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ELEMENTS OF HIGHER HOMOTOPY GROUPS UNDETECTABLE BY POLYHEDRAL APPROXIMATION

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When nontrivial local structures are present in a topological space X, a common approach to characterizing the isomorphism type of the n-th homotopy group $\pi_n(X, x_0)$ is to consider the image of $\pi_n(X, x_0)$ in the n-th Čech homotopy group $\check{\pi}_n(X, x_0)$ under the canonical homomorphism $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$. The subgroup $\ker(\Psi_n)$ is the obstruction to this tactic as it consists of precisely those elements of $\pi_n(X, x_0)$, which cannot be detected by polyhedral approximations to X. In this paper, we use higher dimensional analogues of Spanier groups to characterize $\ker(\Psi_n)$. In particular, we prove that if X is paracompact, Hausdorff, and LC^{n-1}, then $\ker(\Psi_n)$ is equal to the n-th Spanier group of X. We also use the perspective of higher Spanier groups to generalize a theorem of Kozlowski–Segal, which gives conditions ensuring that Ψ_n is an isomorphism.

1. Introduction

When nontrivial local structures are present in a topological space X, a common approach to characterizing the isomorphism type of $\pi_n(X, x_0)$ is to consider the image of $\pi_n(X, x_0)$ in the n-th Čech (shape) homotopy group $\check{\pi}_n(X, x_0)$ under the canonical homomorphism $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$. The *n-th shape kernel* $\ker(\Psi_n)$ is the obstruction to this tactic as it consists of precisely those elements of $\pi_n(X, x_0)$, which cannot be detected by polyhedral approximations to X. This method has proved successful in many situations for both the fundamental group [Cannon and Conner 2006; Eda and Kawamura 1998; Fischer and Guilbault 2005; Fischer and Zastrow 2005] and higher homotopy groups [Brazas 2021; Eda and Kawamura 2000a; 2010; Eda et al. 2013; Kawamura 2003]. In this paper, we study the map Ψ_n and give a characterization the n-th shape kernel in terms of higher-dimensional analogues of Spanier groups.

The subgroups of fundamental groups, which are now commonly referred to as "Spanier groups," first appeared in E.H. Spanier's unique approach [1966] to

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covering space theory. If $\mathscr{U}$ is an open cover of a topological space X and $x_0 \in X$, then the *Spanier group with respect to* $\mathscr{U}$ is the subgroup $\pi_1^{Sp}(\mathscr{U}, x_0)$ of $\pi_1(X, x_0)$ generated by path-conjugates $[\alpha][\gamma][\alpha]^{-1}$ where α is a path starting at x_0 and γ is a loop based at $\alpha(1)$ with image being contained in some element of $\mathscr{U}$. These subgroups are particularly relevant to covering space theory since, when X is locally path-connected, a subgroup $H \leq \pi_1(X, x_0)$ corresponds to a covering map $p:(Y, y_0) \to (X, x_0)$ if and only if $\pi_1^{Sp}(\mathscr{U}, x_0) \leq H$ for some open cover $\mathscr{U}$ [Spanier 1966, 2.5.12]. The intersection $\pi_1^{Sp}(X, x_0) = \bigcap_{\mathscr{U}} \pi_1^{Sp}(\mathscr{U}, x_0)$ is called the *Spanier group of* (X, x_0) [Fischer et al. 2011]. The inclusion $\pi_1^{Sp}(X, x_0) \subseteq \ker(\Psi_1)$ always holds [Fischer and Zastrow 2007, Proposition 4.8]. It is proved in [Brazas and Fabel 2014, Theorem 6.1] that $\pi_1^{Sp}(X, x_0) = \ker(\Psi_1)$ whenever X is paracompact Hausdorff and locally path connected. The upshot of this equality is having a description of level-wise generators (for each open cover $\mathscr{U}$) whereas there may be no readily available generating set for the kernel of a homomorphism induced by a canonical map from X to the nerve $|N(\mathscr{U})|$. Indeed, 1-dimensional Spanier groups have proved useful in persistence theory [Virk 2020]. Since much of applied topology is based on a geometric refinement of polyhedral approximation from shape theory, there seems potential for higher dimensional analogues to be useful as well.

Higher dimensional analogues of Spanier groups recently appeared in [Bahredar et al. 2021] and are defined in a similar way: $\pi_n^{Sp}(\mathscr{U}, x_0)$ is the subgroup of $\pi_n(X, x_0)$ consisting of homotopy classes of path-conjugates $\alpha * f$ where α is a path starting at x_0 and $f : S^n \to X$ is based at $\alpha(1)$ with image being contained in some element of $\mathscr{U}$. Then $\pi_n^{Sp}(X, x_0)$ is the intersection of these subgroups. In this paper, we prove a higher-dimensional analogue of the 1-dimensional equality $\pi_1^{Sp}(X, x_0) = \ker(\Psi_1)$ from [Brazas and Fabel 2014].

A space X is LC^n if for every neighborhood U of a point $x \in X$, there is a neighborhood V of x in U such that every map $f : S^k \to V$, $0 \leq k \leq n$ is null-homotopic in U. When a space is LC^n, "small" maps on spheres of dimension $\leq n$ contract by null-homotopies of relatively the same size. Certainly, every locally n-connected space is LC^n. However, when $n \geq 1$, the converse is not true even for metrizable spaces. Our main result is the following.

Theorem 1.1. *Let $n \geq 1$ and $x_0 \in X$. If X is paracompact, Hausdorff, and LC^{n-1}, then $\pi_n^{Sp}(X, x_0) = \ker(\Psi_n)$.*

This result confirms that higher Spanier groups, like their 1-dimensional counterparts, often identify precisely those elements of $\pi_n(X, x_0)$ which can be detected by polyhedral approximations to X. More precisely, under the hypotheses of Theorem 1.1, $g \in \pi_n^{Sp}(X, x_0)$ if and only if $f_{\#}(g) = 0$ for every map $f : X \to K$ to a polyhedron K. A first countable path-connected space is LC^0 if and only if it is locally path connected. Hence, in dimension $n = 1$, Theorem 1.1 only expands [Brazas and Fabel 2014, Theorem 6.1] to some nonfirst countable spaces.

Regarding the proof of Theorem 1.1, the inclusion $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ was first proved for $n = 1$ in [Fischer and Zastrow 2007, Proposition 4.8] and for $n \geq 2$ in [Bahredar et al. 2021, Theorem 4.14]. We include this proof for the sake of completion (Corollary 3.11). The proof of the inclusion $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$ appears in Section 5 and is more intricate, requiring a carefully chosen sequence of open cover refinements using the LC^{n-1} property. These refinements allow one to recursively extend maps on simplicial complexes skeleton-wise. These extension methods, established in Section 4, are similar to methods found in [Kozlowski and Segal 1977; 1978].

We also put these extension methods to work in Section 6 where we identify conditions that imply Ψ_n is an isomorphism. Kozlowski and Segal [1978], proved that if X is paracompact Hausdorff and LC^n, then Ψ_n is an isomorphism. Fischer and Zastrow [2007], generalized this result in dimension $n = 1$ by replacing "LC^1" with "locally path connected and semilocally simply connected." Similar, to the approach of Fischer and Zastrow, our use of Spanier groups shows that the existence of *small* null-homotopies of small maps $S^n \to X$ (specifically in dimension n) is not necessary to prove that Ψ_n is injective. We say a space X is *semilocally π_n-trivial* if for every $x \in X$ there exists an open neighborhood U of x such that every map $S^n \to U$ is null-homotopic in X. This definition is independent of lower dimensions but certainly $LC^n \Rightarrow (LC^{n-1}$ and semilocally π_n-trivial). Our second result proves Kozlowski–Segal's theorem under a weaker hypothesis and is stated as follows.

Theorem 1.2. *Let $n \geq 1$ and $x_0 \in X$. If X is paracompact, Hausdorff, LC^{n-1}, and semilocally π_n-trivial, then $\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0)$ is an isomorphism.*

The hypotheses in Theorem 1.2 are the homotopical versions of the hypotheses used in [Mardešić 1959] to ensure that the canonical homomorphism $\varphi_* : H_n(X) \to \check{H}_n(X)$ is an isomorphism; see also [Eda and Kawamura 2000b] regarding the surjectivity of φ_*. Examples show that Ψ_n can fail to be an isomorphism if X is semilocally π_n-trivial but not LC^{n-1} (Example 7.4) or if X is LC^{n-1} but not semilocally π_n-trivial (Example 7.5).

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2. Preliminaries and notation

Throughout this paper, X is assumed to be a path-connected topological space with basepoint x_0. The unit interval is denoted I and S^n is the unit n-sphere with basepoint $d_0 = (1, 0, \ldots, 0)$. The n-th homotopy group of (X, x_0) is denoted

$\pi_n(X, x_0)$. If $f : (X, x_0) \to (Y, y_0)$ is a based map, then $f_\# : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is the induced homomorphism.

A *path* in a space X is a map $\alpha : I \to X$ from the unit interval. The *reverse* of α is the path given by $\alpha^-(t) = \alpha(1-t)$ and the concatenation of two paths α, β with $\alpha(1) = \beta(0)$ is denoted $\alpha \cdot \beta$. Similarly, if $f, g : S^n \to X$ are maps based at $x \in X$, then $f \cdot g$ denotes the usual n-loop concatenation and f^- denotes the reverse map. We may write $\prod_{i=1}^m f_i$ to denote an m-fold concatenation $f_1 \cdot f_2 \cdot \cdots \cdot f_m$.

2.1. *Simplicial complexes.* We make heavy use of standard notation and theory of abstract and geometric simplicial complexes, which can be found in texts such as [Mardešić and Segal 1982; Munkres 1984]. We briefly recall relevant notation.

For an abstract (geometric) simplicial complex K and integer $r \geq 0$, K_r denotes the r-skeleton of K. If K is abstract, $|K|$ denotes the geometric realization of K with the weak topology. If K is geometric, then $\mathrm{sd}^m K$ denotes the m-th barycentric subdivision of K and if v is a vertex of K, then $\mathrm{st}(v, K)$ denotes the open star of the vertex v. When $L \subseteq K$ is a subcomplex, $\mathrm{sd}^m L$ is a subcomplex of $\mathrm{sd}^m K$. If $\sigma = \{v_0, v_1, \ldots, v_r\}$ is a r-simplex of K, then $[v_0, v_1, \ldots, v_r]$ denotes the r-simplex of $|K|$ with the indicated orientation.

We frequently make use of the standard n-simplex Δ_n in $\mathbb{R}^n$ spanned by the origin $\boldsymbol{o}$ and standard unit vectors. Since the boundary $\partial\Delta_n = (\Delta_n)_{n-1}$ is homeomorphic to S^{n-1}, we fix a based homeomorphism $\partial\Delta_n \cong S^{n-1}$ that allows us to represent elements of $\pi_n(X, x_0)$ by maps $(\partial\Delta_{n+1}, \boldsymbol{o}) \to (X, x_0)$.

2.2. *The Čech expansion and shape homotopy groups.* We now recall the construction of the first shape homotopy group $\check{\pi}_1(X, x_0)$ via the Čech expansion. For more details; see [Mardešić and Segal 1982].

Let $\mathcal{O}(X)$ be the set of open covers of X directed by refinement; we write $\mathscr{V} \succeq \mathscr{U}$ when $\mathscr{V}$ refines $\mathscr{U}$. Similarly, let $\mathcal{O}(X, x_0)$ be the set of open covers with a distinguished element containing x_0, i.e., the set of pairs $(\mathscr{U}, U_0)$ where $\mathscr{U} \in \mathcal{O}(X)$, $U_0 \in \mathscr{U}$, and $x_0 \in U_0$. We say $(\mathscr{V}, V_0)$ refines $(\mathscr{U}, U_0)$ if $\mathscr{V} \succeq \mathscr{U}$ and $V_0 \subseteq U_0$.

