Pacific Journal of Mathematics

Volume 323 No. 1 March 2023

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.1

COMBINATORIAL PROPERTIES OF NONARCHIMEDEAN CONVEX SETS

ARTEM CHERNIKOV AND ALEX MENNEN

We study combinatorial properties of convex sets over arbitrary valued fields. We demonstrate analogs of some classical results for convex sets over the reals (for example, the fractional Helly theorem and Bárány's theorem on points in many simplices), along with some additional properties not satisfied by convex sets over the reals, including finite breadth and VC dimension. These results are deduced from a simple combinatorial description of modules over the valuation ring in a spherically complete valued field.

1. Introduction

Convexity in the context of nonarchimedean valued fields was introduced in a series of papers by Monna [1946], and has been extensively studied since then in nonarchimedean functional analysis (see for instance the monographs [Perez-Garcia and Schikhof 2010; Schneider 2002] on the subject). Convexity here is defined analogously to the real case, with the role of the unit interval played instead by a valuational unit ball (see Definition 2.1). Convex subsets of $\mathbb{R}^d$ admit rich combinatorial structure, including many classical results around the theorems of Helly, Radon, Carathéodory, Tverberg, etc. — we refer to [De Loera et al. 2019] for a recent survey of the subject. In the case of $\mathbb{R}$, or more generally a real closed field, there is a remarkable parallel between the combinatorial properties of convex and semialgebraic sets (which correspond to definable sets from the point of view of model theory). They share many (but not all) properties in the form of various restrictions on the possible intersection patterns, including the fractional Helly theorem and existence of (weak) ε-nets. A well-studied phenomenon in model theory establishes strong parallels between definable sets in $\mathbb{R}$ and in many nonarchimedean valued fields such as the p-adics $\mathbb{Q}_p$ or various fields of power series (see for instance [van den Dries 2014]). In this paper we focus on the combinatorial study of convex sets over general valued fields, trying to understand

MSC2020: 12J25, 52A01, 52A20, 52A35.
Keywords: nonarchimedean fields, valued fields, combinatorial convexity, Helly theorem, Bárány theorem, VC dimension, breadth.

if there is similarly a parallel theory. On the one hand, we demonstrate valued field analogs of some classical results for convex sets over the reals (e.g., the fractional Helly theorem and Bárány's theorem on points in many simplices). On the other hand, we establish some additional properties not satisfied by convex sets over the reals, including finite breadth and VC dimension. This suggests that in a sense convex sets over valued fields are the best of both worlds combinatorially, and satisfy various properties enjoyed either by convex or by semialgebraic sets over the reals.

We give a quick outline of the paper. Section 2 covers some basics concerning convexity for subsets of K^d over an arbitrary valued field K, in particular discussing the connection to modules over the valuation ring. These results are mostly standard (or small variations of standard results), and can be found in [Perez-Garcia and Schikhof 2010; Schneider 2002] under the unnecessary assumption that K is spherically complete and $(\Gamma, +) \subseteq (\mathbb{R}_{>0}, \times)$; we provide some proofs for completeness. In Section 3 we give a simple combinatorial description of the submodules of K^d over the valuation ring $\mathcal{O}_K$ in the case of a spherically complete field K (Theorem 3.6 and Corollary 3.12), and an analog for finitely generated modules over arbitrary valued fields (Corollary 3.14). We also give an example of a convex set over the field of Puiseux series demonstrating that the assumption of spherical completeness is necessary for our presentation in the nonfinitely generated case (Example 3.11). In Section 4 we use this description of modules to deduce various combinatorial properties of the family of convex subsets Conv_{K^d} of K^d over an arbitrary valued field K. First we show that Conv_{K^d} has breadth d (Theorem 4.3), VC dimension $d+1$ (Theorem 4.8), dual VC dimension d (Theorem 4.10) — in stark contrast, all of these are infinite for the family of convex subsets of $\mathbb{R}^d$ for $d \geq 2$. On the other hand, we obtain valued field analogs of the following classical results: the family Conv_{K^d} has Helly number $d+1$ (Theorem 4.5), fractional Helly number $d+1$ (Theorem 4.14), satisfies a strong form of Tverberg's theorem (Theorem 4.15) and the Boros–Füredi/Bárány theorem on the existence of a common point in a positive fraction of all geometric simplices generated by an arbitrary finite set of points in K^d (Theorem 4.16). Some of the proofs here are adaptations of the classical arguments, and some rely crucially on the finite breadth property specific to the valued field context. Finally, in Section 5A we point out some further applications, for example a valued field analog of the celebrated (p, q)-theorem of Alon and Kleitman [1992] (Corollary 5.1), and that all convex sets over a spherically complete field are externally definable in the sense of model theory (Remark 5.7); as well as pose some questions and conjectures. We also discuss some other notions of convexity over nonarchimedean fields appearing in the literature in Section 5B, and place our work in the context of the study of abstract convexity spaces in discrete geometry and combinatorics in Section 5C.

2. Preliminaries on convexity over valued fields

Notation. For $n \in \mathbb{N}_{\geq 1}$, we write $[n] = \{1, \dots, n\}$ and $\langle \cdot \rangle$ denotes the span in vector spaces. Throughout the paper, K will denote a valued field, with value group $\Gamma = \Gamma_K$, and valuation $\nu = \nu_K : K \to \Gamma_\infty := \Gamma \sqcup \{\infty\}$, valuation ring $\mathcal{O} = \mathcal{O}_K = \nu^{-1}([0, \infty])$, maximal ideal $\mathfrak{m} = \mathfrak{m}_K = \nu^{-1}((0, \infty])$, and residue (class) field $k = \mathcal{O}/\mathfrak{m}$. The residue map $\mathcal{O} \to k$ will be denoted $\alpha \mapsto \bar{\alpha}$. For a ring R, $R^\times$ denotes its group of units.

The following definition of convexity is analogous to the usual one over $\mathbb{R}$, with the unit interval replaced by the (valuational) unit ball.

Definition 2.1. (1) For $d \in \mathbb{N}_{\geq 1}$, a set $X \subseteq K^d$ is *convex* if, for any $n \in \mathbb{N}_{\geq 1}$, $x_1, \dots, x_n \in X$, and $\alpha_1, \dots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \dots + \alpha_n = 1$ we have $\alpha_1 x_1 + \dots + \alpha_n x_n \in X$ (in the vector space K^d).

(2) The family of convex subsets of K^d will be denoted Conv_{K^d}.

It is immediate from the definition that the intersection of any collection of convex subsets of K^d is convex.

Definition 2.2. Given an arbitrary set $X \subseteq K^d$, its *convex hull* $\mathrm{conv}(X)$ is the convex set given by the intersection of all convex sets containing X, equivalently

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N},\ \alpha_i \in \mathcal{O},\ x_i \in X,\ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Definition 2.3. A (valuational) *quasiball* is a set $B = \{x \in K : \nu(x - c) \in \Delta\}$ for some $c \in K$ and an upwards closed subset Δ of Γ_∞. In this case we say that B is *around* c, and refer to Δ as the quasiradius of B. We say that B is a *closed* (respectively, *open*) *ball* if additionally $\Delta = \{\gamma \in \Gamma : \gamma \geq r\}$ (respectively, $\Delta = \{\gamma \in \Gamma : \gamma > r\}$) for some $r \in \Gamma$, and just *ball* if B is either an open or a closed ball (in which case we refer to r as its *radius*).

Remark 2.4. (1) If the value group Γ is Dedekind complete, then every quasiball is a ball (except for K itself, which is a quasiball of quasiradius Γ_∞).

(2) If B is a quasiball of quasiradius Δ around c and $c' \in B$ is arbitrary, then B is also a quasiball of quasiradius Δ around c'.

(3) Thus, any two quasiballs are either disjoint, or one of them contains the other.

Example 2.5. (1) The convex subsets of $K = K^1$ are exactly $\varnothing$ and the quasiballs (see Proposition 2.10 and Example 2.11).

(2) If $e_1, \dots, e_d$ is the standard basis of the vector space K^d, then

$$\mathrm{conv}(\{0, e_1, \dots, e_d\}) = \mathcal{O}^d.$$

(3) The image and the preimage of a convex set under an affine map are convex. In particular, a translate of a convex set is convex, and a projection of a convex set is convex. (Recall that given two vector spaces V, W over the same field K, a map $f : V \to W$ is *affine* if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in V, \alpha, \beta \in K, \alpha + \beta = 1$.)

One might expect, by analogy with real convexity, that the definition of a convex set could be simplified to: if $x, y \in X$, $\alpha, \beta \in \mathcal{O}$ such that $\alpha+\beta=1$, then $\alpha x+\beta y \in X$. The following two propositions show that this is the case if and only if the residue field is not isomorphic to $\mathbb{F}_2$, and that in general we have to require closure under 3-element convex combinations.

Proposition 2.6. *Let K be a valued field and $X \subseteq K^d$. If X is closed under* 3*-element convex combinations* (*in the sense that if $x, y, z \in X$ and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha + \beta + \gamma = 1$, then $\alpha x + \beta y + \gamma z \in X$*), *then X is convex.*

Proof. Suppose X is closed under 3-element convex combinations. We will show by induction on n that then X is closed under n-element convex combinations. Let $n \geq 3$, $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_n \in \mathcal{O}$ such that $\alpha_1 + \cdots + \alpha_n = 1$ be given. Then one of the following two cases holds.

Case 1. $\alpha_1 + \alpha_2 \in \mathcal{O}^\times$. Then $\alpha_1/(\alpha_1 + \alpha_2)$ and $\alpha_2/(\alpha_1 + \alpha_2)$ are elements of $\mathcal{O}$ that sum to 1, so

$$\frac{\alpha_1}{\alpha_1+\alpha_2}x_1 + \frac{\alpha_2}{\alpha_1+\alpha_2}x_2 \in X$$

by assumption. But then

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = (\alpha_1 + \alpha_2)\Big(\frac{\alpha_1}{\alpha_1+\alpha_2}x_1 + \frac{\alpha_2}{\alpha_1+\alpha_2}x_2\Big) + \alpha_3 x_3 + \cdots + \alpha_n x_n \in X$$

by the induction hypothesis, as it is a convex combination of $n-1$ elements of X.

Case 2. $\alpha_1 + \alpha_2 \in \mathfrak{m}$. Then, as $v\big(\sum_{i=1}^n \alpha_i\big) = 0$, there must exist some i with $3 \leq i \leq n$ such that $\alpha_i \in \mathcal{O}^\times$. Hence $\alpha_1 + \alpha_2 + \alpha_i \in \mathcal{O}^\times$, so $\alpha_1/(\alpha_1 + \alpha_2 + \alpha_i)$, $\alpha_2/(\alpha_1 + \alpha_2 + \alpha_i)$, and $\alpha_i/(\alpha_1 + \alpha_2 + \alpha_i)$ are elements of $\mathcal{O}$ that sum to 1. Thus

$$\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_i}x_1 + \frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_i}x_2 + \frac{\alpha_i}{\alpha_1+\alpha_2+\alpha_i}x_i \in X$$

by assumption, and so

$$\begin{aligned}\alpha_1 x_1+\cdots+\alpha_n x_n = (\alpha_1+\alpha_2+\alpha_i)\Big(\frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_i}x_1+\frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_i}x_2+\frac{\alpha_i}{\alpha_1+\alpha_2+\alpha_i}x_i\Big)\\ + \alpha_3 x_3 + \cdots + \alpha_{i-1}x_{i-1} + \alpha_{i+1}x_{i+1} + \cdots + \alpha_n x_n \in X\end{aligned}$$

by the induction hypothesis, as it is a convex combination of $n-2$ elements of X. □

Proposition 2.7. *For any valued field K, the following are equivalent*:

(1) *For every $d \geq 1$, every set in K^d that is closed under* 2*-element convex combinations is convex.*

(2) *The residue field k is not isomorphic to $\mathbb{F}_2$.*

Proof. (1) *implies* (2). If $k = \mathbb{F}_2$, consider the set

$$X := \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathcal{O},\ \exists i \text{ such that } a_i \in \mathfrak{m}\} \subseteq K^3.$$

We claim that X is closed under 2-element convex combinations. That is, given arbitrary $(a_1, a_2, a_3), (b_1, b_2, b_3) \in X$ and $\alpha, \beta \in \mathcal{O}$ with $\alpha + \beta = 1$, we must show that $\alpha(a_1, a_2, a_3) + \beta(b_1, b_2, b_3) \in X$. We have $\bar{\alpha} + \bar{\beta} = 1$ in $k = \mathbb{F}_2$, so necessarily one of $\bar{\alpha}$ and $\bar{\beta}$ is 1 and the other is 0. Without loss of generality $\bar{\alpha} = 1$ and $\bar{\beta} = 0$. Then $\beta \in \mathfrak{m}$. By definition of X, $a_i \in \mathfrak{m}$ for some i. Then $\alpha a_i \in \mathfrak{m}$, and $\beta b_i \in \mathfrak{m}$ as $b_i \in \mathcal{O}$, so $\alpha a_i + \beta b_i \in \mathfrak{m}$. Thus $(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \alpha a_3 + \beta b_3) \in X$. However X is not convex: for an arbitrary $a \in \mathfrak{m}$ we have $(0, 0, 0), (1, 0, 0), (0, 1, 1) \in X$, $1, -1 \in \mathcal{O}$, but $(-1)(0, 0, 0) + 1(1, 0, 0) + 1(0, 1, 1) = (1, 1, 1) \notin X$. (This example can be modified to work in K^2.)

(2) *implies* (1). If $k \not\cong \mathbb{F}_2$, suppose X is closed under 2-element convex combinations. By Proposition 2.6, we only need to check that it is then closed under 3-element convex combinations. Let $x, y, z \in X$, and $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha + \beta + \gamma = 1$. Then one of the following two cases holds.

Case 1. At least one of $\alpha + \beta, \beta + \gamma, \alpha + \gamma$ is an element of $\mathcal{O}^\times$. Without loss of generality, $\alpha + \beta \in \mathcal{O}^\times$. Then $(\alpha/(\alpha + \beta))x + (\beta/(\alpha + \beta))y \in X$ by assumption, and thus

$$\alpha x + \beta y + \gamma z = (\alpha + \beta)\Big(\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y\Big) + \gamma z \in X.$$

Case 2. $\alpha + \beta, \beta + \gamma, \alpha + \gamma \in \mathfrak{m}$. In the residue field, $\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\gamma} = \bar{\alpha} + \bar{\gamma} = 0$, and $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 1$, hence necessarily $\bar{\alpha} = \bar{\beta} = \bar{\gamma} = 1$, and $\operatorname{char}(k) = 2$. Since $k \not\cong \mathbb{F}_2$, there is $\delta \in \mathcal{O}$ such that $\bar{\delta} \notin \{0, 1\}$. Then $\bar{\alpha} + \bar{\delta} = 1 + \bar{\delta} \neq 0$ and $\bar{\beta} - \bar{\delta} + \bar{\gamma} = \bar{\delta} \neq 0$, so

$$\alpha x + \beta y + \gamma z = (\alpha + \delta)\Big(\frac{\alpha}{\alpha+\delta}x + \frac{\delta}{\alpha+\delta}y\Big) + (\beta - \delta + \gamma)\Big(\frac{\beta-\delta}{\beta-\delta+\gamma}y + \frac{\gamma}{\beta-\delta+\gamma}z\Big) \in X. \quad \square$$

The following proposition gives a very strong form of Radon's theorem (not only do we obtain a partition into two sets with intersecting convex hulls, but moreover one of the points is in the convex hull of the other ones).

Proposition 2.8. *Let K be a valued field. For any $d + 2$ points $x_1, \ldots, x_{d+2} \in K^d$, one of them is in the convex hull of the others.*

Proof. There exist $a_1, \ldots, a_{d+2} \in K$, *not all* 0, such that $\sum_{i=1}^{d+2} a_i x_i = 0$ and $\sum_{i=1}^{d+2} a_i = 0$ (because those are $d + 1$ linear equations on $d + 2$ variables, as we are working in K^d). Let $i \in [d + 2]$ be such that $\nu(a_i)$ is minimal among $\nu(a_1), \ldots, \nu(a_{d+2})$, in particular $a_i \neq 0$. Then $x_i = \sum_{j \neq i}(-a_j/a_i)x_j$, and this is a convex combination: for $i \neq j$ we have $-a_j/a_i \in \mathcal{O}$ (as $\nu(-a_j/a_i) = \nu(a_j) - \nu(a_i) \geq 0$ by the choice of i) and $\sum_{j\neq i}(-a_j/a_i) = \left(-\sum_{j\neq i} a_j\right)/a_i = a_i/a_i = 1$. □

By a repeated application of Proposition 2.8 we immediately get a very strong form of Carathéodory's theorem:

Corollary 2.9. *Let K be a valued field. Then the convex hull of any finite set in K^d is already given by the convex hull of at most $d+1$ points from it.*

Convex sets over valued fields have a natural algebraic characterization.

Proposition 2.10. (1) *A subset $C \subseteq K^d$ is an $\mathcal{O}$-submodule of K^d if and only if it is convex and contains* 0*.*

(2) *Nonempty convex subsets of K^d are precisely the translates of $\mathcal{O}$-submodules of K^d.*

Proof. (1) $\mathcal{O}$-submodules of K^d are clearly convex and contain 0. Now suppose $C \subseteq K^d$ is convex and $0 \in C$. Then for any $\alpha \in \mathcal{O}$ and $x \in C$, $\alpha x = \alpha x + (1-\alpha)0 \in C$. And for any $x, y \in C$, $x + y = 1 \cdot x + 1 \cdot y - 1 \cdot 0 \in C$. Therefore C is an $\mathcal{O}$-submodule.

(2) Given a nonempty convex $C \subseteq K^d$, we can choose $a \in K^d$ such that the translate $C + a$ contains 0 and is still convex, hence $C + a$ is an $\mathcal{O}$-submodule of K^d by (1). □

Example 2.11. Let C be an $\mathcal{O}$-submodule of K, and take $\Delta := \nu(C)$. Then Δ is nonempty because it contains $\infty = \nu(0)$, and upward-closed because for $\gamma \in \Delta$ and $\delta > \gamma$, there is $x \in C$ with $\nu(x) = \gamma$, and $\alpha \in K$ with $\nu(\alpha) = \delta - \gamma$; then $\alpha x \in C$ and $\nu(\alpha x) = \delta$. Clearly $C \subseteq \{x \in K \mid \nu(x) \in \Delta\}$ by definition of Δ. To show $C \supseteq \{x \in K \mid \nu(x) \in \Delta\}$, given any $x \in K$ with $\nu(x) \in \Delta$, there is $y \neq 0 \in C$ with $\nu(y) = \nu(x)$, and $x/y \in \mathcal{O}$, so $x = (x/y)y \in C$. Thus $C = \{x \in K \mid \nu(x) \in \Delta\}$ is a quasiball around 0.

Corollary 2.12. *The convex hull of any finite set in K^d is the image of $\mathcal{O}^d$ under an affine map.*

Proof. By Corollary 2.9, the convex hull of a finite subset of K^d is the convex hull of some $d+1$ points $x_0, \ldots, x_d$ from it (possibly with $x_i = x_j$ for some i, j). Let $e_1, \ldots, e_d$ be the standard basis for K^d, and let f be an affine map $f : K^d \to K^d$ such that $f(0) = x_0$ and $f(e_i) = x_i$ for $1 \leq i \leq d$ (we can take f to be the composition of two affine maps: the linear map sending e_i to $x_i - x_0$ for $1 \leq i \leq d$, and translation by x_0). Then we have $\operatorname{conv}(\{x_0, \ldots, x_d\}) = f(\operatorname{conv}\{0, e_1, \ldots, e_d\}) = f(\mathcal{O}^d)$, by Example 2.5(2). □

Proposition 2.13. *For any convex $C \subseteq K^d$ and $a \in K^d$, the translate $C + a := \{x + a \mid x \in C\}$ is either equal to or disjoint from C.*

Proof. If $x \in C \cap (C + a)$, then $y + a = y + x - (x - a) \in C$ for all $y \in C$ since that is a convex combination, and conversely $y = (y + a) - x + (x - a) \in C$ if $y + a \in C$. □

Definition 2.14. Given a valued field K, by a *valued K-vector space* we mean a K-vector space V equipped with a surjective map $\nu = \nu_V : V \to \Gamma_\infty = \Gamma \cup \{\infty\}$ such that $\nu(x) = \infty$ if and only if $x = 0$, $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$ and $\nu(\alpha x) = \nu_K(\alpha) + \nu(x)$ for all $x, y \in V$ and $\alpha \in K$.

Remark 2.15. Here we restrict to the case when V has the same value group as K, and refer to [Fuchs 1975] for a more general treatment (see also [Johnson 2016, Section 6.1.3; Hrushovski 2014, Section 2.5; Aschenbrenner et al. 2017, Section 2.3]).

By a morphism of valued K-vector spaces we mean a morphism of vector spaces preserving valuation. If V and W are valued K-vector spaces, their direct sum $V \oplus W$ is the direct sum of the underlying vector spaces equipped with the valuation $\nu(x, y) := \min\{\nu_V(x), \nu_W(y)\}$. In particular, the vector space K^d is a valued K-vector space with respect to the valuation $\nu_{K^d} : K^d \to \Gamma_\infty$ given by

$$\nu_{K^d}(x_1, \ldots, x_d) := \min\{\nu_K(x_1), \ldots, \nu_K(x_d)\}.$$

Note that for any scalar $\alpha \in K$ and vector $v \in K^d$ we have $\nu_{K^d}(\alpha v) = \nu_K(\alpha) + \nu_{K^d}(v)$. By a (*valuational*) *ball* in K^d we mean a set of the form $\{x \in K^d : \nu_{K^d}(x - c) \square r\}$ for some center $c \in K^d$, radius $r \in \Gamma \cup \{\infty\}$ and $\square \in \{>, \geq\}$ (corresponding to open or closed ball, respectively). The collection of all open balls forms a basis for the *valuation topology* on K^d turning it into a topological vector space. Note that due to the "ultrametric" property of valuations, every open ball is also a closed ball, and vice versa. Equivalently, this topology on K^d is just the product topology induced from the valuation topology on K.

Recall that the *affine span* $\operatorname{aff}(X)$ of a set $X \subseteq K^d$ is the intersection of all affine sets (i.e., translates of vector subspaces of K^d) containing X, equivalently

$$\operatorname{aff}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N}_{\geq 1},\ \alpha_i \in K,\ x_i \in X,\ \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

We have $\operatorname{conv}(X) \subseteq \operatorname{aff}(X)$ for any X.

Proposition 2.16. *Any convex set in K^d is open in its affine span.*

Proof. For $x \in C \subseteq K^d$, C convex, let $d' \leq d$ be the dimension of the affine span of C, and let $y_1, \ldots, y_{d'} \in C$ be such that $x, y_1, \ldots, y_{d'}$ are affinely independent, and thus have the same affine span as C. Then the map $(\alpha_1, \ldots, \alpha_{d'}) \mapsto x + \alpha_1(y_1 - x) + \cdots + \alpha_{d'}(y_{d'} - x)$ is a homeomorphism from $K^{d'}$ to the affine span of C, and sends $\mathcal{O}^{d'}$ (which is open in $K^{d'}$) to a neighborhood of x contained in C. □

Corollary 2.17. *Convex sets in K^d are closed.*

Proof. For convex $C \subseteq K^d$ and $x \in \operatorname{aff}(C) \setminus C$, $C + x$ is an open subset of $\operatorname{aff}(C)$ that is disjoint from C, so C is a closed subset of its affine span, and hence closed in K^d, since affine subspaces are closed. □

3. Classification of $\mathcal{O}$-submodules of K^d

In this section we provide a simple description for the $\mathcal{O}$-submodules of K^d over a spherically complete valued field K (and over an arbitrary valued field K in the finitely generated case). Combined with the description of convex sets in terms of $\mathcal{O}$-submodules from Section 2, this will allow us to establish various combinatorial properties of convex sets over valued fields in the next section. In the following lemma, the construction of the valuation ν is a special case of the standard construction of the quotient norm, when modding out a normed space by a closed subspace, while the second part is more specific to our situation.

Lemma 3.1. *Let K be a valued field, and $V \subseteq K^d$ a subspace. Then the quotient vector space K^d/V is a valued K-vector space equipped with the valuation*

$$\nu(u) := \max\{\nu_{K^d}(v) \mid \pi(v) = u,\ v \in K^d\},$$

for $u \in K^d/V$, where $\pi : K^d \to K^d/V$ is the projection map (and the maximum is taken in Γ_∞). If $\dim(V) = n$, then $K^d/V \cong K^{d-n}$ as valued K-vector spaces, and there is a valuation preserving embedding of K-vector spaces $f : K^d/V \hookrightarrow K^d$ so that $\pi \circ f = \mathrm{id}_{K^d/V}$.

Proof. First we prove the lemma for $n = 1$. Let $V \subseteq K^d$ be one-dimensional. There exists $i \in [d]$ such that $\nu_{K^d}((x_1, \dots, x_d)) = \nu_K(x_i)$ for all $(x_1, \dots, x_d) \in V$ (indeed, if $\nu_K(x_i) = \min\{\nu_K(x_1), \dots, \nu_K(x_d)\}$ for some $(x_1, \dots, x_d) \in V$, then we also have $\nu_K(\alpha x_i) = \nu_K(\alpha) + \nu_K(x_i) = \nu_K(\alpha) + \min\{\nu_K(x_1), \dots, \nu_K(x_d)\} = \min\{\nu_K(\alpha x_1), \dots, \nu_K(\alpha x_d)\}$ for any $\alpha \in K$). Given any $(x_1, \dots, x_d) \in K^d$ with $x_i = 0$ and $(y_1, \dots, y_d) \in V$, we have

$$\begin{aligned}(3\text{-}1)\qquad \nu_{K^d}(x_1 + y_1, \dots, x_d + y_d) &= \min_{j\in[d]}\{\nu_K(x_j + y_j)\}\\ &= \min\{\nu_K(y_i), \min_{j\neq i}\{\nu_K(x_j + y_j)\}\} \le \nu_K(y_i)\\ &= \nu_{K^d}(y_1, \dots, y_d).\end{aligned}$$

Now consider an arbitrary affine translate $x + V$ of V, $x = (x_1, \dots, x_d) \in K^d$. Then there exists $x' = (x'_1, \dots, x'_d) \in x + V$ so that $x'_i = 0$. Indeed, fix any $0 \neq y' \in V$, then $V = \{\alpha y' : \alpha \in K\}$. Take $\alpha' := -x_i/y'_i$ (note that, by the choice of i, $y' \neq 0 \Rightarrow \nu_{K^d}(y') \neq \infty \Rightarrow \nu_K(y'_i) \neq \infty \Rightarrow y'_i \neq 0$), and let $x' := x + \alpha' y'$. We claim that $\nu_{K^d}(x') = \max\{\nu_{K^d}(z) : z \in x + V\}$, thus the valuation ν on K^d/V is well defined. Indeed, $x + V = x' + V$, so fix any $y \in V$. If $\nu_{K^d}(x') < \nu_{K^d}(x' + y)$, we must necessarily have $\nu_{K^d}(x') = \nu_{K^d}(y)$, but by (3-1) we have $\nu_{K^d}(x'+y) \le \nu_{K^d}(y)$, so $\nu_{K^d}(y) < \nu_{K^d}(y)$ — a contradiction; thus $\nu_{K^d}(x') \ge \nu_{K^d}(x' + y)$.

Let $K' := \{(x_1, \dots, x_d) \in K^d \mid x_i = 0\}$, then we have $K^d = V \oplus K'$ as vector spaces, hence the projection of K^d onto K' along V induces an isomorphism between K^d/V

and K', which in turn is naturally isomorphic to K^{d-1}, and these isomorphisms preserve the valuation and give the desired embedding $f : K^d/V \to K^d$. The general case follows by induction on n using the vector space isomorphism theorems. □

We recall an appropriate notion of completeness for valued fields. Recall that a family $\{C_i : i \in I\}$ of subsets of a set X is *nested* if for any $i, j \in I$, either $C_i \subseteq C_j$ or $C_j \subseteq C_i$.

Definition 3.2. A valued field K is *spherically complete* if every nested family of (closed or open) valuational balls has nonempty intersection.

For the following standard fact, see for example Theorem 5 in Section II.3 and Theorem 8 in Section II.6 of [Schilling 1950].

Fact 3.3. *Every valued field K (with valuation ν_K, value group Γ_K and residue field k_K) admits a* ***spherical completion****, i.e., a valued field $\widetilde{K}$ (with valuation $\nu_{\widetilde{K}}$, value group $\Gamma_{\widetilde{K}}$ and residue field $k_{\widetilde{K}}$), so that:*

(1) *$\widetilde{K}$ is an* ***immediate extension*** *of K, i.e., $\widetilde{K}$ is a field extension of K, $\nu_{\widetilde{K}} \restriction_K = \nu_K$, $\Gamma_{\widetilde{K}} = \Gamma_K$ and $k_{\widetilde{K}} = k_K$.*

(2) *$\widetilde{K}$ is spherically complete.*

We remark that in general a valued field might have multiple nonisomorphic spherical completions.

Lemma 3.4. *If K is spherically complete, then every nested family of nonempty convex subsets of K^d has a nonempty intersection.*

Proof. By induction on d. For $d = 1$, let $\{C_i\}_{i\in I}$ be a nested family of nonempty convex sets, so each C_i is a quasiball; see Example 2.5(1). If there exists some $i \in I$ such that C_i is the smallest of these under inclusion, then any element of C_i is in the intersection of the whole family. Hence we may assume that for each $i \in I$ there exists some $i' \in I$ such that $C_{i'} \subsetneq C_i$. Let Δ_i and $\Delta_{i'}$ be the quasiradii of C_i and $C_{i'}$, respectively. We may assume that both quasiballs are around the same point $x_i \in C_{i'}$ (by Remark 2.4), hence necessarily $\Delta_{i'} \subsetneq \Delta_i$. Let $r_i \in \Delta_i \setminus \Delta_{i'}$, and let C'_i be a (open or closed) ball of radius r_i around x_i. We have $C'_i \subseteq C_i$, so if $\bigcap_{i\in I} C'_i$ is nonempty, then so is $\bigcap_{i\in I} C_i$. Hence it is sufficient to show that $\{C'_i\}_{i\in I}$ is nested, and then the intersection is nonempty by spherical completeness of K. By construction for any $i, j \in I$ there exists some $\ell \in I$ such that $C_\ell \subseteq C'_i \cap C'_j$, so C'_i and C'_j have nonempty intersection, and are thus nested as they are balls.

For $d \geq 2$, let $\{C_i\}_{i\in I}$ be a nested family of nonempty convex sets, and let $\pi_1 : K^d \to K$ be the projection onto the first coordinate. Then $\{\pi_1(C_i)\}_{i\in I}$ is a nested family of nonempty convex sets in K, hence has an intersection point x. Then $\{\pi_1^{-1}(x) \cap C_i\}_{i\in I}$ is a nested family of nonempty convex sets in $\pi_1^{-1}(x) \cong K^{d-1}$, which is nonempty by the induction hypothesis. □

Lemma 3.5. *If $C \subseteq K^d$ is an $\mathcal{O}$-module, and $\gamma \in \Gamma_\infty$, then the set*

$$X_\gamma = \{(x_1, \dots, x_{d-1}) \in \mathcal{O}^{d-1} \mid \exists \alpha \in K,\ \nu(\alpha) = \gamma,\ (\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C\}$$

is convex.

Proof. Let $x = (x_1, \dots, x_{d-1}), y = (y_1, \dots, y_{d-1}), z = (z_1, \dots, z_{d-1}) \in X_\gamma$ and $\beta_1, \beta_2, \beta_3 \in \mathcal{O}$ with $\beta_1 + \beta_2 + \beta_3 = 1$ be arbitrary. Then there exist some $\alpha_1, \alpha_2, \alpha_3 \in K$ with $\nu(\alpha_i) = \gamma$, so that

$$(\alpha_1, \alpha_1 x_1, \dots, \alpha_1 x_{d-1}), (\alpha_2, \alpha_2 y_1, \dots, \alpha_2 y_{d-1}), (\alpha_3, \alpha_3 z_1, \dots, \alpha_3 z_{d-1}) \in C.$$

Taking $\alpha := \alpha_1$, we have

$$x' := (\alpha, \alpha x_1, \dots, \alpha x_{d-1}), \quad y' := (\alpha, \alpha y_1, \dots, \alpha y_{d-1}),$$
$$z' := (\alpha, \alpha z_1, \dots, \alpha z_{d-1}) \in C,$$

as for every $i \in [3]$, $\alpha/\alpha_i \in \mathcal{O}$, and hence $(\alpha/\alpha_i)v \in C$ for any $v \in C$ as C is an $\mathcal{O}$-module. Using this and convexity of C we thus have

$$\begin{aligned}(\alpha, \alpha(\beta_1 x_1 + \beta_2 y_1 + \beta_3 z_1), \dots, \alpha(\beta_1 x_{d-1} + \beta_2 y_{d-1} + \beta_3 z_{d-1}))\\ = \beta_1(\alpha, \alpha x_1, \dots, \alpha x_{d-1}) + \beta_2(\alpha, \alpha y_1, \dots, \alpha y_{d-1}) + \beta_3(\alpha, \alpha z_1, \dots, \alpha z_{d-1})\\ = \beta_1 x' + \beta_2 y' + \beta_3 z' \in C.\end{aligned}$$

This shows that $\beta_1 x + \beta_2 y + \beta_3 z \in X_\gamma$, and hence that X_γ is convex using Proposition 2.6. □

Combining the lemmas, we obtain a description of the $\mathcal{O}_K$-submodules of K^d for spherically complete K:

Theorem 3.6. *Suppose K is a spherically complete valued field, $d \in \mathbb{N}_{\geq 1}$, and let $C \subseteq K^d$ be an $\mathcal{O}$-submodule. Then there exists a complete flag of vector subspaces $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = K^d$ and a decreasing sequence of nonempty, upwards closed subsets $\Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_d$ of Γ_∞ such that*

$$C = \{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}.$$

Remark 3.7. If F_i and Δ_i satisfy the conclusion of Theorem 3.6 for C, then $\nu_{K^d}(C \cap F_1) = \nu_{K^d}(C) = \Delta_1$.

Indeed, any $v \in C$ is of the form $v = v_1 + \cdots + v_d$ with $v_i \in F_i,\ \nu(v_i) \in \Delta_i$ and $\Delta_1 \supseteq \Delta_i$ for all $i \in [d]$, hence $\nu(v) \geq \min\{\nu(v_i) : i \in [d]\} \in \Delta_1$, hence $\nu(v) \in \Delta_1$ as Δ_1 is upwards closed, so $\nu(C) \subseteq \Delta_1$. Conversely, assume $\gamma \in \Delta_1$. If $\gamma = \infty$, then $\nu(0) = \infty$ and $0 \in F_1$. So assume $\gamma \in \Gamma$ and let v be any nonzero vector in F_1, and define $\delta := \nu(v) \in \Gamma$. Taking $\alpha \in K$ so that $\nu_K(\alpha) = \gamma - \delta$, we have $\alpha v \in F_1$ and $\nu_{K^d}(\alpha v) = \nu_K(\alpha) + \nu_{K^d}(v) = \gamma$. Note also that $\alpha v = v_1 + \cdots + v_d$

with $v_1 := \alpha v,\ v_i := 0$ for $2 \le i \le d$; in particular $v_i \in F_i$ and $\nu(v_i) \in \Delta_i$, so $\alpha v \in C$, hence $\Delta_1 \subseteq \nu(F_1 \cap C)$.

Proof of Theorem 3.6. By induction on d. For $d = 1$, every $\mathcal{O}$-submodule of K is a quasiball $C = \{x \in K : \nu(x) \in \Delta\}$ for some upwards closed $\Delta \subseteq \Gamma \cup \{\infty\}$ (see Example 2.11), hence we take $F_1 := K$ and $\Delta_1 := \Delta$.

For $d > 1$, let $\Delta_1 := \{\gamma \in \Gamma_\infty \mid \exists v \in C,\ \nu_{K^d}(v) = \gamma\}$. Note that Δ_1 is nonempty because it contains $\infty = \nu(0)$. Then there is some $i \in [d]$ such that every $\gamma \in \Delta_1$ is the valuation of the i-th coordinate of some element of C. To see this, note that for each $i \in [d]$, the set

$$S_i := \{\gamma \in \Gamma_\infty \mid \exists v = (v_1, \dots, v_d) \in C \text{ such that } \nu_{K^d}(v) = \nu(v_i) = \gamma\}$$

is upwards closed in Γ_∞. Indeed, assume $v = (v_1, \dots, v_d) \in C,\ \gamma = \nu(v_i) = \min\{\nu(v_j) : j \in [d]\}$ and $\delta \ge \gamma$ in Γ_∞. Let $\alpha \in K$ be arbitrary with $\nu(\alpha) = \delta - \gamma$, then $\alpha \in \mathcal{O}$, hence $\alpha v \in C$, and so $\nu_{K^d}(\alpha v) = \min\{\nu(\alpha v_j) : j \in [d]\} = \nu(\alpha v_j) = \delta$. As we also have $\Delta_1 = \bigcup_{i \in [d]} S_i$, it follows that $\Delta_1 = S_i$ for some $i \in [d]$ as wanted (and thus Δ_1 is upwards closed in Γ_∞).

Without loss of generality we may assume $i = 1$. Then, given any $\gamma \in \Delta_1$, there is some $(\alpha, y_1, \dots, y_{d-1}) \in C$ such that $\gamma = \nu(\alpha) \le \min\{\nu(y_j) : j \in [d-1]\}$. Taking $x_j := y_j/\alpha \in \mathcal{O}$, we thus have $(\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C$. Hence for any $\gamma \in \Delta_1$, the set

$$X_\gamma := \{(x_1, \dots, x_{d-1}) \in \mathcal{O}^{d-1} \mid \exists \alpha \in K,\ \nu(\alpha) = \gamma \wedge (\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C\}$$

is nonempty and convex (by Lemma 3.5). Note that for $\gamma < \delta \in \Gamma_\infty$ we have $X_\gamma \subseteq X_\delta$, hence $\bigcap_{\gamma \in \Delta_1} X_\gamma \neq \varnothing$ by Lemma 3.4. That is, there exists $(x_1, \dots, x_{d-1}) \in \mathcal{O}^{d-1}$ such that for all $\gamma \in \Delta_1$, there exists $\alpha \in K$ with $\nu(\alpha) = \gamma \wedge (\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C$. Hence

$$\forall \alpha \in K,\ \nu(\alpha) \in \Delta_1 \Longrightarrow (\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C, \tag{3-2}$$

since there exists $\beta \in K$ such that $\nu(\beta) = \nu(\alpha) \wedge (\beta, \beta x_1, \dots, \beta x_{d-1}) \in C$, so $\alpha/\beta \in \mathcal{O}$ and multiplying by it we get $(\alpha, \alpha x_1, \dots, \alpha x_{d-1}) \in C$.

Let $F_1 := \langle (1, x_1, \dots, x_{d-1}) \rangle$. Let $\pi : K^d \twoheadrightarrow K^d/F_1$ be the projection map, $f : K^d/F_1 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 3.1, and $\pi' := f \circ \pi : K^d \to K^d$. Note that $K^d/F_1 \cong K^{d-1}$ as a valued K-vector space by Lemma 3.1, and that $\widetilde{C} := \pi(C)$ is still an $\mathcal{O}$-submodule of K^d/F_1. By the induction hypothesis there is a full flag $\{0\} \subsetneq \widetilde{F}_2 \subsetneq \cdots \subsetneq \widetilde{F}_d = K^d/F_1$ and upwards closed subsets $\nu_{K^d/F_1}(\widetilde{C}) = \Delta_2 \supseteq \cdots \supseteq \Delta_d$ of Γ_∞ satisfying the conclusion of the theorem with respect to $\widetilde{C}$ (the equality $\nu_{K^d/F_1}(\widetilde{C}) = \Delta_2$ is by Remark 3.7). Note that

$$\forall v \in K^d,\ \nu_{K^d}(\pi'(v)) = \nu_{K^d/F_1}(\pi(v)) \ge \nu_{K^d}(v). \tag{3-3}$$

In particular we have $\Delta_1 \supseteq \Delta_2$.

Let the subspaces $F_2, \dots, F_d$ be the preimages of $\widetilde{F}_2, \dots, \widetilde{F}_d$ in K^d. We let $W := f(K^d/F_1) \subseteq K^d$ be the image of the valuation preserving embedding $f : K^d/F_1 \hookrightarrow K^d$. Then we have

$$C = \{v_1 + w \mid v_1 \in F_1,\ \nu_{K^d}(v_1) \in \Delta_1,\ w \in C \cap W\}. \tag{3-4}$$

To see this, given an arbitrary $v \in C$, let $w := \pi'(v)$ and $v_1 := v - w$. As $\pi \circ f = \mathrm{id}_{K^d/F_1}$ by assumption, we have $\pi(w) = \pi(\pi'(v)) = \pi(f(\pi(v))) = \pi(v)$, hence $v_1 \in F_1$. By (3-3) we have $\nu_{K^d}(w) \geq \nu_{K^d}(v)$, and thus $\nu_{K^d}(v_1) \geq \min\{\nu_{K^d}(v), \nu_{K^d}(w)\} \geq \nu_{K^d}(v)$ as well. Thus $\nu_{K^d}(v_1) \in \Delta_1$, and $v_1 \in F_1$, which together with (3-2) and the definition of F_1 implies $v_1 \in C$; hence $w = v - v_1 \in C$ as well. The opposite inclusion is obvious.

Furthermore, applying the isomorphism $f : K^d/F_1 \to W$ to

$$\widetilde{C} = C/F_1 = \{v_2 + \dots + v_d \mid v_i \in \widetilde{F}_i,\ \nu_{K^d/F_1}(v_i) \in \Delta_i\},$$

we get

$$C \cap W = \{v_2 + \dots + v_d \mid v_i \in F_i \cap W,\ \nu_{K^d}(v_i) \in \Delta_i\},$$

which together with (3-4) implies

$$C = \{v_1 + \dots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i,\ v_i \in W \text{ for } i \geq 2\}.$$

Now $C = \{v_1 + \dots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$ follows because for any such vectors $v_1, \dots, v_d$, the vector v_i (for $i \geq 2$) can be moved into W by subtracting an element of F_1 with valuation in Δ_1, and collecting the differences in with v_1. That is, given arbitrary $v_i \in F_i$ with $\nu(v_i) \in \Delta_i$, let $w_i := \pi'(v_i) \in W$ for $i \geq 2$, and let $w_1 := v_1 + (v_2 - \pi'(v_2)) + \dots + (v_d - \pi'(v_d))$. As above, using (3-3), for each $i \geq 2$ we have $\nu_{K^d}(v_i - \pi'(v_i)) \geq \min\{\nu_{K^d}(v_i), \nu_{K^d}(\pi'(v_i))\} \geq \nu_{K^d}(v_i) \in \Delta_i \subseteq \Delta_1$. Hence $\nu_{K^d}(w_1) \geq \min\{v_1, v_2 - \pi'(v_2), \dots, v_d - \pi'(v_d)\} \in \Delta_1$. We also have $\nu_{K^d}(w_i) \geq \nu_{K^d}(v_i) \in \Delta_i$ for $i \geq 2$ by (3-3). Using that f is a one-sided inverse of π as above, we also have $v_i - \pi'(v_i) \in F_1 \subseteq F_i$ for $i \geq 2$. It follows that $w_i \in F_i$ for all $i \in [d]$. Putting all of this together, we get $w_1 + \dots + w_d = v_1 + \dots + v_d$, $w_i \in F_i,\ \nu(w_i) \in \Delta_i$, and $w_i \in W$ for $i \geq 2$. □

Remark 3.8. Note that as $F_d = K^d$ in Theorem 3.6, we have

$$\Delta_d = \{\gamma \in \Gamma_\infty \mid \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C\}.$$

That is, Δ_d is the quasiradius of the largest quasiball around 0 contained in C.

Remark 3.9. Given a convex set $0 \in C \subseteq K^d$ and any F_i and $\Delta_i,\ i \in [d]$ satisfying the conclusion of Theorem 3.6 with respect to it, for every $j \in [d]$ we have

$$C \cap F_j = \{v_1 + \dots + v_j \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i \text{ for all } j \in [i]\}.$$

Indeed, if $x \in C \cap F_j$, then $x = v_1 + \cdots + v_d \in F_j$ for some $v_i \in F_i$ with $\nu(v_i) \in \Delta_i$ for $i \in [d]$. Then, using that the F_i are increasing under inclusion and Δ_i are increasing under inclusion and upwards closed, $v_{j+1} + \cdots + v_d \in F_j$ and taking $v'_j := v_j + \cdots + v_d$ we have $v'_j \in F_j$, $\nu(v'_j) \geq \min\{\nu(v_i) : j \leq i \leq d\} \in \Delta_j$ and $x = v_1 + \cdots + v_{j-1} + v'_j$. Conversely, any element $x = v_1 + \cdots + v_j$ with $v_i \in F_i$, $\nu(v_i) \in \Delta_i$ for $i \in [j]$ can be written as $x = v_1 + \cdots + v_d$ with $v_i := 0 \in F_i$ and $\nu(v_i) = \infty \in \Delta_i$ for $j + 1 \leq i \leq d$. So $x \in C \cap F_j$.

Remark 3.10. (1) It follows from the conclusion of Theorem 3.6 that the subspace F_{d-1} is a linear hyperplane in K^d, and every element of C differs from an element of F_{d-1} (and hence of $F_{d-1} \cap C$ in view of Remark 3.9) by a vector in K^d with valuation in Δ_d (with Δ_d as in Remark 3.8).

(2) Conversely, F_{d-1} can be chosen to be *any* linear hyperplane H in K^d such that every element of C differs from an element of H by a vector in K^d with valuation in Δ_d. To see this, let H be such a hyperplane in K^d. Then $C \cap H$ is a convex subset of $H \cong K^{d-1}$ containing 0, hence an $\mathcal{O}$-submodule of H by Proposition 2.10. Applying Theorem 3.6 to $C \cap H$ in H (with the induced valuation on H), there are $\Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_{d-1}$ and a full flag $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_{d-1} = H$, such that $C \cap H = \{v_1 + \cdots + v_{d-1} \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\}$. Then

$$\{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\} = \{w + v_d \mid w \in C \cap H,\ \nu(v_d) \in \Delta_d\} = C.$$

Example 3.11. The assumption of spherical completeness of K is necessary in Theorem 3.6. For example, let $K := \bigcup_{n \geq 1} k((t^{1/n}))$ be the field of Puiseux series over a field k, and let $\widetilde{K} := k[[t^{\mathbb{Q}}]]$ be the field of Hahn series over k with rational exponents. The field $\widetilde{K}$ is the spherical completion of K (both fields have value group $\mathbb{Q}$ and valuation $\nu(x) = q$ where x has leading term t^q; see [Aschenbrenner et al. 2017, Example 3.3.23] for instance). In particular $\sum_{n \geq 1} t^{1-1/n} \in \widetilde{K} \setminus K$, and let

$$\widetilde{C} := \left\{\alpha\left(1, \sum_{n\geq 1} t^{1-1/n}\right) + v \;\middle|\; \alpha \in \widetilde{K},\ v \in \widetilde{K}^2,\ \nu_{\widetilde{K}}(\alpha) \geq 0,\ \nu_{\widetilde{K}^2}(v) \geq 1\right\} \subseteq \widetilde{K}^2,$$

as well as $C := \widetilde{C} \cap K^2$. Then $\widetilde{C}$ is convex in $\widetilde{K}^2$, and hence C is also convex as a subset of K^2. The basic idea behind why C is not of the form described in Theorem 3.6 is that C is close enough to $\widetilde{C}$, and the subspace F_1 appearing in the conclusion of Theorem 3.6 for $\widetilde{C}$ must be close to $\langle(1, \sum_{n \geq 1} t^{1-1/n})\rangle$; specifically, it must be $\langle(1, x + \sum_{n \geq 1} t^{1-1/n})\rangle$ for some $x \in K^2$ with $\nu(x) \geq 1$, but K^2 contains no such subspaces.

Indeed, by Remark 3.7, given any F_i and Δ_i satisfying the conclusion of Theorem 3.6 with respect to C, the valuation of every element of C must also be the valuation of some element of $F_1 \cap C$. So, to show that C is not of the form described in Theorem 3.6, it suffices to show that C contains elements of valuation arbitrarily close to 0, but that for every 1-dimensional subspace $F_1 \subset K^2$, there is some $q > 0$ in Γ such that every element of $F_1 \cap C$ has valuation at least q (and note that from the definition of C, every element in it has positive valuation).

Claim. *For every $n \in \mathbb{N}_{\geq 1}$, there is some $v \in C$ with $\nu_{K^2}(v) = 1/n$.*

Proof. To see this, note that

$$t^{1/n}\left(1, \sum_{m=1}^{n-1} t^{1-1/m}\right) = t^{1/n}\left(1, \sum_{m\geq 1} t^{1-1/m}\right) - t^{1/n}\left(0, \sum_{m\geq n} t^{1-1/m}\right) \in C$$

as $\nu_K(t^{1/n}) = 1/n \geq 0$ and $\nu_{K^2}\big(t^{1/n}\big(0, \sum_{m\geq n} t^{1-1/m}\big)\big) = 1/n + (1 - 1/n) \geq 1$. $\square$

Claim. *For every 1-dimensional subspace $F_1 \subset K^2$, there is some $n \in \mathbb{N}_{n\geq 1}$ such that every element of $F_1 \cap C$ has valuation at least $1/n$.*

Proof. We prove this by breaking into two cases.

Case 1. $F_1 = \langle(0, 1)\rangle$. Assume $x \in F_1 \cap C$, then $x = (x_1, x_2) = \alpha\big(1, \sum_{n\geq 1} t^{1-1/n}\big) + v$ for some $\alpha \in K$, $v = (v_1, v_2) \in \widetilde{K}^2$ with $\nu_{\widetilde{K}}(\alpha) \geq 0$, $\nu_{\widetilde{K}^2}(v) \geq 1$, and $x_1 = 0$, so $\alpha = -v_1$. But $1 \leq \nu_{\widetilde{K}^2}(v) = \min\{\nu_{\widetilde{K}}(v_1), \nu_{\widetilde{K}}(v_2)\}$, hence $\nu_{\widetilde{K}}(\alpha) \geq 1$ as well. Since $\nu_{\widetilde{K}}\big(\sum_{n\geq 1} t^{1-1/n}\big) = 0$, it follows that

$$\nu_{\widetilde{K}^2}(x) = \min\left\{\nu_{\widetilde{K}}(0), \nu_{\widetilde{K}}\left(\alpha\left(\sum_{n\geq 1} t^{1-1/n}\right)\right)\right\} \geq 1.$$

Thus every element of $F_1 \cap C$ has valuation at least 1.

Case 2. $F_1 = \langle(1, x)\rangle$ for some $x \in K$. Given any $x \in K$, there must exist some $n \in \mathbb{N}$ such that $\nu_{\widetilde{K}}\big(x - \sum_{m\geq 1} t^{1-1/m}\big) \leq 1 - 1/n$. Given any $v \in F_1 \cap C$, we have

$$v = \alpha(1, x) = \beta\left(1, \sum_{m\geq 1} t^{1-1/m}\right) + w$$

for some $\alpha \in K$, some $\beta \in \widetilde{K}$ with $\nu_{\widetilde{K}}(\beta) \geq 0$ and $w = (w_1, w_2) \in \widetilde{K}^2$ with $\nu_{\widetilde{K}^2}(w) \geq 1$. Without loss of generality, $\alpha \neq 0$, so we have

$$x = \frac{\alpha x}{\alpha} = \left(w_2 + \beta \sum_{m\geq 1} t^{1-1/m}\right)(w_1 + \beta)^{-1} = \left(\frac{w_2}{\beta} + \sum_{m\geq 1} t^{1-1/m}\right)\left(1 + \frac{w_1}{\beta}\right)^{-1}.$$

If $\nu_{\widetilde{K}}(\beta) < 1/n$, then

$$\nu_{\widetilde{K}}\left(\frac{w_1}{\beta}\right) > 1 - \frac{1}{n}, \quad \nu_{\widetilde{K}}\left(\frac{w_2}{\beta}\right) > 1 - \frac{1}{n},$$
$$\nu_{\widetilde{K}}\left(\left(1 + \frac{w_1}{\beta}\right)^{-1}\right) = 0, \quad \nu_{\widetilde{K}}\left(\left(1 + \frac{w_1}{\beta}\right)^{-1} - 1\right) > 1 - \frac{1}{n},$$

so

$$\nu\left(x - \sum_{m\geq 1} t^{1-1/m}\right) = \nu\left(\frac{w_2}{\beta}(w_1 + \beta)^{-1} + \left(\sum_{m\geq 1} t^{1-1/m}\right)\left(\left(1 + \frac{w_1}{\beta}\right)^{-1} - 1\right)\right) > 1 - \frac{1}{n},$$

a contradiction to the choice of n. Thus $\nu(\beta) \geq 1/n$, and hence $\nu(v) \geq 1/n$. $\square$

Thus no 1-dimensional subspace F_1 of K^2 can fill its desired role in the presentation for C.

Theorem 3.6 implies the following simple description of convex sets over spherically complete valued fields.

Corollary 3.12. *If K is a spherically complete valued field and $d \in \mathbb{N}_{\geq 1}$, then the nonempty convex subsets of K^d are precisely the affine images of $\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$.*

Proof. Let $C \subseteq K^d$ be an affine image of $\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d)$ for some upwards closed $\Delta_1, \ldots, \Delta_d \subseteq \Gamma_\infty$. Note that $\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d)$ is convex, and an image of a convex set under an affine map is convex (Example 2.5), hence C is convex.

Conversely, let $\varnothing \neq C \subseteq K^d$ be convex. Since the affine images of $\mathcal{O}$-submodules of K^d give us all nonempty convex sets by Proposition 2.10, without loss of generality $0 \in C$ and C is an $\mathcal{O}$-submodule of K^d. Let $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = K^d$ and $\nu_{K^d}(C) = \Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_d$ be as given by Theorem 3.6 for C. Using Lemma 3.1 we can choose $v_1, \ldots, v_d \in K^d$ such that for every $i \in [d]$ we have:

(1) $v_1, \ldots, v_i$ is a basis for F_i.

(2) $\nu(v_i) = 0$.

(3) $\nu(v_i + x) \leq 0$ for all $x \in F_{i-1}$.

Then C is the image of $\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d)$ under the linear map $f : K^d \to K^d$ such that $f(e_i) = v_i$, where e_i is the i-th standard basis vector. Indeed, if $x \in f(\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d))$ then $x = \sum_{i=1}^d c_i v_i$ for some c_i with $\nu(c_i) \in \Delta_i$. Using (2) this implies $\nu(c_i v_i) = \nu(c_i) \in \Delta_i$, and $c_i v_i \in F_i$, hence $x \in C$. Conversely, let x be an arbitrary element of C, then $x = w_1 + \cdots + w_d$ for some $w_i \in F_i$ with $\nu(w_i) \in \Delta_i$. Each w_i is a linear combination of $v_1, \ldots, v_i$, say $w_i = \sum_{j=1}^i c_{i,j} v_j$.

Now we claim that for any $i \in [d]$, $\alpha \in K$ and $v \in F_{i-1}$ we have $\nu(\alpha v_i + v) = \min\{\nu(\alpha v_i), \nu(v)\}$. Indeed, replacing v and α by $\alpha^{-1} v \in F_{i-1}$ and $\alpha^{-1}\alpha \in K$, respectively, changes both sides of the claimed equality by the same amount, hence we may assume that $\alpha = 0$ or $\alpha = 1$. The first case holds trivially, in the second case we need to show that $\nu(v_i + v) = \min\{\nu(v_i), \nu(v)\}$. If $\nu(v_i) \neq \nu(v)$ this holds by the ultrametric inequality, so we assume $\nu(v_i) = \nu(v) = 0$ (using (2)). Then, using (3), $0 \geq \nu(v_i + v) \geq \min\{\nu(v_i), \nu(v)\} = 0$, so $\nu(v_i + v) = 0$ as well.

Applying this claim by induction on $i \in [d]$, we get

$$\nu\left(\sum_{j=1}^{i} c_{i,j} v_j\right) = \min_j\{\nu(c_{i,j} v_j)\},$$

which using (2) implies $\nu(w_i) = \nu\big(\sum_{j=1}^i c_{i,j} v_j\big) = \min_j\{\nu(c_{i,j})\}$ for each $i \in [d]$. As for each $i \in [d]$, we have $\nu(w_i) \in \Delta_i$ and Δ_i is upwards closed, it follows that

$\nu(c_{i,j}) \in \Delta_i$ for all $i \in [d]$, $j \in [i]$. Regrouping the summands $c_{i,j}v_i$, it follows that $x = w_1 + \cdots + w_d$ is a linear combination of $v_1, \ldots, v_d$ where the coefficient of v_i has valuation in Δ_i, hence x belongs to $f(\nu^{-1}(\Delta_1) \times \cdots \times \nu^{-1}(\Delta_d))$. ☐

We can eliminate the assumption of spherical completeness of the field when only considering convex hulls of finite sets. We will say that a convex set is *finitely generated* if it is the convex hull of a finite set of points.

Lemma 3.13. *A subset $C \subseteq K^d$ is a finitely generated $\mathcal{O}$-module if and only if it is a finitely generated convex set and contains* 0*.*

Proof. If an $\mathcal{O}$-module $C \subseteq K^d$ is generated as an $\mathcal{O}$-module by some finite set X, then it is the convex hull of $X \cup \{0\}$. If a set C is the convex hull of some finite set X and contains 0, then it is an $\mathcal{O}$-module by Proposition 2.10, clearly generated as an $\mathcal{O}$-module by X. ☐

We have the following analog of Theorem 3.6 in the finitely generated case over an arbitrary valued field.

