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We prove that the maximum length of an irredundant base for a primitive action of a finite simple group of Lie type is bounded above by a function which is a polynomial in the rank of the group. We give examples to show that this type of upper bound is best possible.

1. Introduction

1A. ***Main results.*** Let G be a group acting on a set Ω. Let ℓ be a nonnegative integer and let $\Lambda = [\omega_1, \dots, \omega_\ell]$ be a sequence of points $\omega_1, \dots, \omega_\ell$ drawn from Ω; we write $G_{(\Lambda)}$ or $G_{\omega_1,\omega_2,\dots,\omega_\ell}$ for the pointwise stabilizer. If $\ell = 0$, so Λ is empty, then we set $G_{(\Lambda)} = G$.

The sequence Λ is called a *base* if $G_{(\Lambda)} = \{1\}$; the sequence Λ is called *irredundant* if

$$G_{\omega_1,\dots,\omega_{k-1}} \gneq G_{\omega_1,\dots,\omega_{k-1},\omega_k}$$

for all $k = 1, \dots, \ell$. The size of the longest possible irredundant base is denoted $\mathrm{I}(G, \Omega)$.

The main result of this paper shows that for any primitive action of a simple group of Lie type, the size of an irredundant base is bounded by a polynomial function of the rank of the group.

Theorem 1. *If G is a simple group of Lie type of rank r acting primitively on a set Ω, then $\mathrm{I}(G, \Omega) \le Cr^8$, where C is an absolute constant. This holds with $C = 174$.*

The degree 8 of the polynomial bound is probably far from sharp but, as discussed in Section 1B, there are examples showing that this degree must be at least 2. Also there is no general complementary lower bound for $\mathrm{I}(G, \Omega)$ that grows with r, as shown by Example 4.5.

An upper bound on $\mathrm{I}(G, \Omega)$ implies an upper bound on a host of other statistics associated with the action of G on Ω. Consider, again, the sequence Λ, defined above. We call Λ a *minimal base* if it is a base and, furthermore, no proper

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subsequence of Λ is a base. We denote the minimum size of a minimal base $\mathrm{b}(G, \Omega)$, and the maximum size of a minimal base $\mathrm{B}(G, \Omega)$.

We say that Λ is *independent* if, for all $k = 1, \dots, \ell$, we have $G_{(\Lambda)} \neq G_{(\Lambda \setminus \omega_k)}$. We define the *height* of G to be the maximum size of an independent sequence, and we denote this quantity $\mathrm{H}(G, \Omega)$.

The last statistic of interest to us is the *relational complexity* of the action of G on Ω, denoted $\mathrm{RC}(G, \Omega)$. The definition of this is slightly involved and can be found in [8] where it is given the name *arity*.

It is easy to verify the inequalities [10]

$$\mathrm{b}(G, \Omega) \leq \mathrm{B}(G, \Omega) \leq \mathrm{H}(G, \Omega) \leq \mathrm{I}(G, \Omega). \tag{1-1}$$

Less obvious, but still rather elementary is the inequality [10]

$$\mathrm{RC}(G, \Omega) \leq \mathrm{H}(G, \Omega) + 1. \tag{1-2}$$

Theorem 1 and inequalities (1-1) and (1-2) immediately yield the following corollary.

Corollary 2. *If G is simple of Lie type of rank r acting primitively on a set Ω, then each of* $\mathrm{b}(G, \Omega)$, $\mathrm{B}(G, \Omega)$, $\mathrm{H}(G, \Omega)$ *and* $\mathrm{I}(G, \Omega)$ *is at most Cr^8 while* $\mathrm{RC}(G, \Omega)$ *is less than $Cr^8 + 1$, where C is as in Theorem 1.*

We can also deduce an upper bound for primitive actions of *almost* simple groups:

Corollary 3. *Let G be an almost simple group, with socle a simple group of Lie type of rank r over $\mathbb{F}_q$, where $q = p^f$ (p prime). If G acts primitively on a set Ω, then*

$$\mathrm{I}(G, \Omega) \leq 177r^8 + \pi(f),$$

where $\pi(f)$ is the number of primes, counted with multiplicity, dividing the integer f.

Example 5.1 shows that the term $\pi(f)$ in the upper bound cannot be avoided.

Our main tool for proving Theorem 1 is the following result on maximal subgroups of finite groups of Lie type. In the statement, we let $G(q) = (\overline{G}^F)'$ be a simple group of Lie type over $\mathbb{F}_q$, where $\overline{G}$ is the corresponding simple adjoint algebraic group over $\overline{\mathbb{F}_q}$ and F is a Frobenius endomorphism. Let p be the characteristic of $\mathbb{F}_q$. For a rational representation $\rho : \overline{G} \mapsto GL_n(\overline{\mathbb{F}_q})$, and a closed subgroup $\overline{H}$ of $\overline{G}$, we define $\deg_\rho(\overline{H})$ to be the degree of the image $\rho(\overline{H})$ as a subvariety of $\mathrm{GL}_n(\overline{\mathbb{F}_q})$. We give some basic definitions and results about degree in Section 2. Also denote by $\overline{H}^0$ the connected component of $\overline{H}$.

Theorem 4. *Let $G(q) = (\overline{G}^F)'$ be a finite simple group of Lie type as above, and let G be an almost simple group with socle $G(q)$. Let M be a maximal subgroup of G, and set $M_0 = M \cap G(q)$. Let $d = \dim \overline{G}$. Then one of the following holds:*

(1) $M_0 = \overline{M}^F \cap G(q)$, *where* $\overline{M}$ *is a closed* F*-stable subgroup of* $\overline{G}$ *of positive dimension*; *moreover*,

(a) $|\overline{M} : \overline{M}^0| \leq |W(\overline{G})|$, *the order of the Weyl group of* $\overline{G}$, *and*

(b) *excluding the cases where* $(\overline{G}, \overline{M}, p) = (C_r, D_r, 2)$ *or* $(C_3, G_2, 2)$, *if we let* $\mathrm{ad} : \overline{G} \mapsto \mathrm{GL}(L(\overline{G}))$ *be the adjoint representation*, *then*

$$\deg_{\mathrm{ad}}(\overline{M}) \leq |W(\overline{G})| \deg_{\mathrm{ad}}(\overline{G}) \leq |W(\overline{G})|\, 2^{d^2}.$$

(2) $M_0 = G(q_0)$, *a subgroup of the same type as* G (*possibly twisted*) *over a subfield* $\mathbb{F}_{q_0}$ *of* $\mathbb{F}_q$.

(3) $|M_0| \leq 2^{d^2}$.

1B. ***Context for, and possible improvements to, Theorem 1.*** We think of Theorem 1 as being a version of the Cameron–Kantor conjecture for irredundant bases. The Cameron–Kantor conjecture, which was stated in [6; 7] and proved in [20], asserts the existence of an absolute upper bound for $\mathrm{b}(G, \Omega)$ for the nonstandard actions of the almost simple groups. (A *standard action* of an almost simple group G with socle S is a transitive action where either $S = A_n$ and the action is on subsets or uniform partitions of $\{1, \dots, n\}$, or G is classical and the action is a subspace action.)

In Section 6 we explain exactly how Theorem 1 is connected to the Cameron–Kantor conjecture and we give a number of examples that clarify why Theorem 1 is, in a certain sense, the best possible "Cameron–Kantor-like statement" that can be made for irredundant bases. In particular, we give examples to show that:

(i) Even for nonstandard actions, the bound Cr^8 in Theorem 1 really needs to depend on r and is not absolute.

(ii) Theorem 1 only holds for primitive actions of *simple* groups of Lie type – it does not extend to actions of almost simple groups in general (although we do prove Corollary 3 for these).

(iii) Likewise, Theorem 1 does not extend to transitive actions of simple groups of Lie type in general.

Although (i) implies that the upper bound given in Theorem 1 is necessarily a function of r, it is undoubtedly true that the particular function of r we have given — $174r^8$ — can be improved. A construction of Freedman, Kelsey and Roney-Dougal (personal communication) implies that any polynomial upper bound must have degree at least 2; our guess is that an upper bound which is quadratic in r may hold in general.

