Pacific Journal of Mathematics

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KANG LI AND EDUARDO SCARPARO

Volume 322 No. 2 February 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 322, No. 2, 2023
https://doi.org/10.2140/pjm.2023.322.369

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Given a commensurated subgroup Λ of a group Γ, we completely characterize when the inclusion $\Lambda \leq \Gamma$ is C^*-irreducible and provide new examples of such inclusions. In particular, we obtain that $\mathrm{PSL}(n, \mathbb{Z}) \leq \mathrm{PGL}(n, \mathbb{Q})$ is C^*-irreducible for any $n \in \mathbb{N}$, and that the inclusion of a C^*-simple group into its abstract commensurator is C^*-irreducible.

The main ingredient that we use is the fact that the action of a commensurated subgroup $\Lambda \leq \Gamma$ on its Furstenberg boundary $\partial_F \Lambda$ can be extended in a unique way to an action of Γ on $\partial_F \Lambda$. Finally, we also investigate the counterpart of this extension result for the universal minimal proximal space of a group.

1. Introduction

A group Γ is said to be *C^*-simple* if its reduced C^*-algebra $C_r^*(\Gamma)$ is simple. After the breakthrough characterizations of C^*-simplicity in [Kalantar and Kennedy 2017; Breuillard et al. 2017], several directions of research applying the new methods in different settings arose.

One of the recent interesting directions is investigating when inclusions of groups $\Lambda \leq \Gamma$ are *C^*-irreducible*, in the sense that every intermediate C^*-algebra B in $C_r^*(\Lambda) \subset B \subset C_r^*(\Gamma)$ is simple. Rørdam [2021] started a systematic study of this property and provided a dynamical criterion for an inclusion of groups to be C^*-irreducible. Together with results in [Amrutam 2021; Ursu 2022; Bédos and Omland 2023], this has provided a complete characterization of C^*-irreducibility of an inclusion in the case that Λ is a normal subgroup of Γ.

Recall that a subgroup Λ of a group Γ is said to be *commensurated* if, for any $g \in \Gamma$, $\Lambda \cap g\Lambda g^{-1}$ has finite index in Λ. This is a much more flexible generalization of normal subgroups and finite-index subgroups. For example, for every $n \geq 2$, $\mathrm{PSL}(n, \mathbb{Z})$ is an infinite-index commensurated subgroup of the simple group $\mathrm{PSL}(n, \mathbb{Q})$.

This project received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 817597).
MSC2020: 22D25, 37B05.
Keywords: commensurated subgroups, C*-simplicity, Furstenberg boundary.

In this work, we generalize the above characterization of C^*-irreducibility to commensurated subgroups (see Theorem 3.5). The main ingredient in our proof is the fact that the action of Λ on its Furstenberg boundary $\partial_F\Lambda$ can be uniquely extended to an action of Γ on $\partial_F\Lambda$ if Λ is a commensurated subgroup in Γ (see Theorem 3.1).

As one of the applications, we show that if Γ is a C^*-simple group, then the inclusion of Γ in its abstract commensurator $\mathrm{Comm}(\Gamma)$ is C^*-irreducible (see Corollary 3.14). To our best knowledge, this is also the first observation of the fact that if Γ is a C^*-simple group, then $\mathrm{Comm}(\Gamma)$ is C^*-simple as well.

Given a subgroup Λ of a group Γ, Ursu [2022] introduced a universal Λ-strongly proximal Γ-boundary $B(\Gamma, \Lambda)$ and showed that if $\Lambda \trianglelefteq \Gamma$, then $B(\Gamma, \Lambda) = \partial_F\Lambda$. In Section 4, we generalize this fact to commensurated subgroups and also observe that, in general, $B(\Gamma, \Lambda)$ is not extremally disconnected.

Finally, we also show that, given a commensurated subgroup Λ of a group Γ, the action of Λ on its universal minimal proximal space $\partial_p\Lambda$ can also be extended in a unique way to an action of Γ on $\partial_p\Lambda$ (see Theorem 5.1). We use this fact for concluding that, for a certain locally finite commensurated subgroup G of Thompson's group V, the resulting action of V on $\partial_p G$ is free (see Example 5.4).

2. Preliminaries

Given a compact Hausdorff space X, we denote by $\mathrm{Prob}(X)$ the space of regular probability measures on X. An action of a group Γ on X by homeomorphisms is said to be *minimal* if X does not contain any nontrivial closed invariant subset, and to be *topologically free* if, for any $g \in \Gamma \setminus \{e\}$, the set $\{x \in X : gx = x\}$ has empty interior (if Γ is countable, then $\Gamma \curvearrowright X$ is topologically free if and only if the set of points in X which are not fixed by any nontrivial element of Γ is dense in X). The action is said to be *proximal* if, given $x, y \in X$, there is a net $(g_i) \subset \Gamma$ such that the nets $(g_i x)$ and $(g_i y)$ converge and $\lim g_i x = \lim g_i y$. We say that the action is *strongly proximal* if the induced action $\Gamma \curvearrowright \mathrm{Prob}(X)$ is proximal. The action is called a *boundary action* (or X is a Γ-*boundary*) if it is both minimal and strongly proximal. We denote by $\partial_F\Gamma$ the *Furstenberg boundary* of Γ, i.e., the universal Γ-boundary (see [Glasner 1976, Section III.1]). The group Γ is C^*-simple if and only if $\Gamma \curvearrowright \partial_F\Gamma$ is free [Breuillard et al. 2017, Theorem 3.1].

