)^4}+\frac{c}{R^4}+\frac{c\delta^2}{R^2}+ck^2+\frac{c}{(\tau-t_0+T)^2} \\ &\qquad +8(k+\lambda_1+\lambda_2+\lambda_3)^2+2\lambda_4^2\\ &\le \tfrac{1}{2}\eta g^2+\frac{c}{R^4}+\frac{c\delta^2}{R^2}+\frac{c}{(\tau-t_0+T)^2}+c(k+\lambda_1+\lambda_2+\lambda_3+\lambda_4)^2.\end{aligned}$$

Since $0\le\eta\le 1$, then

$$\begin{aligned}(\eta g)^2(x_1,t_1) &\le (\eta g^2)(x_1,t_1)\\ &\le \frac{c}{R^4}+\frac{c\delta^2}{R^2}+\frac{c}{(\tau-t_0+T)^2}+c(k+\lambda_1+\lambda_2+\lambda_3+\lambda_4)^2.\end{aligned}$$

Since $\eta(x,\tau)=1$ when $x\in B_{x_0}\big(\frac{R}{2}\big)$ by the proposition (2) in Lemma 2.2 and $R\ge 2$, we obtain

$$\begin{aligned}g(x,\tau)=(\eta g)(x,\tau) &\le (\eta g)(x_1,t_1)\\ &\le \frac{c}{R^2}+\frac{c|\delta|}{R}+\frac{c}{\tau-t_0+T}+c(k+\lambda_1+\lambda_2+\lambda_3+\lambda_4)\\ &\le \frac{c(1+|\delta|)}{R}+\frac{c}{\tau-t_0+T}+c(k+\lambda_1+\lambda_2+\lambda_3+\lambda_4)\end{aligned}$$

for all $x\in B_{x_0}\big(\frac{R}{2}\big)$.

Case II. Assume $x_1\in B_{x_0}(1)\subset B_{x_0}\big(\frac{R}{2}\big)$ when $R\ge 2$. In this case, η is a constant in space direction in $Q_{R/2,T}$. Hence (2.11) can be simplified as

$$2(1-f)\eta g^2 \le \eta_t g+2(k+\lambda_1+\lambda_2+\lambda_3)\eta g+2\lambda_4^2\eta$$

at (x_1,t_1). Since $1-f\ge 1$, this implies

$$2\eta g^2 \le \eta_t g+2(k+\lambda_1+\lambda_2+\lambda_3)\eta g+2\lambda_4^2\eta$$

at (x_1,t_1). Substituting (2.15)–(2.17) into the right hand side of the above inequality, we have

$$(\eta g^2)(x_1,t_1)\le c\Big(\frac{1}{(\tau-t_0+T)^2}+(k+\lambda_1+\lambda_2+\lambda_3)^2+\lambda_4^2\Big).$$

Since $0\le\eta\le 1$, we have

$$(\eta g)(x_1,t_1)\le c\Big(\frac{1}{\tau-t_0+T}+k+\lambda_1+\lambda_2+\lambda_3+\lambda_4\Big).$$

Since $\eta(x,\tau)=1$ whenever $x\in B_{x_0}\big(\frac{R}{2}\big)$,

$$\begin{aligned}g(x,\tau)=(\eta g)(x,\tau)&\le(\eta g)(x_1,t_1)\\&\le c\Big(\frac{1}{\tau-t_0+T}+k+\lambda_1+\lambda_2+\lambda_3+\lambda_4\Big)\end{aligned}$$

for all $x\in B_{x_0}\big(\frac{R}{2}\big)$.

Combining the above two cases, by the definition of g and the fact that $\tau\in(t_0-T,t_0]$ was chosen arbitrarily, we obtain

$$\frac{|\nabla f|}{1-f}(x,t)\le C(n)\bigg(\sqrt{\frac{1+|\delta|}{R}}+\frac{1}{\sqrt{t-t_0+T}}+\sqrt{k+\lambda_1+\lambda_2+\lambda_3+\lambda_4}\bigg)$$

for any $(x,t)\in Q_{R/2,T}$ with $t\ne t_0-T$. Substituting $f=\ln\frac{u}{M}$ into the above estimate completes the proof of theorem. □

Remark 2.3. If $q(x,t)=F(u)=0$, $V=\nabla f$ for some function f, then Theorem 1.1 is the same as Theorem 1.1 in [Wu 2018] for the weighted heat equation. From the proof of Theorem 1.1, we know that the term $\sqrt{(1+|\delta|)/R}$ in (1.4) can be changed into $\frac{1}{R}$ whenever $V\equiv 0$.

