

*Pacific  
Journal of  
Mathematics*

**OPTIMAL  $L^2$  EXTENSION OF SECTIONS FROM  
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# OPTIMAL $L^2$ EXTENSION OF SECTIONS FROM SUBVARIETIES IN WEAKLY PSEUDOCONVEX MANIFOLDS

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**We obtain optimal  $L^2$  extension of holomorphic sections of a holomorphic vector bundle from subvarieties in weakly pseudoconvex Kähler manifolds. Moreover, in the case of a line bundle the Hermitian metric is allowed to be singular.**

## 1. Introduction and main results

The  $L^2$  extension problem is an important topic in several complex variables and complex geometry. Many generalizations and applications (see [Manivel 1993; Ohsawa 1995; 2001; Siu 1996; 1998; Berndtsson 1996; Demailly 2000; 2012a; Siu 2002; McNeal and Varolin 2007; Berndtsson and Păun 2008], etc.) have been obtained since the original work of Ohsawa and Takegoshi [1987]. Recent progress concerns the optimal  $L^2$  extension and its applications (see [Guan et al. 2011; Zhu et al. 2012; Guan and Zhou 2012; 2015a; 2015c; Błocki 2013; Zhou 2015; Ohsawa 2015; Berndtsson and Lempert 2016; Cao 2017; Zhou and Zhu 2018], etc.).

Most recently, several general  $L^2$  extension theorems with optimal estimates were proved in [Guan and Zhou 2015c] for holomorphic sections defined on subvarieties in Stein or projective manifolds. In [Demailly 2016], several  $L^2$  extension theorems were obtained for holomorphic sections defined on subvarieties in weakly pseudoconvex Kähler manifolds.

In this paper, we prove an optimal  $L^2$  extension theorem, which generalizes the main theorems in [Guan and Zhou 2015c] to weakly pseudoconvex Kähler manifolds and a main theorem in [Zhou and Zhu 2018], and optimizes a main theorem in [Demailly 2016] (cf. Theorem 2.8 and Remark 2.9 in [Demailly 2016]).

Let us recall some usual notions and some definitions in [Demailly 2016].

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Zhou was partially supported by the National Natural Science Foundation of China (No. 11688101 and No. 11431013). Zhu was partially supported by the National Natural Science Foundation of China (No. 11201347, No. 11671306 and No. 12022110) and the China Scholarship Council. Zhu is the corresponding author.

*MSC2010:* primary 32D15, 32J25, 32Q15, 32U05, 32W05; secondary 14F18, 32L10.

*Keywords:* optimal  $L^2$  extension, plurisubharmonic function, multiplier ideal sheaf, strong openness, weakly pseudoconvex manifold, Kähler manifold.

**Definition 1.1.** A function  $\psi : X \rightarrow [-\infty, +\infty)$  on a complex manifold  $X$  is said to be quasi-plurisubharmonic if  $\psi$  is locally the sum of a plurisubharmonic function and a smooth function. In addition, we say that  $\psi$  has neat analytic singularities if every point  $x \in X$  possesses an open neighborhood  $U$  on which  $\psi$  can be written as

$$\psi = c \log \sum_{1 \leq j \leq j_0} |g_j|^2 + u,$$

where  $c$  is a nonnegative number,  $g_j \in \mathcal{O}_X(U)$  and  $u \in C^\infty(U)$ .

**Definition 1.2.** If  $\psi$  is a quasi-plurisubharmonic function on a complex manifold  $X$ , the multiplier ideal sheaf  $\mathcal{I}(\psi)$  is the coherent analytic subsheaf of  $\mathcal{O}_X$  defined by

$$\mathcal{I}(\psi)_x = \left\{ f \in \mathcal{O}_{X,x} : \exists U \ni x, \int_U |f|^2 e^{-\psi} d\lambda < +\infty \right\},$$

where  $U$  is an open coordinate neighborhood of  $x$ , and  $d\lambda$  is the Lebesgue measure in the corresponding open chart of  $\mathbb{C}^n$ . We say that the singularities of  $\psi$  are log canonical along the zero variety  $Y = V(\mathcal{I}(\psi))$  if  $\mathcal{I}((1 - \varepsilon)\psi)|_Y = \mathcal{O}_X|_Y$  for every  $\varepsilon > 0$ .

If  $\omega$  is a Kähler metric on  $X$ , we let  $dV_{X,\omega} := \omega^n/n!$  be the corresponding Kähler volume element, where  $n = \dim X$ . In case  $\psi$  has log canonical singularities along  $Y = V(\mathcal{I}(\psi))$ , one can associate in a natural way a measure  $dV_{X,\omega}[\psi]$  on the set  $Y^0 = Y_{\text{reg}}$  of regular points of  $Y$  as follows.

**Definition 1.3.** If  $g \in C_c(Y^0)$  is a compactly supported nonnegative continuous function on  $Y^0$  and  $\tilde{g}$  is a compactly supported nonnegative continuous extension of  $g$  to  $X$  such that  $(\text{supp } \tilde{g}) \cap Y \subset Y^0$ , then we set

$$\int_{Y^0} g dV_{X,\omega}[\psi] = \overline{\lim}_{t \rightarrow -\infty} \int_{\{x \in X : t < \psi(x) < t+1\}} \tilde{g} e^{-\psi} dV_{X,\omega}.$$

**Remark 1.4.** By Hironaka’s desingularization theorem (Theorem 2.8), it is not hard to see that the limit in the above definition does not depend on the extension  $\tilde{g}$  and then  $dV_{X,\omega}[\psi]$  is well defined on  $Y^0$  (see Proposition 4.5 in [Demailly 2016] for a proof).

**Remark 1.5.** The definition of  $dV_{X,\omega}[\psi]$  here has a slight difference with the one in [Guan and Zhou 2015c]. In fact, if we denote the measure in [Guan and Zhou 2015c] by  $d\widehat{V}_{X,\omega}[\psi]$ , the integral  $\int_{Y^0} g dV_{X,\omega}[\psi]$  here is equal to

$$\sum_{1 \leq j \leq n} \frac{\pi^j}{j!} \int_{Y_{n-j}} g d\widehat{V}_{X,\omega}[\psi],$$

where  $Y_{n-j}$  is the  $(n-j)$ -dimensional component of  $Y_{\text{reg}}$ .

We will define a class of functions before the statement of our main theorem.

**Definition 1.6.** Let  $\alpha_0 \in (-\infty, +\infty]$  and  $\alpha_1 \in [0, +\infty)$ . When  $\alpha_0 \neq +\infty$ , let  $\mathfrak{R}_{\alpha_0, \alpha_1}$  be the class of functions defined by

$$\left\{ R \in C^\infty(-\infty, \alpha_0] : R > 0, R \text{ is decreasing near } -\infty, \right. \\ \left. \overline{\lim}_{t \rightarrow -\infty} e^t R(t) < +\infty, \quad C_R := \int_{-\infty}^{\alpha_0} \frac{1}{R(t)} dt < +\infty, \right. \\ (1-1) \quad \left. \int_t^{\alpha_0} \left( \frac{\alpha_1}{R(\alpha_0)} + \int_{t_2}^{\alpha_0} \frac{dt_1}{R(t_1)} \right) dt_2 + \frac{(\alpha_1)^2}{R(\alpha_0)} < R(t) \left( \frac{\alpha_1}{R(\alpha_0)} + \int_t^{\alpha_0} \frac{dt_1}{R(t_1)} \right)^2 \right. \\ \left. \text{for all } t \in (-\infty, \alpha_0) \right\}.$$

When  $\alpha_0 = +\infty$ , we replace  $R \in C^\infty(-\infty, \alpha_0]$  with  $R \in C^\infty(-\infty, +\infty)$  and  $R(+\infty) := \underline{\lim}_{t \rightarrow +\infty} R(t) \in (0, +\infty]$  in the above definition of  $\mathfrak{R}_{\alpha_0, \alpha_1}$ .

**Remark 1.7.** The numbers  $\alpha_0, \alpha_1$  and the function  $R(t)$  are equal to the numbers  $A, 1/\delta$  and the function  $1/(c_A(-t)e^t)$  which are defined just before the main theorems in [Guan and Zhou 2015c]. If  $\alpha_0 \neq +\infty$  and  $R$  is decreasing on  $(-\infty, \alpha_0]$ , then (1-1) holds for all  $t \in (-\infty, \alpha_0)$ . If  $\alpha_0 = +\infty$ , then (1-1) implies that  $\int_t^{+\infty} (\alpha_1/R(+\infty)) dt_2 < +\infty$  for all  $t \in (-\infty, +\infty)$ . Therefore,  $\alpha_1/R(+\infty) = 0$ , i.e.,  $\alpha_1 = 0$  or  $R(+\infty) = +\infty$ .

**Theorem 1.8** (The main theorem). *Let  $R \in \mathfrak{R}_{\alpha_0, \alpha_1}$ . Let  $(X, \omega)$  be a weakly pseudoconvex complex  $n$ -dimensional manifold possessing a Kähler metric  $\omega$ , and  $\psi$  be a quasi-plurisubharmonic function on  $X$  with neat analytic singularities. Let  $Y$  be the analytic subvariety of  $X$  defined by  $Y = V(\mathcal{I}(\psi))$  and assume that  $\psi$  has log canonical singularities along  $Y$ . Let  $L$  (resp.  $E$ ) be a holomorphic line bundle (resp. a holomorphic vector bundle) over  $X$  equipped with a singular Hermitian metric  $h = h_L$  (resp. a smooth Hermitian metric  $h = h_E$ ), which is written locally as  $e^{-\phi_L}$  for some quasi-plurisubharmonic function  $\phi_L$  with respect to a local holomorphic frame of  $L$ . Assume that*

- (i)  $\sqrt{-1}\Theta_h + \sqrt{-1}\partial\bar{\partial}\psi$  is semipositive on  $X \setminus \{\psi = -\infty\}$  in the sense of currents (resp. in the sense of Nakano),

and that there is a continuous function  $\alpha < \alpha_0$  on  $X$  such that the following two assumptions hold:

- (ii)  $\sqrt{-1}\Theta_h + \sqrt{-1}\partial\bar{\partial}\psi + (1/\tilde{\chi}(\alpha))\sqrt{-1}\partial\bar{\partial}\psi$  is semipositive on  $X \setminus \{\psi = -\infty\}$  in the sense of currents (resp. in the sense of Nakano),
- (iii)  $\psi \leq \alpha$ ,

where  $\tilde{\chi}(t)$  is the function

$$(1-2) \quad \frac{\int_t^{\alpha_0} \left( \frac{\alpha_1}{R(\alpha_0)} + \int_{t_2}^{\alpha_0} \frac{dt_1}{R(t_1)} \right) dt_2 + \frac{(\alpha_1)^2}{R(\alpha_0)}}{\frac{\alpha_1}{R(\alpha_0)} + \int_t^{\alpha_0} \frac{dt_1}{R(t_1)}}.$$

Then for every section  $f \in H^0(Y^0, (K_X \otimes L)|_{Y^0})$  (resp.  $f \in H^0(Y^0, (K_X \otimes E)|_{Y^0})$ ) on  $Y^0 = Y_{\text{reg}}$  such that

$$(1-3) \quad \int_{Y^0} |f|_{\omega,h}^2 dV_{X,\omega}[\psi] < +\infty,$$

there exists a section  $F \in H^0(X, K_X \otimes L)$  (resp.  $F \in H^0(X, K_X \otimes E)$ ) such that  $F = f$  on  $Y^0$  and

$$(1-4) \quad \int_X \frac{|F|_{\omega,h}^2}{e^\psi R(\psi)} dV_{X,\omega} \leq \left( \frac{\alpha_1}{R(\alpha_0)} + C_R \right) \int_{Y^0} |f|_{\omega,h}^2 dV_{X,\omega}[\psi].$$

**Remark 1.9.** The case of Theorem 1.8 when  $X$  is Stein or projective was proved in [Guan and Zhou 2015c] (see also Proposition 4.1 in [Zhou and Zhu 2018] for a simplified version). Hence Theorem 1.8 can be regarded as a generalization of the main theorems in [Guan and Zhou 2015c] to weakly pseudoconvex Kähler manifolds. Then it is easy to see from Remark 1.5 and the main theorems in [Guan and Zhou 2015c] that the constant  $\alpha_1/R(\alpha_0) + C_R$  in (1-4) is optimal. Hence Theorem 1.8 gives an optimal version of a main theorem in [Demailly 2016] (cf. Theorem 2.8 and Remark 2.9 in [Demailly 2016]).

**Remark 1.10.** In [Zhou and Zhu 2018], Theorem 1.8 was proved for  $L$  in the special case when  $\psi = m \log |s|^2$ ,  $\alpha_0 = \alpha_1 = 0$  and  $R$  is decreasing on  $(-\infty, 0]$ , where  $s$  is a global holomorphic section of some holomorphic vector bundle of rank  $m$  over  $X$  equipped with a smooth Hermitian metric, and  $s$  is transverse to the zero section. Similarly as in [Zhou and Zhu 2018], a global plurisubharmonic negligible weight can be added to Theorem 1.8 by adding another regularization process to Step 2 in Section 4.

**Remark 1.11.** In order to deal with the singular metric  $h_L$  on the weakly pseudoconvex Kähler manifold  $X$ , not only the regularization Theorem 2.2 and the error term method of solving  $\bar{\partial}$  equations (Lemma 2.1) are needed, but also a limit problem about  $L^2$  integrals with singular weights needs to be solved. We solve the limit problem in Proposition 3.2. Then by using Propositions 3.1, 3.2 and the strong openness property of multiplier ideal sheaves (Theorem 2.7) as the key tools, we construct a family of smooth extensions of  $f$  satisfying some uniform estimates, and overcome the difficulty in dealing with the singular metric (see also [Zhou and Zhu 2018] for the special case).

The rest of this paper are organized as follows. First, we recall some results used in the proof of Theorem 1.8 in Section 2. Then, we prove two key propositions in Section 3 which will be used to deal with the singular metric  $h_L$ . Finally, we prove Theorem 1.8 in Section 4 by using the results in Sections 2 and 3.

## 2. Some results used in the proof of Theorem 1.8

In this section, we recall some results which will be used in the proof of Theorem 1.8.

**Lemma 2.1** [Demailly 2000; 2016]. *Let  $(X, \omega)$  be a complete Kähler manifold equipped with a (not necessarily complete) Kähler metric  $\omega$ , and let  $(Q, h)$  be a holomorphic vector bundle over  $X$  equipped with a smooth Hermitian metric  $h$ . Assume that  $\tau$  and  $A$  are smooth and bounded positive functions on  $X$  and let*

$$\mathbf{B} := [\tau\sqrt{-1}\Theta_{Q,h} - \sqrt{-1}\partial\bar{\partial}\tau - \sqrt{-1}A^{-1}\partial\tau \wedge \bar{\partial}\tau, \Lambda].$$

*Assume that  $\delta \geq 0$  is a nonnegative number such that  $\mathbf{B} + \delta\mathbf{I}$  is semipositive definite everywhere on  $\wedge^{n,q}T_X^* \otimes Q$  for some  $q \geq 1$ . Then given a form  $g \in L^2(X, \wedge^{n,q}T_X^* \otimes Q)$  such that  $D''g = 0$  and*

$$\int_X \langle (\mathbf{B} + \delta\mathbf{I})^{-1}g, g \rangle_{\omega,h} dV_{X,\omega} < +\infty,$$

*there exists an approximate solution  $u \in L^2(X, \wedge^{n,q-1}T_X^* \otimes Q)$  and a correcting term  $v \in L^2(X, \wedge^{n,q}T_X^* \otimes Q)$  such that  $D''u + \sqrt{\delta}v = g$  and*

$$\int_X \frac{|u|_{\omega,h}^2}{\tau + A} dV_{X,\omega} + \int_X |v|_{\omega,h}^2 dV_{X,\omega} \leq \int_X \langle (\mathbf{B} + \delta\mathbf{I})^{-1}g, g \rangle_{\omega,h} dV_{X,\omega}.$$

The following regularization theorem due to Demailly plays an important role in the present paper.