Corollary 3.14. *Let K be an **arbitrary** valued field and C a **finitely generated** convex set containing* 0*. Then there is a full flag $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = K^d$ and an increasing sequence $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_d \in \Gamma_\infty$ such that*

$$C = \{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \geq \gamma_i\}.$$

Proof. Let $C \ni 0$ be the convex hull of some finite set $X \subseteq K^d$. By a repeated application of Proposition 2.8, C is the convex hull of some $d+1$ elements $v_0, \ldots, v_d$ from X (possibly with $x_i = x_j$ for some i, j). As $0 \in C$, we have $0 = \sum_{i=0}^{d} \alpha_i v_i$ for some $\alpha_i \in \mathcal{O}$ with $\sum_{i=0}^{d} \alpha_i = 1$. Let j be such that $\nu(\alpha_j)$ is minimal among $\{\nu(\alpha_i) : 0 \leq i \leq d\}$. In particular $\alpha_j \neq 0$, hence $v_j = \left(1 - \sum_{i \neq j} \alpha_i/\alpha_j\right)0 + \sum_{i \neq j}(\alpha_i/\alpha_j)v_i$. By the choice of j we have $\alpha_i/\alpha_j \in \mathcal{O}$ for all $i \neq j$, hence also $1 - \sum_{i \neq j} \alpha_i/\alpha_j \in \mathcal{O}$, thus $v_j \in \operatorname{conv}(\{0\} \cup \{v_i : i \neq j\})$, and so also $C = \operatorname{conv}(\{0\} \cup \{v_i : i \neq j\})$. Reordering if necessary, we can thus assume that C is the convex hull of some $\{0, v_1, \ldots, v_d\} \subseteq C$ with $\nu(v_1) \leq \nu(v_i)$ for each $i \in [d]$.

Let $F_1 := \langle v_1 \rangle$ and $\gamma_1 := \nu(v_1)$. Let $\pi_1 : K^d \twoheadrightarrow K^d/F_1$ be the projection map, $f_1 : K^d/F_1 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 3.1, $V_1 := f_1(K^d/F_1)$ and $\pi_1' := f_1 \circ \pi_1 : K^d \to K^d$.

For $i \geq 2$, as we explained after (3-4) in the proof of Theorem 3.6, we have $v_i - \pi_1'(v_i) \in F_1$; and by (3-3) and our assumption we have $\nu(\pi_1'(v_i)) \geq \nu(v_i) \geq \nu(v_1)$. So $v_i - \pi_1'(v_i) \in \mathcal{O}v_1$ for all $i \geq 2$, which implies

$$\operatorname{conv}(\{0, v_1, \pi_1'(v_2), \ldots, \pi_1'(v_d)\}) = \operatorname{conv}(\{0, v_1, \ldots, v_d\}) = C.$$

Without loss of generality we suppose $\nu(\pi_1'(v_2)) \leq \nu(\pi_1'(v_i))$ for $i \geq 3$, and let $F_2 := \langle v_1, \pi_1'(v_2) \rangle$ and $\gamma_2 := \nu(\pi_1'(v_2)) \geq \nu(v_1) = \gamma_1$ by assumption. By definition of the valuation on the quotient space, using the properties of f, we have

$$\nu_K(\pi_1'(v_i)) = \nu_{K^d/F_1}(\pi_1(v_i)) = \nu_{K^d/F_1}(\pi_1(\pi_1'(v_i))) \geq \nu_{K^d}(\pi_1'(v_i) + \alpha v_1)$$

for all $\alpha \in K$. As in the proof of Corollary 3.12, this implies $\nu(\beta\pi_1'(v_i) + \alpha v_1) = \min\{\beta\nu(\pi_1'(v_i)), \nu(\alpha v_1))\}$ for all $i \geq 2$ and $\alpha, \beta \in K$. It follows that

$$\{nv_1 + m\pi_1'(v_2) \mid n, m \in \mathcal{O}\} = \{w_1 + w_2 \mid w_i \in F_i,\ \nu(w_i) \geq \gamma_i\}.$$

To see that the set on the right is contained in the set on the left, assume $x = w_1 + w_2$ for some $w_i \in F_i$, $\nu(w_i) \geq \gamma_i$. Then $w_1 = \alpha_1 v_1$ and $w_2 = \alpha_2 v_1 + \beta\pi_1'(v_2)$ for some $\alpha_1, \alpha_2, \beta \in K$, and by the observation above $\gamma_2 \leq \nu(w_2) = \min\{\nu(\alpha_2 v_1), \nu(\beta\pi_1'(v_2))\}$. So $x = (\alpha_1 + \alpha_2)v_1 + \beta\pi_1'(v_2)$, $\nu((\alpha_1 + \alpha_2)v_1) \geq \gamma_1 = \nu(v_1)$, so $(\alpha_1 + \alpha_2) \in \mathcal{O}$, and $\nu(\beta) \geq \gamma_2$, as wanted.

Now we replace v_i by $\pi_1'(v_i)$ for $i \geq 2$, and let $\pi_2 : K^d \twoheadrightarrow K^d/F_2$ be the projection map, $f_2 : K^d/F_2 \hookrightarrow K^d$ the valuation preserving embedding given by Lemma 3.1, $V_2 := f_2(K^d/F_2)$ and $\pi_2' := f_2 \circ \pi_2 : K^d \to K^d$. For $i \geq 3$, $v_i - \pi_2'(v_i) \in F_2$ and $v_i - \pi_2'(v_i) \in \mathcal{O}v_1 + \mathcal{O}v_2$, so again replacing v_i with $\pi_2'(v_i)$ for $i \geq 3$ does not change the convex hull. Again we may assume $\nu(\pi_2'(v_3)) \leq \nu(\pi_2'(v_i))$ for $i \geq 4$, and let $F_3 := \langle v_1, v_2, v_3\rangle$ and $\gamma_3 := \nu(\pi_2'(v_3))$. Repeating this argument as above d times, we have chosen vectors v_i, increasing spaces $F_i = \langle v_1, \ldots, v_i\rangle$ and increasing $\gamma_i = \nu(v_i) \in \Gamma$ for $i \in [d]$, so that

$$\begin{aligned} C = \operatorname{conv}(\{0, v_1, \ldots, v_d\}) &= \{n_1 v_1 + \cdots + n_d v_d \mid n_i \in \mathcal{O}\} \\ &= \{w_1 + \cdots + w_d \mid w_i \in F_i,\ \nu(w_i) \geq \gamma_i\}. \qquad \square \end{aligned}$$

4. Combinatorial properties of convex sets

The following definition is from [Aschenbrenner et al. 2016, Section 2.4].

Definition 4.1. Given a set X and $d \in \mathbb{N}_{\geq 1}$, a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *breadth d* if any nonempty intersection of finitely many sets in $\mathcal{F}$ is the intersection of at most d of them, and d is minimal with this property.

Lemma 4.2. *Let K be a valued field and S a convex subset of K^d.*

(1) *If $0 \in S$ and S is finitely generated, then it is generated as an $\mathcal{O}$-module by a finite linearly independent set of vectors.*

(2) *Let $\widetilde{K}$ be a valued field extension of K and $\widetilde{S} := \operatorname{conv}_{\widetilde{K}^d}(S) \subseteq \widetilde{K}^d$. Then $\widetilde{S} \cap K^d = S$.*

Proof. (1) By Lemma 3.13, S is generated as an $\mathcal{O}$-module by some finite set $v_1, \ldots, v_n \in S$. Assume these vectors are not linearly independent, then $0 = \sum_{i\in[n]} \alpha_i v_i$ for $\alpha_i \in K$ not all 0. Let $i \in [n]$ be such that $\nu(\alpha_i) \leq \nu(\alpha_j)$ for all $j \in [n]$, and $\alpha_i \neq 0$. Then $v_i = \sum_{j\neq i}(\alpha_j/(-\alpha_i))v_j$ and $\nu(\alpha_j/(-\alpha_i)) = \nu(\alpha_j) - \nu(\alpha_i) \geq 0$, hence $\alpha_j/(-\alpha_i) \in \mathcal{O}$ for all $j \neq i$, and S is still generated as an $\mathcal{O}$-module by the set $\{v_j : j \neq i\}$. Repeating this finitely many times, we arrive at a linearly independent set of generators.

(2) Since convexity is invariant under translates, we may assume $0 \in S$. Since every element in the convex hull of a set is in the convex hull of some finite subset, we may also assume that S is finitely generated as an $\mathcal{O}$-module, and by (1) let $v_1, \ldots, v_n \in S$ be a linearly independent (in the vector space K^d, so $n \leq d$) set of its generators. Let $v_{n+1}, \ldots, v_d \in K^d$ be such that $\{v_i : i \in [d]\}$ is a basis of K^d, and say $v_i = (v_{i,j} : j \in [d])$ with $v_{i,j} \in K$. Then the square matrix $A := (v_{i,j} : i, j \in [d]) \in M_{d\times d}(K)$ is invertible, so $A^{-1} \in M_{d\times d}(K) \subseteq M_{d\times d}(\widetilde{K})$, so A is also invertible in $M_{d\times d}(\widetilde{K})$, hence $\{v_i : i \in [d]\}$ are linearly independent vectors in $\widetilde{K}^d$ as well. But now if $\sum_{i\in[n]} \alpha_i v_i = u$ for some $\alpha_i \in \widetilde{K}$ and $u \in K^d$, then necessarily $\alpha_i \in K$ for all i (otherwise we would get a nontrivial linear combination of $v_1, \ldots, v_d$ in $\widetilde{K}^d$). Thus, any element of the $\mathcal{O}_{\widetilde{K}}$-module generated by $v_1, \ldots, v_n$ which is in K^d already belongs to the $\mathcal{O}_K$-module generated by $v_1, \ldots, v_n$, hence $\widetilde{S} \cap K^d = S$. □

We can now demonstrate an (optimal) finite bound on the breadth of the family of convex sets over valued fields. In sharp contrast, over the reals there is no such finite bound already for convex subsets of $\mathbb{R}^2$ (for any n, a convex n-gon in $\mathbb{R}^2$ is the intersection of n half-planes, but not the intersection of any fewer of them).

Theorem 4.3. *Let K be a valued field and $d \geq 1$. Then the family* Conv_{K^d} *has breadth d. That is, any nonempty intersection of finitely many convex subsets of K^d is the intersection of at most d of them.*

Proof. The family Conv_{K^d} cannot have breadth less than d because the d coordinate-aligned hyperplanes are convex, have common intersection $\{0\}$, but any $d-1$ of them intersect in a line.

We now show that Conv_{K^d} has breadth at most d, by induction on d. The case $d = 1$ is clear by Example 2.5(1) since for any two quasiballs, they are either disjoint or one is contained in the other. For $d > 1$, assume $C_1, \ldots, C_n \in \mathrm{Conv}_{K^d}$ with $n \geq d$ are convex and satisfy $\bigcap_{i\in[n]} C_i \neq \varnothing$. Translating, we may assume $0 \in \bigcap_{i\in[n]} C_i$.

We may also assume that K is spherically complete. Indeed, let $\widetilde{K}$ be a spherical completion of K as in Fact 3.3, and let $\widetilde{C}_i := \mathrm{conv}_{\widetilde{K}^d}(C_i) \in \mathrm{Conv}_{\widetilde{K}^d}$. By Lemma 4.2(2), $\widetilde{C}_i \cap K^d = C_i$ for each $i \in [n]$. Hence $\bigcap_{i\in[n]} \widetilde{C}_i \neq \varnothing$, and if $\bigcap_{i\in[n]} \widetilde{C}_i = \bigcap_{i\in S} \widetilde{C}_i$ for some $S \subseteq [n]$ with $|S| \leq d$, then also $\bigcap_{i\in[n]} C_i = \bigcap_{i\in S} C_i$.

Then let the vector subspaces $\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_d = K^d$ and the upwards closed subsets $\Delta_1 \supseteq \Delta_2 \supseteq \cdots \supseteq \Delta_d$ of Γ_∞ be as given by Theorem 3.6 for the convex set

$C := C_1 \cap \cdots \cap C_n$. By Remark 3.8 we have

$$\Delta_d = \{\gamma \in \Gamma_\infty \mid \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C_1 \cap \cdots \cap C_n\}.$$

It follows that there is some $i_d \in [n]$ such that in fact

$$\Delta_d = \{\gamma \in \Gamma_\infty \mid\ \forall v \in K^d,\ \nu(v) = \gamma \implies v \in C_{i_d}\}. \tag{4-1}$$

(Since these are finitely many upwards closed sets in Γ, their intersection is already given by one of them.)

Let $\{0\} \subsetneq F'_1 \subsetneq \cdots \subsetneq F'_d = K^d$ and $\Delta'_1 \supseteq \Delta'_2 \supseteq \cdots \supseteq \Delta'_d$ be as given by Theorem 3.6 for C_{i_d}. By Remark 3.10(1), F'_{d-1} is a linear hyperplane so that every element of C_{i_d} differs from an element of $F'_{d-1} \cap C_{i_d}$ by a vector with valuation in Δ'_d. As $\Delta_d = \Delta'_d$ by (4-1) and $C \subseteq C_{i_d}$, by Remark 3.10(1) we may assume that $F_{d-1} = F'_{d-1}$, hence every element in C_{i_d} differs from an element of $F_{d-1} \cap C_{i_d}$ by a vector with valuation in Δ_d.

Consider $C \cap F_{d-1} = C_1 \cap \cdots \cap C_n \cap F_{d-1} = (C_1 \cap F_{d-1}) \cap \cdots \cap (C_n \cap F_{d-1})$. Note that each $C_i \cap F_{d-1}$ is a convex subset of $F_{d-1} \cong K^{d-1}$, so by induction hypothesis there exist $i_1, \ldots, i_{d-1} \in [n]$ such that

$$C_{i_1} \cap \cdots \cap C_{i_{d-1}} \cap F_{d-1} = C_1 \cap \cdots \cap C_n \cap F_{d-1} = C \cap F_{d-1}. \tag{4-2}$$

Let $x \in C_{i_1} \cap \cdots \cap C_{i_d}$ be arbitrary. As $x \in C_{i_d}$, by the choice of F_{d-1}, $x = w + v_d$ for some $w \in F_{d-1}$ and $v_d \in K^d$ with $\nu(v_d) \in \Delta_d$. By the choice of Δ_d we have $v_d \in C_{i_1} \cap \cdots \cap C_{i_d}$. And as each C_i is a module, it follows that also $w \in C_{i_1} \cap \cdots \cap C_{i_d}$. Combining this with (4-2) and using Remark 3.9 (for $j = d-1$) we thus have

$$\begin{aligned} C_{i_1} \cap \cdots \cap C_{i_d} &= \{w + v_d \mid w \in C_{i_1} \cap \cdots \cap C_{i_d} \cap F_{d-1},\ \nu(v_d) \in \Delta_d\} \\ &= \{w + v_d \mid w \in C \cap F_{d-1},\ \nu(v_d) \in \Delta_d\} = \{v_1 + \cdots + v_d \mid v_i \in F_i,\ \nu(v_i) \in \Delta_i\} \\ &= C_1 \cap \cdots \cap C_n. \quad \square \end{aligned}$$

Definition 4.4. (1) A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *Helly number* $k \in \mathbb{N}_{\geq 1}$ if given any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$, if every k-subset of $\{S_1, \ldots, S_n\}$ has nonempty intersection, then $\bigcap_{i \in [n]} S_i \neq \varnothing$.

(2) The *Helly number of* $\mathcal{F}$ refers to the minimal k with this property (or ∞ if it does not exist).

(3) We say that $\mathcal{F}$ has the *Helly property* if it has a finite Helly number.

Theorem 4.5. *Let K be a valued field and $d \geq 1$. Then the Helly number of* Conv_{K^d} *is $d+1$.*

Proof. The Helly number is bounded by the Radon number minus 1 in an arbitrary convexity space (see Section 5C), but we include a proof for completeness. Let n

be arbitrary, and let $S_1, \ldots, S_n \subseteq K^d$ be convex sets so that any $d+1$ of them have a nonempty intersection. We will show by induction on n that $S_1 \cap \cdots \cap S_n \neq \varnothing$.

Base case: $n = d+2$. By assumption for each $i \in [d+2]$ there exists some $x_i \in K^d$ so that $x_i \in \bigcap_{j \in [d+2] \setminus \{i\}} S_j$. By Proposition 2.8 there exists some $i^* \in [d+2]$ so that $x_{i^*} \in \operatorname{conv}(\{x_i \mid i \neq i^*\})$. By the choice of the x_i we have $x_{i^*} \in S_i$ for all $i \neq i^*$. We also have $x_i \in S_{i^*}$ for all $i \neq i^*$, S_{i^*} is convex and $x_{i^*} \in \operatorname{conv}(\{x_i \mid i \neq i^*\})$, hence $x_{i^*} \in S_{i^*}$. Thus $x_{i^*} \in \bigcap_{i \in [d+2]} S_i$, as wanted.

Inductive step: $n > d+2$. Let $\widetilde{S}_{n-1} := S_{n-1} \cap S_n$; in particular $\widetilde{S}_{n-1}$ is convex. By induction hypothesis, any $n-1$ sets from $\{S_1, \ldots, S_n\}$ have a nonempty intersection. Hence any $n-2$ sets from $\{S_1, \ldots, S_{n-2}, \widetilde{S}_{n-1}\}$ have a nonempty intersection. As $n-2 \geq d+1$ by assumption, applying the induction hypothesis again we get

$$S_1 \cap \cdots \cap S_n = S_1 \cap \cdots \cap S_{n-2} \cap \widetilde{S}_{n-1} \neq \varnothing.$$

This completes the induction, and shows that $\operatorname{Conv}_{K^d}$ has Helly number $d+1$.

It remains to show that $\operatorname{Conv}_{K^d}$ does not have Helly number d. Let $e_i \in K^d$ be the i-th standard basis vector. The set $E := \{0, e_1, \ldots, e_d\}$ is affinely independent, hence the intersection of the affine spans of its $d+1$ maximal proper subsets is empty. The convex hull of a subset of K^d is contained in its affine hull, hence the intersection of the $d+1$ convex hulls of its maximal proper subsets is also empty. But for any d among the $(d+1)$ maximal proper subsets of E, some element of E belongs to their intersection, and hence in particular the intersection of their convex hulls is nonempty. ☐

We recall some terminology around the *Vapnik–Chervonenkis dimension* (and refer to [Aschenbrenner et al. 2016, Sections 1 and 2] for further details).

Definition 4.6. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X.

(1) For a subset $Y \subseteq X$, we let $\mathcal{F} \cap Y := \{S \cap Y : S \in Y\} \subseteq \mathcal{P}(Y)$.

(2) We say that $\mathcal{F}$ *shatters* a subset $Y \subseteq X$ if $\mathcal{F} \cap Y = \mathcal{P}(Y)$.

(3) The *VC dimension* of $\mathcal{F}$, or $\mathrm{VC}(\mathcal{F})$, is the largest $k \in \mathbb{N}$ (if one exists) such that $\mathcal{F}$ shatters some subset of X size k. If $\mathcal{F}$ shatters arbitrarily large finite subsets of X, then it is said to have infinite VC dimension.

(4) The *dual family* $\mathcal{F}^* \subseteq \mathcal{P}(\mathcal{F})$ is given by $\mathcal{F}^* = \{S_x \mid x \in X\}$, where $S_x = \{A \in \mathcal{F} \mid x \in A\}$.

(5) The *dual VC dimension* of $\mathcal{F}$, or $\mathrm{VC}^*(\mathcal{F})$, is the VC dimension of $\mathcal{F}^*$. Equivalently, it is the largest $k \in \mathbb{N}$ (or ∞ if no such k exists) such that there are sets $S_1, \ldots, S_k \in \mathcal{F}$ that generate a Boolean algebra with 2^k atoms, i.e., for any distinct $I, J \subseteq [k]$, $\bigcap_{i \in I} S_i \cap \bigcap_{i \in [k] \setminus I} (X \setminus S_i) \neq \bigcap_{i \in J} S_i \cap \bigcap_{i \in [k] \setminus J} (X \setminus S_i)$.

(6) The *shatter function* $\pi_{\mathcal{F}} : \mathbb{N} \to \mathbb{N}$ of $\mathcal{F}$ is

$$\pi_{\mathcal{F}}(n) := \max\{|\mathcal{F} \cap Y| : Y \subseteq X, |Y| = n\}.$$

(7) By the Sauer–Shelah lemma (see for instance [Aschenbrenner et al. 2016, Lemma 2.1]), if $\mathrm{VC}(\mathcal{F}) \leq d$, then $\pi_{\mathcal{F}}(n) \leq (e/d)^d n^d$ for all $n \geq d$ (and $\pi_{\mathcal{F}}(n) = 2^n$ for all n if $\mathrm{VC}(\mathcal{F}) = \infty$).

(8) The *VC density* of $\mathcal{F}$, or $\mathrm{vc}(\mathcal{F})$, is the infimum of all $r \in \mathbb{R}_{>0}$ such that $\pi_{\mathcal{F}}(n) = O(n^r)$, and ∞ if there is no such r. (In particular $\mathrm{vc}(\mathcal{F}) \leq \mathrm{VC}(\mathcal{F})$.)

(9) Finally, we define the *dual shatter function* $\pi^*_{\mathcal{F}} := \pi_{\mathcal{F}^*}$ and the *dual* VC*-density* $\mathrm{vc}^*(\mathcal{F}) := \mathrm{vc}(\mathcal{F}^*)$ of the family $\mathcal{F}$.

Remark 4.7. Note that if $\mathcal{F} \subseteq \mathcal{P}(X)$ and $Y \subseteq X$, then $\mathrm{VC}(\mathcal{F} \cap Y) \leq \mathrm{VC}(\mathcal{F})$ and $\mathrm{VC}^*(\mathcal{F} \cap Y) \leq \mathrm{VC}^*(\mathcal{F})$.

The following results are in stark contrast with the situation for the family of convex sets over the reals, where already the family of convex subsets of $\mathbb{R}^2$ has infinite VC dimension (e.g., any set of points on a circle is shattered by the family of convex hulls of its subsets).

Theorem 4.8. *Let K be a valued field and $d \geq 1$. Then the family* Conv_{K^d} *has VC dimension $d+1$.*

Proof. We have $\mathrm{VC}(\mathrm{Conv}_{K^d}) \geq d+1$ since the set $E := \{0, e_1, \ldots, e_d\} \subseteq K^d$, with e_i the i-th vector of the standard basis, is shattered by Conv_{K^d}. Indeed, the convex hull of any subset is contained in its affine span, and for any $S \subseteq E$, $\mathrm{aff}(S)$ does not contain any of the points in $E \setminus S$.

On the other hand, $\mathrm{VC}(\mathrm{Conv}_{K^d}) \leq d+1$ as no subset Y of K^d with $|Y| \geq d+2$ can be shattered by Conv_{K^d}. Indeed, by Proposition 2.8, at least one of the points of Y belongs to every convex set containing all the other points of Y. □

The dual VC dimension of a family of sets is bounded by its breadth.

Fact 4.9 [Aschenbrenner et al. 2016, Lemma 2.9]. *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of subsets of X of breadth at most d. Then also* $\mathrm{VC}^*(\mathcal{F}) \leq d$.

Using this fact, we get the following:

Theorem 4.10. *For any valued field K and $d \geq 1$, the family* Conv_{K^d} *has dual VC dimension d.*

Proof. The dual VC dimension of Conv_{K^d} is at least d because the d coordinate-aligned (convex) hyperplanes in K^d generate a Boolean algebra with 2^d atoms.

Conversely, the breadth of Conv_{K^d} is d by Theorem 4.3, hence by Fact 4.9 its dual VC dimension is also at most d. □

Definition 4.11. (1) A family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ has *fractional Helly number* $k \in \mathbb{N}_{\geq 1}$ if for every $\alpha \in \mathbb{R}_{>0}$ there exists $\beta \in \mathbb{R}_{>0}$, so that for any $n \in \mathbb{N}$ and any sets $S_1, \ldots, S_n \in \mathcal{F}$ (possibly with repetitions), if there are at least $\alpha\binom{n}{k}$ k-element subsets of the multiset $\{S_1, \ldots, S_n\}$ with a nonempty intersection, then there are at least βn sets from $\{S_1, \ldots, S_n\}$ with a nonempty intersection.

(2) The *fractional Helly number* of $\mathcal{F}$ refers to the minimal k with this property. We say that $\mathcal{F}$ has the *fractional Helly property* if it has a fractional Helly number.

Note that any finite family of sets trivially has fractional Helly number 1 by choosing β sufficiently small with respect to the size of $\mathcal{F}$. We will use the following theorem of Matoušek.

Fact 4.12 [Matoušek 2004, Theorem 2]. *Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a set system whose dual shatter function satisfies $\pi^*_{\mathcal{F}}(n) = o(n^k)$, i.e., $\lim_{n\to\infty} \pi^*_{\mathcal{F}}(n)/n^k = 0$, where k is a fixed integer. Then $\mathcal{F}$ has fractional Helly number k.*

Remark 4.13. Moreover, if $\mathrm{VC}^*(\mathcal{F}) = d < \infty$, then the fractional Helly number is at most $d+1$, and the β witnessing this can be chosen depending only on d and α (and not on the family $\mathcal{F}$).

Indeed, by Definition 4.6, if $\mathrm{VC}^*(\mathcal{F}) \leq d$, then $\pi^*_{\mathcal{F}}(n) \leq (e/d)^d n^d$ for all $n \geq d$, hence $\pi^*_{\mathcal{F}}(n) \leq cn^d$ for all $n \in \mathbb{N}$, where $c = c(d) := (e/d)^d + 2^d$. We can choose $m = m(d, \alpha)$, so that $\pi^*_{\mathcal{F}}(m) < \frac{1}{4}\alpha\binom{m}{d+1}$. Then it follows from the proof of [Matoušek 2004, Theorem 2] that $\beta = 1/(2m)$ works for all $n \geq m/\beta = 2m^2$, and trivially $\beta = 1/(2m^2)$ works for all $n \leq 2m^2$, hence $\beta := \beta(\alpha, d) := 1/(2m^2)$ works for all $n \in \mathbb{N}$.

Using this, we get the following:

Theorem 4.14. *If K is a valued field, $d \geq 1$, and $X \subseteq K^d$ is an arbitrary subset, then the fractional Helly number of the family*

$$\mathrm{Conv}_{K^d} \cap X = \{C \cap X : C \in \mathrm{Conv}_{K^d}\} \subseteq \mathcal{P}(X)$$

is at most $d+1$. Moreover, β in Definition 4.11 can be chosen depending only on d and α (and not on the field K or set X). And if K is infinite, then the fractional Helly number of the family Conv_{K^d} is exactly $d+1$.

Proof. By Fact 4.12 we have that the fractional Helly number of a set system is at most the smallest integer larger than its dual VC density. Dual VC density is, in turn, at most its dual VC dimension. Also for any set $X \subseteq K^d$ we have $\mathrm{VC}^*(\mathrm{Conv}_{K^d} \cap X) \leq \mathrm{VC}^*(\mathrm{Conv}_{K^d})$ by Remark 4.7. So $\mathrm{Conv}_{K^d} \cap X$ has dual VC density at most d by Theorem 4.10, hence its fractional Helly number is at most $d+1$ by Fact 4.12. And an appropriate β can be chosen depending only on d and α by Remark 4.13.

To show that the fractional Helly number of Conv_{K^d} is at least $d+1$ when K is infinite, we can use the standard example with affine hyperplanes in general position. We include the details for completeness. First note that as the field K is infinite, for any K-vector space V of dimension k and $v \in V \setminus \{0\}$ there exists an infinite set $S \subseteq V$ so that $v \in S$ and any k vectors from S are linearly independent. This is clear for $k = 1$ by taking any infinite set of nonzero vectors, so assume that $k > 1$. By induction on $i \in \mathbb{N}_{\geq k}$ we can find sets S_i such that $v \in S_i$, $|S_i| \geq i$ and every k vectors from S_i are linearly independent, for all i. Let S_k be any basis of V containing v. Assume $i > k$ and S_i satisfies the assumption. Since K is infinite, V is not a union of finitely many proper subspaces; in particular there exists some

$$w \in V \setminus \bigcup_{s \subseteq S_i, |s|=k-1} \langle s \rangle.$$

Let $S_{i+1} := S_i \cup \{w\}$. Since any $s \subseteq S_i$ with $|s| = k-1$ is linearly independent by the inductive assumption, it follows that $s \cup \{w\}$ is also linearly independent, hence S_{i+1} satisfies the assumption. Finally, $S := \bigcup_{i \in \mathbb{N}_{\geq k}} S_i$ is as wanted.

In particular, we can find an infinite set of vectors S in $K^d \times K$ so that any $d+1$ of them are linearly independent and the standard basis vector $e_{d+1} \in S$. Then

$$X := \{\langle v, - \rangle : v \in S\} \subseteq (K^d \times K)^*$$

is an infinite set of dual vectors such that any $d+1$ of them are linearly independent, and it contains the projection map onto the last coordinate $\pi_{d+1} := \langle e_{d+1}, - \rangle : (x_1, \ldots, x_{d+1}) \mapsto x_{d+1}$. Consider the family

$$\mathcal{H} := \{\ker(f) \mid f \in X \setminus \{\pi_{d+1}\}\} \subseteq \mathcal{P}(K^d \times K)$$

of kernels of these dual vectors (excluding the projection map onto the last coordinate), and let

$$\mathcal{H}' := \{\{v \in K^d \mid (v, 1) \in H\} \mid H \in \mathcal{H}\} \subseteq \mathcal{P}(K^d).$$

Then $\mathcal{H}'$ is an infinite family of affine hyperplanes in K^d, and we wish to show that any d elements of $\mathcal{H}'$ intersect in a point, and any $d+1$ elements of $\mathcal{H}'$ have empty intersection. For any pairwise distinct $f_1, \ldots, f_d \in X \setminus \{\pi_{d+1}\}$, by linear independence,

$$\dim(\ker(f_1) \cap \cdots \cap \ker(f_d)) = d + 1 - \dim(\langle f_1, \ldots, f_d \rangle) = 1.$$

And by their linear independence with π_{d+1},

$$\dim(\ker(f_1) \cap \cdots \cap \ker(f_d) \cap \ker(\pi_{d+1})) = 0.$$

That is, $\ker(f_1) \cap \cdots \cap \ker(f_d)$ is a line in $K^d \times K$ that intersects $\ker(\pi_{d+1}) = K^d \times \{0\}$ only at the origin, and thus must also intersect $K^d \times \{1\}$ in a single point;

this shows that every d elements of $\mathcal{H}'$ intersect in a point. And any pairwise distinct $f_1, \dots, f_{d+1} \in X \setminus \{\pi_{d+1}\}$ span $(K^d \times K)^*$ by linear independence, so $\ker(f_1) \cap \cdots \cap \ker(f_{d+1}) = \{0\}$, and thus has empty intersection with $K^d \times \{1\}$. This shows that every $d+1$ elements of $\mathcal{H}'$ have empty intersection.

Using $\alpha = 1$, for any $\beta > 0$, take an arbitrary $n \geq (d+1)/\beta$. Let $H_1, \dots, H_n \in \mathcal{H}'$ be any distinct hyperplanes from this collection. All d-subsets, $\alpha\binom{n}{d}$ of them, of $\{H_1, \dots, H_n\}$ have an intersection point, but there are no $\beta n \geq d+1$ of them with a common intersection point. Therefore Conv_{K^d} does not have fractional Helly number d. □

Note that Theorems 4.5 and 4.14 replicate results for real convex sets, while Theorems 4.3, 4.8, and 4.10 do not: as we have already remarked, $\mathrm{Conv}_{\mathbb{R}^2}$ has infinite breadth, VC dimension, and dual VC dimension. The following result is somewhere in between. The classical Tverberg theorem says that for any $X \subseteq \mathbb{R}^d$ with $|X| \geq (d+1)(r-1)+1$, X can be partitioned into r disjoint subsets $X_1, \dots, X_r$ whose convex hulls intersect: $\mathrm{conv}(X_1) \cap \cdots \cap \mathrm{conv}(X_r) \neq \varnothing$. Over valued fields, we obtain a much stronger version (any element of the nonempty set X_r in the statement of Theorem 4.15 belongs to the convex hulls of each of the sets X_i, $i \in [r]$ — which gives the usual conclusion of Tverberg's theorem over the reals):

Theorem 4.15. *Let K be a valued field and $d, r \in \mathbb{N}_{\geq 1}$. Then any set $X \subseteq K^d$ with*

$$|X| \geq (d+1)(r-1)+1$$

points in K^d can be partitioned into subsets $X_1, \dots, X_r$ such that $|X_i| = d+1$ for $i < r$, $|X_r| = |X| - (d+1)(r-1)$, and $\mathrm{conv}(X_i) \supseteq \mathrm{conv}(X_j)$ for all $i \leq j \in [r]$.

Proof. Since any finitely generated convex set is the convex hull of some $d+1$ points from it by Corollary 2.9, we can find $X_1 \subseteq X$ with $|X_1| = d+1$ and $\mathrm{conv}(X_1) = \mathrm{conv}(X)$, $X_2 \subseteq X \setminus X_1$ with $|X_2| = d+1$ and $\mathrm{conv}(X_2) = \mathrm{conv}(X \setminus X_1)$, and so on: once $X_1, \dots, X_{i-1}$ have been chosen, pick $X_i \subseteq X \setminus \left(\bigcup_{j=1}^{i-1} X_j\right)$ such that $|X_i| = d+1$, $\mathrm{conv}(X_i) = \mathrm{conv}\left(X \setminus \bigcup_{j=1}^{i-1} X_j\right)$, and then let X_r consist of everything left over at the end. □

From this strong Tverberg theorem and the fractional Helly property, we finally get an analog of the result due to Boros and Füredi [1984] and Bárány [1982] on the common points in the intersections of many "simplices" over valued fields. Note that the conclusion is actually stronger than over the reals: the common point comes from the set X itself. This answers a question asked by Kobi Peterzil and Itay Kaplan. Our argument is an adaptation of the second proof in [Matoušek 2002, Theorem 9.1.1].

Theorem 4.16. *For each $d \geq 1$ there is a constant $c = c(d) > 0$ such that for any valued field K and any finite $X \subseteq K^d$ (say $n := |X|$), there is some $a \in X$ contained in the convex hulls of at least $c\binom{n}{d+1}$ of the $\binom{n}{d+1}$ subsets of X of size $d+1$.*

Proof. Let $X \subseteq K^d$ with $|X| = n$ be given, and let

$$\mathcal{F} := \operatorname{Conv}_{K^d} \cap X = \{C \cap X : C \in \operatorname{Conv}_{K^d}\}$$

be the family of all subsets of X cut out by the convex subsets of K^d. Let $(S_i)_{i\in[N]}$ with $S_i \in \operatorname{Conv}_{K^d}$ be the sequence listing all convex hulls of subsets of X of size $d+1$ in an arbitrary order (possibly with repetitions). Then $N = \binom{n}{d+1}$, and for a $(d+1)$-element subset $Y \subseteq X$ we let $g(Y) \in [N]$ be the index at which $\operatorname{conv}(Y)$ appears in this sequence. For each $i \in [N]$ let $S'_i := S_i \cap X \in \mathcal{F}$. It is thus sufficient to show that there exists some $\alpha > 0$, depending only on d, such that at least $\alpha\binom{N}{d+1}$ of the $(d+1)$-element subsets $I \subseteq [N]$ satisfy $\bigcap_{i\in I} S'_i \neq \varnothing$ — as then Theorem 4.14 applied to $\mathcal{F} \subseteq \mathcal{P}(X)$ shows the existence of $c > 0$ depending only on α and d, and hence only on d, so that for some $I \subseteq [N]$ with $|I| \geq cN = c\binom{n}{d+1}$ there exists some $a \in \bigcap_{i\in I} S'_i \subseteq \bigcap_{i\in I} S_i$ (in particular $a \in X$).

Now we find an appropriate α. For any $(d+1)^2$-element subset $Y \subseteq X$, by Theorem 4.15 (with $r := d+1$), we can fix a partition of Y into $d+1$ disjoint parts $Y_1, \dots, Y_{d+1}$, each of which having $d+1$ elements, and so that $\operatorname{conv}(Y_i) \supseteq \operatorname{conv}(Y_j)$ for all $i \leq j \in [d+1]$. In particular any element of the nonempty set $Y_{[d+1]} \subseteq X$ belongs to $\bigcap_{i\in[d+1]}(\operatorname{conv}(Y_i) \cap X) = \bigcap_{i\in[d+1]}(S'_{g(Y_i)})$. As g is a bijection, $Y \mapsto \{g(Y_i) : i \in [d+1]\}$ gives a function f from $(d+1)^2$-element subsets of X to $(d+1)$-element subsets $I \subseteq [N]$, so that $\bigcap_{i\in I} S'_i \neq \varnothing$. Moreover, f is an injection. Indeed, given a set $\{j_i : i \in [d+1]\}$ in the image of f, as g is a bijection, there is a unique set $\{Y_1, \dots, Y_{d+1}\}$ with $Y_i \subseteq X$ disjoint of size $d+1$, so that $g(Y_i) = j_i$ for all $i \in [d+1]$, and there can be only one set $Y \subseteq X$ of size $(d+1)^2$ for which it is a partition. If follows that the number of sets $I \subseteq [N]$ with $\bigcap_{i\in I} S'_i \neq \varnothing$ is at least

$$\binom{n}{(d+1)^2} = \Omega(n^{(d+1)^2}) \geq \alpha\binom{N}{d+1}$$

for some sufficiently small α depending only on d. □

5. Final remarks and questions

5A. ***Some further results and future directions.*** The results of Section 4 imply the following analog of the celebrated (p, q)*-theorem* of Alon and Kleitman [1992] for convex sets over valued fields.

Corollary 5.1. *For any $d, p, q \in \mathbb{N}_{\geq 1}$ with $p \geq q \geq d+1$ there exists $T = T(p, q, d) \in \mathbb{N}$ such that if K is a valued field and $\mathcal{F}$ is a family of convex subsets of K^d such that among every p sets of $\mathcal{F}$, some q have a nonempty intersection, then there exists a T-element set $Y \subseteq K^d$ intersecting all sets of $\mathcal{F}$.*

Corollary 5.1 follows formally by applying [Alon et al. 2002, Theorem 8] since the family $\operatorname{Conv}_{K^d}$ has fractional Helly property (Theorem 4.14) and is closed under intersections. Alternatively, it follows with a slightly better bound on T by combining the fractional Helly property with the existence of ε-nets for families of bounded VC dimension (Theorem 4.8), as outlined at the end of [Matoušek 2004, Section 1]. The problem of determining the optimal bound on $T(p, q, d)$ is widely open over the reals (see [Bárány and Kalai 2022, Section 2.6]), and we expect that it might be easier in the valued fields setting.

Kalai [1984] and Eckhoff [1985] proved that in the fractional Helly property for convex sets over the reals, one can take $\beta(d, \alpha) = 1 - (1-\alpha)^{1/(d+1)}$ (and this bound is sharp).

Problem 5.2. What is the optimal dependence of β on d, α in Theorem 4.14?

Over $\mathbb{R}$, Sierksma's Dutch cheese conjecture predicts a lower bound for the number of Tverberg partitions (see for instance [De Loera et al. 2019, Conjecture 3.12]). We expect the same bound to hold over valued fields:

Conjecture 5.3. For any valued field K and $X \subset K^d$ with $|X| = (r-1)(d+1)+1$, there are at least $((r-1)!)^d$ partitions of X into parts whose convex hulls intersect.

Remark 5.4. In Theorem 4.15, we showed the existence of Tverberg partitions satisfying the stronger property that the convex hulls of the parts are linearly ordered by inclusion. It is not true that for $X \subseteq K^d$ with $|X| = (d+1)(r-1)+1$, there are at least $((r-1)!)^d$ different ways of partitioning X into $X_1, \dots, X_r$ such that $\operatorname{conv}(X_1) \supseteq \cdots \supseteq \operatorname{conv}(X_r)$. Thus any attempt to prove Conjecture 5.3 would have to involve other Tverberg partitions that do not have this property. For an example in K^2 where this bound fails, let $x \in K$ with $\nu(X) \neq 0$, and let $X := \{(x^n, x^{-n}) \mid n \in [3(r-1)+1]\}$. For any partition of X into $X_1, \dots, X_r$ such that $\operatorname{conv}(X_1) \supseteq \cdots \supseteq \operatorname{conv}(X_r)$, for each $i < r$, X_i must consist of the points corresponding to the lowest and highest values of n among all points not already in $X_1 \cup \cdots \cup X_{i-1}$, together with one of the other $3(r-i)-1$ remaining points, and X_r must consist of whatever point is left over. So the number of partitions of X of this form is $\prod_{i=1}^{r-1}(3(r-i)-1) < \prod_{i=1}^{r-1} 3(r-i) = 3^{r-1}(r-1)! < ((r-1)!)^2$ for large enough r.

We expect that the *colorful* Tverberg theorem also holds over valued fields, however the proofs for convex sets over $\mathbb{R}$ rely on topological arguments not readily available in the valued field context:

Conjecture 5.5. For any integers $r, d \geq 2$ there exists $t \geq r$ such that for any valued field K and $X \subseteq K^d$ with $|X| = t(d+1)$, partitioned into $d+1$ color classes $C_1, \dots, C_{d+1}$ each of size t, there exist pairwise disjoint $X_1, \dots, X_r \subseteq X$ with $|X_i \cap C_j| = 1$ for $i \in [r]$ and $j \in [d+1]$, and $\bigcap_{i \in [r]} \operatorname{conv}(X_i) \neq \varnothing$.

It would formally imply (see [Matoušek 2002, Section 9.2]) the "second selection lemma" over valued fields generalizing Theorem 4.16:

Conjecture 5.6. For each $d \in \mathbb{N}_{\geq 1}$ there exist $c, s > 0$ such that for any valued field K, $\alpha \in (0, 1]$ and $n \in \mathbb{N}$, for every $X \subseteq K^d$ with $|X| = n$, and every family $\mathcal{F}$ of $(d + 1)$-element subsets of X with $|\mathcal{F}| \geq \alpha\binom{n}{d+1}$, there is a point contained in the convex hulls of at least $c\alpha^s\binom{n}{d+1}$ of the elements of $\mathcal{F}$.

Corollary 3.12 has the following immediate model-theoretic application.

Remark 5.7. If K is a spherically complete valued field, then every convex subset of K^d is definable in the expansion of the field K by a predicate for each Dedekind cut of the value group (so in particular definable in *Shelah expansion of* K by all externally definable sets [Shelah 2009; Chernikov and Simon 2013]). And conversely, every Dedekind cut of the value group is definable in the expansion of K by a predicate for each $\mathcal{O}$-submodule of K. In particular, if K has value group $\mathbb{Z}$, then all convex subsets of K^d form a definable family.

Example 5.8. In contrast, naming a single (bounded) convex subset of $\mathbb{R}^2$ in the field of reals allows to define the set of integers. Indeed, we can define a continuous and piecewise linear function $f : [0, 1] \to [0, 1]$ such that

$$C := \{(x, y) : x \in [0, 1],\ 0 \leq y \leq f(x)\}$$

is convex but the set of points where f is not differentiable is exactly $\{1/n : n \in \mathbb{N}_{\geq 2}\}$. Now in the field of reals with a predicate for C we can define f and the set of points where it is not differentiable, hence $\mathbb{N}$ is also definable.

5B. ***Other notions of convexity over nonarchimedean fields.*** We briefly overview several other kinds of convexities over nonarchimedean fields considered in the literature. The extension of Hilbert (projective) geometry to convex sets in a generalized sense is a topic of high current interest, see for instance [Guilloux 2016]. In a different spirit, in tropical geometry, convex sets over real closed nonarchimedean fields have been considered (unlike what is done here, this leads to a combinatorial convexity similar to the classical one, since by Tarski's completeness theorem, polyhedral properties of a combinatorial nature are the same over all real closed fields). Moreover, tropical polyhedra are obtained as images of such polyhedra by the nonarchimedean valuation, see for instance [Develin and Yu 2007]. Polytopes and simplexes in p-adic fields are introduced in [Darnière 2017; 2019], and demonstrated to play in p-adically closed fields the role played by real simplexes in the classical results of triangulation of semialgebraic sets over real closed fields. Although we are not aware of any direct link of these results with the present work, we hope for some connections to be found in the future.

5C. ***Abstract convexity spaces.*** Our results here can be naturally placed in the context of abstract convexity spaces; we refer to [van de Vel 1993] for an introduction to the subject. A *convexity space* is a pair $(X, \mathcal{C})$, where X is a set and $\mathcal{C} \subseteq 2^X$ is a family of subsets of X closed under intersection with $\varnothing, X \in \mathcal{C}$. The sets in $\mathcal{C}$ are called *convex*. Given a subset $Y \subseteq X$, the *convex hull* of Y, denoted $\mathrm{conv}(Y)$, is the smallest set in $\mathcal{C}$ containing Y (equivalently, the intersection of all sets in $\mathcal{C}$ containing Y). A convex set $C \in \mathcal{C}$ is called a *half-space* if its complement is also convex. The convexity space $(X, \mathcal{C})$ is *separable* if for every $C \in \mathcal{C}$ and $x \in X \setminus C$, there exists a half-space $H \in \mathcal{C}$ such that $C \subseteq H$ and $x \notin H$ (equivalently, if every convex set is the intersection of all half-spaces containing it). Separability is an abstraction of the hyperplane separation (and more generally Hahn–Banach) theorem. In particular, $(\mathbb{R}^d, \mathrm{Conv}_{\mathbb{R}^d})$ is a separable convexity space (see [Moran and Yehudayoff 2019, Section 1.1] or [van de Vel 1993] for many other examples). The *Radon number*[1] of a convexity space $(X, \mathcal{C})$ is the smallest $k \in \mathbb{N}_{\geq 1}$ (if it exists) such that every $Y \subseteq X$ with $|Y| > k$ can be partitioned into two parts Y_1, Y_2 such that $\mathrm{conv}(Y_1) \cap \mathrm{conv}(Y_2) \neq \varnothing$. The classical Radon's theorem states that the Radon number of $(\mathbb{R}^d, \mathrm{Conv}_{\mathbb{R}^d})$ equals $d+1$. Given $\varnothing \neq Y \subseteq X$, a partition $Y_1, \ldots, Y_r$ of Y is *Tverberg* if $\bigcap_{i=1}^r \mathrm{conv}(Y_i) \neq \varnothing$. The *$r$-th Tverberg number* of $(X, \mathcal{C})$ is the smallest k such that every $Y \subseteq X$ with $|Y| > k$ has a Tverberg partition in $r+1$ parts. Note that the first Tverberg number is the Radon number, and the classical theorem of Tverberg says that the r-th Tverberg number of $(\mathbb{R}^d, \mathrm{Conv}_{\mathbb{R}^d})$ is $r(d+1)$.

Now let K be a valued field and $d \in \mathbb{N}_{\geq 1}$. Then $(K^d, \mathrm{Conv}_{K^d})$ is a convexity space, but we stress that it is *not separable*; in fact, $\varnothing$ and K^d are the only half-spaces. This is because for any nonempty proper convex set C, if we let $x \in C$, $y \in K^d \setminus C$, and $\alpha \in K \setminus \mathcal{O}$, then $z := x + \alpha(y - x) \notin C$, since $y = \alpha^{-1} z + (1 - \alpha^{-1}) x$ is a convex combination. But then $x = (1-\alpha)^{-1}(z - \alpha y)$ is a convex combination of elements of $K^d \setminus C$, so $K^d \setminus C$ is not convex.

Proposition 2.8 implies that the Radon number of $(K^d, \mathrm{Conv}_{K^d})$ is $d+1$. By the Levi inequality in an arbitrary convexity space [van de Vel 1993, Chapter II(1.9)], it follows that the Helly number of Conv_{K^d} (Definition 4.4) is at most $d+1$ (we included a proof in Theorem 4.5 for completeness). It was also recently shown in [Holmsen and Lee 2021] that in any convexity space $(X, \mathcal{C})$ with Radon number k, $\mathcal{C}$ has a fractional Helly number (Definition 4.11) bounded by some function of k. In the case of $(K^d, \mathrm{Conv}_{K^d})$ this general bound is much weaker than the optimal bound $d+1$ given in Theorem 4.14. Corollary 2.9 implies that the Carathéodory number of $(K^d, \mathrm{Conv}_{K^d})$ is $d+1$ (see [van de Vel 1993, Chapter II(1.5)] for the definition). Finally, Theorem 4.15 implies that the r-th Tverberg number of $(K^d, \mathrm{Conv}_{K^d})$ is $r(d+1)$; finiteness of the r-th Tverberg numbers for all r follows from the finiteness of the Radon number in an arbitrary convexity space, with a much weaker bound [van de Vel 1993, Chapter II(5.2)].

[1] An alternative definition uses $\geq$ instead of $>$, leading to a value higher by 1. The definition here follows [van de Vel 1993, Chapter II].

Acknowledgements

We thank the referees for many very helpful literature pointers and suggestions on improving the paper. Sections 5B and 5C were added following their suggestions. We thank Lou van den Dries for pointing out Monna's work to us, Dave Marker for pointing out Example 5.8, and Matthias Aschenbrenner for a helpful conversation. Both authors were partially supported by the NSF CAREER grant DMS-1651321, and Chernikov was additionally supported by a Simons fellowship.

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Received September 16, 2021. Revised January 13, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.31

GENERALISATIONS OF HECKE ALGEBRAS FROM LOOP BRAID GROUPS

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We introduce a generalisation LH_n of the ordinary Hecke algebras informed by the loop braid group LB_n and the extension of the Burau representation thereto. The ordinary Hecke algebra has many remarkable arithmetic and representation theoretic properties, and many applications. We show that LH_n has analogues of several of these properties. In particular we consider a class of local (tensor space/functor) representations of the braid group derived from a meld of the (nonfunctor) Burau representation (1935) and the (functor) Deguchi et al., Kauffman and Saleur, and Martin and Rittenberg representations here called Burau–Rittenberg representations. In its most supersymmetric case somewhat mystical cancellations of anomalies occur so that the Burau–Rittenberg representation extends to a loop Burau–Rittenberg representation. And this factors through LH_n. Let SP_n denote the corresponding (not necessarily proper) quotient algebra, k the ground ring, and $t \in k$ the loop-Hecke parameter. We prove the following:

(1) LH_n is finite dimensional over a field.

(2) The natural inclusion $\mathrm{LB}_n \hookrightarrow \mathrm{LB}_{n+1}$ passes to an inclusion $\mathrm{SP}_n \hookrightarrow \mathrm{SP}_{n+1}$.

(3) Over $k = \mathbb{C}$, SP_n / rad is generically the sum of simple matrix algebras of dimension (and Bratteli diagram) given by Pascal's triangle. (Specifically SP_n / rad $\cong \mathbb{C}S_n/e^1_{(2,2)}$ where S_n is the symmetric group and $e^1_{(2,2)}$ is a $\lambda = (2, 2)$ primitive idempotent.)

(4) We determine the other fundamental invariants of SP_n representation theory: the Cartan decomposition matrix; and the quiver, which is of type-A.

(5) The structure of SP_n is independent of the parameter t, except for $t = 1$.

(6) For $t^2 \neq 1$ then $\mathrm{LH}_n \cong \mathrm{SP}_n$ at least up to rank $n = 7$ (for $t = -1$ they are not isomorphic for $n > 2$; for $t = 1$ they are not isomorphic for $n > 1$).

Finally we discuss a number of other intriguing points arising from this construction in topology, representation theory and combinatorics.

MSC2020: 16T25, 18M15, 20C08, 20F36.

Keywords: Loop braid group, Hecke algebra, charge conservation.

1. Introduction

Until the 1980s, methods to construct linear representations of the braid group B_n were relatively scarce. We have those factoring through the symmetric group and the Burau representation [1935], and those factoring through the Hecke algebra [Hoefsmit 1974] and the Temperley–Lieb algebra [Temperley and Lieb 1971]; and, as for every group, the closure in the monoidal category $\mathrm{Rep}(B_n)$. These proceed essentially through "combinatorial" devices such as Artin's presentation. Then there are some more intrinsically "topological" constructions such as Artin's representation [1947] (and Burau can be recast in this light [Long and Paton 1993]).

In the 80s there were notable steps forward. Algebraic formulations of the Yang–Baxter equation began to yield representations; see e.g., [Baxter 1982]. Jones' discovery [1986] of link invariants from finite dimensional quotients of the group algebra $\mathbb{K}[B_n]$ inspired a revolution in braid group representations and topological invariants [Kauffman 1990; Birman and Wenzl 1989; Murakami 1987; Freyd et al. 1985; Wenzl 1988]. Work of Drinfeld [1987], Reshetikhin and Turaev [1991], Jimbo [1986] and others on quantum groups yielded yet further representations. Enriched through modern category theory [Turaev 1994; Kassel and Turaev 2008; Bakalov and Kirillov 2001; Damiani et al. 2021], constructions are now relatively abundant.

The connections among B_n representations, (2+1)-dimensional topological quantum field theory (see e.g., [Witten 1989]) and statistical mechanics (see e.g., [Baxter 1982; Akutsu and Wadati 1987; Martin 1988; Deguchi 1989; Deguchi and Akutsu 1990]) were already well established in the 1980s. Even more recently, the importance of such representations in topological phases of matter [Freedman et al. 2003; Rowell and Wang 2018] in two spacial dimensions has led to an invigoration of interest, typically focused on unitary representations associated with the 2-dimensional part of a (2+1)-TQFT. In this context the braid group is envisioned as the group of motions of point-like quasiparticles in a disk, with the trajectories of these *anyons* forming the braids in 3-dimensions. Here the braid group generators σ_i correspond to exchanging the positions of the i and $(i+1)$-st anyons. The density of such braid group representations in the group of (special) unitary matrices is intimately related to the universality of quantum computational models built on these topological phases of matter [Freedman et al. 2002a; 2002b], as well as the (classical) computational complexity of the associated link invariants [Rowell 2009]. Indeed, there is a circle of conjectures relating *finite* braid group images [Naidu and Rowell 2011; Rowell et al. 2009], *classical* link invariants, *nonuniversal* topological quantum computers and *localisable* unitary braid group representations [Rowell and Wang 2012; Galindo et al. 2013]. The other side of this conjectured coin relates the holy grail of universal topological quantum computation with powerful 3-manifold invariants through surgery on links in the three sphere.

What is a nontrivial generalisation of the braid group to 3-dimensions? Natural candidates are groups of motions: heuristically, the elements are classes of trajectories of a compact submanifold N inside an ambient manifold M for which the initial and final positions of N are set-wise the same. The group of motions of points in a 3-manifold in effect simply permutes the points, but the motion of circles or more general links in a 3-manifold is highly nontrivial. This motivates the study of these 3-dimensional motion groups, as defined in the mid-20th century by Dahm [1962] and expounded upon by Goldsmith [1981; 1982].

More formally, a motion of N inside M is an ambient isotopy $f_t(x)$ of N in M so that

$$f_0 = \mathrm{id}_M \quad \text{and} \quad f_1(N) = N.$$

Such a motion is *stationary* if $f_t(N) = N$ for all t; and given any motion f, we have the usual notion of the reverse $\bar{f}$. We say two motions f, g are equivalent if the composition of f with $\bar{g}$ (via concatenation) gives a motion endpoint-fixed homotopic to a stationary motion as isotopies

$$M \times [0, 1] \to M.$$

The *motion group* $\mathcal{M}o(M, N)$ is the group of motions modulo this equivalence. When M and N are both oriented we will consider only motions f so that $f_1(N) = N$ as an oriented submanifold, although one may consider the larger groups allowing for orientation reversing motions.

The motion groups of links inside $\mathbb{R}^3$, S^3 or D^3 and their representations are very rich, and only recently explored in the literature [Bellingeri and Bodin 2016; Damiani and Kamada 2019; Kádár et al. 2017; Bullivant et al. 2020; Baez et al. 2007; Bullivant et al. 2019]. Further enticement is provided by the prospect of applications to 3-dimensional topological phases of matter with loop-like excitations (i.e., vortices) [Wang and Levin 2014]. The fruitful symbiosis between braid group representations and 2-dimensional condensed matter systems give us hope that 3-dimensional systems could enjoy a similar relationship with motion group representations, (3+1)-TQFTs, and invariants of surfaces embedded in 4-manifolds; see e.g., [Kamada 2007; Carter et al. 2004].

There are a few hints in the literature that the (3+1)-dimensional story has some key differences from the (2+1)-dimensional situation. Reutter [2020] has shown that semisimple (3+1)-TQFTs cannot detect smooth structures on 4-manifolds. Wang and Qiu [2021] provided evidence that the mapping class group and motion group representations associated with (3+1)-dimensional Dijkgraaf–Witten TQFTs are determined via dimension reduction by the corresponding (2+1)-dimensional DW theory. As the representation theory of motion groups has been largely neglected until very recently, it is hard to speculate on precise statements analogous to the 2-dimensional conjectures and theorems.

In this article we take hints from the classical works [Burau 1935; Hoefsmit 1974], from the braid group revolution [Jones 1987], and more directly from statistical mechanics [Deguchi and Akutsu 1990; Kauffman and Saleur 1991; Martin and Rittenberg 1992; Deguchi and Martin 1992], to study representations of the motion group of free unlinked circles in 3-dimensional space, the *loop braid group* LB_n. Presentations of LB_n are known; see [Fenn et al. 1997; Damiani 2017]. As LB_n contains the braid group B_n as an abstract subgroup, a natural approach to finding linear representations is to extend known B_n representations to LB_n. This has been considered by various authors; see e.g., [Bruillard et al. 2015; Bardakov 2005; Kádár et al. 2017]. Another idea is to look for finite dimensional quotients of the group algebra, mimicking the techniques of [Jones 1987; Birman and Wenzl 1989]. As nontrivial finite-dimensional quotients of the braid group are not so easy to find, we take a hybrid approach: we combine the extension of the Burau representation to LB_n [Burau 1935; Bardakov 2005] with the Hecke algebras $\mathcal{H}_n$ obtained from $\mathbb{Q}(t)[B_n]$ as the quotient by the ideal generated by

$$(\sigma_i + 1)(\sigma_i - t).$$

While the naive quotient of $\mathbb{Q}(t)[\mathrm{LB}_n]$ by this ideal does not provide a finite dimensional algebra, certain additional quadratic relations (satisfied by the extended Burau representation) are sufficient for finite dimensionality, with quotient denoted LH_n. We find a local representation of LH_n that aids in the analysis of its structure — the loop Burau–Rittenberg representation. One important feature of the algebras LH_n is that they are not semisimple; in fact, the image of the loop Burau–Rittenberg representation has a 1-dimensional center, but is far from simple. Its semisimple quotient by the Jacobson radical gives an interesting tower of algebras with Bratteli diagram exactly Pascal's triangle.

Our results suggest new lines of investigation into motion group representations. What other finite dimensional quotients of motion group algebras can we find (see e.g., [Banjo 2013])? What is the role of (non)semisimplicity in such quotients? Can useful topological invariants be derived from these quotients? What do these results say about (3+1)-dimensional TQFTs?

Outline of the paper. In Section 2 we recall the Burau representation and corresponding knot invariants. In Section 3 we introduce loop Hecke algebras and prove they are finite dimensional. In Section 4 we develop arithmetic tools (calculus) that we will need. In Section 5 we construct our local representations and hence prove our main structure Theorems. In Section 6 we apply the results from Section 5 to LH_n, and make several conjectures on the open cases with $t^2 = 1$. We conclude with a discussion of new directions opened up by this work.