A heuristic supporting this guess follows from the fact that $\mathrm{I}(G, \Omega) \leq \ell(G)$, where $\ell(G)$ is the maximum length of a subgroup chain in the simple group of Lie type G. Writing p for the field characteristic, U for a Sylow p-subgroup of G, and Φ^+ for the associated set of positive roots, we know that there exist constants c_1, c_2 such that

$$c_1 r^2 \log_p q \leq |\Phi^+| \log_p q = \ell(U) < \ell(G) < \log_2 |G| \leq c_2 r^2 \log_2 q.$$

More information about $\ell(G)$ can be found in [23].

Theorem 1 is the second recent success in trying to extend well-known results about bases to statements about irredundant bases; the first was achieved by Kelsey and Roney-Dougal [12] extending a result of Liebeck [14].

1C. ***Proofs and the structure of the paper.*** In Section 2 we present a number of definitions and results pertaining to the degree of an affine variety; these include, in particular, a statement of (one version of) Bézout's theorem on the degree of the intersection of a number of algebraic varieties.

In Section 3 we prove Theorem 4. The proof uses various results from the literature on the subgroup structure of algebraic groups [15; 18].

In Section 4 we prove Theorem 1; the proof makes use of both Theorem 4 and Bézout's theorem. Corollary 3 is deduced in Section 5.

The comparison of Theorem 1 with the Cameron-Kantor conjecture, and the relevant examples mentioned above, are given in Section 6.

2. Degree of an affine variety

Our proof of Theorem 1 is carried out by combining Theorem 4 with Bézout's theorem on the degree of the intersection of a number of algebraic varieties. We need a version of Bézout's theorem that holds for affine varieties and is due to Heintz [11].

In what follows we consider subsets of some affine space, $\mathbb{A}^n$, over an algebraically closed field k. A set X in $\mathbb{A}^n$ is called *locally closed* if $X = V \cap W$, where V is open and W is closed (in the Zariski topology). A set X is called *constructible* if it is a finite disjoint union of locally closed sets. Note that the intersection of a finite number of constructible sets is constructible. Note too that any variety in $\mathbb{A}^n$ is constructible. From here on X is a constructible set.

Definition 2.1 [11, Definition 1 and Remark 2]. If X is an irreducible variety of dimension r in $\mathbb{A}^n$, then the *degree* of X, written $\deg(X)$, is defined to be

$$\sup\{|E\cap X| : E \text{ is an } (n-r)\text{-dimensional affine subspace of } \mathbb{A}^n \text{ with } E \cap X \text{ finite}\}.$$

If X is a constructible set and $\mathcal{C}$ is the set of irreducible components of the closure of X, then we define

$$\deg(X) = \sum_{C\in\mathcal{C}} \deg(C). \tag{2-1}$$

Note that if X is an irreducible variety of dimension 0, then we have $\deg(X) = 1$. Thus, if X is any variety of dimension 0, irreducible or not, $\deg(X) = |X|$.

Now the main result that we need concerning degree is the following version of Bézout's theorem.

Proposition 2.2 [11, Theorem 1]. *Let X and Y be constructible sets in $\mathbb{A}^n$. Then*

$$\deg(X \cap Y) \leq \deg(X) \cdot \deg(Y).$$

This proposition obviously generalizes to the intersection of more than two varieties: If $X_1, X_2, \ldots, X_k$ are constructible sets in $\mathbb{A}^n$, then

$$\deg(X_1 \cap X_2 \cap \cdots \cap X_k) \leq \deg(X_1) \cdot \deg(X_2) \cdots \deg(X_k).$$

(We are implicitly using the fact that the intersection of two constructible sets is constructible.)

A useful corollary of Proposition 2.2 is the following fact connecting the degree of an affine variety to the degree of its defining polynomials. We make use of the fact, noted by Heintz [11, p. 247], that the degree of a hypersurface in $\mathbb{A}^n$ is equal to the degree of its defining polynomial.

Lemma 2.3. *Suppose that an affine variety X in $\mathbb{A}^n$ is defined by polynomials $f_1, \ldots, f_r$ of degree at most e. Then*

$$\deg(X) \leq e^r.$$

Proof. By definition $X = V(f_1, \ldots, f_r) = \bigcap_{i=1}^{r} V(f_i)$ where, for $i = 1, \ldots, r$, $V(f_i)$ is the hypersurface defined by the polynomial f_i. We noted that $\deg(V(f_i)) = \deg(f_i)$, and hence Proposition 2.2 implies that

$$\deg(X) \leq \deg(V(f_1)) \cdots \deg(V(f_r)) = \deg(f_1) \cdots \deg(f_r) \leq e^r. \qquad \square$$

As mentioned in the introduction, if $\overline{G}$ is an affine algebraic group over an algebraically closed field k, then for a rational representation $\rho : \overline{G} \mapsto GL_n(k)$, and a closed subgroup $\overline{H}$ of $\overline{G}$, we define $\deg_\rho(\overline{H})$ to be the degree of the image $\rho(\overline{H})$ as a subvariety of $\mathrm{GL}_n(k)$. From (2-1), we have

$$\deg_\rho(\overline{H}) = |\overline{H} : \overline{H}^0| \deg_\rho(\overline{H}^0) \geq \deg_\rho(\overline{H}^0). \tag{2-2}$$

3. Proof of Theorem 4

As in Theorem 4, let $G(q) = (\overline{G}^F)'$ be a simple group of Lie type over $\mathbb{F}_q$, where $\overline{G}$ is a simple algebraic group over $K = \overline{\mathbb{F}_q}$, and let G be an almost simple group with socle $G(q)$. Let M be a maximal subgroup of G, and set $M_0 = M \cap G(q)$. Let $d = \dim \overline{G}$ and let p be the characteristic of $\mathbb{F}_q$. Denote by $\mathrm{Lie}(p)$ the set of finite simple groups of Lie type over fields of characteristic p.

Suppose first that $G(q)$ is a classical group, so that $\overline{G}$ is the corresponding classical algebraic group. Let V be the natural module for $\overline{G}$, and let $n = \dim V$. We shall apply [15, Theorems 1′ and 2]. We postpone consideration of the cases where $G(q) = \mathrm{PSL}_n(q)$, $\mathrm{Sp}_4(2^e)$ or $P\Omega_8^+(q)$ and the group G contains an element in the coset of a graph automorphism (a triality graph automorphism in the last case). Assuming that these cases do not pertain, in [15], six classes $\mathcal{C}_i$ of closed subgroups of $\overline{G}$ are defined, and it is proved that one of the following holds:

(i) $M_0 = \overline{M}^F \cap G(q)$ for some F-stable member $\overline{M} \in \mathcal{C} := \bigcup_1^6 \mathcal{C}_i$.

(ii) $M_0 = G(q_0)$, a subgroup of the same type as $G(q)$ (possibly twisted) over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$.

(iii) M_0 is almost simple, and $F^*(M_0)$ is irreducible on V (and not of the same type as $G(q)$).

In case (ii), conclusion (2) of Theorem 4 holds.

Consider now case (i). The only finite members of $\mathcal{C}$ are:

- Subgroups of type $O_1(K) \operatorname{wr} S_n = 2^n.S_n$ in $O_n(K)$ with $p \neq 2$ (these lie in the class $\mathcal{C}_2$).
- Extraspecial-type subgroups $r^{2m}.\mathrm{Sp}_{2m}(r)$ (r prime, $n = r^m$) or $2^{2m}.O_{2m}^{\pm}(2)$ ($n = 2^m$) (these lie in the class $\mathcal{C}_5$).

A simple check shows that these subgroups have order less than 2^{d^2}, as required for conclusion (3) of Theorem 4.

All the other members of $\mathcal{C}$ are infinite, in which case

$$M_0 = \overline{M}^F \cap G(q), \tag{3-1}$$

where $\overline{M}$ is a maximal closed F-stable subgroup of $\overline{G}$ of positive dimension, as in (1) of Theorem 4.