The nerve of a cover $(\mathscr{U}, U_0) \in \mathcal{O}(X, x_0)$ is the abstract simplicial complex $N(\mathscr{U})$ whose vertex set is $N(\mathscr{U})_0 = \mathscr{U}$ and vertices $A_0, \ldots, A_n \in \mathscr{U}$ span an n-simplex if $\bigcap_{i=0}^n A_i \neq \varnothing$. The vertex U_0 is taken to be the basepoint of the geometric realization $|N(\mathscr{U})|$. Whenever $(\mathscr{V}, V_0)$ refines $(\mathscr{U}, U_0)$, we can construct a simplicial map $p_{\mathscr{U}\mathscr{V}} : N(\mathscr{V}) \to N(\mathscr{U})$, called a *projection*, by sending a vertex $V \in N(\mathscr{V})$ to a vertex $U \in \mathscr{U}$ such that $V \subseteq U$. In particular, we make a convention that $p_{\mathscr{U}\mathscr{V}}(V_0) = U_0$. Any such assignment of vertices extends linearly to a simplicial map. Moreover, the induced map $|p_{\mathscr{U}\mathscr{V}}| : |N(\mathscr{V})| \to |N(\mathscr{U})|$ is unique up to based homotopy. Thus the homomorphism $p_{\mathscr{U}\mathscr{V}\#} : \pi_1(|N(\mathscr{V})|, V_0) \to \pi_1(|N(\mathscr{U})|, U_0)$ induced on fundamental groups is (up to coherent isomorphism) independent of the choice of simplicial map.

Recall that an open cover $\mathscr{U}$ of X is normal if it admits a partition of unity subordinated to $\mathscr{U}$. Let Λ be the subset of $\mathcal{O}(X, x_0)$ (also directed by refinement) consisting of pairs $(\mathscr{U}, U_0)$ where $\mathscr{U}$ is a normal open cover of X and such that there is a partition of unity $\{\phi_U\}_{U\in\mathscr{U}}$ subordinated to $\mathscr{U}$ with $\phi_{U_0}(x_0) = 1$. It is well-known that every open cover of a paracompact Hausdorff space X is normal. Moreover, if $(\mathscr{U}, U_0) \in \mathcal{O}(X, x_0)$, it is easy to refine $(\mathscr{U}, U_0)$ to a cover $(\mathscr{V}, V_0)$ such that V_0 is the only element of $\mathscr{V}$ containing x_0 and therefore $(\mathscr{V}, V_0) \in \Lambda$. Thus, for paracompact Hausdorff X, Λ is cofinal in $\mathcal{O}(X, x_0)$.

The *n-th shape homotopy group* is the inverse limit

$$\check{\pi}_n(X, x_0) = \varprojlim(\pi_n(|N(\mathscr{U})|, U_0), p_{\mathscr{U}\mathscr{V}\#}, \Lambda).$$

This group is also referred to as the n-th Čech homotopy group.

Given an open cover $(\mathscr{U}, U_0) \in \mathcal{O}(X, x_0)$, a map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$ is a (*based*) *canonical map* if $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, N(\mathscr{U}))) \subseteq U$ for each $U \in \mathscr{U}$ and $p_{\mathscr{U}}(x_0) = U_0$. Such a canonical map is guaranteed to exist if $(\mathscr{U}, U_0) \in \Lambda$: find a locally finite partition of unity $\{\phi_U\}_{U\in\mathscr{U}}$ subordinated to $\mathscr{U}$ such that $\phi_{U_0}(x_0) = 1$. When $U \in \mathscr{U}$ and $x \in U$, determine $p_{\mathscr{U}}(x)$ by requiring its barycentric coordinate belonging to the vertex U of $|N(\mathscr{U})|$ to be $\phi_U(x)$. According to this construction, the requirement $\phi_{U_0}(x_0) = 1$ gives $p_{\mathscr{U}}(x_0) = U_0$.

A canonical map $p_{\mathscr{U}}$ is unique up to based homotopy and whenever $(\mathscr{V}, V_0)$ refines $(\mathscr{U}, U_0)$, the compositions $p_{\mathscr{U}\mathscr{V}} \circ p_{\mathscr{V}}$ and $p_{\mathscr{U}}$ are homotopic as based maps. Hence, for $n \geq 1$, the homomorphisms

$$p_{\mathscr{U}\#} : \pi_n(X, x_0) \to \pi_n(|N(\mathscr{U})|, U_0)$$

satisfy $p_{\mathscr{U}\mathscr{V}\#} \circ p_{\mathscr{V}\#} = p_{\mathscr{U}\#}$. These homomorphisms induce the following canonical homomorphism to the limit, which is natural in the continuous maps of based spaces:

$$\Psi_n : \pi_n(X, x_0) \to \check{\pi}_n(X, x_0) \quad \text{given by } \Psi_n([f]) = ([p_{\mathscr{U}} \circ f]).$$

The subgroup $\ker(\Psi_n)$, which we refer to as the *n-th shape kernel* is, in a rough sense, an algebraic measure of the n-dimensional homotopical information lost when approximating X by polyhedra. Since $(p_{\mathscr{U}})$ forms an HPol-expansion of X [Mardešić and Segal 1982, Appendix 1, Sectin 3.2, Theorem 8], we have $[f] \in \pi_n(X, x_0)\backslash \ker(\Psi_n)$ if and only if there exist a polyhedron K and a map $p : (X, x_0) \to (K, k_0)$ such that $p_\#([f]) \neq 0$ in $\pi_n(K, k_0)$. Of utmost importance is the situation when $\ker(\Psi_n) = 0$. In this case, $\pi_n(X, x_0)$ can be understood as a subgroup of $\check{\pi}_n(X, x_0)$, that is, the n-th shape group retains all the data in the n-th homotopy group of X. A space for which $\ker(\Psi_n) = 0$ is said to be π_n*-shape injective*.

3. Higher Spanier groups

To define higher Spanier groups as in [Bahredar et al. 2021], we briefly recall the action of the fundamental groupoid on the higher homotopy groups of a space. Fix a retraction $R : S^n \times I \to S^n \times \{0\} \cup \{d_0\} \times I$. Given a map $f : (S^n, d_0) \to (X, y_0)$ and a path $\alpha : I \to X$ with $\alpha(0) = x_0$ and $\alpha(1) = y_0$, define $F : S^n \times \{0\} \cup \{d_0\} \times I \to X$ so that $g(x, 0) = f(x)$ and $f(d_0, t) = \alpha(1 - t)$. The *path-conjugate of f by α* is the map $\alpha * f : (S^n, d_0) \to (X, x_0)$ given by $\alpha * f(x) = F(R(x, 1))$.

Path-conjugation defines the basepoint-change isomorphism $\varphi_\alpha : \pi_n(X, y_0) \to \pi_n(X, x_0)$, $\varphi_\alpha([f]) = [\alpha * f]$. In particular, $[\alpha * f][\alpha * g] = [\alpha * (f \cdot g)]$. Additionally, if $[\alpha] = [\beta]$, which we write to mean that the paths α and β are homotopic relative to their endpoints, then $[\alpha * f] = [\beta * f]$. Note that when $n = 1$, $f : S^1 \to X$ is a loop and $\alpha * f \simeq \alpha \cdot f \cdot \alpha^-$.

Definition 3.1. Let $n \geq 1$ and $\alpha : (I, 0) \to (X, x_0)$ be a path and U be an open neighborhood of $\alpha(1)$ in X. Define

$$[\alpha] * \pi_n(U) = \{[\alpha * f] \in \pi_n(X, x_0) \mid f(S^n) \subseteq U,\ f(d_0) = \alpha(1)\}.$$

Since $[\alpha * f][\alpha * g] = [\alpha * (f \cdot g)]$, the set $[\alpha] * \pi_n(U)$ is a subgroup of $\pi_n(X, x_0)$.

Definition 3.2. Let $n \geq 1$, $\mathscr{U}$ be an open cover of X, and $x_0 \in X$. The *n-th Spanier group of (X, x_0) with respect to $\mathscr{U}$* is the subgroup $\pi_n^{Sp}(\mathscr{U}, x_0)$ of $\pi_n(X, x_0)$ generated by the subgroups $[\alpha] * \pi_n(U)$ for all pairs (α, U) with $\alpha(1) \in U$ and $U \in \mathscr{U}$. In short

$$\pi_n^{Sp}(\mathscr{U}, x_0) = \langle [\alpha] * \pi_n(U) \mid U \in \mathscr{U}, \alpha(1) \in U \rangle.$$

The *n-th Spanier group of (X, x_0)* is the intersection

$$\pi_n^{Sp}(X, x_0) = \bigcap_{\mathscr{U} \in O(X)} \pi_n^{Sp}(\mathscr{U}, x_0).$$

We may refer to subgroups of the form $\pi_n^{Sp}(\mathscr{U}, x_0)$ as *relative* Spanier groups and to $\pi_n^{Sp}(X, x_0)$ as the *absolute* Spanier group.

Remark 3.3. We note that our definition of n-th Spanier group is the "unbased" definition from [Bahredar et al. 2021]; see also [Fischer et al. 2011] for more on "based" Spanier groups, which is defined using covers of X by *pointed* open sets. The two notions agree for locally path connected spaces. When $n = 1$, Spanier groups (absolute and relative to a cover) are normal subgroups of $\pi_1(X, x_0)$. In the case $n = 1$, Spanier groups have been studied heavily due to their relationship to covering space theory [Spanier 1966].

Remark 3.4 (functorality). Let Top_* denote the category of based topological spaces and based continuous functions and Grp and Ab denote the usual categories of groups and abelian groups respectively. If $f:(X,x_0)\to(Y,y_0)$ is a map and $\mathscr{V}$ is an open cover of Y, then $\mathscr{U}=\{f^{-1}(V)\mid V\in\mathscr{V}\}$ is an open cover of X such that $f_\#(\pi_n(\mathscr{U},x_0))\subseteq\pi_n(\mathscr{V},y_0)$. It follows that $f_\#(\pi_n^{Sp}(X,x_0))\subseteq\pi_n^{Sp}(Y,y_0)$. Thus $(f_\#)|_{\pi_n^{Sp}(X,x_0)}:\pi_n^{Sp}(X,x_0)\to\pi_n^{Sp}(Y,y_0)$ is well-defined showing that $\pi_1^{Sp}:\mathrm{Top}_*\to\mathrm{Grp}$ and $\pi_n^{Sp}:\mathrm{Top}_*\to\mathrm{Ab}$, $n\geq 2$, are functors [Bahredar et al. 2021, Theorem 4.2]. Moreover, if $g:(Y,y_0)\to(X,x_0)$ is a based homotopy inverse of f, then $(f_\#)|_{\pi_n^{Sp}(X,x_0)}$ and $(g_\#)|_{\pi_n^{Sp}(Y,y_0)}$ are inverse isomorphisms. Hence, these functors descend to functors $\mathrm{hTop}_*\to\mathrm{Grp}$ and $\mathrm{hTop}_*\to\mathrm{Ab}$ where hTop_* is the category of based spaces and basepoint-relative homotopy classes of based maps.

Remark 3.5 (basepoint invariance). Suppose $x_0,x_1\in X$ and $\beta:I\to X$ is a path from x_1 to x_0, and $\varphi_\beta:\pi_n(X,x_0)\to\pi_n(X,x_1)$, $\varphi_\beta([g])=[\beta*g]$ is the basepoint-change isomorphism. If $[\alpha*f]$ is a generator of $\pi_n^{Sp}(\mathscr{U},x_0)$, then $\varphi_\beta([\alpha*f])=[(\beta\cdot\alpha)*f]$ is a generator of $\pi_n^{Sp}(\mathscr{U},x_1)$. It follows that $\varphi_\beta(\pi_n^{Sp}(\mathscr{U},x_0))=\pi_n^{Sp}(\mathscr{U},x_1)$. Moreover, in the absolute case, we have $\varphi_\beta(\pi_n^{Sp}(X,x_0))=\pi_n^{Sp}(X,x_1)$. In particular, changing the basepoint of X does not change the isomorphism type of the n-th Spanier group, particularly its triviality.

In terms of our choice of generators, a generic element of $\pi_n^{Sp}(\mathscr{U},x_0)$ is a product $\prod_{i=1}^m[\alpha_i*f_i]$ where each map $f_i:S^n\to X$ has an image in some open set $U_i\in\mathscr{U}$ (see Figure 1). The next lemma identifies how such products might actually appear in practice and motivates the proof of our key technical lemma, Lemma 5.1. Recall that $(\mathrm{sd}^m\,\Delta_{n+1})_n$ is the union of the boundaries of the $(n+1)$-simplices in the m-th barycentric subdivision $\mathrm{sd}^m\,\Delta_{n+1}$.