When V is bounded, we can prove another gradient estimate of (1.1) in any geodesic ball. Its proof is similar to that of Theorem 1.1 except that the V-Laplacian comparison theorem in Theorem 1.1 is replaced by another V-Laplacian comparison theorem [Wu 2018, Theorem 2.2]. We only provide the conclusion and omit the proof.

Theorem 2.4. *Let (M^n,g) be an n-dimensional complete Riemannian manifold. Assume* $\mathrm{Ric}_V\ge -k$ *and $|V|\le a$ in $B_{x_0}(R)$ for some nonnegative constants k and a. Let $0<u\le M$ be a solution of* (1.1) *in $Q_{R,T}$. Then there exists a dimensional constant $C(n)$ such that*

$$|\nabla\ln u|\le C(n)\bigg(\sqrt{\frac{1+a}{R}}+\frac{1}{\sqrt{t-t_0+T}}+\sqrt{k+\lambda_1+\lambda_2+\lambda_3+\lambda_4}\bigg)\Big(1+\ln\frac{M}{u}\Big)\tag{2.18}$$

in $Q_{R/2,T}$ with $t\ne t_0-T$, where $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ are the same as in Theorem 1.1.

As applications of Theorem 1.1, we can derive some corollaries by considering the special cases of (1.1). More precisely,

$$\big(\Delta_V - \partial_t - q(x,t)\big)u = au\ln u, \tag{2.19}$$

$$\big(\Delta_V - \partial_t - q(x,t)\big)u = au^\gamma, \tag{2.20}$$

where a and γ are constants.

When $V \equiv 0$, the elliptic version of (2.19) is closely related to the gradient Ricci soliton; (see [Ma 2006]). In fact, consider the gradient Ricci soliton

$$\mathrm{Ric} + \nabla\nabla f + \lambda g = 0,$$

where λ is a constant. Taking the trace of the above equality, we have

$$R + \Delta f + n\lambda = 0.$$

Using the contracted Bianchi identity and Ricci identity, then

$$|\nabla f|^2 + R - 2\lambda f = c$$

for some constant c. Hence

$$|\nabla f|^2 - \Delta f - 2\lambda f = n\lambda + c.$$

Setting $u = e^{-f}$, we obtain

$$\Delta u - (c + n\lambda)u = -2\lambda u \ln u.$$

When $V \equiv 0$, the elliptic version of (2.20) is related to conformal deformation of the scalar curvature on manifolds. In fact, for any n-dimensional ($n \ge 3$) manifold, consider a conformal metric $\tilde{g} = u^{4/(n-2)}g$ for some positive function u. Then the scalar curvature $\tilde{s}$ of metric $\tilde{g}$ related to the scalar curvature s of metric g is given by

$$\Delta u - \frac{n-2}{4(n-1)}su + \frac{n-2}{4(n-1)}\tilde{s}u^{(n+2)/(n-2)} = 0. \tag{2.21}$$

We have known that if M is compact and $\tilde{s}$ is a constant, the existence of u is the well-known Yamabe problem which has been solved by R. Schoen [1984] (see also [Lee and Parker 1987; Mastrolia et al. 2012] for more details).

Corollary 2.5. *Let (M^n, g) be an n-dimensional complete Riemannian manifold with $\mathrm{Ric}_V \ge -k$ for some constant $k \ge 0$. Let $0 < u \le M$ be a smooth solution of (2.19) with $a \le 0$ in $M^n \times [t_0 - T, t_0]$. Suppose that $q^- \le c_1$ and $\big|\nabla\sqrt{|q|}\big| \le c_2$ for some constants c_1, c_2. Then there exists a dimensional constant $c(n)$ such that*

$$|\nabla \ln u| \le c(n)\left(\frac{1}{\sqrt{t-t_0+T}} + \sqrt{k - a\ln(\max\{M,1\}) - a + 2c_1 + c_2}\right) \cdot \left(1 + \ln\frac{M}{u}\right) \tag{2.22}$$

in $M^n \times (t_0 - T, t_0]$.