Given Γ-boundaries X and Y, if there exists $\varphi \colon X \to Y$ a homeomorphism which is Γ-equivariant (Γ-*isomorphism*), then it follows from [Glasner 1976, Lemma II.4.1] that φ is the unique Γ-isomorphism between X and Y.

Let $\Lambda \leq \Gamma$ be a finite-index subgroup. Then any strongly proximal Γ-action is also Λ-strongly proximal [Glasner 1976, Lemma II.3.1] and any Γ-boundary is also a Λ-boundary [Glasner 1976, Lemma II.3.2]. Furthermore, by [Glasner 1976, Theorem II.4.4], which is stated for the universal minimal proximal space but whose proof also works for the Furstenberg boundary, the action $\Lambda \curvearrowright \partial_F \Lambda$ can be extended to $\Gamma \curvearrowright \partial_F \Lambda$ and $\partial_F \Lambda$ is Γ-isomorphic to $\partial_F \Gamma$. In particular, $\partial_F \Lambda$ and $\partial_F \Gamma$ are also Λ-isomorphic.

Given a group isomorphism $\psi \colon \Gamma_1 \to \Gamma_2$, by universality there is a unique homeomorphism $\tilde{\psi} \colon \partial_F \Gamma_1 \to \partial_F \Gamma_2$ such that $\tilde{\psi}(gx) = \psi(g)\tilde{\psi}(x)$ for any $g \in \Gamma_1$ and $x \in \partial_F \Gamma_1$.

Given a group Γ, let $\mathrm{Sub}(\Gamma)$ be the space of subgroups of Γ endowed with the pointwise convergence topology and with the Γ-action given by conjugation. Given a subgroup $\Lambda \leq \Gamma$, a Λ-*uniformly recurrent subgroup* (URS) is a nonempty closed Λ-invariant minimal set $\mathcal{U} \subset \mathrm{Sub}(\Gamma)$. Moreover, we say that $\mathcal{U}$ is *amenable* if one (equivalently all) of its elements is amenable. By [Kennedy 2020, Theorem 4.1], a group Γ is C^*-simple if and only if it does not admit any nontrivial amenable Γ-uniformly recurrent subgroup.

An inclusion of groups $\Lambda \leq \Gamma$ is said to be *C^*-irreducible* if every intermediate C^*-algebra of $C^*_r(\Lambda) \subset C^*_r(\Gamma)$ is simple.

Given $\Lambda \leq \Gamma$ and $g \in \Gamma$, let $g^\Lambda := \{hgh^{-1} : h \in \Lambda\}$. We say that Γ *is icc relatively to* Λ if, for any $g \in \Gamma \setminus \{e\}$, $|g^\Lambda| < \infty$. The group Γ is said to be *icc* if it is icc relatively to itself.

3. C*-irreducibility of commensurated subgroups

Let Γ be a group. Two subgroups $\Lambda_1, \Lambda_2 \leq \Gamma$ are said to be *commensurable* if $[\Lambda_1 : \Lambda_1 \cap \Lambda_2] < \infty$ and $[\Lambda_2 : \Lambda_1 \cap \Lambda_2] < \infty$. Notice that this is an equivalence relation.

A subgroup $\Lambda \leq \Gamma$ is said to be *commensurated* if, for any $g \in \Gamma$, Λ is commensurable with $g\Lambda g^{-1}$. Equivalently, for any $g \in \Gamma$, $[\Lambda : \Lambda \cap g\Lambda g^{-1}] < \infty$. In this case, we write $\Lambda \leq_c \Gamma$. In the literature, this notion is also referred to by saying that Λ is an *almost normal subgroup* of Γ or that (Γ, Λ) is a *Hecke pair*.

The following result generalizes [Glasner 1976, Theorem II.4.4] and [Ozawa 2014, Lemma 20]:

Theorem 3.1. *Let $\Lambda \leq_c \Gamma$. Then $\Lambda \curvearrowright \partial_F \Lambda$ extends in a unique way to an action of Γ on $\partial_F \Lambda$.*

Proof. Given $g \in \Gamma$, let $\varphi_g \colon \partial_F \Lambda \to \partial_F(\Lambda \cap g\Lambda g^{-1})$ be the $(\Lambda \cap g\Lambda g^{-1})$-isomorphism. Also, let $\psi_g \colon \partial_F(\Lambda \cap g^{-1}\Lambda g) \to \partial_F(\Lambda \cap g\Lambda g^{-1})$ be the homeomorphism such that for all $h \in \Lambda \cap g^{-1}\Lambda g$ and $x \in \partial_F(\Lambda \cap g^{-1}\Lambda g)$ we have $\psi_g(hx) = ghg^{-1}\psi_g(x)$. Let $T_g := (\varphi_g)^{-1}\psi_g\varphi_{g^{-1}} \colon \partial_F \Lambda \to \partial_F \Lambda$. We claim that $g \mapsto T_g$ is a Γ-action which extends $\Lambda \curvearrowright \partial_F \Lambda$.

Given $h \in \Lambda \cap g^{-1}\Lambda g$ and $x \in \partial_F \Lambda$, one can readily check that $T_g(hx) = ghg^{-1}T_g(x)$.