Lemma 3.6. *For $m,n\in\mathbb{N}$, let $\mathscr{U}$ be an open cover of X and $f:((\mathrm{sd}^m\,\Delta_{n+1})_n,\boldsymbol{o})\to(X,x_0)$ be a map such that for every $(n+1)$-simplex σ of $\mathrm{sd}^m\,\Delta_{n+1}$, we have $f(\partial\sigma)\subseteq U$ for some $U\in\mathscr{U}$. Then $f_\#(\pi_n((\mathrm{sd}^m\,\Delta_{n+1})_n,\boldsymbol{o}))\subseteq\pi_n^{Sp}(\mathscr{U},x_0)$.*

Proof. The case $n=1$ is proved in [Brazas and Fabel 2014]. Suppose $n\geq 2$ and set $K=\mathrm{sd}^m\,\Delta_{n+1}$. The set $\mathscr{W}=\{f^{-1}(U)\mid U\in\mathscr{U}\}$ is an open cover of $K_n=(\mathrm{sd}^m\,\Delta_{n+1})_n$ such that $f_\#(\pi_n^{Sp}(\mathscr{W},\boldsymbol{o}))\subseteq\pi_n^{Sp}(\mathscr{U},x_0)$ and for every $(n+1)$-simplex σ in K, we have $\partial\sigma\subseteq f^{-1}(U)$ for some $U\in\mathscr{U}$. Thus it suffices to prove $\pi_n(K_n,\boldsymbol{o})\subset\pi_n^{Sp}(\mathscr{W},\boldsymbol{o})$. Let S be the set of $(n+1)$-simplices of K. Since $n\geq 2$, K_n is simply connected. Standard simplicial homology arguments give that the reduced singular homology groups of K_n are trivial in dimension $<n$ and $H_n(K_n)$ is a finitely generated free abelian group. A set of free generators for $H_n(K_n)$ can be chosen by fixing the homology class of a simplicial map $g_\sigma:\partial\Delta_{n+1}\to K_n$ that sends $\partial\Delta_{n+1}$ homeomorphically onto the boundary of an $(n+1)$-simplex $\sigma\in S$. Thus K_n is $(n-1)$-connected and the Hurewicz homomorphism $h:\pi_k(K_n,\boldsymbol{o})\to H_k(K_n)$ is an isomorphism for all $1\leq k\leq n$. In particular, let $p_\sigma:I\to K_n$ be any path from $\boldsymbol{o}$ to $g_\sigma(\boldsymbol{o})$. Then $\pi_n(K_n,\boldsymbol{o})$ is freely generated by the path-conjugates $[p_\sigma*g_\sigma]$, $\sigma\in S$. By assumption, for every $\sigma\in S$, $[p_\sigma*g_\sigma]$ is a generator of

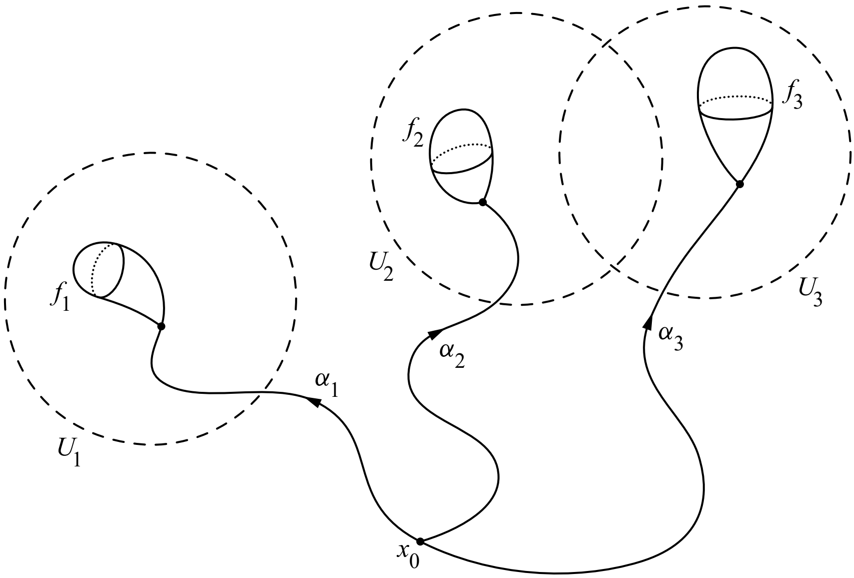

Figure 1. An element of $\pi_2^{Sp}(\mathscr{U}, x_0)$, which is a product of three path-conjugate generators $[\alpha_i * f_i]$.

$\pi_n^{Sp}(\mathscr{W}, \boldsymbol{o})$. Since $\pi_n^{Sp}(\mathscr{W}, \boldsymbol{o})$ contains all the generators of $\pi_n(K_n, \boldsymbol{o})$, the inclusion $\pi_n(K_n, \boldsymbol{o}) \subset \pi_n^{Sp}(\mathscr{W}, \boldsymbol{o})$ follows. □

To characterize the triviality of relative Spanier groups, we establish the following terminology.

Definition 3.7. Let $n \geq 0$ and $x \in X$. We say the space X is:

(1) *Semilocally π_n-trivial at x* if there exists an open neighborhood U of x in X such that every map $S^n \to U$ is null-homotopic in X.

(2) *Semilocally n-connected at x* if there exists an open neighborhood U of x in X such that every map $S^k \to X$, $0 \leq k \leq n$ is null-homotopic in X.

We say X is *semilocally π_n-trivial* (resp. semilocally n-connected) if it has this property at all of its points.

It is straightforward to see that X is semilocally n-connected at x if and only if X is semilocally π_k-trivial at x for all $0 \leq k \leq n$.

Remark 3.8. A space X is semilocally π_n-trivial if and only if X admits an open cover $\mathscr{U}$ such that $\pi_n^{Sp}(\mathscr{U}, x_0)$ is trivial [Bahredar et al. 2021, Theorem 3.7]. Moreover, X is semilocally n-connected if and only if X admits an open cover $\mathscr{U}$ such that $\pi_k^{Sp}(\mathscr{U}, x_0)$ is trivial for all $1 \leq k \leq n$. Note that local path connectivity is independent of the properties given in Definition 3.7.

Attempting a proof of Theorem 1.1, one should not expect the groups $\pi_n^{Sp}(\mathscr{U}, x_0)$ and $\ker(p_{\mathscr{U}\#})$ to agree "on the nose." Indeed, the following example shows that we should not expect the equality $\pi_n^{Sp}(\mathscr{U}, x_0) = \ker(p_{\mathscr{U}\#})$ to hold even in the "nicest" local circumstances.

Example 3.9. Let $X = S^2 \vee S^2$ and W be a contractible neighborhood of d_0 in S^2. Set $U_1 = S^2 \vee W$ and $U_2 = W \vee S^2$ and consider the open cover $\mathscr{U} = \{U_1, U_2\}$ of X. Then $\pi_3^{Sp}(\mathscr{U}, x_0) \cong \mathbb{Z}^2$ is freely generated by the homotopy classes of the two inclusions $i_1, i_2 : S^2 \to X$. However, $\pi_3(X) \cong \mathbb{Z}^3$ is freely generated by $[i_1]$, $[i_2]$, and the Whitehead product $[\![i_1, i_2]\!]$. However $|N(\mathscr{U})|$ is a 1-simplex and is therefore contractible. Thus $\ker(p_{\mathscr{U}\#})$ is equal to $\pi_3(X)$ and contains $[\![i_1, i_2]\!]$. Even though the spaces X, U_1, U_2 are locally contractible and the elements of $\mathscr{U}$ are 1-connected, $\pi_n^{Sp}(\mathscr{U}, x_0)$ is a proper subgroup of $\ker(p_{\mathscr{U}\#})$. One can view this failure as the result of two facts: (1) The sets U_i are not 2-connected and (2) the definition of Spanier group does not allow one to generate homotopy classes by taking Whitehead products of maps $S^2 \to U_i$ in the neighboring elements of $\mathscr{U}$.

First, we show the inclusion $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ holds in full generality. Recall that the intersections $\pi_n^{Sp}(X, x_0) = \bigcap_{\mathscr{U} \in O(X)} \pi_n^{Sp}(\mathscr{U}, x_0)$ and $\ker(\Psi_n) = \bigcap_{(\mathscr{U}, U_0) \in \Lambda} \ker(p_{\mathscr{U}\#})$ are formally indexed by different sets.

Lemma 3.10. *For every open cover $\mathscr{U}$ of X and canonical map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$, there exists a refinement $\mathscr{V} \succeq \mathscr{U}$ such that $\pi_n^{Sp}(\mathscr{V}, x_0) \subseteq \ker(p_{\mathscr{U}\#})$ in $\pi_n(X, x_0)$.*

Proof. Let $\mathscr{U} \in O(X)$. The stars $\mathrm{st}(U, |N(\mathscr{U})|)$, $U \in \mathscr{U}$ form an open cover of $|N(\mathscr{U})|$ by contractible sets and therefore $\mathscr{V} = \{p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \mid U \in \mathscr{U}\}$ is an open cover of X. Since $p_{\mathscr{U}}$ is a canonical map, we have $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \subseteq U$ for all $U \in \mathscr{U}$. Thus $\mathscr{V}$ is a refinement of $\mathscr{U}$. A generator of $\pi_n^{Sp}(\mathscr{V}, x_0)$ is of the form $[\alpha * f]$ for a map $f : S^n \to p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|))$. However, $p_{\mathscr{U}} \circ f$ has image in the contractible open set $\mathrm{st}(U, |N(\mathscr{U})|)$ and is therefore null-homotopic. Thus $p_{\mathscr{U}\#}([\alpha * f]) = 0$. We conclude that $p_{\mathscr{U}\#}(\pi_n^{Sp}(\mathscr{V}, x_0)) = 0$. □

Corollary 3.11 [Bahredar et al. 2021, Theorem 4.14]. *Let $n \geq 1$. For any based space (X, x_0), we have $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$.*

Proof. Suppose $[f] \in \pi_n^{Sp}(X, x_0)$. Given a normal, based open cover $(\mathscr{U}, U_0) \in \Lambda$ and any canonical map $p_{\mathscr{U}} : X \to |N(\mathscr{U})|$, Lemma 3.10 ensures we can find a refinement $\mathscr{V} \succeq \mathscr{U}$ such that $\pi_n^{Sp}(\mathscr{V}, x_0) \subseteq \ker(p_{\mathscr{U}\#})$. Thus $[f] \in \pi_n^{Sp}(\mathscr{V}, x_0) \subseteq \ker(p_{\mathscr{U}\#})$. Since $(\mathscr{U}, U_0)$ is arbitrary, we conclude that $[f] \in \ker(\Psi_n)$. □

Example 3.12 (higher earring spaces). An important space, which we will call upon repeatedly for examples, is the *n-dimensional earring space*

$$\mathbb{E}_n = \bigcup_{j \in \mathbb{N}} \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x} - (1/j, 0, 0, \dots, 0)\| = 1/j\},$$

which is a shrinking wedge (one-point union) of n-spheres with basepoint the origin $\boldsymbol{o}$. It is known that $\mathbb{E}_n$ is $(n-1)$-connected, locally $(n-1)$-connected, and π_n-shape injective for all $n \geq 1$ [Eda and Kawamura 2000a; Morgan and Morrison 1986]. However, $\mathbb{E}_n$ is not semilocally π_n-trivial. Thus $\pi_n^{Sp}(\mathscr{U}, \boldsymbol{o}) \neq 0$ for any open cover $\mathscr{U}$ of $\mathbb{E}_n$ even though in the absolute case $\pi_n^{Sp}(\mathbb{E}_n, \boldsymbol{o})$ is trivial.

Example 3.13. Let $n \geq 3$ and notice that $\mathbb{E}_1 \vee \mathbb{E}_n$ is not semilocally π_1-trivial (since it has $\mathbb{E}_1$ as a retract) and therefore fails to be semilocally $(n-1)$-connected. However, it has recently been shown that $\pi_k(\mathbb{E}_1 \vee \mathbb{E}_n) = 0$ for $2 \leq k \leq n-1$ and that $\mathbb{E}_1 \vee \mathbb{E}_n$ is π_n-shape injective [Brazas 2021]. Thus $\mathbb{E}_1 \vee \mathbb{E}_n$ is semilocally π_k-trivial for all $k \leq n-1$ except $k = 1$ and $\pi_n^{Sp}(\mathbb{E}_1 \vee \mathbb{E}_n, \boldsymbol{o}) = 0$. Thus the failure to be semilocally n-connected can occur at single dimension less than n.

