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REGULARITY FOR FREE MULTIPLICATIVE CONVOLUTION ON THE UNIT CIRCLE

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Suppose that μ_1 and μ_2 are Borel probability measures on the unit circle, both different from unit point masses, and let μ denote their free multiplicative convolution. We show that μ has no continuous singular part (relative to arclength measure) and that its density can only be locally unbounded at a finite number of points, entirely determined by the point masses of μ_1 and μ_2. Analogous results were proved earlier for the free additive convolution on $\mathbb{R}$ and for the free multiplicative convolution of Borel probability measures on the positive half-line.

1. Introduction

It has been known for some time that free convolutions have a strong regularizing effect. The earliest instances of this phenomenon were observed in [Voiculescu 1993; Bercovici and Voiculescu 1998; Biane 1997]. For the additive case (see [Voiculescu 1986; Bercovici and Voiculescu 1993; Voiculescu et al. 1992] for definitions), it was shown in [Belinschi 2008; 2014] that, given Borel probability measures μ_1, μ_2 on $\mathbb{R}$, neither of which is a point mass, the free convolution $\mu = \mu_1 \boxplus \mu_2$ has no singular continuous part relative to the Lebesgue measure, and its density is analytic wherever positive and finite. In addition, this density is locally bounded unless $\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) \geq 1$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$. The atomic part of μ has finite support and was determined earlier [Bercovici and Voiculescu 1998]. Analogous results have been obtained in [Ji 2021] for the free multiplicative convolution of Borel probability measures on $[0, +\infty)$. Despite a strong similarity between these operations, the corresponding result for free multiplicative convolutions of Borel probability measures on the unit circle $\mathbb{T}$ in the complex plane is still missing. Recent results on Denjoy–Wolff points [Belinschi et al. 2022, Corollary 3.3] allow us to rectify this omission in Theorem 3.2.

The necessary background on subordination is given in Section 2, and the main result is proved in Section 3. An application in Section 4 yields a strengthening of

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the results of [Bercovici and Wang 2008] concerning indecomposable measures relative to free convolution.

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2. Analytic subordination for free multiplicative convolution

We begin by recalling the analytical apparatus for the calculation of free multiplicative convolutions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$. An arbitrary Borel probability measure μ on $\mathbb{T}$ is uniquely determined by its *moments*

$$m_n(\mu) = \int_{\mathbb{T}} t^n \, d\mu(t), \quad n \in \mathbb{N},$$

and these moments are encoded in the *moment generating function*

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{tz}{1-tz} \, d\mu(t) = \sum_{n=1}^{\infty} m_n(\mu) z^n.$$

The formal series ψ_μ actually converges for z in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, and

$$\psi_\mu(\mathbb{D}) \subset \left\{ z \in \mathbb{C} \colon \Re z > -\tfrac{1}{2} \right\}.$$

Observe that

$$2\Re\psi_\mu(z) + 1 = \int_{\mathbb{T}} \Re\left(\frac{\bar{\zeta} + z}{\bar{\zeta} - z}\right) d\mu(\zeta) = \int_{\mathbb{T}} \Re\left(\frac{\zeta + z}{\zeta - z}\right) d\mu(\bar{\zeta}), \quad z \in \mathbb{D}, \tag{2-1}$$

and the last term above is precisely a Poisson integral. It follows that μ can be recovered from ψ_μ by taking radial limits

$$2\pi \, d\mu(e^{-i\theta}) = \lim_{r \uparrow 1} \bigl(2\Re\psi_\mu(e^{i\theta}) + 1\bigr) \, d\theta.$$

(See, for instance, [Akhiezer 1965, Chapter 5], [Belinschi and Bercovici 2005, Section 3], and [Garnett 1981, Chapter 1] for details.) In particular, if μ^{s} denotes the singular part of the measure μ, (2-1) shows that

$$\lim_{r \uparrow 1} \Re\psi_\mu(r\bar{\zeta}) = +\infty \quad \text{for } \mu^{\mathrm{s}}\text{-almost all } \zeta \in \mathbb{T}. \tag{2-2}$$

We note for further use the following consequence of (2-1):

Lemma 2.1. *If ψ_μ is a bounded function on $\mathbb{D}$, then μ is absolutely continuous relative to arclength measure and its density is bounded.*