Given $g, h \in \Gamma$, we have that $[\Lambda : \Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh)] < \infty$. Furthermore, given $k \in \Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh)$ and $x \in \partial_F \Lambda$, we have $T_{gh}(kx) = (gh)k(gh)^{-1}T_{gh}(x)$. On the other hand, $T_g T_h(kx) = (gh)k(gh)^{-1}T_g T_h(x)$. In particular, $(T_g T_h)^{-1}T_{gh}$ is a $(\Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh))$-automorphism, hence $T_{gh} = T_g T_h$.

Finally, given $g \in \Lambda$, we have that $x \mapsto g^{-1}T_g(x)$ is a $(\Lambda \cap g^{-1}\Lambda g)$-automorphism, so that $g^{-1}T_g = \mathrm{Id}_{\partial_F \Lambda}$. □

Remark 3.2. The existence part of Theorem 3.1 was shown by Dai and Glasner [2019, Theorem 6.1] using a different method and assuming that Γ is countable.

Given a subset S of a group Γ, let $C_\Gamma(S)$ be the *centralizer* of S in Γ. In the next result, we follow the argument of [Breuillard et al. 2017, Lemma 5.3].

Lemma 3.3. *Let $\Lambda \leq_c \Gamma$ and consider $\Gamma \curvearrowright \partial_F \Lambda$. Given $s \in \Gamma$, if $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$, then* $\mathrm{Fix}(s) = \partial_F \Lambda$. *Conversely, if $\Lambda \curvearrowright \partial_F \Lambda$ is free and* $\mathrm{Fix}(s) \neq \varnothing$, *then* $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$.

Proof. If $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$, then, given $h \in \Lambda \cap s^{-1}\Lambda s$ and $x \in \partial_F \Lambda$, we have $s(hx) = hs(x)$. Since $[\Lambda : \Lambda \cap s^{-1}\Lambda s] < \infty$, we conclude that s acts trivially on $\partial_F \Lambda$.

Suppose now that $\Lambda \curvearrowright \partial_F \Lambda$ is free and $\mathrm{Fix}(s) \neq \varnothing$. Given $t \in A$, with

$$A := \{t \in \Lambda \cap s^{-1}\Lambda s : t\,\mathrm{Fix}(s) \cap \mathrm{Fix}(s) \neq \varnothing\},$$

the actions of sts^{-1} and t coincide on $\mathrm{Fix}(s) \cap t^{-1}\,\mathrm{Fix}(s)$. Since $sts^{-1}, t \in \Lambda$ and $\Lambda \curvearrowright \partial_F \Lambda$ is free, we obtain that $t = sts^{-1}$. Since, by [Breuillard et al. 2017, Lemma 5.1], A generates $\Lambda \cap s^{-1}\Lambda s$, we conclude that $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$. □

The proof of the following result is an adaptation of the argument in [Kennedy 2020, Remark 4.2] and its hypothesis is the same as in [Rørdam 2021, Theorem 5.3 (ii)]:

Proposition 3.4. *Let $\Lambda \leq \Gamma$. Suppose that there exists a Γ-boundary X such that, for any $\mu \in$* $\mathrm{Prob}(X)$, *there exists a net $(g_i) \subset \Lambda$ such that $g_i\mu$ converges to δ_x, for some $x \in X$, on which Γ acts freely. Then Γ does not admit any nontrivial amenable Λ-URS.*

Proof. Suppose $\mathcal{U}$ is a nontrivial amenable Λ-URS, and take $K \in \mathcal{U}$. Since K is amenable, there exists $\mu \in \mathrm{Prob}(X)$ fixed by K. Let $(g_i) \subset \Lambda$ be a net such that $g_i\mu \to \delta_x$, for some $x \in X$, on which Γ acts freely. By taking a subnet, we may assume that $g_i K g_i^{-1} \to L \in \mathrm{Sub}(\Gamma)$. Take $g \in L \setminus \{e\}$ and $(k_i) \subset K$ such that $g_i k_i g_i^{-1} = g$ for i sufficiently big. Then

$$\delta_x = \lim g_i\mu = \lim g_i k_i \mu = \lim g_i k_i g_i^{-1} g_i \mu = g\delta_x,$$

contradicting the fact that Γ acts freely on x. □

The following result generalizes [Ursu 2022, Theorems 1.3 and 1.9] and [Bédos and Omland 2023, Theorem 6.4], as well as the claim about finite-index subgroups in [Rørdam 2021, Theorem 5.3]:

Theorem 3.5. *Let $\Lambda \leq_c \Gamma$. The following conditions are equivalent*:

(1) $\Lambda \leq \Gamma$ *is C^*-irreducible*;

(2) Λ *is C^*-simple and* Γ *is icc relatively to* Λ;

(3) Λ *is C^*-simple and, for any $s \in \Gamma \setminus \{e\}$, we have that $s \notin C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$*;

(4) $\Gamma \curvearrowright \partial_F \Lambda$ *is free*;

(5) *There is no nontrivial amenable Λ-URS of* Γ;

(6) Λ *is C^*-simple and $\Gamma \curvearrowright \partial_F \Lambda$ is faithful.*

Proof. (1) $\Longrightarrow$ (2): Follows from [Rørdam 2021, Remark 3.8 and Proposition 5.1].