Proof. Since $F(u) = au \ln u$ $(a \leq 0)$ and $0 < u \leq M$, by the definitions of λ_i $(i = 1, 2, 3, 4)$, it is easy to obtain

$$\lambda_1 = -a \ln(\max\{M, 1\}), \quad \lambda_2 = -a, \quad \lambda_3 = 2c_1, \quad \lambda_4 = c_2.$$

Applying Theorem 1.1 and setting $R \to \infty$, (2.22) immediately follows. □

Corollary 2.6. *Let (M^n, g), q^- and $\left|\nabla\sqrt{|q|}\right|$ be the same as in Corollary 2.5. Let $0 < u \leq M$ be a smooth solution of* (2.20) *with $\gamma > 1$ in $M^n \times [t_0 - T, t_0]$. Then there exists a dimensional constant $c(n)$ such that*

$$|\nabla \ln u| \leq c(n)\left(\frac{1}{\sqrt{t-t_0+T}} + \sqrt{k + \frac{\operatorname{sgn} a - 1}{2}(a\gamma M^{\gamma-1}) + 2c_1 + c_2}\right) \cdot \left(1 + \ln\frac{M}{u}\right) \tag{2.23}$$

in $M^n \times (t_0 - T, t_0]$, where

$$\operatorname{sgn} a = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

Proof. Since $F(u) = au^\gamma$ $(\gamma > 1)$ and $0 < u \leq M$, by the direct calculations, we have

(i) If $a \geq 0$, then $\lambda_1 = \lambda_2 = 0$.

(ii) If $a < 0$, then $\lambda_1 = -aM^{\gamma-1}$, $\lambda_2 = -a(\gamma - 1)M^{\gamma-1}$.

Using (1.4) and letting $R \to \infty$, the desired result (2.23) follows. □

Next, we apply Theorem 1.1 to prove Theorem 1.3 which analyze the existence of solutions to the parabolic equation (1.5) when the coefficient $q(x)$ and solutions $u(x, t)$ satisfying some growth conditions. Furthermore, we can use Theorem 1.3 to study the problem about conformal deformation of the scalar curvature on complete manifolds.

Proof of Theorem 1.3. From the proof of Theorem 1.1 and Corollary 2.6, since $a \geq 0$, then $\lambda_1 = \lambda_2 = 0$, we get a local elliptic gradient estimate for (1.5):

$$|\nabla \ln u| \leq c(n)\left(\sqrt{\frac{1+|\delta|}{R}} + \frac{1}{\sqrt{t-t_0+T}} + \sqrt{\lambda_3 + \lambda_4}\right)\left(1 + \ln\frac{M}{u}\right) \tag{2.24}$$

for any $(x, t) \in Q_{\frac{R}{2},T}$ with $t \neq t_0 - T$, where λ_3, λ_4 be defined in Theorem 1.1.

For any fixed space-time point (x_0, t_0), by the growth assumptions of $u(x, t)$ and $q(x)$, applying (2.24) to $u(x_0, t_0)$ in the space-time set $Q_{R,R} = B_{x_0}(R) \times [t_0 - R, t_0]$, then

$$|\nabla \ln u(x_0, t_0)| \leq c(n)\left(\sqrt{\frac{1+|\delta|}{R}} + o(R^{-1/2})\right)\left(1 + o(\ln\sqrt{R}) - \ln u(x_0, t_0)\right) \tag{2.25}$$

for sufficiently large $R \geq 2$.

Notice that $\ln u(x_0, t_0)$ is a fixed value, which implies

$$|\nabla u(x_0, t_0)| = 0 \quad \text{as } R \to \infty.$$

Since (x_0, t_0) was chosen arbitrarily, then $u(x, t) \equiv u(t)$, and Equation (1.5) becomes

$$u'(t) = -q(x)u(t) - au^{\gamma}(t). \tag{2.26}$$

Case I. $a = 0$.

In this case, $u'(t) = -q(x)u(t)$. Since $q(x) \neq 0$, we solve this equation and obtain

$$u(t) = Ce^{-q(x)t}, \tag{2.27}$$

where C is an arbitrary constant.

From (2.27), we know $q(x) = c$ for some constant $c > 0$ due to the growth assumption of q^-. Then

$$u(t) = u(0)e^{-ct} = u(0)e^{c|t|},$$

which contradicts the assumption that $u(x, t) = o(r(x)^{1/2} + |t|^{1/2})$ near infinity.