4. Recursive extension lemmas

Toward a proof of the inclusion $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$ for LC^{n-1} space X, we introduce some convenient notation and definitions. If $\mathscr{U}$ is an open cover and $A \subseteq X$, then $\mathrm{St}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} \mid A \cap U \neq \varnothing\}$. Note that if $A \subseteq B$, then $\mathrm{St}(A, \mathscr{U}) \subseteq \mathrm{St}(B, \mathscr{U})$. Also if $\mathscr{V} \succeq \mathscr{U}$, then $\mathrm{St}(A, \mathscr{V}) \subseteq \mathrm{St}(A, \mathscr{U})$. We take the following terminology from [Willard 1970].

Definition 4.1. Let $\mathscr{U}, \mathscr{V} \in O(X)$:

(1) We say $\mathscr{V}$ is a *barycentric-star refinement* of $\mathscr{U}$ if for every $x \in X$, we have $\mathrm{St}(x, \mathscr{V}) \subseteq U$ for some $U \in \mathscr{U}$. We write $\mathscr{V} \succeq_* \mathscr{U}$.

(2) We say $\mathscr{V}$ is a *star refinement* of $\mathscr{U}$ if for every $V \in \mathscr{V}$, we have $\mathrm{St}(V, \mathscr{V}) \subseteq U$ for some $U \in \mathscr{U}$. We write $\mathscr{V} \succeq_{**} \mathscr{U}$.

Note that if $\mathscr{U} \preceq_* \mathscr{V} \preceq_* \mathscr{W}$, then $\mathscr{U} \preceq_{**} \mathscr{W}$.

Lemma 4.2 [Stone 1948]. *A T_1 space X is paracompact if and only if for every $\mathscr{U} \in O(X)$ there exists $\mathscr{V} \in O(X)$ such that $\mathscr{V} \succeq_* \mathscr{U}$.*

Definition 4.3. [Mardešić and Segal 1982, Chapter I, Section 3.2.5] Let $n \in \{0, 1, 2, 3, \dots, \infty\}$. A space X is LC^n at $x \in X$ if for every neighborhood U of x, there exists a neighborhood V of x such that $V \subseteq U$ and such that for all $0 \leq k \leq n$ ($k < \infty$ if $n = \infty$), every map $f : \partial\Delta_{k+1} \to V$ extends to a map $g : \Delta_{k+1} \to U$. We say X is LC^n if X is LC^n at all of its points.

We have the following evident implications for both the point-wise and global properties:

$$X \text{ is locally } n\text{-connected} \Rightarrow X \text{ is } LC^n \Rightarrow X \text{ is semilocally} n\text{-connected.}$$

For first countable spaces, the LC^n property is equivalent to the “n-tame” property in [Brazas 2021] defined in terms of shrinking sequences of maps.

Definition 4.4. Suppose $\mathscr{V} \succeq \mathscr{U}$ in $O(X)$:

(1) We say $\mathscr{V}$ is an *n-refinement of $\mathscr{U}$*, and write $\mathscr{V} \succeq^n \mathscr{U}$, if for all $0 \leq k \leq n$, $V \in \mathscr{V}$, and maps $f : \partial\Delta_{k+1} \to V$, there exists $U \in \mathscr{U}$ with $V \subseteq U$ and a continuous extension $g : \Delta_{k+1} \to U$ of f.

(2) We say $\mathscr{V}$ is an *n-barycentric-star refinement of $\mathscr{U}$*, and write $\mathscr{V} \succeq^n_* \mathscr{U}$, if for every $0 \leq k \leq n$, for every $x \in X$, and every map $f : \partial\Delta_{k+1} \to \mathrm{St}(x, \mathscr{V})$, there exists $U \in \mathscr{U}$ with $\mathrm{St}(x, \mathscr{V}) \subseteq U$ and a continuous extension $g : \Delta_{k+1} \to U$ of f.

Note that if $\mathscr{V} \succeq^n \mathscr{U}$ (resp. $\mathscr{V} \succeq^n_* \mathscr{U}$), then $\mathscr{V} \succeq^k \mathscr{U}$ (resp. $\mathscr{V} \succeq^k_* \mathscr{U}$) for all $0 \leq k \leq n$.

Lemma 4.5. *Suppose X is paracompact, Hausdorff, and LC^n. For every $\mathscr{U} \in O(X)$, there exists $\mathscr{V} \in O(X)$ such that $\mathscr{V} \succeq^n_* \mathscr{U}$.*

Proof. Let $\mathscr{U} \in O(X)$. Since X is LC^n, for every $U \in \mathscr{U}$ and $x \in U$, there exists an open neighborhood $W(U, x)$ such that $W(U, x) \subseteq U$ and such that for all $0 \leq k \leq n$, each map $f : \partial\Delta_{k+1} \to W(U, x)$ extends to a map $g : \Delta_{k+1} \to U$. Let $\mathscr{W} = \{W(U, x) \mid U \in \mathscr{U}, x \in U\}$ and note $\mathscr{W} \succeq^n \mathscr{U}$. Since X is paracompact Hausdorff, by Lemma 4.2, there exists $\mathscr{V} \in O(X)$ such that $\mathscr{V} \succeq_* \mathscr{W}$.

Fix $x' \in X$. Then $\mathrm{St}(x', \mathscr{V}) \subseteq W(U, x)$ for some $x \in U \in \mathscr{U}$. Then $\mathrm{St}(x', \mathscr{V}) \subseteq U$. Moreover, if $0 \leq k \leq n$ and $f : \partial\Delta_{k+1} \to \mathrm{St}(x', \mathscr{V})$ is a map, then since f has image in $W(U, x)$, there is an extension $g : \Delta_{k+1} \to U$. This verifies that $\mathscr{V} \succeq^n_* \mathscr{U}$. □

For the next two lemmas, we fix $n \in \mathbb{N}$, a geometric simplicial complex K with $\dim K = n + 1$, and a subcomplex $L \subseteq K$ with $\dim L \leq n$. Let $M[k] = L \cup K_k$ denote the union of L and the k-skeleton of K. Since $L \subseteq K_n$, $M[n] = K_n$ is the union of the boundaries of the $(n + 1)$-simplices of K. Later we will consider the cases where (1) $K = \mathrm{sd}^m \Delta_{n+1}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ and (2) $K = \mathrm{sd}^m \partial\Delta_{n+2}$ and $L = \{\boldsymbol{o}\}$.

Lemma 4.6 (recursive extensions). *Suppose $1 \leq k \leq n$, $\mathscr{U} \preceq_* \mathscr{V} \preceq^{k-1}_* \mathscr{W}$, $m \in \mathbb{N}$, and $f : M[k - 1] \to X$ is a map such that for every $(n + 1)$-simplex σ of K, we have $f(\sigma \cap M[k - 1]) \subseteq W_\sigma$ for some $W_\sigma \in \mathscr{W}$. Then there exists a continuous extension $g : M[k] \to X$ of f such that for every $(n + 1)$-simplex σ of K, we have $g(\sigma \cap M[k]) \subseteq U_\sigma$ for some $U_\sigma \in \mathscr{U}$.*

Proof. Supposing the hypothesis, we must extend f to the k-simplices of $M[k]$ that do not lie in L. Let τ be a k-simplex of $M[k]$ that does not lie in L and let S_τ be the set of $(n + 1)$-simplices in K that contain τ. By assumption, S_τ is nonempty. We make some general observations first. Since f maps the $(k - 1)$-skeleton of each $(n + 1)$-simplex $\sigma \in S_\tau$ into W_σ and $\partial\tau$ lies in this $(k - 1)$-skeleton, we have

$f(\partial\tau) \subseteq \bigcap_{\sigma\in S_\tau} W_\sigma$. Thus, for all τ, we have

$$f(\partial\tau) \subseteq \bigcap_{\sigma\in S_\tau} \mathrm{St}(W_\sigma, \mathscr{V}).$$

Fix a vertex v_τ of τ and let $x_\tau = f(v_\tau)$. Then $x_\tau \in W_\sigma \subseteq \mathrm{St}(x_\tau, \mathscr{W})$ whenever $\sigma \in S_\tau$. Since $\mathscr{W} \succeq_*^{k-1} \mathscr{V}$, we may find $V_\tau \in \mathscr{V}$ such that $\mathrm{St}(x_\tau, \mathscr{W}) \subseteq V_\tau$ and such that every map $\partial\Delta_k \to \mathrm{St}(x_\tau, \mathscr{W})$ extends to a map $\Delta_k \to V_\tau$. In particular, $f|_{\partial\tau} : \partial\tau \to W_\sigma$ extends to a map $\tau \to V_\tau$. We define $g : M[k] \to X$ so that it agrees with f on $M[k-1]$ and so that the restriction of g to τ is a choice of continuous extension $\tau \to V_\tau$ of $f|_{\partial\tau}$.

We now choose the sets U_σ. Fix an $(n+1)$-simplex σ of K. If the k-skeleton of σ lies entirely in L, we choose any $U_\sigma \in \mathscr{U}$ satisfying $W_\sigma \subseteq U_\sigma$. Suppose there exists at least one k-simplex in σ not in L. Then whenever τ is a k-simplex of σ not in L, we have $W_\sigma \subseteq \mathrm{St}(x_\tau, \mathscr{W}) \subseteq V_\tau$. Fix a point $y_\sigma \in W_\sigma$. The assumption that $\mathscr{V} \succeq_* \mathscr{U}$ implies that there exists $U_\sigma \in \mathscr{U}$ such that $\mathrm{St}(y_\sigma, \mathscr{V}) \subseteq U_\sigma$. In this case, we have $W_\sigma \subseteq V_\tau \subseteq U_\sigma$ whenever τ is a k-simplex of σ not in L.

Finally, we check that g satisfies the desired property. Again, fix an $(n+1)$-simplex σ of K. If τ is a k-simplex of σ not in L, our definition of g gives $g(\tau) \subseteq V_\tau \subseteq U_\sigma$. If τ' is a k-simplex in $\sigma \cap L$, then $g(\tau') = f(\tau') \subseteq W_\sigma \subseteq U_\sigma$. Overall, this shows that $g(\sigma \cap M[k]) \subseteq U_\sigma$ for each $(n+1)$-simplex σ of K. $\square$

A direct, recursive application of the previous lemma is given in the following statement.

Lemma 4.7. *Suppose there is a sequence of open covers*

$$\mathscr{W}_n \preceq_* \mathscr{V}_n \preceq_*^{n-1} \mathscr{W}_{n-1} \preceq_* \cdots \preceq_*^{2} \mathscr{W}_2 \preceq_* \mathscr{V}_2 \preceq_*^{1} \mathscr{W}_1 \preceq_* \mathscr{V}_1 \preceq_*^{0} \mathscr{W}_0$$

and a map $f_0 : M[0] \to X$ such that for every $(n+1)$-simplex σ of K, we have $f_0(\sigma \cap M[0]) \subseteq W$ for some $W \in \mathscr{W}_0$. Then there exists an extension $f_n : M[n] \to X$ of f_0 such that for every $(n+1)$-simplex σ of K, we have $f_n(\partial\sigma) \subseteq U$ for some $U \in \mathscr{W}_n$.

5. A proof of Theorem 1.1

We apply the extension results of the previous section in the case where $K = \mathrm{sd}^m \Delta_{n+1}$ for some $m \in \mathbb{N}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ so that $M[k] = L \cup K_k$ consists of the n-simplices of the boundary of Δ_{n+1} and the k-simplices of $\mathrm{sd}^m \Delta_{n+1}$ not in the boundary. Note that $M[n]$ is the union of the boundaries of the $(n+1)$-simplices of $\mathrm{sd}^m \Delta_{n+1}$.

Lemma 5.1. *Let $n \geq 1$. Suppose X is paracompact, Hausdorff, and LC^{n-1}. Then for every open cover $\mathscr{U}$ of X, there exists $(\mathscr{V}, V_0) \in \Lambda$ such that* $\ker(p_{\mathscr{V}\#}) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$.