(2) $\Longrightarrow$ (3): Suppose that there is $s \in \Gamma \setminus \{e\}$ such that $s \in C_\Gamma(\Lambda \cap s^{-1}\Lambda s)$. Take $g_1, \dots, g_n \in \Lambda$ left coset representatives for $\Lambda/(\Lambda \cap s^{-1}\Lambda s)$. Then

$$s^\Lambda = \{g_i k s k^{-1} g_i^{-1} : 1 \leq i \leq n, k \in \Lambda \cap s^{-1}\Lambda s\} = \{g_i s g_i^{-1} : 1 \leq i \leq n\}$$

is finite.

(3) $\Longrightarrow$ (4): Follows from Lemma 3.3.

(4) $\Longrightarrow$ (1): Follows from [Rørdam 2021, Theorem 5.3].

(5) $\Longrightarrow$ (2): If Λ is not C^*-simple, then it contains a nontrivial amenable Λ-uniformly recurrent subgroup. If Γ is not icc relatively to Λ, there exists $s \in \Gamma \setminus \{e\}$ such that s^Λ is finite. Hence, the Λ-orbit of $\langle s \rangle$ is a finite nontrivial amenable Λ-uniformly recurrent subgroup.

(4) $\Longrightarrow$ (5): Follows from Proposition 3.4.

(3) $\Longleftrightarrow$ (6): Follows from Lemma 3.3. □

Remark 3.6. Rørdam [2021, Theorem 5.3] showed that an inclusion $\Lambda \leq \Gamma$ satisfying the hypothesis of Proposition 3.4 is C^*-irreducible, and asked whether the converse holds. We do not know whether the converse of Proposition 3.4 holds and whether the absence of nontrivial amenable Λ-URS of Γ is equivalent to $\Lambda \leq \Gamma$ being C^*-irreducible in general.

Corollary 3.7. *Given $n \in \mathbb{N}$, the inclusion*

$$\mathrm{PSL}(n, \mathbb{Z}) \leq \mathrm{PGL}(n, \mathbb{Q})$$

is C^*-irreducible.

Proof. It was shown in [Bekka et al. 1994] that $\mathrm{PSL}(n,\mathbb{Z})$ is C^*-simple.

Let $U(n,\mathbb{Z})$ be the group of units of the ring $M_n(\mathbb{Z})$. By [Krieg 1990, Corollary V.5.3], $U(n,\mathbb{Z}) \leq_c \mathrm{GL}(n,\mathbb{Q})$. Since $[U(n,\mathbb{Z}) : \mathrm{SL}(n,\mathbb{Z})] = 2$, we conclude that $\mathrm{SL}(n,\mathbb{Z}) \leq_c \mathrm{GL}(n,\mathbb{Q})$ as well. Since taking quotients preserves being commensurated, it follows that $\mathrm{PSL}(n,\mathbb{Z}) \leq_c \mathrm{PGL}(n,\mathbb{Q})$.

Let $(e_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{Z})$ be the matrix units and fix $[a] \in \mathrm{PGL}(n,\mathbb{Q}) \setminus \{[\mathrm{Id}]\}$. By taking conjugates of $[a]$ by elements of the form $[\mathrm{Id} + m \cdot e_{ij}] \in \mathrm{PSL}(n,\mathbb{Z})$, $m \in \mathbb{Z}$, $1 \leq i \neq j \leq n$, it is easy to see that $[a]^{\mathrm{PSL}(n,\mathbb{Z})}$ is infinite, so that $\mathrm{PGL}(n,\mathbb{Q})$ is icc relatively to $\mathrm{PSL}(n,\mathbb{Z})$.

The conclusion then follows from Theorem 3.5. □

Remark 3.8. Let us sketch a different proof of Corollary 3.7 which gives the stronger statement that $\mathrm{PSL}(n,\mathbb{Z}) \leq \mathrm{PGL}(n,\mathbb{R})$ is C^*-irreducible, where $\mathrm{PGL}(n,\mathbb{R})$ is seen as a discrete group.

Clearly, it suffices to show that, for any countable group Γ such that $\mathrm{PSL}(n,\mathbb{Z}) \leq \Gamma \leq \mathrm{PGL}(n,\mathbb{R})$, the inclusion $\mathrm{PSL}(n,\mathbb{Z}) \leq \Gamma$ is C^*-irreducible. By the argument in [Bryder 2017, Example 3.4.3], the action of $\mathrm{PGL}(n,\mathbb{R})$ on the projective space $P^{n-1}(\mathbb{R})$ is topologically free. Since $\mathrm{PSL}(n,\mathbb{Z}) \curvearrowright P^{n-1}(\mathbb{R})$ is a boundary action, the result follows from [Rørdam 2021, Theorem 5.3].

Corollary 3.9. *Let Λ be a finite-index subgroup of a group Γ. If Γ is C^*-simple, then $\Lambda \leq \Gamma$ is C^*-irreducible. Conversely, if Λ is C^*-simple, then Γ is icc if and only if $\Lambda \leq \Gamma$ is C^*-irreducible.*

Proof. If Γ is C^*-simple, then $\Gamma \curvearrowright \partial_F \Gamma$ is free. Since $\partial_F \Gamma$ is Γ-isomorphic to $\partial_F \Lambda$, it follows that $\Lambda \leq \Gamma$ is C^*-irreducible.