**Theorem 2.2** [Demailly 1994, Theorem 6.1]. *Let  $(X, \omega)$  be a complex manifold equipped with a Hermitian metric  $\omega$ , and  $\Omega \Subset X$  be an open subset. Assume that  $T = \tilde{T} + (\sqrt{-1}/\pi)\partial\bar{\partial}\varphi$  is a closed  $(1, 1)$ -current on  $X$ , where  $\tilde{T}$  is a smooth real  $(1, 1)$ -form and  $\varphi$  is a quasi-plurisubharmonic function. Let  $\gamma$  be a continuous real  $(1, 1)$ -form such that  $T \geq \gamma$ . Suppose that the Chern curvature tensor of  $T_X$  satisfies*

$$(\sqrt{-1}\Theta_{T_X} + \varpi \otimes \text{Id}_{T_X})(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0 \quad (\forall \kappa_1, \kappa_2 \in T_X \text{ with } \langle \kappa_1, \kappa_2 \rangle = 0)$$

*for some continuous nonnegative  $(1, 1)$ -form  $\varpi$  on  $X$ . Then there is a family of closed  $(1, 1)$ -currents  $T_{\zeta,\rho} = \tilde{T} + (\sqrt{-1}/\pi)\partial\bar{\partial}\varphi_{\zeta,\rho}$  defined on a neighborhood of  $\bar{\Omega}$  ( $\zeta \in (0, +\infty)$  and  $\rho \in (0, \rho_1)$  for some positive number  $\rho_1$ ) independent of  $\gamma$ , such that*

- (i)  $\varphi_{\zeta,\rho}$  is quasi-plurisubharmonic on a neighborhood of  $\bar{\Omega}$ , smooth on  $\Omega \setminus E_{\zeta}(T)$ , increasing with respect to  $\zeta$  and  $\rho$  on  $\Omega$ , and converges to  $\varphi$  on  $\Omega$  as  $\rho \rightarrow 0$ ,

(ii)  $T_{\zeta, \rho} \geq \gamma - \zeta \varpi - \delta_\rho \omega$  on  $\Omega$ ,

where  $E_\zeta(T) := \{x \in X : \nu(T, x) \geq \zeta\}$  ( $\zeta > 0$ ) is the  $\zeta$ -upperlevel set of Lelong numbers, and  $\{\delta_\rho\}$  is an increasing family of positive numbers such that  $\lim_{\rho \rightarrow 0} \delta_\rho = 0$ .

**Remark 2.3.** Although Theorem 2.2 is stated in [Demailly 1994] in the case  $X$  is compact, almost the same proof as in [Demailly 1994] shows that Theorem 2.2 holds in the noncompact case while uniform estimates are obtained only on the relatively compact subset  $\Omega$ .

**Lemma 2.4** [Demailly 1982, Theorem 1.5]. *Let  $X$  be a Kähler manifold, and  $Z$  be an analytic subset of  $X$ . Assume that  $\Omega$  is a relatively compact open subset of  $X$  possessing a complete Kähler metric. Then  $\Omega \setminus Z$  carries a complete Kähler metric.*

**Lemma 2.5** [Hörmander 1990, Theorem 4.4.2]. *Let  $\Omega$  be a pseudoconvex open set in  $\mathbb{C}^n$ , and  $\varphi$  be a plurisubharmonic function on  $\Omega$ . For every  $w \in L^2_{(p, q+1)}(\Omega, e^{-\varphi})$  with  $\bar{\partial}w = 0$  there is a solution  $s \in L^2_{(p, q)}(\Omega, \text{loc})$  of the equation  $\bar{\partial}s = w$  such that*

$$\int_{\Omega} \frac{|s|^2}{(1 + |z|^2)^2} e^{-\varphi} d\lambda \leq \int_{\Omega} |w|^2 e^{-\varphi} d\lambda,$$

where  $d\lambda$  is the  $2n$ -dimensional Lebesgue measure on  $\mathbb{C}^n$ .

**Lemma 2.6** [Demailly 1982, Lemma 6.9]. *Let  $\Omega$  be an open subset of  $\mathbb{C}^n$  and  $Z$  be a complex analytic subset of  $\Omega$ . Assume that  $u$  is a  $(p, q-1)$ -form with  $L^2_{\text{loc}}$  coefficients and  $g$  is a  $(p, q)$ -form with  $L^1_{\text{loc}}$  coefficients such that  $\bar{\partial}u = g$  on  $\Omega \setminus Z$  (in the sense of currents). Then  $\bar{\partial}u = g$  on  $\Omega$ .*

Multiplier ideal sheaves have the following new property which was conjectured by Demailly.

**Theorem 2.7** (strong openness property of multiplier ideal sheaves [Guan and Zhou 2015b]). *Let  $\varphi$  be a negative plurisubharmonic function on the unit polydisk  $\Delta^n \subset \mathbb{C}^n$ . Assume that  $F$  is a holomorphic function on  $\Delta^n$  satisfying*

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda < +\infty.$$

Then there exists  $r \in (0, 1)$  and  $\beta \in (0, +\infty)$  such that

$$\int_{\Delta^n_r} |F|^2 e^{-(1+\beta)\varphi} d\lambda < +\infty,$$

where  $\Delta^n_r := \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_k| < r, 1 \leq k \leq n\}$ .

We'll also need the following desingularization theorem due to Hironaka.

**Theorem 2.8** (Hironaka's desingularization theorem [Hironaka 1964; Bierstone and Milman 1991]). *Let  $X$  be a complex manifold, and  $M$  be an analytic subvariety in  $X$ . Then there is a local finite sequence of blow-ups  $\mu_j : X_{j+1} \rightarrow X_j$  ( $X_1 := X$ ,  $j = 1, 2, \dots$ ) with smooth centers  $S_j$  such that:*

- (1) *Each component of  $S_j$  lies either in  $(M_j)_{\text{sing}}$  or in  $M_j \cap E_j$ , where  $M_1 := M$ ,  $M_{j+1}$  denotes the strict transform of  $M_j$  by  $\mu_j$ ,  $(M_j)_{\text{sing}}$  denotes the singular set of  $M_j$ , and  $E_{j+1}$  denotes the exceptional divisor  $\mu_j^{-1}(S_j \cup E_j)$ .*
- (2) *Let  $M'$  and  $E'$  denote the final strict transform of  $M$  and the exceptional divisor respectively. Then:*
  - (a) *The underlying point-set  $|M'|$  is smooth.*
  - (b)  *$|M'|$  and  $E'$  simultaneously have only normal crossings.*

**Remark 2.9.** We say that  $|M'|$  and  $E'$  simultaneously have only normal crossings if, locally, there is a coordinate system in which  $E'$  is a union of coordinate hyperplanes, and  $|M'|$  is a coordinate subspace.

### 3. Key propositions used to deal with the singular metric $h_L$

In order to deal with the singular metric  $h_L$ , we prove two key propositions in this section, which are generalizations of the key propositions in [Zhou and Zhu 2018].

**Proposition 3.1.** *Let  $R$  be a positive continuous function defined on  $(-\infty, 0]$  such that  $\beta_R := \sup_{t \leq 0} (e^t R(t)) < +\infty$  and  $\hat{\beta}_R := \inf_{t \leq 0} R(t) > 0$ . Let  $\Omega \subset \mathbb{C}^n$  be a bounded pseudoconvex domain,  $\phi$  be a plurisubharmonic function on  $\Omega$ , and  $\Upsilon$  be a quasi-plurisubharmonic function defined on a neighborhood on  $\bar{\Omega}$ . Assume that  $\Upsilon$  has neat analytic singularities and the singularities of  $\Upsilon$  are log canonical along the zero variety  $Y = V(\mathcal{I}(\Upsilon))$ . Set*

$$U = \{x \in \Omega : \Upsilon(x) < 0\}.$$

Furthermore, assume that

$$\sqrt{-1} \partial \bar{\partial} \Upsilon \geq -\gamma \sqrt{-1} \partial \bar{\partial} |z|^2$$

on  $\Omega$  for some nonnegative number  $\gamma$ , where  $z := (z_1, \dots, z_n)$  is the coordinate vector in  $\mathbb{C}^n$ . Then for every  $\beta_1 \in (0, 1)$  and every holomorphic  $n$ -form  $f$  on  $U$  satisfying

$$\int_U \frac{|f|^2 e^{-\phi}}{e^{\Upsilon} R(\Upsilon)} d\lambda < +\infty,$$

there exists a holomorphic  $n$ -form  $F$  on  $\Omega$  satisfying  $F = f$  on  $Y$ ,

$$(3-1) \quad \int_U \frac{|F|^2 e^{-\phi} d\lambda}{e^{\Upsilon} R(\Upsilon)} \leq e^{2\gamma \sup_{\Omega} |z|^2} \left( 2 + \frac{72\beta_R}{\beta_1 \hat{\beta}_R} \right) \int_U \frac{|f|^2 e^{-\phi} d\lambda}{e^{\Upsilon} R(\Upsilon)},$$

and

$$(3-2) \quad \int_{\Omega} \frac{|F|^2 e^{-\phi} d\lambda}{(1+e^{\Upsilon})^{1+\beta_1}} \leq e^{2\gamma \sup_{\Omega} |z|^2} \left( \beta_R + \frac{36\beta_R}{\beta_1 2^{\beta_1}} \right) \int_U \frac{|f|^2 e^{-\phi} d\lambda}{e^{\Upsilon} R(\Upsilon)}.$$

*Proof.* This proposition is a modification of a theorem in [Demailly 2012b].

Since  $\Omega$  is a pseudoconvex domain, there is a sequence of pseudoconvex subdomains  $\Omega_k \Subset \Omega$  ( $k = 1, 2, \dots$ ) such that  $\bigcup_{k=1}^{+\infty} \Omega_k = \Omega$ . Then for fixed  $k$ , by convolution we can get a decreasing family of smooth plurisubharmonic functions  $\{\phi_j\}_{j=1}^{+\infty}$  defined on a neighborhood of  $\bar{\Omega}_k$  such that  $\lim_{j \rightarrow +\infty} \phi_j = \phi$ .

Let  $\theta : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\theta = 1$  on  $(-\infty, \frac{1}{4})$ ,  $\theta = 0$  on  $(\frac{3}{4}, +\infty)$  and  $|\theta'| \leq 3$  on  $\mathbb{R}$ .

Fix  $k$  and  $j$ . Set  $\hat{f} = \theta(e^{\Upsilon})f$ . Then the construction of  $\hat{f}$  implies that  $\hat{f}$  is smooth on  $\Omega$  and  $\hat{f} = f$  on  $Y \cap \Omega$ .

Set  $g = \bar{\partial} \hat{f}$ . Then  $g = \theta'(e^{\Upsilon})e^{\Upsilon} \bar{\partial} \Upsilon \wedge f$  on  $\Omega$ .

Let  $\Sigma := \{\Upsilon = -\infty\}$ . Lemma 2.4 implies that  $\Omega_k \setminus \Sigma$  is a complete Kähler manifold. Let  $\Omega_k \setminus \Sigma$  be endowed with the Euclidean metric and let  $Q$  be the trivial line bundle on  $\Omega_k \setminus \Sigma$  equipped with the metric

$$h := e^{-\phi_j - \Upsilon - \beta_1 \log(1+e^{\Upsilon}) - 2\gamma |z|^2}.$$

Then we want to solve a  $\bar{\partial}$  equation on  $\Omega_k \setminus \Sigma$  by applying Lemma 2.1 to the case  $\tau = 1$ ,  $A = 0$  and  $\delta = 0$  (in fact, the case  $\tau = 1$  and  $A = 0$  is the nontwisted version of Lemma 2.1). The key step in applying Lemma 2.1 is to estimate the term

$$\int_{\Omega_k \setminus \Sigma} \langle B^{-1}g, g \rangle_h d\lambda,$$

where  $B := [\sqrt{-1}\Theta_h, \Lambda]$ .

Set  $v = \partial \Upsilon$ . Then  $g = \theta'(e^{\Upsilon})e^{\Upsilon} \bar{v} \wedge f$  on  $\Omega$ .

Since

$$\begin{aligned} & \sqrt{-1}\Theta_h|_{\Omega_k \setminus \Sigma} \\ &= \sqrt{-1}\partial\bar{\partial}\phi_j + \sqrt{-1}\partial\bar{\partial}\Upsilon + \beta_1\sqrt{-1}\partial\bar{\partial}\log(1+e^{\Upsilon}) + 2\gamma\sqrt{-1}\partial\bar{\partial}|z|^2 \\ &= \sqrt{-1}\partial\bar{\partial}\phi_j + \left(1 + \frac{\beta_1 e^{\Upsilon}}{1+e^{\Upsilon}}\right)\sqrt{-1}\partial\bar{\partial}\Upsilon + 2\gamma\sqrt{-1}\partial\bar{\partial}|z|^2 + \frac{\beta_1 e^{\Upsilon}\sqrt{-1}\partial\Upsilon \wedge \bar{\partial}\Upsilon}{(1+e^{\Upsilon})^2} \\ &\geq \frac{\beta_1 e^{\Upsilon}\sqrt{-1}v \wedge \bar{v}}{(1+e^{\Upsilon})^2}, \end{aligned}$$

we get

$$B \geq \frac{\beta_1 e^{\Upsilon}}{(1+e^{\Upsilon})^2} T_{\bar{v}} T_{\bar{v}}^*$$

on  $\Omega_k \setminus \Sigma$ , where  $T_{\bar{v}}$  denotes the operator  $\bar{v} \wedge \bullet$  and  $T_{\bar{v}}^*$  is its Hilbert adjoint operator.

Then we get  $\langle B^{-1}g, g \rangle_h|_{\Omega_k \setminus U} = 0$  and

$$\begin{aligned} \langle B^{-1}g, g \rangle_h|_{(U \cap \Omega_k) \setminus \Sigma} &= \langle B^{-1}(\theta'(e^\Upsilon)e^\Upsilon \bar{v} \wedge f), \theta'(e^\Upsilon)e^\Upsilon \bar{v} \wedge f \rangle_h \\ &\leq \frac{(1+e^\Upsilon)^2}{\beta_1 e^\Upsilon} |\theta'(e^\Upsilon)e^\Upsilon f|^2 e^{-\phi_j - \Upsilon - \beta_1 \log(1+e^\Upsilon) - 2\gamma|z|^2} \\ &= \frac{(1+e^\Upsilon)^{2-\beta_1}}{\beta_1} |\theta'(e^\Upsilon)f|^2 e^{-\phi_j - 2\gamma|z|^2} \leq \frac{36}{\beta_1 2^{\beta_1}} |f|^2 e^{-\phi_j - 2\gamma|z|^2}. \end{aligned}$$

Hence it follows from Lemma 2.1 that there exists  $u_{k,j} \in L^2(\Omega_k \setminus \Sigma, K_\Omega \otimes Q, h)$  such that  $\bar{\partial}u_{k,j} = g = \bar{\partial}\hat{f}$  on  $\Omega_k \setminus \Sigma$  and

$$\int_{\Omega_k \setminus \Sigma} |u_{k,j}|_h^2 d\lambda \leq \int_{\Omega_k \setminus \Sigma} \langle B^{-1}g, g \rangle_h d\lambda.$$

Thus

$$\begin{aligned} (3-3) \quad \int_{\Omega_k \setminus \Sigma} \frac{|u_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^\Upsilon (1+e^\Upsilon)^{\beta_1}} d\lambda &\leq \frac{36}{\beta_1 2^{\beta_1}} \int_{U \cap \Omega_k} |f|^2 e^{-\phi_j - 2\gamma|z|^2} d\lambda \\ &\leq \frac{36\beta_R}{\beta_1 2^{\beta_1}} \int_U \frac{|f|^2 e^{-\phi - 2\gamma|z|^2}}{e^\Upsilon R(\Upsilon)} d\lambda. \end{aligned}$$

Hence we have  $u_{k,j} \in L^2(\Omega_k \setminus \Sigma, K_\Omega)$ . Since  $g \in C^\infty(\Omega_k, \wedge^{n,1} T_\Omega^*)$ , Lemma 2.6 implies that  $\bar{\partial}u_{k,j} = g$  holds on  $\Omega_k$ .

Let  $F_{k,j} := \hat{f} - u_{k,j}$ . Then  $\bar{\partial}F_{k,j} = 0$  on  $\Omega_k$ . Thus  $F_{k,j}$  is holomorphic on  $\Omega_k$ . Hence  $u_{k,j}$  is smooth on  $\Omega_k$ . Then the nonintegrability of  $e^{-\Upsilon}$  along  $Y$  implies that  $u_{k,j} = 0$  on  $Y \cap \Omega_k$ . Therefore,  $F_{k,j} = f$  on  $Y \cap \Omega_k$ .