2. Burau representation, Hecke algebra and invariants of knots

Let $\underline{n} := \{1, 2, \ldots, n\}$. Then the braid group B_n may be identified with the motion group $\mathcal{M}o(\mathbb{R}^2, \underline{n} \times \{0\})$. Artin showed that, for $n \geq 1$, B_n admits the presentation

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \ldots, n-2 \end{array} \right\rangle \tag{2-1}$$

We will write $\mathfrak{A}_n(\sigma)$ for the set of relations here.

We will also need the symmetric group S_n. In a "motion group spirit" this can be identified with $\mathcal{M}o(\mathbb{R}^3, \underline{n} \times \{0\} \times \{0\})$. It can be presented as a quotient of B_n by the relation ${\sigma_1}^2 = 1$ (however since we will often want to have both groups together we will soon rename the S_n generators).

2A. *Burau representation.* We define Burau representation $\varrho{:}B_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ as follows:

$$\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}. \tag{2-2}$$

The Burau representation has Jordan–Holder decomposition into a 1-dimensional representation (the vector $(1, \ldots, 1)^T$ remains fixed) and an $(n-1)$-dimensional irreducible representation known as reduced Burau representation $\bar{\varrho}\colon B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$. The decomposition is not split over $\mathbb{Z}[t, t^{-1}]$ — an inverse of $t + 1$ is needed (see later).

Remark 2.1. One can also use the transpose matrix of (2-2) (depending on orientation choices while building the "carpark cover" of the punctured disc in the homological definition of Burau). The transpose fixes $(1, \ldots, 1, t, t^2, 1, \ldots, 1)^T$.

2B. *Facts about the Burau representation.*

(1) Burau is unfaithful for $n \geq 5$ (Moody [1991] proved unfaithfulness for $n \geq 9$, Long and Paton [1993] for $n \geq 6$, Bigelow [1999] for $n = 5$).

(2) The case $n = 4$ is open, Beridze and Traczyk [2018] recently published some advances toward closing the problem.

(3) It is faithful for $n = 2, 3$ [Magnus and Peluso 1969].

(4) If we consider the braid group in its mapping class group formulation, it has a homological meaning (attached *a posteriori* to it, since Burau [1935] used only combinatorial aspects of matrices). The Burau representation describes the action of braids on the first homology group of the (covering of) the punctured disk. On the other hand the Alexander polynomial is extracted from the presentation matrix of the first homology group of the knot complement (the Alexander matrix). When we close up a braid, each element of homology of the punctured disk on the bottom

becomes identified with its image in the punctured disk at the top. At this point the Alexander matrix of the closed braid is (roughly) the Burau matrix of the braid with the modification of identifying the endpoints.

More specifically, let K be a knot, and b a braid such that $\hat{b}$ is equivalent to K. Then the Alexander polynomial $\Delta_K(t)$ can be obtained by computing:

$$\Delta_K(t) = \frac{\det(\bar{\varrho}(b) - I_{n-1})}{1 + t + \cdots + t^{n-1}}.$$

So one can think of the Alexander polynomial of $K \sim \hat{b}$ as a rescaling of the characteristic polynomial of the image of b in the reduced representation.

Representations of B_n are partially characterised by the eigenvalue spectrum of the image of σ_i. Observe that

$$\varrho({\sigma_i}^2) = (1-t)\varrho(\sigma_i) + t I_n, \tag{2-3}$$

i.e., the eigenvalue spectrum is $\mathrm{Spec}(\varrho(\sigma_i)) = \{1, -t\}$. Recall also that Kronecker products obey $\mathrm{Spec}(A \otimes B) = \mathrm{Spec}(A).\,\mathrm{Spec}(B)$, so $\mathrm{Spec}(\varrho(\sigma_i) \otimes \varrho(\sigma_i) = \varrho \otimes \varrho(\sigma_i)) = \{1, -t, t^2\}$. From this we see that the spectrum is fixed under tensor product only if $t = \pm 1$; see for example [Kauffman and Saleur 1991].

2C. ***Hecke algebras.*** Let R be an integral domain and q_1, q_2 elements of R with q_2 invertible. We define the Hecke algebra $H_n^R(q_1, q_2)$ to be the algebra with generators $\{1, T_1, \ldots, T_{n-1}\}$ and the following defining relations:

$$T_i T_j = T_j T_i \qquad \text{for } |i-j| > 1, \tag{2-4}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad \text{for } i = 1, \ldots, n-2, \tag{2-5}$$

$$T_i^2 = (q_1 + q_2) T_i - q_1 q_2 \qquad \text{for } i = 1, \ldots, n-1. \tag{2-6}$$

Remark 2.2. (1) Relation (2-6) coincides with the characteristic equation of the images of the generators under the Burau representation 2A when $(q_1, q_2) = (1, -t)$. We denote the resulting 1-parameter Iwahori–Hecke algebra by $H_n^R(t)$.

(2) If $t = 1$ then $H_n^R(t)$ is the group algebra $R[S_n]$ (the free R-module RS_n made an R-algebra in the usual way).

(3) There is a map from B_n to $H_n^R(t)$ sending σ_i to T_i. Thus representations of $H_n^R(t)$ are equivalent to representations of B_n for which the generators satisfy relation (2-6). This is described in [Bigelow 2006, Section 3; Jones 1987, Section 4; Martin 1991, Section 5.7] and many other places.

(4) Fixing $R = \mathbb{C}$, point (3) allows us to think of $H_n^R(t)$ as being isomorphic to the quotient $H_n(t) := \mathbb{C}[B_n]\sigma_i^2 = (1-t)\sigma_i + t$.

(5) Using the map in (3) we can represent any element of $H_n(t)$ as a linear combination of braid diagrams. The quadratic relation can be seen as a *skein relation* on elementary crossings. Knowing a basis for $H_n(t)$ makes this fact usable.

Question 2.3. Why these parameters and this quadratic relation?

As noted, Hecke algebras can be defined with two units of R as parameters. We chose to fix these parameters to $(1, -t)$ because from this quotient one should recover the Alexander polynomial. Choosing $(-1, t)$ one should get the quotient on which Ocneanu traces are defined; see [Kassel and Turaev 2008, Chapter 4.2]. With the Ocneanu trace being a 1-parameter family over a 1-parameter algebra, we end up with polynomials in two variables. These polynomials are attached to the braid diagrams that we can see representing elements of $H_n(t)$. Moreover they are defined in such a way to respect Markov moves, so they are invariants for the closures of said braids. Hence, they are knot invariants. The quadratic relation from Remark 2.2(3) translates the trace in a *skein relation*. Through the Ocneanu trace (normalised) the invariant that is obtained is the HOMFLY-PT polynomial, which specialises in both Alexander and Jones. Each specialisation corresponds to factoring through a further quotient of the Hecke algebra (in the case of Jones, this is a quotient of the Temperley–Lieb algebra). Below we "reverse engineer" this process.

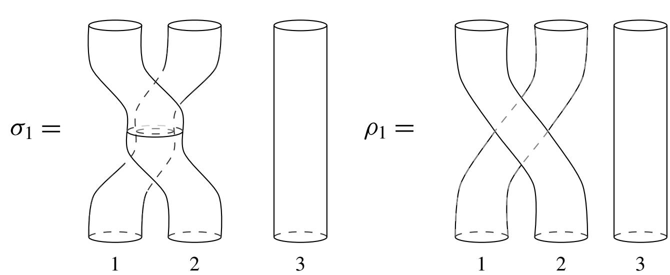
3. Generalising Burau and Hecke to loop braid groups

3A. ***The loop braid group.*** Here S^1 denotes the unit circle. We now consider the loop braid group

$$\mathrm{LB}_n = \mathcal{M}o(\mathbb{R}^3, \underline{n} \times S^1);$$

see e.g., [Goldsmith 1981; Savushkina 1996; Fenn et al. 1997; Brendle and Hatcher 2013; Damiani 2017; Kádár et al. 2017; Bruillard et al. 2015].

Consider the set $\Xi_n = \{\sigma_i, \rho_i, i = 1, 2, \ldots, n-1\}$ and group $\langle \Xi_n \mid \mathfrak{Q}_n \rangle$ presented by generators σ_i and ρ_i, and relations $\mathfrak{Q}_n$ as follows. The generators may be visualised as the "leapfrog" and loop exchange, such as the following depictions of σ_1 and ρ_1 as generators of LB_3 (motions read bottom-to-top):



The σ_i obey the braid relations as in (2-1); the ρ_i obey the braid relations and also

$$\rho_i \rho_i = 1 \tag{3-1}$$

and then there are mixed braid relations

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \tag{3-2}$$

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \tag{3-3}$$

$$\sigma_i \rho_{i\pm j} = \rho_{i\pm j} \sigma_i \qquad (j > 1) \text{ (all distant commutators).} \tag{3-4}$$

Remark 3.1. The first mixed relation (3-2) implies its reversed order counterpart:

$$\sigma_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \sigma_{i+1} \tag{3-5}$$

whereas the reversed order second mixed relation does not hold. The relations also imply

$$\rho_2 \sigma_1 \rho_2 = \rho_1 \sigma_2 \rho_1. \tag{3-6}$$

We have (see e.g., [Fenn et al. 1997]) that

$$\mathrm{LB}_n \cong \langle \Xi_n \mid \mathfrak{Q}_n \rangle. \tag{3-7}$$

It will be convenient to give an *algebra presentation* for the group algebra. Recall that in an algebra presentation inverses are not present automatically by freeness, so we may put them in by hand as formal symbols and then impose the inverse relations. Thus as a presented algebra we have

$$k\langle \Xi_n \mid \mathfrak{Q}_n \rangle = \langle \Xi_n \cup \Xi_n^- \mid \mathfrak{Q}_n, \mathfrak{I}_n \rangle_k;$$

here kG means the group k-algebra of group G, $\langle - \mid - \rangle_k$ means a k-algebra presentation and $\mathfrak{I}_n$ is the set of inverse relations $\sigma_i \sigma_i^- = 1$.

3B. *The loop–Hecke algebra* $\mathbf{LH}_n$. With Section 2 in mind, there *is* a suitable generalisation of the Burau representation to LB_n.

Proposition 3.2 [Vershinin 2001]. *The map on generators of* LB_n *given by*

$$\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}. \tag{3-8}$$

$$\rho_i \mapsto I_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}. \tag{3-9}$$

extends to a representation $\varrho_{GB} : \mathrm{LB}_n \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$.

Proof. Direct calculation. □

This group representation is not faithful for $n \geq 3$ [Bardakov 2005], and corresponds to an Alexander polynomial for welded knots.

We consider a quotient algebra of the group algebra (over a suitable commutative ring) of the group $\langle \Xi_n \mid \mathfrak{Q}_n \rangle$. The quotient algebra is

$$\text{(3-10)} \qquad \mathrm{LH}_n^{\mathbb{Z}} := \mathbb{Z}[t, t^{-1}]\langle \Xi_n \mid \mathfrak{Q}_n \rangle \mathfrak{R}_n = \langle \Xi_n \cup \Xi_n^- \mid \mathfrak{Q}_n, \mathfrak{I}_n, \mathfrak{R}_n \rangle_{\mathbb{Z}[t,t^{-1}]}$$

where $\mathfrak{R}_n$ is the set of (algebra) relations:

$$\text{(3-11)} \qquad \sigma_i^2 = (1-t)\sigma_i + t \qquad \text{(i.e., } (\sigma_i - 1)(\sigma_i + t) = 0\text{)},$$

$$\text{(3-12)} \qquad \rho_i \sigma_i = -t\rho_i + \sigma_i + t \qquad \text{(i.e., } (\rho_i - 1)(\sigma_i + t) = 0\text{)},$$

$$\text{(3-13)} \qquad \sigma_i \rho_i = -\sigma_i + \rho_i + 1 \qquad \text{(i.e., } (\sigma_i - 1)(\rho_i + 1) = 0\text{)}.$$

(NB we already have $(\rho_i - 1)(\rho_i + 1 = 0.)$

Observe that (3-11) yields an inverse for σ_i (the inverse to t is specifically needed), so we have

$$\text{(3-14)} \qquad \mathrm{LH}_n^{\mathbb{Z}} = \langle \Xi_n \mid \mathfrak{Q}_n, \mathfrak{R}_n \rangle_{\mathbb{Z}[t,t^{-1}]}.$$

Observe then that the relations as such do not require an inverse to t, so we could consider the variant algebra over $\mathbb{Z}[t]$.

For any field K that is a $\mathbb{Z}[t, t^{-1}]$ algebra we then define the base change $\mathrm{LH}_n^K = K \otimes_{\mathbb{Z}[t,t^{-1}]} \mathrm{LH}_n^{\mathbb{Z}}$ and, for given $t_c \in \mathbb{C}$,

$$\mathrm{LH}_n(t_c) = \mathrm{LH}_n = \mathrm{LH}_n^{\mathbb{C}}$$

where $\mathbb{C}$ is a $\mathbb{Z}[t]$-algebra by evaluating t at t_c (the choice of which we notationally suppress). Note that there is no reason to suppose that this gives a flat deformation (i.e., the same dimension) in all cases. (It will turn out that it does, at least in low rank, if we can localise at $t^2 - 1$. In particular, perhaps surprisingly, in the variant $t = 0$ is isomorphic to the generic case.)

Remark. The relations (3-11) et seq. are suggested by (2-3) and the following calculations (on σ_1 and ρ_1 in LB_3, noting that blocks work the same way for all generators):

$$\varrho_{GB}(\sigma_1\rho_1) = \begin{pmatrix} t & 1-t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + I_3,$$

$$\varrho_{GB}(\rho_1\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 1-t & t & 0 \\ 0 & 0 & 1 \end{pmatrix} = -t\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + tI_3.$$

3C. ***Notable direct consequences of the relations: Finiteness.*** Given a word in the generators, of form $\sigma_3\sigma_4\rho_2$ say, by a *translate* of it we mean the word obtained by shifting the indices thus: $\sigma_{3+i}\sigma_{4+i}\rho_{2+i}$.

With the $\mathfrak{Q}$ and $\mathfrak{R}$ relations we can derive the following ones, together with the natural translates thereof (here $\overset{*}{=}$ uses (3-1); $\overset{\rho\rho\sigma}{=}$ uses (3-2); $\overset{\sigma\rho}{=}$ uses (3-13), and so on):

$$
\begin{aligned}
\text{(M1)}\quad \boldsymbol{\sigma_2\rho_1\sigma_2} &\overset{*}{=} \sigma_2\rho_2\rho_2\rho_1\sigma_2 \\
&\overset{\rho\rho\sigma}{=} \sigma_2\rho_2\sigma_1\rho_2\rho_1 \\
&\overset{\sigma\rho}{=} -\sigma_2\sigma_1\rho_2\rho_1 + \rho_2\sigma_1\rho_2\rho_1 + \sigma_1\rho_2\rho_1 \\
&\overset{\rho\sigma\sigma,\rho\rho\sigma}{=} -\rho_1\sigma_2\sigma_1\rho_1 + \rho_2\rho_2\rho_1\sigma_2 + \sigma_1\rho_2\rho_1 \\
&\overset{\sigma\rho}{=} \boldsymbol{\sigma_1\rho_2\rho_1 + \rho_1\sigma_2\sigma_1 - \rho_1\sigma_2\rho_1},
\end{aligned}
$$

$$
\begin{aligned}
\text{(M2)}\quad \boldsymbol{\rho_2\sigma_1\sigma_2} &\overset{*}{=} \rho_2\sigma_1\rho_2\rho_2\sigma_2 \\
&\overset{\rho\sigma}{=} -t\rho_2\sigma_1\rho_2\rho_2 + \rho_2\sigma_1\rho_2\sigma_2 + t\rho_2\sigma_1\rho_2 \\
&\overset{*,\rho\sigma\rho}{=} -t\rho_2\sigma_1 + \rho_1\sigma_2\rho_1\sigma_2 + t\rho_1\sigma_2\rho_1 \\
&\overset{M1}{=} -t\rho_2\sigma_1 + \rho_1(\rho_1\sigma_2\sigma_1 - \rho_1\sigma_2\rho_1 + \sigma_1\rho_2\rho_1) + t\rho_1\sigma_2\rho_1 \\
&\overset{*,\rho\sigma}{=} -t\rho_2\sigma_1 + \sigma_2\sigma_1 - \sigma_2\rho_1 + (-t\rho_1 + \sigma_1 + t)\rho_2\rho_1 + t\rho_1\sigma_2\rho_1 \\
&= \sigma_1\rho_2\rho_1 + t\rho_1\sigma_2\rho_1 - t\rho_1\rho_2\rho_1 + \sigma_2\sigma_1 - \sigma_2\rho_1 - t\rho_2\sigma_1 + t\rho_2\rho_1.
\end{aligned}
$$

Definition 3.3. For given n and $m \leq n$ let $\mathrm{LH}^{\langle}_m$ denote the subalgebra of LH_{n+1} generated by Ξ_m (it is a quotient of LH_m, as per the Ψ map formalism in Section 4B).

Lemma 3.4. *For any n let X_i be the vector subspace of LH_n spanned by* $\{1, \sigma_i, \rho_i\}$. *Then* $\mathrm{LH}_{n+1} = \mathrm{LH}^{\langle}_n\, X_n\, \mathrm{LH}^{\langle}_n$.

Proof. It is enough to show that $X_n\, \mathrm{LH}^{\langle}_n\, X_n$ lies in $\mathrm{LH}^{\langle}_n\, X_n\, \mathrm{LH}^{\langle}_n$. We work by induction on n. The case $n = 1$ is clear, since $LH_1 = \mathbb{C}$. Assume true in case $n-1$ and consider case n. We have

$$X_n\, \mathrm{LH}^{\langle}_n\, X_n = X_n\, \mathrm{LH}^{\langle}_{n-1}\, X_{n-1}\, \mathrm{LH}^{\langle}_{n-1}\, X_n$$

by assumption. But $\mathrm{LH}^{\langle}_{n-1}$ and X_n commute so we have $\mathrm{LH}^{\langle}_{n-1}\, X_n X_{n-1} X_n\, \mathrm{LH}^{\langle}_{n-1}$. The inductive step follows from the relations $\mathfrak{Q}$ and $\mathfrak{R}$ and the relations (M1) and (M2) above. □

Corollary 3.5. LH_n *is finite dimensional.* □

Remark 3.6. We may also treat certain other quotients of $\mathbb{C}\,\mathrm{LB}_n$. For example, eliminating either relations (3-12) or (3-13) we still obtain finite dimensional quotients. In particular, if we only include (3-13) and not (3-12) then the analogous proof with X_n replaced by $\{1, \rho_n, \sigma_n, \rho_n\sigma_n\}$ proves finite dimensionality.

3D. ***Refining the spanning set.*** Can we express elements of LH_3 as sums of length-2 words (and hence eventually solve word problem)? We have, for example,

(3-15)
$$\rho_1\rho_2\rho_1 = -1+\rho_2+\frac{(-t-1)}{(t-1)}(-\rho_1+\rho_2\rho_1-\rho_1\rho_2)+\frac{2}{(t-1)}(-\sigma_1+\sigma_2\rho_1-\rho_1\sigma_2)$$

But in general this is not easy. And another problem is that we do not have immediately manifest relationships between different ranks (such as inclusion) that would be useful. With this (and several related points) in mind it would be useful to have a tensor space representation. In what follows we address the construction of such a representation.

4. Basic arithmetic with LH_n

Here we briefly report some basic arithmetic in LH_n that gives the clues we need for our local representation constructions below.

4A. ***Fundamental tools, locality.*** In what follows, B denotes the *braid category*: a strict monoidal category with object monoid $(\mathbb{N}_0, +)$ generated by 1, and $\mathsf{B}(n, n) = B_n$, $\mathsf{B}(n, m) = \varnothing$ otherwise, and monoidal composition is via side-by-side concatenation of suitable braid representatives; see e.g., [Mac Lane 1998, XI.4]. Similarly S is the permutation category (of symmetric groups). Let H denote the ordinary Hecke category — again monoidal, but less obviously so [Humphreys 1990]. (We have not yet shown that LH, the loop-Hecke category, is monoidal.)

Let LB denote the loop-braid category — this is the strict monoidal category analogous to the braid category where the object monoid is $(\mathbb{N}, +)$, $\mathsf{LB}(n, n) = \mathrm{LB}_n$, $\mathsf{LB}(n, m) = \varnothing$ otherwise, and monoidal composition $\otimes$ is side-by-side concatenation of loop-braids.

Suppose C is a strict monoidal category with object monoid $(\mathbb{N}_0, +)$ generated by 1 (for example, LB). Write 1_1 for the unique element of $\mathsf{C}(1, 1)$ and for $x \in \mathsf{C}(n, n)$ define the *translate*

(4-1)
$$x^{(t)} = 1_1^{\otimes t} \otimes x \in \mathsf{C}(n+t, n+t)$$

For k a commutative ring, define translates of elements of $k\,\mathrm{LB}_n$ (i.e., $k\mathsf{LB}(n, n)$), and kS_n and so on, by linear extension.

Caveat. Note that it is a property of the geometric topological construction of loop braids that the composition $\otimes$ in LB makes manifest sense. It requires that side-by-side concatenation of rank n with rank m passes to $n+m$. This is clear by construction. But in groups/algebras defined by generators and relations it would not be intrinsically clear. For example, how do we know that the subalgebra of LH_n generated *in* LH_n by the elements p_i, s_i, $i = 1, 2, \ldots, n-2$ is isomorphic to LH_{n-1}? (Some of our notation requires care at this point since it may lead us to take isomorphism for granted!)

4B. ***The Ψ maps.*** Let $A = \langle X \mid R\rangle_k$ be an algebra presented with generators X and relations R. Then there is a homomorphism from the free algebra generated by any subset X_1 of X to A, taking $s \in X_1$ to its image in A. This factors through the quotient by any relations, R_1 say, expressed only in X_1. We may consider it as a homomorphism from this quotient. But of course the kernel may be bigger — relations induced indirectly by the relations in R. A Ψ map is such a homomorphism:

$$\langle X_1 \mid R_1\rangle_k \xrightarrow{\Psi} \langle X_1 \mid R\rangle_k \hookrightarrow \langle X \mid R\rangle_k$$

Note that arithmetic properties such as idempotency, orthogonality and vanishing are preserved under Ψ maps. Thus for example a decomposition of 1 into orthogonal idempotents in kS_n passes to such a decomposition in LH_n (see (4-3)). However conditions such as primitivity, inequality and even nonzeroness are not preserved in general.

Note that there is a natural (not generally isomorphic) image of

$$kS_n \cong k\langle p_1, \dots, p_i, \dots, p_{n-1} \mid \mathfrak{A}_n(p), p_i p_i = 1\rangle \tag{4-2}$$

in LH_n obtained by the map of generators $p_i \mapsto \rho_i$. Let us call it LH_n^ρ. Thus

$$kS_n \xrightarrow{\Psi} \mathrm{LH}_n^\rho \hookrightarrow \mathrm{LH}_n \tag{4-3}$$

Similarly $H_n = \langle T_1, \dots, T_i, \dots, T_{n-1} \mid \mathfrak{A}_n(T), \dots\rangle_k$ has image LH_n^σ under $T_i \mapsto \sigma_i$:

$$H_n \twoheadrightarrow \mathrm{LH}_n^\sigma \hookrightarrow \mathrm{LH}_n \tag{4-4}$$

Let us consider the image of a primitive idempotent decomposition in kS_n

$$1 = \sum_{\lambda\in\Lambda_n} \sum_{i=1}^{d_\lambda} \boldsymbol{e}_\lambda^i$$

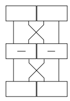
under $\Psi : kS_n \to \mathrm{LH}_n$. Here Λ_n denotes the set of integer partitions of n, and d_λ is the dimension of the S_n irrep. See the Appendix for explicit constructions. We will also write $(\Lambda, \subseteq)$ for the poset of all integer partitions ordered by the usual inclusion as a Young diagram.

Proposition 4.1. *Let k be the field of fractions of $\mathbb{Z}[t, t^{-1}]$:*

(I) *The image $\Psi(\boldsymbol{e}_\lambda^i)$ in LH_n^k of every idempotent with $(2, 2) \subseteq \lambda \in \Lambda_n$ is zero.*

(II) *On the other hand all other $\lambda \in \Lambda_n$, i.e., all hook shapes, give nonzero image.*

Proof. (I) Note that $\boldsymbol{e}^1_\mu$ with $\mu \in \Lambda_m$ is defined in kS_n for $n \geq m$ by $S_m \hookrightarrow S_n$. It is shown for example in [Martin and Rittenberg 1992] that if the relation $\boldsymbol{e}^1_\mu = 0$ is imposed in a quotient of kS_n then $\boldsymbol{e}^i_\nu = 0$ holds for $\mu \subseteq \nu \in \Lambda_n$ (a proof uses $S_{n-1} \hookrightarrow S_n$ restriction rules, from which we see that $\boldsymbol{e}^1_\mu$ is expressible as a sum of orthogonal such idempotents). Consider $\boldsymbol{e}^1_{(2,2)}$ (i.e., with $(2,2) \in \Lambda_4$) which may be expressed as

$\boldsymbol{e}^1_{(2,2)} =$  $\propto (p_1+1)(p_3+1)p_2(p_1-1)(p_3-1)p_2(p_1+1)(p_3+1)$

(using notation and a choice from (A-5)). By a direct calculation in LH_4

$$\Psi((p_1+1)(p_3+1)p_2(p_1-1)(p_3-1)) = 0 \tag{4-5}$$

(NB we know no elegant way to do this calculation; the result holds also for generic t, but *not* for $t = 1$).

(II) This can be verified by evaluation as nonzero in a suitable representation. (For simplicity it is sufficient to work in the "SP quotient" that we give in Theorem 5.2 below, working with Kronecker products. We will omit the explicit calculation.) □

With identity (4-5) in mind, recall that in [Martin and Rittenberg 1992] local representations of ordinary Hecke (and hence S_n) with this property were constructed from spin chains. In Section 5 we will combine this with Burau and thus find the representations of loop-Hecke that we need here.

By Proposition 4.1 we have a decomposition of 1 in LH_n according to hook partitions

$$1 = \sum_{i=0}^{n-1} \sum_{j=1}^{d_{(n-i,1^i)}} \Psi(\boldsymbol{e}^j_{(n-i,1^i)}). \tag{4-6}$$

(NB j varies over idempotents that are equivalent in the sense that they induce isomorphic modules — it will be sufficient to focus on $j = 1$.)

(Left) multiplying by $A = \mathrm{LH}_n$ we thus have a decomposition of the algebra

$$A \cong \oplus_{i=0}^{n-1} \oplus_j A\Psi(\boldsymbol{e}^j_{(n-i,1^i)})$$

as a left-module for itself, into projective summands.

We have not yet shown that these summands are indecomposable. But consider for a moment the action of LH_n on the image under Ψ of

$$Y^n_\pm = \sum_{g \in S_n} (\pm 1)^{\mathrm{len}(g)} g$$

in LH_n (we write Y^n_+ for unnormalised $\boldsymbol{e}^1_{(n)}$ and Y^n_- for $\boldsymbol{e}^1_{(1^n)}$; again, see the Appendix for a review). By abuse of notation we will write $Y^n_\pm$ also for the image. By (3-13) and the classical identities $Y_\pm^{a(1)} Y_\pm^n = a! Y_\pm^n$ (recall $Y_\pm^{a(1)}$ means $Y_\pm^a$ with indices shifted by +1, see (A-2) et seq.) we have

$$\sigma_i Y_+^n = Y_+^n, \quad Y_-^n \sigma_i = -t Y_-^n. \tag{4-7}$$

It follows that Y_+^n spans a 1-d left ideal in LH_n. If we work over a field containing the rationals then it is normalisable as an idempotent, and so we have an indecomposable projective left module

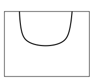
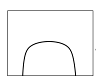
$$P_{(n)} = \mathrm{LH}_n\, Y_+^n = \mathrm{LH}_n\, \boldsymbol{e}_{(n)}^1 = k\boldsymbol{e}_{(n)}^1.$$

5. On local representations

Here Mat is the monoidal category of matrices over a given commutative ring (and Mat_k the case over commutative ring k), with object monoid $(\mathbb{N}, \times)$ and tensor product on morphisms given by a Kronecker product (NB there is a convention choice in defining the Kronecker product). We often focus on the monoidal subcategory Mat^m generated by a single object $m \in \mathbb{N}$ — usually $m = 2$. Then the object monoid $(2^{\mathbb{N}}, \times)$ becomes $(\mathbb{N}, +)$ in the natural way.

In the study of ordinary Hecke algebras (and particularly quantum-group-controlled quotients like Temperley–Lieb) a very useful tool is the beautiful set of local tensor space representations generalising those arising from XXZ spin chains and Schur–Weyl duality. For example we have the following.

Consider the TL diagram category T with object monoid $(\mathbb{N}, +)$ k-linear-monoidally generated by the morphisms represented by diagrams

$\mathsf{u} =$  $\in \mathsf{T}(2, 0)$ and $\mathsf{u}^* =$ .

This has a TQFT F_2 given by $\mathsf{u} \mapsto (0, \tau, \tau^{-1}, 0)$ (the target category is Mat) and taking $*$ to transpose. Of course for $1_1 \in \mathsf{T}(1, 1)$ we have $\mathsf{F}_2(1_1) = I_2$.

To pass to our present topic we note that $1_1 \otimes 1_1 = 1_2$ and that the Yang–Baxter construction $\sigma_1 \mapsto 1_2 - \tau^2 \mathsf{u}^* \mathsf{u}$ gives

$$\sigma_1 \mapsto \mathsf{F}_2(1_2) - \tau^2 \begin{pmatrix} 0 &&& \\ & \tau^2 & 1 & \\ & 1 & \tau^{-2} & \\ &&& 0 \end{pmatrix} = \begin{pmatrix} 1 &&& \\ & 1-\tau^4 & -\tau^2 & \\ & -\tau^2 & 0 & \\ &&& 1 \end{pmatrix} \tag{5-1}$$

thus a representation of the braid category B (note that eigenvalues are 1 and $-\tau^4$ so τ^4 here passes to t in our parametrisation for loop-Hecke). But also note that u, u^* can be used for a Markov trace. And also for idempotent localisation functors: let $\mathsf{U} = \mathsf{u}^* \mathsf{u}$, $\mathsf{U}_1 = \mathsf{U} \otimes 1_{n-2}$, and $T_n = \mathsf{T}(n, n)$ regarded as a k-algebra; then we have the algebra isomorphism $\mathsf{U}_1 T_n \mathsf{U}_1 \cong T_{n-2}$. This naturally gives a category embedding

$\mathsf{G_U}$ of T_{n-2}-mod in T_n-mod. Recall that irreps are naturally indexed by partitions of n into at most two parts: $\lambda = (n-m, m)$, or equivalently (for given n) by "charge" $\lambda_1 - \lambda_2 = n - 2m$, thus by $\Upsilon_n = \{n, n-2, n-4, \dots, 0/1\}$ (depending on n is odd or even). This latter labeling scheme is stable under the embedding. That is, indecomposable projective modules are mapped by $\mathsf{G_U}$ according to $\Upsilon_{n-2} \hookrightarrow \Upsilon_n$.

5A. *Charge conservation.* Another useful property of F_2 is "charge conservation". We may label the row/column index for object 2 in Mat by $\{\varepsilon_1, \varepsilon_2\}$ or $\{+, -\}$. Then $2 \otimes 2$ has index set $\{\varepsilon_1 \otimes \varepsilon_1, \varepsilon_2 \otimes \varepsilon_1, \varepsilon_1 \otimes \varepsilon_2, \varepsilon_2 \otimes \varepsilon_2\}$ (which we may abbreviate to $\{11, 21, 12, 22\}$) and so on. The "charge" ch of an index is $ch = \#1 - \#2$. Note from (5-1) that F_2 does not mix between different charges (hence charge conservation).

For a functor with the charge conservation property the representation of B_n (say) obtained has a direct sum decomposition according to charge, with "Young blocks" β_i of charge $i = n, n-2, \dots, -n$. The dimensions of the blocks are given by Pascal's triangle. It will be convenient to express this with the semiinfinite Toeplitz matrices $\mathcal{U}$ and $\mathcal{T}$:

$$\mathcal{U} = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & \ddots \end{pmatrix}, \quad \mathcal{U}^2 = \begin{pmatrix} 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ & & & & \ddots & \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & & \ddots \end{pmatrix}$$

and semiinfinite vectors $v_1 = (1, 0, 0, 0, \dots)$, $v_2 = (0, 1, 0, 0, \dots)$, ... Thus $v_1 \mathcal{U}^n$ (respectively $v_{n+1}\mathcal{T}^n$) gives the numbers in the $n+1$-th row of Pascal (followed by a tail of zeros). (The two different formulations correspond to two different thermodynamic limits — $\mathcal{T}$ corresponds to the $\Upsilon_{n-2} \hookrightarrow \Upsilon_n$ limit — see later.) Then

$$\dim(\beta_i) = (v_1 \mathcal{U}^n)_{(n-i+2)/2} = (v_{n+1}\mathcal{T}^n)_{n-i+1}. \tag{5-2}$$

In the case of F_2 these blocks are not linearly irreducible in general (the generic irreducible dimensions are given by $v_1 \mathcal{T}^n$). But they still provide a useful framework. We return to this later.

With this construction and Proposition 3.2 in mind, it is natural to ask if we can make a local version of generalised Burau. (Folklore is that this cannot work, and directly speaking it does not. But we now have some more clues at our disposal.)

5B. *Representations of* B. Now we have in mind Proposition 4.1; and brute force calculations in low rank showing (see Section 6) that LH_n is nonsemisimple but has irreducible representations with dimensions given by Pascal's triangle. This is reminiscent of Rittenberg's analysis of the quantum spin chains over Lie superalgebras found in [Deguchi 1989; Deguchi et al. 1989; Deguchi and Akutsu 1990; Kauffman and Saleur 1991; Martin and Rittenberg 1992; Deguchi and Martin 1992]. It is also reminiscent of work of Saleur on "type-B" braids [Martin and Saleur 1994]; emdash but for this see e.g., [Bullivant et al. 2020]. Inspired by this and the Burau representation (and see [Damiani and Florens 2018]) we proceed as follows. Define

$$M_t(\sigma) = \begin{pmatrix} 1 & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}, \quad M'_t(\sigma) = \begin{pmatrix} 1 & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & -t \end{pmatrix} \tag{5-3}$$

as in [Deguchi 1989; Kauffman and Saleur 1991]. Fix a commutative ring k, $\tau \in k^\times$, and $t = \tau^4$. Observe that there is a monoidal functor F_M from the Braid category B to Vect (or at least Mat) given by object 1 mapping to $V = \mathbb{C}\{e_1, e_2\}$ (i.e., to 2 in $\mathsf{Mat}_{\mathbb{Z}[t]}$) and the positive braid σ in $\mathsf{B}(2,2)$ mapping to $M_t(\sigma)$. The conjugation of this matrix to $\mathsf{F}_2(\sigma)$ lifts to a natural isomorphism of functors. Another natural isomorphism class of charge conserving functors has representative functor $\mathsf{F}_{M'}$ given by $M'_t(\sigma)$. (According to the scheme of Deguchi et al., this is the (1,1)-super class; see for example [Deguchi 1989; Kauffman and Saleur 1991]. But note that in extending to LB below, isomorphism will not be preserved, so we are focusing on the specific representative.) In fact some elementary analysis shows that these two classes are all of this form that factor through Hecke (apart from the trivial class).

Let us formulate this in language that will be useful later. First note that (like any invertible matrix) $M_t(\sigma)$ and $M'_t(\sigma)$ extend to monoidal functors from the free monoidal category generated by σ to Mat. Thus, in particular,

$$M'_t(\sigma \otimes 1_1) = M'_t(\sigma) \otimes \mathrm{Id}_2 \in \mathsf{Mat}(2^3, 2^3).$$

Given the form of the construction, proof of the above factoring through B follows from a direct verification of the braid relation in each case. More interestingly we have, again by direct calculation, the stronger result

$$M'_t(\sigma \otimes 1_1) M'_t(1_1 \otimes \sigma) M'_s(\sigma \otimes 1_1) = M'_s(1_1 \otimes \sigma) M'_t(\sigma \otimes 1_1) M'_t(1_1 \otimes \sigma) \tag{5-4}$$

while the tss version of this identity does *not* hold (unless we force $s = 1$, or $s = t$) (NB care must be taken with conventions here.)

To pass back from the basic-algebra/homology to the full algebra we need the dimensions of the irreducibles. For an algebra A with Cartan matrix $C_L(A)$ and a vector $v_L(A)$ giving the dimensions of the irreducible heads of the projectives we have

$$\dim(A) = v_L(A) C_L(A) v_L(A)^T. \tag{5-5}$$

Definition 5.1. Let the $n \times n$ matrix $\overline{\mathcal{M}}_n$ be:

$$\overline{\mathcal{M}}_n = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 1 & 1 \end{pmatrix}$$

We label columns left to right (and rows top to bottom) by the ordered set h_n of hook integer partitions of n:

$$h_n = ((n), (n-1, 1), (n-2, 1^2), \ldots, (1^n)).$$

We will see in Theorem 5.8 that $\overline{\mathcal{M}}_n$ is the left Cartan decomposition matrix of SP_n (it follows that the Ext-matrix is the same except without the main diagonal entries).

5C. *Extending to* LB. Recall we introduced the loop-braid category LB. We write $\sigma \in \mathsf{LB}(2, 2)$ for the positive braid exchange and $\rho \in \mathsf{LB}(2, 2)$ for the symmetric exchange.

Formally extending with elementary transpositions (cf. ϱ_{GB}), the F_M construction fails to satisfy the mixed braid relation (3-3). However the functor $\mathsf{F}_{M'}$ fairs better.

Theorem 5.2. (i) *The $\sigma \mapsto M'_t(\sigma)$ construction extended using the super transposition $\rho \mapsto M'_1(\sigma)$ gives a monoidal functor $F^e_{M'}$ from the loop Braid category* LB *to* Mat.

(ii) *$F^e_{M'}$ factors through* LH.

Proof. The proof is a linear algebra calculation similar to the B cases above, using Kronecker product identities; but also using the appropriate special case of (5-4) for (3-3). □

Definition 5.3. Fix a field k and $t \in k$. Then the k-algebra $\mathrm{SP}_n = k\,\mathrm{LB}_n \,/\, \mathrm{Ann}\, F^e_{M'}$.

We conjecture that the extended super representation, which we call Burau–Rittenberg, or "SP" rep for short, is faithful on LH unless $t^2 = 1$ (see later).

Remark 5.4. As the Hecke algebra is related to the quantum groups $U_q\mathfrak{sl}(k \mid m)$ via Schur–Weyl duality [Jimbo 1986; Deguchi and Akutsu 1990] one naturally wonders if local representations of LH can be obtained from the R-matrices coming from quantum groups, by extension. The results of [Kádár et al. 2017] suggest that R-matrices that extend to local representations of LB_n are in general somewhat rare. The SP representation is of this form: M'_t comes from the super-quantum group $U_q\mathfrak{sl}(1 \mid 1)$. We are not aware of other R-matrices coming from quantum groups that extend to LB_n, but this approach is nevertheless intriguing.

Proposition 5.5. *Fix a field k and $t \in k$, $t \neq 1$. Let $\chi_i = (\sigma_i - \rho_i)/(1-t)$. Then*:

(a) *χ_i and ρ_i ($i = 1, 2, \dots, n-1$) are alternative generators of* SP_n.

(b) *The k-algebra isomorphism class of* SP_n *is independent of t.*

Proof. (a) Elementary. (b) The images of the alternative generators in the defining representation are independent of t. □

5D. ***Towards linear structure of SP.*** Let us work out the linear structure of SP. (i.e., its Artin–Wedderburn linear representation theory over $\mathbb{C}$: simple modules, projective modules and so on. See Section 5E for a review.)

Proposition 5.6. *Suppose $t \neq 1 \in k$. Let $\chi = (\sigma - \rho)/(1-t)$ and $\chi_1 = (\sigma_1 - \rho_1)/(1-t) \in \mathrm{SP}_n$.*

(I) *Then*

$$\chi_1 \, \mathrm{SP}_n \, \chi_1 \cong \mathrm{SP}_{n-1} \tag{5-6}$$

and

$$\mathrm{SP}_n \,/\, \mathrm{SP}_n \, \chi_1 \, \mathrm{SP}_n \cong k. \tag{5-7}$$

(II) *In particular the map $f_\chi : \mathrm{SP}_{n-1} \to \chi_1 \, \mathrm{SP}_n \, \chi_1$ given by $w \mapsto \chi_1 w^{(1)} \chi_1$ (recall the translation notation from* (4-1)) *is an algebra isomorphism.*

Proof. (I) Let us write simply $F = F_n$ for the defining representation $F^e_{M'}$ of SP_n. We write $\{1,2\}^n$ for the basis (i.e., we write simply symbols $1, 2$ for e_1, e_2 and the word 112 for $e_1 \otimes e_1 \otimes e_2$ and so on). Our convention for ordering the basis is given by 11,21,12,22. First observe that the image in F is (here with $n = 3$):

$$(\chi \otimes 1_2) \mapsto \begin{pmatrix} 0 & & & \\ & \boxed{1} & -1 & \\ & & 0 & \\ & & & \boxed{1} \end{pmatrix} \otimes 1_2 = \begin{pmatrix} 0 & & & & & & & \\ & \boxed{1} & -1 & & & & & \\ & & 0 & & & & & \\ & & & \boxed{1} & & & & \\ & & & & 0 & & & \\ & & & & & \boxed{1} & -1 & \\ & & & & & & 0 & \\ & & & & & & & \boxed{1} \end{pmatrix} \tag{5-8}$$

Note that the basis change conjugating by

$$\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix} \otimes 1_2 \tag{5-9}$$

(again this is the example with $n = 3$) brings this into diagonal form, projecting onto the $2\{1,2\}^{n-1}$ subspace (the subspace of $V^n = \mathbb{C}\{1,2\}^n$ spanned by basis elements of form $2w$ with $w \in \{1,2\}^{n-1}$, i.e., of form $e_2 \otimes \ldots$). That is: $\chi \mapsto \binom{0}{1} \otimes 1_2$. Furthermore:

$$(\chi \otimes 1_2)(1_2 \otimes \sigma)(\chi \otimes 1_2) \mapsto \begin{pmatrix} 0 &&&&&&& \\ & \boxed{1} & -1 &&&&& \\ && 0 &&&&& \\ &&& \boxed{1-t} & 0 & \boxed{t} & -t & \\ &&&& 0 &&& \\ &&& \boxed{1} & 0 & \boxed{0} && \\ &&&&&& 0 & \\ &&&&&&& \boxed{-t} \end{pmatrix} \tag{5-10}$$

Note that after the (5-9) basis change this decomposes as a sum of several copies of the 0 module together with the submodule N with basis $2\{1,2\}^{n-1}$. Then the map from N to $\{1,2\}^{n-1}$ given by $2w \mapsto w$ gives

$$\chi_1\sigma_1\chi_1 = -t.\chi_1 \mapsto F_{n-1}(-t.1)$$

and $\chi_1\sigma_2\chi_1 \mapsto F_{n-1}(\sigma_1)$ and $\chi_1\rho_2\chi_1 \mapsto F_{n-1}(\rho_1)$. Also note that χ_1 commutes with σ_i for $i > 2$ so we have

$$\chi_1\sigma_i\chi_1 = \chi_1\sigma_i \mapsto F_{n-1}(\sigma_{i-1}), \quad \chi_1\rho_i\chi_1 \mapsto F_{n-1}(\rho_{i-1}) \quad i > 2. \tag{5-11}$$

Thus the images of the generators under $w \mapsto \chi_1 w \chi_1$ are the generators of SP_{n-1}, establishing (5-6) on generators. To show that the images of the generators span we proceed as follows. From Lemma 3.4 and the (sufficient) symmetry of the relations under $i \mapsto n-i$ on indices, writing $\mathrm{LH}_n = L_n$ for short, we have

$$\begin{aligned} L_{n+1} &= L_n X_n L_n \\ &= L_n^{(1)} X_1 L_n^{(1)} \\ &= L_{n-1}^{(2)} X_2 L_{n-1}^{(2)} X_1 L_{n-1}^{(2)} X_2 L_{n-1}^{(2)} \\ &= L_{n-1}^{(2)} X_2 L_{n-2}^{(3)} X_3 L_{n-2}^{(3)} X_1 X_2 L_{n-1}^{(2)} \\ &= L_{n-1}^{(2)} X_2 X_3 X_1 X_2 L_{n-1}^{(2)}. \end{aligned}$$

Thus

$$\chi_1 L_{n+1} \chi_1 = \chi_1 L_{n-1}^{(2)} X_2 X_3 X_1 X_2 L_{n-1}^{(2)} \chi_1 = L_{n-1}^{(2)} \chi_1 X_2 X_3 X_1 X_2 \chi_1 L_{n-1}^{(2)}.$$

We can show by direct calculations that $\chi_1 X_2 X_3 X_1 X_2 \chi_1$ lies in the algebra generated by the images of the generators. (We can do this even in LH_4. The result then holds in SP_4 since it is a quotient; and then in SP_n by construction. Note however that we have not shown that it holds in LH_n.) Also $L^{(2)}_{n-1}\chi_1$ evidently lies in the algebra generated by the images of the generators, by commutation, so we are done.

Finally (5-7) follows on noting that the quotient corresponds to imposing $\chi_1 = 0$, i.e., $\sigma_1 = \rho_1$. Noting that $t \neq 1$, this gives $\sigma_i = 1$.

(II) Note that f_χ inverses the map from (5-11) above. □

5E. ***Aside on linear/Artinian representation theory.*** Since this paper bridges between topology and linear representation theory it is perhaps appropriate to say a few words on the bridge. While topology focuses on topological invariants, linear rep theory is concerned with invariants such as the spectrum of linear operators (and the generalised "spectrum" of algebras of linear operators). The former is thus of interest for topological quantum field theories, and the latter for usual quantum field theories (where notions such as mass are defined). In this section we recall a few key points of linear/Artinian rep theory that are useful for us. (So of course it can be skipped if you are not interested in this aspect, or are already familiar.)

Recall that every finite dimensional algebra over an algebraically closed field is Morita equivalent to a basic algebra; see e.g., [Nesbitt and Scott 1943; Jacobson 1974; Benson 1991]. This allows us to track separately the combinatorial and homological data of an algebra.

Let A be a finite dimensional algebra over an algebraically closed field k; see, e.g., [Benson 1991]. Let $J(A)$ denote the radical. Let $L = \{L_1, \dots, L_r\}$ be an ordered set of the isomorphism classes of simple A-modules, with projective covers $P_i = Ae_i$ (i.e., the e_is are a set of primitive idempotents). Given an A-module M let $\mathrm{Rad}(M)$ denote the intersection of the maximal proper submodules. Now suppose A is basic. Recall that $\mathrm{Ext}^1_A(L_i, L_j)$ codifies the nonsplit extensions between these modules; i.e., the "atomic" components of nonsemisimplicity. The corresponding "Ext-matrix" $E_L(A)$ is given by

$$(E_L(A))_{ij} = \dim_k \mathrm{Ext}^1_A(L_i, L_j)$$

or equivalently

$$\begin{aligned}\dim_k \mathrm{Ext}^1_A(L_i, L_j) &= \dim_k(\mathrm{Hom}_A(P_j, \mathrm{Rad}(P_i))/\mathrm{Hom}_A(P_j, \mathrm{Rad}^2(P_i))) \\ &= \dim_k(e_j J(A) e_i / e_j J^2(A) e_i).\end{aligned}$$

This perhaps looks technical, but note that $e_j J(A) e_i = e_j A e_i$ when $i \neq j$ and so then is essentially what we study in Section 4B et seq. (and in our case the quotient factor is even conjecturally zero, so in fact we are already studying the Ext-matrix!). Note that the Ext-matrix defines a quiver and hence a path algebra $kE_L(A)$. For any finite dimensional algebra A, basic or otherwise, the Cartan decomposition matrix $C_L(A)$ is given by

$$(C_L(A))_{ij} = \dim_k \mathrm{Hom}_A(P_j, P_i) \tag{5-12}$$

that is, the i-th row gives the number of times each simple module occurs in P_i.

5F. *Linear structure of* $\mathbf{SP}_n$. A corollary of Proposition 5.6 is that we have an embedding of module categories $G_\chi : \mathrm{SP}_{n-1} - \mathrm{mod} \to \mathrm{SP}_n - \mathrm{mod}$. In fact we can use this (together with our earlier calculations) to determine the structure of these algebras. Before giving the structure theorem let us recall the relevant general theory.

Lemma 5.7 (see, e.g., [Green 1980, Section 6.2]). *Let A be an algebra and $e \in A$ an idempotent. Then*:

(i) *The functor $Ae \otimes_{eAe} - : eAe - \mathrm{mod} \to A - \mathrm{mod}$ takes a complete set of inequivalent indecomposable projective left eAe-modules to a set of inequivalent indecomposable projective A-modules that is complete except for the projective covers of simple modules L in which $eL = 0$.* (*There is a corresponding right-module version.*)

(ii) *This functor and the functor $\overline{G}_e : A - \mathrm{mod} \to eAe - \mathrm{mod}$ given by $M \mapsto eM$ form a left-right adjoint pair.*

(iii) *The Cartan decomposition matrix of eAe embeds in that of A according to the labeling of modules in* (*i*). □

Theorem 5.8. (i) *Isomorphism classes of irreps of* SP_n *are naturally indexed by h_n.* (*Indeed* $\mathrm{SP}_n\,/rad \cong \mathbb{Q}S_n/\boldsymbol{e}^1_{2,2}$ *so the dimensions are given by the n-th row of Pascal's triangle*; *see Figure 1.*)

(ii) *The left Cartan decomposition matrix is $\overline{\mathcal{M}}_n$. Note that this determines the structure of* SP_n. *It gives the dimension as*

$$\dim = \binom{2(n-1)}{n-1} + \binom{2(n-1)}{n} = \tfrac{1}{2}\binom{2n}{n}.$$

(iii) *The image of the decomposition* (4-6) *is complete in* SP_n.

Proof. (i) Consider Lemma 5.7(i). In our case, putting

$$A = \mathrm{SP}_n \quad \text{and} \quad e = \chi_1,$$

then by (5-7) there is exactly one module L such that $eL = 0$ (at each n) — the trivial module. Thus by Proposition 5.6 SP_n has one more class of projectives and hence irreps than SP_{n-1}.

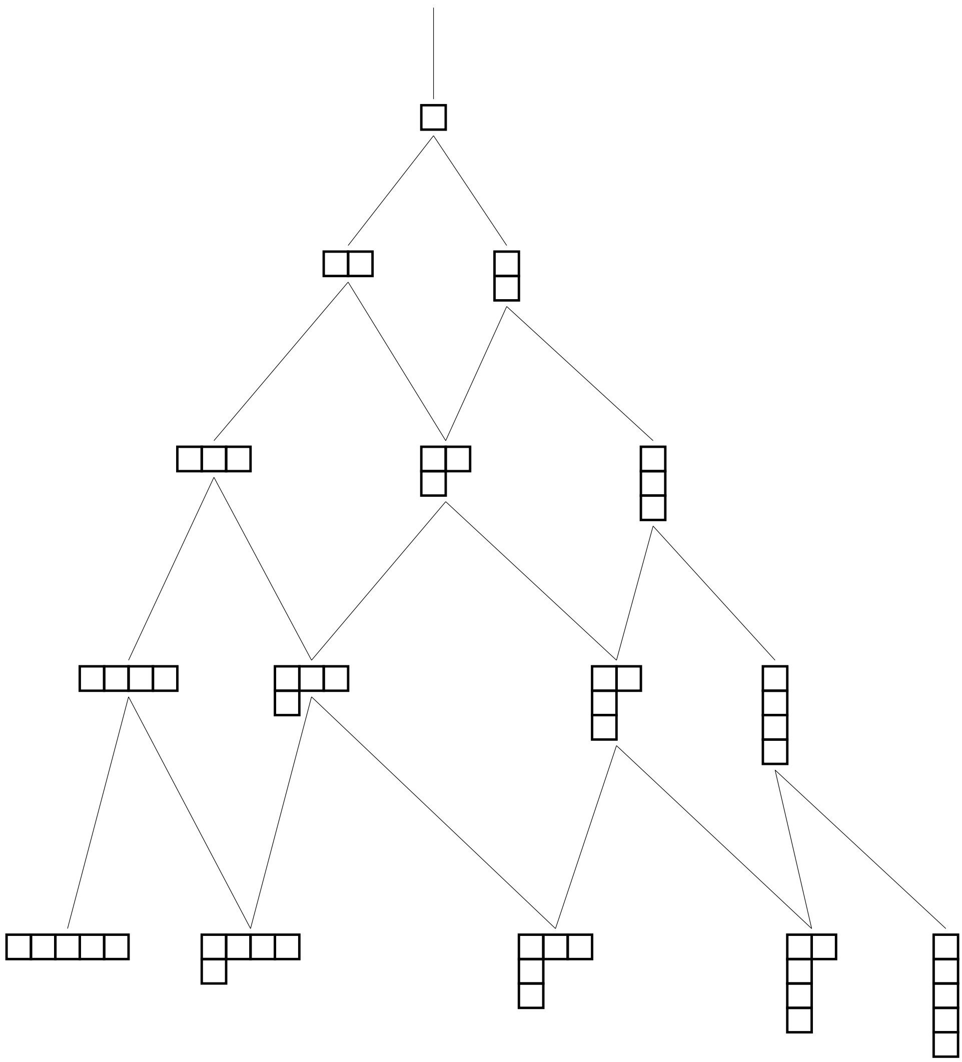


Figure 1. Young graph up to rank 5 with 22-diagrams removed.

In particular write

$$G_\chi : \mathrm{SP}_{n-1} - \mathrm{mod} \to \mathrm{SP}_n - \mathrm{mod}$$

for the functor in our case obtained using (5-6) from Proposition 5.6, that is

$$G_\chi(M) = \mathrm{SP}_n\,\chi \otimes_{\chi\,\mathrm{SP}_n\,\chi} f_\chi M,$$

suppressing the index n, where f_χ is as described above. Then a complete set of indecomposable projectives is

$$\begin{aligned} P^n_n &= \mathrm{SP}_n\, e^1_{(n)}, \\ P^n_{n-1} &= G_\chi(P^{n-1}_{n-1}) = G_\chi(\mathrm{SP}_{n-1}\, e^1_{(n-1)}), \\ P^n_{n-2} &= G_\chi(G_\chi(\mathrm{SP}_{n-2}\, e^1_{(n-2)}), \\ &\vdots \\ P^n_{n-j} &= G^{\circ j}_\chi(\mathrm{SP}_{n-j}\, e^1_{(n-j)}), \\ &\vdots \\ P^n_1 &= G^{\circ n-1}_\chi(k). \end{aligned}$$

It follows that the Cartan decomposition matrix $C(n)$ contains $C(n-1)$ as a sub-matrix, with one new row and column with the label n. The new row gives the simple content of P^n_n. But by (4-7) (noting Theorem 5.2(ii)) this projective is simple. Iterating, we deduce that $C(n)$ is lower-unitriangular.

Working by induction, suppose $C(n)$ is of the claimed form in (ii) at level $n-1$. Then at level n we have

$$C(n) = \left(\begin{array}{c|cccccc} 1 & & & & & & \\ \hline * & 1 & & & & & \\ * & 1 & 1 & & & & \\ * & & 1 & 1 & & & \\ * & & & 1 & 1 & & \\ \vdots & & & & \ddots & \ddots & \\ * & & & & & 1 & 1 \end{array}\right) \tag{5-13}$$

(omitted entries 0). To complete the inductive step we need to compute the $e^1_{(n)} P_{n-j}$ for each j. Write G^m_χ for G_χ and f^m_χ for f_χ at level $m < n$, and note that

$$\begin{aligned} G_\chi(\mathrm{SP}_{n-1}\, e^1_\lambda) &= \mathrm{SP}_n\, \chi_1 \otimes_{\chi\, \mathrm{SP}_n\, \chi} f_\chi(\mathrm{SP}_{n-1}\, e^1_\lambda) \\ &= \mathrm{SP}_n\, \chi_1 \otimes_{\chi\, \mathrm{SP}_n\, \chi} \chi_1\, \mathrm{SP}^{(1)}_{n-1}\, e^{(1)}_\lambda \chi_1 \\ &= \mathrm{SP}_n\, \chi_1\, \mathrm{SP}^{(1)}_{n-1}\, e^{(1)}_\lambda \chi_1 \otimes_{\chi\, \mathrm{SP}_n\, \chi} \chi_1 \\ &\cong \mathrm{SP}_n\, \chi_1\, \mathrm{SP}^{(1)}_{n-1}\, e^{(1)}_\lambda \chi_1 \end{aligned}$$

where we have used that these modules are idempotently generated ideals to apply the tensor product up to isomorphism (and where again we use the notation from (4-1), so $\mathrm{SP}^{(1)}_{n-1}$ is the 1-step translated copy of SP_{n-1} in SP_n). So in particular

$$e^1_{(n)}\, \mathrm{SP}_n\, G_\chi(\mathrm{SP}_{n-1}\, e^1_{(n-1)}) \cong e^1_{(n)}\, \mathrm{SP}_n\, \chi_1\, \mathrm{SP}^{(1)}_{n-1}\, e^{(1)}_{(n-1)} \chi_1 \subseteq e^1_{(n)}\, \mathrm{SP}_n\, \chi_1.$$

It follows from the form of the image of $e^1_{(n)}$ in the SP representation (see [Hamermesh 1962; Martin 1992, Appendix B; Martin and Rittenberg 1992]) that the dimension of $e^1_{(n)}\,\mathrm{SP}_n\,\chi_1$ is 1, so the first $*$ is 1. Specifically we have for example

$$e_2 = \tfrac{1}{2}(1+p_1) \mapsto \left(\begin{array}{c|cc|c} 1 & & & \\ \hline & 1/2 & 1/2 & \\ & 1/2 & 1/2 & \\ \hline & & & 0 \end{array}\right), \quad \chi \mapsto \left(\begin{array}{c|cc|c} 0 & & & \\ \hline & 1 & -1 & \\ & 0 & 0 & \\ \hline & & & 1 \end{array}\right)$$

and

$$e_3 \mapsto \tfrac{1}{3}\left(\begin{array}{c|ccc|ccc|c} 3 & & & & & & & \\ \hline & 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & & & & \\ \hline & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & \\ \hline & & & & & & & 0 \end{array}\right), \quad \chi_1 \mapsto \left(\begin{array}{c|ccc|ccc|c} 0 & & & & & & & \\ \hline & 0 & & & & & & \\ & & 1 & -1 & & & & \\ & & 0 & 0 & & & & \\ \hline & & & & 1 & -1 & 0 & \\ & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 1 & \\ \hline & & & & & & & 1 \end{array}\right)$$

where we have reordered the basis into fixed charge sectors, i.e., as 111, 112, 121, 211, 122, 212, 221, 222 (the charge of a basis element is #(1) − #(2), where #(1) is the number of 1's [Baxter 1982; Martin 1992]). Note from the construction that charge is conserved in SP, so each charge sector is a submodule. We see that in each charge sector except $(n-1, 1)$ we have that either the image of $e^1_{(n)}$ is zero or the image of χ_1 is zero. Finally in the $(n-1, 1)$ sector both of these have rank 1. We deduce that $e^1_n A \chi_1$ is 1-dimensional as required.