Proof. Suppose $\mathscr{U} \in O(X)$. Since X is paracompact, Hausdorff, and LC^{n-1}, we may apply Lemmas 4.2 and 4.5 to first find a sequence of refinements

$$\mathscr{U} = \mathscr{U}_n \preceq_* \mathscr{V}_n \preceq_*^{n-1} \mathscr{U}_{n-1} \preceq_* \cdots \preceq_*^2 \mathscr{U}_2 \preceq_* \mathscr{V}_2 \preceq_*^1 \mathscr{U}_1 \preceq_* \mathscr{V}_1 \preceq_*^0 \mathscr{U}_0$$

and then one last refinement $\mathscr{U}_0 \preceq_* \mathscr{V}_0 = \mathscr{V}$. Let $V_0 \in \mathscr{V}$ be any set containing x_0 and recall that since X is paracompact Hausdorff $(\mathscr{V}, V_0) \in \Lambda$. We will show that $\ker(p_{\mathscr{V}\#}) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$. Note that $p_{\mathscr{V}}^{-1}(\mathrm{st}(V, N(\mathscr{V}))) \subseteq V$ by the definition of canonical map $p_{\mathscr{V}}$.

Suppose $[f] \in \ker(p_{\mathscr{V}\#})$ is represented by a map $f : (|\partial\Delta_{n+1}|, \boldsymbol{o}) \to (X, x_0)$. We will show that $[f] \in \pi_n^{Sp}(\mathscr{U}, x_0)$. Then $p_{\mathscr{V}} \circ f : |\partial\Delta_{n+1}| \to |N(\mathscr{V})|$ is null-homotopic and extends to a map $h : |\Delta_{n+1}| \to |N(\mathscr{V})|$. Set $Y_V = h^{-1}(\mathrm{st}(V, N(\mathscr{V})))$ so that $\mathscr{Y} = \{Y_V \mid V \in \mathscr{V}\}$ is an open cover of $|\Delta_{n+1}|$.

We find a particular simplicial approximation for h using the cover $\mathscr{Y}$ [Munkres 1984, Theorem 16.1]: let λ be a Lebesgue number for $\mathscr{Y}$ so that any subset of Δ_{n+1} of diameter less than λ lies in some element of $\mathscr{Y}$. Find $m \in \mathbb{N}$ such that each simplex in $\mathrm{sd}^m \Delta_{n+1}$ has diameter less than $\lambda/2$. Thus the star $\mathrm{st}(a, \mathrm{sd}^m \Delta_{n+1})$ of each vertex a in $\mathrm{sd}^m \Delta_{n+1}$ lies in a set $Y_{V_a} \in \mathscr{Y}$ for some $V_a \in \mathscr{V}$. The assignment $a \mapsto V_a$ on vertices extends to a simplicial approximation $h' : \mathrm{sd}^m \Delta_{n+1} \to N(\mathscr{V})$ of h, i.e., a simplicial map h' such that

$$h(\mathrm{st}(a, \mathrm{sd}^m \Delta_{n+1})) \subseteq \mathrm{st}(h'(a), N(\mathscr{V})) = \mathrm{st}(V_a, N(\mathscr{V}))$$

for each vertex a [Munkres 1984, Lemma 14.1].

Let $K = \mathrm{sd}^m \Delta_{n+1}$ and $L = \mathrm{sd}^m \partial\Delta_{n+1}$ so that $M[k] = L \cup K_k$. First, we extend $f : L \to X$ to a map $f_0 : M[0] \to X$. For each vertex a in K, pick a point $f_0(a) \in V_a$. In particular, if $a \in L$, take $f_0(a) = f(a)$. This choice is well defined since, for a boundary vertex $a \in L$, we have $p_{\mathscr{V}} \circ f(a) = h(a) \in \mathrm{st}(V_a, |N(\mathscr{V})|)$ and thus $f(a) \in p_{\mathscr{V}}^{-1}(\mathrm{st}(V_a, |N(\mathscr{V}|))) \subseteq V_a$.

Note that h' maps every simplex $\sigma = [a_0, a_1, \ldots, a_k]$ of K to the simplex of $N(\mathscr{V})$ spanned by $\{h'(a_i) \mid 0 \le i \le k\} = \{V_{a_i} \mid 0 \le i \le k\}$. By definition of the nerve, we have $\bigcap\{V_{a_i} \mid 0 \le i \le k\} \neq \varnothing$. Pick a point $x_\sigma \in \bigcap\{V_{a_i} \mid 0 \le i \le k\}$.

By our initial choice of refinements, we have $\mathscr{U}_0 \preceq_* \mathscr{V}$. If $\sigma = [a_0, a_1, \ldots, a_{n+1}]$ is an $(n+1)$-simplex of K, then $\mathrm{St}(x_\sigma, \mathscr{V}) \subseteq U_\sigma$ for some $U_\sigma \in \mathscr{U}$. In particular $\{f_0(a_i) \mid 0 \le i \le n+1\} \subseteq \bigcup\{V_{a_i} \mid 0 \le i \le n+1\} \subseteq U_\sigma$. Thus f_0 maps the 0-skeleton of σ into U_σ. If $1 \le k \le n$, τ is a k-face of $\sigma \cap L$ with $a_i \in \tau$, then $p_{\mathscr{V}} \circ f_0(\mathrm{int}(\tau)) = p_{\mathscr{V}} \circ f(\mathrm{int}(\tau)) = h(\mathrm{int}(\tau)) \subseteq h(\mathrm{st}(a_i, K)) \subseteq \mathrm{st}(V_{a_i}, |N(\mathscr{V})|)$. It follows that

$$f_0(\tau) \subseteq p_{\mathscr{V}}^{-1}(\mathrm{st}(V_{a_i}, |N(\mathscr{V})|)) \subseteq V_{a_i} \subseteq U_\sigma.$$

Thus for every n-simplex in $\sigma \cap L$, we have $f_0(\tau) \subseteq U_\sigma$. We conclude that for every $(n+1)$-simplex σ of K, we have $f_0(\sigma \cap M[0]) \subseteq U_\sigma$.

By our choice of sequence of refinements, we are precisely in the situation to apply Lemma 4.7. Doing so, we obtain an extension $f_n : M[n] \to X$ of f such that for every $(n+1)$-simplex σ of K, we have $f_n(\partial\sigma) \subseteq \boldsymbol{U}_\sigma$ for some $\boldsymbol{U}_\sigma \in \mathscr{U}_n = \mathscr{U}$. By Lemma 3.6, we have $[f] = [f_n|_{\partial\Delta_{n+1}}] \in \pi_n^{Sp}(\mathscr{U}, x_0)$. □

Finally, both inclusions have been established and provide a proof of our main result.

Proof of Theorem 1.1. The inclusion $\pi_n^{Sp}(X, x_0) \subseteq \ker(\Psi_n)$ holds in general by Corollary 3.11. Under the given hypotheses, the inclusion $\ker(\Psi_n) \subseteq \pi_n^{Sp}(X, x_0)$ follows from Lemma 5.1. □

When considering examples relevant to Theorem 1.1, it is helpful to compare π_n-shape injectivity with the following weaker property from [Ghane and Hamed 2009].

Definition 5.2. We say a space X is *n-homotopically Hausdorff at $x \in X$* if no nontrivial element of $\pi_n(X, x)$ has a representing map in every neighborhood of x. We say X is *n-homotopically Hausdorff* if it is n-homotopically Hausdorff at each of its points.

Clearly, π_n-shape injectivity $\Rightarrow$ n-homotopically Hausdorff. The next example, which highlights the effectiveness of Theorem 1.1, shows the converse is not true even for LC^{n-1} Peano continua.

Example 5.3. Fix $n \geq 2$ and let $\ell_j : S^n \to \mathbb{E}_n$ be the inclusion of the j-th sphere and define $f : \mathbb{E}_n \to \mathbb{E}_n$ to be the shift map given by $f \circ \ell_j = \ell_{j+1}$. Let $M_f = \mathbb{E}_n \times [0, 1]/\sim$, $(x, 0) \sim (f(x), 1)$ be the mapping torus of f. We identify $\mathbb{E}_n$ with the image of $\mathbb{E}_n \times \{0\}$ in M_f and take $\boldsymbol{o}$ to be the basepoint of M_f. Let $\alpha : I \to M_f$ be the loop where $\alpha(t)$ is the image of $(\boldsymbol{o}, t)$. Then M_f is locally contractible at all points other than those in the image of α. Also, every point $\alpha(t)$ has a neighborhood that deformation retracts onto a homeomorphic copy of $\mathbb{E}_n$. Thus, since $\mathbb{E}_n$ is LC^{n-1}, so is X. It follows from Theorem 1.1 that $\pi_n^{Sp}(M_f, \boldsymbol{o}) = \ker(\pi_n(M_f, \boldsymbol{o}) \to \check{\pi}_n(M_f, \boldsymbol{o}))$. In particular, the Spanier group of M_f contains all elements $[\alpha^k * g]$ where $g : S^n \to \mathbb{E}_n$ is a based map and $k \in \mathbb{Z}$. Using the universal covering map $E \to M_f$ that “unwinds” α and the relation $[g] = [\alpha * (f \circ g)]$ in $\pi_n(M_f, \boldsymbol{o})$, it is not hard to show that these are, in fact, the only elements of the n-th Spanier group. Hence,

$$\ker(\pi_n(M_f, \boldsymbol{o}) \to \check{\pi}_n(M_f, \boldsymbol{o})) = \{[\alpha^k * g] \mid [g] \in \pi_n(\mathbb{E}_n, \boldsymbol{o}), k \in \mathbb{Z}\},$$

which is an uncountable subgroup. Moreover, since M_f is shape equivalent to the aspherical space S^1, we have $\check{\pi}_n(M_f, \boldsymbol{o}) = 0$ and thus $\pi_n(M_f, \boldsymbol{o}) = \{[\alpha^k * g] \mid [g] \in \pi_n(\mathbb{E}_n, \boldsymbol{o}), k \in \mathbb{Z}\}$.

It follows from this description that, even though M_f is not π_n-shape injective, M_f is n-homotopically Hausdorff. Indeed, it suffices to check this at the points $\alpha(t)$, $t \in I$. We give the argument for $\alpha(0) = \boldsymbol{o}$, the other points are similar. If $0 \neq h \in \pi_n(M_f, \boldsymbol{o})$ has a representative in every neighborhood of $\boldsymbol{o}$ in M_f, then clearly $h \in \ker(\Psi_n)$. Hence, $h = [\alpha^k * g]$ for $[g] \in \pi_n(\mathbb{E}_n, \boldsymbol{o})$ and $k \in \mathbb{Z}$. Since M_f retracts onto the circle parametrized by α, the hypothesis on h can only hold if $k = 0$. However, there is a basis of neighborhoods of $\boldsymbol{o}$ in M_f that deformation retract onto an open neighborhood of $\boldsymbol{o}$ in $\mathbb{E}_n$. Thus $[g]$ has a representative in every neighborhood of $\boldsymbol{o}$ in $\pi_n(\mathbb{E}_n, \boldsymbol{o})$, giving $h = [g] \in \ker(\pi_n(\mathbb{E}_n, \boldsymbol{o}) \to \check{\pi}_n(\mathbb{E}_n, \boldsymbol{o})) = 0$.

It is an important feature of Example 5.3 that M_f is not simply connected and has multiple points at which it is not semilocally π_n-trivial. This motivates the following application of Theorem 1.1, which identifies a partial converse of the implication π_n-shape injective $\Rightarrow$ n-homotopically Hausdorff.

Corollary 5.4. *Let $n \geq 2$ and X be a simply connected, LC^{n-1}, compact Hausdorff space such that X fails to be semilocally π_n-trivial only at a single point $x \in X$. Then for every element $g \in \ker(\Psi_n) \subseteq \pi_n(X, x)$ and neighborhood V of x, g is represented by a map with image in V. In particular, if X is n-homotopically Hausdorff at x, then X is π_n-shape injective.*

Proof. Let $0 \neq g \in \ker(\Psi_n) \subseteq \pi_n(X, x)$. By Theorem 1.1, $g \in \pi_n^{Sp}(X, x)$. Since X is compact Hausdorff, we may replace $O(X)$ by the cofinal subdirected order $O_F(X)$ consisting of finite open covers $\mathscr{U}$ of X with the property that there is a unique $A_{\mathscr{U}} \in \mathscr{U}$ with $x \in A_{\mathscr{U}}$. For each $\mathscr{U} \in O_F(X)$, we can write $g = \prod_{i=1}^{m_{\mathscr{U}}} [\alpha_{\mathscr{U},i} * f_{\mathscr{U},i}]$ where $f_{\mathscr{U},i} : S^n \to U_{\mathscr{U},i}$ is a non-nullhomotopic map for some $U_{\mathscr{U},i} \in \mathscr{U}$ and $\alpha_{\mathscr{U},i}$ is a path from x to $f_{\mathscr{U},i}(d_0)$.