Consider now two Borel probability measures μ_1, μ_2 on $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, and denote by $\mu = \mu_1 \boxtimes \mu_2$ their free multiplicative convolution. This was first defined in [Voiculescu 1987] using the multiplication of *-free unitary operators, and its calculation — in case the two measures have a nonzero first moment — relied on the analytic inverses of the functions ψ_{μ_1} and ψ_{μ_2} in the complex plane (see [Voiculescu et al. 1992] for the technical details). Subsequently, Biane [1998] discovered that ψ_μ is subordinate to ψ_{μ_j}, with $j = 1, 2$, in the sense of Littlewood. This result implies that — at least when μ_1 and μ_2 have nonzero first moments — one can describe the function ψ_μ as the unique solution of a system of implicit equations. This method for the calculation of ψ_μ does in fact extend to arbitrary μ_1 and μ_2, as seen in [Belinschi and Bercovici 2007]. We state the result below because it is instrumental in the proof of Theorem 3.2. We need the additional notation

$$\eta_\mu(z) = \frac{\psi_\mu(z)}{1+\psi_\mu(z)} \quad \text{and} \quad h_\mu(z) = \frac{\eta_\mu(z)}{z}.$$

It is easily seen that $\eta_\mu(\mathbb{D}) \subset \mathbb{D}$, $\eta_\mu(0) = 0$, $\eta'_\mu(0) = m_1(\mu)$, and h_μ extends to an analytic function from $\mathbb{D}$ to $\overline{\mathbb{D}}$. If the function h_μ takes values in $\mathbb{T}$, then it is constant and this happens precisely when μ is a point mass. The following statement combines [Belinschi and Bercovici 2007, Theorem 3.2] and [Belinschi et al. 2022, Corollary 3.3]:

Theorem 2.2. *Consider Borel probability measures μ_1, μ_2 on $\mathbb{T}$ and their free multiplicative convolution $\mu = \mu_1 \boxtimes \mu_2$. There exist unique continuous functions $\omega_1, \omega_2 : \mathbb{D} \cup \mathbb{T} \to \mathbb{D} \cup \mathbb{T}$ that are analytic on $\mathbb{D}$ and, in addition*:

(1) $\omega_1(0) = \omega_2(0) = 0$.

(2) $z\eta_\mu(z) = z\eta_{\mu_1}(\omega_1(z)) = z\eta_{\mu_2}(\omega_2(z)) = \omega_1(z)\omega_2(z)$, $\omega_1(z) = zh_2(\omega_2(z))$, *and* $\omega_2(z) = zh_1(\omega_1(z))$ *for every* $z \in \mathbb{D} \cup \mathbb{T}$. *In particular*, η_μ *extends continuously to* $\mathbb{T}$. *When either* $\omega_1(z)$ *or* $\omega_2(z)$ *belongs to* $\mathbb{T}$, *the values* $\eta_{\mu_j}(\omega_j(z))$ *are understood as radial limits*, *that is*,

$$\eta_{\mu_j}(\omega_j(z)) = \lim_{r \uparrow 1} \eta_{\mu_j}(r\omega_j(z)).$$

(3) *If* $m_1(\mu_1) = m_1(\mu_2) = 0$, *the functions* η_μ, ψ_μ, ω_1, *and* ω_2 *are identically zero.*

3. Boundedness and the lack of a singular continuous part

We are ready now to identify the singular behavior of a free multiplicative convolution on $\mathbb{T}$. Of course, part (1) was proved in [Belinschi 2003].

Lemma 3.1. *Suppose that μ_1 and μ_2 are Borel probability measures on $\mathbb{T}$, neither of which is a unit point mass, set $\mu = \mu_1 \boxtimes \mu_2$, and let $\alpha \in \mathbb{T}$.*

(1) *If* $\mu(\{\alpha\}) > 0$, *then there exist* $\alpha_1, \alpha_2 \in \mathbb{T}$ *such that* $\alpha_1\alpha_2 = \alpha$ *and*

$$\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) = 1 + \mu(\{\alpha\}).$$

(2) *If* ψ_μ *is unbounded near* $1/\alpha$, *then there exist* $\alpha_1, \alpha_2 \in \mathbb{T}$ *such that* $\alpha_1\alpha_2 = \alpha$ *and*

$$\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) \geq 1.$$

Proof. We only prove (2). As already mentioned, if $m_1(\mu_1) = m_1(\mu_2) = 0$, then μ is the Haar measure on $\mathbb{T}$, which has no singular part and a density identically equal to $1/2\pi$. Indeed, by Theorem 2.2 (3), ψ_μ is identically zero; in particular, bounded. For the remainder of the proof, we assume that at least one of $m_1(\mu_1), m_1(\mu_2)$ is nonzero, and thus the functions $\psi_\mu, \omega_1, \omega_2$ of Theorem 2.2 are not constant. Suppose now that $\beta = 1/\alpha$ is such that $\eta_\mu(\beta) = 1$ or, equivalently,

$$\psi_\mu(\beta) = \lim_{r\uparrow 1} \psi_\mu(r\beta) = \infty.$$

Setting $\alpha_1 = \omega_1(\beta)$ and $\alpha_2 = \omega_2(\beta)$,Theorem 2.2 (2) yields the equality $\alpha_1\alpha_2 = \beta$. Since $|\alpha_j| \le 1$, it follows that, in fact, $\alpha_j \in \mathbb{T}$ for $j = 1, 2$. The subordination in Theorem 2.2 (2) also yields

$$\lim_{z\to\beta} \eta_{\mu_j}(\omega_j(z)) = \eta_\mu(\beta) = 1, \quad j = 1, 2,$$

and then

$$\lim_{r\uparrow 1} \eta_{\mu_j}(r\alpha_j) = 1, \quad j = 1, 2,$$

by Lindelöf's Theorem (see [Collingwood and Lohwater 1966, Theorem 2.3]).