Similarly we have to consider

$$\begin{aligned} G_\chi G_\chi^{n-1}(\mathrm{SP}_{n-2}\,e^1_{(n-2)}) &\cong \mathrm{SP}_n\,\chi_1 f_\chi f_\chi^{n-1}(\mathrm{SP}_{n-2}\,e^1_{(n-2)}) \\ &\cong \mathrm{SP}_n\,\chi_1 f_\chi(\chi_1\,\mathrm{SP}^{(1)}_{n-2}\,e^{(1)}_{(n-2)}\chi_1) \\ &= \mathrm{SP}_n\,\chi_1\chi_1\chi_1^{(1)}\,\mathrm{SP}^{(2)}_{n-2}\,e^{(2)}_{(n-2)}\chi_1^{(1)}\chi_1, \end{aligned}$$

(NB $\chi_1^{(1)} = \chi_2$) giving

$$e^1_{(n)}\,\mathrm{SP}_n\,G_\chi G_\chi(\mathrm{SP}_{n-2}\,e^1_{(n-2)}) \cong e^1_{(n)}\,\mathrm{SP}_n\,f_\chi f_\chi(\mathrm{SP}_{n-2}\,e^1_{(n-2)}) = e^1_{(n)}\,\mathrm{SP}_n\,\chi_1\chi_2 \ldots.$$

We have, in the charge block basis,

$$\chi_2 \mapsto \left(\begin{array}{c|ccc|ccc|c} 0 & & & & & & & \\ \hline & 1 & -1 & & & & & \\ & & 0 & 0 & & & & \\ & & 0 & 0 & & & & \\ \hline & & & & 1 & 0 & 0 & \\ & & & & 0 & 1 & -1 & \\ & & & & 0 & 0 & 0 & \\ \hline & & & & & & & 1 \end{array}\right), \qquad \chi_1\chi_2 \mapsto \left(\begin{array}{c|ccc|ccc|c} 0 & & & & & & & \\ \hline & 0 & 0 & & & & & \\ & & 0 & 0 & & & & \\ & & 0 & 0 & & & & \\ \hline & & & & 1 & -1 & 1 & \\ & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & \\ \hline & & & & & & & 1 \end{array}\right)$$

(in general for a nonzero entry in $\chi_1\chi_2$ we need basis elements with 2 in the first and second position) so

$$e^1_{(n)}\,\mathrm{SP}_n\,\chi_1\chi_2 = 0.$$

Remark. Indeed we can verify that $e^1_{(n)}\chi_2\chi_1 = 0$ holds in LH_n) so the second $*$ and indeed the other $*$s in (5-13) are all zero. We have verified the inductive step for (ii).

Statement (iii) may be deduced from (i,ii) as follows. Note that we have n isomorphism classes in the decomposition, and their multiplicities are the dimensions of the hook irreps of S_n in the natural order. On the other hand the $n{+}1$ charge blocks of the SP representation are each either an irrep or contains two irreps, since each contains one or two irreps upon restricting to S_n. The first is an irrep (since dimension 1). By the proof of (ii) the second contains the first irrep, so two irreps in total, and the other again has the same dimension as the corresponding S_n hook representation. Furthermore no other block contains the first irrep so this block must be indecomposable (else the SP representation could not be faithful, which it is by definition). Proceeding through the blocks then by (ii) the first n of them are a complete set of projective modules, so each one except the first and last contains two simple modules ("adjacent" in the hook order). But then by the construction of the Pascal triangle and (ii) these simple modules have the same dimension as the corresponding S_n irreps, and (iii) follows. □

6. On representation theory of LH_n

Combining (5-2) with (5-5) and Theorem 5.8 we have

$$\dim(\mathrm{SP}_n(t \neq 1)) = v_1\mathcal{U}^{n-1} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} (v_1\mathcal{U}^{n-1})^T = \binom{2n-1}{n-1} = \tfrac{1}{2}\binom{2n}{n}.$$

n	$t=1$ dim	$t=-1$ dim	$t^2 \neq 1$ dim	$t \neq 1$ ss dim	irreps/dimensions –6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6
1	1	1	1	1							1						
2	3	3	3	2						1		1					
3	15	11	10	6					1		2		1				
4	114	42	35	20				1		3		3		1			
5	1170	163	126	70			1		4		6		4		1		
6	15570	638	462	252		1		5		10		10		5		1	
7		2510	1716	924	1		6		15		20		15		6		1

Table 1. A summary of what we learn for the algebra dimensions, and irreducible reps, of LH_n. The irrep labels here are given by $(n-i, 1^i) \mapsto n-2i-1$.

NB we have used the obvious "global" limit of all the Cartan matrices (it is a coincidence that this and the $\mathcal{U}$ matrix are similar).

Given a vector v we write $\mathrm{Diag}(v)$ for the diagonal matrix with v down the diagonal. Let p^n be the vector with the n-th row of Pascal's triangle as the entries, thus for example $p^4 = (1, 3, 3, 1)$. We have

$$\mathcal{M}_n^p := \mathrm{Diag}(p^n)\overline{\mathcal{M}}_n \,\mathrm{Diag}(p^n)$$

(examples are given in (6-2) below) and the dimension is the sum of all the entries. The closed form follows readily from this. Also from Theorem 5.8 we have:

Corollary 6.1. *For $t \neq 1$ the Morita class of SP_n is of the path algebra with A_n quiver (directed $1 \to 2 \to \cdots \to n$) and relations given by vanishing of all proper paths of length 2. In particular the radical-squared vanishes.*

6A. ***Properties determined from Theorem 5.8 and direct calculation in low rank.*** Our results for LH_n may be neatly given as follows. Firstly,

Proposition 6.2. *For $t^2 \neq 1$ and $n < 8$,*

$$\mathrm{LH}_n \cong \mathrm{SP}_n\,.$$

Proof. Here we can compute dimensions directly, which saturates the bound on the kernel. □

Conjecture 6.3. *For $t^2 \neq 1$,*

$$\mathrm{LH}_n \cong \mathrm{SP}_n\,.$$

Combining (5-2) with (5-5), Theorems 5.8 and 6.3 we have the conjecture

$$\dim(\mathrm{LH}_n(t^2 \neq 1)) = v_1 \mathcal{U}^{n-1} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} (v_1 \mathcal{U}^{n-1})^T = \binom{2n-1}{n-1} = \tfrac{1}{2}\binom{2n}{n}.$$

For $t = -1$ we note that SP_n is generally a proper quotient of LH_n, and that LH_n has larger radical (the square does not vanish). We define the semiinfinite matrix

$$C(\mathrm{LH}(t=-1)) = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and conjecture that the Cartan matrix $C(\mathrm{LH}_n(t=-1))$ is this truncated at $n \times n$ (i.e., the quiver is the same as the generic case, but without quotient relations); and thus we conjecture

$$\text{(6-1)}\quad \dim(\mathrm{LH}_n(t=-1)) = v_1 \mathcal{U}^{n-1} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} (v_1 \mathcal{U}^{n-1})^T = \frac{n^2 + \binom{2n-2}{n-1}}{2};$$

see OEIS A032443. Note that our calculations verify this for $n \leq 7$.

For $t = 1$ we see that $\mathrm{LH}_n(t=1)$ has semisimple quotient at least as big as $\mathbb{C}S_n$, which is of dimension $n!$. Indeed, in this case the quotient by the relation $\sigma_i = \rho_i$ is precisely $\mathbb{C}S_n$, since in this case $\sigma_i^2 = 1$. For $n \leq 4$ we have computationally verified that the semisimple subalgebra of $\mathrm{LH}_n(t=1)$ is precisely $\mathbb{C}S_n$, and we conjecture that this is the case for all n. The Jacobson radical grows quite quickly however, and we do not have a conjecture on the general structure.

Observe that the numbers in Table 1 follow the conjectured patterns. Since the vector v_1 has finite support the nominally infinite sums above are all finite. To inspect the supported part, in the generic case consider matrices $\mathcal{M}_n^p$ ($n = 2, 3, 4, 5$)

$$\text{(6-2)}\qquad \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ 2 & 2^2 & \\ & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ 3 & 3^3 & & \\ & 9 & 3^3 & \\ & & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & & \\ 4 & 4^2 & & & \\ & 24 & 6^2 & & \\ & & 24 & 4^2 & \\ & & & 4 & 1 \end{pmatrix}.$$

Here the semisimple dimension is given by the sum down the diagonal and the radical dimension is given by the sum in the off-diagonal.

For $t = -1$

$$\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ 2 & 2^2 & \\ 1 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ 3 & 3^3 & & \\ 3 & 9 & 3^3 & \\ 1 & 3 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & & \\ 4 & 4^2 & & & \\ 6 & 24 & 6^2 & & \\ 4 & 16 & 24 & 4^2 & \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

6B. ***On χ elements.*** Let us define

$$\chi^{(m+1)} = (\sigma_1 - \rho_1)(\sigma_2 - \rho_2)\dots(\sigma_m - \rho_m), \tag{6-3}$$

understood as an element in LH_n with $n > m$. Thus in particular $\chi^{(2)} = \chi_1$. Similarly for sequence $X = (x_1, x_2, \dots, x_k)$ define

$$\chi^{(X)} = (\sigma_{x_1} - \rho_{x_1})(\sigma_{x_2} - \rho_{x_2})\cdots(\sigma_{x_k} - \rho_{x_k}), \tag{6-4}$$

and

$$\chi_-^{(m+1)} = (\sigma_m - \rho_m)(\sigma_{m-1} - \rho_{m-1})\cdots(\sigma_2 - \rho_2)(\sigma_1 - \rho_1). \tag{6-5}$$

It is easy to verify that if X is nonincreasing then $\chi^{(X)}\chi^{(X)} = (1-t)^k\chi^{(X)}$. Thus (for $t \neq 1$) the nonincreasing cases can all be normalised as idempotents. However it is also easy to check that no increasing case can. (A nice illustration of the "chirality" present in the defining relations.)

Observe that imposing the relation $\sigma_1 = \rho_1$ in LH_n forces $\sigma_1 = 1$, unless $t = 1$. Thus the quotient algebra

$$\mathrm{LH}_n / \chi^{(2)} \cong k, \quad t \neq 1 \tag{6-6}$$

i.e., only the trivial, or label $\lambda = +n$, irrep survives. And the same holds for SP_n. The following has been checked up to rank 5.

Conjecture 6.4. *The structure of the quotient* $\mathrm{LH}_n / \chi^{(j+1)}$ *is given by the* $j \times j$ *truncation of* $\mathcal{M}_n^p$.

7. Discussion and avenues for future work

Above we give answers to the main structural questions for SP_n and LH_n. But exploration of generalisations is also well-motivated, since these algebras (even taken together with the constructions discussed in [Kádár et al. 2017]) cover a relatively small quotient inside $\mathrm{Rep}(\mathrm{LB}_n)$. With this in mind, there are a number of other questions worth addressing around SP_n and LH_n, offering clues on generalisation, and hence towards understanding more of the structure of the group algebra. Remark 3.6 suggests that for most values of t we obtain larger finite dimensional quotients by eliminating one of the local relations (3-12) or (3-13). Computational experiments suggest that for $t = 0$ eliminating (3-13) yields infinite dimensional algebras. This parameter-dependence should be further explored.

In light of the results of [Reutter 2020] the nonsemisimplicity of LH_n is an important feature, rather than a shortcoming. Extracting topological information from the nonsemisimple part requires some further work, as Markov traces typically "see" the semisimple part. Another aspect of our work is the (conjectural) localisation of the regular representation of LH_n. It is worth pointing out that localisations of *unitary* sequences of B_n representations are relatively rare, conjecturally corresponding to representations with finite braid group image [Rowell and Wang 2012; Galindo et al. 2013]. Since LH_n is nonsemisimple and hence nonunitary this does not contradict this conjectural relationship, but gives us some hope that localisations are possible for other parameter choices and other quotients.

The quotient of LB_n by the relation ${\sigma_i}^2 = 1$ is a potentially interesting infinite group, which we call the mixed double symmetric group MDS_n. The reason for this nomenclature is that MDS_n is a quotient of the free product of two copies of the symmetric group. In particular, MDS_n surjects onto S_n by $\sigma_i \to \rho_i$. It is of special interest here as $\mathrm{LH}_n(1)$ is a quotient of $\mathbb{Z}[\mathrm{MDS}_n]$. We expect it could be of quite general interest.

In [Kádár et al. 2017] constructions are developed based on BMW algebras, but still starting from "classical" precepts. It would be very interesting to meld the super-Burau–Rittenberg construction to the KMRW construction. For example, one might try to use cubic local (eigenvalue) relations among the generators ρ_i, σ_i to obtain finite dimensional quotients, possibly inspired by the relations satisfied by a subsequence of LB_n lifts of BMW algebra representations.

Appendix: Preparatory arithmetic and notation for left ideals

AA. ***Symmetric group and Hecke algebra arithmetic.*** Recall Young's (anti) symmetrisers in kS_n. Unnormalised in $\mathbb{Z}S_n$ they are

$$Y^n_{\pm} = \sum_{g \in S_n} (\pm 1)^{\mathrm{len}(g)} g \tag{A-1}$$

where $\mathrm{len}(g)$ is the usual Coxeter length function. If k has characteristic 0 then kS_n is semisimple and these elements are simply the (unnormalised) idempotents corresponding to the trivial and alternating representations respectively. Note that exactly the same classical construction works for the Hecke algebra over any field where it is semisimple. (The corresponding idempotents are sometimes called Jones–Wenzl projectors.) Specifically (see e.g., [Curtis and Reiner 1981, Section 9B])

$$X^n_{\pm} = \sum_{g \in S_n} (-\lambda_{\mp})^{-\mathrm{len}(g)} T_g, \quad \text{i.e., } X^2_{-} = 1 - \sigma_1,\ X^2_{+} = 1 + t^{-1}\sigma_1, \ldots$$

where for us $\lambda_- = -t$ and $\lambda_+ = 1$ (the apparent flip of labels is just because we use non-Lusztig scaling), and T_g is the product of generators obtained by writing g in reduced form then applying $\rho_i \mapsto T_i$.

Working in kS_{n+m} we understand $Y^n_\pm$ and translates such as $Y^{n(1)}_+$ in the obvious way. Note then that we have many identities like

$$Y^2_+ Y^n_+ = 2Y^n_+, \quad Y^{a(1)} Y^n_+ = a! Y^n_+ \quad (a < n). \tag{A-2}$$

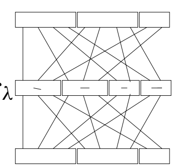
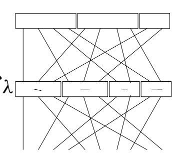
Recall Λ_n denotes the set of integer partitions of n. Over the rational field we have a decomposition of $1 \in kS_n$ into primitive central idempotents

$$1 = \sum_{\lambda \in \Lambda_n} \epsilon_\lambda \tag{A-3}$$

where each ϵ_λ is a known unique element; see e.g., [Cohn 1977, Section 7.6] or [Curtis and Reiner 1981] for gentle expositions. There is a further (not generally unique) decomposition of each ϵ_λ into primitive orthogonal idempotents

$$\epsilon_\lambda = \sum_{i=1}^{\dim_\lambda} \boldsymbol{e}^i_\lambda \tag{A-4}$$

where $\dim_\lambda$ is the number of walks from the root to λ on the directed Young graph. The elements $\boldsymbol{e}^i_\lambda$ are conjugate to each other. The elements $\boldsymbol{e}^i_\lambda$ are not uniquely defined in general. Two possible constructions of one for each λ are exemplified pictorially by (case $\lambda = 442$)

(A-5) $e^1_\lambda = c_\lambda$ , $\quad \hat{e}^1_\lambda = c_\lambda$ 

where an undecorated box is a symmetriser and a "−" decorated box an antisymmetriser, and the factor c_λ is just a scalar. (NB For the moment we write e^1_λ instead of $\boldsymbol{e}^1_\lambda$ for this specific choice.) In particular though, $\boldsymbol{e}^1_{(n)}$ *is* unique: $\boldsymbol{e}^1_{(n)} = \frac{1}{n!} Y^n_+$. (The whole story lifts to the Hecke case; see e.g., [Martin 1991] for a gentle exposition.)

An idempotent decomposition of 1 in a subalgebra B of an algebra A is of course a decomposition in A. Thus in particular we can take an idempotent in kS_n and consider it as an idempotent in kS_{n+1} by the inclusion that is natural from the presentation ($p_i \mapsto p_i$). Understanding $\boldsymbol{e}^1_\lambda$ with $\lambda \vdash n$ in kS_{n+1} in this way, a useful property in our $k = \mathbb{C}$ case will be

$$\boldsymbol{e}^j_\lambda = \sum_{\mu \in \lambda+} \boldsymbol{e}'_\mu \tag{A-6}$$

where $\lambda+$ denotes the set of partitions obtained from λ by adding a box, and the prime indicates that we identify this idempotent only up to equivalence. (Various proofs exist. For example note that the existence of such a decomposition follows from the induction rules for $S_n \hookrightarrow S_{n+1}$.) For example

$$\boldsymbol{e}^1_{(2,2)} = \boldsymbol{e}'_{(3,2)} + \boldsymbol{e}'_{(2,2,1)}.$$

Acknowledgements

Damiani and Martin would like to thank the Leverhulme Trust for support under research project grant RPG-2018-029: "Emergent Physics From Lattice Models of Higher Gauge Theory"; and João Faria Martins (PI) for encouragement. Damiani would like to thank Paolo Bellingeri and Emmanuel Wagner for their guidance during the time this idea was being born. Martin would like to thank Raphael Bennett-Tennenhaus and Paula Martin for useful conversations. We all thank Alex Bullivant (and hence EPSRC), and the University of Leeds, for partial support of Rowell's visit to Leeds. Rowell is grateful for insightful comments from Vaughan Jones. Rowell is partially supported by US NSF grant DMS-1664359, a Presidential Impact Fellowship from Texas A&M University and a Simons Foundation Fellowship. This research was carried out while Rowell was participating in a semester-long program on Quantum Symmetries at MSRI, a research institute partially supported by US NSF grant DMS-1440140.

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Received March 3, 2021. Revised July 11, 2022.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.67

BACKSTRÖM ALGEBRAS

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We introduce *Backström pairs* and *Backström rings*, study their derived categories and construct for them a sort of *categorical resolutions*. For the latter we define the global dimension, construct a sort of semiorthogonal decomposition of the derived category and deduce that the derived dimension of a Backström ring is at most 2. Using this semiorthogonal decomposition, we define a description of the derived category as the category of elements of a special bimodule. We also construct a partial tilting for a Backström pair to a ring of triangular matrices and define the global dimension of the latter.

Introduction

Backström orders were introduced in [Ringel and Roggenkamp 1979], where it was shown that their representations are in correspondence with those of quivers or species. A special class of Backström orders are *nodal orders*, which appeared (without this name) in [Drozd 1990] as such pure noetherian algebras that the classification of their finitely generated modules is tame. In [Burban and Drozd 2004] tameness was also proved for the derived categories of nodal orders. Global analogues of nodal algebras, called *nodal curves*, were considered in [Burban and Drozd 2011; Drozd and Voloshyn 2012; Voloshyn and Drozd 2013]. Namely, in [Burban and Drozd 2011] a sort of tilting theory for such curves was developed, which related them to some quasihereditary finite dimensional algebras. In [Drozd and Voloshyn 2012] a criterion was found for a nodal curve to be tame with respect to the classification of vector bundles, and in [Voloshyn and Drozd 2013] it was proved that the same class of curves has tame derived categories. It was clear that the tilting theory of [Burban and Drozd 2011] can be extended to a general situation, namely, to *Backström curves*, i.e., noncommutative curves having Backström orders as their localizations. Nodal orders and related gentle algebras appear in studying mirror symmetry, see for instance, [Lekili and Polishchuk 2018]. Finite dimensional

This work was accomplished during the visits of the author to the Max-Plank-Institut of Mathematics and the University of Paderborn, and he is grateful to these institutions for their hospitality and financial support.
MSC2020: primary 16E35; secondary 16E10, 18G80.
Keywords: Backström rings, Backström pairs, derived categories, partial tilting.

analogues of nodal orders, called *nodal algebras*, were introduced in [Drozd and Zembyk 2013; Zembyk 2014]. In the latter paper their structure was completely described. In [Zembyk 2015] it was shown that certain important classes of algebras, such as gentle and skewed-gentle algebras, are nodal. In [Burban and Drozd 2017] a tilting theory was developed for nodal algebras, which was applied to the study of derived categories of gentle and skewed-gentle algebras.

This paper is devoted to a tilting theory for *Backström rings*, which are a straightforward generalization of Backström orders and algebras.

In Section 1, we propose a variant of partial tilting, which generalizes the technique of minors from [Burban et al. 2017].

In Section 2, we introduce *Backström pairs*, which are pairs of semiperfect rings $\boldsymbol{H} \supseteq \boldsymbol{A}$ with a common radical; (piecewise) *Backström rings* are likewise introduced as those rings $\boldsymbol{A}$ that occur in (piecewise) Backström pairs with (piecewise) hereditary $\boldsymbol{H}$. We construct the *Auslander envelope* $\tilde{\boldsymbol{A}}$ of a Backström pair and calculate its global dimension. It turns out that this global dimension only depends on the global dimension of $\boldsymbol{H}$. In particular, Auslander envelopes for Backström rings are of global dimension at most 2.

In Section 3, we apply the tilting technique to show that the derived category of the algebra $\boldsymbol{A}$ is connected by a recollement with the derived category of its Auslander envelope. This implies that the derived dimension of $\boldsymbol{A}$ in the sense of [Rouquier 2008] is not greater than that of the Auslander envelope.

In Section 4, we consider a recollement between the derived categories of the algebra $\boldsymbol{H}$ and of the Auslander envelope. It is used to calculate the derived dimension of the Auslander envelope, thus obtaining an upper bound for the derived dimension of the algebra $\boldsymbol{A}$. In particular, we prove that the derived dimension of a Backström or piecewise Backström algebra is at most 2. Moreover, if $\boldsymbol{A}$ is a Backström or piecewise Backström algebra of Dynkin type, then either it is piecewise hereditary of Dynkin type, so $\operatorname{der.dim} \boldsymbol{A} = 0$, or else $\operatorname{der.dim} \boldsymbol{A} = 1$.

In Section 5, we establish an equivalence between the category $\mathscr{D}(\tilde{\boldsymbol{A}})$ and a bimodule category. This gives a useful instrument for calculations in this derived category. (See, for instance, [Bekkert et al. 2003; Bekkert and Merklen 2003; Burban and Drozd 2004; 2006; 2017; Voloshyn and Drozd 2013].)

In Section 6, we consider another partial tilting for the Auslander envelope $\tilde{\boldsymbol{A}}$ of a Backström pair, relating its derived category by a recollement to the derived category of an algebra $\boldsymbol{B}$ of triangular matrices which looks simpler than the Auslander algebra. In this case, we calculate explicitly the global dimension of $\boldsymbol{B}$ and the kernel of the partial tilting functor

$$\mathsf{F} : \mathscr{D}(\boldsymbol{B}) \to \mathscr{D}(\boldsymbol{A}).$$

1. Partial tilting

Let $\mathscr{T}$ be a triangulated category, $\mathfrak{R} \subseteq \operatorname{Ob}\mathscr{T}$. We denote by $\operatorname{Tri}(\mathfrak{R})$ the smallest strictly full triangulated subcategory containing $\mathfrak{R}$ that is closed under coproducts (this means that if a coproduct of objects from $\operatorname{Tri}(\mathfrak{R})$ exists in $\mathscr{T}$, it belongs to $\operatorname{Tri}(\mathfrak{R})$). For a DG-category $\mathscr{R}$ we denote by $\mathscr{D}(\mathscr{R})$ its derived category [Keller 1994]. The following result is a generalization of [Lunts 2010, Proposition 2.6]:

Theorem 1.1. *Let $\mathfrak{R}$ be a subset of the set of compact objects of* $\operatorname{Ob}\mathscr{D}(\mathscr{A})$, *where $\mathscr{A}$ is a Grothendieck category. We consider the DG-category $\mathscr{R}$ with the set of objects $\mathfrak{R}$ and the sets of morphisms $\mathscr{R}(T,R) = \mathbb{R}\mathrm{Hom}(T,R)$. Define the functor $\mathsf{F} : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{R}^{\mathrm{op}})$ by mapping a complex C to the DG-module $\mathsf{F}C = \mathbb{R}\mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(-,C)$ restricted onto $\mathfrak{R}$.*

(1) *The restriction of* F *onto* $\operatorname{Tri}(\mathfrak{R})$ *is an equivalence* $\operatorname{Tri}(\mathfrak{R}) \to \mathscr{D}(\mathscr{R}^{\mathrm{op}})$.

(2) *There is a recollement diagram in the sense of* [Beĭlinson et al. 1982, 1.4.3]

$$\operatorname{Ker}\mathsf{F}\ \begin{matrix}\xleftarrow{\ \mathsf{I}^*\ }\\ \xrightarrow{\ \mathsf{I}\ }\\ \xleftarrow{\ \mathsf{I}^!\ }\end{matrix}\ \mathscr{D}(\mathscr{A})\ \begin{matrix}\xleftarrow{\ \mathsf{F}^*\ }\\ \xrightarrow{\ \mathsf{F}\ }\\ \xleftarrow{\ \mathsf{F}^!\ }\end{matrix}\ \mathscr{D}(\mathscr{R}^{\mathrm{op}}), \tag{1-1}$$

where I *is the embedding.*[1]

Recall that this means that the following conditions hold:

(a) *F and I are exact.*

(b) $\mathsf{F}\mathsf{I} = 0$.

(c) *F^* and $F^!$ are left and right adjoint functors to F, respectively.*

(d) *Both adjunction morphisms $\eta : \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})} \to \mathsf{F}\mathsf{F}^*$ and $\zeta : \mathsf{F}\mathsf{F}^! \to \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}$ are isomorphisms.*

(e) *The same holds for the triple* $(\mathsf{I}, \mathsf{I}^*, \mathsf{I}^!)$.

(Note that Condition 1.4.3.4 from [Beĭlinson et al. 1982] is a consequence of the other ones; see [Neeman 2001, 9.2].)

If $\mathfrak{R}$ generates $\mathscr{D}(\mathscr{A})$, we obtain an equivalence $\mathscr{D}(\mathscr{A}) \simeq \mathscr{D}(\mathscr{R}^{\mathrm{op}})$, as in [Lunts 2010]. *If $\mathfrak{R}$ consists of one object R, we obtain an equivalence* $\operatorname{Tri}(R) \simeq \mathscr{D}(\boldsymbol{R}^{\mathrm{op}})$, *where* $\boldsymbol{R} = \mathbb{R}\mathrm{Hom}(R,R)$.

Proof. (1) We identify $\mathscr{D}(\mathscr{A})$ with the homotopy category $\mathscr{I}(\mathscr{A})$ of *K-injective complexes*, i.e., complexes I such that $\mathrm{Hom}(C,I)$ is acyclic for every acyclic complex C, and suppose that $\mathfrak{R} \subseteq \mathscr{I}(\mathscr{A})$. Then, $\mathbb{R}\mathrm{Hom}$ coincides with Hom within the category $\mathscr{I}(\mathscr{A})$; so, for $C \in \mathscr{I}(\mathscr{A})$, $\mathsf{F}C = \mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(-,C))$ restricted onto $\mathfrak{R}$. The full subcategory of $\mathscr{I}(\mathscr{A})$ consisting of complexes C such that the natural map $\mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(R,C) \to \mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(\mathsf{F}R, \mathsf{F}C)$ is bijective for all $R \in \mathfrak{R}$ contains $\mathfrak{R}$, is strictly full, triangulated and closed under coproducts, since all objects from $\mathscr{R}$ are

[1] Note that $\mathfrak{R}$ is not necessarily *recollement-defining* in the sense of [Nicolás and Saorín 2009].

compact. Therefore, it contains $\mathrm{Tri}(\mathfrak{R})$. Quite analogously, the full subcategory of complexes C such that the natural map $\mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(C, C') \to \mathrm{Hom}\,\mathscr{I}(\mathscr{A})(\mathsf{F}C, \mathsf{F}C')$ is bijective for every $C' \in \mathrm{Tri}(\mathfrak{R})$ also contains $\mathrm{Tri}(\mathfrak{R})$. Hence, the restriction of F onto $\mathrm{Tri}(\mathfrak{R})$ is fully faithful. Moreover, as the functors $\mathrm{Hom}_{\mathscr{R}}(-, R)$, where R runs through $\mathfrak{R}$, generate $\mathscr{D}(\mathscr{R}^{\mathrm{op}})$, the functor F is essentially surjective. Therefore, restricted to $\mathrm{Tri}(\mathfrak{R})$, it gives an equivalence $\mathrm{Tri}(\mathfrak{R}) \to \mathscr{D}(\mathscr{R})$.

(2) Note that $\mathscr{D}(\mathscr{R}^{\mathrm{op}})$ is cocomplete and compactly generated, hence satisfies the Brown representability theorem [Neeman 2001, Theorem 8.3.3]. Therefore, it is true for $\mathrm{Tri}(\mathfrak{R})$ too. Then, [Neeman 2001, Proposition 9.1.19] implies that a Bowsfield localization functor exists for $\mathrm{Tri}(\mathfrak{R}) \subseteq \mathscr{D}(\mathscr{A})$ and [Neeman 2001, Proposition 9.1.18] implies that the embedding $\mathsf{E} : \mathrm{Tri}(\mathfrak{R}) \to \mathscr{D}(\mathscr{A})$ has a right adjoint $\Theta : \mathscr{D}(\mathscr{A}) \to \mathrm{Tri}(\mathfrak{R})$. Let $\mathsf{F}' : \mathscr{D}(\mathscr{R}^{\mathrm{op}}) \to \mathrm{Tri}(\mathfrak{R})$ be a quasi-inverse to the restriction of F onto $\mathrm{Tri}(\mathfrak{R})$. In particular, F' is a left adjoint to this restriction and the adjunction $\mathsf{F}\mathsf{F}' \to \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}$ is an isomorphism. Then,

$$\mathsf{F}C = \mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(-, C)|_{\mathfrak{R}} \simeq \mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(-, \Theta C)|_{\mathfrak{R}} = \mathsf{F}\Theta C.$$

Set $\mathsf{F}^* = \mathsf{E}\mathsf{F}'$. Since $\mathsf{F}'M \in \mathrm{Tri}(\mathfrak{R})$ for every $M \in \mathscr{D}(\mathscr{R}^{\mathrm{op}})$,

$$\begin{aligned}\mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(\mathsf{F}^*M, C) &\simeq \mathrm{Hom}_{\mathrm{Tri}(\mathfrak{R})}(\mathsf{F}'M, \Theta C)\\ &\simeq \mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(M, \mathsf{F}\Theta C) \simeq \mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(M, \mathsf{F}C),\end{aligned}$$

for any $M \in \mathscr{D}(\mathscr{R}^{\mathrm{op}})$ and $C \in \mathsf{I}(\mathscr{A})$. Hence, F^* is a left adjoint to F. If, moreover, $C \in \mathrm{Tri}(\mathfrak{R})$, we obtain

$$\mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(\mathsf{F}\mathsf{F}^*M, \mathsf{F}C) \simeq \mathrm{Hom}_{\mathscr{I}(\mathscr{A})}(\mathsf{F}^*M, C) \simeq \mathrm{Hom}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}(M, \mathsf{F}C).$$

As F is essentially surjective, this implies that $\eta : \mathsf{F}\mathsf{F}^* \to \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}$ is an isomorphism. As all objects from $\mathfrak{R}$ are compact, F respects coproducts, hence has a right adjoint $\mathsf{F}^!$ [Neeman 2001, Theorem 8.4.4]. Now it follows from [Burban et al. 2017, Corollary 2.3] that $\zeta : \mathsf{F}\mathsf{F}^! \to \mathrm{Id}_{\mathscr{D}(\mathscr{R}^{\mathrm{op}})}$ is an isomorphism and there is a recollement diagram (1-1). □

Note that $\mathrm{Im}\,\mathsf{F}^* = \mathrm{Tri}(\mathfrak{R})$ by construction, but usually $\mathrm{Im}\,\mathsf{F}^! \neq \mathrm{Tri}(\mathfrak{R})$, though it is equivalent to $\mathrm{Tri}(\mathfrak{R})$.

Corollary 1.2. *Under the conditions and notations of the preceding theorem, suppose that* $\mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(R, T[m]) = 0$ *for* $R, T \in \mathfrak{R}$ *and* $m \neq 0$. *Then, the functor* F *induces an equivalence* $\mathrm{Tri}(R) \xrightarrow{\sim} \mathscr{D}(\mathscr{R}^{\mathrm{op}})$, *where* $\mathscr{R}$ *is the category with the set of objects* $\mathfrak{R}$ *and the sets of morphisms* $\mathscr{R}(A, B) = \mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(A, B)$.

In this situation, we call the functor F *a partial tilting functor.*

2. Backström pairs

Recall from [Bass 1960; Lambek 1976] that a *semiperfect ring* is a ring A such that $A/\operatorname{rad} A$ is a semisimple artinian ring and idempotents can be lifted modulo rad A. Equivalently, as a left (or as a right) A-module, A decomposes into a direct sum of modules with local endomorphism rings.

Definition 2.1. (1) A *Backström pair* is a pair of semiperfect rings $H \supseteq A$ such that rad A = rad H. We denote by $C(H, A)$ the *conductor* of H in A:

$$C(H, A) = \{\alpha \in A \mid H\alpha \subseteq A\} = \operatorname{ann}(H/A)_A$$

(the right subscript $_A$ means that we consider H/A as a right A-module). Obviously, $C(H, A) \supseteq \operatorname{rad} A$, so both A/C and H/C are semisimple rings.

(2) We call a ring A a (left) *Backström ring* (resp. *piecewise Backström ring*) if there is a Backström pair $H \supseteq A$, where the ring H is left hereditary (resp. *left piecewise hereditary* [Happel 1988], i.e., derived equivalent to a left hereditary ring). If, moreover, both A and H are finite dimensional algebras over a field $\Bbbk$, we call A a *Backström algebra* (resp. *piecewise Backström algebra*).

Remark 2.2. If e is an idempotent in A, then $\operatorname{rad}(eAe) = e(\operatorname{rad} A)e$, hence, if $H \supseteq A$ is a Backström pair, so is $eHe \supseteq eAe$. This implies that if P is a finitely generated projective A-module, $A' = \operatorname{End}_A P$ and $H' = \operatorname{End}_H(H \otimes_A P)$, then $H' \supseteq A'$ is also a Backström pair. Note that if H is left hereditary (or piecewise hereditary), so is H', hence A' is a Backström ring (piecewise Backström ring) whenever A is. In particular, the notion of Backström (or piecewise Backström) ring is Morita invariant. Note also that if H is left hereditary and noetherian, it is also right hereditary, so A^{op} is also a Backström ring (piecewise Backström ring).

Examples 2.3. (1) An important example of Backström algebras are *nodal algebras* introduced in [Drozd and Zembyk 2013; Zembyk 2014]. By definition, they are finite dimensional algebras such that there is a Backström pair $H \supseteq A$, where H is a hereditary algebra and $\operatorname{length}_A(H \otimes_A U) \le 2$ for every simple A-module U. Their structure was completely described in [Zembyk 2014].

(2) Recall that a $\Bbbk$-algebra A is called *gentle* [Assem and Skowroński 1987] if $A \simeq \Bbbk\Gamma/J$, where Γ is a finite quiver (oriented graph) and J is an ideal in the path algebra $\Bbbk\Gamma$ such that $(J_+)^2 \supseteq J \supseteq (J_+)^k$ for some k, where J_+ is the ideal generated by all arrows, and the following conditions hold:

(a) For every vertex $i \in \operatorname{Ver}\Gamma$ there are at most two arrows starting at i and at most two arrows ending at i.

(b) If an arrow a starts at i (resp. ends at i) and arrows b_1, b_2 end at i (resp. start at i), then either $ab_1 = 0$ or $ab_2 = 0$ (resp. either $b_1a = 0$ or $b_2a = 0$), but not both.

(c) The ideal J is generated by products of arrows of the sort ab.

It is proved in [Zembyk 2015] that such algebras are nodal, hence Backström algebras. The same is true for skewed-gentle algebras [Geißand de la Peña 1999] obtained from gentle algebras by blowing up some vertices.

(3) *Backström orders* are orders $\boldsymbol{A}$ over a discrete valuation ring such that there is a Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, where $\boldsymbol{H}$ is a hereditary order. They were considered in [Ringel and Roggenkamp 1979].

(4) Let $\boldsymbol{H} = T(n, \Bbbk)$ be the ring of upper triangular $n \times n$ matrices over a field $\Bbbk$ and $\boldsymbol{A} = \mathrm{UT}(n, \Bbbk)$ be its subring of unitriangular matrices M, i.e., such that all diagonal elements of M are equal. Then, $\boldsymbol{H}$ is hereditary and $\operatorname{rad} \boldsymbol{H} = \operatorname{rad} \boldsymbol{A}$, hence $\boldsymbol{A}$ is a Backström algebra. In this case, $\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A}) = \operatorname{rad} \boldsymbol{A}$.

(5) $\Lambda_n = \Bbbk[x_1, x_2, \dots, x_n]/(x_1, x_2, \dots, x_n)^2$ embeds into $\boldsymbol{H} = \prod_{i=1}^n \Bbbk\Gamma_i$, where $\Gamma_i = \cdot \xrightarrow{a_i} \cdot$ (x_i maps to a_i). Obviously, under this embedding $\operatorname{rad} \Lambda_n = \operatorname{rad} \boldsymbol{H}$, so Λ_n is a Backström algebra.

We consider a fixed Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$, set $\mathfrak{r} = \operatorname{rad} \boldsymbol{A} = \operatorname{rad} \boldsymbol{H}$ and denote by $\boldsymbol{C}$ the conductor $\boldsymbol{C}(\boldsymbol{H}, \boldsymbol{A})$. Obviously, $\boldsymbol{C}$ is a two-sided $\boldsymbol{A}$-ideal and the biggest left $\boldsymbol{H}$-ideal contained in $\boldsymbol{A}$. Actually, it even turns out to be a two-sided $\boldsymbol{H}$-ideal and its definition is left-right symmetric.

Lemma 2.4. *Let $R \subseteq S$ be semisimple rings, $I = \{\alpha \in R \mid S\alpha \subseteq R\}$. Then, I is a two-sided S-ideal.*

Proof. Obviously, I is a left S-ideal and a two-sided R-ideal. As R is semisimple, $I = Re$ for some central idempotent $e \in R$. Then, $Se \subseteq Re$, so $Se = Re = eR$ and $(1-e)Se = 0$. Hence, $eS(1-e)$ is a left ideal in S and $(eS(1-e))^2 = 0$, so $eS(1-e) = 0$ and $I = Se = eS$ is also a right S-ideal. □

Proposition 2.5. *$\boldsymbol{C}$ is a two-sided $\boldsymbol{H}$-ideal. It is the biggest $\boldsymbol{H}$-ideal contained in $\boldsymbol{A}$. Therefore, it coincides with the set $\{\alpha \in \boldsymbol{A} \mid \alpha\boldsymbol{H} \subseteq \boldsymbol{A}\}$ or with $\operatorname{ann}_{\boldsymbol{A}}(\boldsymbol{H}/\boldsymbol{A})$ considered as a left $\boldsymbol{A}$-module.*

Proof. It follows from the preceding lemma applied to the rings $\boldsymbol{A}/\operatorname{rad} \boldsymbol{A}$ and $\boldsymbol{H}/\operatorname{rad} \boldsymbol{H}$. □

In what follows we assume that $\boldsymbol{A} \neq \boldsymbol{H}$, so $\boldsymbol{C} \neq \boldsymbol{A}$. To calculate $\boldsymbol{C}$, we consider a decomposition $\boldsymbol{A} = \bigoplus_{i=1}^m A_i$, where A_i are indecomposable projective left $\boldsymbol{A}$-modules. Arrange them so that $\boldsymbol{H}A_i \neq A_i$ for $1 \le i \le r$ and $\boldsymbol{H}A_i = A_i$ for $r < i \le m$, and set $A^0 = \bigoplus_{i=1}^r A_i$, $H^0 = \boldsymbol{H}A^0$ and $A^1 = \bigoplus_{i=r+1}^m A_i = \boldsymbol{H}A^1$. Then, $\boldsymbol{A} = A^0 \oplus A^1$ and $\boldsymbol{H} = H^0 \oplus A^1$ (possibly, $r = m$, so $A^0 = \boldsymbol{A}$ and $H^0 = \boldsymbol{H}$). Let $A^0 = \boldsymbol{A}e_0$ and $A^1 = \boldsymbol{A}e_1$, where e_0 and e_1 are orthogonal idempotents and $e_0 + e_1 = 1$. Set $A_b^a = e_b\boldsymbol{A}e_a$ and $H_b^a = e_b\boldsymbol{H}e_a$, where $a, b \in \{0, 1\}$. Note that $A_b^1 = H_b^1$ and $A_1^0 = H_1^0$. As A^0 and A^1 have no isomorphic direct summands,

$A_b^a \subseteq \operatorname{rad} \boldsymbol{A}$ if $a \neq b$. Hence, if we set $\mathfrak{r}^a = \operatorname{rad} A^a\,(a = 0, 1)$ and consider the Pierce decomposition of the ring $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} A_0^0 & A_0^1 \\ A_1^0 & A_1^1 \end{pmatrix},$$

the Pierce decomposition of the ideal $\mathfrak{r}$ becomes

$$\mathfrak{r} = \begin{pmatrix} \mathfrak{r}_0^0 & A_0^1 \\ A_1^0 & \mathfrak{r}_1^1 \end{pmatrix},$$

where $\mathfrak{r}_a^a = \operatorname{rad} A_a^a,\ a = 0, 1$. This implies that H^0 and H^1 have no isomorphic direct summands, the Pierce decomposition of $\boldsymbol{H}$ is

$$\boldsymbol{H} = \begin{pmatrix} H_0^0 & A_0^1 \\ A_1^0 & A_1^1 \end{pmatrix}$$

and $\mathfrak{r}_0^0 = \operatorname{rad} H_0^0$. Now, one easily sees that an element $a = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ belongs to $\boldsymbol{C}$ if and only if $H^0\alpha \subseteq A^0$. We claim that in that case $H^0\alpha \subseteq \operatorname{rad} A^0$. Otherwise $H^0\alpha$ contains an idempotent, hence a direct summand of A^0, which is isomorphic to some A_i with $1 \leq i \leq r$. This is impossible, since $\boldsymbol{H} A_i \neq A_i$. Therefore, $\alpha \in \mathfrak{r}_0^0$ and we obtain the following result:

Proposition 2.6. *The Pierce decomposition of the ideal* $\boldsymbol{C}$ *is*

$$\boldsymbol{C} = \begin{pmatrix} \mathfrak{r}_0^0 & A_0^1 \\ A_1^0 & A_1^1 \end{pmatrix}.$$

Definition 2.7. Analogously to [Burban and Drozd 2011], we define the *Auslander envelope* of the Backström pair $\boldsymbol{H} \supseteq \boldsymbol{A}$ as the ring $\tilde{\boldsymbol{A}}$ of 2×2 matrices of the form

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{H} \end{pmatrix}$$

with the usual matrix multiplication.

Using Pierce decompositions of $\boldsymbol{A}$, $\boldsymbol{H}$ and $\boldsymbol{C}$, we also present $\tilde{\boldsymbol{A}}$ as the ring of 4×4 matrices

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} A_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \\ \mathfrak{r}_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \end{pmatrix}. \tag{2-1}$$

We also define $\widetilde{\boldsymbol{H}}$ as the ring of 4×4 matrices of the form

$$\widetilde{\boldsymbol{H}} = \begin{pmatrix} \boldsymbol{H} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{H} \end{pmatrix} \quad \text{or} \quad \widetilde{\boldsymbol{H}} = \begin{pmatrix} H_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \\ \mathfrak{r}_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \end{pmatrix}.$$

Obviously, $\operatorname{rad}\widetilde{\boldsymbol{H}} = \operatorname{rad}\tilde{\boldsymbol{A}}$, so $\widetilde{\boldsymbol{H}} \supseteq \tilde{\boldsymbol{A}}$ is also a Backström pair. $\tilde{\boldsymbol{A}}$ is left noetherian if and only if $\boldsymbol{A}$ is left noetherian and $\boldsymbol{H}$ is finitely generated as a left $\boldsymbol{A}$-module.

In the noetherian case one can calculate the global dimensions of $\tilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{H}}$. It turns out that it only depends on $\boldsymbol{H}$.

Theorem 2.8. *Suppose that either $\boldsymbol{A}$ (hence also $\boldsymbol{H}$) is left perfect or $\boldsymbol{A}$ is left noetherian and $\boldsymbol{H}$ is finitely generated as a left $\boldsymbol{A}$-module (hence also left noetherian). Then*

$$\begin{aligned} \operatorname{l.gl.dim}\tilde{\boldsymbol{A}} &= 1+\max(1+\operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^0, \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^1) \\ &= \begin{cases} 1+\operatorname{l.gl.dim}\boldsymbol{H} & \text{if } \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^0 \geq \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^1, \\ \operatorname{l.gl.dim}\boldsymbol{H} & \text{if } \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^0 < \operatorname{pr.dim}_{\boldsymbol{H}}\mathfrak{r}^1 \end{cases} \end{aligned}$$

and

$$\operatorname{l.gl.dim}\widetilde{\boldsymbol{H}} = \operatorname{l.gl.dim}\boldsymbol{H},$$

where we set $\operatorname{pr.dim} 0 = -1$. *In particular, if $\boldsymbol{A}$ is a Backström ring, so is $\tilde{\boldsymbol{A}}$, and if $\boldsymbol{A}$ is not left hereditary, then* $\operatorname{l.gl.dim}\tilde{\boldsymbol{A}} = 2$.[2]

For instance, this is the case for nodal (in particular, gentle or skewed-gentle) algebras (Examples 2.3).

Proof. Under these conditions $\tilde{\boldsymbol{A}}$ and $\widetilde{\boldsymbol{H}}$ are either left perfect or left noetherian. We recall that if a ring Λ is left perfect or left noetherian and semiperfect, then $\operatorname{l.gl.dim}\Lambda = \operatorname{pr.dim}_\Lambda(\Lambda/\operatorname{rad}\Lambda) = 1+\operatorname{pr.dim}_\Lambda \operatorname{rad}\Lambda$. The 4×4 matrix presentation (2-1) of $\tilde{\boldsymbol{A}}$ implies that the corresponding presentation of $\operatorname{rad}\tilde{\boldsymbol{A}}$ is

$$\operatorname{rad}\tilde{\boldsymbol{A}} = \begin{pmatrix} \mathfrak{r}_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & \mathfrak{r}_1^1 & A_1^0 & \mathfrak{r}_1^1 \\ \mathfrak{r}_0^0 & A_0^1 & \mathfrak{r}_0^0 & A_0^1 \\ A_1^0 & \mathfrak{r}_1^1 & A_1^0 & \mathfrak{r}_1^1 \end{pmatrix}. \tag{2-2}$$

An $\tilde{\boldsymbol{A}}$-module M is given by a quadruple (M', M'', ϕ, ψ), where M' is an $\boldsymbol{A}$-module, M'' is an $\boldsymbol{H}$-module, $\psi : M'' \to M'$ is a homomorphism of $\boldsymbol{A}$-modules and $\phi : \boldsymbol{C}\otimes_{\boldsymbol{A}} M' \to M''$ is a homomorphism of $\boldsymbol{H}$-modules. Namely, $M' = e'M$, $M'' = e''M$, where $e' = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, $e'' = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, $\psi(m'') = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)m''$ and $\phi(c\otimes m') = \left(\begin{smallmatrix}0&0\\c&0\end{smallmatrix}\right)m'$.

[2]Note that if $\tilde{\boldsymbol{A}}$ is left hereditary, so is $\boldsymbol{A} = e'\tilde{\boldsymbol{A}}e'$ [Sandomierski 1969].

We frequently write $M = \binom{M'}{M''}$, not mentioning ϕ and ψ. For an $\boldsymbol{H}$-module N we define the $\tilde{\boldsymbol{A}}$-module $N^+ = \binom{N}{N}$. Then, $N \mapsto N^+$ is an exact functor mapping projective modules to projective ones, since $\boldsymbol{H}^+ = \binom{\boldsymbol{H}}{\boldsymbol{H}}$ is a projective $\tilde{\boldsymbol{A}}$-module.

We denote by L^i and by R^i the i-th column of the presentations (2-1) and (2-2), respectively. Then, $R^1 = (\mathfrak{r}^0)^+$ and $R^2 = R^4 = (\mathfrak{r}^1)^+$, where $\mathfrak{r}^a = \mathfrak{r}e_a$. If

$$\cdots \to F_k \to \cdots \to F_1 \to F_0 \to N \to 0$$

is a minimal projective resolution of an $\boldsymbol{H}$-module N,

$$\cdots \to F_k^+ \to \cdots \to F_1^+ \to F_0^+ \to N^+ \to 0$$

is a minimal projective resolution of N^+, so $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} N^+ = \operatorname{pr.dim}_{\boldsymbol{H}} N$. In particular, $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} R^1 = \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0$ and $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} R^2 = \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^1$. For the module R^3 we have an exact sequence

$$0 \to (\mathfrak{r}^0)^+ \to R^3 \to \begin{pmatrix} H^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} \to 0. \tag{2-3}$$

Note that $H^0/\mathfrak{r}^0$ is a semisimple $\boldsymbol{A}$-module and $e_1(H^0/\mathfrak{r}^0) = 0$, hence it contains the same simple direct summands as $A^0/\mathfrak{r}^0$. The same is true for

$$\begin{pmatrix} H^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} = L^1/R^1.$$

Hence,

$$\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} \begin{pmatrix} H^0/\mathfrak{r}^0 \\ 0 \end{pmatrix} = 1 + \operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} R^1 = 1 + \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0.$$

Therefore, the exact sequence (2-3) shows that $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} R^3 = 1 + \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0$ and

$$\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} \operatorname{rad} \tilde{\boldsymbol{A}} = \max(1 + \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^0, \operatorname{pr.dim}_{\boldsymbol{H}} \mathfrak{r}^1),$$

which gives the necessary result for $\tilde{\boldsymbol{A}}$. On the other hand, R^3 is a projective $\widetilde{\boldsymbol{H}}$-module, whence $\operatorname{l.gl.dim} \widetilde{\boldsymbol{H}} = \operatorname{l.gl.dim} \boldsymbol{H}$. □

3. The structure of derived categories

In what follows we denote by $\mathscr{D}(\boldsymbol{A})$ the derived category $\mathscr{D}(\boldsymbol{A}\text{-Mod})$. We denote by $\mathscr{D}_f(\boldsymbol{A})$ the full subcategory of $\mathscr{D}(\boldsymbol{A})$ consisting of complexes quasi-isomorphic to complexes of finitely generated projective modules. If $\boldsymbol{A}$ is left noetherian, it coincides with the derived category of the category $\boldsymbol{A}$-mod of finitely generated $\boldsymbol{A}$-modules. We also use the usual superscripts $+, -, b$. By $\operatorname{Perf}(\boldsymbol{A})$ we denote the full subcategory of perfect complexes from $\mathscr{D}(\boldsymbol{A})$, i.e., complexes quasi-isomorphic to finite complexes of finitely generated projective modules. It coincides with the full subcategory of compact objects in $\mathscr{D}(\boldsymbol{A})$ [Rouquier 2008]. If $\boldsymbol{A}$ is left noetherian, an $\boldsymbol{A}$-module M belongs to $\operatorname{Perf}(A)$ if and only if it is finitely generated and of finite projective dimension.

There are close relations between the categories $\mathscr{D}(\boldsymbol{A})$, $\mathscr{D}(\boldsymbol{H})$ and $\mathscr{D}(\tilde{\boldsymbol{A}})$ based on the following construction [Burban et al. 2017]:

Let $\boldsymbol{P} = \binom{\boldsymbol{A}}{\boldsymbol{C}}$. It is a projective $\tilde{\boldsymbol{A}}$-module and $\operatorname{End}\boldsymbol{P} \simeq \boldsymbol{A}^{\mathrm{op}}$, so it can be considered as a right $\boldsymbol{A}$-module. Consider the functors

$$\begin{aligned}
\mathsf{F} &= \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\boldsymbol{P}, -) \simeq \boldsymbol{P}^\vee \otimes_{\tilde{\boldsymbol{A}}} - : \tilde{\boldsymbol{A}}\text{-Mod} \to \boldsymbol{A}\text{-Mod},\\
\mathsf{F}^* &= \boldsymbol{P} \otimes_{\boldsymbol{A}} - : \boldsymbol{A}\text{-Mod} \to \tilde{\boldsymbol{A}}\text{-Mod},\\
\mathsf{F}^! &= \operatorname{Hom}_{\boldsymbol{A}}(\boldsymbol{P}^\vee, -) : \boldsymbol{A}\text{-Mod} \to \tilde{\boldsymbol{A}}\text{-Mod},
\end{aligned}$$

where $\boldsymbol{P}^\vee = \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(P, \tilde{\boldsymbol{A}}) \simeq (\boldsymbol{A}\ \ \boldsymbol{H})$ is the dual right projective $\tilde{\boldsymbol{A}}$-module, the functor F is exact, F^* is its left adjoint and $\mathsf{F}^!$ is its right adjoint. Moreover, the adjunction morphisms $\mathsf{F}\mathsf{F}^* \to \mathrm{Id}_{\boldsymbol{A}\text{-Mod}}$ and $\mathrm{Id}_{\boldsymbol{A}\text{-Mod}} \to \mathsf{F}\mathsf{F}^!$ are isomorphisms [Burban et al. 2017, Theorem 4.3]. The functors F^* and $\mathsf{F}^!$ are fully faithful and F is essentially surjective, i.e., every $\boldsymbol{A}$-module is isomorphic to $\mathsf{F}M$ for some $\tilde{\boldsymbol{A}}$-module M. $\operatorname{Ker}\mathsf{F}$ is a Serre subcategory of $\tilde{\boldsymbol{A}}$-Mod equivalent to $\overline{\boldsymbol{H}}$-Mod, where $\overline{\boldsymbol{H}} = \boldsymbol{H}/\boldsymbol{C} \simeq \tilde{\boldsymbol{A}}/\left(\begin{smallmatrix}\boldsymbol{A} & \boldsymbol{H}\\ \boldsymbol{C} & \boldsymbol{C}\end{smallmatrix}\right)$. The embedding functor $\mathsf{I} : \operatorname{Ker}\mathsf{F} \to \tilde{\boldsymbol{A}}$-Mod has a left adjoint I^* and a right adjoint $\mathsf{I}^!$ and we obtain a recollement diagram

$$\operatorname{Ker}\mathsf{F}\ \begin{matrix}\xleftarrow{\ \ \mathsf{I}^*\ \ }\\ \xrightarrow{\ \ \mathsf{I}\ \ }\\ \xleftarrow[\ \ \mathsf{I}^!\ \ ]{}\end{matrix}\ \tilde{\boldsymbol{A}}\text{-Mod}\ \begin{matrix}\xleftarrow{\ \ \mathsf{F}^*\ \ }\\ \xrightarrow{\ \ \mathsf{F}\ \ }\\ \xleftarrow[\ \ \mathsf{F}^!\ \ ]{}\end{matrix}\ \boldsymbol{A}\text{-Mod}.$$

As the functor F is exact, it extends to the functor between the derived categories $\mathsf{DF} : \mathscr{D}(\tilde{\boldsymbol{A}}) \to \mathscr{D}(\boldsymbol{A})$ acting on complexes componentwise. The derived functors LF^* and $\mathsf{RF}^!$ are its left and right adjoints, respectively, the adjunction morphisms $\mathrm{Id}_{\mathscr{D}(\boldsymbol{A})} \to \mathsf{DF}\cdot\mathsf{LF}^*$ and $\mathsf{DF}\cdot\mathsf{LF}^* \to \mathrm{Id}_{\mathscr{D}(\boldsymbol{A})}$ are again isomorphisms and we have a recollement diagram

$$\operatorname{Ker}\mathsf{DF}\ \begin{matrix}\xleftarrow{\ \ \mathsf{LI}^*\ \ }\\ \xrightarrow{\ \ \mathsf{DI}\ \ }\\ \xleftarrow[\ \ \mathsf{RI}^!\ \ ]{}\end{matrix}\ \mathscr{D}(\tilde{\boldsymbol{A}})\ \begin{matrix}\xleftarrow{\ \ \mathsf{LF}^*\ \ }\\ \xrightarrow{\ \ \mathsf{DF}\ \ }\\ \xleftarrow[\ \ \mathsf{RF}^!\ \ ]{}\end{matrix}\ \mathscr{D}(\boldsymbol{A}).$$

(It also follows from Corollary 1.2.) Here $\operatorname{Ker}\mathsf{DF} = \mathscr{D}_{\overline{\boldsymbol{H}}}(\tilde{\boldsymbol{A}})$, the full subcategory of complexes whose cohomologies are $\overline{\boldsymbol{H}}$-modules, i.e., are annihilated by the ideal $\left(\begin{smallmatrix}\boldsymbol{A} & \boldsymbol{H}\\ \boldsymbol{C} & \boldsymbol{C}\end{smallmatrix}\right)$. Note that, as a rule, it is not equivalent to $\mathscr{D}(\overline{\boldsymbol{H}})$. From the definition of F it follows that

$$\operatorname{Ker}\mathsf{DF} = \boldsymbol{P}^\perp = \{C \in \mathscr{D}(\tilde{\boldsymbol{A}}) \mid \operatorname{Hom}_{\mathscr{D}(\tilde{\boldsymbol{A}})}(\boldsymbol{P}, C[k]) = 0 \ \text{ for all } \ k\}.$$

Obviously, DF maps $\mathscr{D}^\sigma(\tilde{\boldsymbol{A}})$ to $D^\sigma(\boldsymbol{A})$ for $\sigma \in \{+, -, b\}$, LF^* maps $\mathscr{D}^-(\boldsymbol{A})$ to $\mathscr{D}^-(\tilde{\boldsymbol{A}})$ and $\mathsf{RF}^!$ maps $\mathscr{D}^+(\boldsymbol{A})$ to $\mathscr{D}^+(\tilde{\boldsymbol{A}})$. If $\tilde{\boldsymbol{A}}$ is left noetherian, DF maps $\mathscr{D}_f(\tilde{\boldsymbol{A}})$ to $\mathscr{D}_f(\boldsymbol{A})$ and LF^* maps $\mathscr{D}_f(\boldsymbol{A})$ to $\mathscr{D}_f(\tilde{\boldsymbol{A}})$. Finally, both DF and LF^* have right adjoints, hence map compact objects (i.e., perfect complexes) to compact ones.