Now consider case (iii) above. If $F^*(M) \notin \mathrm{Lie}(p)$, then an unpublished manuscript of Weisfeiler [24], subsequently improved and developed in [9], shows that $|M| < n^4(n+2)!$, which is less than 2^{d^2}, as in (3) of Theorem 4. And if $F^*(M) \in \mathrm{Lie}(p)$, then [22, Theorem 1] shows that (3-1) holds.

To complete the proof of Theorem 4 in the case where G is classical (apart from the postponed cases), it remains to prove the bounds for $|\overline{M} : \overline{M}^0|$, $\deg_{\mathrm{ad}}(\overline{M})$ and $\deg_{\mathrm{ad}}(\overline{G})$ for $\overline{M}$ in (1) of Theorem 4. The bound $|\overline{M} : \overline{M}^0| \leq |W(\overline{G})|$ follows by simply inspecting the structure of the members of $\mathcal{C}$; equality occurs when $\overline{M} = N_{\overline{G}}(T)$, where T is a maximal torus (these subgroups are in class $\mathcal{C}_2$ for $\mathrm{SL}(V)$ and $\mathrm{SO}(V)$).

To establish the degree bounds, we first prove:

Claim: Let $M_0 = \overline{M}^F \cap G(q)$ be as in (3-1). Then with two exceptions, $\overline{M}^0$ acts reducibly on some $\overline{G}$-composition factor of the adjoint module $L(\overline{G})$. The two exceptions are $(\overline{G}, \overline{M}, p) = (\mathrm{Sp}_n, \mathrm{SO}_n, 2)$ or $(\mathrm{Sp}_6, G_2, 2)$.

Proof of Claim. The composition factors of $L(\overline{G})$ are given in [16, Proposition 1.10]. Also $L(\overline{M}) \subseteq L(\overline{G})$. First consider $M_0 = \overline{M}^F \cap G(q)$ as in (i). Inspecting $\overline{M}^0$ for $\overline{M} \in \mathcal{C}$, we see that $L(\overline{M})$ maps to a proper subspace of some composition factor of $L(\overline{G})$, with the exception of $(\overline{G}, \overline{M}, p) = (\mathrm{Sp}_n, \mathrm{SO}_n, 2)$, proving the claim for M_0 as in (i). Finally, for M_0 as in (iii), the group $\overline{M}^0$ is simple, and [16, Theorem 4] shows that the only case where $L(\overline{M})$ does not map to a proper subspace of some composition factor of $L(\overline{G})$ is $(\overline{G}, \overline{M}, p) = (\mathrm{Sp}_6, G_2, 2)$. This completes the proof of the Claim.

We now use the Claim to deduce the required degree bounds. Let M, $\overline{M}$ be as in (3-1), and exclude the exceptions in the Claim, so that $\overline{M}^0$ acts reducibly on some composition factor of $L(\overline{G})$. If also $\overline{M}$ is reducible, then as it is maximal there is a subspace W of $L(\overline{G})$ such that

$$\overline{M} = \mathrm{stab}_{\overline{G}}(W).$$

This defines $\overline{M}$ by the polynomials defining $\overline{G}$ in the adjoint representation, together with some linear equations, and hence by Lemma 2.3, we have

$$\deg_{\mathrm{ad}}(\overline{M}) \leq \deg_{\mathrm{ad}}(\overline{G}).$$

On the other hand, if $\overline{M}$ acts irreducibly on every composition factor of $L(\overline{G})$, then by the Claim, there is a composition factor V such that $V \downarrow \overline{M}^0 = \bigoplus_1^t V_i$, where each V_i is irreducible for $\overline{M}^0$ and $t \geq 2$. Set

$$\overline{M}^1 = \bigcap_1^t \mathrm{stab}(V_i),$$

so that $\overline{M}^0 \leq \overline{M}^1 \lhd \overline{M}$. As above we see that $\deg_{\mathrm{ad}}(\overline{M}^1) \leq \deg_{\mathrm{ad}}(\overline{G})$, and so by the remarks after Lemma 2.3, we have $\deg_{\mathrm{ad}}(\overline{M}) \leq |\overline{M} : \overline{M}^1| \deg_{\mathrm{ad}}(\overline{G})$. We have seen that $|\overline{M} : \overline{M}^0| \leq |W(\overline{G})|$, so it follows that

$$\deg_{\mathrm{ad}}(\overline{M}) \leq |W(\overline{G})| \deg_{\mathrm{ad}}(\overline{G}),$$

as required for (1) of Theorem 4. Finally, in the adjoint representation, $\overline{G}$ is defined by d^2 quadratic polynomials expressing preservation of the Lie bracket on $L(\overline{G})$, so $\deg_{\mathrm{ad}}(\overline{G}) \leq 2^{d^2}$. Note that the exceptional cases $(\mathrm{Sp}_n, \mathrm{SO}_n, 2)$, $(\mathrm{Sp}_6, G_2, 2)$ in the Claim are also excepted in part (i)(b) of Theorem 4. Hence the proof of the theorem for G classical is now complete, apart from the postponed cases where $G(q) = \mathrm{PSL}_n(q)$, $\mathrm{Sp}_4(2^e)$ or $P\Omega_8^+(q)$ and G contains an element in the coset of a graph automorphism.

Now consider the excluded cases. Suppose first that $G(q) = \mathrm{PSL}_n(q)$. In this case, the collection $\mathcal{C}$ is extended in [15] to a collection $\mathcal{C}'$, and it is proved that conclusion (i), (ii) or (iii) above holds, with $\mathcal{C}'$ replacing $\mathcal{C}$. The only subgroups in $\mathcal{C}' \setminus \mathcal{C}$ are stabilizers of pairs $\{U, W\}$ of subspaces of V such that either $U \subseteq W$ or $V = U \oplus W$. The above proof shows that these subgroups satisfy (1) of Theorem 4. In the other cases, where $G(q) = \mathrm{Sp}_4(2^e)$ or $P\Omega_8^+(q)$, the maximal subgroups of G are listed in [1, Tables 8.14, 8.50]. Inspection of these lists shows that (1), (2) or (3) of Theorem 4 holds (using the same argument as above to bound the degree of $\overline{M}$). This completes the proof of Theorem 4 for $G(q)$ a classical group.

Suppose finally that $G(q)$ is an exceptional group of Lie type. The proof runs along similar lines. First we use [17, Theorem 8], which gives the possibilities for the maximal subgroup M. These are:

(i) $M_0 = \overline{M}^F \cap G(q)$, where $\overline{M}$ is a maximal closed F-stable subgroup of $\overline{G}$ of positive dimension.

(ii) $M_0 = G(q_0)$, a subgroup of the same type as G (possibly twisted) over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$.

(iii) M_0 is an “exotic local” subgroup:

$$3^3.\mathrm{SL}_3(3) < F_4, \quad 3^{3+3}.\mathrm{SL}_3(3) < E_6, \quad 5^3.\mathrm{SL}_3(5) < E_8 \quad \text{or} \quad 2^{5+10}.\mathrm{SL}_5(2) < E_8.$$

(iv) M_0 is the “Borovik subgroup” $(\mathrm{Alt}_5 \times \mathrm{Alt}_6).2^2 < E_8$.

(v) M_0 is almost simple with socle M_1, and one of the following holds:

(a) $M_1 \notin \mathrm{Lie}(p)$: the possibilities for M_0 are listed in [17, Theorem 4].

(b) $M_1 = M(q_1) \in \mathrm{Lie}(p)$, $\mathrm{rank}(M_1) \le \frac{1}{2}\,\mathrm{rank}(\overline{G})$ satisfying

- $q_1 \le 9$,
- $M_1 = A_2^{\pm}(16)$,
- M_1 has rank 1 and $q_1 \le (2, p-1)\cdot t(\overline{G})$, where $t(\overline{G}) = 12, 68, 124, 388, 1312$ according to $\overline{G} = G_2, F_4, E_6, E_7, E_8$, respectively.