An application of the dominated convergence theorem shows that

$$\lim_{r\uparrow 1}(1-r)\psi_{\mu_j}(r\alpha_j) = \mu\Big(\Big\{\frac{1}{\alpha_j}\Big\}\Big) \in [0, 1), \quad j = 1, 2.$$

In terms of the functions η_{μ_j}, this amounts to

$$\lim_{r\uparrow 1} \frac{\eta_{\mu_j}(r\alpha_j) - 1}{r - 1} = \frac{1}{\mu_j(\{1/\alpha_j\})}, \quad j = 1, 2,$$

where the right-hand side is understood as ∞ if $\mu_j(\{1/\alpha_j\}) = 0$. Using Julia–Carathéodory derivatives (see, for instance, [Garnett 1981, Chapter I, Exercise 7]) this relation can be rewritten as $\eta'_\mu(\omega_1(\alpha)) = 1/(\mu_j(\{1/\alpha_j\}))$. Properties of this derivative imply now that

$$\begin{aligned}
\frac{1}{\mu_1(\{1/\alpha_1\})} - 1 &= \liminf_{w\to\alpha_1} \frac{|\eta_{\mu_1}(w)| - 1}{|w| - 1} - 1 \\
&= \liminf_{w\to\alpha_1} \frac{|\eta_{\mu_1}(w)| - |w|}{|w| - 1} \\
&\le \liminf_{z\to\beta} \frac{|\eta_{\mu_1}(\omega_1(z))| - |\omega_1(z)|}{|\omega_1(z)| - 1} \quad \text{(substituting } w = \omega_1(z)\text{)} \\
&= \liminf_{z\to\beta} \frac{|\omega_1(z)|}{|z|}\frac{|\omega_2(z)| - |z|}{|\omega_1(z)| - 1} \quad \text{(using Theorem 2.2)} \\
&= \liminf_{z\to\beta} \frac{|\omega_2(z)| - |z|}{|\omega_1(z)| - 1} \\
&\le \liminf_{z\to\beta} \frac{1 - |\omega_2(z)|}{1 - |\omega_1(z)|}.
\end{aligned}$$

Switching the roles of μ_1 and μ_2, we obtain

$$\begin{aligned}\frac{1}{\mu_2(\{1/\alpha_2\})}-1 &\le \liminf_{z\to\beta}\frac{1-|\omega_1(z)|}{1-|\omega_2(z)|}\\ &=\left[\limsup_{z\to\beta}\frac{1-|\omega_2(z)|}{1-|\omega_1(z)|}\right]^{-1}\\ &\le\left[\liminf_{z\to\beta}\frac{1-|\omega_2(z)|}{1-|\omega_1(z)|}\right]^{-1}\\ &\le\left[\frac{1}{\mu_1(\{1/\alpha_1\})}-1\right]^{-1}.\end{aligned}$$

A simple calculation shows now that the inequality

$$\left(\frac{1}{\mu_2(\{1/\alpha_2\})}-1\right)\left(\frac{1}{\mu_1(\{1/\alpha_1\})}-1\right)\le 1$$

is equivalent to $\mu_1(\{1/\alpha_1\})+\mu_2(\{1/\alpha_2\})\ge 1$, thus concluding the proof. □

We are now ready to state and prove the main result of this paper.

Theorem 3.2. *Consider the Borel probability measures μ_1, μ_2 on $\mathbb{T}$ and their free multiplicative convolution $\mu=\mu_1\boxtimes\mu_2$. Suppose that neither μ_1 nor μ_2 is a point mass. Then*:

(1) *The singular continuous part of μ relative to the arclength measure is zero.*

(2) *If we have*

$$\max\bigl\{\mu_1(\{\alpha_1\})+\mu_2(\{\alpha_2\}) : \alpha_1,\alpha_2\in\mathbb{T}\bigr\}\le 1, \tag{3-1}$$

then μ is absolutely continuous relative to the arclength measure.

(3) *If* (3-1) *is strict, then the density of μ relative to the arclength measure is bounded.*

Remark 3.3. It is remarkable that, for all free convolutions (see [Belinschi 2014; Ji 2021]), only the atomic parts of μ_1, μ_2 have an impact on the local boundedness of the density of their convolution.