Case II. $a > 0$.

In this case, (2.26) can be regraded as a one-order linear ordinary equation which has a general solution

$$u^{1-\gamma}(t) = Ce^{(\gamma-1)q(x)t} - \frac{a}{q(x)}, \tag{2.28}$$

where C is an arbitrary constant.

By the same way, we know $q(x) = c$ for some constant $c > 0$, then

$$u^{1-\gamma}(t) = \Big(u^{1-\gamma}(0) + \frac{a}{c}\Big)e^{(\gamma-1)ct} - \frac{a}{c}.$$

Since $a, c, \gamma - 1$ and $u(0)$ are positive constants, which imply

$$u^{1-\gamma}(t) \to -\frac{a}{c} < 0 \quad \text{as } t \to -\infty,$$

this is impossible since $u > 0$.

As for the case $V \equiv 0$, the term $\sqrt{(1 + |\delta|)/R}$ in (2.24) can be changed into $\frac{1}{R}$. Since $u(x, t) = o(r(x) + |t|^{1/2})$ near infinity, we apply (2.24) to $u(x_0, t_0)$ in $Q_{R,R^2} = B_{x_0}(R) \times [t_0 - R^2, t_0]$ and the proof is almost the same as before except that (2.25) is replaced by

$$|\nabla \ln u(x_0, t_0)| \leq c(n)\Big(\frac{1}{R} + o(R^{-1/2})\Big)\big(1 + o(\ln R) - \ln u(x_0, t_0)\big). \qquad \square$$

As an application of Theorem 1.3, we discuss the Yamabe type problem of complete Riemannian manifolds and immediately obtain the following corollary.

Corollary 2.7. *Let* (M^n, g) *be an n-dimensional* $(n \geq 3)$ *complete Riemannian manifold with* Ric ≥ 0 *and the scalar curvature s of g satisfying*

$$\sup_{B_{x_0}(R)} |\nabla \sqrt{s}| = o(R^{-1})$$

as $R \to \infty$. *Then there does not exist complete metric*

$$\tilde{g} \in \left\{ u^{4/(n-2)} g \mid 0 < u \in C^\infty(M) \text{ and } u(x) = o(r(x)^{1/2}) \right\},$$

such that the scalar curvature $\tilde{s}$ *of* $\tilde{g}$ *is some nonpositive constant.*

Proof. It is equivalent to prove that if $\tilde{s}$ is some nonpositive constant, then there does not exist any positive solution to (2.21) satisfying $u(x) = o(r(x)^{1/2})$. In Theorem 1.3, let

$$u(x,t) = u(x), \quad V = 0, \quad q(x) = \frac{n-2}{4(n-1)} s \geq 0,$$
$$a = -\frac{n-2}{4(n-1)} \tilde{s} \geq 0, \quad \gamma = \frac{n+2}{n-2} > 1,$$

we know that $q(x)$ satisfies the growth conditions in Theorem 1.3 due to the assumptions on s, hence the conclusion follows. □

We also apply Theorem 1.1 to derive the parabolic Liouville theorem for the V-heat equation which extends some known results.

Proof of Theorem 1.5. Since $F(u) = q = 0$, using Theorem 1.1, we have

$$|\nabla \ln u| \leq C(n) \left(\sqrt{\frac{1+|\delta|}{R}} + \frac{1}{\sqrt{t - t_0 + T}} \right) \left(1 + \ln \frac{M}{u} \right). \tag{2.29}$$

(i) By the assumption of $u(x,t)$, we have

$$\ln u = o(r^{1/2}(x) + |t|^{1/2})$$

near infinity. For any space-time point (x_0, t_0), we apply (2.29) to $u(x_0, t_0)$ in the space-time set $Q_{R,R} = B_{x_0}(R) \times [t_0 - R, t_0]$, then

$$\frac{|\nabla u(x_0, t_0)|}{u(x_0, t_0)} \leq \frac{C(n, \delta)}{\sqrt{R}} \left(1 + o(\sqrt{R}) - \ln u(x_0, t_0) \right)$$

for sufficiently large $R \geq 2$.