In cases (iii), (iv) and (v) we check that $|M_0| < 2^{d^2}$, as in (3) of Theorem 4; and case (ii) is (2) of the theorem. Finally, in case (i), the list of possibilities for $\overline{M}$ is given in [17, Theorem 8]. We can check that $|\overline{M} : \overline{M}^0| \le |W(\overline{G})|$, and also that $\overline{M}^0$ acts reducibly on some $\overline{G}$-composition factor of $L(\overline{G})$ (see also [19] for this). Now we can argue exactly as in the classical case to obtain the required bounds on $\deg_{\mathrm{ad}}(\overline{G})$ for $\overline{M}$ for (1) of Theorem 4. This completes the proof of Theorem 4. □

4. Proof of Theorem 1

Let G be a simple group of Lie type of rank r over $\mathbb{F}_q$ with $G = (\overline{G}^F)'$, where $\overline{G}$ is the corresponding simple algebraic group over $\overline{\mathbb{F}_q}$ and F is a Frobenius endomorphism. Let $d = \dim \overline{G}$ and $p = \mathrm{char}(\mathbb{F}_q)$.

We write G_1 for a maximal subgroup of G. We consider the action of G on Ω, the set of cosets of G_1. We suppose that we have a stabilizer chain,

$$G > G_1 > G_2 > \cdots > G_k = \{1\} \tag{4-1}$$

where $G_i = G_{i-1} \cap G_1^{g_i}$ for some $g_i \in G$ $(i = 1, \ldots, k)$.

Theorem 4 gives three possibilities for G_1.

4A. ***Case 1 of Theorem 4.*** In this case we have $G_1 = \overline{G_1}^F \cap G$ where $\overline{G_1}$ is a closed F-stable subgroup of $\overline{G}$ of positive dimension. We start by proving three lemmas where, in fact, the maximality assumption for G_1 is not necessary.

Set ρ to be a rational representation of $\overline{G}$ and let c be an upper bound for $\deg_\rho(\overline{G_1})$; note that, by (2-2), we also have $|\overline{G_1} : (\overline{G_1})^0| \le c$.

For each $i = 2, \ldots, k$, we define $\overline{G_i} = \overline{G_{i-1}} \cap \overline{G_1}^{g_i}$ where g_i is the element of G mentioned above. Thus we have a chain of subgroups

$$\overline{G} > \overline{G_1} \ge \overline{G_2} \ge \cdots \ge \overline{G_k}. \tag{4-2}$$

Lemma 4.1. *The subgroups $G_1, \ldots, G_k$ in (4-1) satisfy $G_i = \overline{G_i}^F \cap G$ for each $i = 1, \ldots, k$.*

Proof. We proceed by induction on i. The result is true for $i = 1$. We assume the result is true for i and prove it for $i+1$. Note that $G_{i+1} = G_i \cap G_1^{g_{i+1}}$ and $\overline{G_{i+1}} = \overline{G_i} \cap \overline{G_1}^{g_{i+1}}$.

Let $x \in \overline{G_{i+1}}^F \cap G$. This is equivalent to

$$\begin{aligned} x \in (\overline{G_i} \cap \overline{G_1}^{g_{i+1}})^F \cap G &\Leftrightarrow x \in (\overline{G_i}^F \cap (\overline{G_1}^{g_{i+1}})^F) \cap G \\ &\Leftrightarrow x \in (\overline{G_i}^F \cap G) \cap ((\overline{G_1}^{g_{i+1}})^F \cap G) \\ &\Leftrightarrow x \in (\overline{G_i}^F \cap G) \cap ((\overline{G_1}^F)^{g_{i+1}} \cap G) \\ &\Leftrightarrow x \in (\overline{G_i}^F \cap G) \cap (\overline{G_1}^F \cap G)^{g_{i+1}} \\ &\Leftrightarrow x \in G_i \cap G_1^{g_{i+1}} = G_{i+1}. \qquad \square \end{aligned}$$

The lemma implies, in particular, that all of the containments in (4-2) are proper. Let $d_1 = \dim(\overline{G_1})$. Then of course $d_1 < d = \dim \overline{G}$. Note that $\overline{G_1}$ is the largest group in the chain (4-2) of dimension d_1.

Now let $k_1, \ldots, k_s$ be the points in the chain (4-2) where the dimension drops: that is, $k_1 = 1$, and for each $i \ge 2$, $\overline{G_{k_i}}$ is the largest group in the chain such that $\dim \overline{G_{k_i}} < \dim \overline{G_{k_i - 1}}$. Obviously $s \le d_1 + 1 \le d$.

Lemma 4.2. *We have* $\deg_\rho \overline{G_{k_i}} \le c^i$.

Proof. We proceed by induction on i. For $i = 1$, $\overline{G_{k_1}} = \overline{G_1}$ and this has degree at most c. We assume the result is true for i and prove it for $i+1$. In particular, this means that $\overline{G_{k_i}}$ has degree at most c^i. Consider the chain

$$\overline{G_{k_i}} > \overline{G_{k_i+1}} > \overline{G_{k_i+2}} > \cdots > \overline{G_{k_{i+1}}}.$$

Notice that, all but the last listed group have the same dimension, and so have the same identity component; what is more the number of components decreases as we descend the chain from $\overline{G_{k_i}}$ to $\overline{G_{k_{i+1}-1}}$. Thus (2-2) implies that

$$\deg_\rho(\overline{G_{k_{i+1}-1}}) \leq \deg_\rho(\overline{G_{k_i}}) \leq c^i.$$

Now $\overline{G_{k_{i+1}}}$ is the intersection of $\overline{G_{k_{i+1}-1}}$ and a conjugate of $\overline{G_1}$. The former has degree at most c^i, and the latter has degree at most c. Hence Proposition 2.2 implies that $\deg_\rho(\overline{G_{k_{i+1}}}) \leq c^{i+1}$, as required. □

Lemma 4.3. *The length k of the stabilizer chain (4-1) satisfies*

$$k \leq d + \tfrac{1}{2}d(d+1)\log_2 c.$$

Proof. The previous lemma asserts that the degree of $\overline{G_{k_i}}$ is at most c^i and so we also know that $|\overline{G_{k_i}} : (\overline{G_{k_i}})^0| \leq c^i$. Now, for each $i = 1, \dots, s$, we know that

$$\overline{G_{k_i}} > \overline{G_{k_i+1}} > \overline{G_{k_i+2}} > \cdots > \overline{G_{k_{i+1}-1}} \geq (\overline{G_{k_i}})^0,$$

where $\overline{G_{k_i}}^0$ is the identity component of all of the groups in this chain. Since $|\overline{G_{k_i}} : (\overline{G_{k_i}})^0| \leq c^i$, the length of the chain

$$\overline{G_{k_i}} > \overline{G_{k_i+1}} > \overline{G_{k_i+2}} > \cdots > \overline{G_{k_{i+1}-1}}$$

is at most $\log_2(c^i) = i \log_2 c$; in particular, for $i = 1, \dots, s$, the length of the chain from $\overline{G_{k_i}}$ to $\overline{G_{k_{i+1}}}$ is at most $i \log_2 c + 1$. There are two further parts of the chain that we have not considered.

First, at the top of the chain, the containment $G > G_1 = G_{k_1}$ adds 1 to the total length. Second, at the bottom of the chain, $\overline{G_{k_s}}$ is of dimension 0 and degree at most c^s; in other words $\overline{G_{k_s}}$ has cardinality at most c^s and so there at most $\log_2(c^s)$ further containments at the end of the chain from $\overline{G_{k_s}}$ to $\{1\}$.