Let V be an open neighborhood of x. We check that g is represented by a map with image in V. Since X is LC^0 at x, there exists an open neighborhood V' of x such that any two points of V' may be connected by a path in V. Fix $\mathscr{U}_0 \in O_F(X)$ such that $A_{\mathscr{U}_0} \subseteq V'$. Then $A_{\mathscr{V}} \subseteq V'$ whenever $\mathscr{V} \in O_F(X)$ refines $\mathscr{U}_0$.

We claim that for sufficiently refined $\mathscr{V}$, all of the maps $f_{\mathscr{V},i}$ have image in V'. Suppose, to obtain a contradiction, there is a subset $T \subseteq \{\mathscr{V} \in O_F(X) \mid \mathscr{V} \succeq \mathscr{U}_0\}$, which is cofinal in $O_F(X)$ and such that for every $\mathscr{V} \in T$ there exists $i_{\mathscr{V}} \in \{1, 2, \ldots, m_{\mathscr{V}}\}$ and $d_{\mathscr{V}} \in S^n$ such that $f_{\mathscr{V},i_{\mathscr{V}}}(d_{\mathscr{V}}) \in U_{\mathscr{V},i} \backslash V' \subseteq U_{\mathscr{V},i} \backslash A_{\mathscr{U}_0}$. Since X is compact, we may replace $\{f_{\mathscr{V},i_{\mathscr{V}}}(d_{\mathscr{V}})\}$ with a subnet $\{x_j\}_{j \in J}$ that converges to a point $y \in X$. Here, $x_j = f_{\mathscr{V}_j,i_{\mathscr{V}_j}}(d_{\mathscr{V}_j})$ for some directed set J and monotone, final function $J \to T$ given by $j \mapsto \mathscr{V}_j$. Let Y be an open neighborhood of y in X. Find $\mathscr{W} \in O_F(X)$ such that there exists $W_0 \in \mathscr{W}$ such that $y \in W_0$ and $\mathrm{St}(W_0, \mathscr{W}) \subseteq Y$. Since $\{x_j\}$ is subnet that converges to y, there exists $k \in J$ such that $\mathscr{V}_k \succeq \mathscr{W}$ and $x_k \in W_0$. We have $x_k \in \mathrm{Im}(f_{\mathscr{V}_k,i_{\mathscr{V}_k}}) \subseteq U_{\mathscr{V}_k,i_{\mathscr{V}_k}} \subseteq W$ for some $W \in \mathscr{W}$ and thus $\mathrm{Im}(f_{\mathscr{V}_k,i_{\mathscr{V}_k}}) \subseteq U_{\mathscr{V}_k,i} \subseteq \mathrm{St}(W_0, \mathscr{W}) \subseteq Y$. However, for every $\mathscr{V} \in O_F(X)$, $f_{\mathscr{V},i_{\mathscr{V}}}$ is not null-homotopic in X. Thus, since Y represents an arbitrary neighborhood of y, X is not semilocally π_n-trivial at y. By assumption, we must have $x = y$. Since $\{x_j\} \to x$, the same argument, but where Y is replaced by V', shows that there exists sufficiently refined $\mathscr{V}$ for which $\mathrm{Im}(f_{\mathscr{V},i_{\mathscr{V}}}) \subseteq V'$; a contradiction. Since the claim is proved, there exists $\mathscr{U}_1 \succeq \mathscr{U}_0$ in $O_F(X)$ such that whenever $\mathscr{V} \succeq \mathscr{U}_1$, we have $\mathrm{Im}(f_{\mathscr{V},i}) \subseteq V'$ for all $i \in \{1, 2, \dots, m_{\mathscr{V}}\}$.

Fix $\mathscr{V} \succeq \mathscr{U}_1$ in $O_F(X)$. For all $i \in \{1, 2, \dots, m_{\mathscr{V}}\}$, we may find a path $\beta_{\mathscr{V},i} : I \to V$ from x to $f_{\mathscr{V},i}(d_0)$. Since X is simply connected, we have $[\alpha_{\mathscr{V},i} * f_{\mathscr{U},i}] = [\beta_{\mathscr{V},i} * f_{\mathscr{U},i}]$ for all i. Thus g is represented by $\prod_{i=1}^{m_{\mathscr{V}}} \beta_{\mathscr{V},i} * f_{\mathscr{V},i}$, which has image in V. □

Remark 5.5 (topologies on homotopy groups). Given a group G and a collection of subgroups $\{N_j \mid j \in J\}$ of G such that for all $j, j' \in J$, there exists $k \in J$ such that $N_k \subseteq N_j \cap N_{j'}$, we can generate a topology on G by taking the set $\{gN_j \mid j \in J, g \in G\}$ of left cosets as a basis. We can apply this to both the collection of Spanier subgroups $\pi_n^{Sp}(\mathscr{U}, x_0)$ and the collection of kernels $\ker(p_{\mathscr{U}\#})$ to define two natural topologies on $\pi_n(X, x_0)$:

(1) The *Spanier topology* on $\pi_n(X, x_0)$ is generated by the left cosets of Spanier groups $\pi_n(\mathscr{U}, x_0)$ for $\mathscr{U} \in O(X)$.

(2) The *shape topology* on $\pi_n(X, x_0)$ is generated by left cosets of the kernels $\ker(p_{\mathscr{U}\#})$ where $(\mathscr{U}, U_0) \in \Lambda$. Equivalently, the shape topology is the initial topology with respect to the map Ψ_n where the groups $\pi_n(|N(\mathscr{U})|, U_0)$ are given the discrete topology and $\check{\pi}_n(X, x_0)$ is given the inverse limit topology.

Lemma 3.10 ensures the Spanier topology is always finer than the shape topology. Lemma 5.1 then implies that, whenever X is paracompact, Hausdorff, and LC^{n-1}, the two topologies agree. Moreover, $\pi_n(X, x_0)$ is Hausdorff in the shape topology if and only if X is π_n-shape injective.

6. When is Ψ_n an isomorphism?

It is a result of Kozlowski and Segal [1978] that if X is paracompact Hausdorff and LC^n, then $\Psi_n : \pi_n(X, x) \to \check{\pi}_n(X, x)$ is an isomorphism. This result was first proved for compact metric spaces in [Kuperberg 1975]. The assumption that X is LC^n assumes that small maps $S^n \to X$ may be contracted by small null-homotopies. However, if $\mathbb{E}_n$ is the n-dimensional earring space, then the cone $C\mathbb{E}_n$ is LC^{n-1} but not LC^n. However, $C\mathbb{E}_n$ is contractible and so Ψ_n is an isomorphism of trivial groups. Certainly, many other examples in this range exist. Our Spanier group-based approach allows us to generalize Kozlowski–Segal's theorem in a way that includes this example by removing the need for "small" homotopies in dimension n. In this section, when $\mathscr{U}$ is an open cover of a space X and a distinguished element

$U_0 \in \mathscr{U}$ containing the basepoint x_0 has been established or is clear from context, we will often write $\mathscr{U}$ to represent the pair $(\mathscr{U}, U_0) \in \Lambda$.

Lemma 6.1. *Let $n \geq 1$. Suppose that X is paracompact, Hausdorff, and LC^{n-1}. If $([f_{\mathscr{U}}])_{\mathscr{U}\in\Lambda} \in \check{\pi}_1(X, x_0)$, then for every $\mathscr{U} \in \Lambda$, there exists $[g] \in \pi_n(X, x)$ such that $(p_{\mathscr{U}})_\#([g]) = [f_{\mathscr{U}}]$.*

Proof. With $(\mathscr{U}, U_0) \in \Lambda$ and $p_{\mathscr{U}}$ fixed, consider a representing map

$$f_{\mathscr{U}} : (|\partial\Delta_{n+1}|, \boldsymbol{o}) \to (|N(\mathscr{U})|, U_0).$$

Let $\mathscr{U}' = \{p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \mid U \in \mathscr{U}\}$. Since $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \subseteq U$ for all $U \in \mathscr{U}$, we have $\mathscr{U} \preceq \mathscr{U}'$. Applying Lemmas 4.2 and 4.5 we can choose the following sequence of refinements of $\mathscr{U}'$. First, we choose a star refinement $\mathscr{U}' \preceq_{**} \mathscr{W}$ so that for every $W \in \mathscr{W}$, there exists $U' \in \mathscr{U}'$ such that $\mathrm{St}(W, \mathscr{W}) \subseteq U'$. In this case, we can choose the projection map $p_{\mathscr{U}'\mathscr{W}} : |N(\mathscr{W})| \to |N(\mathscr{U}')|$ so that if $p_{\mathscr{U}'\mathscr{W}}(W) = U'$ on vertices, then $\mathrm{St}(W, \mathscr{W}) \subseteq U'$ in X. This choice will be important near the end of the proof.

To construct g, we must take further refinements. First, we choose a sequence of a refinements

$$\mathscr{W} = \mathscr{W}_n \preceq_* \mathscr{V}_n \preceq_*^{n-1} \mathscr{W}_{n-1} \preceq_* \cdots \preceq_*^2 \mathscr{W}_2 \preceq_* \mathscr{V}_2 \preceq_*^1 \mathscr{W}_1 \preceq_* \mathscr{V}_1 \preceq_*^0 \mathscr{W}_0$$

followed by one last refinement $\mathscr{W}_0 \preceq_* \mathscr{V}_0 = \mathscr{V}$. Let $V_0 \in \mathscr{V}$ be any set containing x_0 and recall that since X is paracompact Hausdorff $(\mathscr{V}, V_0) \in \Lambda$. For some choice of canonical map $p_{\mathscr{V}}$, we have $p_{\mathscr{V}}^{-1}(\mathrm{st}(V, N(\mathscr{V}))) \subseteq V$ for all $V \in \mathscr{V}$.

Recall that we have assumed the existence of a map

$$f_{\mathscr{V}} : (\partial\Delta_{n+1}, \boldsymbol{o}) \to (|N(\mathscr{V})|, V_0)$$

such that $p_{\mathscr{U}\mathscr{V}\#}([f_{\mathscr{V}}]) = [f_{\mathscr{U}}]$. Set $Y_V = f_{\mathscr{V}}^{-1}(\mathrm{st}(V, N(\mathscr{V})))$ so that $\mathscr{Y} = \{Y_V \mid V \in \mathscr{V}\}$ is an open cover of $\partial\Delta_{n+1}$. As before, we find a simplicial approximation for $f_{\mathscr{V}}$. Find $m \in \mathbb{N}$ such that the star $\mathrm{st}(a, \mathrm{sd}^m\,\partial\Delta_{n+1})$ of each vertex a in $\mathrm{sd}^m\,\partial\Delta_{n+1}$ lies in a set $Y_{V_a} \in \mathscr{Y}$ for some $V_a \in \mathscr{V}$. Since $f_{\mathscr{V}}(\boldsymbol{o}) = V_0$, we may take $V_{\boldsymbol{o}} = V_0$. The assignment $a \mapsto V_a$ on vertices extends to a simplicial approximation $f' : \mathrm{sd}^m\,\partial\Delta_{n+1} \to |N(\mathscr{V})|$ of $f_{\mathscr{V}}$, i.e., a simplicial map f' such that

$$f_{\mathscr{V}}(\mathrm{st}(a, \mathrm{sd}^m\,\partial\Delta_{n+1})) \subseteq \mathrm{st}(f'(a), |N(\mathscr{V})|) = \mathrm{st}(V_a, |N(\mathscr{V})|)$$

for each vertex a.