It follows from (3-3) that

$$\begin{aligned} \int_{U \cap \Omega_k} \frac{|u_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^\Upsilon R(\Upsilon)} d\lambda &\leq \frac{2^{\beta_1}}{\hat{\beta}_R} \int_{U \cap \Omega_k} \frac{|u_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^\Upsilon (1+e^\Upsilon)^{\beta_1}} d\lambda \\ &\leq \frac{36\beta_R}{\beta_1 \hat{\beta}_R} \int_U \frac{|f|^2 e^{-\phi - 2\gamma|z|^2}}{e^\Upsilon R(\Upsilon)} d\lambda. \end{aligned}$$

Since

$$|F_{k,j}|^2|_{U \cap \Omega_k} \leq 2|\hat{f}|^2 + 2|u_{k,j}|^2 \leq 2|f|^2 + 2|u_{k,j}|^2,$$

we get

$$\begin{aligned} (3-4) \quad \int_{U \cap \Omega_k} \frac{|F_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^\Upsilon R(\Upsilon)} d\lambda &\leq 2 \int_{U \cap \Omega_k} \frac{(|f|^2 + |u_{k,j}|^2) e^{-\phi_j - 2\gamma|z|^2}}{e^\Upsilon R(\Upsilon)} d\lambda \\ &\leq \left(2 + \frac{72\beta_R}{\beta_1 \hat{\beta}_R}\right) \int_U \frac{|f|^2 e^{-\phi - 2\gamma|z|^2}}{e^\Upsilon R(\Upsilon)} d\lambda. \end{aligned}$$

Since

$$(3-5) \quad \langle \kappa_1 + \kappa_2, \kappa_1 + \kappa_2 \rangle \leq \langle \kappa_1, \kappa_1 \rangle + \langle \kappa_2, \kappa_2 \rangle + c \langle \kappa_1, \kappa_1 \rangle + \frac{1}{c} \langle \kappa_2, \kappa_2 \rangle$$

for any inner product space  $(H, \langle \cdot, \cdot \rangle)$ , where  $\kappa_1, \kappa_2 \in H$ , we get

$$|F_{k,j}|^2|_{U \cap \Omega_k} \leq (|f| + |u_{k,j}|)^2 \leq (1 + e^\Upsilon)|f|^2 + \left(1 + \frac{1}{e^\Upsilon}\right)|u_{k,j}|^2.$$

Then

$$\frac{|F_{k,j}|^2}{(1 + e^\Upsilon)^{1+\beta_1}} \Big|_{U \cap \Omega_k} \leq |f|^2 + \frac{|u_{k,j}|^2}{e^\Upsilon(1 + e^\Upsilon)^{\beta_1}}.$$

Since  $|F_{k,j}|^2|_{\Omega_k \setminus U} = |u_{k,j}|^2$ , we get

$$\frac{|F_{k,j}|^2}{(1 + e^\Upsilon)^{1+\beta_1}} \Big|_{\Omega_k \setminus U} \leq \frac{|u_{k,j}|^2}{e^\Upsilon(1 + e^\Upsilon)^{\beta_1}}.$$

Hence it follows from the two inequalities above and (3-3) that

$$(3-6) \quad \int_{\Omega_k} \frac{|F_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{(1 + e^\Upsilon)^{1+\beta_1}} d\lambda \leq \int_U |f|^2 e^{-\phi - 2\gamma|z|^2} d\lambda + \int_{\Omega_k} \frac{|u_{k,j}|^2 e^{-\phi_j - 2\gamma|z|^2}}{e^\Upsilon(1 + e^\Upsilon)^{\beta_1}} d\lambda \\ \leq \left(\beta_R + \frac{36\beta_R}{\beta_1 2^{\beta_1}}\right) \int_U \frac{|f|^2 e^{-\phi - 2\gamma|z|^2}}{e^\Upsilon R(\Upsilon)} d\lambda.$$

Since  $e^{-2\gamma \sup_{\Omega} |z|^2} \leq e^{-2\gamma|z|^2} \leq 1$  on  $\Omega$ , the desired holomorphic  $n$ -form  $F$  on  $\Omega$  and the  $L^2$  estimates (3-1) and (3-2) can be obtained from (3-4) and (3-6) by applying Montel's theorem and extracting weak limits of  $\{F_{k,j}\}_{k,j}$ , first as  $j \rightarrow +\infty$  and then as  $k \rightarrow +\infty$ . □

**Proposition 3.2.** *Let  $X, \psi, Y$  and  $Y^0$  be as in Theorem 1.8. Let  $U \Subset V \Subset \Omega$  be three local coordinate balls in  $X, \phi$  be a plurisubharmonic function on  $\Omega$  such that  $\sup_{\Omega} \phi < +\infty$ , and  $v$  be a nonnegative continuous function on  $\Omega$  with  $\text{supp } v \subset U$ . Let  $C, \beta, c_1$  and  $c_2$  be positive numbers, and let  $\beta_1$  be a small enough positive number. Assume that  $f$  is a holomorphic function on  $\Omega \cap Y$  satisfying*

$$(3-7) \quad \int_{\Omega \cap Y^0} |f|^2 e^{-\phi} d\lambda[\psi] < +\infty,$$

and that  $f_t \in \mathcal{O}(\Omega)$  ( $t \in (-\infty, 0)$ ) are a family of holomorphic functions such that for all  $t \in (-\infty, 0)$ ,  $f_t = f$  on  $\Omega \cap Y$ ,

$$(3-8) \quad \sup_V |f_t|^2 \leq C e^{-\beta_1 t},$$

$$(3-9) \quad \frac{1}{e^t} \int_{\Omega \cap \{\psi < t + c_2\}} |f_t|^2 e^{-(1+\beta)\phi} d\lambda \leq C.$$

Then

$$(3-10) \quad \overline{\lim}_{t \rightarrow -\infty} \int_{U \cap \{t - c_1 < \psi < t + c_2\}} \frac{e^t v |f_t|^2 e^{-\phi}}{(e^\psi + e^t)^2} d\lambda \leq \int_{U \cap Y^0} v |f|^2 e^{-\phi} d\lambda[\psi].$$

**Remark 3.3.** One of the key points in the proof of Proposition 3.2 is to verify that the upper limit in (3-10) produces the zero measure on the singular set of  $Y$ , i.e., we have (3-16). Then the key uniform estimates in Step 2 of the proof are obtained.

In order to prove Proposition 3.2, we prove the following lemma at first.

**Lemma 3.4.** *Let  $r_1, r_2$  and  $\gamma$  be positive numbers such that  $r_1 < r_2 < \gamma$ . Let  $\varphi$  be a bounded negative subharmonic function on  $\Delta_\gamma$ , where  $\Delta_\gamma := \{w \in \mathbb{C} : |w| < \gamma\}$ . Assume that  $\{v_t\}_{t \in (-\infty, 0)}$  are nonnegative continuous functions defined on  $\Delta_\gamma$  such that*

$$(3-11) \quad \lim_{t \rightarrow -\infty} \sup_{\{w \in \mathbb{C} : e^t (r_1)^{2\alpha} < |w|^{2\alpha} < e^t (r_2)^{2\alpha}\}} |v_t(w) - v_0| = 0,$$

where  $\alpha \in [1, +\infty)$  and  $v_0 \in [0, +\infty)$ . Let

$$P_t := \int_{\{w \in \mathbb{C} : e^t (r_1)^{2\alpha} < |w|^{2\alpha} < e^t (r_2)^{2\alpha}\}} \frac{e^t |w|^{2\alpha-2} v_t(w) e^{-\varphi(w)}}{(|w|^{2\alpha} + e^t)^2} d\lambda(w).$$

Then

$$(3-12) \quad \overline{\lim}_{t \rightarrow -\infty} P_t \leq \frac{\pi v_0 e^{-\varphi(0)}}{\alpha}.$$

*Proof.* Put

$$S_{\delta,t} = \{z \in \Delta_\gamma : \varphi(e^{t/(2\alpha)} z) < (1 + \delta)\varphi(0)\}, \quad \delta \in (0, +\infty), \quad t \in (-\infty, 0).$$

Denote by  $\lambda(S_{\delta,t})$  the 2-dimensional Lebesgue measure of  $S_{\delta,t}$ .

Since  $\varphi(w)$  is a negative upper semicontinuous function on  $\Delta_\gamma$  and  $\varphi(0) > -\infty$ , we have that for every  $\varepsilon \in (0, 1)$ , there exists  $t_\varepsilon \in (-\infty, 0)$  such that

$$\varphi(e^{t/(2\alpha)} z) \leq (1 - \varepsilon)\varphi(0)$$

for all  $z \in \Delta_\gamma$  when  $t \in (-\infty, t_\varepsilon)$ .

Since  $\varphi(e^{t/(2\alpha)} z)$  is subharmonic on  $\Delta_\gamma$  with respect to  $z$  for any  $t \in (-\infty, t_\varepsilon)$ , it follows from the mean value inequality that, for all  $t \in (-\infty, t_\varepsilon)$ ,

$$\begin{aligned} \varphi(0) &\leq \frac{1}{\pi \gamma^2} \int_{z \in \Delta_\gamma} \varphi(e^{t/(2\alpha)} z) d\lambda(z) \\ &= \frac{1}{\pi \gamma^2} \int_{z \in \Delta_\gamma \setminus S_{\delta,t}} \varphi(e^{t/(2\alpha)} z) d\lambda(z) + \frac{1}{\pi \gamma^2} \int_{z \in S_{\delta,t}} \varphi(e^{t/(2\alpha)} z) d\lambda(z) \\ &\leq \frac{(1 - \varepsilon)\varphi(0)(\pi \gamma^2 - \lambda(S_{\delta,t}))}{\pi \gamma^2} + \frac{(1 + \delta)\varphi(0)\lambda(S_{\delta,t})}{\pi \gamma^2} \\ &= \varphi(0) \left( 1 - \varepsilon + \frac{(\delta + \varepsilon)\lambda(S_{\delta,t})}{\pi \gamma^2} \right). \end{aligned}$$

Then  $\varphi(0) < 0$  implies that

$$\lambda(S_{\delta,t}) \leq \frac{\pi\gamma^2\varepsilon}{\delta + \varepsilon} \leq \frac{\pi\gamma^2}{\delta}\varepsilon$$

when  $t \in (-\infty, t_\varepsilon)$ . Hence

$$(3-13) \quad \lim_{t \rightarrow -\infty} \lambda(S_{\delta,t}) = 0 \quad \text{for all } \delta \in (0, +\infty).$$

Since  $\varphi$  is bounded, we have

$$-\varphi \leq C_1$$

for some positive number  $C_1$ .

Equation (3-11) implies that

$$\sup_{\{w \in \mathbb{C} : e^t(r_1)^{2\alpha} < |w|^{2\alpha} < e^t(r_2)^{2\alpha}\}} v_t(w) \leq C_2$$

for some positive number  $C_2$  independent of  $t$  when  $t$  is small enough.

Then by the change of variables  $w = e^{t/(2\alpha)}z$ , we have

$$\begin{aligned} P_t &= \int_{\{z \in \mathbb{C} : r_1 < |z| < r_2\}} \frac{|z|^{2\alpha-2} v_t(e^{t/(2\alpha)}z) e^{-\varphi(e^{t/(2\alpha)}z)}}{(|z|^{2\alpha} + 1)^2} d\lambda(z) \\ &= \int_{\{r_1 < |z| < r_2\} \cap S_{\delta,t}} \frac{|z|^{2\alpha-2} v_t(e^{t/(2\alpha)}z) e^{-\varphi(e^{t/(2\alpha)}z)}}{(|z|^{2\alpha} + 1)^2} d\lambda(z) \\ &\quad + \int_{\{r_1 < |z| < r_2\} \setminus S_{\delta,t}} \frac{|z|^{2\alpha-2} v_t(e^{t/(2\alpha)}z) e^{-\varphi(e^{t/(2\alpha)}z)}}{(|z|^{2\alpha} + 1)^2} d\lambda(z) \\ &\leq \frac{(r_2)^{2\alpha-2} C_2 e^{C_1}}{((r_1)^{2\alpha} + 1)^2} \cdot \lambda(S_{\delta,t}) \\ &\quad + \left( \sup_{r_1 < |z| < r_2} v_t(e^{t/(2\alpha)}z) \right) e^{-(1+\delta)\varphi(0)} \int_{\{r_1 < |z| < r_2\}} \frac{|z|^{2\alpha-2}}{(|z|^{2\alpha} + 1)^2} d\lambda(z). \end{aligned}$$

Since

$$\int_{\{r_1 < |z| < r_2\}} \frac{|z|^{2\alpha-2}}{(|z|^{2\alpha} + 1)^2} d\lambda(z) \leq \frac{\pi}{\alpha},$$

we obtain from (3-11), (3-13) that

$$\overline{\lim}_{t \rightarrow -\infty} P_t \leq \frac{\pi v_0 e^{-(1+\delta)\varphi(0)}}{\alpha}.$$

Since  $\delta$  is an arbitrary positive number, we get (3-12). □

Now we begin to prove Proposition 3.2.

*Proof.* Let  $\beta_v := \sup_U v$ .

Without loss of generality, we may suppose that  $\phi$  is negative on  $\Omega$ .

We will use Hironaka’s desingularization theorem (Theorem 2.8) to deal with the measure  $d\lambda[\psi]$ . This idea comes from [Demailly 2016].

At first we use Theorem 2.8 on  $X$  to resolve the singularities of  $Y$  and we denote the corresponding proper modification by  $\mu_1$ . Next, we make a blow-up  $\mu_2$  along  $|Y'|$ . Then we use Theorem 2.8 again to resolve the singularities of  $\Sigma$  and we denote the corresponding proper holomorphic modification by  $\mu_3$ , where  $\Sigma$  denote the strict transform of  $\{\psi = -\infty\}$  by  $\mu_1 \circ \mu_2$ . Finally, we make a blow-up  $\mu_4$  along  $|\Sigma'|$ . Thus we can get a proper holomorphic map  $\mu : \tilde{X} \rightarrow X$ , which is locally a finite composition of blow-ups with smooth centers and is equal to  $\mu_1 \circ \mu_2 \circ \mu_3 \circ \mu_4$ . Moreover,  $\tilde{Y}$  and the divisor  $\mu^{-1}(\{\psi = -\infty\}) \setminus \tilde{Y}$  simultaneously have only normal crossings in  $\tilde{X}$ , where  $\tilde{Y}$  denotes the strict transform of  $\mu_2^{-1}(|Y'|)$  by  $\mu_3 \circ \mu_4$ .

**Step 1:** Representing the measure  $|f|^2 d\lambda[\psi]$  on  $Y^0 \cap U$  explicitly as an integral on  $\tilde{Y}$  (see (3-15)).

For any  $\tilde{x} \in \overline{\mu^{-1}(U)} \cap \mu^{-1}(\{\psi = -\infty\})$ , there exists a relatively compact coordinate ball  $(W; w_1, \dots, w_n)$  contained in  $\mu^{-1}(V)$  centered at  $\tilde{x}$  such that  $w^b = 0$  is the zero divisor of the Jacobian  $J_\mu$ , and  $\psi \circ \mu$  can be written on  $W$  as

$$\psi \circ \mu(w) = c \log |w^a|^2 + \tilde{u}(w),$$

where  $c$  is a positive number,  $w := (w_1, \dots, w_n)$ ,  $\tilde{u} \in C^\infty(\overline{W})$ ,  $w^a := \prod_{p=1}^n w_p^{a_p}$  and  $w^b := \prod_{p=1}^n w_p^{b_p}$  for some nonnegative integers  $a_p$  and  $b_p$ .

Let  $D_p := \{w_p = 0\}$ . Then as proved in [Demailly 2016], the multiplier ideal sheaf  $\mathcal{I}(\psi)$  is given by the direct image formula

$$\mathcal{I}(\psi) = \mu_* \mathcal{O}_{\tilde{X}} \left( - \sum_{p=1}^n \lfloor ca_p - b_p \rfloor_+ D_p \right),$$

where  $\lfloor ca_p - b_p \rfloor_+$  denotes the minimal nonnegative integer bigger than  $ca_p - b_p - 1$ . Since  $\psi$  has log canonical singularities, by the construction of  $\mu$  and Theorem 2.8, one of the following cases is true on  $W$ :

- (A)  $\tilde{Y}$  is given on  $W$  precisely by  $D_{p_0}$  (if  $W$  is small enough) for some  $p_0$  satisfying  $ca_{p_0} - b_{p_0} = 1$ , and  $ca_p - b_p \leq 1$  for  $p \neq p_0$ ;
- (B)  $\tilde{Y} \cap W = \emptyset$ , and  $ca_p - b_p \leq 1$ .