Our total chain length is, then, at most

$$1 + \sum_{i=1}^{s-1}(i \log_2 c + 1) + s \log_2 c = s + \tfrac{1}{2}s(s+1)\log_2 c.$$

Since $s \leq d$, the conclusion follows. □

We are ready to complete the proof of Theorem 1 in this case. We reinstate the maximality supposition on G_1. We consider the adjoint representation, ad, of $\overline{G}$ and we set

$$c = |W(\overline{G})| \cdot 2^{d^2}.$$

For the moment we exclude the exceptional cases $(\overline{G}, \overline{G_1}, p) = (C_n, D_n, 2)$ or $(C_3, G_2, 2)$ in Theorem 4(1)(b); then, by Theorem 4(1), c is an upper bound for $\deg_{\mathrm{ad}}(\overline{G_1})$ and also, by (2-2), for $|\overline{G_1} : (\overline{G_1})^0|$.

Recall that r is the rank of $\overline{G}$, and that $d = \dim \overline{G}$, so that $d \le 4r^2$. Also

$$c = |W(\overline{G})| \cdot 2^{d^2} \le 2^{r^2+d^2} \le 2^{r^2+16r^4}.$$

Hence Lemma 4.3 gives

$$k \le 4r^2 + \tfrac{1}{2}(4r^2)(4r^2+1)(r^2+16r^4).$$

The right-hand side is at most Cr^8 with $C = 174$, as required for Theorem 1.

It remains to deal with the excluded cases $(\overline{G}, \overline{G_1}, p) = (C_n, D_n, 2)$ or $(C_3, G_2, 2)$. In the former case [10, Lemma 6.11] implies that $\mathrm{I}(G, \Omega) \le 2r+1$ and the conclusion holds. In the latter case the action of $G = C_3(q)$ on $\Omega = (C_3(q) : G_2(q))$ is contained in $(D_4(q) : (D_4(q) : B_3(q))$, since there is a factorization $D_4(q) = AB$, where $A \cong B \cong B_3(q)$ and $A \cap B \cong G_2(q)$ (see [21, p. 105]). For this action of $X := D_4(q)$, we have $\mathrm{I}(X, \Omega) \le 15$ by [12, Theorem 3.1]. Hence $\mathrm{I}(G, \Omega) \le 15$.

This completes the proof of Theorem 1. $\square$

4B. *Case 2 of Theorem 4.* In this case we have $G_1 = G(q_0)$, a subgroup of the same type as G (possibly twisted) over a subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$. Writing $G = (\overline{G}^F)'$ as before, there is a Frobenius endomorphism F_0 of $\overline{G}$ such that $G_1 = \overline{G}^{F_0} \cap G$, where $F_0^r = F$ for some integer $r \ge 2$.

Lemma 4.4. *For $x \in G$ we have*

$$G_1 \cap G_1^x = C_{G_1}(x^{-1}x^{F_0}) = (C_{\overline{G}}(x^{-1}x^{F_0}))^{F_0}.$$

Note that the group $C_{\overline{G}}(x^{-1}x^{F_0})$ may not be F_0-stable.

Proof. We have

$$\begin{aligned} g \in G_1 \cap G_1^x \Leftrightarrow g, g^{x^{-1}} \in G_1 &\Leftrightarrow g^{F_0} = g \text{ and } (xgx^{-1})^{F_0} = xgx^{-1} \\ &\Leftrightarrow g^{F_0} = g \text{ and } x^{F_0} g x^{-F_0} = xsx^{-1} \\ &\Leftrightarrow g \in C_{G_1}(x^{-1}x^{F_0}). \end{aligned}$$ $\square$

Recall that we have a stabilizer chain $G > G_1 > G_2 > \cdots > G_k = 1$, where $G_i = G_{i-1} \cap G_1^{g_i}$ for each i, and $g_i \in G$. Define

$$\overline{G_1} = \overline{G}, \quad \overline{G_2} = C_{\overline{G}}(g_2^{-1} g_2^{F_0}),$$

and for $2 \le j \le k$,

$$\overline{G_j} = \bigcap_{i=2}^{j} C_{\overline{G}}(g_i^{-1} g_i^{F_0}).$$

Then by Lemma 4.4, we have $G_j = \overline{G_j}^{F_0}$ for $1 \le j \le k$, and so

$$\overline{G} = \overline{G_1} > \overline{G_2} > \cdots > \overline{G_k}.$$

Given $x \in \overline{G}$, we of course have $C_{\overline{G}}(x) = \{g \in \overline{G} : gx = xg\}$, so this centralizer consists of solutions of a system of linear equations in the entries of g, and hence $\deg_{\mathrm{ad}} C_{\overline{G}}(x) \le \deg_{\mathrm{ad}} \overline{G}$. Now we can bound the length k of the chain exactly as in Case 1, and the proof is complete. □

4C. *Case 3 of Theorem 4.* This case is a triviality: clearly if $|G_1| \le 2^{d^2}$, then a stabilizer chain has length at most d^2. This observation completes the proof of Theorem 1. □

Example 4.5. Here is an example that shows there is no general complementary *lower bound* to go with the upper bound given in Theorem 1. Let $G = \mathrm{SL}_r(2)$ acting on Ω, the set of cosets of H where H is the normalizer of a Singer cycle, with r an odd prime. Then $H \cong C_{2^r-1} \rtimes C_r$ and H is maximal in G for $r \ge 13$ (see [13, Table 3.5A]). Since distinct conjugates of the Singer cycle C_{2^r-1} intersect trivially, it follows that for this action we have $\mathrm{I}(G, \Omega) \le 3$. In particular, $\mathrm{I}(G, \Omega)$ does not necessarily grow as the rank increases, even when G is simple and the action is primitive.

Remark 4.6. It is possible to improve the polynomial bound of Theorem 1 in particular cases. For example, consider parabolic actions of $G = \mathrm{PSL}_n(q)$, i.e., transitive actions for which the stabilizer G_1 is a parabolic subgroup. Set $\overline{G} = \mathrm{SL}_n(\overline{\mathbb{F}_q})$ and let ρ be the usual n-dimensional rational representation. In this situation, parabolic subgroups $\overline{G}_1$ satisfy $\deg_\rho(\overline{G}_1) \le n$ and so Lemma 4.3 gives $\mathrm{I}(G, \Omega) \le n^4 \log_2 n$.

5. Almost simple groups: proof of Corollary 3

Let G be an almost simple group, with socle $S = G(q)$, a simple group of Lie type of rank r over $\mathbb{F}_q$, where $q = p^f$ (p prime). Let G act primitively on a set Ω, with point-stabilizer G_1, and let $M_1 = G_1 \cap S$. Note that $G = G_1 S$, and so $G_1/M_1 \cong G/S$, a subgroup of $\mathrm{Out}(S)$.

Now let $G > G_1 > G_2 > \cdots > G_k = \{1\}$ be a stabilizer chain, where $G_i = G_{\alpha_1 \cdots \alpha_i}$ for $1 \le i \le k$. Define $M_i = G_i \cap S$. We obtain two chains:

$$S > M_1 \ge M_2 \ge \cdots \ge M_k = \{1\},$$
$$G/S = G_1/M_1 \ge G_2/M_2 \ge \cdots \ge G_k/M_k = \{1\}.$$

Observe that, for $i = 1, \ldots, k-1$, if $M_i = M_{i+1}$, then $G_i/M_i > G_{i+1}/M_{i+1}$. By [13, Tables 5.1A, 5.1B], the order of $\mathrm{Out}(S)$ divides kf, for some integer $k \le 6r$, and hence a proper subgroup chain in G/S has length at most $\log_2(6r)+\pi(f)$. Now define

$$I=\{i : 1\le i\le k-1 \text{ and } M_i > M_{i+1}\}$$

and write $I=\{i_1,\ldots,i_{\ell-1}\}$ where $i_j < i_{j+1}$ for $j=1,\ldots,\ell-2$. Setting $i_\ell = k$ we have, firstly, that

$$\ell \ge k-\log_2(6r)-\pi(f) \tag{5-1}$$

and, secondly, that

$$S > M_{i_1} > M_{i_2} > \cdots > M_{i_\ell} = \{1\}. \tag{5-2}$$

Note that $i_1 = 1$, and (5-2) is the stabilizer chain $S > S_{\alpha_1} > S_{\alpha_1\alpha_{i_2}} > \cdots$ for the action of S on Ω.