We begin to define g with the constant map $\{\boldsymbol{o}\} \to X$ sending $\boldsymbol{o}$ to x_0. In preparation for applications of Lemma 4.6, set $K = \mathrm{sd}^m\,\partial\Delta_{n+1}$ and $L = \{\boldsymbol{o}\}$ so that $K[k] = K_k$. First, we define a map $g_0 : M[0] \to X$ by picking, for each vertex $a \in K_0$, a point $g_0(a) \in V_a$. In particular, set $g_0(\boldsymbol{o}) = x_0$. This choice is well defined since we have $p_{\mathscr{V}}(x_0) = V_0 \in \mathrm{st}(V_{\boldsymbol{o}}, N(\mathscr{V}))$ and thus $g_0(\boldsymbol{o}) = x_0 \in p_{\mathscr{V}}^{-1}(\mathrm{st}(V_{\boldsymbol{o}}, N(\mathscr{V}))) \subseteq V_{\boldsymbol{o}}$. Note that f' maps every simplex $\sigma = [a_0, a_1, \ldots, a_k]$ of K to the simplex of $|N(\mathscr{V})|$ spanned by $\{V_{a_i} \mid 0 \leq i \leq k\}$. By definition of the nerve, we have $\bigcap\{V_{a_i} \mid 0 \le i \le k\} \neq \varnothing$. Pick a point $x_\sigma \in \bigcap\{V_{a_i} \mid 0 \le i \le k\}$. By our initial choice of refinements, we have $\mathscr{U}_0 \preceq_* \mathscr{V}$. If $\sigma = [a_0, a_1, \dots, a_n]$ is a n-simplex of K, then $\mathrm{St}(x_\sigma, \mathscr{V}) \subseteq U_{0,\sigma}$ for some $U_{0,\sigma} \in \mathscr{U}_0$. In particular $\{g_0(a_i) \mid 0 \le i \le n+1\} \subseteq \bigcup\{V_{a_i} \mid 0 \le i \le n\} \subseteq U_{0,\sigma}$. Thus g_0 maps the 0-skeleton of σ into $U_{0,\sigma}$. If $\boldsymbol{o} \in \sigma$, then $g_0(\boldsymbol{o}) \in p_{\mathscr{V}}^{-1}(\mathrm{st}(V_{\boldsymbol{o}}, N(\mathscr{V}))) \subseteq V_{\boldsymbol{o}} \subseteq U_{0,\sigma}$. Hence, for every n-simplex σ of K, we have $g_0(\sigma \cap M[0]) \subseteq U_{0,\sigma}$.

We are now in the situation to recursively apply Lemma 4.6. This is similar to the application in the proof of Lemma 5.1 with the dimension $n+1$ shifted down by one so we omit the details. Recalling that $M[n] = \mathrm{sd}^m\, \partial\Delta_{n+1}$, we obtain an extension $g : K = M[n] \to X$ of g_0 such that for every n-simplex σ of K, we have $g(\sigma) \subseteq W_\sigma$ for some $W_\sigma \in \mathscr{W} = \mathscr{U}_n$.

With g being defined, we seek show that $f_{\mathscr{U}} \simeq p_{\mathscr{U}} \circ g$. Since $f' \simeq f_{\mathscr{V}}$ (by simplicial approximation), $p_{\mathscr{U}\mathscr{V}} \simeq p_{\mathscr{U}\mathscr{U}'} \circ p_{\mathscr{U}'\mathscr{W}} \circ p_{\mathscr{W}\mathscr{V}}$ (for any choice of projection maps), and $p_{\mathscr{U}\mathscr{V}} \circ f_{\mathscr{V}} \simeq f_{\mathscr{U}}$ (for any choice of projection $p_{\mathscr{U}\mathscr{V}}$), it suffices to show that $p_{\mathscr{U}\mathscr{U}'} \circ p_{\mathscr{U}'\mathscr{W}} \circ p_{\mathscr{W}\mathscr{V}} \circ f' \simeq p_{\mathscr{U}} \circ g$. We do this by proving that the simplicial map $F = p_{\mathscr{U}\mathscr{U}'} \circ p_{\mathscr{U}'\mathscr{W}} \circ p_{\mathscr{W}\mathscr{V}} \circ f' : K \to |N(\mathscr{U})|$ is a simplicial approximation for $p_{\mathscr{U}} \circ g$. Recall that this can be done by verifying the "star-condition" $p_{\mathscr{U}} \circ g(\mathrm{st}(a, K)) \subseteq \mathrm{st}(F(a), |N(\mathscr{U})|)$ for any vertex $a \in K$ [Munkres 1984, Chapter 2, Section 14]. Since $n \ge 1$, we have $\mathscr{W} \preceq_{**} \mathscr{V}$. Hence, just like our choice of $p_{\mathscr{U}'\mathscr{W}}$, we may choose $p_{\mathscr{W}\mathscr{V}}$ so that whenever $p_{\mathscr{W}\mathscr{V}}(V) = W$, then $\mathrm{St}(V, \mathscr{V}) \subseteq W$. Also, we choose $p_{\mathscr{U}\mathscr{U}'}$ to map $p_{\mathscr{U}}^{-1}(\mathrm{st}(U, |N(\mathscr{U})|)) \mapsto U$ on vertices.

Fix a vertex $a_0 \in K$. To check the star-condition, we'll check that $p_{\mathscr{U}} \circ g(\sigma) \subseteq \mathrm{st}(F(a_0), |N(\mathscr{U})|)$ for each n-simplex σ having a_0 as a vertex. Pick an n-simplex $\sigma = [a_0, a_1, \dots, a_n] \subseteq K$ having a_0 as a vertex. Recall that $f'(a_i) = V_{a_i}$ for each i. Set $p_{\mathscr{W}\mathscr{V}}(V_{a_i}) = W_i$ and $p_{\mathscr{U}'\mathscr{W}}(W_i) = p_{\mathscr{U}}^{-1}(\mathrm{st}(U_i, |N(\mathscr{U})|)) \in \mathscr{U}'$ for some $U_i \in \mathscr{U}$. Then $F(a_i) = U_i$ for all i. It now suffices to check that $p_{\mathscr{U}} \circ g(\sigma) \subseteq \mathrm{st}(U_0, |N(\mathscr{U})|)$. Recall that by our choice of $p_{\mathscr{U}'\mathscr{W}}$, we have $\mathrm{St}(W_0, \mathscr{W}) \subseteq p_{\mathscr{U}}^{-1}(\mathrm{st}(U_0, |N(\mathscr{U})|))$. Thus it is enough to check that $g(\sigma) \subseteq \mathrm{St}(W_0, \mathscr{W})$. By construction of g, we have $g(\sigma) \subseteq W_\sigma$ for some $W_\sigma \in \mathscr{W}$. Since $g(a_0) \in W_0 \cap W_\sigma$, we have $g(\sigma) \subseteq W_\sigma \subseteq \mathrm{St}(W_0, \mathscr{W})$, completing the proof. □

Finally, we prove our second result, Theorem 1.2.

Proof of Theorem 1.2. Since X is paracompact, Hausdorff, LC^{n-1}, we have $\pi_n^{Sp}(X, x_0) = \ker(\Psi_n)$ by Theorem 1.1. Since X is semilocally π_n-trivial, we have $\pi_n^{Sp}(\mathscr{U}, x_0) = 1$ for some $\mathscr{U} \in \Lambda$. It follows that Ψ_n is injective. Moreover, by Lemma 5.1, we may find $\mathscr{V} \in \Lambda$ with $\ker(p_{\mathscr{V}\#}) \subseteq \pi_n^{Sp}(\mathscr{U}, x_0)$. Thus $p_{\mathscr{V}\#} : \pi_n(X, x_0) \to \pi_n(|N(\mathscr{V})|, V_0)$ is injective. Let $([f_{\mathscr{U}}])_{\mathscr{U}\in\Lambda} \in \check{\pi}_n(X, x_0)$. By Lemma 6.1, for each $\mathscr{U} \in \Lambda$, there exists $[g_{\mathscr{U}}] \in \pi_n(X, x_0)$ such that $p_{\mathscr{U}}([g_{\mathscr{U}}]) = [f_{\mathscr{U}}]$. If $\mathscr{V} \preceq \mathscr{W}$, then we have

$$p_{\mathscr{V}\#}([g_{\mathscr{V}}]) = [f_{\mathscr{V}}] = p_{\mathscr{V}\mathscr{W}\#}([f_{\mathscr{W}}]) = p_{\mathscr{V}\mathscr{W}\#} \circ p_{\mathscr{W}\#}([g_{\mathscr{W}}]) = p_{\mathscr{V}\#}([g_{\mathscr{W}}]).$$

Since $p_{\mathscr{V}\#}$ is injective, it follows that $[g_{\mathscr{W}}]=[g_{\mathscr{V}}]$ whenever $\mathscr{V}\preceq\mathscr{W}$. Setting $[g]=[g_{\mathscr{V}}]$ gives $\Psi_n([g])=([f_{\mathscr{U}}])_{\mathscr{U}\in\Lambda}$. Hence, Ψ_n is surjective. □

7. Examples

Example 7.1. Fix $n\geq 2$. When X is a metrizable LC^{n-1} space, the cone CX and unreduced suspension SX are LC^{n-1} and semilocally π_n-trivial but need not be LC^n. This occurs in the case $X=\mathbb{E}_n$ or if $X=Y\vee\mathbb{E}_n$ where Y is a CW-complex. In such cases, $\Psi_n:\pi_n(SX)\to\check{\pi}_n(SX)$ is an isomorphism. One point unions of such cones and suspensions, e.g., $CX\vee CY$ or $CX\vee SY$ also meet the hypotheses of Theorem 1.2 (checking this is fairly technical [Brazas 2021]) but need not be LC^n.

Example 7.2. The converse of Theorem 1.2 does not hold. For $n\geq 2$, $\mathbb{E}_n$ is LC^{n-1} but is not semilocally π_n-trivial at the wedgepoint x_0. However, $\Psi_n:\pi_n(\mathbb{E}_n,x_0)\to\check{\pi}_n(\mathbb{E}_n,x_0)$ is an isomorphism where both groups are canonically isomorphic to $\mathbb{Z}^{\mathbb{N}}$ [Eda and Kawamura 2000a]. Additionally, for the infinite direct product $\prod_{\mathbb{N}}S^n$, $\Psi_k:\pi_k\big(\prod_{\mathbb{N}}S^n,x_0\big)\to\check{\pi}_k\big(\prod_{\mathbb{N}}S^n,x_0\big)$ is an isomorphism for all $k\geq 1$ even though $\prod_{\mathbb{N}}S^n$ is not LC^{k-1} when $k-1\geq n$.

Example 7.3. We can also modify the mapping torus M_f from Example 5.3 so that Ψ_n becomes an isomorphism (recall that $n\geq 2$ is fixed). Let $X=M_f\cup C\mathbb{E}_n$ be the mapping cone of the inclusion $\mathbb{E}_n\to M_f$. For the same reason M_f is LC^{n-1}, the space X is LC^{n-1}. Moreover, if U is a neighborhood of $\alpha(t)$ that deformation retracts onto a homeomorphic copy of $\mathbb{E}_n$, then any map $S^n\to U$ may be freely homotoped "around" the torus and into the cone. It follows that X is semilocally π_n-trivial. We conclude from Theorem 1.2 that $\Psi_n:\pi_n(X)\to\check{\pi}_n(X)$ is an isomorphism. Since sufficiently fine covers of X always give nerves homotopy equivalent to $S^1\vee S^{n+1}$, we have $\check{\pi}_n(X)=0$. Thus $\pi_n(X)=0$.

Example 7.4. Let $n\geq 2$ and $X=\mathbb{E}_1\vee S^n$ (see Figure 2). Note that because $\mathbb{E}_1$ is aspherical [Cannon et al. 2002; Curtis and Fort 1957], X is semilocally π_n-trivial. However, X is not LC^1 because it has $\mathbb{E}_1$ as a retract. It is shown in [Brazas 2021] that $\pi_n(X)\cong\bigoplus_{\pi_1(\mathbb{E}_1)}\pi_n(S^n)\cong\bigoplus_{\pi_1(\mathbb{E}_1)}\mathbb{Z}$ and that $\Psi_n:\pi_n(X)\to\check{\pi}_n(X)$ is injective. In particular, we may represent elements of $\pi_n(X)$ as finite-support sums $\sum_{\beta\in\pi_1(\mathbb{E}_1)}m_\beta$ where $m_\beta\in\mathbb{Z}$. We show that Ψ_n is not surjective.