On the contrary, usually LF^* does not map $\mathscr{D}^b(\boldsymbol{A})$ to $\mathscr{D}^b(\tilde{\boldsymbol{A}})$. For instance, it is definitely so if $\operatorname{l.gl.dim} \tilde{\boldsymbol{A}} < \infty$ while $\operatorname{l.gl.dim} \boldsymbol{A} = \infty$ as in Examples 2.3 (4, 5). If $\operatorname{l.gl.dim} \boldsymbol{H}$ is finite, so is $\operatorname{l.gl.dim} \tilde{\boldsymbol{A}}$, thus this recollement can be considered as a sort of categorical resolution of the category $\mathscr{D}(\boldsymbol{A})$. In any case, it is useful for studying the categories $\boldsymbol{A}$-Mod and $\mathscr{D}(\boldsymbol{A})$ if we know the structure of the categories $\tilde{\boldsymbol{A}}$-Mod and $\mathscr{D}(\tilde{\boldsymbol{A}})$. For instance, it is so if we are interested in the *derived dimension*, i.e., the dimension of the category $\mathscr{D}^b_f(\boldsymbol{A})$ in the sense of [Rouquier 2008].

Definition 3.1. Let $\mathscr{T}$ be a triangular category and $\mathfrak{M}$ be a set of objects from $\mathscr{T}$.

(1) We denote by $\langle\mathfrak{M}\rangle$ the smallest full subcategory of $\mathscr{T}$ containing $\mathfrak{M}$ and closed under direct sums, direct summands and shifts (not closed under cones, so not a triangulated subcategory).

(2) If $\mathfrak{N}$ is another subset of $\mathscr{T}$, we denote by $\mathfrak{M} \dagger \mathfrak{N}$ the set of objects C from $\mathscr{T}$ such that there is an exact triangle $A \to B \to C \xrightarrow{+}$, where $A \in \mathfrak{M}$, $B \in \mathfrak{N}$.

(3) We define $\langle\mathfrak{M}\rangle_k$ recursively, setting $\langle\mathfrak{M}\rangle_1 = \langle\mathfrak{M}\rangle$ and $\langle\mathfrak{M}\rangle_{k+1} = \langle\langle\mathfrak{M}\rangle \dagger \langle\mathfrak{M}\rangle_k\rangle$.

(4) The *dimension* $\dim \mathscr{T}$ of $\mathscr{T}$ is the smallest k such that there is a finite set of objects $\mathfrak{M}$ such that $\langle\mathfrak{M}\rangle_{k+1} = \mathscr{T}$ (if it exists). We call the dimension $\dim \mathscr{D}^b_f(\boldsymbol{A})$ the *derived dimension* of the ring $\boldsymbol{A}$ and denote it by $\operatorname{der.dim} \boldsymbol{A}$.

As the functor F is exact and essentially surjective, the next result is evident:

Proposition 3.2. *We have* $\operatorname{der.dim} \boldsymbol{A} \le \operatorname{der.dim} \tilde{\boldsymbol{A}}$. *Namely, if* $\mathscr{D}^b_f(\tilde{\boldsymbol{A}}) = \langle\mathfrak{M}\rangle_{k+1}$, *then* $\mathscr{D}^b_f(\boldsymbol{A}) = \langle \mathsf{DF}(\mathfrak{M})\rangle_{k+1}$.

4. Semiorthogonal decomposition

There is another recollement diagram for $\mathscr{D}(\tilde{\boldsymbol{A}})$ related to the projective module $\boldsymbol{Q} = \binom{\boldsymbol{H}}{\boldsymbol{H}}$ with $\operatorname{End} \boldsymbol{Q} \simeq \boldsymbol{H}^{\mathrm{op}}$. Namely, we set

$$\begin{aligned}
\mathsf{G} &= \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\boldsymbol{Q}, -) \simeq \boldsymbol{Q}^\vee \otimes_{\tilde{\boldsymbol{A}}} - : \tilde{\boldsymbol{A}}\text{-Mod} \to \boldsymbol{H}\text{-Mod},\\
\mathsf{G}^* &= \boldsymbol{Q} \otimes_{\boldsymbol{H}} - : \boldsymbol{H}\text{-Mod} \to \tilde{\boldsymbol{A}}\text{-Mod},\\
\mathsf{G}^! &= \operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{Q}^\vee, -) : \boldsymbol{H}\text{-Mod} \to \tilde{\boldsymbol{A}}\text{-Mod},
\end{aligned}$$

where $\boldsymbol{Q}^\vee = \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\boldsymbol{Q}, \tilde{\boldsymbol{A}}) \simeq (\boldsymbol{C}\ \ \boldsymbol{H})$,

$$\begin{aligned}
&\mathsf{DG} : \mathscr{D}(\tilde{\boldsymbol{A}}) \to \mathscr{D}(\boldsymbol{H}) \ \text{ is } \mathsf{G} \text{ applied componentwise},\\
&\mathsf{LG}^* : \mathscr{D}(A) \to \mathscr{D}(\tilde{\boldsymbol{A}}) \ \text{ is the left adjoint of } \mathsf{DG},\\
&\mathsf{RG}^! : \mathscr{D}(A) \to \mathscr{D}(\tilde{\boldsymbol{A}}) \ \text{ is the right adjoint of } \mathsf{DG}.
\end{aligned}$$

We also set $\bar{\boldsymbol{A}} = \boldsymbol{A}/\boldsymbol{C} \simeq \tilde{\boldsymbol{A}}/\left(\begin{smallmatrix} \boldsymbol{C} & \boldsymbol{H} \\ \boldsymbol{C} & \boldsymbol{H} \end{smallmatrix}\right)$. Then, we have recollement diagrams

$$\operatorname{Ker}\mathsf{G} \underset{\mathsf{J}^!}{\overset{\mathsf{J}^*}{\leftleftarrows}} \xrightarrow{\ \mathsf{J}\ } \tilde{\boldsymbol{A}}\text{-Mod} \underset{\mathsf{G}^!}{\overset{\mathsf{G}^*}{\leftleftarrows}} \xrightarrow{\ \mathsf{G}\ } \boldsymbol{H}\text{-Mod}$$

and

$$\operatorname{Ker}\mathsf{DG} \underset{\mathsf{RJ}^!}{\overset{\mathsf{LJ}^*}{\leftleftarrows}} \xrightarrow{\ \mathsf{DJ}\ } \mathscr{D}(\tilde{\boldsymbol{A}}) \underset{\mathsf{RG}^!}{\overset{\mathsf{LG}^*}{\leftleftarrows}} \xrightarrow{\ \mathsf{DG}\ } \mathscr{D}(\boldsymbol{H}),$$

where $\operatorname{Ker}\mathsf{G} \simeq \bar{\boldsymbol{A}}$-Mod. Since the $\tilde{\boldsymbol{A}}$-ideal $(\boldsymbol{C}\ \boldsymbol{H})$ is projective as a right $\tilde{\boldsymbol{A}}$-module, [Burban et al. 2017, Theorem 4.6] implies that $\operatorname{Ker}\mathsf{DG} \simeq \mathscr{D}(\bar{\boldsymbol{A}})$.

As usual, this recollement diagram gives semiorthogonal decompositions [Burban et al. 2017, Corollary 2.6]

$$\mathscr{D}(\tilde{\boldsymbol{A}}) = (\operatorname{Ker}\mathsf{DG}, \operatorname{Im}\mathsf{LG}^*) = (\operatorname{Im}\mathsf{RG}^!, \operatorname{Ker}\mathsf{DG}) \tag{4-1}$$

with $\operatorname{Ker}\mathsf{DG} \simeq \mathscr{D}(\bar{\boldsymbol{A}})$ and $\operatorname{Im}\mathsf{LG}^* \simeq \operatorname{Im}\mathsf{RG}^! \simeq \mathscr{D}(\boldsymbol{H})$ (though usually $\operatorname{Im}\mathsf{LG}^* \neq \operatorname{Im}\mathsf{RG}^!$).

Recall from [Kuznetsov and Lunts 2015] that a *semiorthogonal decomposition* $\mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2)$, where $\mathscr{T}_1$, $\mathscr{T}_2$ are full triangulated subcategories of $\mathscr{T}$, means that

$$\operatorname{Hom}_{\mathscr{T}}(T_2, T_1) = 0 \quad \text{for all } T_1 \in \mathscr{T}_1 \text{ and } T_2 \in \mathscr{T}_2,$$

and for every object $T \in \mathscr{T}$ there is an exact triangle $T_1 \to T_2 \to T \xrightarrow{+}$, where $T_i \in \mathscr{T}_i$.

Lemma 4.1.[3] *If $\mathscr{T} = (\mathscr{T}_1, \mathscr{T}_2)$ is a semiorthogonal decomposition of a triangulated category $\mathscr{T}$, then*

$$\dim\mathscr{T} \le \dim\mathscr{T}_1 + \dim\mathscr{T}_2 + 1.$$

Proof. First we show that for any subsets $\mathfrak{M}$, $\mathfrak{N}$ of objects of the category $\mathscr{T}$

$$\langle\mathfrak{M}\rangle_{k+1} \dagger \mathfrak{N} \subseteq \langle\mathfrak{M}\rangle \dagger \langle\langle\mathfrak{M}\rangle_k \dagger N\rangle \subseteq \underbrace{\langle\mathfrak{M}\rangle \dagger \langle\langle\mathfrak{M}\rangle \dagger \langle\langle\mathfrak{M}\rangle \dagger \cdots \langle\langle\mathfrak{M}\rangle}_{k+1} \dagger \mathfrak{N}\rangle \cdots\rangle\rangle. \tag{4-2}$$

Indeed, let $C \in \langle\mathfrak{M}\rangle_{k+1} \dagger \mathfrak{N}$, i.e., there is an exact triangle $A \to B \to C \xrightarrow{+}$, where $A \in \langle\mathfrak{M}\rangle_{k+1}$, $B \in \mathfrak{N}$. There is also an exact triangle $A_1 \to A \to A_2 \xrightarrow{+}$, where $A_1 \in \langle\mathfrak{M}\rangle_k$, $A_2 \in \langle\mathfrak{M}\rangle$. The octahedron axiom implies that there are exact triangles $A_1 \to B \to B' \xrightarrow{+}$ and $A_2 \to B' \to C \xrightarrow{+}$. Therefore, $B' \in \langle\mathfrak{M}\rangle_k \dagger \mathfrak{N}$ and $C \in \langle\mathfrak{M}\rangle \dagger \langle\langle\mathfrak{M}\rangle_k \dagger \mathfrak{N}\rangle$.

Now, let $\langle\mathfrak{M}\rangle_{k+1} = \mathscr{T}_1$ and $\langle\mathfrak{N}\rangle_{l+1} = \mathscr{T}_2$. Then, for every $T \in \mathscr{T}$ there is an exact triangle $T_1 \to T_2 \to T \xrightarrow{+}$, where $T_1 \in \langle\mathfrak{M}\rangle_{k+1}$, $T_2 \in \langle\mathfrak{N}\rangle_{l+1}$. But, according to (4-2), $\langle\mathfrak{M}\rangle_{k+1} \dagger \langle\mathfrak{N}\rangle_{l+1} \subseteq \langle\mathfrak{M}\cup\mathfrak{N}\rangle_{k+l+2}$, so $\mathscr{T} = \langle\mathfrak{M}\cup\mathfrak{N}\rangle_{k+l+2}$ and $\dim\mathscr{T} \le k+l+1$. $\square$

[3] In [Psaroudakis 2014, Theorem 7.4] this result is proved in the case when this decomposition arises from a recollement.

Since $\bar{A}$ is semisimple, any indecomposable object from $\mathscr{D}(\bar{A})$ is just a shifted simple module, so $\mathscr{D}^b_f(\bar{A}) = \langle \bar{A} \rangle$ and der.dim $\bar{A} = 0$. If H is hereditary, every indecomposable object from $\mathscr{D}^b_f(H)$ is a shift of a module. For every module M there is an exact sequence $0 \to P' \to P \to M \to 0$ with projective modules P, P' and, since H is semiperfect, every indecomposable projective H-module is a direct summand of H. Hence, $\mathscr{D}^b_f(H) = \langle H \rangle_2$ and der.dim $H \le 1$.

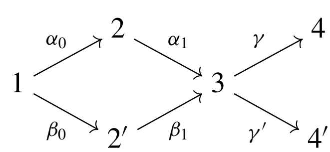
Corollary 4.2. *We have* der.dim $A \le$ der.dim $H + 1$. *In particular*, *if A is a Backström* (*or piecewise Backström*) *ring*, der.dim $A \le 2$.

A finite dimensional hereditary algebra is said to be of *Dynkin type* if it has finitely many isomorphism classes of indecomposable modules. Such algebras, up to Morita equivalence, correspond to Dynkin diagrams [Dlab and Ringel 1976; Gabriel 1972]. If the derived category of an algebra H is equivalent to the derived category of a hereditary algebra of Dynkin type, we say that H is *piecewise hereditary of Dynkin type*.[4] We say that a Backström (or piecewise Backström) algebra A is of *Dynkin type* if there is a Backström pair $H \supseteq A$, where H is a hereditary (piecewise hereditary) algebra of Dynkin type. For instance, it is so if A is a gentle or skewed-gentle algebra [Zembyk 2015], or the algebra $\mathrm{UT}(n\Bbbk)$ of unitriangular matrices over a field (Examples 2.3 (4)), or the algebra Λ_n from Examples 2.3 (5). In this case, $\mathscr{D}^b_f(H) = \langle M_1, M_2, \dots, M_m \rangle_1$, where $M_1, M_2, \dots, M_m$ are all pairwise nonisomorphic indecomposable H-modules, so der.dim $H = 0$.

In [Chen et al. 2008] it was proved that der.dim $A = 0$ for a finite dimensional algebra A if and only if A is a piecewise hereditary algebra of Dynkin type.

Corollary 4.3. *If A is a Backström* (*or piecewise Backström*) *algebra of Dynkin type* (*for instance*, *gentle or skewed-gentle*), *but is not piecewise hereditary of Dynkin type*, *then* der.dim $A = 1$.

Example 4.4. The path algebra of the commutative quiver



$$\alpha_1\alpha_0 = \beta_1\beta_0$$

is a tilted (hence piecewise hereditary) algebra of type $\widetilde{D}_5$. At the same time it is a Backström algebra of type A_4. Namely, it is a skewed-gentle algebra obtained from the path algebra of the quiver $1 \to 2 \to 3 \to 4$ by blowing up vertices 2 and 4.[5]

[4] It is proved in [Happel 1988] that piecewise hereditary algebras of Dynkin type are just iterated tilted algebras of Dynkin type.

[5] See [Zembyk 2014] for the construction of blowing up and its relation to nodal algebras.

5. Relation to bimodule categories

In this section, we explain how a semiorthogonal decomposition allows us to apply to calculations in a triangulated category the technique of matrix problems, namely, of bimodule categories, as in [Drozd 2010].

Let $\mathscr{A}$ and $\mathscr{B}$ be additive categories, $\mathcal{U}$ be an $\mathscr{A}$-$\mathscr{B}$-bimodule, i.e., a biadditive functor $\mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{Ab}$. Recall from [Drozd 2010] that the *bimodule category* or the *category of elements* of the bimodule $\mathcal{U}$ is the category $\mathsf{El}(\mathcal{U})$ whose set of objects is $\bigcup_{A\in\mathscr{A},B\in\mathscr{B}} \mathcal{U}(A, B)$ and whose morphisms from $u \in \mathcal{U}(A, B)$ to $v \in \mathcal{U}(A', B')$ are the pairs (α, β) such that $u\alpha = \beta v$, where $\alpha : A' \to A$, $\beta : B \to B'$. Here, as usual, we wrote $u\alpha$ and βv instead of $\mathcal{U}(\alpha, 1_B)u$ and $\mathcal{U}(1_{A'},\beta)v$. Bimodule categories appear when there is a semiorthogonal decomposition of a triangulated category.

Theorem 5.1. *Let $(\mathscr{A}, \mathscr{B})$ be a semiorthogonal decomposition of a triangulated category $\mathscr{C}$. Consider the $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathcal{U}$ such that $\mathcal{U}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$, $A \in \mathscr{A}$, $B \in \mathscr{B}$. For every $f : A \to B$ fix a cone Cf, that is, an exact triangle $A \xrightarrow{f} B \xrightarrow{f_1} Cf \xrightarrow{f_2} A[1]$. The map $f \mapsto Cf$ induces an equivalence of categories $\mathsf{C} : \mathsf{El}(\mathcal{U}) \xrightarrow{\sim} \mathscr{C}/\mathscr{J}$, where $\mathscr{J}$ is the ideal of $\mathscr{C}$ consisting of morphisms η such that there are factorizations $\eta = \eta'\xi = \zeta\eta''$, where the source of η' is in $\mathscr{A}$ and the target of η'' is in $\mathscr{B}$. Moreover, $\mathscr{J}^2 = 0$, so C induces a bijection between isomorphism classes of objects from $\mathsf{El}(\mathcal{U})$ and from $\mathscr{C}$.*[6]

Proof. As $(\mathscr{A}, \mathscr{B})$ is a semiorthogonal decomposition of $\mathscr{C}$, every object from $\mathscr{C}$ occurs in an exact triangle $A \xrightarrow{f} B \to C \xrightarrow{+}$, where $A \in \mathscr{A}$, $B \in \mathscr{B}$, so f is an object from $\mathsf{El}(\mathcal{U})$ and $C \simeq Cf$. Let $f' : A' \to B'$ be another object of $\mathsf{El}(\mathcal{U})$ and $(\alpha, \beta) : f \to f'$ be a morphism from $\mathsf{El}(\mathcal{U})$. Fix a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{f_1} & Cf & \xrightarrow{f_2} & A[1] \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha[1]} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{f'_1} & Cf' & \xrightarrow{f'_2} & A'[1] \end{array} \tag{5-1}$$

It exists, though is not unique. Let γ' be another morphism making the diagram (5-1) commutative and set $\eta = \gamma - \gamma'$. Then, $\eta f_1 = 0$, hence η factors through f_2, and $f'_2\eta = 0$, hence η factors through f'_1. Thus, $\eta \in \mathscr{J}$. On the other hand, if $\eta : Cf \to Cf'$ is in $\mathscr{J}$, the decomposition $\eta = \eta'\xi$ implies that $\eta f_1 = \eta'\xi f_1 = 0$ and the decomposition $\eta = \zeta\eta''$ implies that $f'_2\eta = f'_2\zeta\eta'' = 0$, hence the morphism $\gamma' = \gamma + \eta$ makes the diagram (5-1) commutative. Therefore, the class $\mathsf{C}(\alpha, \beta)$ of γ modulo $\mathscr{J}$ is uniquely defined, so the maps $f \mapsto Cf$ and $(\alpha, \beta) \mapsto \mathsf{C}(\alpha, \beta)$ define a functor $\mathsf{C} : \mathsf{El}(\mathcal{U}) \to \mathscr{C}/\mathscr{J}$.

[6] This theorem is a partial case of [Drozd 2010, Theorem 1.1].

Let now $\gamma : Cf \to Cf'$ be any morphism. Then, $f_2'\gamma f_1 = 0$, so $\gamma f_1 = f_1'\beta$ for some $\beta : B \to B'$. Hence, there is a morphism $\alpha : A \to A'$ making the diagram (5-1) commutative, i.e., defining a morphism $(\alpha, \beta) : f \to f'$ such that $\gamma \equiv \mathsf{C}(\alpha, \beta) \pmod{\mathscr{J}}$. If (α', β') is another such morphism, $f_1'(\beta - \beta') = 0$, so $\beta - \beta' = f'\xi$ for some $\xi : B \to A$. But $\xi = 0$, so $\beta = \beta'$. In the same way $\alpha = \alpha'$. Hence, the functor C is fully faithful. As we have already noticed, it is essentially surjective, and therefore defines an equivalence $\mathsf{El}(\mathcal{U}) \xrightarrow{\sim} \mathscr{C}/\mathscr{J}$. The equality $\mathscr{J}^2 = 0$ follows immediately from the definition and the conditions of the theorem. □

We apply Theorem 5.1 to Backström pairs $\boldsymbol{H} \subseteq \boldsymbol{A}$ such that $\boldsymbol{A}$ is left noetherian and $\boldsymbol{H}$ is left hereditary and finitely generated as a left $\boldsymbol{A}$-module. For instance, it is so in the case of Backström algebras or Backström orders. Then, the ring $\tilde{\boldsymbol{A}}$ is also noetherian and $\boldsymbol{C}$ is projective as a left $\boldsymbol{H}$-module. According to (4-1), $(\operatorname{Ker} \mathsf{DG}, \operatorname{Im} \mathsf{LG}^*)$ is a semiorthogonal decomposition of $\mathscr{D}(\tilde{\boldsymbol{A}})$. Moreover, both G and G^* map finitely generated modules to finitely generated modules, so the same is valid if we consider their restrictions onto $\mathscr{D}_f(\tilde{\boldsymbol{A}})$ and $\mathscr{D}_f(\boldsymbol{H})$. Note also that G^* is exact, so G^* can be applied to complexes componentwise. The $\tilde{\boldsymbol{A}}$-module G^*M can be identified with the module of columns $M^+ = \binom{M}{M}$ with the action of $\tilde{\boldsymbol{A}}$ given by matrix multiplication. It gives an equivalence of $\mathscr{D}(\boldsymbol{H})$ with $\operatorname{Im} \mathsf{LG}^*$. As $\boldsymbol{H}$ is left hereditary, every complex from $\mathscr{D}(\boldsymbol{H})$ is equivalent to a direct sum of shifted modules (see [Keller 2007, Section 2.5]). On the other hand, $\operatorname{Ker} \mathsf{DG} \simeq \mathscr{D}(\bar{\boldsymbol{A}})$ and $\bar{\boldsymbol{A}}$ is semisimple, since $\boldsymbol{C} \supseteq \boldsymbol{r}$. Hence, every complex from $\mathscr{D}(\bar{\boldsymbol{A}})$ is isomorphic to a direct sum of shifted simple $\bar{\boldsymbol{A}}$-modules, which are direct summands of $\bar{\boldsymbol{A}}$. So, to calculate the bimodule $\mathcal{U}$, we only have to calculate $\operatorname{Ext}^i_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+)$, where M is an $\boldsymbol{H}$-module. Note also that $\boldsymbol{C}^+$ is a projective $\tilde{\boldsymbol{A}}$-module, since $\boldsymbol{C}$ is a projective $\boldsymbol{H}$-module. Therefore, a projective resolution of $\bar{\boldsymbol{A}}$ is $0 \to \boldsymbol{C}^+ \xrightarrow{\varepsilon} \boldsymbol{P} \to \bar{\boldsymbol{A}} \to 0$ and $\operatorname{pr.dim}_{\tilde{\boldsymbol{A}}} \bar{\boldsymbol{A}} = 1$. Hence, we only have to calculate $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+)$ and $\operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+)$.

Theorem 5.2. (1) $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+) \simeq \operatorname{ann}_M \boldsymbol{C} = \{u \in M \mid \boldsymbol{C}u = 0\}$.

(2) $\operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+) \simeq \operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)/(M/\operatorname{ann}_M \boldsymbol{C})$, *where the quotient* $M/\operatorname{ann}_M \boldsymbol{C}$ *embeds into* $\operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)$ *if we map an element* $u \in M$ *to the homomorphism* $\mu_u : c \mapsto cu$.

Proof. (1) $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+)$ is identified with the set of homomorphisms $\phi : \boldsymbol{P} \to M^+$ such that $\phi\varepsilon = 0$. A homomorphism $\phi : \boldsymbol{P} \to M^+$ is uniquely defined by an element $u \in M$ such that $\phi\binom{1}{0} = \binom{u}{0}$. Namely, $\phi\binom{a}{c} = \binom{au}{cu}$. Obviously, $\phi\varepsilon = 0$ if and only if $\boldsymbol{C}u = 0$, i.e., $u \in \operatorname{ann}_M \boldsymbol{C}$.

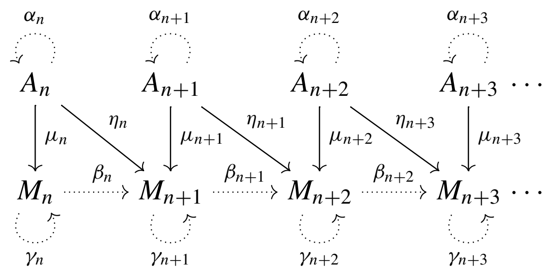
(2) $\operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(\bar{\boldsymbol{A}}, M^+) \simeq \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(C^+, M^+)/\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(\boldsymbol{P}, M^+)\varepsilon$. As the functor G^* is fully faithful, $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(C^+, M^+) \simeq \operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)$. Namely, $\psi : \boldsymbol{C} \to M$ induces

$\psi^+ : \boldsymbol{C}^+ \to M^+$ mapping $\binom{a}{b}$ to $\binom{\psi(a)}{\psi(b)}$. Let $\phi : \boldsymbol{P} \to M^+$ correspond, as above, to an element $u \in M$. Then,

$$\phi\varepsilon \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} au \\ cu \end{pmatrix},$$

so it equals μ_u, and $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(P, M^+)\varepsilon$ is identified with $M/\operatorname{ann}_M \boldsymbol{C}$ embedded into $\operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, M)$ as above. □

Actually, in our case an object E from the category $\mathsf{El}(\mathcal{U})$ (therefore, also an object from $\mathscr{D}^b(\tilde{\boldsymbol{A}})$) is given by the vertices and solid arrows of a diagram



(of arbitrary length), where A_i are $\bar{\boldsymbol{A}}$-modules, M_i are $\boldsymbol{H}$-modules, μ_i belongs to $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(A_i, M_i^+)$ and η_i belongs to $\operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(A_i, M_{i-1}^+)$. A morphism between E and E' is given by the dotted arrows, where

$$\begin{aligned} \alpha_i &\in \operatorname{Hom}_{\bar{\boldsymbol{A}}}(A_i, A_i') \simeq \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(A_i, A_i'), \\ \gamma_i &\in \operatorname{Hom}_{\boldsymbol{H}}(M_i, M_i') \simeq \operatorname{Hom}_{\tilde{\boldsymbol{A}}}(M_i^+, (M_i')^+), \\ \beta_i &\in \operatorname{Ext}^1_{\boldsymbol{H}}(M_i, M_{i+1}') \simeq \operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(M_i^+, (M_{i+1}')^+). \end{aligned}$$

These morphisms must satisfy the relations

$$\mu_i'\alpha_i = \gamma_i\mu_i, \quad \eta_i'\alpha_i = \gamma_{i+1}\eta_i + \beta_i\mu_i.$$

6. Partial tilting for Backström pairs

Let $\boldsymbol{H} \subseteq \boldsymbol{A}$ be a Backström pair. Consider the ring $\boldsymbol{B}$ of triangular matrices of the form

$$\boldsymbol{B} = \begin{pmatrix} \bar{\boldsymbol{A}} & \bar{\boldsymbol{H}} \\ 0 & \boldsymbol{H} \end{pmatrix}.$$

Let $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and let $B_1 = \boldsymbol{B}e_1$ and $B_2 = \boldsymbol{B}e_2$ be projective $\boldsymbol{B}$-modules given by the first and the second column of $\boldsymbol{B}$, i.e.,

$$B_1 = \begin{pmatrix} \bar{\boldsymbol{A}} \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \bar{\boldsymbol{H}} \\ \boldsymbol{H} \end{pmatrix}.$$

A $\boldsymbol{B}$-module M is defined by a triple $\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\ \chi_M\right)$, where $M_1 = e_1 M$ is an $\bar{\boldsymbol{A}}$-module, $M_2 = e_2 M$ is an $\boldsymbol{H}$-module and $\chi_M : M_2 \to M_1$ is an $\boldsymbol{A}$-homomorphism such that $\operatorname{Ker} \chi_M \supseteq \boldsymbol{C} M_2$ (it is necessary since $\boldsymbol{C} M_1 = 0$). Namely, χ_M is multiplication by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. We write an element $u \in M$ as a column $\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$, where $u_1 = e_1 u$, $u_2 = e_2 u$. Then,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a u_1 + \chi_M(b u_2) \\ c u_2 \end{pmatrix}.$$

A homomorphism $\alpha : M \to N$ is defined by two homomorphisms $\alpha_1 : M_1 \to N_1$ and $\alpha_2 : M_2 \to N_2$ such that $\alpha_1 \chi_M = \chi_N \alpha_2$. We write $\alpha = \left(\begin{smallmatrix} \alpha_1 \\ \alpha_2 \end{smallmatrix}\right)$.

Proposition 6.1. *We have* $\operatorname{l.gl.dim} \boldsymbol{B} = \max(\operatorname{l.gl.dim} \boldsymbol{H}, \operatorname{w.dim} \overline{\boldsymbol{H}}_{\boldsymbol{H}} + 1)$.

In particular, if $\boldsymbol{H}$ is left hereditary and $\overline{\boldsymbol{H}}$ is not flat as a right $\boldsymbol{H}$-module, then $\operatorname{l.gl.dim} \boldsymbol{B} = 2$.

Proof. [Palmér and Roos 1973, Theorem 5] shows that $\operatorname{l.gl.dim} \boldsymbol{B} \le n$ if and only if

$$\operatorname{l.gl.dim} \boldsymbol{H} \le n \text{ and } \mathbb{R}^n \operatorname{Hom}_{\bar{\boldsymbol{A}}}(\overline{\boldsymbol{H}} \otimes_{\boldsymbol{H}} -\,, -) = 0.$$

As the ring $\bar{\boldsymbol{A}}$ is semisimple,

$$\mathbb{R}^n \operatorname{Hom}_{\bar{\boldsymbol{A}}}(\overline{\boldsymbol{H}} \otimes_{\boldsymbol{H}} -\,, -) = \operatorname{Hom}_{\bar{\boldsymbol{A}}}(\operatorname{Tor}_n^{\boldsymbol{H}}(\overline{\boldsymbol{H}}, -), -).$$

This implies the first assertion. The second is obvious, since $\operatorname{Tor}_1^{\boldsymbol{H}}(\overline{\boldsymbol{H}}, -) = 0$ if and only if $\overline{\boldsymbol{H}}$ is flat as a right $\boldsymbol{H}$-module. □

We denote by R the $\boldsymbol{B}$-module given by the triple $\left(\begin{smallmatrix} \boldsymbol{H}/\boldsymbol{A} \\ \boldsymbol{H} \end{smallmatrix}\ \pi\right)$, where $\pi : \boldsymbol{H} \to \boldsymbol{H}/\boldsymbol{A}$ is the natural surjection.

Proposition 6.2. (1) $\operatorname{End}_{\boldsymbol{B}} R \simeq \boldsymbol{A}^{\mathrm{op}}$.

(2) $\operatorname{pr.dim}_{\boldsymbol{B}} R = 1$.

(3) $\operatorname{Ext}^1_{\boldsymbol{B}}(R, R) = 0$.

Recall that conditions (2) *and* (3) *mean that R is a partial tilting $\boldsymbol{B}$-module.*

Proof. The minimal projective resolution of R is

$$0 \to B_1 \xrightarrow{\varepsilon} B_2 \to R \to 0,$$

where ε is the embedding, which gives (2). Any endomorphism γ of R induces a commutative diagram:

$$\begin{array}{ccc} B_1 & \xrightarrow{\varepsilon} & B_2 \\ {\scriptstyle\gamma_1}\downarrow & & \downarrow{\scriptstyle\gamma_2} \\ B_1 & \xrightarrow{\varepsilon} & B_2 \end{array}$$

As $\operatorname{End}_{\boldsymbol{B}} B_2 \simeq \boldsymbol{H}^{\mathrm{op}}$, γ_2 is given by multiplication with an element $h \in \boldsymbol{H}$ on the right. If there is a commutative diagram as above, necessarily $h \in \boldsymbol{A}$, which proves (1).

Finally, a homomorphism $\alpha : B_1 \to R$ maps the generator $\binom{1}{0}$ of B_1 to an element $\binom{\bar{h}}{0} \in R$. If h is a preimage of $\bar{h}$ in $\boldsymbol{H}$, then α extends to the homomorphism $B_2 \to R$ that maps the generator $\binom{0}{1}$ of B_2 to $\binom{0}{h} \in R$. This implies (3). □

Now Theorem 1.1 applied to the module R gives the following result:

Theorem 6.3. (1) *The functor* $\mathsf{F} = \mathbb{R}\mathrm{Hom}(R, -)$ *induces an equivalence*

$$\mathrm{Tri}(R) \xrightarrow{\sim} \mathscr{D}(\boldsymbol{A}).$$

(2) $\operatorname{Ker}\mathsf{F}$ *consists of complexes* C *such that the map* $\chi_{H^k(C)}$ *is bijective for all* k.

(3) *There is a recollement diagram*

$$\operatorname{Ker}\mathsf{F} \;\underset{\mathsf{I}^!}{\overset{\mathsf{I}^*}{\leftleftarrows}}\!\!\xrightarrow{\;\mathsf{I}\;}\; \mathscr{D}(\boldsymbol{B}) \;\underset{\mathsf{F}^!}{\overset{\mathsf{F}^*}{\leftleftarrows}}\!\!\xrightarrow{\;\mathsf{F}\;}\; \mathscr{D}(\boldsymbol{A}).$$

Actually, claim (2) *means that a complex* C *is in* $\operatorname{Ker}\mathsf{F}$ *if and only if its cohomologies are direct sums of* $\boldsymbol{B}$*-modules of the form* $\left(\begin{smallmatrix} U \\ U \end{smallmatrix}\ 1_U\right)$, *where* U *is a simple* $\overline{\boldsymbol{H}}$*-module.*

F is a partial tilting functor in the sense of Corollary 1.2.

Proof. (1) and (3) follow from Proposition 6.2 and Theorem 1.1, since the complex $P : 0 \to B_1 \xrightarrow{\varepsilon} B_2 \to 0$ is perfect, hence compact, and isomorphic to R in $\mathscr{D}(\mathscr{B})$. To find $\operatorname{Ker}\mathsf{F}$, consider a complex

$$C : \cdot \to C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \to \cdots,$$

where C^k is defined by a triple $\left(\begin{smallmatrix} C_1^k \\ C_2^k \end{smallmatrix}\ \chi_k\right)$ and $d^k = \left(\begin{smallmatrix} d_1^k \\ d_2^k \end{smallmatrix}\right)$, where $d_1^k\chi_k = \chi_{k+1}d_2^k$ for all k. Note that $C_i = (C_i^k, d_i^k)$ $(i = 1, 2)$ are complexes, (χ_k) is a homomorphism of complexes and $H^k(C) = \left(\begin{smallmatrix} H^k(C_1) \\ H^k(C^2) \end{smallmatrix}\ \bar{\chi}_k\right)$, where $\bar{\chi}_k = \chi_{H^k(C)}$ is induced by χ_k. A homomorphism $P \to C[k]$ is a pair of homomorphisms $\alpha : B_2 \to C^k$, $\beta : B_1 \to C^{k-1}$ such that $\alpha_1\pi = \chi_k\alpha_2$, $\beta_2 = 0$, $d_i^k\alpha_i = 0$ $(i = 1, 2)$ and $d^{k-1}\beta_1 = \alpha_1|_{\bar{A}}$. Let $\alpha_2(1) = x \in C_2^k$ and $\beta_1(1) = y \in C_1^{k-1}$. These values completely define α and β. The conditions for α and β mean that $d_2^k x = 0$ and $d^{k-1}y = \chi_k x$.

This morphism is homotopic to zero if and only if there are maps $\sigma : B_2 \to C^{k-1}$ and $\tau : B_1 \to C^{k-2}$ such that $\alpha = d^{k-1}\sigma$ and $\beta = \sigma\varepsilon + d^{k-2}\tau$. Again, σ is defined by the element $z = \sigma_2(1) \in C_2^{k-1}$ and τ is defined by the element $t = \tau_1(1) \in C_1^{k-2}$. Then, the conditions for α and β mean that $x = d_2^{k-1}z$ and $y = \chi_{k-1}z + d_1^{k-2}t$.

Suppose that any homomorphism $P \to C[k]$ is homotopic to zero. Let $\bar{x}$ in $H^k(C^2)$ be such that $\bar{\chi}_k(\bar{x}) = 0$ and $x \in \operatorname{Ker} d_2^k$ be a representative of $\bar{x}$. Then, $\chi_k(x) = d_1^{k-1}y$ for some $y \in C^{k-1}$, so the pair (x, y) defines a homomorphism $P \to C[k]$. Therefore, there must be $z \in C_2^{k-1}$ such that $x = d^{k-1}z$; thus $\bar{x} = 0$ and $\bar{\chi}_k$ is injective. Let now $\bar{y} \in H^{k-1}(C_2)$ and $y \in C_2^{k-1}$ be its representative. Then, the pair $(0, y)$ defines a homomorphism $P \to C[k]$, so there must be elements $z \in C_2^{k-1}$

and $t \in C_1^{k-2}$ such that $d_1^{k-1}z = 0$ and $y = \chi_{k-1}z + d_1^{k-2}t$. Hence, $\bar{y} = \bar{\chi}_{k-1}(\bar{z})$, so $\bar{\chi}_{k-1}$ is surjective. As this holds for all k, we have that all maps $\bar{\chi}_k$ are bijective.

On the contrary, suppose that all $\bar{\chi}_k$ are bijective. If a pair (x, y) defines a homomorphism $P \to C[k]$, then $\chi_k(x) = d_1^{k-1}y$, so $\bar{\chi}_k(x) = 0$. Therefore, $\bar{x} = 0$, i.e., $x = d_2^{k-1}z$ for some $z \in C_2^{k-1}$ and $\chi_k x = d_1^{k-1}\chi_{k-1}z$. Then, $d_1^{k-1}(y - \chi_{k-1}z) = 0$, hence there is an element $z' \in C_2^{k-1}$ such that $d_2^{k-1}z' = 0$ and the cohomology class of $y - \chi_{k-1}z$ equals $\bar{\chi}_{k-1}\bar{z}'$, i.e., $y - \chi_{k-1}z = \chi_{k-1}z' + d_1^{k-2}t$ for some t. Then, $x = d_2^{k-1}(z + z')$ and $y = \chi_{k-1}(z + z') + d_1^{k-2}t$, so this homomorphism is homotopic to zero. □

As usual, we identify the category $\boldsymbol{A}$-Mod with the full subcategory of $\mathscr{D}(\boldsymbol{A})$ consisting of the complexes C concentrated in degree 0. The following result shows how the partial titling functor F behaves with respect to modules:

Corollary 6.4. *Let a $\boldsymbol{B}$-module M be given by the triple $\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\ \chi_M\right)$.*

(1) $\mathsf{F}M \in \boldsymbol{A}$-Mod *if and only if χ_M is surjective. Namely, then* $\mathsf{F}M \simeq \operatorname{Ker} \chi_M$.

(2) $\mathsf{F}M \in \boldsymbol{A}$-Mod[1] *if and only if χ_M is injective. Namely, then* $\mathsf{F}M \simeq \operatorname{Cok} \chi_M[1]$.

Proof. Note that $\operatorname{Hom}_{\boldsymbol{B}}(B_1, M) \simeq M_1$, $\operatorname{Hom}_{\boldsymbol{B}}(B_2, M) \simeq M_2$ and if $\phi : B_2 \to M$ maps $\binom{0}{1}$ to $\binom{0}{x}$, then $\phi\varepsilon$ maps $\binom{1}{0}$ to $\binom{\chi_M(x)}{0}$. Therefore, $\mathbb{R}\operatorname{Hom}_{\boldsymbol{B}}(R, M)$ is the complex

$$0 \to M_2 \xrightarrow{\chi_M} M_1 \to 0,$$

which proves the claim. □

Remark 6.5. There are several derived equivalences related to $\tilde{\boldsymbol{A}}$.

(1) If $\boldsymbol{A}$ is a Backström order, it is known (see [Burban et al. 2017]) that the complex $T = B_1[1] \oplus \boldsymbol{H}^+$, where $B_1 = \binom{\bar{\boldsymbol{A}}}{0}$, is a tilting complex for $\tilde{\boldsymbol{A}}$ and $(\operatorname{End}_{\mathscr{D}(\tilde{\boldsymbol{A}})})^{\mathrm{op}} T \simeq \boldsymbol{B}$, hence $\tilde{\boldsymbol{A}}$ is derived equivalent to $\boldsymbol{B}$. Nevertheless, in the general situation of Backström rings (even of Backström algebras) this is not so. First of all, $\operatorname{Hom}_{\tilde{\boldsymbol{A}}}(B_1, \boldsymbol{H}^+) \simeq \operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C}$, so it can happen that $\operatorname{Hom}_{\mathscr{D}(\tilde{\boldsymbol{A}})}(T, T[1]) \neq 0$. This is so, for instance, for the pair $(T(n, \Bbbk), \mathrm{UT}(n, \Bbbk))$ from Equation (2-3) (4), since in this case the matrix unit e_{nn} belongs to $\operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C}$. This is also so for Equation (2-3) (5). Moreover, even if $\operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C} = 0$, one can see that $\overline{\boldsymbol{H}}' = \operatorname{Ext}^1_{\tilde{\boldsymbol{A}}}(B_1, \boldsymbol{H}^+) \simeq \boldsymbol{C}^{-1}/{}_{\boldsymbol{C}}\boldsymbol{H}$, where $\boldsymbol{C}^{-1} = \operatorname{Hom}_{\boldsymbol{H}}(\boldsymbol{C}, \boldsymbol{H})$ and ${}_{\boldsymbol{C}}\boldsymbol{H} = \boldsymbol{H}/\operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C}$ is naturally embedded into $\boldsymbol{C}^{-1}$. Therefore, in this case,

$$(\operatorname{End}_{\mathscr{D}(\tilde{\boldsymbol{A}})} T)^{\mathrm{op}} \simeq \boldsymbol{B}' = \begin{pmatrix} \bar{\boldsymbol{A}} & \overline{\boldsymbol{H}}' \\ 0 & \boldsymbol{H} \end{pmatrix},$$

which need not coincide with $\boldsymbol{B}$ (see Example 6.6 below). If $\boldsymbol{H}$ is a hereditary order, then $\operatorname{ann}_{\boldsymbol{H}} \boldsymbol{C} = 0$ and $\overline{\boldsymbol{H}}' \simeq \overline{\boldsymbol{H}}$, hence $\boldsymbol{B}' \simeq \boldsymbol{B}$, in accordance with [Burban et al. 2017].

(2) On the other hand, set $T' = \left(\begin{smallmatrix} A & H/A \\ C & \overline{H} \end{smallmatrix}\right)$ considered as a left $\tilde{A}$-module. One can check it is a tilting module for $\tilde{A}$ and

$$(\operatorname{End}_{\mathscr{D}(\tilde{A})} T')^{\mathrm{op}} \simeq \widetilde{B} = \begin{pmatrix} A & H/A \\ 0 & \overline{A} \end{pmatrix},$$

hence $\tilde{A}$ is derived equivalent to $\widetilde{B}$. Unfortunately, this ring can be not so good from the homological point of view. At least, it is not better than A itself. Namely, as one can easily check,

$$\operatorname{l.gl.dim} \widetilde{B} = \max(\operatorname{l.gl.dim} A,\ 1 + \operatorname{pr.dim}_A(H/A)),$$

which is either $\operatorname{l.gl.dim} A$ or (more often) $\operatorname{l.gl.dim} A + 1$.

(3) One more observation: Consider the right $\tilde{A}$-modules $(\overline{A}\ 0)$ and $(C\ H)$. One can check that $T'' = (\overline{A}\ 0)[1] \oplus (C\ H)$ is a tilting complex for $\mathscr{D}(\tilde{A}^{\mathrm{op}})$ and

$$\operatorname{End}_{\mathscr{D}(\tilde{A}^{\mathrm{op}})} T'' \simeq B'' = \begin{pmatrix} \overline{A} & 0 \\ \overline{H} & H \end{pmatrix},$$

hence $\tilde{A}^{\mathrm{op}}$ is derived equivalent to $(B'')^{\mathrm{op}}$.

Note that the functor $P \mapsto \operatorname{Hom}_R(P, R)$ induces an exact duality

$$\operatorname{Perf}(R) \to \operatorname{Perf}(R^{\mathrm{op}})$$

for any ring R. Hence, $\operatorname{Perf}(\tilde{A}) \simeq \operatorname{Perf}(B'')$.

Example 6.6. Let $H = \mathsf{T}(3, \Bbbk)$ and $A = \{(a_{ij}) \in H \mid a_{11} = a_{22}\}$. Set $H_i = He_{ii}$ and $U_i = H_i/\operatorname{rad} H_i$. Then, $C = \{(a_{ij}) \in H \mid a_{11} = a_{22} = 0\}$, hence $\overline{H} = U_1 \oplus U_2$. On the other hand, $C = \operatorname{rad} H_2 \oplus H_3 \simeq H_1 \oplus H_3$, so $C^{-1} = \operatorname{Hom}_H(C, H)$ can be identified with the set of 3×2 matrices (b_{ij}) such that $b_{12} = b_{22} = 0$. One can check that ${}_CH$ is identified with the subset $\{(b_{ij}) \mid b_{11} = 0\} \subset C^{-1}$ and $\overline{H}' \simeq U_2 \not\simeq \overline{H}$ (even $\dim_\Bbbk \overline{H}' \neq \dim_\Bbbk \overline{H}$).

Acknowledgement

The author is thankful to the referee for important remarks that allowed an essential improvement to the paper.

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Received June 28, 2022. Revised February 14, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.89

RIGIDITY OF 3D SPHERICAL CAPS VIA μ-BUBBLES

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By using Gromov's μ-bubble technique, we show that the 3-dimensional spherical caps are rigid under perturbations that do not reduce the metric, the scalar curvature, and the mean curvature along its boundary. Several generalizations of this result will be discussed.

1. Introduction

In recent decades, a lot of progress has been made toward understanding the scalar curvature of a Riemannian manifold; see [Gromov 2023]. A particular medium for gaining such understanding is to answer whether one can perturb the metric of a "model space" in certain ways without reducing its scalar curvature. This viewpoint was famously represented by the positive mass theorem and its various generalizations and analogues. One analogue, which motivated the current work, is the following conjecture proposed by Min-Oo around 1995; see [Min-Oo 1998, Theorem 4].

Conjecture 1.1 (Min-Oo). *Suppose that g is a smooth Riemannian metric on the (topological) hemisphere $\boldsymbol{S}^n_+$ $(n \geq 3)$ with the properties*:

(1) *The scalar curvature R_g satisfies $R_g \geq n(n-1)$ on $\boldsymbol{S}^n_+$.*

(2) *The boundary $\partial \boldsymbol{S}^n_+$ is totally geodesic with respect to g.*

(3) *The induced metric on $\partial \boldsymbol{S}^n_+$ agrees with the standard metric on $\boldsymbol{S}^{n-1}$.*

Then g is isometric to the standard metric on $\boldsymbol{S}^n_+$.

Unlike its counterparts modeled on $\mathbb{R}^n$ and $\boldsymbol{H}^n$,[1] Min-Oo's conjecture turned out to admit counterexamples; see [Brendle et al. 2011]. Yet, its statement remains interesting, especially when it is compared with the following theorem of Llarull [1998, Theorem A].

Theorem 1.2 (Llarull). *Let $(\boldsymbol{S}^n, \hat{g})$ be the standard n-sphere $(n \geq 3)$. Suppose that g is another Riemannian metric on $\boldsymbol{S}^n$ satisfying $g \geq \hat{g}$ and $R_g \geq R_{\hat{g}}$. Then $g = \hat{g}$.*

MSC2020: primary 53C21; secondary 53C24.
Keywords: Llarull's theorem, spherical cap, μ-bubble.

[1] See [Schoen and Yau 1979, Corollary 2; Gromov and Lawson 1983, Theorem A; Min-Oo 1989; Andersson et al. 2008].

A side-by-side view of Min-Oo's conjecture and Llarull's theorem suggests the following.

Conjecture 1.3. *Let $(S^n_+, \hat{g})$ be the standard n-dimensional hemisphere. Then Conjecture 1.1 holds under the additional assumption*: $g \geq \hat{g}$.

Our first result in this article is that Conjecture 1.3 holds when $n = 3$; here is a more precise statement; also see Corollary 3.12 below.

Theorem 1.4. *Let $(\boldsymbol{S}^3_+, \hat{g})$ be the standard 3-dimensional hemisphere. Suppose that g is another Riemannian metric on $\boldsymbol{S}^3_+$ with the properties*:

(1) $g \geq \hat{g}$ *and* $R_g \geq R_{\hat{g}}$ *on* $\boldsymbol{S}^3_+$.

(2) *the mean curvature* H_g *on* $\partial \boldsymbol{S}^3_+$ *satisfies* $H_g \geq 0$.[2]

(3) *The induced metrics on* $\partial \boldsymbol{S}^3_+$ *satisfy* $g_{\partial \boldsymbol{S}^3_+} = \hat{g}_{\partial \boldsymbol{S}^3_+}$.

Then $g = \hat{g}$.

As we will see below, Theorem 1.4 admits a somewhat direct proof. With more technical work, we can generalize it in the following aspects: (i) the assumption (3) in Theorem 1.4 will be removed; and (ii) the model space will not need to be the standard hemisphere — it can be a "spherical cap" or, more generally, a geodesic ball inside a space form. To make these points explicit, we now state our main result; also see Theorem 5.3 below.

Theorem 1.5. *For any suitable constants κ, μ, let $(\boldsymbol{B}_{\kappa,\mu}, \hat{g}_\kappa)$ be a geodesic ball in the 3-dimensional space form with sectional curvature κ such that $\partial \boldsymbol{B}_{\kappa,\mu}$ has mean curvature μ. Suppose that g is another Riemannian metric on $\boldsymbol{B}_{\kappa,\mu}$ satisfying*

$$g \geq \hat{g}_\kappa, \quad R_g \geq 6\kappa \text{ on } \boldsymbol{B}_{\kappa,\mu} \quad \text{and} \quad H_g \geq \mu \text{ on } \partial \boldsymbol{B}_{\kappa,\mu}.$$

Then $g = \hat{g}_\kappa$.

In Gromov's first preprint of [2019a], a (more general) version of Theorem 1.5 was stated as a "nonexistence" result (see [Gromov 2019b, Theorem 1]); an outline of proof was sketched, which relied on a "generalized Llarull's theorem". Following Gromov's main idea, we present a detailed and purely variational proof of Theorem 1.5; this theorem also confirms, in the case of $n = 3$, a rigidity statement mentioned in [Gromov 2019b, Remark (d)] without proof.

A simple modification of the proof of Theorem 1.5 yields the following; also see Theorem 5.1.

[2]Given a domain Ω in a Riemannian manifold, unless we specify otherwise, we shall adopt the (sign) convention for the mean curvature of $\partial\Omega$ to be $H = \mathrm{tr}(\nabla \nu)$, where ν is the *outward* unit normal along $\partial\Omega$. Under this convention, the mean curvature of the boundary of the unit ball in $\mathbb{R}^n$ is $n - 1$.

Theorem 1.6. *Let $(\boldsymbol{S}^3 \setminus \{O, O'\}, \hat{g})$ be the standard 3-sphere with a pair of antipodal points removed, and let $h \geq 1$ be a smooth function on $\boldsymbol{S}^3 \setminus \{O, O'\}$. Suppose that g is another Riemannian metric on $\boldsymbol{S}^3 \setminus \{O, O'\}$ satisfying*

$$g \geq h^4 \hat{g} \quad \text{and} \quad R_g \geq h^{-2} R_{\hat{g}}.$$

Then $h \equiv 1$, and $g = \hat{g}$.

When $h \equiv 1$, Theorem 1.6 is a special case of Gromov's theorem of "extremality of doubly punctured spheres" (see [Gromov 2023, Sections 5.5 and 5.7]), and it implies Theorem 1.2 in the case of $n = 3$. We also remark that Theorem 1.6 would fail without the assumption $h \geq 1$ (see Remark 5.2 below). We tend to believe that the conclusion of Theorem 1.6 still holds when the condition $g \geq h^4 \hat{g}$ is replaced by $g \geq h^2 \hat{g}$; a condition such as $\inf h > 0$ would still be needed, otherwise, the metric in Remark 5.2 would serve as a counterexample.

Before sketching our technical ingredients, let us remind the reader that since the early 1980s, two different approaches — variational and spinorial — have been developed for studying the scalar curvature. Yet, for more than two decades, extensions of Llarull's rigidity theorem, like Llarull's original proof, had been mainly carried out from the spinorial approach. See, for example, [Goette and Semmelmann 2002; Herzlich 2005; Listing 2009; Cecchini and Zeidler 2022, especially Theorem 1.15, Corollary 1.17; Lott 2021; Su et al. 2022; Zhang 2020]. It is relatively recent that variational methods have also become available for proving results of Llarull type.[3] A key in this new development, which is also a main tool for the current paper, is Gromov's *μ-bubble* technique [2023, Section 5].

Roughly speaking, given a function μ on a Riemannian manifold (M^n, g), a μ-bubble is a minimizer (and a critical point) of the functional

$$\Omega \mapsto \mathrm{vol}_{n-1}(\partial\Omega) - \int_\Omega \mu \tag{1-1}$$

defined for suitable subsets $\Omega \subset M$; given a μ-bubble, useful geometric information can be extracted from its first and second variation formulae. In order to guarantee that a nondegenerate μ-bubble exists, (M, g) is often assumed to be a *Riemannian band*,[4] and μ is often required to satisfy a *barrier condition* (see (2-2) below), which prevents minimizing sequences from collapsing either to a point or into ∂M.

In some cases, even without the assumption of either a Riemannian band or a barrier condition, a μ-bubble may still be found by direct observation of the functional (1-1). This is the case with our proof of Theorem 1.4. In fact, if we

[3] To our best knowledge, a purely variational proof of Llarull's original theorem remains to be found.

[4] See Section 2A below for definition, and see [Gromov 2018; Räde 2021] for related discussion.

modify (1-1) by considering the new functional

$$\Omega \mapsto \mathrm{vol}_{n-1}(\partial\Omega) + \int_{S^3_+\setminus\Omega} \mu, \tag{1-2}$$

the variational properties remain unchanged; in our situation, the new functionals associated to g and $\hat{g}$ admit an inequality, which becomes an equality when $\Omega = S^3_+$, and then direct comparison shows that S^3_+ is a μ-bubble (see the proof of Corollary 3.12). We note that this argument crucially relies on the assumption (3) in Theorem 1.4.

Now let us continue to take Theorem 1.4 as an example to explain how to obtain rigidity results from having an "initial" μ-bubble Ω. Although Ω need not be S^3_+, we do, for a technical reason, require that $\partial\Omega$ has a connected component Σ_0 whose projection onto S^2 has *nonzero degree* (see (3-6)) — for simplicity, let us call such a Σ_0 a "good component". By using the second variation and the Gauss–Bonnet formulae, we show that, under certain extra assumptions, Σ_0 must be a 2-sphere parallel (with respect to $\hat{g}$) to the equator ∂S^3_+; furthermore, along Σ_0 the ambient metric g must agree with $\hat{g}$ (Proposition 3.4). This obtained, a standard foliation lemma (Lemma 3.8) and minimality of Ω imply that g must agree with $\hat{g}$ in a *neighborhood* of Σ_0 (Lemma 3.10). Finally, with an "open-closed" argument and standard facts in geometric measure theory, we show that such a neighborhood can be extended to the whole manifold, thus completing the proof (Proposition 3.11).

In the more general setting of Theorem 1.5, the existence of an "initial" μ-bubble becomes less direct to prove. For simplicity, let us still assume that the model space is the standard hemisphere. Although (S^3_+, g) is *not* a Riemannian band, we may consider creating one from it by removing a small geodesic ball centered at the north pole $O \in (S^3_+, \hat{g})$, but an immediate problem arises: the natural choice $\mu = \hat{H}$ (see (3-3)), which corresponds to the mean curvature of the geodesic spheres centered at O with respect to $\hat{g}$, may not satisfy the barrier condition.

To address this problem, we construct a sequence of perturbations μ_ϵ (see (4-3); also see [Zhu 2021, Section 3]) of $\hat{H}$ that *do* satisfy the barrier conditions on a corresponding sequence of Riemannian bands $M_\epsilon \subset S^3_+$. In particular, in each M_ϵ there exists a μ_ϵ-bubble Ω_ϵ (Lemma 4.1). By construction, μ_ϵ tends to $\hat{H}$, and M_ϵ tends to S^3_+, as ϵ approaches 0. However, two new questions arise:

(a) *As ϵ tends to 0, do the Ω_ϵ subconverge to an $\hat{H}$-bubble Ω in (S^3_+, g)?*

(b) *If so, does $\partial\Omega$ possess a component whose projection to S^2 has nonzero degree?*

To put these in a slightly different way, regarding (a), we worry that Ω_ϵ may become degenerate in the limit; regarding (b), we worry that the "good components" of $\partial\Omega_\epsilon$ may either approach the north pole O and thus lose the "degree" property, or "meet and cancel" each other so that none of them is actually preserved in the limit.

In Sections 4C and 4D, we answer both questions (a) and (b) in the affirmative. A key step is to argue that each $\partial\Omega_\epsilon$ not only possesses a "good component" Σ_0^ϵ, but such a component must be disjoint from a fixed neighborhood of $O \in S^3_+$ provided that ϵ is small (Proposition 4.7), which is, again, enforced by the Gauss–Bonnet theorem. This step allows us to obtain a universal upper bound for the norm of the second fundamental form on Σ_0^ϵ, which is then used to prove the existence of a limiting hypersurface Σ_0 that is indeed a component of $\partial\Omega$ (Lemma 4.11).

Once having an "initial" μ-bubble, one may complete the proof of Theorem 1.5 by the foliation argument described above.