Now Theorem 4 tells us that S_{α_1} satisfies (1), (2) or (3) of the conclusion of that theorem. Hence, arguing exactly as in the proof of Theorem 1 we obtain that $\ell \le 174r^8$. Combining this bound with (5-1) yields $k\le 174r^8+\log_2(6r)+\pi(f)$, which is less than $177r^8+\pi(f)$. This completes the proof of Corollary 3. □

Example 5.1. Here is an example that shows that the term $\pi(f)$ in the upper bound in Corollary 3 cannot be avoided.

Let $G=\mathrm{P\Gamma L}_2(q)$ with $q=p^f$, and consider the action of G on the set of 1-subspaces of $V=(\mathbb{F}_q)^2$. We claim that $\mathrm{I}(G,\Omega)=3+\pi(f)$. To see this, write the prime factorization of f as $f=r_1r_2\cdots r_\ell$ where $\ell=\pi(f)$, write $\{e_1,e_2\}$ for the natural basis of V over $\mathbb{F}_q$, and consider the stabilizer chain obtained by successively stabilizing the following 1-spaces (in order):

$$\langle e_1\rangle, \langle e_2\rangle, \langle e_1+e_2\rangle, \langle e_1+\zeta_1e_2\rangle, \langle e_1+\zeta_2e_2\rangle, \ldots, \langle e_1+\zeta_\ell e_2\rangle,$$

where, for $i=1,2,\ldots,\ell$, ζ_i is a primitive element of $\mathbb{F}_{p^{r_1r_2\cdots r_i}}$. This stabilizer chain establishes that $\mathrm{I}(G,\Omega)\ge 3+\pi(f)$; on the other hand, the 3-transitivity of the action of G implies that the stabilizer of any 3 distinct points is isomorphic to C_f and this implies that $\mathrm{I}(G,\Omega)\le 3+\pi(f)$.

It seems possible, however, that one could do better for $\mathrm{B}(G,\Omega)$ and/or $\mathrm{H}(G,\Omega)$. In the proof of Lemma 6.3 below we shall show that there exists a primitive action of $G=\mathrm{P\Gamma L}_2(q)$ for which $\mathrm{B}(G,\Omega)\ge \pi_d(f)$, where $\pi_d(f)$ is the number of *distinct* primes dividing the integer f.

Conjecture 5.2. *There exists a function $g:\mathbb{Z}^+\to\mathbb{Z}^+$ such that if G is an almost simple group of Lie type of rank r over a field of order p^f acting primitively on a set Ω, then*

$$\mathrm{B}(G,\Omega)\le \mathrm{H}(G,\Omega) < g(r)+\pi_d(f),$$

where $\pi_d(f)$ is the number of distinct primes dividing the integer f.

6. Theorem 1 and the Cameron–Kantor conjecture

The Cameron–Kantor conjecture (now a theorem due to Liebeck and Shalev [20]) asserts the following:

> *There exists a constant* $c > 0$ *such that if* G *is an almost simple primitive nonstandard permutation group on a set* Ω, *then* $\mathrm{b}(G, \Omega) \leq c$.

(A *standard action* of an almost simple group G with socle S is a transitive action where either $S = A_n$ and the action is on subsets or uniform partitions of $\{1, \dots, n\}$, or G is classical and the action is a subspace action; see [2] for more detail.) This statement is now known to be true with $c = 7$, by [2; 3; 4; 5].

Colva Roney-Dougal asked us whether a statement like the Cameron–Kantor conjecture might be true for any of the statistics $\mathrm{B}(G, \Omega)$, $\mathrm{H}(G, \Omega)$ or $\mathrm{I}(G, \Omega)$ and Theorem 1 was our answer to this question. One naturally wonders, though, whether it is possible to do better — to investigate this, given (1-1), the first question one should ask is whether a stronger statement can be proved for $\mathrm{B}(G, \Omega)$ (since any such statement for $\mathrm{H}(G, \Omega)$ or $\mathrm{I}(G, \Omega)$ is necessarily true for $\mathrm{B}(G, \Omega)$). To investigate this we need to clarify some things.

Primitivity and transitivity. Suppose that G is a transitive permutation group on Ω and identify Ω with $(G : H)$ where H is the stabilizer of a point. Now let $F \leq H$ and let $\Gamma = (G : F)$. Then it is true that $\mathrm{b}(G, \Gamma) \leq \mathrm{b}(G, \Omega)$ and hence, in particular, the Cameron–Kantor conjecture gives information about all transitive almost simple permutation groups G for which a point-stabilizer is a subgroup of a maximal subgroup that is a point stabilizer for a nonstandard primitive action.

Things are more complicated for us because it is not necessarily true that $\mathrm{B}(G, \Gamma) \leq \mathrm{B}(G, \Omega)$, that $\mathrm{H}(G, \Gamma) \leq \mathrm{H}(G, \Omega)$ or that $\mathrm{I}(G, \Gamma) \leq \mathrm{I}(G, \Omega)$; the examples below demonstrate this. Hence in investigating how to extend the statement of the Cameron–Kantor conjecture we need to distinguish between statements involving primitive groups and those involving transitive groups.

Rank-dependent constant versus absolute constant. Our investigations will focus on almost simple groups with socle a group of Lie type. Our first example will establish that it is not possible to give an absolute upper bound for $\mathrm{B}(G, \Omega)$, even for nonstandard actions. In light of this it is worth clarifying what the Cameron–Kantor conjecture implies with regard to a rank-dependent upper bound:

> *For every positive integer* r *there exists a constant* $c > 0$ *such that if* G *is an almost simple primitive permutation group on a set* Ω, *with socle a group of Lie type of rank at most* r, *then* $\mathrm{b}(G, \Omega) \leq c$.

The point we are making here is that, if we allow our upper bound to be rank-dependent, then we do not need to distinguish between standard and nonstandard actions — it is easy enough to establish that the standard actions also satisfy the given statement. (For the $\mathcal{C}_8$ standard actions of $\mathrm{Sp}_{2m}(q)$ this follows from [10, Lemma 6.11]; for the $\mathcal{C}_1$ standard actions of the classical groups this follows from [12, Theorem 3.1].)

Note, finally, that we have not considered the question of Cameron–Kantor-like statements for irredundant bases of primitive actions of the alternating groups.

6A. ***Simple, primitive, absolute upper bound.*** In this subsection we show that the following possible extension of the Cameron–Kantor conjecture is false:

> *There exists a constant* $c > 0$ *such that if* G *is a simple primitive nonstandard permutation group on a set* Ω*, then* $\mathrm{B}(G, \Omega) \leq c$.

The key point here is that an upper bound on $\mathrm{B}(G, \Omega)$ in this setting must depend on r.

Lemma 6.1. *For every* $n \geq 13,\ q \geq 5$*, there exists a nonstandard primitive action* $(\mathrm{PSL}_n(q), \Omega)$ *such that* $\mathrm{B}(\mathrm{PSL}_n(q), \Omega) \geq n - 1$.

Proof. We consider the action of $G = \mathrm{SL}_n(q)$ acting on the cosets of a $\mathcal{C}_2$-maximal subgroup that is the normalizer of a split torus. For $q \geq 5,\ n \geq 13$ this induces a primitive nonstandard action of $\mathrm{PSL}_n(q)$ (see [13, Table 3.5A]); furthermore this action of G is equivalent to the action of G on decompositions of $V = (\mathbb{F}_q)^n$ as a direct sum of n 1-dimensional subspaces.