By definition, the measure  $|f|^2 d\lambda[\psi]$  can be defined as

$$(3-14) \quad g \mapsto \overline{\lim}_{t \rightarrow -\infty} \int_{\{t < c \log |w^a|^2 + \tilde{u}(w) < t+1\}} \frac{|\tilde{f} \circ \mu|^2 (\tilde{g} \circ \mu) \xi e^{-\tilde{u}}}{|w^{ca-b}|^2} d\lambda(w),$$

where  $d\lambda(w) :=$  the Lebesgue measure with respect to the coordinate vector  $w$ ,  $\tilde{f}$  is a holomorphic extension of  $f$  to  $\Omega$ ,  $g$  and  $\tilde{g}$  are defined as in Definition 1.3, and  $\xi$  is the

smooth positive function  $|J_\mu|^2/|w^b|^2$  (as stated in [Demailly 2016], one would still have to take into account a partition of unity on the various coordinate charts covering the fibers of  $\mu$ , but we will avoid this technicality for the simplicity of notation).

In Case (A), let us denote  $w = (w', w_{p_0}) \in \mathbb{C}^{n-1} \times \mathbb{C}$ ,  $a = (a', a_{p_0})$ ,  $b = (b', b_{p_0})$  and  $d\lambda(w) = d\lambda(w')d\lambda(w_{p_0})$ . Then (3-14) becomes

$$g \mapsto \overline{\lim}_{t \rightarrow -\infty} \int_{\{t < c \log |w^a|^2 + \tilde{u}(w) < t+1\}} \frac{|\tilde{f} \circ \mu|^2}{|(w')^{ca'-b'}|^2} \cdot \frac{(\tilde{g} \circ \mu)\xi e^{-\tilde{u}}}{|w_{p_0}|^2} d\lambda(w).$$

Since the domain of integration can be written as

$$\{e^{t-\tilde{u}(w)} |(w')^{a'}|^{-2c} < |w_{p_0}|^{2ca_{p_0}} < e^{t+1-\tilde{u}(w)} |(w')^{a'}|^{-2c}\},$$

the mapping (3-14) becomes

$$(3-15) \quad g \mapsto \frac{\pi}{ca_{p_0}} \int_{w' \in D_{p_0}} \frac{|f \circ \mu|^2}{|(w')^{ca'-b'}|^2} \cdot (g \circ \mu)\xi e^{-\tilde{u}} d\lambda(w').$$

Set  $\kappa = \{p : ca_p - b_p = 1\}$ .

If  $p \in \kappa \setminus \{p_0\}$ , then Theorem 2.8 and the construction of  $\mu$  imply that an image of  $D_p$  under a finite sequence of blow-ups in the desingularization process must be contained in a smooth center contained in  $Y$  or  $\mu_2^{-1}(|Y'|)$ . Hence the images of  $D_p$  and  $D_p \cap D_{p_0}$  coincide under the composition of these blow-ups.

Since it is implied from (3-7) and (3-15) that  $f \circ \mu|_{D_p \cap D_{p_0}} = 0$ , we obtain that

$$(3-16) \quad f \circ \mu|_{D_p} = 0$$

holds for all  $p \in \kappa \setminus \{p_0\}$  in Case (A).

Similarly, we can get that (3-16) holds for all  $p \in \kappa$  in Case (B). Then (3-14) is the zero measure in Case (B).

Therefore, we represent the measure  $|f|^2 d\lambda[\psi]$  on  $Y^0 \cap U$  explicitly as in (3-15).

**Step 2:** Obtaining some uniform estimates for  $f_t \circ \mu$ .

By Cauchy’s inequality for holomorphic functions, it follows from (3-8) that

$$(3-17) \quad \sup_{U_1} |\partial^\gamma f_t|^2 \leq C_1 \sup_V |f_t|^2 \leq C_1 C e^{-\beta_1 t}$$

for any  $t \in (-\infty, 0)$  and any multi-index  $\gamma$  satisfying  $|\gamma| \leq n$ , where  $U_1 \Subset V$  is a neighborhood of  $\bar{U}$ , and  $C_1$  is a positive number independent of  $t$  and  $\gamma$ .

Let  $W_t := W \cap \mu^{-1}(U) \cap \{\psi \circ \mu < t + c_2\}$ .

In Case (A), by applying the mean value theorem to  $f_t \circ \mu$  successively along the directions in  $\kappa$ , we get from (3-17) and (3-16) that for any  $w = (w', w_{p_0}) \in W_t$ ,

$$(3-18) \quad |f_t \circ \mu(w', w_{p_0}) - f_t \circ \mu(w', 0)|^2 \leq C_2 \prod_{p \in \kappa} |w_p|^2 \sup_{|\gamma| \leq |\kappa|} \sup_{\mu^{-1}(U_1)} |\partial^\gamma f_t|^2 \\ \leq C_3 e^{-\beta_1 t} \prod_{p \in \kappa} |w_p|^2$$

and

$$(3-19) \quad |f_t \circ \mu(w', 0)|^2 = |f \circ \mu(w', 0)|^2 \leq C_4 \prod_{p \in \kappa \setminus \{p_0\}} |w_p|^2$$

when  $t$  is small enough, where  $C_2, C_3$  and  $C_4$  are positive numbers independent of  $t$ .

In Case (B), if  $\kappa \neq \emptyset$ , take  $p_1 \in \kappa$  and denote  $w = (w'', w_{p_1})$ . Since  $f_t \circ \mu(w'', 0) = f \circ \mu(w'', 0) = 0$ , by a similar method we have that

$$(3-20) \quad |f_t \circ \mu(w'', w_{p_1})|^2 \leq C_5 e^{-\beta_1 t} \prod_{p \in \kappa} |w_p|^2$$

for any  $w = (w'', w_{p_1}) \in W_t$  when  $t$  is small enough, where  $C_5$  is a positive number independent of  $t$ . If  $\kappa = \emptyset$ , (3-8) implies that

$$(3-21) \quad |f_t \circ \mu(w)|^2 \leq C e^{-\beta_1 t}$$

for any  $w \in W_t$ .

**Step 3:** The proof of (3-10).

Let  $j$  be a positive integer. Then (3-9) implies that

$$\frac{1}{e^t} \int_{\{\phi \leq -j\} \cap U \cap \{\psi < t + c_2\}} |f_t|^2 e^{-\phi} d\lambda \leq \frac{1}{e^t} \int_{\{\phi \leq -j\} \cap U \cap \{\psi < t + c_2\}} |f_t|^2 e^{-(1+\beta)\phi - \beta j} d\lambda \\ \leq C e^{-\beta j}$$

for any  $t \in (-\infty, 0)$ .

Therefore, for every  $\varepsilon \in (0, 1)$ , there exists a positive integer  $j_\varepsilon$  such that

$$(3-22) \quad \int_{\{\phi \leq -j_\varepsilon\} \cap U \cap \{t - c_1 < \psi < t + c_2\}} \frac{e^t v |f_t|^2 e^{-\phi}}{(e^\psi + e^t)^2} d\lambda \\ \leq \frac{1}{(e^{-c_1} + 1)^2 e^t} \int_{\{\phi \leq -j_\varepsilon\} \cap U \cap \{\psi < t + c_2\}} v |f_t|^2 e^{-\phi} d\lambda \leq \frac{\beta_v C e^{-\beta j_\varepsilon}}{(e^{-c_1} + 1)^2} < \frac{\varepsilon}{2}$$

for any  $t \in (-\infty, 0)$ .

Set  $\phi_\varepsilon = \max\{\phi, -j_\varepsilon\}$ . We want to prove

$$(3-23) \quad \overline{\lim}_{t \rightarrow -\infty} \int_{U \cap \{t - c_1 < \psi < t + c_2\}} \frac{e^t v |f_t|^2 e^{-\phi_\varepsilon}}{(e^\psi + e^t)^2} d\lambda \leq \int_{U \cap Y^0} v |f|^2 e^{-\phi_\varepsilon} d\lambda[\psi].$$

Set

$$I_0 = \overline{\lim}_{t \rightarrow -\infty} \int_{W \cap \mu^{-1}(U) \cap \{t-c_1 < \psi \circ \mu < t+c_2\}} \frac{e^t (v \circ \mu) |f_t \circ \mu|^2 e^{-\phi_\varepsilon \circ \mu} |J_\mu|^2}{(e^{\psi \circ \mu} + e^t)^2} d\lambda.$$

Then by Step 1, it suffices to prove that

$$(3-24) \quad I_0 \leq \frac{\pi}{ca_{p_0}} \int_{W \cap \mu^{-1}(U) \cap D_{p_0}} \frac{(v \circ \mu) |f \circ \mu|^2 \xi e^{-\tilde{u} - \phi_\varepsilon \circ \mu}}{|(w')^{ca'-b'}|^2} d\lambda(w')$$

in Case (A) and  $I_0 = 0$  in Case (B), where  $\xi$  is the smooth positive function  $|J_\mu|^2/|w^b|^2$  defined in Step 1.

In Case (A), let

$$\Phi_t(w') := \int_{W_{t,w'}} \frac{e^t (v \circ \mu) |f_t \circ \mu|^2 e^{-\phi_\varepsilon \circ \mu} |J_\mu|^2}{(e^{\psi \circ \mu} + e^t)^2} d\lambda(w_{p_0})$$

and

$$\Phi(w') := \frac{\pi}{ca_{p_0}} \cdot \frac{v \circ \mu(w', 0) |f \circ \mu(w', 0)|^2 \xi(w', 0) e^{-\tilde{u}(w', 0) - \phi_\varepsilon \circ \mu(w', 0)}}{|(w')^{ca'-b'}|^2},$$

where  $W_{t,w'}$  is the 1-dimensional open set

$$\{e^{t-c_1-\tilde{u}(w',w_{p_0})} |(w')^{a'}|^{-2c} < |w_{p_0}|^{2ca_{p_0}} < e^{t+c_2-\tilde{u}(w',w_{p_0})} |(w')^{a'}|^{-2c}\} \cap W \cap \mu^{-1}(U)$$

for every fixed  $t$  and  $w'$  ( $w' \in D_{p_0} \setminus \bigcup_{p \neq p_0} D_p$ ). Then

$$(3-25) \quad I_0 = \overline{\lim}_{t \rightarrow -\infty} \int_{W \cap \mu^{-1}(U) \cap D_{p_0}} \Phi_t(w') d\lambda(w').$$

Since  $-c_1 < \psi \circ \mu - t < c_2$  holds on  $W_{t,w'}$ , we obtain from (3-18) and (3-19) that

$$\begin{aligned} \Phi_t(w') &\leq C_6 \int_{W_{t,w'}} \frac{(v \circ \mu) |f_t \circ \mu|^2 e^{-\phi_\varepsilon \circ \mu} |J_\mu|^2}{e^{\psi \circ \mu}} d\lambda(w_{p_0}) \\ &\leq C_7 \int_{W_{t,w'}} \frac{|f_t \circ \mu|^2}{|w^{ca-b}|^2} d\lambda(w_{p_0}) \\ &\leq C_8 \int_{W_{t,w'}} \frac{\prod_{p \in \kappa} |w_p|^2}{|w^{(1+\beta_1)ca-b}|^2} d\lambda(w_{p_0}) + C_8 \int_{W_{t,w'}} \frac{\prod_{p \in \kappa \setminus \{p_0\}} |w_p|^2}{|w^{ca-b}|^2} d\lambda(w_{p_0}), \end{aligned}$$

where  $C_7$  and  $C_8$  are positive numbers independent of  $t$ .

Since it is easy to prove that the right-hand side of the above inequality is dominated by a function of  $w'$  which is independent of  $t$  and which belongs to  $L^1(W \cap \mu^{-1}(U) \cap D_{p_0})$  when

$$\beta_1 < \min_{\{p: a_p \neq 0\}} \frac{1 - (ca_p - b_p) + \lfloor ca_p - b_p \rfloor_+}{ca_p},$$

it follows from (3-25) and Fatou's lemma that

$$(3-26) \quad I_0 \leq \int_{W \cap \mu^{-1}(U) \cap D_{p_0}} \overline{\lim}_{t \rightarrow -\infty} \Phi_t(w') d\lambda(w').$$

Since (3-18) implies that

$$\lim_{t \rightarrow -\infty} \sup_{w_{p_0} \in W_{t, w'}} |f_t \circ \mu(w', w_{p_0}) - f \circ \mu(w', 0)| = 0$$

for every fixed  $w' \in (W \cap \mu^{-1}(U) \cap D_{p_0}) \setminus \bigcup_{p \neq p_0} (D_{p_0} \cap D_p)$  when  $\beta_1 < 1/ca_{p_0}$ , it follows from Lemma 3.4 that

$$\overline{\lim}_{t \rightarrow -\infty} \Phi_t(w') \leq \Phi(w') \quad \text{for all } w' \in (W \cap \mu^{-1}(U) \cap D_{p_0}) \setminus \bigcup_{p \neq p_0} (D_{p_0} \cap D_p).$$

Hence (3-24) follows from (3-26). Similarly, we can obtain from (3-20) and (3-21) that  $I_0 = 0$  in Case (B) when

$$\beta_1 < \min_{\{p: a_p \neq 0\}} \frac{1 - (ca_p - b_p) + \lfloor ca_p - b_p \rfloor_+}{ca_p}.$$

Thus we get (3-23).

It is easy to see that (3-10) follows from (3-22) and (3-23). Thus we finish the proof of Proposition 3.2.  $\square$

#### 4. Proof of Theorem 1.8

Without loss of generality, we can suppose that  $f$  is not 0 identically.

Let  $h_0$  be any fixed smooth metric of  $L$  on  $X$ . Then  $h = h_0 e^{-\phi}$  for some global function  $\phi$  on  $X$ , which is quasi-plurisubharmonic by the assumption in the theorem.

Since  $X$  is weakly pseudoconvex, there exists a smooth plurisubharmonic exhaustion function  $P$  on  $X$ . Let  $X_k := \{P < k\}$  ( $k = 1, 2, \dots$ , we choose  $P$  such that  $X_1 \neq \emptyset$ ).

Our proof consists of several steps. We will discuss for fixed  $k$  until the end of Step 5.

We will give the proof for the line bundle  $L$  in the first five steps, and we will give the proof for the vector bundle  $E$  in Step 6.

**Step 1:** Construction of a family of special smooth extensions  $\tilde{f}_t$  of  $f$  to a neighborhood of  $\overline{X}_k \cap Y$  in  $X$ .

In order to deal with singular metrics of holomorphic line bundles on weakly pseudoconvex Kähler manifolds, we construct in this step a family of smooth extensions  $\tilde{f}_t$  of  $f$  satisfying some special estimates by using the results in Section 3.

Let  $\varepsilon \in (0, \frac{1}{2})$ .

For the sake of clarity, we divide this step into four parts.

**Part I:** Construction of local coordinate patches  $\{\Omega_i\}_{i=1}^N$ ,  $\{U_i\}_{i=1}^N$  and a partition of unity  $\{\xi_i\}_{i=1}^{N+1}$ .

For any point  $x \in Y$ , we can find a local coordinate ball  $\Omega'_x$  in  $X$  centered at  $x$  such that there exists a local holomorphic frame of  $L$  on  $\Omega'_x$  and such that  $\phi$  can be written as a sum of a smooth function and a plurisubharmonic function on  $\Omega'_x$ . Moreover, we assume that  $\psi$  can be written on  $\Omega'_x$  as

$$(4-1) \quad \psi = c_x \log \sum_{1 \leq j \leq j_0} |g_{x,j}|^2 + u_x,$$

where  $c_x$  is a positive number,  $g_{x,j} \in \mathcal{O}_X(\Omega'_x)$  and  $u_x \in C^\infty(\Omega'_x)$ .

Let  $U_x \Subset V_x \Subset \Omega_x \Subset \Omega'_x$  be three small coordinate balls.

Since  $\overline{X}_k \cap Y$  is compact, there exist points  $x_1, x_2, \dots, x_N \in \overline{X}_k \cap Y$  such that  $\overline{X}_k \cap Y \subset \bigcup_{i=1}^N U_{x_i}$ .

For simplicity, we denote  $\Omega'_{x_i}, \Omega_{x_i}, U_{x_i}, V_{x_i}$  and  $u_{x_i}$  by  $\Omega'_i, \Omega_i, U_i, V_i$  and  $u_i$ , respectively. We denote the local expression (4-1) on  $\Omega'_i$  by

$$\psi = \Upsilon_i + u_i.$$

Choose an open set  $U_{N+1}$  in  $X$  such that  $\overline{X}_k \cap Y \subset X \setminus \overline{U_{N+1}} \Subset \bigcup_{i=1}^N U_i$ . Set  $U = X \setminus \overline{U_{N+1}}$ .

Let  $\{\xi_i\}_{i=1}^{N+1}$  be a partition of unity subordinate to the cover  $\{U_i\}_{i=1}^{N+1}$  of  $X$ . Then  $\text{supp } \xi_i \Subset U_i$  for  $i = 1, \dots, N$  and  $\sum_{i=1}^N \xi_i = 1$  on  $U$ .