Proof. The set $\{(\alpha_1,\alpha_2)\in\mathbb{T}^2 : \mu_1(\{\alpha_1\})+\mu_2(\{\alpha_2\})\ge 1\}$ is obviously finite. By Lemma 3.1 (2), the set $S=\{\alpha\in\mathbb{T} : \eta_\mu(\{1/\alpha\})=1\}$ is finite as well. Since (2-2) implies that the support of the singular summand of μ is contained in S, it follows that this summand is a finite sum of point masses. This proves (1). Suppose now that (3-1) holds. Then Lemma 3.1 (1) shows that μ is absolutely continuous. Finally, suppose that (3-1) is strict. Then Lemma 3.1 (2) implies that η_μ does not take the value 1 at any point on $\mathbb{T}$. Since η_μ is continuous on $\overline{\mathbb{D}}$, it must be bounded away from 1. Thus $\psi_\mu=\eta_\mu/(1-\eta_\mu)$ is a bounded function. Then (3) follows from Lemma 2.1. □

Remark 3.4. Suppose that $\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) = 1$ for some $\alpha_1, \alpha_2 \in \mathbb{T}$. It was shown in [Belinschi 2003] that, setting $\beta_j = 1/\alpha_j$ and $\beta = \beta_1\beta_2$, we have $\omega_j(\beta) = \beta_j$ for $j = 1, 2$, but, of course, $\mu(\{1/\beta\}) = 0$. (This can also be proved using the results of [Belinschi et al. 2022] and the "chain rule" for Julia–Carathéodory derivatives.) In all computable examples, the density of μ is unbounded near $1/\beta$. We suspect that this is true in full generality.

4. An application

The following statement extends the main result of [Bercovici and Wang 2008] for probability measures on the circle. Nearly identical proofs yield the corresponding extensions for free additive convolutions and for free multiplicative convolutions on the positive half-line. For these two convolutions, it is not necessary to assume that one of the convolved measures has more than two points in its support. The condition $\eta_\mu(\alpha) = 1$ in the statement amounts to the requirement that either γ is an atom of μ, or the density of μ is unbounded near γ (or both).

Theorem 4.1. *Consider Borel probability measures μ_1, μ_2 on $\mathbb{T}$, different from point masses, and set $\mu = \mu_1 \boxtimes \mu_2$. Suppose that $J \subset \mathbb{T}$ is an open arc such that each endpoint α of J satisfies $\eta_\mu(\alpha) = 1$. If either μ_1 or μ_2 has more than two points in its support, then $\mu(J) > 0$.*

Proof. Let α and β be the two endpoints of J, and let ω_j denote the subordination function of η_μ relative to η_{μ_j}. By Lemma 3.1, the points $\alpha_j = \omega_j(\alpha)$ and $\beta_j = \omega_j(\beta)$ satisfy

$$\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) \geq 1 \quad \text{and} \quad \mu_1(\{\beta_1\}) + \mu_2(\{\beta_2\}) \geq 1.$$

The hypothesis implies that either $\alpha_1 = \beta_1$ or $\alpha_2 = \beta_2$. Indeed, otherwise, it would follow that the support of μ_j is $\{\alpha_j, \beta_j\}$, for $j = 1, 2$. Switching, if necessary, the roles of μ_1 and μ_2, we may assume that $\alpha_1 = \beta_1$, so $\omega_1(\alpha) = \omega_1(\beta)$.

Suppose now that $\mu(J) = 0$. Then $|\eta_\mu(\zeta)| = 1$ for every $\zeta \in J$. The equation $\eta_\mu(\zeta) = \eta_{\mu_1}(\omega_1(\zeta))$ and the Schwarz lemma (which applies because $\eta_\mu(0) = 0$), imply that

$$|\eta_\mu(z)| \leq |\omega_1(z)|$$

for every $z \in \mathbb{D}$. Letting z approach a point $\zeta \in J$, we see that $|\omega_1(\zeta)| = 1$. Now, ω_1 is not constant, and therefore $\omega_1(\zeta)$ moves counterclockwise as $\zeta \in J$ does so. By the Schwarz reflection principle, ω_1 is analytic and, thanks to the Julia–Carathéodory Theorem, it is locally injective on J. The equation $\omega_1(\alpha) = \omega_1(\beta)$ allows us to conclude that $\omega_1(J) \supseteq \mathbb{T} \setminus \{\omega_1(\alpha)\}$. Moreover, the fact that $|\eta_{\mu_1}(\omega_1(\zeta))| = 1$ for $\zeta \in J$ shows that the support of μ_1 is contained in $\mathbb{T} \setminus \omega_1(J) \subseteq \{\omega_1(\alpha)\}$, contrary to the hypothesis. This contradiction yields the desired conclusion that $\mu(J) \neq 0$. □

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