Let $\{e_1, \dots, e_n\}$ be a basis of V over $\mathbb{F}_q$. For $i = 1, \dots, n-1$, we define a decomposition $\mathcal{D}_i$ of V as

$$\mathcal{D}_i = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots \oplus \langle e_{i-1} \rangle \oplus \langle e_i + e_{i+1} \rangle \oplus \langle e_{i+1} \rangle \oplus \langle e_{i+2} \rangle \oplus \cdots \oplus \langle e_n \rangle.$$

Suppose, first, that $g \in G$ fixes $\mathcal{D}_1, \dots, \mathcal{D}_{n-1}$. This implies that g fixes the space $\langle e_n \rangle$ (since it is the only 1-space appearing in all $n-1$ decompositions); similarly, for $j = 1, \dots, n-1,\ g$ fixes the space $\langle e_j \rangle$ (since it is the only 1-space appearing in all $n-1$ decompositions except for $\mathcal{D}_j$). Thus, for $j = 1, \dots, n$, there exists $\lambda_j \in \mathbb{F}_q$ such that $e_j^g = \lambda_j e_j$. But now, for $j = 1, \dots, n-1$, the space $\langle e_j + e_{j+1} \rangle$ occurs in decomposition $\mathcal{D}_j$ and no others, hence this 1-space too is fixed by g. This implies, finally, that, for $j = 1, \dots, n-1,\ \lambda_j = \lambda_{j+1}$ and so g acts as a scalar. In particular, the set $\{\mathcal{D}_1, \dots, \mathcal{D}_{n-1}\}$ is a base for the induced action of $\mathrm{PSL}_n(q)$.

On the other hand, for $j \in 1, \dots, n-1$, define

$$\Lambda_j = \{\mathcal{D}_1, \dots, \mathcal{D}_{j-1}, \mathcal{D}_{j+1}, \dots, \mathcal{D}_{n-1}\}$$

and set g_j to be an element of G that swaps $\langle e_j \rangle$ and $\langle e_n \rangle$ while fixing $\langle e_i \rangle$ for $i = 1, \dots, j-1, j+1, \dots, n-1$. It is straightforward to check that g fixes all of the decompositions in Λ_j. We conclude that Λ is a minimal base for this action of size $n-1$. $\square$

In light of this lemma our remaining investigations will focus on almost simple groups where the socle is a group of Lie type of bounded rank.

6B. ***Simple, transitive, rank-dependent upper bound.*** In this subsection we show that the following possible extension of the Cameron–Kantor conjecture is false:

> *For every positive integer r there exists a constant $c > 0$ such that if G is a simple transitive permutation group on a set Ω, with socle a group of Lie type of rank at most r, then* $\mathrm{B}(G, \Omega) \leq c$.

The next lemma does the job:

Lemma 6.2. *For every integer $c > 1$, there exists a transitive action* $(\mathrm{SL}_2(2^c), \Omega)$, *such that* $\mathrm{B}(\mathrm{SL}_2(2^c), \Omega) \geq c$.

Proof. Let $q = 2^c$, let $G = \mathrm{SL}_2(q)$, let U be a Sylow 2-subgroup of G, let H be an index 2 subgroup of U and let Ω be the set of right cosets of H in G. Since $H = 2^{c-1}$ it is clear that $\mathrm{B}(G, \Omega) \leq \mathrm{I}(G, \Omega) \leq c$. We claim that, in fact, $\mathrm{B}(G, \Omega) = c$.

To show this, let $B = N_G(U)$ and let Δ be the set of right cosets of H in B. Since $\mathrm{B}(B, \Delta) \leq \mathrm{B}(G, \Omega)$ it is sufficient to show that $\mathrm{B}(B, \Delta) \geq c$.

Consider U as a c-dimensional vector space over $\mathbb{F}_2$. The action of B on Δ is isomorphic to the action of B on the set of all *affine hyperplanes* — these are the usual linear hyperplanes as well as their translates. Since we are working over $\mathbb{F}_2$, each hyperplane has 2 cosets (itself and one other) thus $|\Delta| = 2(q-1)$.

Observe that if H_1 is a linear hyperplane, then the stabilizer of H_1 in B is H_1 itself (in particular, H_1 is a conjugate of H). Let $e_1, \ldots, e_c$ be the usual vectors in the natural basis of U (so e_i has 0's in all places except the i-th where the entry is 1). For $i = 1, \ldots, c$, define

$$H_i := \langle e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_c \rangle.$$

Then $H_1, \ldots, H_c$ are linear hyperplanes in U hence are elements of Δ and conjugates of H. For $i = j, \ldots, c$, define $\Lambda_j = \{H_1, \ldots, H_{j-1}, H_{j+1}, \ldots, H_c\}$ and observe that $B_{(\Lambda_j)} = \langle e_j \rangle$. Thus $\Lambda = \{H_1, \ldots, H_c\}$ is a minimal base of size c. $\square$

6C. ***Almost simple, primitive, rank-dependent upper bound.*** Here we show that the following possible extension of the Cameron–Kantor conjecture is false:

> *For every positive integer r there exists a constant $c > 0$ such that if G is an almost simple primitive permutation group on a set Ω, with socle a group of Lie type of rank at most r, then* $\mathrm{B}(G, \Omega) \leq c$.

The next lemma does the job:

Lemma 6.3. *For all $c > 0$, there exists a nonstandard primitive action* $(\mathrm{P\Gamma L}_2(q), \Omega)$, *for some q, such that* $\mathrm{B}(\mathrm{P\Gamma L}_2(q), \Omega) > c$.

Proof. Let $G = \Gamma\mathrm{L}_2(q)$ and consider the action on cosets of the normalizer of a split torus. For $q > 11$ this induces a primitive nonstandard action of $\mathrm{P\Gamma L}_2(q)$; furthermore, this action of G is equivalent to the action of G on decompositions of $V = (\mathbb{F}_q)^2$ as a direct sum of two 1-dimensional subspaces. Let $q = p^d$ and assume that $d = f_1 \cdots f_k$ where $k \geq 3$ and $f_1, \dots, f_k$ are distinct primes.

Let $\{e_1, e_2\}$ be the natural basis for V over $\mathbb{F}_q$: $e_1 = (1\ 0)$ and $e_2 = (0\ 1)$. We define decompositions $\mathcal{D}_i$ for $i = 1, \dots, k$ as

$$\mathcal{D}_i : \langle e_1 \rangle \oplus \langle e_1 + \zeta_i e_2 \rangle,$$

where ζ_i is a primitive element in $\mathbb{F}_{p^{f_i}}$. To see that $\mathcal{D}_1, \dots, \mathcal{D}_k$ form an independent set we consider the action $F = \langle \sigma \rangle < G$ where σ is the field automorphism that acts on vectors by raising each entry to the p-th power.

For $j \in 1, \dots, k$, define $\Lambda_j = \{\mathcal{D}_1, \dots, \mathcal{D}_{j-1}, \mathcal{D}_{j+1}, \dots, \mathcal{D}_k\}$. The pointwise-stabilizer of Λ_j in F is $\langle \sigma^{d/f_j} \rangle$ and so the pointwise-stabilizers of Λ_j are distinct for $j = 1, \dots, k$; in particular we obtain that $\Lambda = \{\mathcal{D}_1, \dots, \mathcal{D}_k\}$ is an independent set of size k.