**Part II:** Construction of local holomorphic extensions  $\hat{f}_{i,t}$  ( $1 \leq i \leq N$ ) of  $f$  to  $\Omega_i \cap \{\psi < t + c_2\}$ , where  $c_2$  will be defined in this part.

By Remark 1.9,  $f$  has local  $L^2$  extensions to local coordinate balls around every point in  $Y$ . Hence  $f$  is indeed a holomorphic section well defined on  $Y$  (not only on  $Y^0$ ). By Step 1 (see (3-15)) in the proof of Proposition 3.2, (1-3) is equivalent to

$$\int_{D_{\rho_0}} \frac{|f \circ \mu|_{\omega, h_0}^2 \xi e^{-\tilde{u} - \phi \circ \mu}}{|(w')^{ca' - b'}|^2} d\lambda(w') < +\infty.$$

Hence by Theorem 2.7, there exists a positive number  $\beta \in (0, 1)$  such that

$$(4-2) \quad \int_{\Omega_i \cap Y^0} |f|_{\omega, h_0}^2 e^{-(1+\beta)\phi} dV_{X, \omega}[\psi] < +\infty \quad (1 \leq i \leq N).$$

Let  $\tilde{\alpha}_0 < \alpha_0$  be a fixed number such that  $R$  is decreasing on  $(-\infty, \tilde{\alpha}_0]$ . Then set  $R_0(t) = R(\tilde{\alpha}_0)e^{-\beta_2(t-\tilde{\alpha}_0)}$ ,  $t \in (-\infty, \tilde{\alpha}_0]$ , where  $\beta_2$  is a positive number which will be determined later in Step 4. Let

$$R_1(t) := \min\{R_0(t + \tilde{\alpha}_0), R(t + \tilde{\alpha}_0)\}, \quad t \in (-\infty, 0].$$

Then  $R_1$  is decreasing and thereby satisfies all the requirements for the functions in  $\mathfrak{R}_{0, \alpha_1}$  except that  $R_1$  is only continuous.

Let  $c_1 = c_2 := \log((2 - \varepsilon)/\varepsilon)$ ,  $m_i := \inf_{\Omega_i} u_i$  and  $M_i := \sup_{\Omega_i} u_i$ .

For each fixed  $t \in (-\infty, 0)$ , by Remark 1.9, we apply Theorem 1.8 to the Stein manifold  $\Omega_i \cap \{\Upsilon_i < t + c_2 - m_i\}$ , to the negative plurisubharmonic function  $\Upsilon_i - t - c_2 + m_i$ , to the holomorphic section  $f$  on  $\Omega_i \cap Y^0$  with the  $L^2$  condition (4-2) and to the function  $R_1$  ( $R_1$  is only needed to be continuous by the remark after Theorem 2.1 in [Guan and Zhou 2015c]), and then we obtain  $L^2$  extensions of  $f$  from  $\Omega_i \cap Y^0$  to

$$\Omega_i \cap \{\Upsilon_i < t + c_2 - m_i\},$$

where we equip the line bundle  $L$  with the singular metric  $h_0 e^{-(1+\beta)\phi}$ . More precisely, there exists a uniform positive number  $C_1$  (independent of  $t$ ) and holomorphic extensions  $\hat{f}_{i,t}$  ( $1 \leq i \leq N$ ) of  $f$  from  $\Omega_i \cap Y^0$  to  $\Omega_i \cap \{\Upsilon_i < t + c_2 - m_i\}$  such that

$$(4-3) \quad \int_{\Omega_i \cap \{\Upsilon_i < t + c_2 - m_i\}} \frac{|\hat{f}_{i,t}|_{\omega, h_0}^2 e^{-(1+\beta)\phi}}{e^{\Upsilon_i - t - c_2 + m_i} R_1(\Upsilon_i - t - c_2 + m_i)} dV_{X, \omega} \\ \leq C_1 \int_{\Omega_i \cap Y^0} |f|_{\omega, h_0}^2 e^{-(1+\beta)\phi} dV_{X, \omega}[\Upsilon_i - t - c_2 + m_i] \\ \leq C_2 e^t \int_{\Omega_i \cap Y^0} |f|_{\omega, h_0}^2 e^{-(1+\beta)\phi} dV_{X, \omega}[\psi],$$

where  $C_2$  is a positive number independent of  $t$ . Furthermore, we get that  $f$  is in fact holomorphic on  $\Omega_i \cap Y$  and  $\hat{f}_{i,t} = f$  on  $\Omega_i \cap Y$ .

**Part III:** Construction of local holomorphic extensions  $\tilde{f}_{i,t}$  ( $1 \leq i \leq N$ ) of  $f$  to  $\Omega_i$ .

For each fixed  $t$ , applying Proposition 3.1 to the local extensions  $\hat{f}_{i,t}$  ( $1 \leq i \leq N$ ) with the weight  $(1 + \beta)\phi$  and to the case  $\Upsilon = \Upsilon_i - t - c_2 + m_i$ ,  $\Omega = \Omega_i$  and some small positive number  $\beta_1$  which will be determined later in Step 4, we obtain

from (4-3) holomorphic sections  $\tilde{f}_{i,t}$  ( $1 \leq i \leq N$ ) on  $\Omega_i$  satisfying  $\tilde{f}_{i,t} = \hat{f}_{i,t} = f$  on  $\Omega_i \cap Y^0$ ,

$$(4-4) \quad \int_{\Omega_i \cap \{\Upsilon_i < t + c_2 - m_i\}} \frac{|\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-(1+\beta)\phi}}{e^{\Upsilon_i - t - c_2 + m_i} R_1(\Upsilon_i - t - c_2 + m_i)} dV_{X, \omega} \leq C_3 e^t,$$

and

$$(4-5) \quad \int_{\Omega_i} \frac{|\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-(1+\beta)\phi}}{(1 + e^{\Upsilon_i - t - c_2 + m_i})^{1+\beta_1}} dV_{X, \omega} \leq C_3 e^t$$

for some positive number  $C_3$  independent of  $t$ .

Since  $\sup_{t \leq 0} (e^t R_1(t)) < +\infty$ , it follows from (4-4) that

$$(4-6) \quad \int_{\Omega_i \cap \{\psi < t + c_2\}} |\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-(1+\beta)\phi} dV_{X, \omega} \leq C_4 e^t$$

for any  $t$ , where  $C_4$  is a positive number independent of  $t$ .

Since  $\Upsilon_i$  is bounded above on  $\Omega_i$ , it follows from (4-5) that

$$(4-7) \quad \int_{\Omega_i} |\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-(1+\beta)\phi} dV_{X, \omega} \leq C_5 e^{-\beta_1 t}$$

for any  $t$ , where  $C_5$  is a positive number independent of  $t$ .

Since  $|\tilde{f}_{i,t}|^2$  is plurisubharmonic on  $\Omega_i$ , by mean value inequality, we get from (4-7) that

$$(4-8) \quad \sup_{V_i} |\tilde{f}_{i,t}|_{\omega, h_0}^2 \leq C_6 e^{-\beta_1 t}$$

for any  $t$ , where  $C_6$  is a positive number independent of  $t$ .

Since (4-6) and (4-8) imply that the assumptions in Proposition 3.2 hold for  $\tilde{f}_{i,t}$ , we apply Proposition 3.2 to  $\tilde{f}_{i,t}$  ( $1 \leq i \leq N$ ) and get

$$(4-9) \quad \overline{\lim}_{t \rightarrow -\infty} \int_{U_i \cap \{t - c_1 < \psi < t + c_2\}} \frac{e^t \xi_i |\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-\phi}}{(e^\psi + e^t)^2} dV_{X, \omega} \\ \leq \int_{U_i \cap Y^0} \xi_i |f|_{\omega, h_0}^2 e^{-\phi} dV_{X, \omega}[\psi],$$

which will be used in Step 4.

**Part IV:** Construction of a family of smooth extensions  $\tilde{f}_t$  of  $f$  to a neighborhood of  $\overline{X_k} \cap Y$  in  $X$ .

Define  $\tilde{f}_t = \sum_{i=1}^N \xi_i \tilde{f}_{i,t}$  for all  $t$ .

Since

$$\tilde{f}_t|_{U_j} = \sum_{i=1}^N \xi_i \tilde{f}_{j,t} + \sum_{i=1}^N \xi_i (\tilde{f}_{i,t} - \tilde{f}_{j,t}) = \tilde{f}_{j,t} + \sum_{i=1}^N \xi_i (\tilde{f}_{i,t} - \tilde{f}_{j,t})$$

for any  $j = 1, \dots, N$ , we have

$$(4-10) \quad |D'' \tilde{f}_t|_{\omega, h_0}|_{U_j} = \left| \sum_{i=1}^N \bar{\partial} \xi_i \wedge (\tilde{f}_{i,t} - \tilde{f}_{j,t}) \right|_{\omega, h_0} \quad \text{for all } t.$$

Let  $\mu$  and  $W$  be as in the beginning of the proof of Proposition 3.2 (here  $W$  is centered at a point  $\tilde{x} \in \mu^{-1}(U_i \cap U_j) \cap \{\psi = -\infty\}$ ). For similar reasons as in (3-18), (3-20) and (3-21), we get from (4-8) that

$$(4-11) \quad |\tilde{f}_{i,t} \circ \mu - \tilde{f}_{j,t} \circ \mu|_{\omega, h_0}^2|_{W_{i,j,t}} \leq C_7 e^{-\beta_1 t} \prod_{p \in \kappa} |w_p|^2$$

when  $\kappa \neq \emptyset$  and  $t$  is small enough, and that

$$(4-12) \quad |\tilde{f}_{i,t} \circ \mu - \tilde{f}_{j,t} \circ \mu|_{\omega, h_0}^2|_{W_{i,j,t}} \leq C_7 e^{-\beta_1 t}$$

when  $\kappa = \emptyset$  and  $t$  is small enough, where

$$W_{i,j,t} := W \cap \mu^{-1}(U_i \cap U_j) \cap \{\psi \circ \mu < t + c_2\}$$

and  $C_7$  is a positive number independent of  $t$ .

**Step 2:** Singularity attenuation process for the current  $\sqrt{-1} \partial \bar{\partial} \phi$ .

Since the singularities of  $\sqrt{-1} \partial \bar{\partial} \psi$  obstruct the application of Theorem 2.2, we work on  $\tilde{X}$  first and then go back to  $X$ . Some ideas in this step come from [Yi 2012].

Let  $\mu : \tilde{X} \rightarrow X$  be as in the beginning of the proof of Proposition 3.2. Let  $\tilde{X}_{k+1} := \mu^{-1}(X_{k+1})$ ,  $\tilde{X}_k := \mu^{-1}(X_k)$  and  $\tilde{\Sigma}_0 := \mu^{-1}(\Sigma_0)$ , where  $\Sigma_0 := \{\psi = -\infty\}$ . Then

$$\gamma_1 := \sqrt{-1} \partial \bar{\partial} (\psi \circ \mu) - \sum_j q_j [D_j]$$

is a smooth real  $(1, 1)$ -form for some positive numbers  $q_j$ , where  $(D_j)$  are the irreducible components of  $\tilde{\Sigma}_0$ . It is not hard to prove the following lemma and we won't give its proof.

**Lemma 4.1.** *There exists a positive number  $\tilde{n}_k$  such that*

$$\tilde{\omega}_{k+1} := \tilde{n}_k \mu^* \omega + \sqrt{-1} \partial \bar{\partial} (\psi \circ \mu) - \sum_j q_j [D_j]$$

is a Kähler metric on  $\tilde{X}_{k+1}$ .

Since  $\mu : \tilde{X} \setminus \tilde{\Sigma}_0 \rightarrow X \setminus \Sigma_0$  is biholomorphic and  $\sum_j q_j [D_j]|_{\tilde{X} \setminus \tilde{\Sigma}_0} = 0$ , the curvature assumptions (i) and (ii) in Theorem 1.8 imply that

$$\sqrt{-1}\partial\bar{\partial}(\phi \circ \mu)|_{\tilde{X} \setminus \tilde{\Sigma}_0} + \gamma_2|_{\tilde{X} \setminus \tilde{\Sigma}_0} \geq 0 \quad \text{and} \quad \sqrt{-1}\partial\bar{\partial}(\phi \circ \mu)|_{\tilde{X} \setminus \tilde{\Sigma}_0} + \gamma_3|_{\tilde{X} \setminus \tilde{\Sigma}_0} \geq 0$$

hold on  $\tilde{X} \setminus \tilde{\Sigma}_0$ , where

$$\gamma_2 := \sqrt{-1}\mu^*\Theta_{L,h_0} + \gamma_1, \quad \gamma_3 := \sqrt{-1}\mu^*\Theta_{L,h_0} + \left(1 + \frac{1}{\tilde{\chi}(\alpha \circ \mu)}\right)\gamma_1.$$

Since  $\gamma_2$  and  $\gamma_3$  are continuous on  $\tilde{X}$ , and  $\phi \circ \mu$  is quasi-plurisubharmonic on  $\tilde{X}$ , we get that

$$(4-13) \quad \sqrt{-1}\partial\bar{\partial}(\phi \circ \mu) + \gamma_2 \geq 0$$

and

$$(4-14) \quad \sqrt{-1}\partial\bar{\partial}(\phi \circ \mu) + \gamma_3 \geq 0$$

hold on  $\tilde{X}$ . Since there must exist a continuous nonnegative  $(1, 1)$ -form  $\varpi_{k+1}$  on the Kähler manifold  $(\tilde{X}_{k+1}, \tilde{\omega}_{k+1})$  such that

$$(\sqrt{-1}\Theta_{T_{\tilde{X}_{k+1}}} + \varpi_{k+1} \otimes \text{Id}_{T_{\tilde{X}_{k+1}}})(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geq 0 \quad (\text{for all } \kappa_1, \kappa_2 \in T_{\tilde{X}_{k+1}})$$

holds on  $\tilde{X}_{k+1}$ , by Theorem 2.2, we obtain from (4-13) and (4-14) a family of functions  $\{\tilde{\phi}_{\varsigma, \rho}\}_{\varsigma > 0, \rho \in (0, \rho_1)}$  on a neighborhood of the closure of  $\tilde{X}_k$  such that

- (i)  $\tilde{\phi}_{\varsigma, \rho}$  is quasi-plurisubharmonic on a neighborhood of the closure of  $\tilde{X}_k$ , smooth on  $\tilde{X}_k \setminus E_{\varsigma}(\phi \circ \mu)$ , increasing with respect to  $\varsigma$  and  $\rho$  on  $\tilde{X}_k$ , and converges to  $\phi \circ \mu$  on  $\tilde{X}_k$  as  $\rho \rightarrow 0$ ,
- (ii)  $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\tilde{\phi}_{\varsigma, \rho} \geq -\frac{\gamma_2}{\pi} - \varsigma\varpi_{k+1} - \delta_{\rho}\tilde{\omega}_{k+1}$  on  $\tilde{X}_k$ ,
- (iii)  $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\tilde{\phi}_{\varsigma, \rho} \geq -\frac{\gamma_3}{\pi} - \varsigma\varpi_{k+1} - \delta_{\rho}\tilde{\omega}_{k+1}$  on  $\tilde{X}_k$ ,

where  $E_{\varsigma}(\phi \circ \mu) := \{x \in \tilde{X} : v(\phi \circ \mu, x) \geq \varsigma\}$  ( $\varsigma > 0$ ) is the  $\varsigma$ -upperlevel set of Lelong numbers of  $\phi \circ \mu$ , and  $\{\delta_{\rho}\}$  is an increasing family of positive numbers such that  $\lim_{\rho \rightarrow 0} \delta_{\rho} = 0$ .