Regarding Theorem 1.6, we may consider Riemannian bands in $S^3 \setminus \{O, O'\}$ bounded by small geodesic spheres in $(S^3, \hat{g})$ centered at O and O', but because of the lack of mean curvature information with respect to g along those boundaries, perturbations of the form (4-3) are no longer adequate for meeting the barrier condition. To address this issue, we construct new functions μ_α by composing the function $\hat{H}$ with dilations of $S^3 \setminus \{O, O'\}$ in the "longitude" direction, and then μ_α will satisfy the desired barrier conditions; see Section 5 for more detail. The rest of the proof is similar to the other cases.

Remark 1.7. After our paper was submitted, an analogous result of Theorem 1.5 for higher dimensional spherical domains was proved in [Lee and Tam 2022]. Relying on harmonic maps flow and Ricci flow their argument works only for the case of compact domains in sphere for the time being.

2. Elements of Gromov's μ-bubble technique

In this section we recall some elements of Gromov's μ-bubble technique. Our discussion follows Section 5 of [Gromov 2023], Section 2 of [Zhu 2021] and Section 3 of [Zhou and Zhu 2020].

2A. ***μ-bubbles in a Riemannian band.*** Let (M^n, g) be a compact Riemannian manifold whose boundary ∂M is expressed as a disjoint union $\partial M = \partial_- \sqcup \partial_+$ where both ∂_- and ∂_+ are closed hypersurfaces. Such a quadruple $(M, g; \partial_-, \partial_+)$ is called a *Riemannian band*. Given a Riemannian band, let $\Omega_0 \subset M$ be a fixed smooth Caccioppoli set that contains a neighborhood of ∂_- and is disjoint from a neighborhood of ∂_+;[5] we call such an Ω_0 a *reference set*. Let $\mathcal{C}_{\Omega_0}$ denote the collection of Caccioppoli sets $\Omega \subset M$ satisfying $\Omega \Delta \Omega_0 \Subset \mathring{M}$ ("$\Subset$" reads "is compactly contained in"); here $\Omega\Delta\Omega_0$ denotes the symmetric difference between Ω and Ω_0, and $\mathring{M}$ stands for the interior of M.

[5] Also known as "sets of locally finite perimeter"; see [Giusti 1984] for details.

Let μ be either a smooth function on M, or a smooth function defined on $\mathring{M}$ satisfying $\mu \to \pm\infty$ on $\partial_{\mp}$. For $\Omega \in \mathcal{C}_{\Omega_0}$ consider the *brane action*

$$\mathcal{A}^{\mu}_{\Omega_0}(\Omega) := \mathcal{H}^{n-1}(\partial\Omega) - \mathcal{H}^{n-1}(\partial\Omega_0) - \int_M (\chi_\Omega - \chi_{\Omega_0})\mu \, d\mathcal{H}^n \tag{2-1}$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure induced by g and χ_Ω denotes the characteristic function associated to Ω. A minimizer Ω of (2-1) is called a *μ-bubble*.

Remark 2.1. (1) For $\Omega_1, \Omega_2 \in \mathcal{C}_{\Omega_0}$, we have $\mathcal{A}^{\mu}_{\Omega_0}(\Omega_2) - \mathcal{A}^{\mu}_{\Omega_1}(\Omega_2) = \mathcal{A}^{\mu}_{\Omega_0}(\Omega_1)$; thus, in a sense, minimizers are independent of the choice of a reference set. (2) The brane action (2-1) may be defined on manifolds that are not necessarily Riemannian bands; in those cases, one may replace $\mathcal{H}^{n-1}(\partial\Omega)$ by $\mathcal{H}^{n-1}(\partial(\Omega \cap K))$ and similarly for $\mathcal{H}^{n-1}(\partial\Omega_0)$, where K is a compact set such that $\Omega\Delta\Omega_0 \subset K$.

2B. ***Existence and regularity.***

Definition 2.2. Given a Riemannian band $(M, g; \partial_-, \partial_+)$, a function μ is said to satisfy the *barrier condition* if either $\mu \in C^\infty(\mathring{M})$ with $\mu \to \pm\infty$ on $\partial_{\mp}$, or $\mu \in C^\infty(M)$ with

$$\mu|_{\partial_-} > H_{\partial_-}, \quad \mu|_{\partial_+} < H_{\partial_+} \tag{2-2}$$

where H_{∂_-} is the mean curvature of ∂_- with respect to the inward normal and H_{∂_+} is the mean curvature of ∂_+ with respect to the outward normal.

Lemma 2.3 [Zhu 2021, Proposition 2.1]. *Let $(M^n, g; \partial_-, \partial_+)$ be a Riemannian band with $n \le 7$, and let Ω_0 be a reference set. If μ satisfies the barrier condition, then there exists an $\Omega \in \mathcal{C}_{\Omega_0}$ with smooth boundary such that*

$$\mathcal{A}^{\mu}_{\Omega_0}(\Omega) = \inf_{\Omega' \in \mathcal{C}_{\Omega_0}} \mathcal{A}^{\mu}_{\Omega_0}(\Omega').$$

Remark 2.4. In Lemma 2.3 the smooth hypersurface $\Sigma := \partial\Omega \setminus \partial_-$ is homologous to ∂_+.

2C. ***Variational properties.*** Let Ω be a smooth μ-bubble in a Riemannian band $(M^n, g; \partial_-, \partial_+)$, and let $\Sigma = \partial\Omega \setminus \partial_-$. One may derive variation formulae for $\mathcal{A}^\mu$ at Ω; see (2.3) in [Zhu 2021] and the unnumbered equation above it. Specifically, the first variation implies that the mean curvature of Σ (with its outward normal ν) is equal to $\mu|_\Sigma$; the second variation implies that the Jacobi operator

$$J_\Sigma := -\Delta_\Sigma + \tfrac{1}{2}(R_\Sigma - R_g - \mu^2 - |\mathrm{II}|^2) - \nu(\mu) \tag{2-3}$$

is nonnegative, where Δ_Σ and R_Σ are respectively the g-induced Laplacian and scalar curvature of Σ; R_g is the scalar curvature of (M, g); and II is the second fundamental form of Σ.

Definition 2.5. Let μ be a smooth function on a Riemannian manifold (M^n, g). A smooth two-sided hypersurface $\mathcal{S} \subset M$ with unit normal ν is said to be a *μ-hypersurface* if its mean curvature taken with respect to ν is equal to $\mu|_{\mathcal{S}}$.

Clearly, (2-3) also makes sense when Σ is replaced by a μ-hypersurface; this motivates the following notion of stability.

Definition 2.6. A μ-hypersurface $\mathcal{S} \subset M$ with unit normal ν is said to be *stable* if $J_{\mathcal{S}}$ is nonnegative on $C_0^\infty(\mathcal{S})$.

Remark 2.7. If μ satisfies the barrier condition, then for any μ-bubble Ω each connected component of $\partial\Omega \setminus \partial_-$ with its outward unit normal is a stable μ-hypersurface.

Let $\mathcal{S}$ be a μ-hypersurface. Following [Gromov 2023, Section 5.1] we consider the operator

$$L_{\mathcal{S}} := -\Delta_{\mathcal{S}} + \tfrac{1}{2}(R_{\mathcal{S}} - R_+^\mu) \tag{2-4}$$

where

$$R_+^\mu := R_g + \frac{n}{n-1}\mu^2 - 2|\mathrm{d}\mu|_g. \tag{2-5}$$

In fact, $L_{\mathcal{S}}$ is obtained from applying the obvious inequalities

$$-\partial_\nu \mu \le |\mathrm{d}\mu|_g, \quad |\mathrm{II}|^2 \ge \frac{1}{n-1}\mu^2 \tag{2-6}$$

to $J_{\mathcal{S}}$. One can easily verify that the following holds when $\mathcal{S}$ is stable:

$$L_{\mathcal{S}} \ge J_{\mathcal{S}} \ge 0. \tag{2-7}$$

Example 2.8. Consider $\boldsymbol{S}^2 \times [t_1, t_2]$ $(0 < t_1 < t_2 < \pi)$ equipped with the metric $g = (\sin^2 t) g_{\boldsymbol{S}^2} + \mathrm{d}t^2$ where $g_{\boldsymbol{S}^2}$ is the standard metric on $\boldsymbol{S}^2$. This represents an annular region in the standard $\boldsymbol{S}^3$. Take $\mu(t) = 2\cot t$. It is easy to see that each t-level set S_t, with the unit normal $\nu = \partial_t$, is a μ-hypersurface. Moreover, on S_t we have

$$R_g = 6, \quad R_{S_t} = \frac{2}{\sin^2 t}, \quad |\mathrm{II}|^2 = 2\cot^2 t, \quad \nu(\mu) = \mu'(t) = -\frac{2}{\sin^2 t}.$$

In this case, both J_{S_t} and L_{S_t} reduce to $-\Delta_{S_t}$.

The following lemma is a direct consequence of Theorem 3.6 in [Zhou and Zhu 2020].

Lemma 2.9. *Let (M^n, g) be a closed Riemannian manifold with $2 \le n \le 6$, and let $\mu \in C^\infty(M)$. Let $\mathcal{S}$ be an immersed stable μ-hypersurface contained in an open subset $V \subset M$ and satisfying $\partial\mathcal{S} \cap V = \varnothing$. If* $\operatorname{area}(\mathcal{S}) \le C$ *for some constant C, then there exists a constant $C_1 = C_1(M, n, \|\mu\|_{C^3(M)}, C)$ such that*

$$|\mathrm{II}|^2(x) \le \frac{C_1}{\operatorname{dist}_g^2(x, \partial V)} \quad \textit{for all } x \in \mathcal{S}. \tag{2-8}$$

2D. *Comparison with a warped-product metric.* Given a Riemannian manifold (N^{n-1}, g_N), an interval I (with coordinate t) and a function $\varphi : I \to \mathbb{R}_+$, consider the warped product metric defined on $\hat{N} := N \times I$

$$\hat{g} := \varphi(t)^2 g_N + \mathrm{d}t^2. \tag{2-9}$$

A standard calculation shows that the mean curvature on each slice $N \times \{t\}$ with respect to the ∂_t-direction is

$$\hat{H}(t) = (n-1)\frac{\varphi'(t)}{\varphi(t)}; \tag{2-10}$$

moreover, one may verify that the scalar curvature $R_{\hat{g}}$ of $\hat{g}$ satisfies

$$0 = -R_{\hat{g}} + \frac{1}{\varphi^2}R_N - \frac{n}{n-1}\hat{H}^2 - 2\frac{d\hat{H}}{dt}, \tag{2-11}$$

where R_N is the scalar curvature of (N, g_N).

Now suppose that $f : M \to \hat{N}$ is a smooth map from a Riemannian band $(M, g; \partial_-, \partial_+)$ to $\hat{N}$. By pulling back all functions in (2-11) via f and adding the resulting equation with (2-5), we obtain

$$R_+^\mu = \frac{1}{\varphi^2}R_N + (R_g - R_{\hat{g}}) + \frac{n}{n-1}(\mu^2 - \hat{H}^2) - 2(\partial_t\hat{H} + |\mathrm{d}\mu|_g), \tag{2-12}$$

where pull-back symbols are omitted for clarity. The expression (2-12) will be useful in our analysis of μ-bubbles.

3. Rigidity of 3D spherical caps

A spherical cap of radius $T \in (0, \pi)$ in the standard $\boldsymbol{S}^3$ may be represented by the closed ball $\boldsymbol{B}_T := \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| \le T\}$ equipped with the metric

$$\hat{g} = \varphi(t)^2 g_{\boldsymbol{S}^2} + \mathrm{d}t^2 \quad \text{with } \varphi(t) = \sin t, \tag{3-1}$$

where $t \in [0, T]$ serves as the radial coordinate on $\boldsymbol{B}_T$ and $g_{\boldsymbol{S}^2}$ is the standard metric on $\boldsymbol{S}^2$. For $t \in (0, T]$, let $S_t := \partial\boldsymbol{B}_t$. For $0 < t_1 < t_2 \le T$, let $\boldsymbol{B}_{[t_1,t_2]} := \boldsymbol{B}_{t_2} \setminus \mathring{\boldsymbol{B}}_{t_1}$; similarly, let $\boldsymbol{B}_{(t_1,t_2]} := \boldsymbol{B}_{t_2} \setminus \boldsymbol{B}_{t_1}$. Given a domain $\Omega \subset \boldsymbol{B}_T$ with smooth boundary Σ, the outward normal along Σ with respect to the metric $\hat{g}$ will be denoted by $\hat{\nu}$.

The objective of this section and the next is to prove the following rigidity theorem.

Theorem 3.1. *Let $(\boldsymbol{B}_T, \hat{g})$ be the 3-dimensional spherical cap of radius $T \in (0, \pi)$. Suppose that g is another Riemannian metric on $\boldsymbol{B}_T$ satisfying*

$$g \geq \hat{g}, \quad R_g \geq R_{\hat{g}} \text{ on } \boldsymbol{B}_T, \quad \text{and} \quad H_g \geq H_{\hat{g}} = 2\cot T \text{ on } \partial \boldsymbol{B}_T. \tag{3-2}$$

Then $g = \hat{g}$.

Our proof begins by establishing a key ingredient: certain stable μ-hypersurfaces are necessarily t-level sets in $\boldsymbol{B}_T$ (Proposition 3.4), the justification of which hinges on an integral inequality (see (3-14)) involving an application of the Gauss–Bonnet formula. This result is followed by a classical foliation lemma (Lemma 3.8). Under a suitable "minimality" assumption (Assumption 3.9), each leaf in that foliation turns out to be stable, which implies local rigidity of the metric (Lemma 3.10). Section 3 culminates at Proposition 3.11, which justifies Theorem 3.1 assuming the existence of an "initial" minimizer (Assumption 3.9); this assumption will be verified in Section 4 via a perturbation argument (see Proposition 4.12).

3A. ***Stable μ-hypersurfaces and t-level sets.*** The metric (3-1) is of the form (2-9); thus, (2-10) applies to give

$$\hat{H}(t) = 2\cot t. \tag{3-3}$$

It will be useful to define, for $\mu = \mu(t)$, the function (see the last two terms in (2-12))

$$\begin{aligned} Z_\mu(t) &:= \tfrac{3}{2}(\mu(t)^2 - \hat{H}(t)^2) - 2(\hat{H}'(t) - \mu'(t)) \\ &= \tfrac{3}{2}\mu(t)^2 + 2\mu'(t) - 6\cot^2 t + \frac{4}{\sin^2 t}. \end{aligned} \tag{3-4}$$

Notice, in particular, that $Z_{\hat{H}}(t) \equiv 0$. As t is a coordinate on $\boldsymbol{B}_T$, we may regard μ and Z_μ as functions defined on $\boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$.

Lemma 3.2. *Let $\mu(t)$ be a smooth, decreasing function defined on $(0, T]$, and let g be a Riemannian metric on $\boldsymbol{B}_T$ satisfying (3-2). At a point $q \in \boldsymbol{B}_T$, if $Z_\mu \geq 0$, then $R_+^\mu \geq 2/\varphi^2 > 0$.*

Proof. On the right-hand side of (2-12), the second term is nonnegative by assumption. Moreover, $g \geq \hat{g}$ implies

$$|\mathrm{d}\mu|_g \leq |\mathrm{d}\mu|_{\hat{g}} = |\partial_t \mu| = -\mu'(t). \tag{3-5}$$

Substituting this in the last term of (2-12) and noticing that $R_{S^2} = 2$, we obtain the desired inequality. □

Now let Σ_0 be a hypersurface in $\boldsymbol{B}_{(0,T]}$, and let Φ denote the projection map from Σ_0 to $\boldsymbol{S}^2$, namely,

$$\Phi : \Sigma_0 \hookrightarrow \boldsymbol{B}_{(0,T]} \cong (0, T] \times \boldsymbol{S}^2 \to \boldsymbol{S}^2. \tag{3-6}$$

Lemma 3.3. *Let $d\sigma_{\hat{g}}$ be the area form on Σ_0 induced by $\hat{g}$. We have*

$$d\sigma_{\hat{g}} \geq \varphi^2 |\Phi^* \, d\sigma_{\boldsymbol{S}^2}| \tag{3-7}$$

where the absolute-value sign is put to eliminate the ambiguity of orientation.

Proof. Let (θ_α) $(\alpha = 1, 2)$ be local coordinates on $\boldsymbol{S}^2$, and write $g_{\boldsymbol{S}^2} = h_{\alpha\beta} \mathrm{d}\theta_\alpha \mathrm{d}\theta_\beta$. We get

$$\Phi^*(g_{\boldsymbol{S}^2}) = h_{\alpha\beta} \, \mathrm{d}\theta_\alpha \, \mathrm{d}\theta_\beta \leq \frac{1}{\varphi^2}(\mathrm{d}t^2 + \varphi^2 h_{\alpha\beta} \, \mathrm{d}\theta_\alpha \, \mathrm{d}\theta_\beta) = \frac{1}{\varphi^2} \hat{g}_{\Sigma_0}, \tag{3-8}$$

where the functions and forms are restricted to Σ_0. The conclusion follows. □

Proposition 3.4. *Let $\mu(t)$ be a smooth, decreasing function defined on $(0, T]$. Suppose that $\Sigma_0 \hookrightarrow (\boldsymbol{B}_T \setminus \{\boldsymbol{0}\}, g)$ is a stable, closed μ-hypersurface with unit normal ν, where g satisfies $g \geq \hat{g}$ and $R_g \geq R_{\hat{g}}$. Moreover, suppose that $Z_\mu \geq 0$ on Σ_0 and that the projection Φ from Σ_0 to $\boldsymbol{S}^2$ has nonzero degree. Then*:

(a) $\Sigma_0 = S_\tau$ *for some* $\tau \in (0, T]$.

(b) $J_{\Sigma_0} = L_{\Sigma_0} = -\Delta_{\Sigma_0}$; *see* (2-3), (2-4).

(c) $\Sigma_0 \subset (\boldsymbol{B}_T, g)$ *is umbilic with constant mean curvature* $\mu(\tau)$.

(d) $g(p) = \hat{g}(p)$ *at all points* $p \in \Sigma_0$; *in particular*, $g_{\Sigma_0} = \hat{g}_{\Sigma_0} = (\sin^2)\tau g_{\boldsymbol{S}^2}$.

(e) *On* Σ_0, $\nu = \partial_t$.

(f) *On* Σ_0, $R_+^\mu = 2/\varphi^2$ *and* $Z_\mu = 0$.

We prepare our proof of this proposition with the following two lemmas.

Lemma 3.5. *Under the assumption of Proposition 3.4*, Σ_0 *is homeomorphic to* $\boldsymbol{S}^2$.

Proof. By stability, the operator L_{Σ_0} defined by (2-4) is nonnegative. Let $u \in C^\infty(\Sigma_0)$ be a principal eigenfunction of L_{Σ_0}, and let $\lambda_1 \geq 0$ be the corresponding eigenvalue. By the maximum principle, we can always choose u to be strictly positive. Thus,

$$-u^{-1} \Delta_{\Sigma_0} u + \tfrac{1}{2}(R_{\Sigma_0} - R_+^\mu) = \lambda_1 \geq 0. \tag{3-9}$$

Expanding

$$\mathrm{div}(u^{-1} \nabla_{\Sigma_0} u) = -u^{-2} |\nabla_{\Sigma_0} u|^2 + u^{-1} \Delta_{\Sigma_0} u, \tag{3-10}$$

applying it in the previous equation and integrating over Σ_0, we obtain

$$\tfrac{1}{2}\int_{\Sigma_0}(R_{\Sigma_0}-R^{\mu}_{+})\,d\sigma_g=\int_{\Sigma_0}(\lambda_1+u^{-2}|\nabla_{\Sigma_0}u|^2)\,d\sigma_g\geq 0. \tag{3-11}$$

From (3-11), the Gauss–Bonnet formula, and Lemma 3.2, we deduce

$$4\pi\,\chi(\Sigma_0)=\int_{\Sigma_0}R_{\Sigma_0}\,d\sigma_g\geq\int_{\Sigma_0}R^{\mu}_{+}\,d\sigma_g>0; \tag{3-12}$$

since Σ_0 is a connected oriented surface, it is homeomorphic to $\boldsymbol{S}^2$. □

Remark 3.6. Lemma 3.5 remains true if we assume $R^{\mu}_{+}>0$ instead of $Z_{\mu}\geq 0$ on Σ_0.

Lemma 3.7. *Under the assumption of Proposition 3.4, if* $J_{\Sigma_0}=L_{\Sigma_0}=-\Delta_{\Sigma_0}$, *then*:

(i) $\Sigma_0\subset(\boldsymbol{B}_T,g)$ *is umbilic.*

(ii) $\Sigma_0=S_\tau$ *for some* $\tau\in(0,T]$.

(iii) $\mu|_{\Sigma_0}=\mu(\tau)$.

Proof. By assumption, (2-6) must be equalities. In particular, the traceless part of II_{Σ_0} must vanish, and thus $\Sigma_0\subset(\boldsymbol{B}_T,g)$ is umbilic, justifying (i). Moreover, $-\nu(\mu)=|\mathrm{d}\mu|_g$, and so ν must be parallel to $\nabla_g\mu$. Thus, for any tangent vector $X\in T\Sigma_0$, we have that $\mathrm{d}\mu(X)=g(\nabla_g\mu,X)$ is proportional to $g(\nu,X)=0$; this implies that μ is constant along Σ_0. Combining with the fact that $\Sigma_0\cong\boldsymbol{S}^2$ (Lemma 3.5), we conclude that Σ_0 is a level set S_τ, justifying (ii), and (iii) immediately follows. □

Proof of Proposition 3.4. The assumption $g\geq\hat{g}$ implies the relation between area forms on Σ_0:

$$d\sigma_g\geq d\sigma_{\hat{g}}. \tag{3-13}$$

We deduce

$$\begin{aligned}\int_{\Sigma_0}R^{\mu}_{+}\,d\sigma_g&\geq\int_{\Sigma_0}\frac{2}{\varphi^2}\,d\sigma_{\hat{g}}\\&\geq 2\int_{\Sigma_0}|\Phi^*\,d\sigma_{\boldsymbol{S}^2}|\\&\geq 2\left|\int_{\Sigma_0}\Phi^*\,d\sigma_{\boldsymbol{S}^2}\right|\\&=2k\int_{\boldsymbol{S}^2}d\sigma_{\boldsymbol{S}^2}\\&=8k\pi,\end{aligned} \tag{3-14}$$

where $k := |\deg(\Phi)| \geq 1$ by assumption. In (3-14), the first inequality is due to (3-13) and Lemma 3.2; the second inequality follows from Lemma 3.3; the remaining (in)equalities are obvious.

On combining (3-12) with (3-14), we obtain

$$8\pi = \int_{\Sigma_0} R_{\Sigma_0}\, d\sigma_g \geq \int_{\Sigma_0} R_+^\mu\, d\sigma_g \geq 8k\pi, \quad (k \geq 1). \tag{3-15}$$

This enforces the two inequalities in (3-15) to become equalities. Saturation of the first inequality, which we deduced from (3-11), implies that $\lambda_1 = 0$ and that u is a constant; hence, by (3-9), $R_{\Sigma_0} = R_+^\mu$; then, by (2-4), $L_{\Sigma_0} = -\Delta_{\Sigma_0}$. With this established, the relation (2-7) would enforce that $J_{\Sigma_0} = L_{\Sigma_0} = -\Delta_{\Sigma_0}$, justifying (b). By Lemma 3.7, (a) and (c) follow.

Next consider saturation of the second inequality in (3-15), or rather (3-14). Because we have already deduced that Σ_0 is a t-level set, the second and third inequalities in (3-14) automatically become equalities. Saturation of the first inequality in (3-14), on the other hand, has two implications:

$$d\sigma_g = d\sigma_{\hat{g}} \quad \text{and} \quad R_+^\mu = \frac{2}{\varphi(\tau)^2}.$$

The former, along with $g \geq \hat{g}$, implies that

$$g_{\Sigma_0} = \hat{g}_{\Sigma_0} = \varphi(\tau)^2 g_{S^2}; \tag{3-16}$$

the latter, along with the proof of Lemma 3.2, implies that $Z_\mu(\tau) = 0$ and $|\mathrm{d}\mu|_g = |\mathrm{d}\mu|_{\hat{g}}$, which is just $-\nu(\mu) = |\partial_t \mu|$ (see the proof of Lemma 3.7). Hence, $\nu = \partial_t + X$ for some vector field X on $\Sigma_0 = S_\tau$. Note that

$$1 = |\nu|_g \geq |\nu|_{\hat{g}} = \sqrt{|\partial_t|_{\hat{g}}^2 + |X|_{\hat{g}}^2} = \sqrt{1 + |X|_{\hat{g}}^2}; \tag{3-17}$$

we have $X = 0$ and $\nu = \partial_t$. Combining this with (3-16), we get $g(p) = \hat{g}(p)$ for all $p \in \Sigma_0$. This justifies (d), (e) and (f), completing the proof. □

3B. ***Foliation, minimality and rigidity.*** The following "foliation" lemma is standard; see [Ye 1991; Andersson et al. 2008; Nunes 2013; Zhu 2021].

Lemma 3.8. *Suppose that $\Sigma_0 \subset (\boldsymbol{B}_T, g)$ is a μ-hypersurface (with unit normal ν) on which the stability operator J (see (2-3)) reduces to $-\Delta_{\Sigma_0}$. Then there exists an interval I and a map $\phi : \Sigma_0 \times I \to \boldsymbol{B}_T$ such that:*[6]

(1) *ϕ is a diffeomorphism onto a neighborhood of $\Sigma_0 \subset \boldsymbol{B}_T$.*

(2) *The family $\Sigma_s = \phi(\Sigma_0, s)$ is a normal variation of Σ_0 with $\partial_s \phi = \nu$ along Σ_0.*

(3) *On each Σ_s, the difference $H_{\Sigma_s} - \mu$ is a constant k_s.*

[6] If $0 < \tau < T$, I can be taken to be an open interval containing 0; if $\tau = T$, I is of the form $(a, 0]$; and if $\tau = \delta$, I is of the form $[0, b)$.

Proof. The proof is the same as that of Lemma 3.4 in [Zhu 2021], except for the extra step: once having obtained a foliation, we reexpress it as a normal variation by using a vector field normal to all its leaves; see [Andersson et al. 2008, page 6, second paragraph]. □

Before proceeding further, let us state a recurring assumption.

Assumption 3.9. Let g be a metric on $\boldsymbol{B}_T$ satisfying (3-2), and let $\Omega \subset (\boldsymbol{B}_T, g)$ be a Caccioppoli set such that $\partial\Omega \setminus \{\boldsymbol{0}\}$ is smooth and embedded. Define the class $\mathcal{C}_\Omega$ of Caccioppoli sets by

$$\mathcal{C}_\Omega := \{\Omega' \subset \boldsymbol{B}_T \text{ Caccioppoli set} : \Omega' \Delta \Omega \Subset \boldsymbol{B}_T \setminus \{\boldsymbol{0}\}\}. \tag{3-18}$$

Suppose that Ω is a minimizer in the sense that for any $\Omega' \in \mathcal{C}_\Omega$, we have $\mathcal{A}_\Omega^{\hat{H}}(\Omega') \geq 0$; and assume that there is a connected component $\Sigma_0 \subset \partial\Omega$ that is a stable $\hat{H}$-hypersurface,[7] disjoint from $\boldsymbol{0} \in \boldsymbol{B}_T$ and with nonzero-degree projection onto $\boldsymbol{S}^2$. Assume that $\operatorname{dist}_g(\Sigma_0, \partial\Omega \setminus \Sigma_0) > 0$.

Lemma 3.10 (compare to [Gromov 2023, Section 5.7]). *If Assumption 3.9 holds, then*:

(1) *There exists a constant $\tau \in (0, T]$ such that $\Sigma_0 = S_\tau$ with outward normal ∂_t.*

(2) *There exists an open neighborhood $\mathcal{U}$ of $\Sigma_0 = S_\tau$, disjoint from $\partial\Omega \setminus \Sigma_0$, on which $g = \hat{g}$.*

Proof. Since Σ_0 is assumed to be a stable, closed $\hat{H}$-hypersurface, and since $Z_{\hat{H}} \equiv 0$ (see (3-4)), Proposition 3.4 applies and yields (1).

To prove (2), first note that Proposition 3.4 and Lemma 3.8 together imply that a neighborhood $\mathcal{U}$ of Σ_0 is foliated by a normal variation $\{\Sigma_s\}$ ($s \in I$) of Σ_0; moreover, on each leaf Σ_s the difference $H_{\Sigma_s} - \hat{H}$ is a constant k_s. Since $\boldsymbol{0} \notin \Sigma_0$ and $\operatorname{dist}_g(\Sigma_0, \partial\Omega \setminus \Sigma_0) > 0$, $\mathcal{U}$ can be chosen to be disjoint from both $\partial\Omega \setminus \Sigma_0$ and $\boldsymbol{0}$.

For $s_1, s_2 \in I$ with $s_1 < s_2$ define $\Sigma_{[s_1,s_2]} \subset \boldsymbol{B}_T$ to be the (compact) subset with boundary $\Sigma_{s_1} \cup \Sigma_{s_2}$; then consider Ω_s defined by

$$\Omega_s := \begin{cases} \Omega \cup \Sigma_{[0,s]} & \text{if } s \geq 0, \\ \Omega \setminus \Sigma_{[-s,0]} & \text{if } s < 0. \end{cases} \tag{3-19}$$

Clearly, these Ω_s belong to the class $\mathcal{C}_\Omega$. Let us denote $\mathcal{A}_\Omega^{\hat{H}}(\Omega_s)$ by $\mathcal{A}(s)$ for brevity, and write $u_s = \langle \partial_s \phi, \nu_s \rangle > 0$ where ν_s is the (suitably oriented) unit normal along Σ_s. By Lemma 3.8 and the first variation formula,

$$\mathcal{A}'(s) = \int_{\Sigma_s} k_s u_s. \tag{3-20}$$

[7] We allow Σ_0 to overlap with $\partial \boldsymbol{B}_T$.

Since $\mathcal{A}(0)$ attains the minimum, it is necessary that:

(i) Either $\mathcal{A}(s) \equiv 0$ for all $s \geq 0$, or $\mathcal{A}'(s) > 0$ (equivalently, $k_s > 0$) for some $s > 0$.

(ii) Either $\mathcal{A}(s) \equiv 0$ for all $s \leq 0$, or $\mathcal{A}'(s) < 0$ (equivalently, $k_s < 0$) for some $s < 0$.

To complete the proof, it suffices to show that $\mathcal{A}(s) \equiv 0$ for all $s \in I$. If this does not hold, first suppose that $k_s > 0$ for some $s > 0$. Then on the Riemannian band $\Sigma_{[0,s]}$ with $\partial_- = \Sigma_0$ and $\partial_+ = \Sigma_s$ define the function

$$\tilde{\mu}(t) = \hat{H}(t) + \frac{\epsilon}{\sin^3 t}, \tag{3-21}$$

which is smooth and decreasing in t. By choosing sufficiently small ϵ, we can arrange that $\tilde{\mu} > H_{\Sigma_0}$ on Σ_0 and that $\tilde{\mu} < H_{\Sigma_s}$ on Σ_s. Thus, by Lemma 2.3, there exists a $\tilde{\mu}$-bubble $\tilde{\Omega}$ in $\Sigma_{[0,s]}$; in particular, $\tilde{\Sigma} = \partial\tilde{\Omega} \setminus \Sigma_0$ has a component $\tilde{\Sigma}_0$ whose projection to $\boldsymbol{S}^2$ has nonzero degree. However, by a direct calculation using (3-4), we get

$$Z_{\tilde{\mu}}(t) = \frac{3\epsilon^2}{2\sin^6 t} > 0, \tag{3-22}$$

contradicting Proposition 3.4(f).

The case when $k_s < 0$ for some $s < 0$ may be similarly and independently ruled out; it suffices to consider $\Sigma_{[s,0]}$ with $\partial_- = \Sigma_s$ and $\partial_+ = \Sigma_0$ and the following analogue of (3-21): $\tilde{\mu}(t) = \hat{H}(t) - \epsilon \sin^{-3} t$.

Finally, since we have proved that all Ω_s are $\mathcal{A}^{\hat{H}}$-minimizing in the class $\mathcal{C}_\Omega$, each Σ_s must be a t-level set. By Proposition 3.4(d), $g = \hat{g}$ on $\mathcal{U}$, and this completes the proof. □

Proposition 3.11. *If Assumption 3.9 holds, then $g = \hat{g}$ on $\boldsymbol{B}_T$.*

Proof. By Lemma 3.10, $\Sigma_0 = S_\tau$ for some $\tau \in (0, T]$, and its outward normal is ∂_t. Without loss of generality, we assume $\tau \in (0, T)$. Let $I = (t_1, t_2)$ be the maximum open interval containing τ such that $\boldsymbol{B}_{(t_1,t_2)}$ is disjoint from $\partial\Omega \setminus \Sigma_0$ and that $g = \hat{g}$ on $\boldsymbol{B}_{(t_1,t_2)}$. For $t \in I$, let Ω_t denote $\Omega \setminus \boldsymbol{B}_{(t,\tau]}$ if $t < \tau$ and $\Omega \cup \boldsymbol{B}_{[\tau,t]}$ if $t \geq \tau$. In particular, $\partial\Omega_t = (\partial\Omega \setminus \Sigma_0) \cup S_t$.

It suffices to show that $t_1 = 0$ and $t_2 = T$, and we argue by contradiction. First suppose that $t_1 > 0$. Then Ω_{t_1} is in the class $\mathcal{C}_\Omega$, and it satisfies $\mathcal{A}_\Omega^{\hat{H}}(\Omega_{t_1}) = 0$. If S_{t_1} were disjoint from $\partial\Omega \setminus \Sigma_0$, then, by Lemma 3.10, the interval I can be extended further, violating its maximality. On the other hand, if S_{t_1} were to touch a connected component $\Sigma' \subset \partial\Omega \setminus \Sigma_0$, then by smoothness and embeddedness $\partial\Omega_{t_1} \setminus \{\boldsymbol{0}\}$ (see [Zhou and Zhu 2020, Theorem 2.2]), Σ' must be equal to Σ_{t_1} but with the opposite outward normal, violating Proposition 3.4(e). Thus, we conclude that $t_1 = 0$. The proof of $t_2 = T$ is similar. □

With Proposition 3.11, it becomes clear that Theorem 3.1 would follow if one can verify Assumption 3.9. To illustrate this point, we now discuss a special case of Theorem 3.1 which admits a more direct proof. (The general situation is more subtle and will be addressed in the next section.)

Corollary 3.12. *Let $(\boldsymbol{B}_T, \hat{g})$ be the 3-dimensional spherical cap of radius $T \in (0, \pi/2]$. Suppose that g is another Riemannian metric on $\boldsymbol{B}_T$ satisfying $g \geq \hat{g}$ and $R_g \geq R_{\hat{g}}$ on $\boldsymbol{B}_T$; in addition, suppose that $H_g \geq H_{\hat{g}} = 2\cot T$ and $g_{\partial \boldsymbol{B}_T} = \hat{g}_{\partial \boldsymbol{B}_T}$ on $\partial \boldsymbol{B}_T$. Then $g = \hat{g}$.*

Proof. Take $\mu = \hat{H}$, which is in $L^1(\boldsymbol{B}_T)$. Since adding a constant to a functional does not affect its variational properties, we may consider, instead of (2-1),

$$\mathcal{B}^{\mu}(\Omega) := \mathcal{H}^2(\partial\Omega) + \int_{\boldsymbol{B}_T \setminus \Omega} \mu \, d\mathcal{H}^3, \tag{3-23}$$

for all smooth Caccioppoli sets $\Omega \subset \boldsymbol{B}_T$ with $\Omega \Delta \boldsymbol{B}_T \Subset \boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$, and underlying metrics will be specified in subscripts. Since $\hat{H} = \operatorname{div}(\partial_t)$ on $\boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$, we have

$$\mathcal{B}^{\mu}_{\hat{g}}(\Omega) = \mathcal{H}^2_{\hat{g}}(\partial\Omega) - \int_{\partial\Omega} \langle \partial_t, \hat{\nu} \rangle_{\hat{g}} \, d\mathcal{H}^2_{\hat{g}} + \mathcal{H}^2_{\hat{g}}(S_T) \geq \mathcal{H}^2_{\hat{g}}(S_T) = \mathcal{B}^{\mu}_{\hat{g}}(\boldsymbol{B}_T), \tag{3-24}$$

where the first equality is an application of the divergence formula, and the inequality is derived from the relation $\langle \partial_t, \hat{\nu} \rangle_{\hat{g}} \leq 1$. Now, since $g \geq \hat{g}$ and $\mu \geq 0$ on $\boldsymbol{B}_T$ ($T \leq \pi/2$), we have $\mathcal{B}^{\mu}_{g}(\Omega) \geq \mathcal{B}^{\mu}_{\hat{g}}(\Omega)$; moreover, by $g_{\partial \boldsymbol{B}_T} = \hat{g}_{\partial \boldsymbol{B}_T}$, we have $\mathcal{B}^{\mu}_{\hat{g}}(\boldsymbol{B}_T) = \mathcal{B}^{\mu}_{g}(\boldsymbol{B}_T)$. Combining these with (3-24) gives $\mathcal{B}^{\mu}_{g}(\Omega) \geq \mathcal{B}^{\mu}_{g}(\boldsymbol{B}_T)$; and using $H_g \geq 2\cot T = \hat{H}|_{\partial \boldsymbol{B}_T}$, we deduce that $H_g = 2\cot T$ and hence, for any $\phi \in Lip(S_T)$ and $\phi \geq 0$, we have

$$D^2\mathcal{A}(\phi, \phi) := \int_{S_T} |\nabla \phi|^2 + \int_{S_T} (R_{S_T} - R_g - \hat{H}^2 - |\mathrm{II}|^2 - 2\nu(\hat{H}))\phi^2 \geq 0,$$

and then clearly, for all $\varphi \in C^{\infty}(S_T)$ we have

$$D^2\mathcal{A}(\varphi, \varphi) \geq D^2\mathcal{A}(|\varphi|, |\varphi|) \geq 0,$$

hence, S_T is a stable $\hat{H}$-hypersurface. Now it is easy to see that the pair $(\boldsymbol{B}_T, S_T)$ satisfies Assumption 3.9. The conclusion then follows from Proposition 3.11. □

4. Existence of an initial minimizer

Throughout this section, let g be a Riemannian metric on $\boldsymbol{B}_T$ satisfying (3-2). Our goal is to obtain an "initial" minimizer Ω and a connected component $\Sigma_0 \subset \partial\Omega$ which satisfy Assumption 3.9. To achieve this, we consider perturbations μ_ϵ of $\hat{H} = 2\cot t$ (see (4-3)). For each ϵ, we find a Riemannian band $M_\epsilon \subset \boldsymbol{B}_T$ on which μ_ϵ satisfies the barrier condition; thus, a μ_ϵ-bubble Ω_ϵ exists, and $\partial\Omega_\epsilon \cap \mathring{M}_\epsilon$ has a component Σ_0^ϵ which projects onto $\boldsymbol{S}^2$ with nonzero degree. One may wonder whether this "degree" property is preserved in the limit as $\epsilon \to 0$; this led us to find that each Σ_0^ϵ must be disjoint from a fixed open neighborhood of $\mathbf{0} \in \boldsymbol{B}_T$, provided ϵ is small (Proposition 4.7). Then we verify Assumption 3.9 by analyzing the limits of Ω_ϵ and Σ_0^ϵ (Proposition 4.12).

4A. ***A choice of μ_ϵ.*** Let $\epsilon > 0$ be a small constant, and define

$$t_c := \min\{\tfrac{\pi}{4}, \tfrac{T}{2}\}. \tag{4-1}$$

Moreover, we shall fix a function $\beta \in C^\infty((0, T])$ which is strictly decreasing and satisfies

$$\beta(t) = \cot t \text{ on } (0, t_c] \quad \text{and} \quad \beta(T) = -1; \tag{4-2}$$

such a β clearly exists. Now consider the function defined on $(0, T]$:

$$\mu_\epsilon(t) \equiv \hat{H}(t) + \epsilon\beta(t) = 2\cot t + \epsilon\beta(t). \tag{4-3}$$

Writing Z^ϵ for Z_{μ_ϵ}, we have (see (3-4))

$$Z^\epsilon(t) = \tfrac{3}{2}[\epsilon\beta(t)]^2 + 2\epsilon\beta'(t) + 6\epsilon(\cot t)\beta(t), \tag{4-4}$$

and, in particular,

$$Z^\epsilon(t) = \frac{\epsilon}{2\sin^2 t}[(3\epsilon + 12)\cos^2 t - 4] > 0 \quad \text{for } t \in (0, t_c]. \tag{4-5}$$

Moreover, by (4-4), it is clear that there exists a constant $b_0 > 0$, depending only on β, such that

$$Z^\epsilon(t) \geq -\epsilon\, b_0 \quad \text{for } t \in (0, T]. \tag{4-6}$$

4B. ***Existence of a μ_ϵ-bubble.*** Let $S(r, g)$ (resp., $B(r, g)$) denote the geodesic sphere (resp., open geodesic ball) of radius r, taken with respect to the metric g and centered at $\mathbf{0} \in \boldsymbol{B}_T$. An asymptotic expansion of the mean curvature function (see Lemma 3.4 of [Fan et al. 2009]) gives: for small $r > 0$ and all $q \in S(r, g)$,

$$H_{S(r,g)}(q) = \frac{2}{r} + O(r), \quad \hat{H}(q) = \frac{2}{t(q)} + O(t(q)). \tag{4-7}$$

Since $g \geq \hat{g}$, we have $r \geq t(q)$; then by (4-3) and (4-2), as long as $r < t_c$, we have

$$\mu_\epsilon(t(q)) = \frac{2+\epsilon}{t(q)} + O(t(q)) \geq \frac{2+\epsilon}{r} + O(t(q)), \quad q \in S(r, g). \tag{4-8}$$

It is now clear that there exists an $r_\epsilon < \epsilon$ such that $\mu_\epsilon > H_{S(r_\epsilon, g)}$ on $S(r_\epsilon, g)$. On the other hand, we have $H_g \geq 2\cot T > \mu_\epsilon(T)$ on S_T, where the first inequality is part of (3-2), and the second inequality is due to the choice of μ_ϵ and β. Therefore, μ_ϵ satisfies the barrier condition (see Definition 2.2) applied to the Riemannian band

(M_ϵ, g), where $M_\epsilon = \boldsymbol{B}_T \setminus B(r_\epsilon, g)$, with the distinguished boundaries: $\partial_- = S(r_\epsilon, g)$ and $\partial_+ = S_T$. The lemma below follows directly from Lemma 2.3.

Lemma 4.1. *In the Riemannian band $(M_\epsilon, g; S(r_\epsilon, g), S_T)$ there exists a minimal μ_ϵ-bubble Ω_ϵ; moreover, $\partial\Omega_\epsilon \setminus S(r_\epsilon, g)$ is disjoint from S_T, and it has a connected component Σ_0^ϵ whose projection onto $\boldsymbol{S}^2$ has nonzero degree.*

Lemma 4.2. *$\Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_c,T]}$ is nonempty.*

Proof. Otherwise, $Z^\epsilon > 0$ on Σ_0^ϵ, which contradicts Proposition 3.4(f). □

4C. ***A "no-crossing" property of $\boldsymbol{\Sigma_0^\epsilon}$.*** From now on, let $t_* \in (0, t_c)$ be fixed. We will begin by assuming that $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*}$ were nonempty; consequences of this hypothesis will be developed progressively with three lemmas (Lemmas 4.3, 4.5 and 4.6). Based on these lemmas, we prove that Σ_0^ϵ must be disjoint from $\boldsymbol{B}_{t_*}$ for small enough ϵ (Proposition 4.7).

In the following, let $\hat{\nu}$ denote the outward-pointing unit normal on Σ_0^ϵ with respect to $\hat{g}$, and let Φ denote the projection map from Σ_0^ϵ to $\boldsymbol{S}^2$; see (3-6).

Lemma 4.3. *If $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*}$ were nonempty, then there would exist a point $q \in \Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*,T]}$ such that the angle $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [\alpha, \pi - \alpha]$ at q, where*

$$\alpha = \min\left\{\arctan\left(\frac{t_c - t_*}{2\pi}\right), \frac{\pi}{4}\right\}. \tag{4-9}$$

Proof. We argue by contradiction, so let us assume that $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [0, \alpha) \cup (\pi - \alpha, \pi]$ everywhere on $\Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*,T]}$. Because Σ_0^ϵ is connected and intersects both S_{t_*} (by assumption) and S_{t_c} (by Lemma 4.2), the image of $t|_{\Sigma_0^\epsilon}$ contains the interval $[t_*, t_c]$.

Let $t' \in (t_*, t_c)$ be a regular value of $t|_{\Sigma_0^\epsilon}$ that is sufficiently close to t_*. Because Σ_0^ϵ is connected, there exists a connected component $\mathcal{E} \subset \Sigma_0^\epsilon \cap \boldsymbol{B}_{(t',t_c]}$ whose closure $\bar{\mathcal{E}}$ intersects both $S_{t'}$ and S_{t_c}. On $\mathcal{E}$, the angle $\angle_{\hat{g}}(\hat{\nu}, \partial_t)$ can only take value in one of the intervals $[0, \alpha)$ and $(\pi - \alpha, \pi]$, but not both. Without loss of generality, let us assume that $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [0, \alpha)$ on $\mathcal{E}$.

Since t' is a regular value of $t|_{\Sigma_0^\epsilon}$, $\bar{\mathcal{E}}$ meets $S_{t'}$ transversely. In particular, $\mathscr{C} := \bar{\mathcal{E}} \cap S_{t'}$ is a disjoint union of finitely many circles. It is easy to see that $S_{t'} \setminus \mathscr{C} = U_1 \cup U_2$ for some open subsets $U_i \subset S_{t'}$ with $\partial U_i = \mathscr{C}$ $(i = 1, 2)$.

Both U_i and $\mathcal{E}$ are oriented, and the orientations are associated to the respective normal directions, ∂_t and $\hat{\nu}$, by the right-hand rule. The orientation on $\mathscr{C}$ induced by $\mathcal{E}$ must completely agree with that induced by either U_1 or U_2; otherwise, gluing $\mathcal{E}$ with either U_1 or U_2 along $\mathscr{C}$ and smoothing would yield a nonorientable closed surface embedded in $\boldsymbol{B}_T$, which is impossible.

Thus, we can assume that U_1 and $\mathcal{E}$ induce *opposite* orientations on $\mathscr{C}$. Since $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [0, \alpha)$ on $\mathcal{E}$, it is easy to see that the restriction of Φ to $\bar{\mathcal{E}} \cup U_1$ is a local homeomorphism to $\boldsymbol{S}^2$. Since $\bar{\mathcal{E}} \cup U_1$ is compact, $\Phi|_{\bar{\mathcal{E}} \cup U_1}$ is a covering map;

this map must be a homeomorphism, since $\boldsymbol{S}^2$ is simply connected and $\bar{\mathcal{E}} \cup U_1$ is connected.

Pick any $x \in \mathcal{E} \cap S_{t_c}$. Choose a shortest (regular) curve $\Gamma : [0, 1] \to \Phi(\bar{\mathcal{E}})$ connecting $\Gamma(0) = \Phi(x)$ and $\partial(\Phi(\mathcal{E}))$; in particular,

$$\text{length}_{g_{S^2}}(\Gamma) \le \pi. \tag{4-10}$$

Now let $\gamma = (\Phi|_{\bar{\mathcal{E}}})^{-1} \circ \Gamma$, and write its tangent vectors γ' as the sum of γ'_N (parallel to ∂_t) and γ'_T (tangent to t-level sets). By $\hat{g} \le g_{S^2} + \mathrm{d}t^2$ and the hypothesis $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in [0, \alpha) \cup (\pi - \alpha, \pi]$, we obtain the estimate

$$|\gamma'_N|_{\hat{g}} \le (\tan\alpha)|\gamma'_T|_{\hat{g}} \le (\tan\alpha)|\mathrm{d}\Phi(\gamma')|_{g_{S^2}}. \tag{4-11}$$

Hence,

$$t_c - t' \le \int_\gamma |\gamma'_N|_{\hat{g}} \le (\tan\alpha) \cdot \text{length}_{g_{S^2}}(\Phi(\gamma)) \le \pi \tan\alpha \le \tfrac{1}{2}(t_c - t_*), \tag{4-12}$$

where the first inequality holds because $\gamma(0) \in S_{t_c}$ and $\gamma(1) \in S_{t'}$; the second and third inequalities are due to (4-11) and (4-10), respectively; the last inequality holds by the choice of α. Since t' is close to t_*, (4-12) is a contradiction. □

Corollary 4.4. *In Lemma 4.3 we can choose q such that*: $\angle_{\hat{g}}(\hat{\nu}, \partial_t) = \alpha$ *or* $\pi - \alpha$ *at q.*

Proof. In Σ_0^ϵ there exists a point at which t attains global maximum. At that point $\hat{\nu} = \pm\partial_t$. Thus, by continuity of angle, there exists a point $q \in \Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*, T]}$ at which the angle between $\hat{\nu}$ and ∂_t is equal to either α or $\pi - \alpha$. □

Lemma 4.5. *Let α be defined by* (4-9). *If $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*}$ were nonempty, then there would exist a constant* $S = S(g, \hat{g}, \beta, t_*) > 0$, *independent of ϵ, and an open subset $U_\epsilon \subset \Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*/2, T]}$ such that*:

(1) *At each point $q \in U_\epsilon$, $\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in (\alpha/2, 2\alpha) \cup (\pi - 2\alpha, \pi - \alpha/2)$.*

(2) $\int_{U_\epsilon} |\Phi^* \, d\sigma_{S^2}| \ge S$.

Proof. To begin with, let q be as in Corollary 4.4. For any unit tangent vector X (with respect to $\hat{g}$) of Σ_0^ϵ, we have

$$|X\langle \hat{\nu}, \partial_t\rangle_{\hat{g}}| = |\langle \hat{\nabla}_X \hat{\nu}, \partial_t\rangle_{\hat{g}} + \langle \hat{\nu}, \hat{\nabla}_X \partial_t\rangle_{\hat{g}}| \le |\hat{\mathrm{II}}|_{\hat{g}} + |\hat{\nabla}\partial_t|_{\hat{g}}. \tag{4-13}$$

where $\hat{\nabla}$ is the connection of $\hat{g}$. It is clear that there exists a constant $C = C(\hat{g}, t_*)$ such that $|\hat{\nabla}\partial_t|_{\hat{g}} \le C$ on $\boldsymbol{B}_{[t_*/2, T]}$. Moreover, by applying Lemma 2.9 (if necessary, extend g to a smooth metric on $\boldsymbol{B}_{T+\delta_0}$ for some fixed $\delta_0 > 0$, and let $V = \boldsymbol{B}_{(t_*/2, T+\delta_0)}$) and by comparing between $|\mathrm{II}|_g$ and $|\hat{\mathrm{II}}|_{\hat{g}}$, it is not difficult to see that there exists a constant $C' = C'(g, \hat{g}, \beta, t_*)$ such that $|\hat{\mathrm{II}}|_{\hat{g}} \le C'$ on $\Sigma_0^\epsilon \cap \boldsymbol{B}_{[t_*/2, T]}$ for all sufficiently small ϵ. Thus, there exists a constant $\rho = \rho(g, \hat{g}, \beta, t_*) > 0$ such that on the geodesic ball

$$U_\epsilon := \{x \in \Sigma_0^\epsilon : \mathrm{dist}_{\hat{g}_{\Sigma_0^\epsilon}}(x, q) \le \rho\}$$

we have

$$\angle_{\hat{g}}(\hat{\nu}, \partial_t) \in \left(\tfrac{\alpha}{2}, 2\alpha\right) \cup \left(\pi - 2\alpha, \pi - \tfrac{\alpha}{2}\right). \tag{4-14}$$

It is easy to see that $\Phi(U_\epsilon)$ contains a ball $\mathscr{B}$ of radius $\cos(2\alpha)\rho$ in $\boldsymbol{S}^2$. The proof is complete by taking $S := \mathrm{area}_{g_{\boldsymbol{S}^2}}(\mathscr{B})$. ☐

Lemma 4.6. *If $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*}$ were nonempty, then we would have*

$$\int_{\Sigma_0^\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} - 2\int_{\Sigma_0^\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}| \ge A_0 \tag{4-15}$$

for some positive constant A_0 that is independent of ϵ.

Proof. Up to sign, the area form $d\sigma_{\hat{g}}$ induced by $\hat{g}$ on each tangent space of Σ_0^ϵ is equal to

$$\frac{1}{\cos(\angle_{\hat{g}}(\hat{\nu}, \partial_t))}\varphi^2 \Phi^* d\sigma_{\boldsymbol{S}^2}$$

provided that $\hat{\nu}$ is not orthogonal to ∂_t. Thus, by Lemma 4.5, we have

$$\begin{aligned} \int_{U_\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} &\ge \int_{U_\epsilon} \frac{2}{\varphi^2} \frac{1}{\cos(\alpha/2)} \varphi^2 |\Phi^*\, d\sigma_{\boldsymbol{S}^2}| \\ &\ge 2S\left(\frac{1}{\cos(\alpha/2)} - 1\right) + 2\int_{U_\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}|. \end{aligned} \tag{4-16}$$

On the other hand, by Lemma 3.3,

$$\int_{\Sigma_0^\epsilon \setminus U_\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} \ge 2\int_{\Sigma_0^\epsilon \setminus U_\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}|. \tag{4-17}$$

Adding (4-16) with (4-17) and rearranging terms, we get

$$\int_{\Sigma_0^\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} - 2\int_{\Sigma_0^\epsilon} |\Phi^*\, d\sigma_{\boldsymbol{S}^2}| \ge 2S\left(\frac{1}{\cos(\alpha/2)} - 1\right). \tag{4-18}$$

The proof is complete by taking A_0 to be the right-hand side of (4-18). ☐

Proposition 4.7. *For sufficiently small ϵ, Σ_0^ϵ must be disjoint from the set $\boldsymbol{B}_{t_*} \subset \boldsymbol{B}_T$.*

Proof. By (4-6) and the proof of Lemma 3.2, we obtain

$$R_+^{\mu_\epsilon} \ge \frac{2}{\varphi^2} - 2b_0\epsilon \quad \text{on } \Sigma_0^\epsilon. \tag{4-19}$$

For small ϵ, Remark 3.6 and Lemma 3.5 imply that Σ_0^ϵ is homeomorphic to $\boldsymbol{S}^2$. Moreover, since Ω_ϵ is a μ_ϵ-bubble, the area of Σ_0^ϵ with respect to g has an upper bound $C_0 > 0$, which can be chosen to depend only on the metric g and not on ϵ.

Now suppose that $\Sigma_0^\epsilon \cap \boldsymbol{B}_{t_*} \neq \varnothing$. Then from (4-19), (3-13) and (4-15), we obtain

$$\int_{\Sigma_0^\epsilon} R_+^{\mu_\epsilon}\, d\sigma_g \geq \int_{\Sigma_0^\epsilon} \frac{2}{\varphi^2}\, d\sigma_{\hat{g}} - 2\epsilon b_0 C_0 \geq (A_0 - 2\epsilon b_0 C_0) + 2\int_{\Sigma_0^\epsilon} |\Phi^* \, d\sigma_{\boldsymbol{S}^2}|. \tag{4-20}$$

For small enough ϵ, $A_0 > 2\epsilon b_0 C_0$; by stability of Σ_0^ϵ, the analogue of (3-12) reads

$$4\pi\chi(\boldsymbol{S}^2) = \int_{\Sigma_0^\epsilon} R_{\Sigma_0^\epsilon}\, d\sigma_g \geq \int_{\Sigma_0^\epsilon} R_+^{\mu_\epsilon}\, d\sigma_g > 2\int_{\Sigma_0^\epsilon} |\Phi^* \, d\sigma_{\boldsymbol{S}^2}| \geq 8\pi; \tag{4-21}$$

a contradiction. □

Remark 4.8. There is another way to get (4-21), which does not rely on the assumption of an upper bound C_0 of $\text{area}_g(\Sigma_0^\epsilon)$ but does rely on the fact that $\varphi \leq 1$. In fact, (4-19) implies that $R_+^{\mu_\epsilon} \geq 2\varphi^{-2}(1 - b_0\epsilon)$, and again by (3-13), (4-15) and the degree assumption we have

$$\int_{\Sigma_0^\epsilon} R_+^{\mu_\epsilon}\, d\sigma_g \geq (1 - b_0\epsilon)\Big(A_0 + 2\int_{\Sigma_0^\epsilon} |\Phi^* \, d\sigma_{\boldsymbol{S}^2}|\Big) \geq (1 - b_0\epsilon)(A_0 + 8\pi) > 8\pi$$

for small enough ϵ.

4D. ***Existence of a minimizer.*** Let M_ϵ, Ω_ϵ and Σ_0^ϵ be as in Lemma 4.1. We now study how Ω_ϵ and Σ_0^ϵ behave as $\epsilon \to 0$.

Recall from (4-1) the definition of t_c, and let $t_* \in (0, t_c)$ be fixed. By considering small enough ϵ, we can assume Σ_0^ϵ to be homeomorphic to $\boldsymbol{S}^2$ and disjoint from $\boldsymbol{B}_{t_*}$.

For a fixed ϵ, since Σ_0^ϵ is disjoint from S_T, the Jordan–Brouwer separation theorem applies. As a result, $\boldsymbol{B}_T \setminus \Sigma_0^\epsilon$ has exactly two connected components, say $\mathcal{U}_-^\epsilon$ and $\mathcal{U}_+^\epsilon$. Without loss of generality, let us assume that ν points away from $\mathcal{U}_-^\epsilon$ along Σ_0^ϵ. Given any constant $\delta > 0$, let us define

$$\begin{aligned} W_{-\delta}^\epsilon &:= \{x \in \mathcal{U}_-^\epsilon : \text{dist}_g(x, \Sigma_0^\epsilon) \leq \delta\}, \\ W_{+\delta}^\epsilon &:= \{x \in \mathcal{U}_+^\epsilon : \text{dist}_g(x, \Sigma_0^\epsilon) \leq \delta\}, \end{aligned} \tag{4-22}$$

where distance is taken in $(\boldsymbol{B}_T, g)$.