For the fixed value $\ln u(x_0, t_0)$, setting $R \to \infty$ in the above inequality, we get

$$|\nabla u(x_0, t_0)| = 0.$$

Then u is only a time-dependent function due to (x_0, t_0) being arbitrary. Moreover, u is a constant by using (1.6).

(ii) Let $M_R = \sup_{Q_{\sqrt{R},\sqrt{R}}} |u|$. Considering the function $U = u + 2M_{2R}$, then

$$M_{2R} \le U(x,t) \le 3M_{2R}$$

whenever $(x,t) \in Q_{2\sqrt{R},\,2\sqrt{R}}$. For any fixed point (x_0, t_0), applying (2.29) to U, we have

$$\frac{|\nabla u(x_0,t_0)|}{u(x_0,t_0)+2M_{2R}} \le \frac{C(n,\delta)}{\sqrt{R}}$$

for sufficiently large $R \ge 2$. By the assumption of $u(x,t)$, we have $M_{2R} = o(R^{1/4})$. The conclusion immediately follows by taking $R \to \infty$. □

3. Global elliptic gradient estimate

In this section, we follow the arguments of Kotschwar [2007] and Wu [2015] to prove Theorem 1.7. The key is to derive a local elliptic gradient estimate which is different from Souplet–Zhang's gradient estimate. Our proof is based on the technique of Shi [1989] from the estimation of derivatives of curvature under the Ricci flow. Firstly, we give the following lemma.

Lemma 3.1. *Define*

$$G(x,t) := (4M^2 + u^2)|\nabla u|^2.$$

Under the same assumptions as in Theorem 1.7, we have

$$(\partial_t - \Delta_V)G \le \left(\tfrac{5}{2}k + \tfrac{7}{2}\alpha\right)G - \frac{2}{125M^4}G^2 + 50\beta^2 M^4. \tag{3.1}$$

Proof. By straightforward calculations, we obtain

$$(\partial_t - \Delta_V)u^2 = -2|\nabla u|^2 - 2qu^2.$$
$$(\partial_t - \Delta_V)|\nabla u|^2 \le -2|\nabla\nabla u|^2 - 2u\langle\nabla u, \nabla q\rangle - 2q|\nabla u|^2 + 2k|\nabla u|^2.$$

Then,

$$\begin{aligned}(\partial_t - \Delta_V)G &= (\partial_t - \Delta_V)u^2\cdot|\nabla u|^2 + (4M^2+u^2)(\partial_t - \Delta_V)|\nabla u|^2 \\ &\qquad - 2\langle\nabla u^2, \nabla|\nabla u|^2\rangle \\ &\le (-2|\nabla u|^2 - 2qu^2)|\nabla u|^2 \\ &\quad + (4M^2+u^2)\big(-2|\nabla\nabla u|^2 - 2u\langle\nabla u, \nabla q\rangle - 2q|\nabla u|^2 + 2k|\nabla u|^2\big) \\ &\qquad - 8u\nabla\nabla u(\nabla u, \nabla u).\end{aligned} \tag{3.2}$$

Since

$$-8u\nabla\nabla u(\nabla u,\nabla u) \le 10u^2|\nabla\nabla u|^2 + \tfrac{8}{5}|\nabla u|^4, \quad 5u^2 \le 4M^2 + u^2 \le 5M^2,$$

the inequality (3.2) can be simplified as

$$(3.3)\quad (\partial_t-\Delta_V)G\le 10kM^2|\nabla u|^2-\tfrac{2}{5}|\nabla u|^4-2qu^2|\nabla u|^2 \\ -2(4M^2+u^2)u\langle\nabla u,\nabla q\rangle-2(4M^2+u^2)q|\nabla u|^2.$$

Using the Young's inequality, then

$$\begin{aligned}-2(4M^2+u^2)u\langle\nabla u,\nabla q\rangle &\le 10M^2u|\nabla u||\nabla q|\\ &\le 2|q|u^2|\nabla u|^2+\tfrac{25}{2}\frac{|\nabla q|^2}{|q|}M^4\\ &=2|q|u^2|\nabla u|^2+50\big|\nabla\sqrt{|q|}\big|^2M^4.\end{aligned}$$