Since  $\tilde{\omega}_{k+1}$  is a Kähler metric on  $\tilde{X}_{k+1}$  by Lemma 4.1 and  $\tilde{X}_k$  is relatively compact in  $\tilde{X}_{k+1}$ , there exists a positive number  $n_k > 1$  such that  $n_k\tilde{\omega}_{k+1} \geq \varpi_{k+1}$  holds on  $\tilde{X}_k$ . Take  $\varsigma = \delta_{\rho}$  and denote  $\tilde{\phi}_{\delta_{\rho}, \rho}$  simply by  $\tilde{\phi}_{\rho}$ . Then  $\tilde{\phi}_{\rho}$  is quasi-plurisubharmonic on a neighborhood of the closure of  $\tilde{X}_k$ , smooth on  $\tilde{X}_k \setminus E_{\delta_{\rho}}(\phi \circ \mu)$ , increasing with respect to  $\rho$  on  $\tilde{X}_k$ , and converges to  $\phi \circ \mu$  on  $\tilde{X}_k$  as  $\rho \rightarrow 0$ . Furthermore,

$$\sqrt{-1}\partial\bar{\partial}\tilde{\phi}_{\rho} + \gamma_2 + 2\pi n_k \delta_{\rho}\tilde{\omega}_{k+1} \geq 0 \quad \text{and} \quad \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_{\rho} + \gamma_3 + 2\pi n_k \delta_{\rho}\tilde{\omega}_{k+1} \geq 0$$

hold on  $\tilde{X}_k$ . Since  $\mu : \tilde{X}_k \setminus \tilde{\Sigma}_0 \rightarrow X_k \setminus \Sigma_0$  is biholomorphic, we get that

$$\sqrt{-1}\partial\bar{\partial}(\tilde{\phi}_\rho \circ \mu^{-1}) + (\mu^{-1})^*\gamma_2 + 2\pi n_k \delta_\rho (\mu^{-1})^*\tilde{\omega}_{k+1} \geq 0$$

and

$$\sqrt{-1}\partial\bar{\partial}(\tilde{\phi}_\rho \circ \mu^{-1}) + (\mu^{-1})^*\gamma_3 + 2\pi n_k \delta_\rho (\mu^{-1})^*\tilde{\omega}_{k+1} \geq 0$$

hold on  $X_k \setminus \Sigma_0$ . Then, replacing  $\gamma_2$ ,  $\gamma_3$  and  $\tilde{\omega}_{k+1}$  with their definitions, we obtain that

$$(4-15) \quad \sqrt{-1}\partial\bar{\partial}(\tilde{\phi}_\rho \circ \mu^{-1}) + \sqrt{-1}\Theta_{L,h_0} + (1+2\pi n_k \delta_\rho)\sqrt{-1}\partial\bar{\partial}\psi \geq -2\pi n_k \tilde{n}_k \delta_\rho \omega$$

and

$$(4-16) \quad \sqrt{-1}\partial\bar{\partial}(\tilde{\phi}_\rho \circ \mu^{-1}) + \sqrt{-1}\Theta_{L,h_0} + \left(1 + 2\pi n_k \delta_\rho + \frac{1}{\tilde{\chi}(\alpha)}\right)\sqrt{-1}\partial\bar{\partial}\psi \geq -2\pi n_k \tilde{n}_k \delta_\rho \omega$$

hold on  $X_k \setminus \Sigma_0$ .

Since  $E_{\delta_\rho}(\phi \circ \mu)$  is an analytic set in  $\tilde{X}$ , Remmert's proper mapping theorem implies that

$$\Sigma_\rho := \mu(E_{\delta_\rho}(\phi \circ \mu))$$

is an analytic set in  $X$ . By Lemma 2.4,  $X_k \setminus (\Sigma_0 \cup \Sigma_\rho)$  is a complete Kähler manifold.

It follows from the properties of  $\tilde{\phi}_\rho$  that  $\tilde{\phi}_\rho \circ \mu^{-1}$  is smooth on  $X_k \setminus (\Sigma_0 \cup \Sigma_\rho)$ , increasing with respect to  $\rho$  on  $X_k \setminus \Sigma_0$ , uniformly bounded above on  $X_k \setminus \Sigma_0$  with respect to  $\rho$ , and converges to  $\phi$  on  $X_k \setminus \Sigma_0$  as  $\rho \rightarrow 0$ .

In Step 3, we will use  $\tilde{\phi}_\rho \circ \mu^{-1}$  to construct a smooth metric of  $L$  on  $X_k \setminus (\Sigma_0 \cup \Sigma_\rho)$ .

**Step 3:** Construction of additional weights and twist factors.

Let  $\zeta$ ,  $\chi$  and  $\eta$  be the solution to the following system of ODEs defined on  $(-\infty, \alpha_0)$ :

$$(4-17) \quad \chi(t)\zeta'(t) - \chi'(t) = 1,$$

$$(4-18) \quad (\chi(t) + \eta(t))e^{\zeta(t)} = \left(\frac{\alpha_1}{R(\alpha_0)} + C_R\right)R(t),$$

$$(4-19) \quad \frac{(\chi'(t))^2}{\chi(t)\zeta''(t) - \chi''(t)} = \eta(t),$$

where we assume that  $\zeta$ ,  $\chi$  and  $\eta$  are smooth on  $(-\infty, \alpha_0)$ , and that  $\inf_{t < \alpha_0} \zeta(t) = 0$ ,  $\inf_{t < \alpha_0} \chi(t) = \alpha_1$ ,  $\eta > 0$ ,  $\zeta' > 0$  and  $\chi' < 0$  on  $(-\infty, \alpha_0)$ . If  $\alpha_0 = +\infty$ , we replace the assumption  $\inf_{t < \alpha_0} \chi(t) = \alpha_1$  by  $\chi > 0$ . By a similar calculation as in [Guan

and Zhou 2015c] or [Zhou and Zhu 2018], we can solve the system of ODEs and the solution is

$$\begin{aligned}\chi(t) &= \tilde{\chi}(t), \\ \zeta(t) &= \log\left(\frac{\alpha_1}{R(\alpha_0)} + C_R\right) - \log\left(\frac{\alpha_1}{R(\alpha_0)} + \int_t^{\alpha_0} \frac{dt_1}{R(t_1)}\right), \\ \eta(t) &= R(t)\left(\frac{\alpha_1}{R(\alpha_0)} + \int_t^{\alpha_0} \frac{dt_1}{R(t_1)}\right) - \tilde{\chi}(t),\end{aligned}$$

where  $\tilde{\chi}(t)$  is defined by (1-2).

Let  $\varepsilon \in (0, \frac{1}{2})$  be as in Step 1 and put  $\sigma_t = \log(e^\psi + e^t) - \varepsilon$ . Then there exists a negative number  $t_\varepsilon$  such that  $\sigma_t \leq \alpha - \frac{\varepsilon}{2}$  on  $\bar{X}_k$  for any  $t \in (-\infty, t_\varepsilon)$ .

Let  $h_{\rho,t}$  be the new metric on the line bundle  $L$  over  $X_k \setminus (\Sigma_0 \cup \Sigma_\rho)$  defined by

$$h_{\rho,t} := h_0 e^{-\tilde{\phi}_\rho \circ \mu^{-1} - (1+2\pi n_k \delta_\rho)\psi - \zeta(\sigma_t)}.$$

Let  $\tau_t := \chi(\sigma_t)$  and  $A_t := \eta(\sigma_t)$ . Set  $B_{\rho,t} = [\Theta_{\rho,t}, \Lambda]$  on  $X_k \setminus (\Sigma_0 \cup \Sigma_\rho)$ , where

$$\Theta_{\rho,t} := \tau_t \sqrt{-1} \Theta_{L, h_{\rho,t}} - \sqrt{-1} \partial \bar{\partial} \tau_t - \sqrt{-1} \frac{\partial \tau_t \wedge \bar{\partial} \tau_t}{A_t}.$$

Set  $v_t = \partial \sigma_t$ . We want to prove

$$(4-20) \quad \Theta_{\rho,t} \Big|_{X_k \setminus (\Sigma_0 \cup \Sigma_\rho)} \geq \frac{e^t}{e^\psi} \sqrt{-1} v_t \wedge \bar{v}_t - 2\pi n_k \tilde{n}_k \chi(\sigma_t) \delta_\rho \omega.$$

It follows from (4-17) and (4-19) that

$$\begin{aligned}\Theta_{\rho,t} \Big|_{X_k \setminus (\Sigma_0 \cup \Sigma_\rho)} &= \chi(\sigma_t) (\sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \bar{\partial} (\tilde{\phi}_\rho \circ \mu^{-1}) + (1+2\pi n_k \delta_\rho) \sqrt{-1} \partial \bar{\partial} \psi) \\ &\quad + (\chi(\sigma_t) \zeta'(\sigma_t) - \chi'(\sigma_t)) \sqrt{-1} \partial \bar{\partial} \sigma_t \\ &\quad + \left( \chi(\sigma_t) \zeta''(\sigma_t) - \chi''(\sigma_t) - \frac{(\chi'(\sigma_t))^2}{\eta(\sigma_t)} \right) \sqrt{-1} \partial \sigma_t \wedge \bar{\partial} \sigma_t \\ &= \chi(\sigma_t) (\sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \bar{\partial} (\tilde{\phi}_\rho \circ \mu^{-1}) + (1+2\pi n_k \delta_\rho) \sqrt{-1} \partial \bar{\partial} \psi) + \sqrt{-1} \partial \bar{\partial} \sigma_t \\ &= \chi(\sigma_t) (\sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \bar{\partial} (\tilde{\phi}_\rho \circ \mu^{-1}) + (1+2\pi n_k \delta_\rho) \sqrt{-1} \partial \bar{\partial} \psi) \\ &\quad + \frac{e^t}{e^\psi} \sqrt{-1} v_t \wedge \bar{v}_t + \frac{e^\psi}{e^\psi + e^t} \sqrt{-1} \partial \bar{\partial} \psi.\end{aligned}$$

Since  $\chi$  is decreasing and  $\chi = \tilde{\chi}$ , it follows from (4-15) and (4-16) that

$$\begin{aligned}\chi(\sigma_t) (\sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \bar{\partial} (\tilde{\phi}_\rho \circ \mu^{-1}) + (1+2\pi n_k \delta_\rho) \sqrt{-1} \partial \bar{\partial} \psi) &+ \frac{e^\psi}{e^\psi + e^t} \sqrt{-1} \partial \bar{\partial} \psi \\ &= \chi(\sigma_t) (\sqrt{-1} \Theta_{L, h_0} + \sqrt{-1} \partial \bar{\partial} (\tilde{\phi}_\rho \circ \mu^{-1}) + (1+2\pi n_k \delta_\rho) \sqrt{-1} \partial \bar{\partial} \psi + 2\pi n_k \tilde{n}_k \delta_\rho \omega) \\ &\quad - 2\pi n_k \tilde{n}_k \chi(\sigma_t) \delta_\rho \omega + \frac{\chi(\alpha) e^\psi}{e^\psi + e^t} \cdot \frac{\sqrt{-1} \partial \bar{\partial} \psi}{\chi(\alpha)}\end{aligned}$$

$$\begin{aligned} &\geq \frac{\chi(\alpha)e^\psi}{e^\psi+e^t} \left( \sqrt{-1}\Theta_{L,h_0} + \sqrt{-1}\partial\bar{\partial}(\tilde{\phi}_\rho \circ \mu^{-1}) + (1+2\pi n_k \delta_\rho) \sqrt{-1}\partial\bar{\partial}\psi \right. \\ &\qquad \qquad \qquad \left. + 2\pi n_k \tilde{n}_k \delta_\rho \omega + \frac{\sqrt{-1}\partial\bar{\partial}\psi}{\chi(\alpha)} \right) - 2\pi n_k \tilde{n}_k \chi(\sigma_t) \delta_\rho \omega \\ &\geq -2\pi n_k \tilde{n}_k \chi(\sigma_t) \delta_\rho \omega \end{aligned}$$

on  $X_k \setminus (\Sigma_0 \cup \Sigma_\rho)$ . Hence we get (4-20) as desired.

Let  $\beta$  be as in Step 1. Let  $\beta_0$  and  $\beta_3$  be two positive numbers which will be determined later in Step 4. We choose an increasing family of positive numbers  $\{\rho_t\}_{t \in (-\infty, t_\varepsilon)}$  such that  $\lim_{t \rightarrow -\infty} \rho_t = 0$  and for any  $t$ ,

$$(4-21) \qquad 2\pi n_k \tilde{n}_k \chi(t-1) \delta_{\rho_t} < e^{\beta_0 t},$$

$$(4-22) \qquad 2\pi n_k \delta_{\rho_t} < \beta_3,$$

$$(4-23) \qquad \left( \frac{\varepsilon}{2-\varepsilon} e^t \right)^{2\pi n_k \delta_{\rho_t}} > \frac{1}{1+\varepsilon}.$$

Since  $\sigma_t \geq t-1$  on  $X_k$  and  $\chi$  is decreasing, we have  $\chi(\sigma_t) \leq \chi(t-1)$  on  $X_k$ . Then it follows from (4-20) and (4-21) that

$$\Theta_{\rho_t, t} \Big|_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})} \geq \frac{e^t}{e^\psi} \sqrt{-1} v_t \wedge \bar{v}_t - e^{\beta_0 t} \omega.$$

Hence

$$(4-24) \qquad B_{\rho_t, t} + e^{\beta_0 t} \mathbf{I} \geq \left[ \frac{e^t}{e^\psi} \sqrt{-1} v_t \wedge \bar{v}_t, \Lambda \right] = \frac{e^t}{e^\psi} T_{\bar{v}_t} T_{\bar{v}_t}^* \geq 0$$

on  $X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})$  as an operator on  $(n, 1)$ -forms, where  $T_{\bar{v}_t}$  denotes the operator  $\bar{v}_t \wedge \bullet$  and  $T_{\bar{v}_t}^*$  is its Hilbert adjoint operator.

**Step 4:** Construction of suitably truncated forms and solving  $\bar{\partial}$  globally with  $L^2$  estimates.

In this step and Step 5, we denote  $B_{\rho_t, t}$  and  $h_{\rho_t, t}$  simply by  $B_t$  and  $h_t$  respectively.

Let  $\varepsilon \in (0, \frac{1}{2})$  be as in Step 1. It is easy to construct a smooth function  $\theta : \mathbb{R} \rightarrow [0, 1]$  such that  $\theta = 0$  on  $(-\infty, \frac{\varepsilon}{2}]$ ,  $\theta = 1$  on  $[1 - \frac{\varepsilon}{2}, +\infty)$  and  $|\theta'| \leq (1+\varepsilon)/(1-\varepsilon)$  on  $\mathbb{R}$ .

Define  $g_t = D''(\theta(e^t/(e^\psi + e^t))\tilde{f}_t)$ , where  $\tilde{f}_t$  is constructed in Step 1. Then  $D''g_t = 0$  and

$$g_t = -\theta' \left( \frac{e^t}{e^\psi + e^t} \right) \frac{e^{\psi+t}}{(e^\psi + e^t)^2} \bar{\partial}\psi \wedge \tilde{f}_t + \theta \left( \frac{e^t}{e^\psi + e^t} \right) D''\tilde{f}_t = g_{1,t} + g_{2,t},$$

where  $g_{1,t}$  and  $g_{2,t}$  denote

$$-\bar{v}_t \wedge \theta' \left( \frac{e^t}{e^\psi + e^t} \right) \frac{e^t}{e^\psi + e^t} \tilde{f}_t \quad \text{and} \quad \theta \left( \frac{e^t}{e^\psi + e^t} \right) D''\tilde{f}_t,$$

respectively.

Then

$$\text{supp } g_{1,t} \subset \{t - c_1 < \psi < t + c_2\} \quad \text{and} \quad \text{supp } g_{2,t} \subset \{\psi < t + c_2\},$$

where  $c_1$  and  $c_2$  are defined as in Step 1.