Identify $\pi_1(X)$ with $\pi_1(\mathbb{E}_1)$ and recall from [Eda 1992] that we can represent the elements of $\pi_1(\mathbb{E}_1)$ as countably infinite reduced words indexed by a countable linear order (with a countable alphabet $\beta_1,\beta_2,\beta_3,\dots$). Here β_j is represented by a loop $S^1\to\mathbb{E}_1$ going once around the j-th circle. Let X_j be the union of S^n and the largest j circles of $\mathbb{E}_1$ so that $X=\varprojlim_j X_j$. Identify $\pi_1(X_j)$ with the free group F_j on generators $\beta_1,\beta_2,\dots\beta_j$ and note that $\pi_n(X_j)\cong\bigoplus_{F_j}\mathbb{Z}$. Thus we may view an element of $\pi_n(X_j)$ as a finite-support sums $\sum_{w\in F_j}m_w$ of integers indexed over reduced words in F_j. Let $d_{j+1,j}:F_{j+1}\to F_j$ be the homomorphism that deletes the

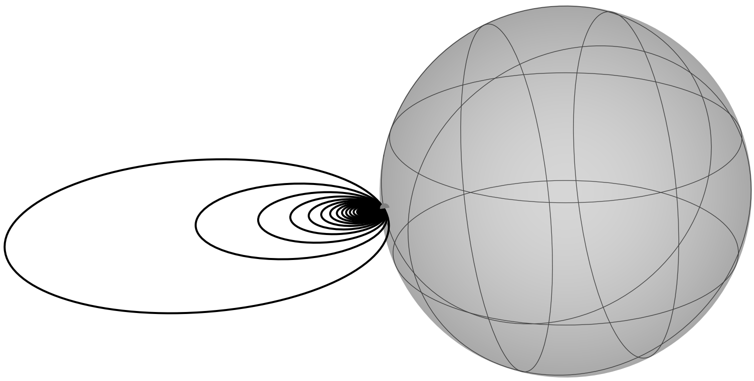

Figure 2. The one point union $\mathbb{E}_1 \vee S^2$.

letter β_{j+1}. Consider the inverse limit $\check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$. The map $X \to X_j$ that collapses all but the first j-circles of $\mathbb{E}_1$ induces a homomorphism $d_j : \pi_1(X) \to F_j$. There is a canonical homomorphism $\phi : \pi_1(X) \to \check{\pi}_1(X) = \varprojlim_j (F_j, d_{j+1,j})$ given by $\phi(\beta) = (d_1(\beta), d_2(\beta), \dots)$, which is known to be injective [Morgan and Morrison 1986] but not surjective. For example, if $x_k = \prod_{j=1}^{k} [\beta_1, \beta_j]$, then $(x_1, x_2, x_3, x_4, \dots)$ is an element of $\check{\pi}_1(X)$ not in the image of ϕ.

The bonding map $b_{j+1,j} : \pi_n(X_{j+1}) \to \pi_n(X_j)$ sends a sum $\sum_{w \in F_{j+1}} m_w$ to $\sum_{v \in F_j} p_v$ where $p_v = \sum_{d_{j+1,j}(w)=v} m_w$. Similarly, projection map $b_j : \pi_n(X) \to \pi_n(X_j)$ sends the sum $\sum_{\beta \in \pi_1(X)} n_\beta$ to $\sum_{v \in F_j} m_v$ where $m_v = \sum_{d_j(\beta)=v} m_\beta$. Let $y_j \in \pi_n(X)$ be the sum whose only nonzero coefficient is the x_j-coefficient, which is 1. Since $d_{j+1,j}(x_{j+1}) = x_j$, it's clear that $(y_1, y_2, y_3, \dots) \in \check{\pi}_n(X)$. Suppose $\Psi_n\big(\sum_\beta m_\beta\big) = (y_1, y_2, y_3, \dots)$. Writing $\sum_\beta m_\beta$ as a finite sum $\sum_{i=1}^{r} m_{\beta_i}$ for nonzero m_{β_i}, we must have $\sum_{d_j(\beta_i)=x_j} m_{\beta_i} = 1$ for all $j \in \mathbb{N}$. Since there are only finitely many β_i involved, there must exist at least one i for which $d_j(\beta_i) = x_j$ for infinitely many j. For such i, we have $\phi(\beta_i) = (x_1, x_2, x_3, \dots)$, which, as mentioned above, is impossible. Hence Ψ_n is not surjective.

The previous example shows why we cannot remove the LC^{n-1} hypothesis in Theorem 1.2. Since we weakened the hypothesis from [Kozlowski and Segal 1978] in dimension n and no hypothesis in dimension n is required for Theorem 1.1, one might suspect that we might be able to remove the dimension n hypothesis completely. The next example, which is a higher analogue of the harmonic archipelago [Bogley and Sieradski 1998; Conner et al. 2015; Karimov and Repovš 2012] shows why this is not possible.

Example 7.5. Let $n \geq 2$ and $\ell_j : S^n \to \mathbb{E}_n$ be the inclusion of the j-th n-sphere in $\mathbb{E}_n$. Let X be the space obtained by attaching $(n+1)$-cells to $\mathbb{E}_n$ using the attaching maps ℓ_j. Since $\mathbb{E}_n$ is LC^{n-1}, it follows that X is LC^{n-1}. However, X is not semilocally π_n-trivial at the wedgepoint $\boldsymbol{o}$ of $\mathbb{E}_n$. Indeed, the infinite concatenation maps $\prod_{j\geq k}\ell_j=\ell_k\cdot\ell_{k+1}\cdots$ are not null-homotopic (using a standard argument that works for the harmonic archipelago) but are all homotopic to each other. Thus, $\pi_n(X,\boldsymbol{o})\neq 0$. However, for sufficiently fine open covers $\mathscr{U}\in O(X)$, $|N(\mathscr{U})|$ is homotopy equivalent to a wedge of $(n+1)$-spheres and thus $\check{\pi}_n(X,\boldsymbol{o})=0$. Therefore, despite X being LC^{n-1}, Ψ_n is not an isomorphism. In fact, $\pi_n(X,\boldsymbol{o})=\pi_n^{Sp}(X,\boldsymbol{o})=\ker(\Psi_n)$. The reader might also note that since $\mathbb{E}_{n-1}$ is $(n-1)$-connected and $\pi_n(\mathbb{E}_n,\boldsymbol{o})\cong H_n(\mathbb{E}_n)\cong\mathbb{Z}^{\mathbb{N}}$, X will also be $(n-1)$-connected. A Meyer–Vietoris sequence argument similar to that in [Karimov and Repovš 2012] can then be used to show $\pi_n(X,\boldsymbol{o})\cong H_n(X)\cong\mathbb{Z}^{\mathbb{N}}/\bigoplus_{\mathbb{N}}\mathbb{Z}$.

References

[Bahredar et al. 2021] A. A. Bahredar, N. Kouhestani, and H. Passandideh, "The n-dimensional Spanier group", *Filomat* **35**:9 (2021), 3169–3182. MR

[Bogley and Sieradski 1998] W. A. Bogley and A. J. Sieradski, "Universal path spaces", preprint, 1998.

[Brazas 2021] J. Brazas, "Sequential n-connectedness and infinite factorization in higher homotopy groups", preprint, 2021. arXiv 2103.13456

[Brazas and Fabel 2014] J. Brazas and P. Fabel, "Thick Spanier groups and the first shape group", *Rocky Mountain J. Math.* **44**:5 (2014), 1415–1444. MR Zbl

[Cannon and Conner 2006] J. W. Cannon and G. R. Conner, "On the fundamental groups of one-dimensional spaces", *Topology Appl.* **153**:14 (2006), 2648–2672. MR Zbl

[Cannon et al. 2002] J. W. Cannon, G. R. Conner, and A. Zastrow, "One-dimensional sets and planar sets are aspherical", *Topology Appl.* **120**:1-2 (2002), 23–45. MR Zbl

[Conner et al. 2015] G. R. Conner, W. Hojka, and M. Meilstrup, "Archipelago groups", *Proc. Amer. Math. Soc.* **143**:11 (2015), 4973–4988. MR Zbl

[Curtis and Fort 1957] M. L. Curtis and M. K. Fort, Jr., "Homotopy groups of one-dimensional spaces", *Proc. Amer. Math. Soc.* **8** (1957), 577–579. MR Zbl

[Eda 1992] K. Eda, "Free σ-products and non-commutatively slender groups", *J. Algebra* **148**:1 (1992), 243–263. MR Zbl

[Eda and Kawamura 1998] K. Eda and K. Kawamura, "The fundamental groups of one-dimensional spaces", *Topology Appl.* **87**:3 (1998), 163–172. MR Zbl

[Eda and Kawamura 2000a] K. Eda and K. Kawamura, "Homotopy and homology groups of the n-dimensional Hawaiian earring", *Fund. Math.* **165**:1 (2000), 17–28. MR Zbl

[Eda and Kawamura 2000b] K. Eda and K. Kawamura, "The surjectivity of the canonical homomorphism from singular homology to Čech homology", *Proc. Amer. Math. Soc.* **128**:5 (2000), 1487–1495. MR Zbl

[Eda and Kawamura 2010] K. Eda and K. Kawamura, "On the asphericity of one-point unions of cones", *Topology Proc.* **36** (2010), 63–75. MR Zbl

[Eda et al. 2013] K. Eda, U. H. Karimov, D. Repovš, and A. Zastrow, "On snake cones, alternating cones and related constructions", *Glas. Mat. Ser. III* **48(68)**:1 (2013), 115–135. MR Zbl

[Fischer and Guilbault 2005] H. Fischer and C. R. Guilbault, "On the fundamental groups of trees of manifolds", *Pacific J. Math.* **221**:1 (2005), 49–79. MR Zbl

[Fischer and Zastrow 2005] H. Fischer and A. Zastrow, "The fundamental groups of subsets of closed surfaces inject into their first shape groups", *Algebr. Geom. Topol.* **5** (2005), 1655–1676. MR Zbl

[Fischer and Zastrow 2007] H. Fischer and A. Zastrow, "Generalized universal covering spaces and the shape group", *Fund. Math.* **197** (2007), 167–196. MR Zbl

[Fischer et al. 2011] H. Fischer, D. Repovš, Ž. Virk, and A. Zastrow, "On semilocally simply connected spaces", *Topology Appl.* **158**:3 (2011), 397–408. MR Zbl

[Ghane and Hamed 2009] F. Ghane and Z. Hamed, "n-homotopically Hausdorff spaces", lecture notes, Univ. Kurdistan, 2009, available at https://profdoc.um.ac.ir/paper-abstract-1012363.html.

[Karimov and Repovš 2012] U. H. Karimov and D. Repovš, "On the homology of the harmonic archipelago", *Cent. Eur. J. Math.* **10**:3 (2012), 863–872. MR Zbl

[Kawamura 2003] K. Kawamura, "Low dimensional homotopy groups of suspensions of the Hawaiian earring", *Colloq. Math.* **96**:1 (2003), 27–39. MR Zbl

[Kozlowski and Segal 1977] G. Kozlowski and J. Segal, "Locally well-behaved paracompacta in shape theory", *Fund. Math.* **95**:1 (1977), 55–71. MR Zbl

[Kozlowski and Segal 1978] G. Kozlowski and J. Segal, "Local behavior and the Vietoris and Whitehead theorems in shape theory", *Fund. Math.* **99**:3 (1978), 213–225. MR Zbl

[Kuperberg 1975] K. Kuperberg, "Two Vietoris-type isomorphism theorems in Borsuk's theory of shape, concerning the Vietoris–Cech homology and Borsuk's fundamental groups", pp. 285–313 in *Studies in topology* (Charlotte, NC, 1974), edited by N. M. Stavrakas and K. R. Allen, Academic Press, New York, 1975. MR Zbl

[Mardešić 1959] S. Mardešić, "Comparison of singular and Čech homology in locally connected spaces", *Michigan Math. J.* **6** (1959), 151–166. MR Zbl

[Mardešić and Segal 1982] S. Mardešić and J. Segal, *Shape theory: the inverse system approach*, North-Holland Math. Lib. **26**, North-Holland, Amsterdam, 1982. MR Zbl

[Morgan and Morrison 1986] J. W. Morgan and I. Morrison, "A van Kampen theorem for weak joins", *Proc. Lond. Math. Soc.* (3) **53**:3 (1986), 562–576. MR Zbl

[Munkres 1984] J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley, Menlo Park, CA, 1984. MR Zbl

[Spanier 1966] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR Zbl

[Stone 1948] A. H. Stone, "Paracompactness and product spaces", *Bull. Amer. Math. Soc.* **54** (1948), 977–982. MR Zbl

[Virk 2020] Ž. Virk, "1-dimensional intrinsic persistence of geodesic spaces", *J. Topol. Anal.* **12**:1 (2020), 169–207. MR Zbl

[Willard 1970] S. Willard, *General topology*, Addison-Wesley, Reading, MA, 1970. MR Zbl

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