If Γ is icc, then, since $[\Gamma : \Lambda] < \infty$, it is also icc relatively to Λ, hence $\Lambda \leq \Gamma$ is C^*-irreducible by Theorem 3.5. The last implication is immediate. □

Example 3.10. The inclusion given by the Sanov subgroup $\mathbb{F}_2 \leq \mathrm{PSL}(2,\mathbb{Z})$ is finite-index, hence it is C^*-irreducible by Corollary 3.9.

Free groups. Fix $m, n \in \mathbb{N}$ such that $2 \leq m < n$ and consider the free groups $\mathbb{F}_m = \langle a_1, \dots, a_m \rangle \leq \langle a_1, \dots, a_n \rangle = \mathbb{F}_n$. Rørdam [2021, Example 5.4] observed that $\mathbb{F}_m \leq \mathbb{F}_n$ is C^*-irreducible. Notice that $\mathbb{F}_m$ is far from being commensurated in $\mathbb{F}_n$. In fact, given $g \in \mathbb{F}_n \setminus \mathbb{F}_m$, we have that $\mathbb{F}_m \cap g\mathbb{F}_m g^{-1} = \{e\}$ (i.e., $\mathbb{F}_m$ is *malnormal* in $\mathbb{F}_n$). In particular, this example is not covered by Theorems 3.1 and 3.5. Nonetheless, there does exist an extension to $\mathbb{F}_n$ of the action $\mathbb{F}_m \curvearrowright \partial_F \mathbb{F}_m$, but it is far from being unique, since the generators $a_{m+1}, \dots, a_n$ can be mapped into any homeomorphisms on $\partial_F \mathbb{F}_m$.

Furthermore, we claim that $\mathbb{F}_m \leq \mathbb{F}_n$ satisfies condition (5) in Theorem 3.5. We will prove this by using Proposition 3.4.

Let

$$\partial\mathbb{F}_n := \Big\{(x_i) \in \prod_{\mathbb{N}}\{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\} : \forall i \in \mathbb{N},\ x_{i+1} \neq x_i^{-1}\Big\}$$

be the Gromov boundary of $\mathbb{F}_n$, and consider the action of $\mathbb{F}_n$ on $\partial\mathbb{F}_n$ by left multiplication. Fix $\mu \in \mathrm{Prob}(\partial\mathbb{F}_n)$, and we will show that there is $w \in \partial\mathbb{F}_n$ on which $\mathbb{F}_n$ acts freely and such that $\delta_w \in \overline{\mathbb{F}_m\mu}$.

Let $z_+ := (a_1)_{i\in\mathbb{N}} \in \partial\mathbb{F}_n$, and let $z_- := (a_1^{-1})_{i\in\mathbb{N}} \in \partial\mathbb{F}_n$. Notice that, for all $y \in \partial\mathbb{F}_n \setminus \{z_-\}$, we have that, as $k \to +\infty$, $a_1^k y \to z_+$. Furthermore, a_1 fixes z_-.

It follows from the dominated convergence theorem that

$$a_1^k\mu \to \mu(\{z_-\})\delta_{z_-} + (1-\mu(\{z_-\}))\delta_{z_+},$$

as $k \to +\infty$. In particular, $\nu := \mu(\{z_-\})\delta_{z_-} + (1-\mu(\{z_-\}))\delta_{z_+} \in \overline{\mathbb{F}_n\mu}$.

Let $w := a_1a_2^1a_1a_2^2a_1a_2^3\cdots a_1a_2^la_1a_2^{l+1}\cdots \in \partial\mathbb{F}_n$. Since w is not eventually periodic, we have that $\mathbb{F}_n$ acts freely on w. Given $k \in \mathbb{N}$, let $g_k := w_1\cdots w_ka_2 \in \mathbb{F}_m$. We have that $g_kz_\pm = w_1\cdots w_ka_2z_\pm \to w$, as $k \to +\infty$. Therefore, $\delta_w \in \overline{\mathbb{F}_m\nu} \subset \overline{\mathbb{F}_m\mu}$, thus showing the claim.

Abstract commensurator. Let Γ be a group and Ω be the set of isomorphisms between finite-index subgroups of Γ. Given $\alpha, \beta \in \Omega$, we say that $\alpha \sim \beta$ if there exists a finite-index subgroup $H \le \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta)$ such that $\alpha|_H = \beta|_H$. Recall that the *abstract commensurator* of Γ, denoted by $\mathrm{Comm}(\Gamma)$, is the group whose underlying set is $\Omega/{\sim}$, with product given by composition (defined up to finite-index subgroup).

Let Λ be a commensurated subgroup of Γ. Given $g \in \Gamma$, let

$$\beta_g\colon \Lambda \cap g^{-1}\Lambda g \to \Lambda \cap g\Lambda g^{-1}, \quad h \mapsto ghg^{-1},$$

and $j_\Lambda^\Gamma\colon \Gamma \to \mathrm{Comm}(\Lambda)$ be the homomorphism given by $j_\Lambda^\Gamma(g) := [\beta_g]$. In order to ease the notation, we will sometimes denote j_Λ^Γ simply by j, and it will always be clear from the context what the involved groups are. Let us now collect a few elementary facts about j.