It follows from (3-5) and (4-24) that

$$\begin{aligned} (4-25) \quad & \langle (\mathbf{B}_t + 2e^{\beta_0 t} \mathbf{I})^{-1} g_t, g_t \rangle_{\omega, h_t} \Big|_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})} \\ & \leq (1 + \varepsilon) \langle (\mathbf{B}_t + 2e^{\beta_0 t} \mathbf{I})^{-1} g_{1,t}, g_{1,t} \rangle_{\omega, h_t} + \frac{1 + \varepsilon}{\varepsilon} \langle (\mathbf{B}_t + 2e^{\beta_0 t} \mathbf{I})^{-1} g_{2,t}, g_{2,t} \rangle_{\omega, h_t} \\ & \leq (1 + \varepsilon) \langle (\mathbf{B}_t + e^{\beta_0 t} \mathbf{I})^{-1} g_{1,t}, g_{1,t} \rangle_{\omega, h_t} + \frac{1 + \varepsilon}{\varepsilon} \left\langle \frac{1}{e^{\beta_0 t}} g_{2,t}, g_{2,t} \right\rangle_{\omega, h_t}. \end{aligned}$$

By (4-24), we have

$$\langle (\mathbf{B}_t + e^{\beta_0 t} \mathbf{I})^{-1} g_{1,t}, g_{1,t} \rangle_{\omega, h_t} \Big|_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})} \leq \frac{e^\psi}{e^t} \left| \theta' \left( \frac{e^t}{e^\psi + e^t} \right) \frac{e^t}{e^\psi + e^t} \tilde{f}_t \right|_{\omega, h_t}^2.$$

Then  $\zeta > 0$  implies that

$$\begin{aligned} I_{1,t} & := \int_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})} \langle (\mathbf{B}_t + e^{\beta_0 t} \mathbf{I})^{-1} g_{1,t}, g_{1,t} \rangle_{\omega, h_t} dV_{X,\omega} \\ & \leq \frac{(1 + \varepsilon)^2}{(1 - \varepsilon)^2} \int_{X_k \cap \{t - c_1 < \psi < t + c_2\}} \frac{e^t |\tilde{f}_t|_{\omega, h_0}^2 e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1}}}{(e^\psi + e^t)^2 e^{2\pi n_k \delta_{\rho_t} \psi}} dV_{X,\omega}. \end{aligned}$$

Since  $\tilde{\phi}_{\rho_t} \circ \mu^{-1} \geq \phi$  on  $X_k \setminus \Sigma_0$ , it follows from (4-23) that

$$\begin{aligned} I_{1,t} & \leq \frac{(1 + \varepsilon)^2}{(1 - \varepsilon)^2} \int_{X_k \cap \{t - c_1 < \psi < t + c_2\}} \frac{e^t |\tilde{f}_t|_{\omega, h_0}^2 e^{-\phi} dV_{X,\omega}}{(e^\psi + e^t)^2 \left(\frac{\varepsilon}{2 - \varepsilon} e^t\right)^{2\pi n_k \delta_{\rho_t}}} \\ & \leq \frac{(1 + \varepsilon)^3}{(1 - \varepsilon)^2} \int_{X_k \cap \{t - c_1 < \psi < t + c_2\}} \frac{e^t |\tilde{f}_t|_{\omega, h_0}^2 e^{-\phi} dV_{X,\omega}}{(e^\psi + e^t)^2}. \end{aligned}$$

Since

$$|\tilde{f}_t|_{\omega, h_0}^2 \Big|_U = \left| \sum_{i=1}^N \sqrt{\xi_i} \cdot \sqrt{\xi_i} \tilde{f}_{i,t} \right|_{\omega, h_0}^2 \leq \left( \sum_{i=1}^N \xi_i \right) \left( \sum_{i=1}^N \xi_i |\tilde{f}_{i,t}|_{\omega, h_0}^2 \right) = \sum_{i=1}^N \xi_i |\tilde{f}_{i,t}|_{\omega, h_0}^2$$

by the Cauchy–Schwarz inequality, we have

$$I_{1,t} \leq \frac{(1 + \varepsilon)^3}{(1 - \varepsilon)^2} \sum_{i=1}^N \int_{X_k \cap \{t - c_1 < \psi < t + c_2\}} \frac{e^t \xi_i |\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-\phi} dV_{X,\omega}}{(e^\psi + e^t)^2}.$$

Then it follows from (4-9) that

$$\begin{aligned} \overline{\lim}_{t \rightarrow -\infty} I_{1,t} &\leq \sum_{i=1}^N \overline{\lim}_{t \rightarrow -\infty} \left( \frac{(1+\varepsilon)^3}{(1-\varepsilon)^2} \int_{X_k \cap \{t-c_1 < \psi < t+c_2\}} \frac{e^t \xi_i |\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-\phi} dV_{X, \omega}}{(e^\psi + e^t)^2} \right) \\ &\leq \sum_{i=1}^N \frac{(1+\varepsilon)^3}{(1-\varepsilon)^2} \int_{U_i \cap Y^0} \xi_i |f|_{\omega, h_0}^2 e^{-\phi} dV_{X, \omega}[\psi] \\ &\leq \frac{(1+\varepsilon)^3}{(1-\varepsilon)^2} \int_{Y^0} |f|_{\omega, h_0}^2 e^{-\phi} dV_{X, \omega}[\psi]. \end{aligned}$$

Then

$$(4-26) \quad I_{1,t} \leq \frac{(1+\varepsilon)^4}{(1-\varepsilon)^2} \int_{Y^0} |f|_{\omega, h_0}^2 e^{-\phi} dV_{X, \omega}[\psi]$$

when  $t$  is small enough.

Since  $\zeta(\sigma_t) > 0$  and  $\tilde{\phi}_{\rho_t} \circ \mu^{-1} \geq \phi$  on  $X_k \setminus \Sigma_0$ , by (4-22), we have

$$\begin{aligned} I_{2,t} &:= \int_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})} \left\langle \frac{1}{e^{\beta_0 t}} g_{2,t}, g_{2,t} \right\rangle_{\omega, h_t} dV_{X, \omega} \\ &\leq \frac{1}{e^{\beta_0 t}} \int_{X_k \cap \{\psi < t+c_2\}} \frac{|D'' \tilde{f}_t|_{\omega, h_0}^2 e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1}}}{e^{(1+2\pi n_k \delta_{\rho_t})\psi}} dV_{X, \omega} \\ &\leq \frac{1}{e^{\beta_0 t}} \int_{X_k \cap \{\psi < t+c_2\}} \frac{|D'' \tilde{f}_t|_{\omega, h_0}^2 e^{-\phi}}{e^{(1+\beta_3)\psi}} dV_{X, \omega}. \end{aligned}$$

Then it follows from (4-10) and the Cauchy–Schwarz inequality that  $I_{2,t}$  is bounded by the sum of the terms

$$\frac{C_8}{e^{\beta_0 t}} \int_{U_i \cap U_j \cap \{\psi < t+c_2\}} \frac{|\tilde{f}_{i,t} - \tilde{f}_{j,t}|_{\omega, h_0}^2 e^{-\phi}}{e^{(1+\beta_3)\psi}} dV_{X, \omega} \quad (1 \leq i, j \leq N),$$

where  $C_8$  is some positive number independent of  $t$ .

By the definition of  $R_1$  (see Part II in Step 1), (4-4) implies that for  $i = 1, \dots, N$ ,

$$(4-27) \quad \int_{\Omega_i \cap \{\psi < t+c_2\}} \frac{|\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-(1+\beta)\phi}}{e^\psi R_0(\psi)} dV_{X, \omega} \leq C_9$$

for some positive number  $C_9$  independent of  $t$  when  $t$  is small enough. Then by

the Hölder inequality, we get

$$\begin{aligned} & \int_{U_i \cap U_j \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t} - \tilde{f}_{j,t}|_{\omega, h_0}^2 e^{-\phi}}{e^{(1+\beta_3)\psi}} dV_{X, \omega} \\ & \leq \left( \int_{U_i \cap U_j \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t} - \tilde{f}_{j,t}|_{\omega, h_0}^2 e^{-(1+\beta)\phi}}{e^\psi R_0(\psi)} dV_{X, \omega} \right)^{\frac{1}{1+\beta}} \\ & \quad \times \left( \int_{U_i \cap U_j \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t} - \tilde{f}_{j,t}|_{\omega, h_0}^2 (R_0(\psi))^{\frac{1}{\beta}}}{e^{(1+\beta_3 \cdot \frac{1+\beta}{\beta})\psi}} dV_{X, \omega} \right)^{\frac{\beta}{1+\beta}} \\ & \leq C_{10} \left( \int_{U_i \cap U_j \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t} - \tilde{f}_{j,t}|_{\omega, h_0}^2}{e^{(1+\beta_3 \cdot \frac{1+\beta}{\beta} + \beta_2 \cdot \frac{1}{\beta})\psi}} dV_{X, \omega} \right)^{\frac{\beta}{1+\beta}} \end{aligned}$$

when  $t$  is small enough, where  $C_{10}$  is a positive number independent of  $t$ .

We will estimate the last integral above by estimating its pull-back under  $\mu$ . We cover  $\mu^{-1}(U_i \cap U_j) \cap \{\psi \circ \mu < t + c_2\}$  by a finite number of coordinate balls such as  $W$  in Step 1 in the proof of Proposition 3.2. It follows from (4-11) and (4-12) that for each  $W$ ,

$$\int_{W_{i,j,t}} \frac{|\tilde{f}_{i,t} \circ \mu - \tilde{f}_{j,t} \circ \mu|_{\omega, h_0}^2 |J\mu|^2}{e^{(1+\beta_3 \cdot \frac{1+\beta}{\beta} + \beta_2 \cdot \frac{1}{\beta})\psi \circ \mu}} d\lambda(w) \leq C_{11} \int_{W_{i,j,t}} \frac{1}{\prod_{p=1}^n |w_p|^{2\beta_{5,p}}} d\lambda(w),$$

where

$$\begin{aligned} \beta_{5,p} & := \beta_4 c a_p + (c a_p - b_p) - \lfloor c a_p - b_p \rfloor_+, \\ \beta_4 & := \beta_3 \cdot \frac{1+\beta}{\beta} + \beta_2 \cdot \frac{1}{\beta} + \beta_1, \end{aligned}$$

and  $C_{11}$  is a positive number independent of  $t$ .

Since

$$(W \cap \{\psi \circ \mu < t + c_2\}) \subset \bigcup_{p=1}^n (\{|w_p| < e^{(t+c_2-m)/(2c|a|)}\} \cap W),$$

where  $m := \inf_W \tilde{u}(w)$ , we obtain

$$\begin{aligned} \int_{W_{i,j,t}} \frac{1}{\prod_{p=1}^n |w_p|^{2\beta_{5,p}}} d\lambda(w) & \leq \sum_{p=1}^n \int_{\{|w_p| < e^{(t+c_2-m)/(2c|a|)}\} \cap W} \frac{1}{\prod_{p=1}^n |w_p|^{2\beta_{5,p}}} d\lambda(w) \\ & \leq C_{12} \sum_{p=1}^n e^{((1-\beta_{5,p})/c|a|)t} \end{aligned}$$

when  $\max_{1 \leq p \leq n} \beta_{5,p} < 1$ , where  $C_{12}$  is a positive number independent of  $t$ .

Let  $\beta_1$  be a positive number such that

$$(4-28) \quad \beta_1 < \min_{\{p: a_p \neq 0\}} \frac{1 - (ca_p - b_p) + \lfloor ca_p - b_p \rfloor_+}{3ca_p}.$$

Take  $\beta_2 = \beta_1\beta$ ,  $\beta_3 = \beta_1\beta/(1 + \beta)$ . Then  $\beta_4 = 3\beta_1$  and  $\max_{1 \leq p \leq n} \beta_{5,p} < 1$ .  
Let  $\beta_0$  be a positive number such that

$$\beta_0 < \min_{1 \leq p \leq n} \frac{\beta(1 - \beta_{5,p})}{2(1 + \beta)c|a|}$$

for every  $W$ . Then we have

$$(4-29) \quad I_{2,t} \leq C_{13} \cdot e^{\beta_0 t},$$

where  $C_{13}$  is a positive number independent of  $t$ .

Therefore, it follows from (4-25), (4-26) and (4-29) that

$$\int_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})} \langle (\mathbf{B}_t + 2e^{\beta_0 t} \mathbf{I})^{-1} g_t, g_t \rangle_{\omega, h_t} dV_{X, \omega} \leq (1 + \varepsilon) I_{1,t} + \frac{1 + \varepsilon}{\varepsilon} I_{2,t} \leq C(t),$$

where

$$C(t) := \frac{(1 + \varepsilon)^5}{(1 - \varepsilon)^2} \int_{Y_0} |f|_{\omega, h_0}^2 e^{-\phi} dV_{X, \omega}[\psi] + \frac{1 + \varepsilon}{\varepsilon} C_{13} \cdot e^{\beta_0 t}.$$

Then by Lemma 2.1, there exists  $u_{k, \varepsilon, t} \in L^2(X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t}), K_X \otimes L, h_t)$  and  $v_{k, \varepsilon, t} \in L^2(X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t}), \wedge^{n,1} T_X^* \otimes L, h_t)$  such that

$$(4-30) \quad D'' u_{k, \varepsilon, t} + \sqrt{2e^{\beta_0 t}} v_{k, \varepsilon, t} = g_t$$

on  $X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})$  and

$$(4-31) \quad \int_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})} \frac{|u_{k, \varepsilon, t}|_{\omega, h_0}^2 e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1} - (1 + 2\pi n_k \delta_{\rho_t}) \psi - \zeta(\sigma_t)}}{\tau_t + A_t} dV_{X, \omega} \\ + \int_{X_k \setminus (\Sigma_0 \cup \Sigma_{\rho_t})} |v_{k, \varepsilon, t}|_{\omega, h_0}^2 e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1} - (1 + 2\pi n_k \delta_{\rho_t}) \psi - \zeta(\sigma_t)} dV_{X, \omega} \leq C(t).$$

Since  $\{\tilde{\phi}_{\rho_t} \circ \mu^{-1}\}$  are uniformly bounded above on  $X_k \setminus \Sigma_0$  with respect to  $t$  as obtained in Step 2, we have

$$(4-32) \quad e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1}} \geq C_{14}$$

on  $X_k \setminus \Sigma_0$  for any  $t$ , where  $C_{14}$  is a positive number independent of  $t$ . Since  $t - \varepsilon \leq \sigma_t \leq \alpha - \frac{\varepsilon}{2}$  on  $\overline{X_k}$  and  $\psi$  is upper semicontinuous on  $X$ , we have that  $\psi$ ,  $\zeta(\sigma_t)$  and  $\tau_t + A_t$  are all bounded above on  $\overline{X_k}$  for each fixed  $t$ . Then it follows

from (4-31) that  $u_{k,\varepsilon,t} \in L^2$  and  $v_{k,\varepsilon,t} \in L^2$ . Hence it follows from (4-30) and Lemma 2.6 that

$$(4-33) \quad D''u_{k,\varepsilon,t} + \sqrt{2e^{\beta_0 t}}v_{k,\varepsilon,t} = D''\left(\theta\left(\frac{e^t}{e^\psi + e^t}\right)\tilde{f}_t\right)$$

holds on  $X_k$ . Furthermore, (4-31) and (4-18) imply that

$$(4-34) \quad \int_{X_k} \frac{|u_{k,\varepsilon,t}|_{\omega,h_0}^2 e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1}}}{\left(\frac{\alpha_1}{R(\alpha_0)} + C_R\right)e^\psi R(\sigma_t)} dV_{X,\omega} + \int_{X_k} |v_{k,\varepsilon,t}|_{\omega,h_0}^2 e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1} - \psi - \zeta(\sigma_t)} dV_{X,\omega} \leq e^{2\pi n_k \delta_{\rho_t} M_\psi} C(t),$$

where  $M_\psi := \sup_{X_k} \psi$ .

Define

$$F_{k,\varepsilon,t} = -u_{k,\varepsilon,t} + \theta\left(\frac{e^t}{e^\psi + e^t}\right)\tilde{f}_t.$$

Then (4-33) implies that  $D''F_{k,\varepsilon,t} = \sqrt{2e^{\beta_0 t}}v_{k,\varepsilon,t}$  on  $X_k$ . Since  $\tilde{\phi}_{\rho_t} \circ \mu^{-1} \geq \phi$  on  $X_k \setminus \Sigma_0$ , it follows from (3-5) and (4-34) that

$$(4-35) \quad \int_{X_k} \frac{|F_{k,\varepsilon,t}|_{\omega,h_0}^2 e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1}}}{e^\psi \max\{R(\psi - \varepsilon), R(\sigma_t)\}} dV_{X,\omega} \leq (1 + \varepsilon) \int_{X_k} \frac{|u_{k,\varepsilon,t}|_{\omega,h_0}^2 e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1}}}{e^\psi R(\sigma_t)} dV_{X,\omega} + \frac{1 + \varepsilon}{\varepsilon} \int_{X_k} \frac{\left|\theta\left(\frac{e^t}{e^\psi + e^t}\right)\tilde{f}_t\right|_{\omega,h_0}^2 e^{-\tilde{\phi}_{\rho_t} \circ \mu^{-1}}}{e^\psi R(\psi - \varepsilon)} dV_{X,\omega} \leq (1 + \varepsilon)e^{2\pi n_k \delta_{\rho_t} M_\psi} \left(\frac{\alpha_1}{R(\alpha_0)} + C_R\right)C(t) + \tilde{C}(t)$$

when  $t$  is small enough, where

$$\tilde{C}(t) := \frac{1 + \varepsilon}{\varepsilon} \int_{X_k \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_t|_{\omega,h_0}^2 e^{-\phi}}{e^\psi R(\psi - \varepsilon)} dV_{X,\omega}.$$

Now we want to prove

$$(4-36) \quad \lim_{t \rightarrow -\infty} \tilde{C}(t) = 0.$$

As in (4-27), we can obtain from (4-4) that for  $i = 1, \dots, N$ ,

$$\int_{\Omega_i \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t}|_{\omega,h_0}^2 e^{-(1+\beta)\phi}}{e^\psi R(\psi - \varepsilon)} dV_{X,\omega} \leq C_{15}$$

for some positive number  $C_{15}$  independent of  $t$  when  $t$  is small enough. Then by the Hölder inequality, we have that

$$\begin{aligned} & \int_{U_i \cap X_k \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-\phi}}{e^{\psi} R(\psi - \varepsilon)} dV_{X, \omega} \\ & \leq \left( \int_{U_i \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t}|_{\omega, h_0}^2 e^{-(1+\beta)\phi}}{e^{\psi} R(\psi - \varepsilon)} dV_{X, \omega} \right)^{\frac{1}{1+\beta}} \\ & \quad \times \left( \int_{U_i \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t}|_{\omega, h_0}^2}{e^{\psi} R(\psi - \varepsilon)} dV_{X, \omega} \right)^{\frac{\beta}{1+\beta}} \\ & \leq C_{15}^{\frac{1}{1+\beta}} \left( \int_{U_i \cap \{\psi < t + c_2\}} \frac{|\tilde{f}_{i,t}|_{\omega, h_0}^2}{e^{\psi} R(\psi - \varepsilon)} dV_{X, \omega} \right)^{\frac{\beta}{1+\beta}} \end{aligned}$$

when  $t$  is small enough.