Notice that

$$-2(4M^2+u^2)q|\nabla u|^2\le 10M^2q^-|\nabla u|^2,$$

and

$$|q|-q=2q^-.$$

Hence, (3.3) can be written as

$$\begin{aligned}(\partial_t-\Delta_V)G &\le 10kM^2|\nabla u|^2-\tfrac{2}{5}|\nabla u|^4+4q^-u^2|\nabla u|^2+10M^2q^-|\nabla u|^2\\ &\qquad +50\big|\nabla\sqrt{|q|}\big|^2M^4\\ &\le -\tfrac{2}{5}|\nabla u|^4+(10k+14\alpha)M^2|\nabla u|^2+50\beta^2M^4,\end{aligned}$$

where we used the assumptions $q^-\le\alpha$ and $\big|\nabla\sqrt{|q|}\big|\le\beta$.

By the definition of G, we know

$$4M^2|\nabla u|^2\le G\le 5M^2|\nabla u|^2.$$

Hence, the inequality (3.1) follows. □

Now, applying Lemma 3.1, we give a proof of Theorem 1.7.

Proof of Theorem 1.7. As in [Li and Yau 1986], we take a cut-off function $\bar\phi(s)$ which is defined in $[0,\infty)$ such that $0\le\bar\phi(s)\le 1$ and

$$\bar\phi(s)=1\quad\text{for } s\in\big[0,\tfrac{1}{2}\big],\qquad \bar\phi(s)=0\quad\text{for } s\in[1,\infty).$$

$\bar\phi(s)$ also satisfies

$$-c_1\le\frac{\bar\phi'(s)}{\bar\phi^{1/2}(s)}\le 0,\quad \bar\phi''(s)\ge -c_2$$

for positive absolute constants c_1 and c_2.

Let $\phi(x)=\bar\phi\big(\frac{r(x)}{R}\big)$ for $R\ge 2$, where $r(x)$ denotes the distance from the fixed point x_0 to x. Using the argument of Calabi [1958], we may assume $\phi(x)\in C^2(M)$

with support in $B_{x_0}(R)$. By direct calculations, we have

$$\frac{|\nabla\phi|^2}{\phi} \le \frac{c_3}{R^2}, \quad \Delta_V\phi = \frac{\bar{\phi}'\Delta_V r}{R} + \frac{\bar{\phi}''}{R^2} \tag{3.4}$$

for some positive absolute constant c_3.

Considering $t\phi G$ in $B_{x_0}(R)\times[0,T]$, using Lemma 3.1, we get

$$(\partial_t - \Delta_V)(t\phi G) \le \phi G + t\phi\Big(\big(\tfrac{5}{2}k + \tfrac{7}{2}\alpha\big)G - \frac{2}{125M^4}G^2 + 50\beta^2M^4\Big) - tG\Delta_V\phi - 2t\langle\nabla\phi, \nabla G\rangle \tag{3.5}$$

Assume

$$(t\phi G)(x_1,t_1) = \max_{B_{x_0}(R)\times(0,T]}(t\phi G).$$

If $t\phi G$ is not identically zero (i.e., u is not a constant in $\operatorname{supp}\phi$), then

$$(t\phi G)(x_1,t_1) > 0.$$

By the maximum principle, at (x_1,t_1),

$$\nabla(t\phi G) = 0, \quad (\partial_t - \Delta_V)(t\phi G) \ge 0.$$

In the following, we will estimate the each term on the right hand side of (3.5) at (x_1,t_1).

Case I. Assume $x_1 \in B_{x_0}(1) \subset B_{x_0}\big(\frac{R}{2}\big)$ because of $R \ge 2$.

In this case, $\phi \equiv 1$ implies $\nabla\phi = \Delta_V\phi = 0$. The inequality (3.5) can be simplified as

$$0 \le G + t\Big(\big(\tfrac{5}{2}k + \tfrac{7}{2}\alpha\big)G - \frac{2G^2}{125M^4} + 50\beta^2M^4\Big).$$

It is equivalent to

$$\frac{2}{125M^4}tG^2 - \big(\big(\tfrac{5}{2}k + \tfrac{7}{2}\alpha\big)t + 1\big)G - 50\beta^2M^4t \le 0$$

at (x_1,t_1). Since $0 < t \le T$, we have

$$\frac{2}{125M^4}(tG)^2 - \big(\big(\tfrac{5}{2}k + \tfrac{7}{2}\alpha\big)T + 1\big)tG - 50\beta^2M^4T^2 \le 0.$$

At this time, (x_1,t_1) is also the maximum point of tG in $B_{x_0}\big(\frac{R}{2}\big)\times(0,T]$. Hence, we obtain

$$(tG)(x,t) \le (tG)(x_1,t_1) \le CM^4\big(1 + (k+\alpha+\beta)T\big)$$

in $B_{x_0}\big(\frac{R}{2}\big)\times[0,T]$.