Lemma 3.11. *Let Γ be a group. Then $j_\Gamma^\Gamma(\Gamma) \le_c \mathrm{Comm}(\Gamma)$.*

Proof. Fix $[\alpha] \in \mathrm{Comm}(\Gamma)$. Given $g \in \mathrm{dom}(\alpha)$, we have that $[\alpha]j(g)[\alpha]^{-1} = j(\alpha(g))$. In particular, $j(\Gamma) \cap [\alpha]j(\Gamma)[\alpha]^{-1} \supset j(\mathrm{Im}(\alpha))$. Since $[\Gamma : \mathrm{Im}(\alpha)] < \infty$, we conclude that $\big[j(\Gamma) : j(\Gamma) \cap [\alpha]j(\Gamma)[\alpha]^{-1}\big] < \infty$. □

Lemma 3.12. *Let $\Lambda \le_c \Gamma$. Then* $\ker j_\Lambda^\Gamma = \{g \in \Gamma : |g^\Lambda| < \infty\}$.

Proof. Given $g \in \ker j$, there exists a finite-index subgroup $H \le \Lambda \cap g^{-1}\Lambda g$ such that, for all $h \in H$, $ghg^{-1} = h$, which implies that $|g^\Lambda| < \infty$. Conversely, if $|g^\Lambda| < \infty$, then $H := \{k \in \Lambda : kg = gk\}$ is a finite-index subgroup of Λ and $g \in \ker j$. □

As a consequence of Lemma 3.12, if Γ is an icc group, then $j:\Gamma\to\mathrm{Comm}(\Gamma)$ is injective [Kida 2011, Lemma 3.8 (i)]. The next result is known [Kida 2011, Lemma 3.8 (iii)]. For the convenience of the reader, we provide the proof here.

Lemma 3.13. *If* Γ *is an icc group, then* $\mathrm{Comm}(\Gamma)$ *is icc relatively to* Γ.

Proof. Given $[\alpha]\in\mathrm{Comm}(\Gamma)$ and $g\in\mathrm{dom}(\alpha)$, we have

$$j(g)[\alpha]j(g^{-1})=j(g\alpha(g^{-1}))[\alpha].$$

If $[\alpha]\neq e$, then $H:=\{g\in\mathrm{dom}(\alpha):g=\alpha(g)\}$ has infinite-index in $\mathrm{dom}(\alpha)$. Given $g_1,g_2\in\mathrm{dom}(\alpha)$ such that $g_1H\neq g_2H$, one can readily check that $g_1\alpha(g_1)^{-1}\neq g_2\alpha(g_2)^{-1}$. From this, it follows immediately that $[\alpha]^{\Gamma}$ is infinite. □

Bédos and Omland [2023, Corollary 6.6] showed that if Γ is a C^*-simple group, then $\Gamma\leq\mathrm{Aut}(\Gamma)$ is C^*-irreducible. The same conclusion holds when we consider the abstract commensurator:

Corollary 3.14. *Given a* C^**-simple group* Γ, *we have that* $\Gamma\leq\mathrm{Comm}(\Gamma)$ *is* C^**-irreducible.*

Proof. Recall that any C^*-simple group is icc (this follows, e.g., from Theorem 3.5). The result is then a consequence of Theorem 3.5 and Lemma 3.13. □

Remark 3.15. Corollary 3.14 generalizes the fact proven in [Le Boudec and Matte Bon 2018, Corollary 4.4] that, if Thompson's group F is C^*-simple, then $\mathrm{Comm}(F)$ is C^*-simple.

Remark 3.16. Let $\mathbb{F}_n$ be a nonabelian free group of finite rank. Then Corollary 3.14 implies that $\mathrm{Comm}(\mathbb{F}_n)$ is C^*-simple. In particular, it does not admit any nontrivial amenable normal subgroup. It is an open problem whether $\mathrm{Comm}(\mathbb{F}_n)$ is a simple group [Caprace and Monod 2018, Problem 7.2].

4. Relative boundaries

Given groups $\Lambda\leq\Gamma$, Ursu [2022, Proposition 4.1] introduced a Λ-strongly proximal Γ-boundary $B(\Gamma,\Lambda)$ which is universal with these properties.

Consider $\Gamma:=\mathrm{PSL}(2,\mathbb{Z})$ and the boundary action $\Gamma\curvearrowright\mathbb{R}\cup\{\infty\}$. The stabilizer Γ_∞ of ∞ is isomorphic to $\mathbb{Z}$ and consists of the translations $g_n(x):=x+n$, $n\in\mathbb{Z}$, $x\in\mathbb{R}$.

Proposition 4.1. *The action of* $\Gamma=\mathrm{PSL}(2,\mathbb{Z})$ *on* $B(\Gamma,\Gamma_\infty)$ *is topologically free but nonfree. In particular,* $B(\Gamma,\Gamma_\infty)$ *is not extremally disconnected.*

Proof. For any $x\in\mathbb{R}\cup\{\infty\}$, we have $g_n(x)\to\infty$ as $n\to+\infty$. As a consequence of the dominated convergence theorem, it follows easily that $\Gamma_\infty\curvearrowright\mathbb{R}\cup\{\infty\}$ is strongly proximal. Hence, there is a Γ-equivariant map $B(\Gamma,\Gamma_\infty)\to\mathbb{R}\cup\{\infty\}$. Since $\Gamma_\infty\curvearrowright B(\Gamma,\Gamma_\infty)$ is strongly proximal, it follows from amenability of Γ_∞ that