Lemma 4.9. *There exists a constant $\delta > 0$, independent of ϵ, such that for all small enough ϵ we have $W_{-\delta}^\epsilon \subset \mathring{\Omega}_\epsilon$ and $W_{+\delta}^\epsilon \cap \Omega_\epsilon = \varnothing$.*

Proof. Since in $\boldsymbol{B}_{[t_*/2, T]}$ all derivatives of μ_ϵ are uniformly bounded, it follows from Lemma 2.9 that the norm of the second fundamental form of $\partial\Omega_\epsilon \cap \boldsymbol{B}_{[t_*/2, T]}$ is also uniformly bounded. If some other component Σ' in $\partial\Omega_\epsilon$ were to get arbitrarily close to Σ_0^ϵ, then a suitable surgery (i.e., a connected sum of Σ_0^ϵ and Σ'

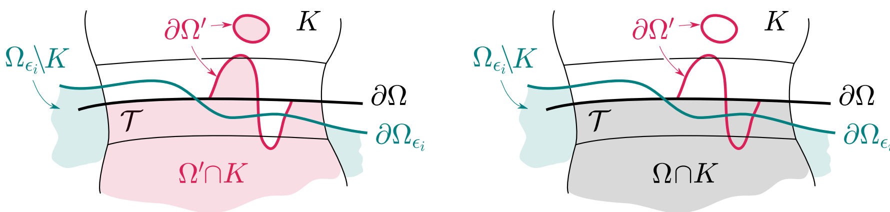

Figure 1. The shaded regions represent Ω'_{ϵ_i} (left figure) and $\Omega^*_{\epsilon_i}$ (right figure).

performed within M_ϵ) would yield a Caccioppoli set that has strictly less brane action, contradicting the minimality of Ω_ϵ. $\square$

Now we fix a sequence $\{\epsilon_i\} \to 0$ and corresponding sequences of Ω_{ϵ_i} and $\Sigma_0^{\epsilon_i}$.

Lemma 4.10. *The sequence $\{\Omega_{\epsilon_i}\}$ subconverges to a Caccioppoli set $\Omega \subset \boldsymbol{B}_T$ where convergence is interpreted via the characteristic functions with respect to the L^1_{loc}-norm. Moreover*:

(1) *$\partial\Omega \setminus \{\boldsymbol{0}\}$ is smooth and embedded.*

(2) *Ω is a minimizer in the sense that $\mathcal{A}^{\hat{H}}_{\Omega}(\Omega') \geq 0$ for any Caccioppoli set Ω' with $\Omega'\Delta\Omega \Subset \boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$.*

Proof. The existence of a convergent subsequence and that of Ω follow from standard theory of BV functions (see [Giusti 1984, Theorem 1.20]), and let us replace $\{\Omega_{\epsilon_i}\}$ by that subsequence.

Now let $K \subset \boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$ be any compact domain. For sufficiently large i, the second fundamental form of $\partial\Omega_{\epsilon_i} \cap K$ has a uniform upper bound, and thus $\partial\Omega_{\epsilon_i} \cap K$ subconverges to a smooth hypersurface $\mathcal{S} \subset K$ in the graph sense. By using Lemma 4.9, it is easy to see that $\mathcal{S}$ is embedded and $\mathcal{S} = \partial\Omega \cap K$. Since K is arbitrary, we conclude (1).

To show that Ω is a minimizer, we argue by contradiction. Suppose that there exists a Caccioppoli set Ω' and a constant $c > 0$ such that $\Omega'\Delta\Omega \Subset \boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$ and $\mathcal{A}^{\hat{H}}_{\Omega}(\Omega') \leq -c < 0$. Let us choose a compact domain $K \subset \boldsymbol{B}_T \setminus \{\boldsymbol{0}\}$ with smooth boundary such that $\Omega'\Delta\Omega \Subset \mathring{K}$. Consider a thin tubular neighborhood $\mathcal{T}$ of $\partial\Omega \cap K$ that is generated by the unit normal field along $\partial\Omega \cap K$; as $\mathcal{T}$ is diffeomorphic to $(\partial\Omega \cap K) \times I$ for some interval I, we may modify K such that the image of $(\partial\Omega \cap \partial K) \times I$ is equal to $\partial\mathcal{T} \cap \partial K$ (in particular, $\partial\Omega$ is transversal to ∂K). Note that for large i, $S(r_{\epsilon_i}, g)$ would be disjoint from K, and $\partial\Omega_{\epsilon_i} \cap K$ would be completely contained in $\mathcal{T}$.

Now consider the following Caccioppoli sets (see Figure 1):

$$\Omega'_{\epsilon_i} := (\Omega_{\epsilon_i} \setminus K) \cup (\Omega' \cap K), \qquad \Omega^*_{\epsilon_i} := (\Omega_{\epsilon_i} \setminus K) \cup (\Omega \cap K). \tag{4-23}$$

We claim that, for sufficiently large i,

$$\mathcal{A}^{\mu_{\epsilon_i}}_{\Omega_{\epsilon_i}}(\Omega^*_{\epsilon_i}) \le \frac{c}{4}. \tag{4-24}$$

To see this, note that $\chi_{\Omega^*_{\epsilon_i}} - \chi_{\Omega_{\epsilon_i}}$ is just $\chi_{\Omega_{\epsilon_i}\cap K} - \chi_{\Omega\cap K}$; since $\mu_{\epsilon_i}|_K$ is uniformly bounded and $\chi_{\Omega_{\epsilon_i}} \to \chi_\Omega$ in L^1, we have

$$\int_{\boldsymbol{B}_T} (\chi_{\Omega^*_{\epsilon_i}} - \chi_{\Omega_{\epsilon_i}})\mu_{\epsilon_i} \to 0 \quad (i \to \infty). \tag{4-25}$$

Moreover, it is easy to see that

$$\mathcal{H}^2(\partial\Omega^*_{\epsilon_i}) - \mathcal{H}^2(\partial\Omega_{\epsilon_i}) \le [\mathcal{H}^2(\partial\Omega\cap K) - \mathcal{H}^2(\partial\Omega_{\epsilon_i}\cap K)] + \mathcal{H}^2(\partial\mathcal{T}\cap\partial K). \tag{4-26}$$

Thus, by graph convergence of $\partial\Omega_{\epsilon_i}\cap K$, we can choose $\mathcal{T}$ and i such that

$$\mathcal{H}^2(\partial\Omega^*_{\epsilon_i}) - \mathcal{H}^2(\partial\Omega_{\epsilon_i}) \le \frac{c}{8}. \tag{4-27}$$

On combining (4-25) and (4-27), we obtain (4-24) for large i.

Now, since $\mu_{\epsilon_i} \to \mu$ in $L^1(K)$ and $\Omega'_{\epsilon_i}\Delta\Omega^*_{\epsilon_i} = \Omega\Delta\Omega' \Subset \mathring{K}$, we have, for sufficiently large i,

$$\mathcal{A}^{\mu_{\epsilon_i}}_{\Omega^*_{\epsilon_i}}(\Omega'_{\epsilon_i}) \le -\frac{c}{2}. \tag{4-28}$$

On comparing (4-24) and (4-28), we get $\mathcal{A}^{\mu_{\epsilon_i}}_{\Omega_{\epsilon_i}}(\Omega'_{\epsilon_i}) \le -c/4 < 0$, contradicting the minimality of Ω_{ϵ_i}. This proves (2). □

Lemma 4.11. *Let Ω be as in Lemma 4.10. The sequence $\{\Sigma_0^{\epsilon_i}\}$ subconverges to a smooth, closed stable $\hat{H}$-hypersurface $\Sigma_0 \subset \boldsymbol{B}_{[t_*,T]}$, which is a t-level set in $\boldsymbol{B}_T$; moreover, $\Sigma_0 \subset \partial\Omega$ and $\partial\Omega\setminus\Sigma_0 \Subset \boldsymbol{B}_T\setminus\Sigma_0$.*

Proof. By our choice of $\{\epsilon_i\}$, all $\Sigma_0^{\epsilon_i}$ are contained in the compact set $\boldsymbol{B}_{[t_*,T]}$ and have a uniform upper bound on their second fundamental form. Thus, by standard minimal surface theory (see [Colding and Minicozzi 2011, Proposition 7.14]), $\{\Sigma_0^{\epsilon_i}\}$ subconverges to a smooth closed hypersurface Σ_0 whose projection onto $\boldsymbol{S}^2$ has nonzero degree. Now recall that each $\Sigma_0^{\epsilon_i}$ is a stable μ_{ϵ_i}-hypersurface. Since all derivatives of μ_{ϵ_i} respectively and uniformly converge to those $\hat{H}$, by passing stability to limit, Σ_0 is a stable $\hat{H}$-hypersurface; hence, Σ_0 is a t-level set, by Proposition 3.4.

To see that $\Sigma_0 \subset \partial\Omega$, first suppose that $\Sigma_0 \ne S_T$; in this case, it suffices to show that each open neighborhood of any $x \in \Sigma_0$ must intersect both $\mathring{\Omega}$ and $\boldsymbol{B}_T\setminus\Omega$, and this can be easily deduced from Lemma 4.9. The case of $\Sigma_0 = S_T$ is similar. Also by Lemma 4.9, Σ_0 has a tubular neighborhood that is disjoint from all other components of $\partial\Omega$, hence $\operatorname{dist}_g(\Sigma_0, \partial\Omega\setminus\Sigma_0) > 0$. □

On combining Lemmas 4.10 and 4.11, we immediately get the following.

Proposition 4.12. *Let g be a Riemannian metric on $\boldsymbol{B}_T$ satisfying* (3-2). *Then there exists a Caccioppoli set $\Omega \subset \boldsymbol{B}_T$ and a connected component $\Sigma_0 \subset \partial\Omega$ that satisfy Assumption 3.9.*

Theorem 3.1 follows directly from Propositions 3.11 and 4.12.

5. Generalizations

In this section we discuss a few variants of Theorem 3.1.

To begin with, we consider a version of Gromov's rigidity theorem for the doubly punctured sphere (see [Gromov 2023, Sections 5.5 and 5.7]), restricted to the 3-dimensional case.

Theorem 5.1. *Let $(\boldsymbol{S}^3 \setminus \{O, O'\}, \hat{g})$ be the standard 3-sphere with a pair of antipodal points removed, and let $h \geq 1$ be a smooth function on $\boldsymbol{S}^3 \setminus \{O, O'\}$. Suppose that g is another Riemannian metric on $\boldsymbol{S}^3 \setminus \{O, O'\}$ satisfying*

$$g \geq h^4 \hat{g} \quad \text{and} \quad R_g \geq h^{-2} R_{\hat{g}}. \tag{5-1}$$

Then $h \equiv 1$, and $g = \hat{g}$.

Proof. For convenience, let us use slightly different notations than those introduced at the beginning of Section 3 by representing $\boldsymbol{S}^3 \setminus \{O, O'\}$ as $\mathbb{B}_{(-\pi/2,\pi/2)} \cong \boldsymbol{S}^2 \times (-\pi/2, \pi/2)$ with t being the coordinate on $(-\pi/2, \pi/2)$. Under this representation we have $\varphi(t) = \cos t$ and

$$\hat{H}(t) = -2\tan t \tag{5-2}$$

instead of (3-3). Now for $\alpha \in (0, \pi/2)$ sufficiently close to $\pi/2$, consider the Riemannian band $\mathscr{B}_\alpha := (\mathbb{B}_{[-\alpha,\alpha]}, g; S_{-\alpha}, S_\alpha)$ and the functions

$$t_\alpha = \frac{t}{\alpha} \cdot \frac{\pi}{2} \quad \text{and} \quad \mu_\alpha = -2\tan t_\alpha \text{ on } \mathbb{B}_{(-\alpha,\alpha)}, \tag{5-3}$$

and consider the problem of finding μ_α-bubbles in $\mathscr{B}_\alpha$. Since $\mu_\alpha \to \pm\infty$ as $t \to \mp\alpha$, μ_α satisfies the barrier condition; thus, there exists a μ_α-bubble $\Omega_\alpha \subset \mathscr{B}_\alpha$, which satisfies analogous properties as described in Lemma 4.1. Let Σ_0^α be a connected component of $\partial\Omega_\alpha \setminus S_{-\alpha}$ whose projection to $\boldsymbol{S}^2$ has nonzero degree; Σ_0^α is a stable μ_α-hypersurface, on which

$$\begin{aligned} R_+^{\mu_\alpha} &= R_g + \tfrac{3}{2}(\mu_\alpha)^2 - 2|\mathrm{d}\mu_\alpha|_g \\ &\geq \frac{1}{h^2}\left(\frac{R_{S^2}}{\varphi^2} - \frac{3}{2}\hat{H}^2 + 2|\mathrm{d}\hat{H}|_{\hat{g}}\right) + \frac{3}{2}(\mu_\alpha)^2 - \frac{2}{h^2}|\mathrm{d}\mu_\alpha|_{\hat{g}} \\ &\geq \frac{1}{h^2}\left(\frac{R_{S^2}}{\varphi^2} + Z_{\mu_\alpha}\right) \end{aligned} \tag{5-4}$$

where the last step follows from the assumption $h \geq 1$ and the definition

$$Z_{\mu_\alpha} := \tfrac{3}{2}(\mu_\alpha^2 - \hat{H}^2) + 2(\partial_t \mu_\alpha - \partial_t \hat{H}).$$

By a careful estimate of Z_{μ_α} using the mean value theorem, it is not difficult to show that there exists a constant $t_c > 0$ such that

$$Z_{\mu_\alpha} > 0 \quad \text{for } t \in (-\alpha, -t_c) \cup (t_c, \alpha) \quad \text{and} \quad \varphi^2 Z_{\mu_\alpha} \geq C(\alpha) \quad \text{for } t \in (-\alpha, \alpha), \tag{5-5}$$

where $C(\alpha) < 0$ is a constant depending only on α and satisfies $C(\alpha) \to 0$ as $\alpha \to \pi/2$. Similar to the proof of Proposition 4.7, here (5-5) implies that Σ_0^α is contained in a fixed compact domain in $\mathbb{B}_{(-\pi/2,\pi/2)}$ that is independent of the choice of α. Thus, as $\alpha \to \pi/2$, such Σ_0^α subconverge to a stable $\hat{H}$-hypersurface, and an analogue of Proposition 4.12 can be obtained. An analogue of Proposition 3.4 and a foliation argument yield that $h \equiv 1$ and $g = \hat{g}$. ☐

Remark 5.2. The assumption $h \geq 1$ is important for Theorem 5.1 to hold. Without this assumption, one may let $g = \cos^2 t(\mathrm{d}t^2 + g_{S^2}) \neq \hat{g}$ on $S^3 \setminus \{O, O'\} \cong S^2 \times (-\pi/2, \pi/2)$ and take $h = (\cos t)^{1/2}$, and it is easy to check that (5-1) is satisfied — in particular, $R_g = (2 + 4\cos^2 t)(\cos t)^{-4}$ and $h^{-2} R_{\hat{g}} = 6(\cos t)^{-1}$, so $R_g \geq h^{-2} R_{\hat{g}}$.

Theorem 3.1 has Euclidean and hyperbolic analogues. Putting together, let us take

$$\hat{g}_\kappa = \varphi_\kappa(t)^2 g_{S^2} + \mathrm{d}t^2 \quad \text{on } \boldsymbol{B}_T \tag{5-6}$$

where

$$\varphi_\kappa(t) = \begin{cases} \sin \sqrt{\kappa} t, & \kappa > 0, \\ t, & \kappa = 0, \\ \sinh \sqrt{-\kappa} t, & \kappa < 0, \end{cases}$$

and $T \in (0, \pi/\sqrt{\kappa})$ if $\kappa > 0$; $T > 0$ if $\kappa \leq 0$. In particular, $\sec(\hat{g}_\kappa) = \kappa$, and $\hat{H}_\kappa(t) = 2\varphi_\kappa'(t)/\varphi_\kappa(t)$.

Theorem 5.3. *Let $\boldsymbol{B}_T$, $\hat{g}_\kappa$ be as above. Let g be a Riemannian metric on $\boldsymbol{B}_T$ satisfying*

$$g \geq h^4 \hat{g}_\kappa, \quad R_g \geq h^{-2} R_{\hat{g}_\kappa}, \quad H_{\partial \boldsymbol{B}_T} \geq \hat{H}_\kappa(T),$$

for some smooth function $h \geq 1$ defined on $\boldsymbol{B}_T$. Then $h \equiv 1$, and $g = \hat{g}_\kappa$.

As pointed out by Gromov [2023, Section 5.5], a key fact that allows the different cases (corresponding to different choices of κ) in Theorem 5.3 to be treated similarly is that the function $\varphi_\kappa(t)$ is "log-concave" — in other words, $\hat{H}_\kappa(t)$ is strictly decreasing in t; see Lemma 3.2 and Proposition 3.4. Having this in mind, the proof proceeds as that of either Theorem 3.1 or 5.1, and we leave the details to the interested reader.

Remark 5.4. When $\kappa \leq 0$ and $T = +\infty$, whether Theorem 5.3 holds remains unknown to us.

Acknowledgements

We would like to thank Prof. Weiping Zhang for his interest in this work; we also thank Dr. Jintian Zhu and Dr. Tin Yau Tsang for helpful conversations and suggestions. Y. Shi and P. Liu are grateful for the support of the National Key R&D Program of China Grant 2020YFA0712800, and Y. Hu is grateful for the support of the China Postdoctoral Science Foundation Grant 2021TQ0014.

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Received May 27, 2022. Revised January 23, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.115

THE DEFORMATION SPACE OF DELAUNAY TRIANGULATIONS OF THE SPHERE

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We determine the topology of the spaces of convex polyhedra inscribed in the unit 2-sphere and the spaces of strictly Delaunay geodesic triangulations of the unit 2-sphere. These spaces can be regarded as discretized groups of diffeomorphisms of the unit 2-sphere. Hence, it is natural to conjecture that these spaces have the same homotopy types as those of their smooth counterparts. The main result of this paper confirms this conjecture for the unit 2-sphere. It follows from an observation on the variational principles on triangulated surfaces developed by I. Rivin.

On the contrary, the similar conjecture does not hold in the cases of flat tori and convex polygons. We will construct simple examples of flat tori and convex polygons such that the corresponding spaces of Delaunay geodesic triangulations are not connected.

1. Introduction

One of the fundamental problems in low dimensional topology is to identify the homotopy types of groups of diffeomorphisms of a smooth manifold. Smale [1959] proved that the group of orientation preserving diffeomorphisms of the 2-sphere is homotopy equivalent to SO(3).

This paper studies two types of finite dimensional spaces which could be considered as discrete analogues of the group of orientation preserving diffeomorphisms of the 2-sphere. They are the deformation spaces of Delaunay triangulations of the unit 2-sphere and the deformation spaces of convex polyhedra inscribed in the unit 2-sphere. The main results of this paper show that these discrete analogues are homotopy equivalent to SO(3).

Theorem 1.1. *The deformation space of Delaunay triangulations of the unit* 2-*sphere is homeomorphic to* $\mathrm{SO}(3) \times \mathbb{R}^k$ *for some* $k > 0$.

The authors were supported in part by NSF 1737876, NSF 1760471, NSF DMS FRG 1760527 and NSF DMS 1811878.

MSC2020: 05C62, 58D10.

Keywords: geodesic triangulations, angle structures, Delaunay triangulations, spaces of embeddings.

Theorem 1.2. *The deformation space of the convex polyhedra inscribed in the unit* 2*-sphere whose faces are all triangles is homeomorphic to* $\mathrm{SO}(3) \times \mathbb{R}^k$ *for some* $k > 0$.

However, we will construct explicit examples of spaces of Delaunay triangulations of convex polygons and flat tori which have different homotopy types from their smooth counterparts. Specifically, we show the spaces of Delaunay triangulations of some flat tori and spaces of Delaunay triangulations of some convex polygons are not connected.

Let $T = (V, E, F)$ denote a 2-dimensional simplicial complex, where V is the set of vertices, E is the set of edges, and F is the set of triangles. Any edge in E is identified with the closed interval [0, 1], and any triangle in F is identified with a Euclidean equilateral triangle with unit length. Denote $T^{(1)}$ as the 1-skeleton of T, and $|T|$ as the underlying space of T homeomorphic to a surface possibly with boundary.

Delaunay triangulations of the unit sphere. Let $\mathbb{S}^2$ be the unit sphere as a Riemannian surface. Assume $|T|$ is homeomorphic to $\mathbb{S}^2$. An embedding $\varphi : T^{(1)} \to \mathbb{S}^2$ is called a *geodesic triangulation* of $\mathbb{S}^2$ if the restriction of φ on each edge is a geodesic parametrized with constant speed. A geodesic triangulation φ naturally divides $\mathbb{S}^2$ into spherical geodesic triangles. For our convenience, we will only consider the geodesic triangulations where all the spherical triangles are convex. A geodesic triangulation φ of $\mathbb{S}^2$ is called a *convex geodesic triangulation* if any spherical triangle in φ is contained in some open hemisphere. Such a convex geodesic triangulation φ is uniquely determined by the images of the vertices of T.

A convex geodesic triangulation φ is called *Delaunay* if it satisfies the empty circle property, meaning that for any pair of adjacent spherical triangles $\triangle ABC$ and $\triangle ABD$, D is not inside the circumcircle of $\triangle ABC$. This condition is equivalent to the following condition on the angles of a convex geodesic triangulation:

$$b + c + b' + c' - a - a' \geq 0, \tag{1}$$

where a, b, c, a', b', c' are the inner angles of two neighbored triangles as in Figure 1. Similarly, a convex geodesic triangulation is called *strictly Delaunay* if for any pair of adjacent spherical triangles $\triangle ABC$ and $\triangle ABD$, D is strictly outside the circumcircle of $\triangle ABC$. This condition is equivalent to the following condition on the angles of a convex geodesic triangulation:

$$b + c + b' + c' - a - a' > 0. \tag{2}$$

Delaunay and strictly Delaunay triangulations naturally appear in the study of discrete differential geometry and geometry processing. They are widely investigated

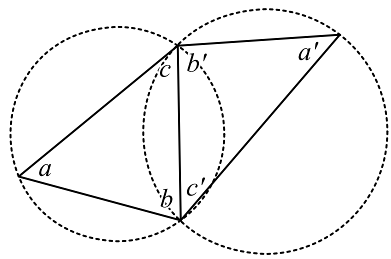

Figure 1. The edge invariant.

and implemented in practice. See [Devadoss and O'Rourke 2011; Edelsbrunner 2001] for example. We will focus on strictly Delaunay triangulations in this paper.

Given an embedding $\psi : T^{(1)} \to \mathbb{S}^2$, we define *the deformation space of Delaunay triangulations of the unit sphere* determined by ψ, denoted by $X(T, \psi)$, as the set of all strictly Delaunay convex geodesic triangulations that are isotopic to ψ in $\mathbb{S}^2$. Then $X(T, \psi)$ is naturally a manifold of dimension $2|V|$ without boundary, if $X(T, \psi)$ is not empty. Notice that $X(T, \psi)$ could be empty for some T since there are 3-connected graphs that cannot be realized as the 1-skeleton of a convex polyhedron with vertices on the unit 2-sphere. See [Steinitz 1928] for noninscribable polytopes.

Theorem 1.1 can be rephrased as:

Theorem 1.3. *Given a strictly Delaunay convex geodesic triangulation ψ, $X(T, \psi)$ is homeomorphic to $\mathbb{R}^{2|V|-3} \times \mathrm{SO}(3)$.*

Notice that by the assumption, $X(T, \psi)$ is not empty in Theorem 1.3.

The topology of spaces of geodesic triangulations of surfaces has been studied since Cairns [1944] first proved the connectivity of the spaces of geodesic triangulations of the 2-sphere. It was conjectured that for constant curvature surfaces they are homotopy equivalent to their smooth counterparts by Connelly et al. [1983]. This conjecture has been confirmed by Bloch, Connelly and Henderson [Bloch et al. 1984] for convex polygons, and a new proof based on Tuttes' embedding theorem was provided by Luo [2022]. Recently, this conjecture was proved for the cases of flat tori and closed surfaces of negative curvature (see the work of Erickson and Lin [2021] and Luo, Wu and Zhu [2021b; 2021a]).

For the case of the unit sphere, Awartani and Henderson [1987] identified the homotopy type of a subspace of the space of geodesic triangulations on the unit 2-sphere, but the general case remains open. Theorem 1.3 provides an affirmative evidence about this conjecture, and we hope that it could be an intermediate step to prove the conjecture for the unit sphere.

Convex polyhedra inscribed in the unit sphere. Assume $|T|$ is homeomorphic to $\mathbb{S}^2$. An embedding $\varphi : |T| \to \mathbb{R}^3$ is called a *polyhedral realization inscribed in the unit sphere* if φ maps any vertex to the unit sphere and maps any face linearly to a Euclidean triangle. Such a polyhedral realization φ is called (strictly) convex if for any triangle $\sigma \in F$, $\varphi(\sigma)$ is a face of the boundary of the convex hull of $\varphi(V)$ in $\mathbb{R}^3$. Given T, denote $Y(T)$ as the set of convex polyhedral realization inscribed in the unit sphere.

We say a point q is *inside* a convex polyhedral surface P if q is in the interior of the convex hull of P. Given a point q in the unit open ball, denote $p_q : \mathbb{R}^3 \backslash \{q\} \to \mathbb{S}^2$ as the radial projection centered at q to the unit sphere. We say two convex polyhedral realizations φ_1, φ_2 in $Y(T)$ *have the same orientation* if and only if $p_{q_1} \circ \varphi_1$ is isotopic to $p_{q_2} \circ \varphi_2$ on $\mathbb{S}^2$, for q_1 inside $\varphi_1(|T|)$ and q_2 inside $\varphi_2(|T|)$. It is straightforward to check that the choice of q_1 and q_2 does not matter.

Given a convex realization polyhedral realization ψ, we define *the deformation space of convex polyhedra inscribed in the sphere* determined by ψ, denoted by $Y(T, \psi) \subset Y(T)$, as the set of all convex realizations φ of $\mathbb{S}^2$ having the same orientation with ψ. Then $Y(T, \psi)$ is naturally a manifold of dimension $2|V|$ without boundary. Theorem 1.2 can be rephrased as

Theorem 1.4. *Given a convex realization ψ, $Y(T, \psi)$ is homeomorphic to* $\mathbb{R}^{2|V|-3} \times \mathrm{SO}(3)$.

The space of inscribed convex polyhedra in the unit sphere is closely related to realization spaces of polytopes with a fixed combinatorial type. Steinitz [1922] proved that every planar 3-connected graph is the 1-skeleton of a convex polyhedron in $\mathbb{R}^3$. Moreover, his proof implies that the realization space of polyhedra is a cell after the normalization by affine transformations. See [Richter-Gebert 1996] for a detailed discussion about the realization spaces.

Connections between the two spaces. Denote $Y_0(T)$ as the subset of $Y(T)$ containing all the convex realizations φ such that the origin $O = (0, 0, 0)$ is inside $\varphi(|T|)$. Given a convex realization ψ, denote $Y_0(T, \psi) = Y(T, \psi) \cap Y_0(T)$. If $\varphi \in Y_0$, then the radial projection p_O maps the triangulation structure on $\varphi(|T|)$ to a strictly Delaunay convex geometric triangulation of $\mathbb{S}^2$. This naturally produces a homeomorphism from $Y_0(T, \psi)$ to $X(T, p_O \circ \psi|_{T^{(1)}})$ for any convex realization ψ. Therefore, Theorem 1.3 can be reformulated as

Theorem 1.5. *Given a convex realization $\psi \in Y_0$, $Y_0(T, \psi)$ is homeomorphic to* $\mathbb{R}^{2|V|-3} \times \mathrm{SO}(3)$.

Organization. In Section 2, we will review the concept of angle structures. In Section 3, we will determine the topology of the spaces of Delaunay triangulations of convex polygons with fixed angles. In Section 4, we will prove Theorem 1.4

and Theorem 1.5. In Section 5, we will provide examples showing the homotopy types of spaces of Delaunay triangulations of flat tori and convex polygons could be disconnected.

2. Angle structures on triangulated surfaces

The tool to study the topology of spaces of Delaunay triangulations on $\mathbb{S}^2$ is the concept of angle structure or angle system on triangulated surfaces. This concept was proposed by Colin de Vedière [1991], and developed by Rivin [1994], Leibon [2002], Luo [2006], Bobenko and Springborn [2004], Springborn [2008], and others. We briefly summarize the theory in the following.

Angle structures on triangulated surfaces. Assume $|T|$ is a 2-dimensional manifold possibly with boundary. A *corner* in T is defined as a vertex-face pair (v, f) in T such that the face f contains v. It represents the inner angle of the face f at the vertex v. A *Euclidean angle structure* θ, or an angle structure in short, on T is a positive function on the set of the corners such that $\theta_1 + \theta_2 + \theta_3 = \pi$ for the three angles in every face f. Every angle structure can be presented as a positive vector in $\mathbb{R}^{3|F|}$. Denote $V_b \subset V$ as the set of boundary vertices, and then the *edge invariant* $\alpha = \alpha(\theta) \in \mathbb{R}^{E \cup V_b}$ is defined as:

(a) $\alpha_e = \theta_1 + \theta_2$, if e is an inner edge, and θ_1 and θ_2 are the two angles opposite to e.

(b) $\alpha_e = \theta_1$, if e is a boundary edge, and θ_1 is the angle opposite to e.

(c) $\alpha_v = \sum_i \theta_i$, if v is a boundary vertex, and θ_i's are the angles at v.

Denote the set of angle structures realizing a prescribed edge invariant $\bar{\alpha} \in \mathbb{R}^{E \cup V_b}$ as $\mathcal{A}(T, \bar{\alpha})$.

Given an edge length function $l \in \mathbb{R}^E$ satisfying the triangle inequalities, we can naturally determine a piecewise Euclidean metric on T and induce an angle structure $\theta(l)$ using the inner angles in this piecewise Euclidean metric. Notice that not every angle structure can be induced from a piecewise Euclidean metric, and there are holonomy conditions on the angle structures so that we can glue the Euclidean triangles determined by the angles to form a Euclidean triangle mesh. We will see that these geometric angle structures can be found by the following variational principles on $\mathcal{A}(T, \bar{\alpha})$.

Variational principles of angle structures. Variational methods are introduced to find piecewise Euclidean surfaces with a prescribed edge invariant. The functionals in these variational principles have elegant geometric interpretations in terms of volumes of polyhedra in the hyperbolic 3-space $\mathbb{H}^3$.

For each face f in F, an energy functional is defined in terms of three angles at the corners of the face in an angle structure. For a face in a Euclidean angle structure

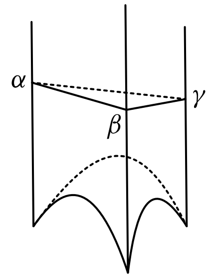

Figure 2. The volume of an ideal tetrahedron.

with three angles (α, β, γ), the energy functional is the volume of ideal hyperbolic tetrahedron whose horospherical section is similar to a Euclidean triangle with three angles (α, β, γ). See Figure 2. The volume is given by

$$V(\alpha, \beta, \gamma) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where Λ is the *Lobachevsky function*

$$\Lambda(x) = -\int_0^x \log(2\sin\theta)\, d\theta.$$

The total energy for a given angle structure is defined as the sum of functionals on each face

$$\mathcal{E}(\theta) = \sum_{f_i \in F} V_i(\alpha_i, \beta_i, \gamma_i).$$

The variational principles for these energy functionals can be summarized as follows.

Theorem 2.1 [Rivin 1994]. *Assume $\bar{\alpha} \in (0, \pi]^{E \cup V_b}$ and $\mathcal{A}(T, \bar{\alpha})$ is nonempty, then*:

(a) *The energy functional $\mathcal{E}$ is strictly concave down on $\mathcal{A}(T, \bar{\alpha})$.*

(b) *There exists a unique critical point $\theta = \Theta(\bar{\alpha})$ of $\mathcal{E}$ in $\mathcal{A}(T, \bar{\alpha})$.*

(c) *$\Theta(\bar{\alpha})$ is the unique angle structure in $\mathcal{A}(T, \bar{\alpha})$ that could be induced from a piecewise Euclidean metric on T.*

Denote $\mathcal{A}_0(T)$ as the set of angle structures θ such that $\alpha(\theta) \in (0, \pi)^{E \cup V_b}$ and the angle sum $\sum_i \theta_i$ around any interior vertex is 2π. Denote $\mathcal{A}_E(T)$ as the set of angle structures θ in $\mathcal{A}_0(T)$ that can be induced from a piecewise Euclidean metric on T. Notice that the angle structure induced from a Delaunay triangulation of a convex polygon in the plane belongs to $\mathcal{A}_E(T)$. Then by Theorem 2.1, we have the following.

Lemma 2.2. *If $\mathcal{A}_E(T)$ is nonempty, then $\mathcal{A}_E(T)$ is homeomorphic to $\mathbb{R}^k$ for some $k \geq 0$.*

Proof. If $\mathcal{A}_E(T)$ is nonempty, then $\mathcal{A}_0(T)$ is nonempty. From the definition we can see that $\mathcal{A}_0(T)$ is an open convex subset in an affine subspace of $\mathbb{R}^{3|F|}$. Then its image $\alpha(\mathcal{A}_0(T))$ under the edge invariant map α, which is a linear map, is an open convex subset of an affine subspace of $\mathbb{R}^{E\cup V_b}$. Hence, $\alpha(\mathcal{A}_0(T))$ is homeomorphic to $\mathbb{R}^k$ for some $k \geq 0$.

It remains to show that $\bar{\alpha} \mapsto \Theta(\bar{\alpha})$ is a homeomorphism from $\alpha(\mathcal{A}_0(T))$ to $\mathcal{A}_E(T)$. It is straightforward to show that such a map is continuous from $\alpha(\mathcal{A}_0(T))$ to $\mathbb{R}^{3|F|}$. Moreover, $\bar{\alpha} \mapsto \Theta(\bar{\alpha}) \mapsto \alpha(\Theta(\bar{\alpha}))$ is the identity map on $\alpha(\mathcal{A}_0(T))$. By Theorem 2.1, $\theta \mapsto \alpha(\theta) \mapsto \Theta(\alpha(\theta))$ is the identity map on $\mathcal{A}_E(T)$. Then we only need to show that the image $\Theta(\bar{\alpha})$ is in $\mathcal{A}_E(T)$. By the definition we only need to verify that for any interior vertex v, the angle sum around v in $\Theta(\bar{\alpha})$ is equal to the angle sum around v in θ. This is because the angle sum of an angle structure θ around an interior vertex v is determined by the edge invariant $\alpha(\theta)$ as the following.

$$\sum_{f\in F: f\ni v} \theta_{v,f} = \sum_{f\in F: f\ni v} \pi - \sum_{e\in E: e\ni v} \alpha_e. \qquad \square$$

The dimension of the space $\mathcal{A}_E(T)$ can be explicitly computed in the next section.

3. Delaunay Triangulations of Convex Polygons

Assume that $|T|$ is homeomorphic to a closed disk, an embedding $\varphi : |T| \to \mathbb{R}^2$ is called a *triangulation of a polygon* if φ is linear on any triangle of T. Further such φ is called a *triangulation of a convex polygon* if the inner angle of the polygon $\varphi(|T|)$ at $\varphi(v_i)$ is less than π for any boundary vertex v_i of T. Such φ is called *strictly Delaunay* if for any pair of adjacent triangles $\triangle ABC$ and $\triangle ABD$ in $\varphi(T)$, D is strictly outside the circumcircle of $\triangle ABC$. This condition is equivalent to that $a + a' < \pi$, where a, a' are the inner angles of two neighbored triangles as in Figure 1.

Denote $\theta(\varphi)$ as the angle structure induced from the triangulation φ, and $Z(T) = \{\varphi : \theta(\varphi) \in \mathcal{A}_E(T)\}$ as the set of strictly Delaunay triangulations of a convex polygon. We say two embeddings φ, ψ from $|T|$ to $\mathbb{R}^2$ have the same orientation if $\psi \circ \varphi^{-1}$ is an orientation preserving map on $\varphi(|T|)$. Given a triangulation ψ of a polygon, denote $Z(T, \psi)$ as the set of strictly Delaunay triangulations φ of a convex polygon that have the same orientation with ψ.

Furthermore, if we are given a directed edge e_{ij} of T, denote $Z(T, \psi, e_{ij})$ as the set of strictly Delaunay triangulations $\varphi \in Z(T, \psi)$ satisfying that $\varphi(j) - \varphi(i) = (\lambda, 0)$ for some $\lambda > 0$. Then it is elementary to see that a triangulation in $Z(T, \psi, e_{ij})$ is uniquely determined by the induced angle structure $\theta(\varphi)$, $\varphi(i)$ and $\varphi(j) - \varphi(i)$. Therefore, $\varphi \mapsto (\theta(\varphi), \varphi(i), \varphi(j) - \varphi(i))$ gives a homeomorphism from $Z(T, \psi, e_{ij})$ to $\mathcal{A}_E(T) \times \mathbb{R}^2 \times \mathbb{R}_+$. On the other hand, the space $Z(T, \psi, e_{ij})$ is a $(2|V| - 1)$-dimensional manifold if not empty, then we have the following from Lemma 2.2.

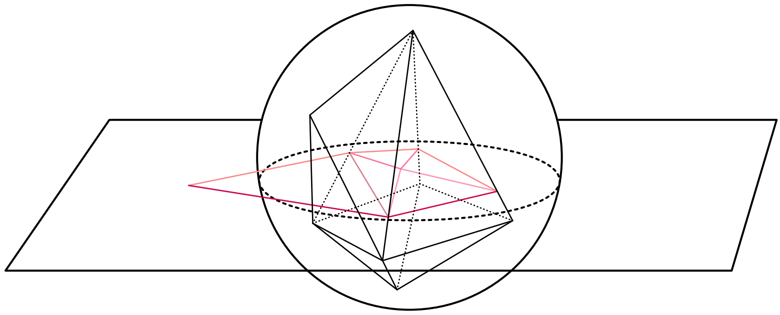

Figure 3. The stereographic projection of an inscribed convex polyhedron.

Corollary 3.1. *Given any Delaunay triangulation of a convex polygon ψ, and a directed edge e_{ij}, $Z(T, \psi, e_{ij})$ is homeomorphic to $\mathbb{R}^{2|V|-1}$.*

In the next section, we will reduce the spaces of Delaunay triangulations on the sphere and the spaces of convex polyhedra inscribed in the sphere to the space $Z(T, \psi, e_{ij})$.

4. Proof of the main theorems

We will prove Theorems 1.4 and 1.5 in this section using the stereographic projection. It is well known that the stereographic projection

$$\pi : (x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

gives an angle-preserving diffeomorphism from $\mathbb{S}^2 \backslash \{(0, 0, 1)\}$ to $\mathbb{R}^2$. For a circle Γ on $\mathbb{S}^2$, the stereographic projection maps Γ to a circle on $\mathbb{R}^2$ if Γ does not contain $(0, 0, 1)$, and maps $\Gamma \backslash \{(0, 0, 1)\}$ to a straight line in $\mathbb{R}^2$ if Γ contains $(0, 0, 1)$.

Assume $|T|$ is homeomorphic to $\mathbb{S}^2$, and v_0 is a vertex of T, and $\psi \in Y(T)$ is a convex realization inscribed in the unit sphere, then denote $Y(T, \psi, v_0)$ (resp. $Y(T, v_0)$, $Y_0(T, \psi, v_0)$, $Y_0(T, v_0)$) as the set of $\varphi \in Y(T, \psi)$ (resp. $\varphi \in Y(T)$, $Y_0(T, \psi)$, $Y_0(T)$) with $\varphi(v_0) = (0, 0, 1)$.

Lemma 4.1. *Assume $|T|$ is homeomorphic to $\mathbb{S}^2$, v_0 is a vertex of T, T_0 denotes the subcomplex of T obtained by removing the open 1-ring neighborhood of v_0, and e_{ij} is a directed edge in T_0:*

(a) *There exists a map $\tilde{\pi} : Y(T, v_0) \to Z(T_0)$ induced by π such that $\phi = \tilde{\pi}(\varphi)$ is the strictly Delaunay triangulation of a convex polygon determined by $\phi(v) = \pi(\varphi(v))$ for any vertex v of T_0; see Figure 3.*

(b) *There exists a map $\tilde{\eta} : Z(T_0) \to Y(T, v_0)$ induced by π^{-1} such that $\varphi = \tilde{\eta}(\phi)$ is the convex realization determined by $\varphi(v) = \pi^{-1}(\phi(v))$ for any vertex v of T_0.*

(c) *$\tilde{\pi}$ and $\tilde{\eta}$ are inverse to each other and then $\tilde{\pi}$ is a homeomorphism from $Y(T, v_0)$ to $Z(T_0)$.*

(d) *Given a convex realization $\psi \in Y(T, v_0)$, $\tilde{\pi}$ gives a homeomorphism from $Y(T, \psi, v_0)$ to $Z(T_0, \tilde{\pi}(\psi))$.*

(e) *If $\phi \in \tilde{\pi}(Y_0(T, v_0))$:*

  (i) *The origin $(0, 0)$ is in the interior of $\phi(|T_0|)$.*
  (ii) *$\lambda\phi$ is also in $\tilde{\pi}(Y_0(T, v_0))$ for any $\lambda \in (0, 1)$.*

(f) *For any $\varphi \in Y(T, \psi)$, there exists a unique $\varphi_0 \in Y(T, \psi, v_0)$ and $g \in \mathrm{SO}(3)$, such that $\varphi = g \circ \varphi_0$ and $\tilde{\pi}(\varphi_0) \in Z(T_0, \tilde{\pi}(\psi), e_{ij})$. Then $Y(T, \psi)$ is homeomorphic to $Z(T_0, \tilde{\pi}(\psi), e_{ij}) \times \mathrm{SO}(3)$.*

(g) *For any $\varphi \in Y_0(T, \psi)$, there exists a unique $\varphi_0 \in Y_0(T, \psi, v_0)$ and $g \in \mathrm{SO}(3)$, such that $\varphi = g \circ \varphi_0$ and $\tilde{\pi}(\varphi_0) \in Z(T_0, \tilde{\pi}(\psi), e_{ij})$. Then $Y_0(T, \psi)$ is homeomorphic to $(\tilde{\pi}(Y_0(T, v_0)) \cap Z(T_0, \tilde{\pi}(\psi), e_{ij})) \times \mathrm{SO}(3)$.*

Proof. (a) and (b) are true by the empty circle property of the (strict) Delaunay triangulations and the fact that the stereographic projection preserves circles.

(c) This is a direct consequence from the definition.

(d) Given a convex realization $\varphi \in Y(T, v_0)$ and q_1 inside $\psi(|T|)$ and q_2 inside $\varphi(|T|)$, the following elementary facts related to orientations are equivalent by the definition and properties of stereographic projections:

(i) $\varphi \in Y(T, \psi, v_0)$.

(ii) ψ and φ have the same orientation.

(iii) $\pi_{q_1} \circ \psi$ is isotopic to $\pi_{q_2} \circ \varphi$ in $\mathbb{S}^2$.

(iv) $\pi_{q_1} \circ \psi$ and $\pi_{q_2} \circ \varphi$ have the same orientation.

(v) $\tilde{\pi}(\psi)$ and $\tilde{\pi}(\varphi)$ have the same orientation.

(vi) $\tilde{\pi}(\varphi) \in Z(T_0, \tilde{\pi}(\psi))$.

(e) If $\varphi \in Y_0(T, v_0)$, then the origin is inside $\varphi(|T|)$. Then the ray starting from the north pole passing through the origin intersects with $\varphi(|T|)$ at a unique point q in the interior of $\varphi(|T_0|)$. So part (i) is true. We prove part (ii) by contradiction. If $\lambda\phi$ is not in $\tilde{\pi}(Y_0(T, v_0))$ for some $\lambda \in (0, 1)$, then the origin $(0, 0, 0)$ is not inside $\tilde{\eta}(\lambda\phi)$ and there is an open hemisphere H on $\mathbb{S}^2$ not intersecting $\tilde{\eta}(\lambda\phi)(V)$. Notice that H does not contain the north pole so $\pi(H)$ is well-defined. Then $\pi(H)$ is an open disk containing $(0, 0)$ or an open half plane with $(0, 0)$ on its boundary, and $\pi(H)$ does not intersect $(\lambda\phi)(V)$. So $\pi(H)$ does not intersect $\phi(V)$, meaning that H does not intersect $\tilde{\eta}(\phi)(V)$. So $(0, 0, 0)$ is not inside $\tilde{\eta}(\phi)(|T|)$, but this contradicts with that $\phi \in \tilde{\pi}(Y_0(T, v_0))$.

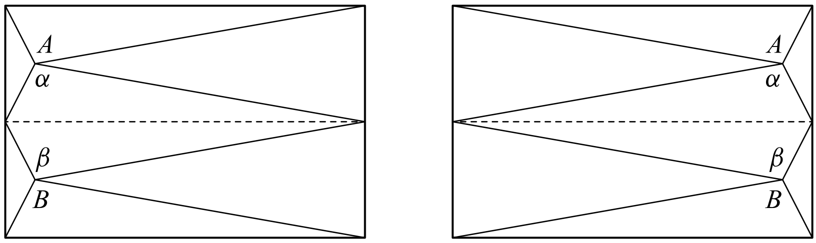

Figure 4. Counterexample, a convex polygon.

(f) and (g) Follow from the fact that the rotation along the z-axis (or the origin in the xy-plane) is invariant under the stereographic projection. □

Proof of Theorem 1.4. This is an immediate consequence of Corollary 3.1 and part (f) of Lemma 4.1. □

Proof of Theorem 1.5. By part (g) of Lemma 4.1, we only need to show that $\tilde{\pi}(Y_0(T, v_0)) \cap Z(T_0, \tilde{\pi}(\psi), e_{ij})$ is homeomorphic to $\mathbb{R}^{2|V|-3}$. By part (e) of Lemma 4.1 it is elementary to verify that

$$\varphi \in \tilde{\pi}(Y_0(T, v_0)) \cap Z(T_0, \tilde{\pi}(\psi), e_{ij})$$

is uniquely determined by $\theta(\varphi)$, $\varphi^{-1}(0,0)$ and $d(\varphi)$, where $d(\varphi)$ is the Euclidean diameter of $\varphi(|T|)$ and describes the scaling transformation needed to determine φ. So $\varphi \mapsto (\theta(\varphi), \varphi^{-1}(0,0), d(\varphi))$ gives a continuous injective map from $\tilde{\pi}(Y_0(T, v_0)) \cap Z(T_0, \tilde{\pi}(\psi), e_{ij})$ to $\mathcal{A}_E(T_0) \times \mathrm{int}(|T_0|) \times (0, \infty)$, where $\mathrm{int}(|T_0|) = |T_0| \backslash \partial(|T_0|)$ is homeomorphic to $\mathbb{R}^2$. Then by Lemma 2.2 and a dimension counting, we complete the proof. □

5. Delaunay triangulations of other surfaces

In this section, we will discuss the space of Delaunay geodesic triangulations of convex polygons and flat tori.

Convex polygons. A convex polygon P in the plane is determined by the position of a sequence of cyclically ordered vertices. The following simple example in Figure 4 shows that for a fixed convex polygon P in the plane with a triangulation $\psi : T \to P$, denote the space of Delaunay triangulations of P which are isotopic to ψ and have the same orientation with ψ as $X(T, \psi)$. Notice that $X(T, \psi)$ is different from the space $Z(T, \psi)$ in Section 3, since the positions of boundary vertices of T for elements in X are fixed.

The following example shows that $X(T, \psi)$ may not be connected.

In Figure 4, there are nine interior edges in the triangulation, eight of which are Delaunay. The dashed edge might not be Delaunay. In Figure 4, if the vertices A

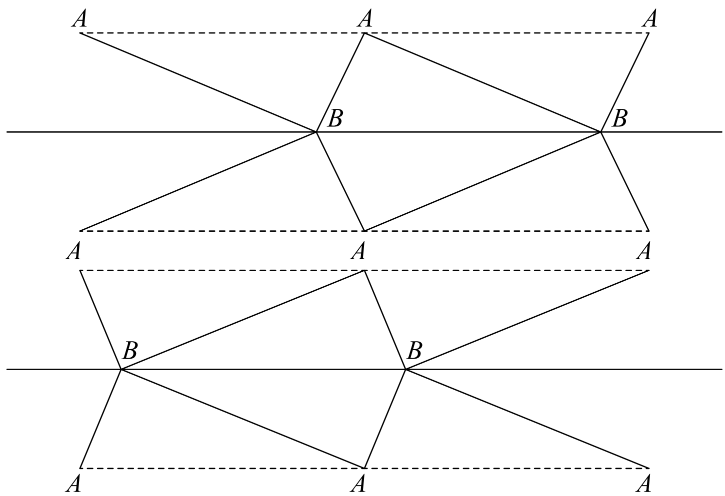

Figure 5. Counterexample, a flat torus.

and B are close to the vertical boundaries, then α and β are both acute, so we can construct two Delaunay triangulations τ_1 and τ_2 on the left and right. If there is a family of Delaunay triangulations connecting τ_1 and τ_2, the vertex A or B will pass the perpendicular bisector of the horizontal boundary of this rectangle. If the rectangle is flat enough, the angle sum $\alpha + \beta > \pi$ when one of A and B lies on the perpendicular bisector. This shows that $X(T, \psi)$ for this rectangle P is not connected.

Delaunay triangulations on flat tori. Assume $|T|$ is homeomorphic to the torus $\mathbb{T}^2$ with a marking homeomorphism whose restriction on $T^{(1)}$ is denoted as ψ. An embedding $\varphi : T^{(1)} \to \mathbb{T}^2$ is a *Delaunay geodesic triangulation* with the combinatorial type (T, ψ) satisfying:

(a) The restriction φ_{ij} of φ on each edge e_{ij}, identified with a unit interval $[0, 1]$, is a geodesic parametrized with constant speed.

(b) φ is homotopic to ψ.

(c) Equation (2) is satisfied for all edges in T.

Let $X = X(T, \psi)$ denote the set of all such geodesic triangulations, which is called the *deformation space of Delaunay geodesic triangulations of* $\mathbb{T}^2$ of combinatorial type (T, ψ).

The following example shows that the space of Delaunay geodesic triangulations $X = X(T, \psi)$ may not be connected.

In Figure 5, we draw two geodesic triangulations τ_1 and τ_2 on a flat torus. For each geodesic triangulation, we draw two fundamental domains of this torus. The triangulation has two vertices and six edges. Fixing the vertex A at a point in the universal covering, we can see that the position of the vertex B determines a geodesic triangulation of this flat torus. Notice that τ_1 and τ_2 are both Delaunay, since all the angles in these two triangulations are acute when B is sufficiently close to the vertical line connecting two adjacent copies of A in the universal covering.

We can choose the shape of the fundamental domain of the flat torus as shown in the picture. Then τ_1 and τ_2 are in two different connected components of the space of Delaunay triangulations of this flat torus. This observation is based on the following fact: any path connecting τ_1 and τ_2 needs to move the vertex B from the right to the left. However, we can choose a flat enough fundamental domain such that when B passes the perpendicular bisector of the dashed edge, the dashed edge is never Delaunay. This implies that the space $X = X(T, \psi)$ for this flat torus is not connected.

Acknowledgments

The authors would like to thank Professor Jeff Erickson for his insightful comments.

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Received April 22, 2022. Revised November 21, 2022.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.129

NONEXISTENCE OF NEGATIVE WEIGHT DERIVATIONS OF THE LOCAL k-TH HESSIAN ALGEBRAS ASSOCIATED TO ISOLATED SINGULARITIES

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A new conjecture about the nonexistence of negative weight derivations of the k-th Hessian algebras for weighted homogeneous isolated hypersurface singularities is proposed. We verify this conjecture up to dimension three.

1. Introduction

A holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is called quasihomogeneous if $f \in J(f)$, where $J(f) := (\partial f/\partial z_0, \partial f/\partial z_1, \dots, \partial f/\partial z_n)$ is the Jacobian ideal. A polynomial $f(z_0, \dots, z_n)$ is called weighted homogeneous of type $(\alpha_0, \dots, \alpha_n; d)$, where $\alpha_0, \dots, \alpha_n$ and d are fixed positive integers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $\alpha_0 i_0 + \cdots + \alpha_n i_n = d$. According to a beautiful theorem of Saito [1971], if $V = V(f)$ has isolated singularities, then f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if f is quasihomogeneous. The order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity, denoted by $\mathrm{mult}(f)$, of the singularity $(V, 0)$.

For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ that is defined by the holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, one has the moduli algebra $A(V) := \mathcal{O}_{n+1}/(f, \partial f/\partial z_0, \dots, \partial f/\partial z_n)$ which is finite dimensional. The well-known Mather–Yau theorem [1982] states that: Let $(V_1, 0)$ and $(V_2, 0)$ be two isolated hypersurface singularities, and let $A(V_1)$ and $A(V_2)$ be their respective moduli algebras, then $(V_1, 0) \cong (V_2, 0) \iff A(V_1) \cong A(V_2)$. In 1983, Yau introduced the Lie algebra of derivations of $A(V)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. The finite dimensional Lie algebra $L(V)$ is called the Yau algebra, and its dimension $\lambda(V)$ is called the Yau number in ([Khimshiashvili 2006; Yu 1996]). The Yau algebra plays an important role in singularity theory and was used to distinguish complex analytic structures of isolated hypersurface singularities [Seeley and Yau 1990]. Yau and

Zuo was supported by the NSFC (12271280) and the Tsinghua University Initiative Scientific Research Program. Yau was supported by the Tsinghua University Education Foundation (042202008).
MSC2020: 14B05, 32S05.
Keywords: derivations, Hessian algebra, isolated singularity, weighted homogeneous.

his collaborators have been systematically studying the Yau algebras of isolated hypersurface singularities and their generalizations from the eighties (see [Yau 1986; Xu and Yau 1996; Seeley and Yau 1990; Chen et al. 2019; Hussain et al. 2021]). In [Hussain et al. 2021] and [Chen et al. 2020], many new derivation Lie algebras that arise from isolated hypersurface singularities are introduced. These Lie algebras are more subtle invariants of singularities compared with previous Lie algebras. These Lie algebras are defined as follows: For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ defined by the holomorphic function $f(z_0, z_1, \dots, z_n)$, let $\operatorname{Hess}(f)$ be the Hessian matrix (f_{ij}) of the second-order partial derivatives of f and $h(f)$ be the Hessian of f, i.e., the determinant of the matrix $\operatorname{Hess}(f)$. More generally, for each k satisfying $0 \le k \le n+1$, we denote by I_k the ideal in $\mathcal{O}_{n+1}$ generated by all $k \times k$-minors in the matrix $\operatorname{Hess}(f)$. In particular, the ideal $I_{n+1} = (h(f))$ is a principal ideal. For each k as above, consider the graded k-th Hessian algebra of the polynomial f defined by

$$H_k(f) = \mathcal{O}_{n+1}/\big((f) + J(f) + I_k\big).$$

In particular, $H_0(f)$ is exactly the well-known moduli algebra $A(V)$.

It is easy to check that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is contact invariant of f, i.e., it depends only on the isomorphism class of the germ $(V, 0)$ [Dimca and Sticlaru 2015].

In particular, $H_{n+1}(f)$ has a geometric meaning. We recall the following beautiful characterization theorem of zero-dimensional isolated complete intersection singularities:

Theorem 1.1 [Dimca 1984]. *Two zero-dimensional isolated complete intersection singularities X and Y are isomorphic if and only if their singular subspaces* $\operatorname{Sing}(X)$ *and* $\operatorname{Sing}(Y)$ *are isomorphic.*

Remark 1.2. Let $V = V(f)$ be an isolated quasihomogeneous hypersurface singularity. It follows that X, defined by $(\partial f/\partial z_0, \dots, \partial f/\partial z_n)$, is a zero-dimensional isolated complete intersection singularity. In this case, $\operatorname{Sing}(X)$ is defined by

$$\left(f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}, h(f)\right).$$

Theorem 1.1 implies that to study the analytic isomorphism type of the zero-dimensional isolated complete intersection singularity X, we only need to consider the Artinian local algebra $H_{n+1}(f)$, which is the coordinate ring of $\operatorname{Sing}(X)$.

Combining Theorem 1.1 with the Mather–Yau theorem, we know that $H_{n+1}(f)$ is a complete invariant of quasihomogeneous isolated hypersurface singularities (i.e., $H_{n+1}(f)$ determines and is determined by the analytic isomorphism type of the singularity). In [Chen et al. 2020], we also call $H_{n+1}(f)$ the generalized moduli algebra of V. As a generalization of the Yau algebra, it is natural to introduce the following new Lie algebras for isolated hypersurface singularities:

Definition 1.3. Let $V = \{f = 0\}$ be a germ of the isolated hypersurface singularity at the origin of $\mathbb{C}^{n+1}$ defined by $f(z_0, z_1, \ldots, z_n)$, with $n \geq 1$. The series of new derivation Lie algebras arising from the isolated hypersurface singularity $(V, 0)$ is defined as $L_k(V) := \mathrm{Der}(H_k(f), H_k(f))$, where $0 \leq k \leq n+1$, or $\mathrm{Der}(H_k(f))$ for short. The dimension of $L_k(V)$ is denoted by $\lambda_k(V)$.

It is known that the Yau algebra cannot characterize the ADE singularities completely. In fact, Elashvili and Khimshiashvili [2006] proved the following result: If X and Y are two simple singularities except for the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic. However, we have proven that the ADE singularities are characterized completely by the new Lie algebra $L_{n+1}(V)$ as follows. We have reasons to believe that the new Lie algebras $L_k(V)$ and numerical invariants $\lambda_k(V)$, where $1 \leq k \leq n+1$, will also play an important role in the study of singularities.

Theorem 1.4 [Chen et al. 2020]. *If X and Y are two n-dimensional ADE singularities, then $L_{n+1}(X) \cong L_{n+1}(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

Let A be a weighted zero-dimensional complete intersection, i.e., a commutative algebra of the form

$$A = \mathbb{C}[z_0, z_1, \ldots, z_n]/I,$$

where the ideal I is generated by a regular sequence of length $n+1$, $(f_0, f_1, \ldots, f_n)$. Here, the variables have strictly positive integral weights, denoted by $\mathrm{wt}(z_i) = \alpha_i$, where $0 \leq i \leq n$, and the equations are weighted homogeneous with respect to these weights. Consequently, the algebra A is graded and one may speak about its homogeneous degree k derivations, where k is an integer. Recall that a linear map $D : A \to A$ is a derivation if $D(ab) = D(a)b + aD(b)$, for any $a, b \in A$. The map D belongs to $\mathrm{Der}^k(A)$ if $D : A^* \to A^{*+k}$.

On the one hand, one of the most prominent open problems in rational homotopy theory is related to the vanishing of the above derivations in strictly negative degrees.