We claim that, in fact, Λ is a minimal base. To see this, we must prove that the pointwise-stabilizer of Λ is trivial. Let $g \in G_{(\Lambda)}$ and write $g = \sigma^r x$ where r is some positive integer and $x \in \mathrm{GL}_2(q)$; without loss of generality we can assume that r divides d. It is clear that $\langle e_1 \rangle^g = \langle e_1 \rangle$, so there exists $\lambda_0 \in \mathbb{F}_q$ such that

$$\lambda_0 e_1 = e_1^g = e_1^{\sigma^r x} = e_1^x.$$

Similarly, for $i = 1, \dots, k$, there exists $\lambda_i \in \mathbb{F}_q$ such that

$$\lambda_i(e_1 + \zeta_i e_2) = (e_1 + \zeta_i e_2)^g = e_1^g + (\zeta_i e_2)^g = e_1^x + \zeta_i^{\sigma^r} e_2^x = \lambda_0 e_1 + \zeta_i^{p^r} e_2^x.$$

Rearranging we obtain that

$$e_2^x = \lambda_i \zeta_i^{1-p^r} e_2 + \zeta_i^{-p^r} (\lambda_i - \lambda_0)\, e_1.$$

We conclude that, for distinct $i, j \in \{1, \dots, k\}$ we have

$$\lambda_i \zeta_i^{1-p^r} = \lambda_j \zeta_j^{1-p^r} \quad \text{and} \quad \zeta_i^{-p^r} (\lambda_i - \lambda_0) = \zeta_j^{-p^r} (\lambda_j - \lambda_0).$$

The latter equation yields that

$$\lambda_i = \left(\frac{\zeta_i}{\zeta_j}\right)^{p^r} \lambda_j + \left(1 - \left(\frac{\zeta_i}{\zeta_j}\right)^{p^r}\right)\lambda_0,$$

while the former yields that

$$\lambda_i = \frac{\zeta_i^{p^r-1}}{\zeta_j^{p^r-1}}\lambda_j.$$

Combining these two identities and rearranging yields

$$\left(\frac{\zeta_j/\zeta_i - 1}{(\zeta_j/\zeta_i)^{p^r}-1}\right)\lambda_j = \lambda_0.$$

If we fix j and choose $\ell, m \in \{1, \dots, k\}$ such that j, ℓ and m are all distinct, then we obtain that

$$\frac{\zeta_j/\zeta_\ell - 1}{(\zeta_j/\zeta_\ell)^{p^r}-1} = \frac{\zeta_j/\zeta_m - 1}{(\zeta_j/\zeta_m)^{p^r}-1}$$

and, rearranging, we have

$$\left(\frac{\zeta_j/\zeta_\ell - 1}{\zeta_j/\zeta_m - 1}\right)^{p^r-1} = 1.$$

We claim that the smallest field containing the quantity in parenthesis is either $\mathbb{F}_{p^{f_j f_\ell f_m}}$ or $\mathbb{F}_{p^{f_\ell f_m}}$. To see this, denote this quantity η and suppose that η is contained in $\mathbb{F}_{p^{f_j f_\ell}}$. Rearranging we obtain

$$\zeta_m = \frac{\zeta_j\zeta_\ell\eta}{\zeta_j - \zeta_\ell + \zeta_\ell\eta} \in \mathbb{F}_{p^{f_j f_\ell}},$$

a contradiction. A similar argument allows us to conclude that this quantity is not contained in $\mathbb{F}_{p^{f_j f_m}}$ and the claim follows.

We obtain that r is divisible by both f_ℓ and f_m. Repeating this argument we obtain that r is divisible by all primes $f_1, \dots, f_k$ and thus $g = x$. But this implies that $\lambda_i = \lambda_j = \lambda_0$ for all $i, j = 1, \dots, k$ and g is a scalar, as required.

We conclude that Λ is a minimal base for this action. Since $|\Lambda| = k$, we need only choose $k > c$ to obtain that $\mathrm{B}(G, \Omega) \geq k > c$ as required. □

6D. ***Simple, primitive, rank-dependent upper bound.*** In light of the examples given in the preceding sections, this is the only setting where a direct extension of Cameron–Kantor conjecture is possible. As mentioned above, if we allow our upper bound to be rank-dependent, then we can ignore the distinction between standard and nonstandard actions, and hence the statement we end up with has the form of Theorem 1.

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References

[1] J. N. Bray, D. F. Holt, and C. M. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*, Lond. Math. Soc. Lect. Note Ser. **407**, Cambridge Univ. Press, 2013. MR Zbl

[2] T. C. Burness, "On base sizes for actions of finite classical groups", *J. Lond. Math. Soc.* (2) **75**:3 (2007), 545–562. MR Zbl

[3] T. C. Burness, M. W. Liebeck, and A. Shalev, "Base sizes for simple groups and a conjecture of Cameron", *Proc. Lond. Math. Soc.* (3) **98**:1 (2009), 116–162. MR Zbl

[4] T. C. Burness, E. A. O'Brien, and R. A. Wilson, "Base sizes for sporadic simple groups", *Israel J. Math.* **177** (2010), 307–333. MR Zbl

[5] T. C. Burness, R. M. Guralnick, and J. Saxl, "On base sizes for symmetric groups", *Bull. Lond. Math. Soc.* **43**:2 (2011), 386–391. MR Zbl

[6] P. J. Cameron, "Some open problems on permutation groups", pp. 340–350 in *Groups*, *combinatorics and geometry* (Durham, 1990), edited by M. Liebeck and J. Saxl, Lond. Math. Soc. Lect. Note Ser. **165**, Cambridge Univ. Press, 1992. MR Zbl

[7] P. J. Cameron and W. M. Kantor, "Random permutations: some group-theoretic aspects", *Combin. Probab. Comput.* **2**:3 (1993), 257–262. MR Zbl

[8] G. L. Cherlin, G. A. Martin, and D. H. Saracino, "Arities of permutation groups: wreath products and k-sets", *J. Combin. Theory Ser. A* **74**:2 (1996), 249–286. MR Zbl

[9] M. J. Collins, "Modular analogues of Jordan's theorem for finite linear groups", *J. Reine Angew. Math.* **624** (2008), 143–171. MR Zbl

[10] N. Gill, B. Lodà, and P. Spiga, "On the height and relational complexity of a finite permutation group", *Nagoya Math. J.* **246** (2022), 372–411. MR Zbl

[11] J. Heintz, "Definability and fast quantifier elimination in algebraically closed fields", *Theoret. Comput. Sci.* **24**:3 (1983), 239–277. MR Zbl

[12] V. Kelsey and C. M. Roney-Dougal, "On relational complexity and base size of finite primitive groups", *Pacific J. Math.* **318**:1 (2022), 89–108. MR Zbl

[13] P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, Lond. Math. Soc. Lect. Note Ser. **129**, Cambridge Univ. Press, 1990. MR Zbl

[14] M. W. Liebeck, "On minimal degrees and base sizes of primitive permutation groups", *Arch. Math.* (*Basel*) **43**:1 (1984), 11–15. MR Zbl

[15] M. W. Liebeck and G. M. Seitz, "On the subgroup structure of classical groups", *Invent. Math.* **134**:2 (1998), 427–453. MR Zbl

[16] M. W. Liebeck and G. M. Seitz, "On the subgroup structure of exceptional groups of Lie type", *Trans. Amer. Math. Soc.* **350**:9 (1998), 3409–3482. MR Zbl

[17] M. W. Liebeck and G. M. Seitz, "A survey of maximal subgroups of exceptional groups of Lie type", pp. 139–146 in *Groups*, *combinatorics and geometry* (Durham, 2001), edited by A. A. Ivanov et al., World Sci., River Edge, NJ, 2003. MR Zbl

[18] M. W. Liebeck and G. M. Seitz, *The maximal subgroups of positive dimension in exceptional algebraic groups*, Mem. Amer. Math. Soc. **802**, Amer. Math. Soc., Providence, RI, 2004. MR Zbl

[19] M. W. Liebeck and G. M. Seitz, "Subgroups of exceptional algebraic groups which are irreducible on an adjoint or minimal module", *J. Group Theory* **7**:3 (2004), 347–372. MR Zbl

[20] M. W. Liebeck and A. Shalev, "Simple groups, permutation groups, and probability", *J. Amer. Math. Soc.* **12**:2 (1999), 497–520. MR Zbl

[21] M. W. Liebeck, C. E. Praeger, and J. Saxl, *The maximal factorizations of the finite simple groups and their automorphism groups*, Mem. Amer. Math. Soc. **432**, Amer. Math. Soc., Providence, RI, 1990. MR Zbl

[22] G. M. Seitz and D. M. Testerman, "Extending morphisms from finite to algebraic groups", *J. Algebra* **131**:2 (1990), 559–574. MR Zbl

[23] R. Solomon and A. Turull, "Chains of subgroups in groups of Lie type, III", *J. Lond. Math. Soc.* (2) **44**:3 (1991), 437–444. MR Zbl

[24] B. Weisfeiler, "On the size and structure of finite linear groups", preprint, 2012. Zbl arXiv 1203.1960

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