Γ_∞ fixes some point in $B(\Gamma, \Gamma_\infty)$. In particular, $\Gamma \curvearrowright B(\Gamma, \Gamma_\infty)$ is not free. On the other hand, since $\Gamma \curvearrowright \mathbb{R} \cup \{\infty\}$ is topologically free, it follows from [Breuillard et al. 2017, Lemma 3.2] that $\Gamma \curvearrowright B(\Gamma, \Gamma_\infty)$ is topologically free. As a consequence of [Frolík 1971, Theorem 3.1], $B(\Gamma, \Gamma_\infty)$ is not extremally disconnected. □

Remark 4.2. Let Γ be a group. One of the key properties in the applications of $\partial_F \Gamma$ to C^*-simplicity of Γ is the fact that $C(\partial_F \Gamma)$ is injective, shown in [Kalantar and Kennedy 2017, Theorem 3.12]. Proposition 4.1 implies that $C(B(\Gamma, \Lambda))$ is not injective, in general. We believe that this is evidence that $B(\Gamma, \Lambda)$ is not likely to play the same role as the Furstenberg boundary in C^*-algebraic applications.

Our next aim is to show that, given $\Lambda \leq_c \Gamma$, it holds that $B(\Gamma, \Lambda) = \partial_F \Lambda$. We start with a result which we believe has its own interest.

Theorem 4.3. *Let $\Lambda \leq_c \Gamma$ and $\Gamma \curvearrowright X$ be a minimal action on a compact space such that $\Lambda \curvearrowright X$ is proximal. Then $\Lambda \curvearrowright X$ is minimal as well.*

Proof. Let $M \subset X$ be a closed nonempty Λ-invariant set. For any $g \in \Gamma$, we have that gM is $g\Lambda g^{-1}$-invariant.

Fix $g_1, \ldots, g_n \in \Gamma$. We have that $H := \Lambda \cap g_1 \Lambda g_1^{-1} \cap \cdots \cap g_n \Lambda g_n^{-1}$ has finite index in Λ. In particular, $H \curvearrowright X$ is proximal and admits a unique minimal component K. Since each $g_i M$ is $g_i \Lambda g_i^{-1}$-invariant, we conclude that $K \subset \bigcap_{i=1}^n g_i M$.

By compactness of X, we obtain that $L := \bigcap_{g \in \Gamma} gM \neq \varnothing$. Since L is Γ-invariant, we have $X = L \subset M$. □

The following is an immediate consequence of the previous theorem:

Corollary 4.4. *Let $\Lambda \leq_c \Gamma$. If X is a Γ-boundary which is also Λ-strongly proximal, then X is a Λ-boundary.*

By arguing as in [Ursu 2022, Corollary 4.3], we conclude the following:

Corollary 4.5. *If $\Lambda \leq_c \Gamma$, then $B(\Gamma, \Lambda) = \partial_F \Lambda$.*

5. Commensurated subgroups and proximal actions

Given a group Γ, there exists a universal minimal proximal Γ-space $\partial_p \Gamma$ [Glasner 1976, Theorem II.4.2]. It was shown in [Frisch et al. 2019, Proposition 2.12] and [Glasner et al. 2021, Theorem 1.5] that a countable group Γ is icc if and only if $\Gamma \curvearrowright \partial_p \Gamma$ is faithful if and only if $\Gamma \curvearrowright \partial_p \Gamma$ is free.

One can easily check that the statements of Theorem 3.1 and Lemma 3.3 hold with $\partial_p \Lambda$ instead of $\partial_F \Lambda$, with the exact same proofs (in particular, [Breuillard et al. 2017, Lemma 5.1], which is needed in the proof of Lemma 3.3, uses only proximality). Thus, we obtain:

Theorem 5.1. *Let $\Lambda \leq_c \Gamma$. Then $\Lambda \curvearrowright \partial_p \Lambda$ extends in a unique way to an action of Γ on $\partial_p \Lambda$. Furthermore, given $s \in \Gamma$, if $s \in \mathrm{C}_\Gamma(\Lambda \cap s^{-1} \Lambda s)$, then $\mathrm{Fix}(s) = \partial_p \Lambda$. Conversely, if $\Lambda \curvearrowright \partial_p \Lambda$ is free and $\mathrm{Fix}(s) \neq \varnothing$, then $s \in \mathrm{C}_\Gamma(\Lambda \cap s^{-1} \Lambda s)$.*

As a consequence, we obtain the following:

Theorem 5.2. *Let $\Lambda \leq_c \Gamma$ and suppose that $\Lambda \curvearrowright \partial_p \Lambda$ is free. The following conditions are equivalent*:

(1) *Γ is icc relatively with Λ;*

(2) *for any $s \in \Gamma \setminus \{e\}$, we have that $s \notin C_\Gamma(\Lambda \cap s^{-1} \Lambda s)$;*

(3) *$\Gamma \curvearrowright \partial_p \Lambda$ is free;*

(4) *$\Gamma \curvearrowright \partial_p \Lambda$ is faithful.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are proven as in Theorem 3.5.