We cover  $\mu^{-1}(U_i) \cap \{\psi \circ \mu < t + c_2\}$  by a finite number of coordinate balls such as  $W$  in Step 1 in the proof of Proposition 3.2. Then, in order to prove  $\lim_{t \rightarrow -\infty} \tilde{C}(t) = 0$ , it suffices to prove

$$\lim_{t \rightarrow -\infty} \int_{W_{i,t}} \frac{|\tilde{f}_{i,t} \circ \mu|_{\omega, h_0}^2 |J\mu|^2}{e^{\psi \circ \mu} R(\psi \circ \mu - \varepsilon)} d\lambda(w) = 0,$$

where

$$W_{i,t} := W \cap \mu^{-1}(U_i) \cap \{\psi \circ \mu < t + c_2\}.$$

Then by (3-18)–(3-21), it suffices to prove

$$(4-37) \quad \lim_{t \rightarrow -\infty} \int_{W_{i,t}} \frac{d\lambda(w)}{R(\psi \circ \mu - \varepsilon) |w_{p_0}|^2 \prod_{1 \leq p \leq n, p \neq p_0} |w_p|^{2(ca_p - b_p) - 2\lfloor ca_p - b_p \rfloor_+}} = 0$$

in Case (A) and

$$(4-38) \quad \lim_{t \rightarrow -\infty} \int_{W_{i,t}} \frac{d\lambda(w)}{R(\psi \circ \mu - \varepsilon) \prod_{p=1}^n |w_p|^{2\beta_1 ca_p + 2(ca_p - b_p) - 2\lfloor ca_p - b_p \rfloor_+}} = 0$$

in Case (A) and Case (B).

Applying Fubini's theorem with respect to  $(w', w_{p_0})$  and then using change of variables, we can obtain that

$$\begin{aligned} & \lim_{t \rightarrow -\infty} \int_{W_{i,t}} \frac{d\lambda(w)}{R(\psi \circ \mu - \varepsilon) |w_{p_0}|^2 \prod_{1 \leq p \leq n, p \neq p_0} |w_p|^{2(ca_p - b_p) - 2\lfloor ca_p - b_p \rfloor_+}} \\ & \leq C_{16} \lim_{t \rightarrow -\infty} \int_{-\infty}^{t+c_2-m} \frac{ds}{R(s + M - \varepsilon)} = 0, \end{aligned}$$

where  $M := \sup_W \tilde{u}(w)$ ,  $m := \inf_W \tilde{u}(w)$  and  $C_{16}$  is a positive number independent of  $t$ . Hence we get (4-37).

Similarly, it is easy to see that (4-28) implies that (4-38).

Therefore, we obtain (4-36).

Let  $\widehat{\alpha}_k := \sup_{X_k} \alpha$ . Then

$$e^\psi \max\{R(\psi - \varepsilon), R(\sigma_t)\} \leq e^\varepsilon \sup_{t \leq \widehat{\alpha}_k} (e^t R(t)).$$

Hence it follows from (4-32) and (4-35) that

$$(4-39) \quad \int_{X_k} |F_{k,\varepsilon,t}|_{\omega,h_0}^2 dV_{X,\omega} \leq C_{17}$$

for some positive number  $C_{17}$  independent of  $t$  when  $t$  is small enough.

Since the positive continuous function  $R$  is decreasing near  $-\infty$ , it is easy to see that  $\max\{R(\psi - \varepsilon), R(\sigma_t)\}$  is equal to  $R(\psi - \varepsilon)$  near  $\{\psi = -\infty\}$  and converges uniformly to  $R(\psi - \varepsilon)$  on  $\overline{X_k}$  as  $t \rightarrow -\infty$ .

Since  $\widetilde{\phi}_{\rho_t} \circ \mu^{-1}$  is increasing with respect to  $t$  and converges to  $\phi$  on  $X_k \setminus \Sigma_0$  as  $t \rightarrow -\infty$ , by extracting weak limits of  $\{F_{k,\varepsilon,t}\}$  as  $t \rightarrow -\infty$ , we get from (4-39) and (4-35) a sequence  $\{t_j\}_{j=1}^{+\infty}$  and  $F_{k,\varepsilon} \in L^2$  such that  $\lim_{j \rightarrow +\infty} t_j = -\infty$ ,  $F_{k,\varepsilon,t_j} \rightarrow F_{k,\varepsilon}$  weakly in  $L^2$  as  $j \rightarrow +\infty$  and

$$(4-40) \quad \int_{X_k} \frac{|F_{k,\varepsilon}|_{\omega,h_0}^2 e^{-\phi}}{e^\psi R(\psi - \varepsilon)} dV_{X,\omega} \leq \frac{(1 + \varepsilon)^6}{(1 - \varepsilon)^2} \left( \frac{\alpha_1}{R(\alpha_0)} + C_R \right) \int_{Y^0} |f|_{\omega,h_0}^2 e^{-\phi} dV_{X,\omega}[\psi].$$

Since  $\sigma_t \leq \alpha - \frac{\varepsilon}{2}$  on  $X_k$ ,  $\widehat{\alpha}_k := \sup_{X_k} \alpha$  and  $\zeta$  is increasing, we get

$$(4-41) \quad e^{-\zeta(\sigma_t)} \geq e^{-\zeta(\widehat{\alpha}_k - \frac{\varepsilon}{2})}$$

on  $X_k$ . Then (4-34), (4-32) and (4-41) imply that

$$\int_{X_k} |v_{k,\varepsilon,t}|_{\omega,h_0}^2 dV_{X,\omega} \leq e^{\zeta(\widehat{\alpha}_k - \frac{\varepsilon}{2}) + (1 + 2\pi n_k \delta_{\rho_t}) M_\psi} C_{14}^{-1} C(t).$$

Hence  $\sqrt{2e^{\beta_0 t}} v_{k,\varepsilon,t_j} \rightarrow 0$  in  $L^2$  as  $j \rightarrow +\infty$ . Since  $D'' F_{k,\varepsilon,t} = \sqrt{2e^{\beta_0 t}} v_{k,\varepsilon,t}$  on  $X_k$ , we get  $D'' F_{k,\varepsilon} = 0$  on  $X_k$ . Then  $F_{k,\varepsilon}$  is a holomorphic section of  $K_X \otimes L$  on  $X_k$ . In Step 5, we will prove that  $F_{k,\varepsilon} = f$  on  $X_k \cap Y^0$  by solving  $\bar{\partial}$  locally.

**Step 5:** Solving  $\bar{\partial}$  locally with  $L^2$  estimates and the end of the proof for the line bundle  $L$ .

For any  $x \in X_k \cap Y$ , let  $\Omega_x$  be as in Step 1. Let

$$\widehat{\Omega}_x \Subset (X_k \cap \Omega_x)$$

be a coordinate ball with center  $x$ . Since the bundle  $L$  is trivial on  $\Omega_x$ ,  $u_{k,\varepsilon,t}$  and  $v_{k,\varepsilon,t}$  can be regarded as forms on  $\Omega_x$  with values in  $\mathbb{C}$  and the metric  $h_0$  of  $L$  on  $\Omega_x$  can be regarded as a positive smooth function.

It is easy to see that  $C(t) \leq C_{18}$  for some positive number  $C_{18}$  independent of  $t$  when  $t$  is small enough. Then it follows from (4-34), (4-41) and (4-32) that

$$\int_{\widehat{\Omega}_x} |v_{k,\varepsilon,t}|^2 e^{-\psi} d\lambda \leq C_{19} C_{18}$$

for some positive number  $C_{19}$  independent of  $t$  when  $t$  is small enough.

Since  $\bar{\partial} v_{k,\varepsilon,t} = 0$  on  $\widehat{\Omega}_x$  by (4-33), applying Lemma 2.5 to the  $(n, 1)$ -form

$$\sqrt{2e^{\beta_0 t}} v_{k,\varepsilon,t} \in L^2_{(n,1)}(\widehat{\Omega}_x, e^{-\psi}),$$

we get an  $(n, 0)$ -form  $s_{k,\varepsilon,t} \in L^2_{(n,0)}(\widehat{\Omega}_x, e^{-\psi})$  such that

$$\bar{\partial} s_{k,\varepsilon,t} = \sqrt{2e^{\beta_0 t}} v_{k,\varepsilon,t}$$

on  $\widehat{\Omega}_x$  and

$$(4-42) \quad \int_{\widehat{\Omega}_x} |s_{k,\varepsilon,t}|^2 e^{-\psi} d\lambda \leq C_{20} \int_{\widehat{\Omega}_x} |\sqrt{2e^{\beta_0 t}} v_{k,\varepsilon,t}|^2 e^{-\psi} d\lambda \leq 2C_{20} C_{19} C_{18} e^{\beta_0 t}$$

for some positive number  $C_{20}$  independent of  $t$ . Hence

$$(4-43) \quad \int_{\widehat{\Omega}_x} |s_{k,\varepsilon,t}|^2 d\lambda \leq C_{21} e^{\beta_0 t}$$

for some positive number  $C_{21}$  independent of  $t$ .

Now define

$$G_{k,\varepsilon,t} = -u_{k,\varepsilon,t} - s_{k,\varepsilon,t} + \theta \left( \frac{e^t}{e^\psi + e^t} \right) \tilde{f}_t$$

on  $\widehat{\Omega}_x$ . Then  $G_{k,\varepsilon,t} = F_{k,\varepsilon,t} - s_{k,\varepsilon,t}$  and  $\bar{\partial} G_{k,\varepsilon,t} = 0$ . Hence  $G_{k,\varepsilon,t}$  is holomorphic in  $\widehat{\Omega}_x$ . Therefore,  $u_{k,\varepsilon,t} + s_{k,\varepsilon,t}$  is smooth in  $\widehat{\Omega}_x$ . Furthermore, we get from (4-39) and (4-43) that

$$(4-44) \quad \int_{\widehat{\Omega}_x} |G_{k,\varepsilon,t}|^2 d\lambda \leq 2 \int_{\widehat{\Omega}_x} |F_{k,\varepsilon,t}|^2 d\lambda + 2 \int_{\widehat{\Omega}_x} |s_{k,\varepsilon,t}|^2 d\lambda \leq C_{22}$$

for some positive number  $C_{22}$  independent of  $t$  when  $t$  is small enough.

We get from (4-32) and (4-34) that

$$\int_{\widehat{\Omega}_x} \frac{|u_{k,\varepsilon,t}|^2 e^{-\psi}}{R(\sigma_t)} d\lambda \leq C_{23} C(t) \leq C_{23} C_{18}$$

for some positive number  $C_{23}$  independent of  $t$  when  $t$  is small enough. Since  $R(\sigma_t) \leq R(t - \varepsilon)$  on  $\widehat{\Omega}_x$  when  $t$  is small enough, we have that

$$\int_{\widehat{\Omega}_x} |u_{k,\varepsilon,t}|^2 e^{-\psi} d\lambda \leq C_{23} C_{18} R(t - \varepsilon).$$

Therefore, combining the last inequality and (4-42), we obtain that

$$\int_{\widehat{\Omega}_x} |u_{k,\varepsilon,t} + s_{k,\varepsilon,t}|^2 e^{-\psi} d\lambda \leq 2C_{23}C_{18}R(t - \varepsilon) + 4C_{20}C_{19}C_{18}e^{\beta_0 t}.$$

Then the nonintegrability of  $e^{-\psi}$  along  $\widehat{\Omega}_x \cap Y$  and the smoothness of  $u_{k,\varepsilon,t} + s_{k,\varepsilon,t}$  in  $\widehat{\Omega}_x$  show that  $u_{k,\varepsilon,t} + s_{k,\varepsilon,t} = 0$  on  $\widehat{\Omega}_x \cap Y$  for any  $t$ . Hence  $G_{k,\varepsilon,t} = f$  on  $\widehat{\Omega}_x \cap Y^0$  for any  $t$ .

Since  $s_{k,\varepsilon,t_j} \rightarrow 0$  in  $L^2_{(n,0)}(\widehat{\Omega}_x)$  by (4-43) and  $F_{k,\varepsilon,t_j} \rightharpoonup F_{k,\varepsilon}$  weakly in  $L^2_{(n,0)}(\widehat{\Omega}_x)$  as  $j \rightarrow +\infty$ ,  $G_{k,\varepsilon,t_j} \rightharpoonup F_{k,\varepsilon}$  weakly in  $L^2_{(n,0)}(\widehat{\Omega}_x)$  as  $j \rightarrow +\infty$ . Hence it follows from (4-44) and routine arguments applying Montel's theorem that a subsequence of  $\{G_{k,\varepsilon,t_j}\}_{j=1}^{+\infty}$  converges to  $F_{k,\varepsilon}$  uniformly on compact subsets of  $\widehat{\Omega}_x$ . Then  $F_{k,\varepsilon} = f$  on  $\widehat{\Omega}_x \cap Y^0$  and thereby on  $X_k \cap Y^0$ .

Since the positive continuous function  $R$  is decreasing near  $-\infty$ ,  $e^t R(t)$  is bounded above near  $-\infty$  and  $\phi$  is locally bounded above, applying Montel's theorem and extracting weak limits of  $\{F_{k,\varepsilon}\}_{k,\varepsilon}$ , first as  $\varepsilon \rightarrow 0$ , and then as  $k \rightarrow +\infty$ , we get from (4-40) a holomorphic section  $F$  on  $X$  with values in  $K_X \otimes L$  such that  $F = f$  on  $Y^0$  and

$$\int_X \frac{|F|_{\omega,h}^2}{e^\psi R(\psi)} dV_{X,\omega} \leq \left( \frac{\alpha_1}{R(\alpha_0)} + C_R \right) \int_{Y^0} |f|_{\omega,h}^2 dV_{X,\omega}[\psi].$$

Theorem 1.8 is thus proved for the line bundle  $L$ .

**Step 6:** The proof for the vector bundle  $E$ .

The proof for  $E$  is similar but simpler. We only point out the main modifications by examining the proof for  $L$ .

In Step 1, we don't need to construct a family of special smooth extensions  $\tilde{f}_t$  of  $f$  since  $h_E$  is smooth. Hence the strong openness property and the key propositions are not needed. Delete Parts II and III in Step 1 and replace the family of sections  $\tilde{f}_{i,t}$  with a fixed local holomorphic extension  $\tilde{f}_i$ . Then  $\tilde{f}_i$  becomes a fixed smooth extension  $\tilde{f} = \sum_{i=1}^N \xi_i \tilde{f}_i$ . Then it is easy to see that (4-9)–(4-12) hold for  $\tilde{f}_{i,t} = \tilde{f}_i$ ,  $\tilde{f}_t = \tilde{f}$  and  $\beta_1 = 0$ .

Step 2 is not needed since  $h_E$  is already smooth.

In Step 3, the negative term will not appear on the right-hand side of (4-20) since  $\delta_\rho = 0$ .

In Step 4, it is easy to prove the estimate (4-26) for  $I_{1,t}$  by the modified (4-9). It is also not hard to prove the estimate (4-29) for  $I_{2,t}$  by the modified (4-10)–(4-12). Equation (4-36) can be easily obtained since  $h_E$  is smooth.

Step 5 for  $E$  is almost the same and Theorem 1.8 is thus proved for the vector bundle  $E$ .

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Received September 27, 2019. Revised June 5, 2020.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

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Volume 309 No. 2 December 2020

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