Case II. Assume $x_1 \notin B_{x_0}(1)$.

In this case, since $\mathrm{Ric}_V \geq -k$, $d(x_1, x_0) \geq 1$ and $R \geq 2$, by the V-Laplacian comparison theorem, we have

$$\Delta_V r(x_1) \leq \delta + k(R-1) \leq |\delta| + k(R-1),$$

where $\delta = \max_{\{x \mid d(x,x_0)=1\}} \Delta_V r(x)$. Hence,

$$\Delta_V \phi \geq -\frac{c_1}{R}\big(|\delta| + k(R-1)\big) - \frac{c_2}{R^2}. \tag{3.6}$$

By using Lemma 3.1, (3.4) and (3.6), at (x_1, t_1), we have

$$\begin{aligned} 0 &\leq (\partial_t - \Delta_V)(t\phi G) \\ &= \phi G + t\phi(\partial_t - \Delta_V)G - tG\Delta_V\phi - 2t\langle \nabla\phi, \nabla G\rangle \\ &= \Big(\phi + 2t\frac{|\nabla\phi|^2}{\phi} - t\Delta_V\phi\Big)G + t\phi(\partial_t - \Delta_V)G - 2\Big\langle \nabla(t\phi G), \frac{\nabla\phi}{\phi}\Big\rangle \\ &\leq \bigg(1 + \frac{c_4 t}{R^2} + \frac{c_1 t}{R}\big(|\delta| + k(R-1)\big)\bigg)G \\ &\qquad + t\phi\Big(\big(\tfrac{5}{2}k + \tfrac{7}{2}\alpha\big)G - \frac{2G^2}{125M^4} + 50\beta^2 M^4\Big) \\ &\leq -\frac{c_5}{M^4}t\phi G^2 + \Big(1 + c_6\Big(\frac{1+|\delta|}{R} + k + \alpha\Big)T\Big)G + c_7\beta^2 M^4 T. \end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by $t\phi$ and using $t\phi \leq T$, we have

$$\frac{c_5}{M^4}(t\phi G)^2 - \Big(1 + c_6\Big(\frac{1+|\delta|}{R} + k + \alpha\Big)T\Big)(t\phi G) - c_7\beta^2 M^4 T^2 \leq 0.$$

We solve this inequality and obtain that

$$(t\phi G)(x_1, t_1) \leq c_8 M^4\Big(1 + \Big(\frac{1+|\delta|}{R} + k + \alpha + \beta\Big)T\Big).$$

Notice that the above constants $c_i\,(i = 1, 2, \ldots, 8)$ are all absolute positive constants. Consequently,

$$\begin{aligned} (tG)(x,t) = (t\phi G)(x,t) &\leq (t\phi G)(x_1, t_1) \\ &\leq c_8 M^4\Big(1 + \Big(\frac{1+|\delta|}{R} + k + \alpha + \beta\Big)T\Big) \end{aligned}$$

for any $(x,t) \in B_{x_0}\big(\frac{R}{2}\big) \times [0, T]$.

Combining the above two cases, we obtain

$$(tG)(x,t) \leq CM^4\Big(1 + \Big(\frac{1+|\delta|}{R} + k + \alpha + \beta\Big)T\Big)$$

for any $(x,t) \in B_{x_0}\big(\frac{R}{2}\big) \times [0, T]$.

Since $G \geq 4M^2|\nabla u|^2$, the theorem follows by taking $R \to \infty$. □

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JIAN-HONG WANG
SCHOOL OF STATISTICS AND MATHEMATICS
SHANGHAI LIXIN UNIVERSITY OF ACCOUNTING AND FINANCE
SHANGHAI
CHINA
jianhongwang@163.com

YU ZHENG
SCHOOL OF MATHEMATICAL SCIENCES
EAST CHINA NORMAL UNIVERSITY
SHANGHAI
CHINA
zhyu@math.ecnu.edu.cn

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