(4) $\Longrightarrow$ (1): Suppose that there is $g \in \Gamma \setminus \{e\}$ such that $|g^\Lambda| < \infty$. Then it follows that $H := \{h \in \Lambda : gh = hg\}$ is a finite-index subgroup of Λ, hence $H \curvearrowright \partial_p \Lambda$ is also minimal and proximal. Since the homeomorphism on $\partial_p \Lambda$ given by g is H-equivariant, we conclude that g acts trivially on $\partial_p \Lambda$. ☐

Remark 5.3. Given a group Γ, let $L(\Gamma)$ be its *group von Neumann algebra*. Given $\Lambda \leq \Gamma$, it follows from [Rørdam 2021, Proposition 5.1] and [Bédos and Omland 2023, Corollary 4.3] that Γ is icc relatively to Λ if and only if any intermediate von Neumann algebra of $L(\Lambda) \subset L(\Gamma)$ is a factor if and only if any intermediate C^*-algebra of $C^*_r(\Lambda) \subset C^*_r(\Gamma)$ is prime.

Let us now apply Theorem 5.2 to a certain locally finite commensurated subgroup of Thompson's group V.

Example 5.4. Let $X := \{0, 1\}$ and, given $n \geq 0$, let X^n be the set of words in X of length n. Given $w \in X^n$, let $\mathcal{C}(w) := \{(s_n) \in X^{\mathbb{N}} : s_{[1,n]} = w\}$. Recall that Thompson's group V is the group of homeomorphisms on $X^{\mathbb{N}}$ consisting of elements g for which there exist two partitions $\{\mathcal{C}(w_1), \dots, \mathcal{C}(w_m)\}$ and $\{\mathcal{C}(z_1), \dots, \mathcal{C}(z_m)\}$ of $\{0, 1\}^{\mathbb{N}}$ such that $g(w_i s) = z_i s$ for every $1 \leq i \leq m$ and $s \in X^{\mathbb{N}}$.

Let us define inductively groups G_n acting by permutations on X^n. Let $G_1 := \mathbb{Z}_2$ acting nontrivially on X and, for $n \in \mathbb{N}$,

$$G_{n+1} := \Big(\bigoplus_{w \in X^n} \mathbb{Z}_2 \Big) \rtimes G_n,$$

where the action of G_{n+1} on X^{n+1} is defined as follows: given $v \in X^n$, $x \in X$, $\sigma \in G_n$ and $f \in \bigoplus_{X^n} \mathbb{Z}_2$,

$$(f, \sigma)(vx) := \sigma(v) f_{\sigma(v)}(x).$$

Let $G := \lim_{n\in\mathbb{N}} G_n$. Then G acts faithfully on $X^{\mathbb{N}}$ and, as observed in [Le Boudec 2017, Proposition 7.11], $G \leq_c V$.

We claim that V is icc relatively with G. Given $u \in X^n$, let the *rigid stabilizer* of u, denoted by $\mathrm{rist}_G(u)$, be the subgroup of G consisting of the elements which, for every $v \in X^n \setminus \{u\}$, act as the identity on $\mathcal{C}(v)$. Given $g \in G$, there is $\tilde{g} \in \mathrm{rist}_G(u)$ such that $\tilde{g}(us) = ug(s)$ for any $s \in X^{\mathbb{N}}$. Clearly, the map $g \mapsto \tilde{g}$ is an isomorphism from G to $\mathrm{rist}_G(u)$. Fix $h \in V \setminus \{e\}$ and take $w \in X^n$ and $z \in X^m$ such that $w \neq z$, $n \geq m$ and $h(ws) = zs$ for any $s \in X^{\mathbb{N}}$. Furthermore, take $v \in X^{n-m}$ such that $zv \neq w$. Given $s \in X^{\mathbb{N}}$, we have that

$$\{\tilde{g}h\tilde{g}^{-1}(wvs) : \tilde{g} \in \mathrm{rist}_G(zv)\} = \{zvg(s) : g \in G\}. \tag{1}$$

Since $G \curvearrowright X^{\mathbb{N}}$ is faithful, it follows from (1) that $|h^G| = \infty$, thus proving the claim.

From [Glasner et al. 2021, Theorem 1.5], we obtain that $G \curvearrowright \partial_p G$ is free and from Theorem 5.2, we conclude that $V \curvearrowright \partial_p G$ is free.

Remark 5.5. Le Boudec and Matte Bon [2018, Theorem 1.5] showed that Thompson's group V is C^*-simple, hence $V \curvearrowright \partial_F V$ is free. However, their proof is done by showing that V does not admit nontrivial amenable URS, not by exhibiting a concrete topologically free V-boundary. It seems as an interesting problem to determine whether $V \curvearrowright \partial_p G$ is strongly proximal, thus providing an alternative proof of C^*-simplicity of V.

Remark 5.6. In [Breuillard et al. 2017, Theorem 1.4], it was shown that the class of C^*-simple groups is closed by taking normal subgroups. Obviously, this class is not closed by taking commensurated subgroups, since any finite subgroup is commensurated. Moreover, Example 5.4 shows that, given $\Lambda \leq_c \Gamma$ such that Γ is icc relatively to Λ, C^*-simplicity of Γ does not pass to Λ in general.

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Received December 30, 2022.

KANG LI
FRIEDRICH-ALEXANDER-UNIVERSITÄT ERLANGEN-NÜRNBERG
ERLANGEN
GERMANY
kang.li@fau.de

EDUARDO SCARPARO
CENTER FOR ENGINEERING
FEDERAL UNIVERSITY OF PELOTAS
PELOTAS
BRAZIL
eduardo.scarparo@ufpel.edu.br

PACIFIC JOURNAL OF MATHEMATICS

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 322 No. 2 February 2023