Halperin Conjecture [Meier 82; Chen et al. 2019]. *Let*

$$A = \mathbb{C}[z_0, z_1, \ldots, z_n]/I,$$

where the ideal I is generated by a regular sequence of length $n+1$, $(f_0, f_1, \ldots, f_n)$. Here, the variables have strictly positive even integer weights, denoted by $\mathrm{wt}(z_i) = \alpha_i$, $0 \leq i \leq n$, and the equations are weighted homogeneous with respect to these weights. Then $\mathrm{Der}^{<0}(A) = 0$.

The Halperin Conjecture has been verified in several particular cases, see [Papadima and Paunescu 1996; Thomas 1981]. For recent progress, please see [Chen et al. 2019].

Let $(V, 0) = \{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \colon f(z_0, z_1, \dots, z_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(z_0, z_1, \dots, z_n)$ of type $(\alpha_0, \alpha_1, \dots, \alpha_n; d)$. Then by a well-known result of Saito [1971], we can always assume, without loss of generality, that $d \geq 2\alpha_i > 0$ for all $0 \leq i \leq n$. We give the variable z_i weight α_i for $0 \leq i \leq n$, thus the moduli algebra $A(V)$ is a graded algebra, i.e., $A(V) = \bigoplus_{i=0}^{\infty} A_i(V)$, and the Lie algebra of derivations Der($A(V)$) is also graded. Thus, $L(V)$ is graded.

On the other hand, Yau discovered independently the following conjecture on the nonexistence of the negative weight derivation, which is a special case of the Halperin Conjecture.

Yau Conjecture (see [Chen 1995; Chen et al. 1995]). *Consider the isolated singularity*

$$(V, 0) = \left\{(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \colon f(z_0, z_1, \dots, z_n) = 0\right\}$$

defined by the weighted homogeneous polynomial $f(z_0, z_1, \dots, z_n)$ *of weight type* $(\alpha_0, \alpha_1, \dots, \alpha_n; d)$. *Assume that* $d \geq 2\alpha_0 \geq 2\alpha_1 \geq \cdots \geq 2\alpha_n > 0$, *without loss of generality. Then there is no nonzero negative weight derivation on the moduli algebra* (= *Milnor algebra*)

$$A(V) = \mathbb{C}[z_0, z_1, \dots, z_n] \Big/ \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right),$$

i.e., $L(V)$ *is nonnegatively graded.*

This conjecture is still open and was only proved in the low-dimensional case $n \leq 3$ by explicit calculations [Chen 1995; Chen et al. 1995]. It was also proved for the high-dimensional singularities under certain conditions [Yau and Zuo 2016] and for homogeneous singularities (see Proposition 2.1).

Theorem 1.5 [Chen 1995, Theorem 2.1]. *Let* $f(z_0, z_1, z_2, z_3)$ *be a weighted homogeneous polynomial of type* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *with an isolated singularity at the origin. Assume that* $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2 \geq 2\alpha_3 > 0$, *without loss of generality. Let* D *be a derivation of the moduli algebra*

$$A(V) = \mathbb{C}[z_0, z_1, z_2, z_3] \Big/ \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_3}\right).$$

Then $D \equiv 0$ *if* D *is negatively weighted.*

Assume that f is a weighted homogeneous polynomial, then the k-th Hessian algebra $H_k(V)$ and $L_k(V)$ are also naturally graded. It is natural to propose the following new conjecture:

Conjecture 1.6. *Let* $(V,0)=\{(z_0,z_1,\dots,z_n)\in\mathbb{C}^{n+1}\colon f(z_0,z_1,\dots,z_n)=0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* f *of weight type* $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$. *Assume that* $d\ge 2\alpha_0\ge 2\alpha_1\ge\cdots\ge 2\alpha_n>0$, *without loss of generality. Let* $H_k(V)$ *be the* k*-th Hessian algebra. Furthermore, in the case of* $1<k\le n$ (*respectively*, $k=1$), *we need to assume that* $\mathrm{mult}(f)\ge 4$ (*respectively*, 5). *Then for any* $0\le k\le n+1$, *there is no nonzero*, *negative weight derivation on the* $H_k(V)$, *i.e.*, $L_k(V)$ *is nonnegatively graded.*

This Conjecture 1.6 seems very hard to verify in general, in fact, when $k=0$, it is exactly the long-standing Yau Conjecture above. When $k=n+1$, it was also verified for $n\le 3$ as follows:

Theorem 1.7 [Ma et al. 2020]. *Consider the isolated singularity*

$$(V,0)=\left\{(z_0,z_1,\dots,z_n)\in\mathbb{C}^{n+1}\colon f(z_0,z_1,\dots,z_n)=0\right\}$$

defined by the weighted homogeneous polynomial f *of weight type* $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$, *where* $1\le n\le 3$. *Assume that* $d\ge 2\alpha_0\ge 2\alpha_1\ge\cdots\ge 2\alpha_n>0$, *without loss of generality. Let* D *be a derivation of the algebra*

$$\mathbb{C}[z_0,z_1,\dots,z_n]\Big/\left(\frac{\partial f}{\partial z_0},\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n},\det\left(\frac{\partial^2 f}{\partial z_i\partial z_j}\right)_{0\le i,j\le n}\right).$$

Then $D\equiv 0$, *if* D *has negative weight, i.e.*, $L_{n+1}(V)$ *is nonnegatively graded for* $1\le n\le 3$.

In this paper, we shall verify Conjecture 1.6 for the case $n=1,2$, with $1\le k\le n$, and $n=3$, with $1<k\le 3$ (the case $n=0$ is trivial). The proof of the case where $n=3$ and $k=1$ is completely different and long. It will appear in our subsequent paper. In this paper, we obtain the following main result which partially verifies the Conjecture 1.6:

Main Theorem. *Let* $(V,0)=\{(z_0,z_1,\dots,z_n)\in\mathbb{C}^{n+1}\colon f(z_0,z_1,\dots,z_n)=0\}$ *be an isolated singularity defined by the weighted homogeneous polynomial* f *of weight type* $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$, *where* $1\le n\le 3$. *Assume, without loss of generality, that* $d\ge 2\alpha_0\ge 2\alpha_1\ge\cdots\ge 2\alpha_n>0$. *Let* $L_k(V)$ *be the derivation Lie algebra of the* k*-th Hessian algebra* $H_k(V)$ *and* $D_k\in L_k(V)$.

(a) *For* $n=1$, *if* D_1 *is of negative weight, then* $D_1\equiv 0$.

(b) *For* $n=2$, *if* D_1 (*respectively*, D_2) *is of negative weight*, *then* $D_1\equiv 0$ (*respectively*, $D_2\equiv 0$). *In this case, we need the assumption* $\mathrm{mult}(f)\ge 4$, *see Example 1.8.*

(c) *For* $n=3$, *if* D_2 (*respectively*, D_3) *is of negative weight*, *then* $D_2\equiv 0$ (*respectively*, $D_3\equiv 0$). *In this case, we need the assumption* $\mathrm{mult}(f)\ge 4$, *see Example 1.9.*

Example 1.8. We need to add the condition $\mathrm{mult}(f) \geq 4$ in Main Theorem (b) due to the following two examples:

(a) Let $f = z_0^3 + z_0z_1z_2^2 + z_1^3z_2 + z_2^5$ with weighted type (5, 4, 3; 15). We have

$$\frac{\partial f}{\partial z_0} = 3z_0^2 + z_1z_2^2, \quad \frac{\partial f}{\partial z_1} = z_0z_2^2 + 3z_1^2z_2, \quad \frac{\partial f}{\partial z_2} = 2z_0z_1z_2 + z_1^3 + 5z_2^4$$

and

$$\frac{\partial^2 f}{\partial z_0^2} = 6z_0, \qquad \frac{\partial^2 f}{\partial z_1^2} = 6z_1z_2, \qquad \frac{\partial f}{\partial z_2^2} = 2z_0z_1 + 20z_2^3,$$
$$\frac{\partial^2 f}{\partial z_0\partial z_1} = z_2^2, \quad \frac{\partial^2 f}{\partial z_0\partial z_2} = 2z_1z_2, \quad \frac{\partial f}{\partial z_1\partial z_2} = 2z_0z_2 + 3z_1^2.$$

It is easy to check that $D_1 = z_2(\partial/\partial z_1)$ is a negative weight derivation (weighted degree of D_1 is -1, i.e., $\mathrm{wt}(D_1) = -1$) of

$$\mathbb{C}[z_0, z_1, z_2] \Big/ \left(\frac{\partial^2 f}{\partial z_0^2}, \frac{\partial^2 f}{\partial z_1^2}, \frac{\partial^2 f}{\partial z_2^2}, \frac{\partial^2 f}{\partial z_0\partial z_1}, \frac{\partial^2 f}{\partial z_0\partial z_2}, \frac{\partial^2 f}{\partial z_1\partial z_2}\right),$$

i.e., $D_1 \in L_1(V(f))$.

(b) Let $f = z_0^2z_2 + z_0z_2^5 + z_1^3$ with weighted type (4, 3, 1; 9). We have

$$\frac{\partial f}{\partial z_0} = 2z_0z_2 + z_2^5, \quad \frac{\partial f}{\partial z_1} = 3z_1^2, \quad \frac{\partial f}{\partial z_2} = z_0^2 + 5z_0z_2^4$$

and

$$\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix} = \begin{vmatrix} 2z_2 & 0 & 2z_0+5z_2^4 \\ 0 & 6z_1 & 0 \\ 2z_0+5z_2^4 & 0 & 20z_0z_2^3 \end{vmatrix}.$$

Thus $I_2 = \langle f_1, f_2, f_3, f_4 \rangle$, where

$$f_1 = \begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} = 12z_1z_2, \qquad f_2 = \begin{vmatrix} f_{01} & f_{02} \\ f_{11} & f_{12} \end{vmatrix} = -12z_0z_1 - 30z_1z_2^4,$$
$$f_3 = \begin{vmatrix} f_{00} & f_{02} \\ f_{02} & f_{22} \end{vmatrix} = 20z_0z_2^4 - 4z_0^2 - 25z_2^8, \quad f_4 = \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{12} \end{vmatrix} = 120z_0z_1z_2^3.$$

It is easy to check that $D_2 = z_1(\partial/\partial z_0) \in L_2(V(f))$ is a negative weight derivation ($\mathrm{wt}(D_2) = -1$).

Example 1.9. We need to add the condition $\mathrm{mult}(f) \geq 4$ in Main Theorem (c) due to the following example:

Let $f = z_0^2z_2 + z_2^3z_0 + z_1^3 + z_3^5$ with weighted type (6, 5, 3, 3; 15). We have

$$\frac{\partial f}{\partial z_0} = 2z_0z_2 + z_2^3, \quad \frac{\partial f}{\partial z_1} = 3z_1^2, \quad \frac{\partial f}{\partial z_2} = z_0^2 + 3z_2^2z_0, \quad \frac{\partial f}{\partial z_3} = 5z_3^4$$

and

$$\begin{vmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{01} & f_{11} & f_{12} & f_{13} \\ f_{02} & f_{12} & f_{22} & f_{23} \\ f_{03} & f_{13} & f_{23} & f_{33} \end{vmatrix} = \begin{vmatrix} 2z_2 & 0 & 2z_0+3z_2^2 & 0 \\ 0 & 6z_1 & 0 & 0 \\ 2z_0+3z_2^2 & 0 & 6z_0z_2 & 0 \\ 0 & 0 & 0 & 20z_3^3 \end{vmatrix}.$$

Thus $I_2 = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8 \rangle$, where

$$f_1 = \begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} = 12z_1z_2, \qquad f_2 = \begin{vmatrix} f_{01} & f_{02} \\ f_{11} & f_{12} \end{vmatrix} = -12z_0z_1 - 18z_1z_2^2,$$

$$f_3 = \begin{vmatrix} f_{00} & f_{02} \\ f_{02} & f_{22} \end{vmatrix} = -4z_0^2 - 9z_2^4, \qquad f_4 = \begin{vmatrix} f_{00} & f_{03} \\ f_{03} & f_{33} \end{vmatrix} = 40z_2z_3^3,$$

$$f_5 = \begin{vmatrix} f_{02} & f_{23} \\ f_{03} & f_{33} \end{vmatrix} = 40z_0z_3 + 60z_2^2z_3, \qquad f_6 = \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} = 36z_0z_1z_2,$$

$$f_7 = \begin{vmatrix} f_{11} & f_{13} \\ f_{13} & f_{33} \end{vmatrix} = 120z_1z_3^3, \qquad f_8 = \begin{vmatrix} f_{22} & f_{23} \\ f_{23} & f_{33} \end{vmatrix} = 120z_0z_2z_3^3.$$

It is easy to check that $D_2 := z_1(\partial/\partial z_0) \in L_2(V(f))$ is a negative weight derivation ($\mathrm{wt}(D_2) = -1$).

Remark 1.10. Examples 1.8 and 1.9 are interesting because one cannot find such examples when $k = 0$ (see the Yau Conjecture) and $k = n+1$ (see Theorem 1.7).

Xu and Yau [1996] used the property of nonexistence of negative derivations of the moduli algebra $A(V)$ to obtain a characterization of quasihomogeneous singularities (see [Xu and Yau 1996, Theorem 3.2] for details). We believe this characterization can be generalized by using the Lie algebra of derivations of the k-th Hessian algebra. The Main Theorem in this paper provides evidence for the generalization.

2. Proof of the Main Theorem

Firstly, we recall the following known results which will be used in proof of the Main Theorem frequently:

Proposition 2.1 [Xu and Yau 1996, Proposition 2.6]. *Let $A = \bigoplus_{i=0}^{k} A_i$ be a graded commutative Artinian local algebra with $A_0 = \mathbb{C}$. Suppose the maximal ideal of A is generated by A_j for some $j > 0$. Then $L(A)$ is a graded Lie algebra without negative weight.*

Lemma 2.2 [Yau 1986]. *Let $(A, \mathfrak{m})$ be a commutative local Artinian algebra ($\mathfrak{m}$ is the unique maximal ideal of A and $D \in L(A)$ is the derivation of A). Then D preserves the m-adic filtration of A, i.e., $D(\mathfrak{m}) \subset \mathfrak{m}$.*

Lemma 2.3 [Chen et al. 1995, Lemma 2.1]. *Let f be a weighted homogeneous polynomial with isolated singularity in the variables $z_0, \dots, z_n$ of type $(\alpha_0, \dots, \alpha_n; d)$. Assume* $\mathrm{wt}(z_0) = \alpha_0 \geq \mathrm{wt}(z_1) = \alpha_1 \geq \cdots \geq \mathrm{wt}(z_n) = \alpha_n$. *Then f must be as in one of the following two cases*:

Case 1: *Let* $f = z_0^m + a_1(z_1, \dots, z_n) z_0^{m-1} + \cdots + a_{m-1}(z_1, \dots, z_n) z_0 + a_m(z_1, \dots, z_n)$.

Case 2: *Let* $f = z_0^m z_i + a_1(z_1, \dots, z_n) z_0^{m-1} + \cdots + a_{m-1}(z_1, \dots, z_n) z_0 + a_m(z_1, \dots, z_n)$.

Lemma 2.4 [Chen 1995, Lemma 1.2]. *Let f be a weighted homogeneous polynomial in $z_0, \dots, z_n$ which defines an isolated singularity at the origin. Then there is a term of the form $z_i^{a_i}$ or $z_i^{a_i} z_j$ in f for any i ($a_i \geq 2$ in the case $z_i^{a_i}$ and $a_i \geq 1$ otherwise).*

Remark 2.5. When we talk about the weight of an element in an ideal, we always assume that the element is nonzero.

Now we begin to prove the Main Theorem.

Proof of the Main Theorem. Let

$$A_\nu := \mathbb{C}[z_0, z_1, \dots, z_n] \Big/ \left(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}, I_\nu \right)$$

and

$$B := \mathbb{C}[z_0, z_1, \dots, z_n] \Big/ \left(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right).$$

It is clear that $A_\nu = B/(I_\nu)$, and A_ν is a commutative Artinian algebra. Let $D_\nu \in L(A_\nu)$ be a derivation of A_ν, and let D_ν be an A_ν-linear combination of $\partial/\partial z_0, \partial/\partial z_1, \dots, \partial f/\partial z_n$. By Lemma 2.2, we know that $D_\nu(\mathfrak{m}) \subset \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal $(z_0, \dots, z_n)$, thus the coefficients of $\partial/\partial z_0, \partial/\partial z_1, \dots, \partial f/\partial z_n$ do not contain the constant term. Moreover, D_ν has negative weight, thus we write

$$D_\nu = p_0(z_1, \dots, z_n) \frac{\partial}{\partial z_0} + p_1(z_2, \dots, z_n) \frac{\partial}{\partial z_1} + \cdots + p_{n-2}(z_{n-1}, z_n) \frac{\partial}{\partial z_{n-2}} + c z_n^k \frac{\partial}{\partial z_{n-1}},$$

where $k \geq 1$ and c is a constant. Observe that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_0}\right) = d - \alpha_0, \ \mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = d - \alpha_1, \dots, \ \mathrm{wt}\left(\frac{\partial f}{\partial z_n}\right) = d - \alpha_n,$$

so we have $0 < \mathrm{wt}(\partial/\partial z_0) \leq \mathrm{wt}(\partial/\partial z_1) \leq \cdots \leq \mathrm{wt}(\partial f/\partial z_n)$. Since D_ν is a derivation of A_ν, we have $D_\nu(J_\nu) \subset J_\nu$, where $J_\nu = (\partial f/\partial z_0, \partial f/\partial z_1, \dots, \partial f/\partial z_n, I_\nu)$. Moreover, $\mathrm{wt}(D_\nu(\partial f/\partial z_0)) < \mathrm{wt}(\partial f/\partial z_0)$ implies that $D_\nu(\partial f/\partial z_0)$ does not contain any linear combination of $\partial f/\partial z_0, \partial f/\partial z_1, \dots, \partial f/\partial z_n$.

We divide the proof of the main theorem into four propositions.

Proposition 2.6. *Let $f(z_0, z_1)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1; d)$ with an isolated singularity at the origin. Assume that $d \geq 2\alpha_0 \geq 2\alpha_1$. Let D be a derivation of the algebra*

$$\mathbb{C}[z_0, z_1] \Big/ \left(\frac{\partial^2 f}{\partial z_0^2}, \frac{\partial^2 f}{\partial z_0 \partial z_1}, \frac{\partial^2 f}{\partial z_1^2} \right).$$

Then $D \equiv 0$, if D is of negative weight.

Proof. It is clear that $D(\partial^2 f/\partial z_0^2) = 0$. We have $D = z_1^k(\partial/\partial z_0)$, where $k \geq 1$ and $\mathrm{wt}(D) = k\alpha_1 - \alpha_0 < 0$. Let

$$f(z_0, z_1) = \sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} c_{(n_0,n_1)} z_0^{n_0} z_1^{n_1}.$$

Then we have

$$\begin{aligned} D\left(\frac{\partial^2 f}{\partial z_0^2}\right) &= z_1^k \frac{\partial}{\partial z_0} \left(\sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} n_0(n_0 - 1) c_{(n_0,n_1)} z_0^{n_0-2} z_1^{n_1} \right) \\ &= \sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} n_0(n_0 - 1)(n_0 - 2) c_{(n_0,n_1)} z_0^{n_0-3} z_1^{n_1+k} = 0. \end{aligned}$$

So, when $n_0 \geq 3$, $c_{(n_0,n_1)} = 0$, i.e.,

$$f(z_0, z_1) = c_{(2,p)} z_0^2 z_1^r + c_{(1,p)} z_0 z_1^p + c_{(0,q)} z_1^q,$$

where $d = 2\alpha_0 + r\alpha_1 = \alpha_0 + p\alpha_1 = q\alpha_1$.

If $c_{(2,p)} = 0$, then in order for f to have isolated singularity at the origin, we need $p = 1$. So

$$f(z_0, z_1) = c_{(1,p)} z_0 z_1 + c_{(0,q)} z_1^q \quad \text{and} \quad \frac{\partial^2 f}{\partial z_0 \partial z_1} = c_{(1,p)}.$$

So, $D = z_1^k(\partial/\partial z_0)$ is a zero derivation on $\mathbb{C}[z_0, z_1]/(\partial^2 f/\partial z_0^2, \partial^2 f/\partial z_0 \partial z_1, \partial^2 f/\partial z_1^2)$.

If $c_{(2,p)} \neq 0$, then by Lemma 2.4, we obtain that $r = 0$ or $r = 1$. If $r = 0$, then $\partial^2 f/\partial z_0^2 = 2c_{(2,p)}$. If $r = 1$, then $\partial^2 f/\partial z_0^2 = 2c_{(2,p)} z_1$. Hence, $D = z_1^k(\partial/\partial z_0)$ is a zero derivation on $\mathbb{C}[z_0, z_1]/(\partial^2 f/\partial z_0^2, \partial^2 f/\partial z_0 \partial z_1, \partial^2 f/\partial z_1^2)$. □

Proposition 2.7. *Let $(V, 0) = \{(z_0, z_1, z_2) \in \mathbb{C}^3 : f(z_0, z_1, z_2) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_0, \alpha_1, \alpha_2; d)$ with* $\mathrm{mult}(f) \geq 4$. *Assume that $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2 > 0$, without loss of generality. Let $H_1(V)$ be the first Hessian algebra. Let D_1 be a derivation of the algebra $H_1(V)$, i.e., $D_1 \in L_1(V)$, then $D_1 \equiv 0$, if D_1 is of negative weight.*

Proof. For simplicity, we use D to denote D_1. It is clear that $D(\partial^2 f/\partial z_0^2) = 0$. We have $D = p(z_1, z_2)(\partial/\partial z_0) + c z_2^k(\partial/\partial z_1)$, where c is a constant. There are two cases: $c = 0$ or $c \neq 0$.

Case 1: Assume $c = 0$. In this case, $D = p(z_1, z_2)(\partial/\partial z_0)$. By Lemma 2.3, we separate it into two cases.

Case 1.1: Let $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$. Then

$$D\left(\frac{\partial^2 f}{\partial z_0^2}\right) = p(z_1, z_2)\big[m(m-1)(m-2)z_0^{m-3} + (m-1)(m-2)(m-3)a_1(z_1, z_2)z_0^{m-4} + \cdots + 6a_{m-3}(z_1, z_2)\big] = 0,$$

which implies $p(z_1, z_2) = 0$.

Case 1.2: Let $f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2)$. Then

$$D\left(\frac{\partial^2 f}{\partial z_0^2}\right) = p(z_1, z_2)\big[m(m-1)(m-2)z_0^{m-3}z_i + (m-1)(m-2)(m-3)a_1(z_1, z_2)z_0^{m-3} + \cdots + 6a_{m-3}(z_1, z_2)],$$

which implies $p(z_1, z_2) = 0$, i.e., $D \equiv 0$.

Case 2: Assume $c \neq 0$. According to Lemma 2.3, we also need to separate it into two cases.

Case 2.1: Let $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$. Then

$$\begin{aligned} D\left(\frac{\partial^2 f}{\partial z_0^2}\right) &= p(z_1, z_2)\big[m(m-1)(m-2)z_0^{m-3} + (m-1)(m-2)(m-3)a_1(z_1, z_2)z_0^{m-4} + \cdots + 6a_{m-3}(z_1, z_2)\big] \\ &\quad + cz_2^k\left[(m-1)(m-2)\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-3} + (m-2)(m-3)\frac{\partial a_2(z_1, z_2)}{\partial z_1}z_0^{m-4} + \cdots + 2\frac{\partial a_{m-2}(z_1, z_2)}{\partial z_1}\right]. \end{aligned}$$

Because $D(\partial^2 f/\partial z_0^2) = 0$ and $m \geq 4$, we have

$$mp(z_1, z_2) = -cz_2^k\frac{\partial a_1(z_1, z_2)}{\partial z_1}.$$

We construct the coordinate transformation

$$\begin{cases} z_0 = z_0' - \dfrac{1}{m}a_1(z_1', z_2'), \\ z_1 = z_1', \\ z_2 = z_2'. \end{cases}$$

Then

$$\begin{aligned} D &= -\frac{1}{m}cz_2^k\frac{\partial a_1(z_1, z_2)}{\partial z_1}\frac{\partial}{\partial z_0} + cz_2^k\frac{\partial}{\partial z_1} \\ &= cz_2^k\left(-\frac{1}{m}\frac{\partial a_1(z_1, z_2)}{\partial z_1}\frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1}\right) = c(z_2')^k\frac{\partial}{\partial z_1'}. \end{aligned}$$

Letting $g(z_0', z_1', z_2') = f(z_0, z_1, z_2)$, we know that g is also a weighted homogeneous polynomial and

$$g = (z_0')^m + b_1(z_1', z_2')(z_0')^{m-1} + \cdots + b_m(z_1', z_2').$$

By the same argument as before, we have

$$D\left(\frac{\partial^2 g}{\partial {z_0'}^2}\right) = 0.$$

So we assume that

$$D\left(\frac{\partial^2 g}{\partial (z_0')^2}\right) = c(z_2')^k \frac{\partial}{\partial z_1'}\left(\frac{\partial^2 g}{\partial (z_0')^2}\right) = 0.$$

So

$$\frac{\partial b_1(z_1', z_2')}{\partial z_1'} = \cdots = \frac{\partial b_{m-2}(z_1', z_2')}{\partial z_1'} = 0.$$

Furthermore,

$$D\left(\frac{\partial^2 g}{\partial z_0' \partial z_1'}\right) = c(z_2')^k \frac{\partial^3 g}{\partial z_0' \partial (z_1')^2} = c(z_2')^k \frac{\partial^2 b_{m-1}(z_1', z_2')}{\partial (z_1')^2}$$

belongs to the principal ideal generated by $\partial^2 g/\partial (z_0')^2$. Hence,

$$c = 0 \quad \text{or} \quad \frac{\partial^2 b_{m-1}(z_1', z_2')}{\partial (z_1')^2} = 0.$$

If $c = 0$, then we have already finished it. In the following, we assume that $\partial^2 b_{m-1}(z_1', z_2')/\partial (z_1')^2 = 0$. So we have

$$\frac{\partial^2 g}{\partial z_0' \partial z_1'} = \frac{\partial b_{m-1}(z_1', z_2')}{\partial z_1'}.$$

Then, it is easy to see that

$$D\left(\frac{\partial^2 g}{\partial (z_1')^2}\right) = c(z_2')^k \frac{\partial^3 g}{\partial (z_1')^3} = c(z_2')^k \frac{\partial^3 b_m(z_1', z_2')}{\partial (z_1')^3}$$

belongs to the ideal generated by $\partial^2 g/\partial (z_0')^2$, $\partial^2 g/(\partial z_0' \partial z_1')$ and $\partial^2 g/(\partial z_0' \partial z_2')$. If one of $b_1, \ldots, b_{m-2}$ is not zero, then $D(\partial^2 g/\partial (z_1')^2)$ belongs to the principal ideal generated by $\partial^2 g/(\partial z_0' \partial z_1')$, i.e., there exists a polynomial $h(z_2, z_3)$ such that

$$D\left(\frac{\partial^2 g}{\partial (z_1')^2}\right) = h \frac{\partial^2 g}{\partial z_0' \partial z_1'},$$

i.e.,

$$c(z_2')^k \frac{\partial^3 b_m(z_1', z_2')}{\partial (z_1')^3} = h(z_1', z_2') \frac{\partial b_{m-1}(z_1', z_2')}{\partial z_1'}. \tag{1}$$

Let $b_{m-1}(z_1', z_2') = pz_1'(z_2')^s + q(z_2')^t$, where $p \neq 0$. By Lemma 2.4, we obtain that at least one of $(z_1')^{l_1}$ and $(z_1')^{l_2}z_2'$ is contained in $b_m(z_1', z_2')$. If $(z_1')^{l_1}$ is contained in $b_m(z_1', z_2')$ as a monomial, then $\partial^3 b_m(z_1', z_2')/\partial(z_1')^3$ is not divisible by z_2'. Hence, $k \geq s$ by (2). Moreover, since $s\alpha_2 + \alpha_1 + \alpha_0 = m\alpha_0$, we easily obtain

$$(m-1)\alpha_0 - \alpha_1 = s\alpha_2 \leq k\alpha_2 < \alpha_1,$$

i.e., $(m-1)\alpha_0 < 2\alpha_0$ which is in contradiction with $m \geq 4$. Hence, $(z_1')^{l_2}z_2'$ must be contained in $b_m(z_1', z_2')$. Then $\partial^3 b_m(z_1', z_2')/\partial(z_1')^3$ is not divisible by $(z_2')^2$. By (1), it is easy to see that $k+1 \geq s$, which implies

$$(m-1)\alpha_0 - \alpha_1 - \alpha_2 = (s-1)\alpha_2 \leq k\alpha_2 < \alpha_1,$$

i.e., $(m-1)\alpha_0 < 2\alpha_1 + \alpha_2$, which is in contradiction with $m \geq 4$.

In the following, we assume that $b_1 = \cdots = b_{m-2} = 0$, then

$$f = (z_0')^m + b_{m-1}(z_1', z_2')z_0' + b_m(z_1', z_2').$$

Hence there exist two polynomial h_1 and h_2 such that

$$c(z_2')^k \frac{\partial^3 b_m(z_1', z_2')}{\partial(z_1')^3} = h_1(z_1', z_2')\frac{\partial b_{m-1}(z_1', z_2')}{\partial z_1'} + h_2(z_1', z_2')\frac{\partial b_{m-1}(z_1', z_2')}{\partial z_2'}. \tag{2}$$

The weight of the left-hand side of (2) is equal to $k\alpha_2 + m\alpha_0 - 3\alpha_1$. The weight of the right-hand side of (2) is equal to $\mathrm{wt}(h_2) + (m-1)\alpha_0 - \alpha_2$. Hence, $\mathrm{wt}(h_2) = k\alpha_2 + \alpha_0 - 3\alpha_1 + \alpha_2 \geq \alpha_2$, which implies that $\alpha_0 \geq 3\alpha_1 - k\alpha_2 > 2\alpha_1$. Let $b_{m-1}(z_1', z_2') = pz_1'(z_2')^s + q(z_2')^t$, where $p \neq 0$. By Lemma 2.4, we obtain that at least one of $(z_1')^{l_1}$ and $(z_1')^{l_2}z_2'$ is contained in $b_m(z_1', z_2')$. If $(z_1')^{l_1}$ is contained in $b_m(z_1', z_2')$ as a monomial, then $\partial^3 b_m(z_1', z_2')/\partial(z_1')^3$ is not divisible by z_2'. Hence, we obtain $k \geq s-1$ by (2). Moreover, since $s\alpha_2 + \alpha_1 + \alpha_0 = m\alpha_0$, we easily obtain

$$(m-1)\alpha_0 - \alpha_1 = s\alpha_2 \leq (k+1)\alpha_2 < \alpha_1 + \alpha_2,$$

i.e., $(m-1)\alpha_0 < 3\alpha_0$, which is in contradiction with $m \geq 4$. Hence, $(z_1')^{l_2}z_2'$ must be contained in $b_m(z_1', z_2')$. Then $\partial^3 b_m(z_1', z_2')/\partial(z_1')^3$ is not divisible by $(z_2')^2$. By (2), it is easy to see that $k+1 \geq s-1$, which implies

$$(m-1)\alpha_0 - \alpha_1 - \alpha_2 = (s-1)\alpha_2 \leq (k+1)\alpha_2 < \alpha_1 + \alpha_2,$$

i.e., $(m-1)\alpha_0 < 2\alpha_1 + 2\alpha_2 < 3\alpha_0$, which is in contradiction with $m \geq 4$.

Case 2.2: Let $f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$, with $m \geq 3$.

If $i = 1$, then we have

$$\begin{aligned}0 &= D\left(\frac{\partial^2 f}{\partial z_0^2}\right)\\ &= p(z_1, z_2)\big[m(m-1)(m-2)z_0^{m-3}z_1 + (m-1)(m-2)(m-3)a_1(z_1, z_2)z_0^{m-4} + \cdots\big]\\ &\qquad + cz_2^k\left[m(m-1)z_0^{m-2} + (m-1)(m-2)\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-3} + \cdots\right].\end{aligned}$$

This forces $c = 0$ which contradicts our hypothesis. So this case does not occur.

If $i = 2$, then we have

$$\begin{aligned}0 &= D\left(\frac{\partial^2 f}{\partial z_0^2}\right)\\ &= p(z_1, z_2)\frac{\partial}{\partial z_0}\left(\frac{\partial^2 f}{\partial z_0^2}\right) + cz_2^k\frac{\partial}{\partial z_1}\left(\frac{\partial^2 f}{\partial z_0^2}\right)\\ &= \frac{\partial^2}{\partial z_0^2}\left[p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right],\end{aligned} \tag{3}$$

and similarly $D((\partial^2 f)/\partial z_0 \partial z_1)$ is a multiple of $\partial^2 f/\partial z_0^2$.

$$\begin{aligned}D\left(\frac{\partial^2 f}{\partial z_0 \partial z_1}\right) &= p(z_1, z_2)\frac{\partial}{\partial z_0}\left(\frac{\partial^2 f}{\partial z_0 \partial z_1}\right) + cz_2^k\frac{\partial}{\partial z_1}\left(\frac{\partial^2 f}{\partial z_0 \partial z_1}\right)\\ &= \frac{\partial^2}{\partial z_0 \partial z_1}\left(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) - \frac{\partial p(z_1, z_2)}{\partial z_1}\cdot\frac{\partial^2 f}{\partial z_0^2}\\ &= h\frac{\partial^2 f}{\partial z_0^2}.\end{aligned} \tag{4}$$

Equation (4) implies that

$$\frac{\partial^2}{\partial z_0 \partial z_1}\left(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) = \tilde{h}\,\frac{\partial^2 f}{\partial z_0^2}.$$

From (3), we know that the left-hand side of this equation is independent of z_0 variable. Since $m \geq 3$, the right-hand side of this equation is independent of z_0 variable only if $\tilde{h} = 0$. Thus we have

$$\frac{\partial^2}{\partial z_0 \partial z_1}\left(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) = 0.$$

So we have

$$\frac{\partial}{\partial z_0}\left(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) = uz_2^l,$$

where either $u = 0$ or $u \neq 0$ and $l > k$. This is

$$(5)\quad \frac{\partial}{\partial z_0}\Big[p(z_1,z_2)\big(mz_2z_0^{m-1}+(m-1)a_1(z_1,z_2)z_0^{m-2}+\cdots+a_{m-1}(z_1,z_2)\big) + cz_2^k\Big(\frac{\partial a_1(z_1,z_2)}{\partial z_1}z_0^{m-1}+\frac{\partial a_2(z_1,z_2)}{\partial z_1}z_0^{m-2}+\cdots+\frac{\partial a_m(z_1,z_2)}{\partial z_1}\Big)\Big] = uz_2^l.$$

As $m \geq 4$, (5) implies

$$mp(z_1,z_2)z_2 + cz_2^k\frac{\partial a_1(z_1,z_2)}{\partial z_1} = 0.$$

If $cz_2^k(\partial a_1(z_1,z_2)/\partial z_1) = 0$, then $p(z_1,z_2) = 0$ and (5) becomes

$$\frac{\partial}{\partial z_0}\Big[cz_2^k\Big(\frac{\partial a_1(z_1,z_2)}{\partial z_1}z_0^{m-1}+\frac{\partial a_2(z_1,z_2)}{\partial z_1}z_0^{m-2}+\cdots+\frac{\partial a_m(z_1,z_2)}{\partial z_1}\Big)\Big] = uz_2^l.$$

Since $c \neq 0$ and $m \geq 3$, we have

$$\frac{\partial a_1(z_1,z_2)}{\partial z_1} = \frac{\partial a_2(z_1,z_2)}{\partial z_1} = \cdots = \frac{\partial a_{m-2}(z_1,z_2)}{\partial z_1} = 0$$

and

$$\frac{\partial a_{m-1}(z_1,z_2)}{\partial z_1} = ez_2^{l-k},$$

where $e \neq 0$. Hence,

$$a_{m-1}(z_1,z_2) = ez_1z_2^{l-k} + e'z_2^{l'}.$$

Now we consider

$$\frac{\partial^2 f}{\partial z_1^2} = cz_2^k\frac{\partial^3 a_m(z_1,z_2)}{\partial z_1^3}.$$

We can do a similar computation as in Case 2.1 and get a contradiction.

If $cz_2^k(\partial a_1(z_1,z_2)/\partial z_1) \neq 0$. Then

$$p(z_1,z_2) = z_2^{k-1}q(z_1,z_2),$$

where

$$q(z_1,z_2) = -\frac{c}{m}\frac{\partial a_1(z_1,z_2)}{\partial z_1}.$$

So we have

$$\frac{\partial}{\partial z_0}\Big(q(z_1,z_2)\frac{\partial f}{\partial z_0} + cz_2\frac{\partial f}{\partial z_1}\Big) = uz_2^{l-k+1}.$$

We write

$$q(z_1,z_2) = \alpha z_1^s + z_2\gamma(z_1,z_2).$$

We claim that $\alpha = 0$. Suppose on the contrary that $\alpha \neq 0$. If we rewrite f in the form

$$f = b_0(z_0, z_1)z_2^n + b_1(z_0, z_1)z_2^{n-1} + \cdots + b_n(z_0, z_1),$$

then

$$\begin{aligned}(6)\quad &\frac{\partial}{\partial z_0}\Bigg[(\alpha z_1^s + z_2\gamma(z_1, z_2))\left(\frac{\partial b_0(z_0, z_1)}{\partial z_0}z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_0}\right)\\ &\qquad + cz_2\left(\frac{\partial b_0(z_0, z_1)}{\partial z_1}z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1}\right)\Bigg]\\ &= (\alpha z_1^s + z_2\gamma(z_1, z_2))\left(\frac{\partial^2 b_0(z_0, z_1)}{\partial z_0^2}z_2^n + \cdots + \frac{\partial^2 b_n(z_0, z_1)}{\partial z_0^2}\right)\\ &\qquad + cz_2\left(\frac{\partial^2 b_0(z_0, z_1)}{\partial z_0\partial z_1}z_2^n + \cdots + \frac{\partial^2 b_n(z_0, z_1)}{\partial z_0\partial z_1}\right)\\ &= uz_2^{l-k+1}.\end{aligned}$$

Considering the coefficient of z_1^s, we know that $\partial^2 b_n(z_0, z_1)/\partial z_0^2 = 0$, and hence from (6) again, we have

$$\begin{aligned}(7)\quad &(\alpha z_1^s + z_2\gamma(z_1, z_2))\left(\frac{\partial^2 b_0(z_0, z_1)}{\partial z_0^2}z_2^{n-1} + \cdots + \frac{\partial^2 b_{n-1}(z_0, z_1)}{\partial z_0^2}\right)\\ &\qquad + c\left(\frac{\partial^2 b_0(z_0, z_1)}{\partial z_0\partial z_1}z_2^n + \cdots + \frac{\partial^2 b_n(z_0, z_1)}{\partial z_0\partial z_1}\right) = uz_2^{l-k}.\end{aligned}$$

Recall that either $u = 0$ or $u \neq 0$ and $l > k$. Equation (7) implies

$$\alpha z_1^s \frac{\partial^2 b_{n-1}(z_0, z_1)}{\partial z_0^2} = -c\frac{\partial^2 b_n(z_0, z_1)}{\partial z_0\partial z_1}.$$

Since $\partial^2 b_n(z_0, z_1)/\partial z_0^2 = 0$, we have

$$\frac{\partial^2 b_{n-1}(z_0, z_1)}{\partial z_0^2} = c' z_1^{s'},$$

where $c' \neq 0$.

If $s' = 0$, then $b_{n-1}(z_0, z_1) = c' z_0^2 + \cdots$ and $z_0^2 z_2$ occur in f which is in contradiction with our assumption.

If $s' > 0$, then

$$b_{n-1}(z_0, z_1) = c' z_1^{s'} z_0^2 + c'' z_1^{s''} z_0 + \tilde{u} z_1^{\tau},$$

where $s' > 0$ and $\tau \geq 0$. Now $\partial^2 b_n(z_0, z_1)/\partial z_0^2 = 0$ implies $b_n(z_0, z_1) = wz_0 z_1^t + w' z_1^{t'}$, where $t > 1$. Notice that

$$\frac{\partial f}{\partial z_2} = nb_0(z_0, z_1)z_2^{n-1} + (n-1)b_1(z_0, z_1)z_2^{n-2} + \cdots + b_{n-1}(z_0, z_1).$$

It follows that f is singular along the z_0-axis. The contradiction comes from our hypothesis that $\alpha \neq 0$. Thus, $\alpha = 0$.

Now we have

$$q(z_1, z_2) = z_2\gamma(z_1, z_2) \quad \text{and} \quad \gamma(z_1, z_2)\frac{\partial^2 f}{\partial z_0^2} + c\frac{\partial^2 f}{(\partial z_0 \partial z_1)} = u z_2^{l-k}.$$

So we have

$$\gamma(z_1, z_2)\big[m(m-1)z_0^{m-2}z_2 + (m-1)(m-2)a_1(z_1, z_2)z_0^{m-3} + \cdots + 2a_{m-2}(z_1, z_2)\big] + c\left[(m-1)\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-2} + \cdots + \frac{\partial a_{m-1}(z_1, z_2)}{\partial z_1}\right] = u z_2^{l-k}.$$

This implies that

$$m\gamma(z_1, z_2)z_2 + c\frac{\partial a_1(z_1, z_2)}{\partial z_1} = 0.$$

Hence, $\partial a_1(z_1, z_2)/\partial z_1$ is divisible by z_2. Let $a_1'(z_1, z_2)$ be a weighted homogeneous polynomial satisfying $\partial a_1'(z_1, z_2)/\partial z_1 = (\partial a_1/\partial z_1)/z_2$. Consider the coordinate transformation

$$\begin{cases} z_0 = z_0' - \dfrac{1}{m}a_1'(z_1', z_2'), \\ z_1 = z_1', \\ z_2 = z_2'. \end{cases}$$

After this coordinate transformation, we have

$$\frac{\partial^2}{\partial (z_0')^2} = \frac{\partial^2}{\partial z_0^2} \quad \text{and} \quad \frac{\partial^2}{\partial z_0' \partial z_1'} = -\frac{1}{m}\frac{\partial a_1'}{\partial z_1'}\frac{\partial^2}{\partial z_0^2} + \frac{\partial^2}{\partial z_0 \partial z_1}.$$

Hence,

$$\begin{aligned} \gamma(z_1, z_2)\frac{\partial^2 f}{\partial z_0^2} + c\frac{\partial^2 f}{\partial z_0 \partial z_1} &= -\frac{c}{m}\frac{\partial a_1'}{\partial z_1}\frac{\partial^2 f}{\partial z_0^2} + c\frac{\partial^2 f}{\partial z_0 \partial z_1} \\ &= c\left(-\frac{1}{m}\frac{\partial a_1'}{\partial z_1}\frac{\partial^2 f}{\partial z_0^2} + \frac{\partial^2 f}{\partial z_0 \partial z_1}\right) \\ &= c\frac{\partial^2 f}{\partial z_0' \partial z_1'} \\ &= u(z_2')^{l-k}. \end{aligned}$$

For simplicity of notation, we still use (z_0, z_1, z_2) to represent the coordinates after coordinate transformation. Without loss of generality, we assume

$$f = z_0^m z_2 + a_1(z_2)z_0^{m-1} + \cdots + a_{m-2}(z_2)z_0^2 + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2).$$

Now we have $cz_2^k(\partial a_1/\partial z_1) = 0$. From a similar discussion as above, we obtain the conclusion. □

Proposition 2.8. *Let $f(z_0, z_1, z_2)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1, \alpha_2; d)$ with isolated singularity at the origin. Assume $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$. Let D_2 be a derivation of the algebra $\mathbb{C}[z_0, z_1, z_2]/(I, I_2)$. Then $D_2 \equiv 0$, if D_2 is of negative weight.*

First, we need the following lemma:

Lemma 2.9. *The smallest weight of an element in I_2 is greater than or equal to the weight of $\partial f/\partial z_0$ when $m \geq 3$, where m is the exponent of z_0 in Lemma 2.3.*

Proof. It is obvious that

$$I_2 = \left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}, \begin{vmatrix} f_{00} & f_{02} \\ f_{01} & f_{12} \end{vmatrix}, \dots, \begin{vmatrix} f_{01} & f_{02} \\ f_{12} & f_{22} \end{vmatrix}, \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} \right)$$

and

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} \right) = 2d - 2\alpha_0 - 2\alpha_1.$$

Note that $\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}$ is an element with the smallest weight in I_2. We obtain that

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} \right) \geq \mathrm{wt}\left(\frac{\partial f}{\partial z_0} \right)$$

if and only if $2d - 2\alpha_0 - 2\alpha_1 \geq d - \alpha_0$, which is equivalent to $d \geq \alpha_0 + 2\alpha_1$.

Case 1: Let $f = z_0^m + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_m(z_1, z_2)$. In this case, $d = m\alpha_0$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 3$.

Case 2: Let $f = z_0^m z_1 + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_m(z_1, z_2)$. In this case, $d = m\alpha_0 + \alpha_1$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 + \alpha_1 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 2$.

Case 3: Let $f = z_0^m z_2 + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_m(z_1, z_2)$. In this case, $d = m\alpha_0 + \alpha_2$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 + \alpha_2 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 3$. ☐

Proof of Proposition 2.8. By Lemma 2.9, we obtain that $D_2(\partial f/\partial z_0) = 0$ when $m \geq 3$. We only need to consider the following two cases:

Case 1: Assume $c = 0$. In this case, $D = p(z_1, z_2)(\partial/\partial z_0)$. By Lemma 2.3, we have to consider two subcases.

Case 1.1: Let $f = z_0^m + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2) z_0 + a_m(z_1, z_2)$. Then

$$D\left(\frac{\partial f}{\partial z_0} \right) = p(z_1, z_2)\big[m(m-1) z_0^{m-2} + (m-1)(m-2) a_1(z_1, z_2) z_0^{m-3} + \cdots + 2a_{m-2}(z_1, z_2) \big] = 0,$$

which implies $p(z_1, z_2) = 0$. If not, the above equation holds only if $m = 1$, which is absurd in view of our assumption. So we must have $p(z_1, z_2) = 0$, i.e., $D \equiv 0$.

Case 1.2: Let $f = z_0^m z_i + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_m(z_1, z_2)$. Then

$$D\left(\frac{\partial f}{\partial z_0}\right) = p(z_1, z_2)\big[m(m-1)z_0^{m-2} z_i + (m-1)(m-2)a_1(z_1, z_2) z_0^{m-3} + \cdots + 2a_{m-2}(z_1, z_2)\big],$$

which implies $p(z_1, z_2) = 0$. If not, the above equation holds only if $m = 1$, which is absurd in view of our assumption. So we must have $p(z_1, z_2) = 0$, i.e., $D \equiv 0$.

Case 2: Assume $c \neq 0$. According to Lemma 2.3, we also need to divide it into two subcases.

Case 2.1: Let $f = z_0^m + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2) z_0 + a_m(z_1, z_2)$. Then

$$\begin{aligned} D\left(\frac{\partial f}{\partial z_0}\right) &= p(z_1, z_2)\big[m(m-1)z_0^{m-2} + (m-1)(m-2)a_1(z_1, z_2) z_0^{m-3} + \cdots + 2a_{m-1}(z_1, z_2)\big] \\ &\quad + c z_2^k \left[(m-1)\frac{\partial a_1(z_1, z_2)}{\partial z_1} z_0^{m-2} + (m-2)\frac{\partial a_2(z_1, z_2)}{\partial z_1} z_0^{m-3} + \cdots + \frac{\partial a_{m-1}(z_1, z_2)}{\partial z_1}\right]. \end{aligned}$$

Because $D(\partial f/\partial z_0) = 0$ and $m \geq 3$, we have

$$m p(z_1, z_2) = -c z_2^k \frac{\partial a_1(z_1, z_2)}{\partial z_1}.$$

We construct the coordinate transformation

$$\begin{cases} z_0 = z_0' - \dfrac{1}{m} a_1(z_1', z_2'), \\ z_1 = z_1', \\ z_2 = z_2'. \end{cases}$$

Then

$$\begin{aligned} D &= -\frac{1}{m} c z_2^k \frac{\partial a_1(z_1, z_2)}{\partial z_1} \frac{\partial}{\partial z_0} + c z_2^k \frac{\partial}{\partial z_1} \\ &= c z_2^k \left(-\frac{1}{m} \frac{\partial a_1(z_1, z_2)}{\partial z_1} \frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1}\right) \\ &= c (z_2')^k \frac{\partial}{\partial z_1'}. \end{aligned}$$

Letting $g(z_0', z_1', z_2') = f(z_0, z_1, z_2)$, we obtain that g is also a weighted homogeneous polynomial and

$$g = (z_0')^m + b_1(z_1', z_2')(z_0')^{m-1} + \cdots + b_m(z_1', z_2').$$

By the same argument as before, we have $D(\partial g/\partial z_0') = 0$. So

$$D\left(\frac{\partial g}{\partial z_0'}\right) = c (z_2')^k \frac{\partial}{\partial z_1'}\left(\frac{\partial g}{\partial z_0'}\right) = 0.$$

Thus,

$$\frac{\partial b_1(z_1', z_2')}{\partial z_1'} = \cdots = \frac{\partial b_{m-1}(z_1', z_2')}{\partial z_1'} = 0.$$

Consider

$$D\left(\frac{\partial g}{\partial z_1'}\right) = c(z_2')^k \frac{\partial^2 g}{\partial (z_1')^2} = c(z_2')^k \frac{\partial^2 b_m(z_1', z_2')}{\partial (z_1')^2}.$$

Since $\mathrm{wt}(D(\partial g/\partial z_1')) < \mathrm{wt}(\partial g/\partial z_1') = m\alpha_0 - \alpha_1$ and

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right) = (2m-2)\alpha_0 - 2\alpha_1 \geq m\alpha_0 - \alpha_1 = \mathrm{wt}\left(\frac{\partial g}{\partial z_1'}\right),$$

where $m \geq 3$, $D(\partial g/\partial z_1')$ is a multiple of $\partial g/\partial z_0'$:

$$D\left(\frac{\partial g}{\partial z_1'}\right) = c(z_2')^k \frac{\partial^2 g}{\partial (z_1')^2} = c(z_2')^k \frac{\partial^2 b_m(z_1', z_2')}{\partial (z_1')^2},$$
$$\frac{\partial g}{\partial z_0'} = m(z_0')^{m-1} + \cdots + b_{m-1}(z_1', z_2').$$

Further, $D(\partial g/\partial z_1')$ is a multiple of $\partial g/\partial z_0'$, i.e., there exists h such that

$$c(z_2')^k \frac{\partial^2 b_m(z_1', z_2')}{\partial (z_1')^2} = h\frac{\partial g(z_1', z_2')}{\partial z_0'} = h\big[m(z_0')^{m-1} + \cdots + b_{m-1}(z_1', z_2')\big].$$

If $\partial^2 b_m(z_1', z_2')/\partial (z_1')^2 \neq 0$, then we have $m = 1$, which is absurd. So we have $\partial^2 b_m(z_1', z_2')/\partial (z_1')^2 = 0$. This implies

$$b_m(z_1', z_2') = d_1 z_1'(z_2')^{l_1} + d_2(z_2')^{l_2},$$

where d_1, d_2 are constants. Then g has an isolated singularity at 0 only if $l_1 = 1$. Because $b_m(z_1', z_2')$ is weighted homogeneous, we have $d = \alpha_1 + \alpha_2$. It follows from the assumption $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ that $\alpha_0 = \alpha_1 = \alpha_2$. So g is a homogeneous polynomial. It follows from Proposition 2.1 that $D \equiv 0$.

Case 2.2: Let $f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$.

Case 2.2.1: Let $i = 1$. Then we have

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = p(z_1, z_2)\big[m(m-1)z_0^{m-2}z_1 + (m-1)(m-2)a_1(z_1, z_2)z_0^{m-3} + \cdots\big]$$
$$+ cz_2^k\left[mz_0^{m-1} + (m-1)\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-2} + \cdots\right].$$

This forces $c = 0$, which is in contradicts with our hypothesis. So this case does not occur.

Case 2.2.2: Let $i = 2$. In this case, $m \geq 3$ by our assumption. Then

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = p(z_1, z_2)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_0}\right) + cz_2^k\frac{\partial}{\partial z_1}\left(\frac{\partial f}{\partial z_0}\right) = \frac{\partial}{\partial z_0}\left[p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right]. \tag{8}$$

Similarly, we obtain that $D(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$:

$$D\left(\frac{\partial f}{\partial z_1}\right) = p(z_1, z_2)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_1}\right) + cz_2^k\frac{\partial}{\partial z_1}\left(\frac{\partial f}{\partial z_1}\right) = \frac{\partial}{\partial z_1}\left(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) - \frac{\partial p(z_1, z_2)}{\partial z_1}\frac{\partial f}{\partial z_0} = h\frac{\partial f}{\partial z_0}. \tag{9}$$

Equation (9) implies that

$$\frac{\partial}{\partial z_1}\left[p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right] = \tilde{h}\frac{\partial f}{\partial z_0}.$$

From (8), we know that the left-hand side of this equation is independent of the variable z_0. Since $m > 1$, the right-hand side of this equation is independent of the variable z_0 only if $\tilde{h} = 0$. Thus, we have

$$\frac{\partial}{\partial z_1}\left(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) = 0.$$

So we have

$$p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1} = uz_2^l,$$

where either $u = 0$ or $u \neq 0$ and $l > k$. This is

$$p(z_1, z_2)\big(mz_2z_0^{m-1} + (m-1)a_1(z_1, z_2)z_0^{m-2} + \cdots + a_{m-1}(z_1, z_2)\big) + cz_2^k\left[\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-1} + \frac{\partial a_2(z_1, z_2)}{\partial z_1}z_0^{m-2} + \cdots + \frac{\partial z_m(z_1, z_2)}{\partial z_1}\right] = uz_2^l. \tag{10}$$

As $m > 1$, (10) implies

$$mp(z_1, z_2)z_2 + cz_2^k\frac{\partial a_1(z_1, z_2)}{\partial z_1} = 0.$$

If $cz_2^k(\partial a_1(z_1, z_2)/\partial z_1) = 0$, then $p(z_1, z_2) = 0$ and (10) becomes

$$cz_2^k\left(\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-1} + \frac{\partial a_2(z_1, z_2)}{\partial z_1}z_0^{m-2} + \cdots + \frac{\partial a_m(z_1, z_2)}{\partial z_1}\right) = uz_2^l.$$

Since $c \neq 0$, we have

$$\frac{\partial a_1(z_1, z_2)}{\partial z_1} = \frac{\partial a_2(z_1, z_2)}{\partial z_1} = \cdots = \frac{\partial a_{m-1}(z_1, z_2)}{\partial z_1} = 0,$$
$$\frac{\partial a_m(z_1, z_2)}{\partial z_1} = e z_2^{l-k},$$

where $e \neq 0$. Hence,

$$a_m(z_1, z_2) = e z_1 z_2^{l-k} + z_2^k.$$

We must have $l - k = 1$ and $e \neq 0$, otherwise f will be singular along the z_1-axis, which is a contradiction. Since the term $z_1 z_2$ appears in f, we conclude that $\alpha_1 + \alpha_2 = d$. The assumption $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies that $\alpha_0 = \alpha_1 = \alpha_2$. So f is a homogeneous polynomial. By Proposition 2.1, we have that $D \equiv 0$.

If $c z_2^k (\partial a_1(z_1, z_2)/\partial z_1) \neq 0$, then

$$p(z_1, z_2) = z_2^{k-1} q(z_1, z_2),$$

where

$$q(z_1, z_2) = -\frac{c}{m} \frac{\partial a_1(z_1, z_2)}{\partial z_1}.$$

So we have

$$q(z_1, z_2) \frac{\partial f}{\partial z_0} + c z_2 \frac{\partial f}{\partial z_1} = u z_2^{l-k+1}.$$

Write

$$q(z_1, z_2) = \alpha z_1^s + z_2 \gamma(z_1, z_2).$$

We claim that $\alpha = 0$; suppose on the contrary that $\alpha \neq 0$. If we rewrite f in the form

$$f = b_0(z_0, z_1) z_2^n + b_1(z_0, z_1) z_2^{n-1} + \cdots + b_n(z_0, z_1),$$

then

$$(11) \quad [\alpha z_1^s + z_2 \gamma(z_1, z_2)] \left(\frac{\partial b_0(z_0, z_1)}{\partial z_0} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_0} \right) + c z_2 \left(\frac{\partial b_0(z_0, z_1)}{\partial z_1} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1} \right) = u z_2^{l-k+1}.$$

Considering the coefficient of z_1^s, we know that $\partial b_n(z_0, z_1)/\partial z_0 = 0$, and hence from (11) again, we have

$$(12) \quad [\alpha z_1^s + z_2 \gamma(z_1, z_2)] \left(\frac{\partial b_0(z_0, z_1)}{\partial z_0} z_2^{n-1} + \cdots + \frac{\partial b_{n-1}(z_0, z_1)}{\partial z_0} \right) + c \left(\frac{\partial b_0(z_0, z_1)}{\partial z_1} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1} \right) = u z_2^{l-k}.$$

Recall that either $u = 0$ or $u \neq 0$ and $l > k$. Equation (12) implies

$$\alpha z_1^s \frac{\partial b_{n-1}(z_0, z_1)}{\partial z_0} = -c \frac{\partial b_n(z_0, z_1)}{\partial z_1}.$$

Since $\partial b_n(z_0, z_1)/\partial z_0 = 0$, we have

$$\frac{\partial b_{n-1}(z_0, z_1)}{\partial z_0} = c' z_1^{s'},$$

where $c' \neq 0$.

If $s' = 0$, then $b_{n-1}(z_0, z_1) = c' z_0 + c'' z_1$, and hence $z_0 z_2$ and $z_1 z_2$ occur in f. It follows again that $\alpha_0 = \alpha_1 = \alpha_2$, and we are finished.

If $s' > 0$, then

$$b_{n-1}(z_0, z_1) = c' z_1^{s'} z_0 + \tilde{u} z_1^{\tau},$$

where $s' > 0$ and $\tau > 0$. Now $\partial b_n(z_0, z_1)/\partial z_0 = 0$ implies $b_n(z_0, z_1) = w z_1^t$, where $t > 1$. Notice that

$$\frac{\partial f}{\partial z_2} = n b_0(z_0, z_1) z_2^{n-1} + (n-1) b_1(z_0, z_1) z_2^{n-2} + \cdots + b_{n-1}(z_0, z_1).$$

It follows that f is singular along the z_0-axis. The contradiction comes from our hypothesis that $\alpha \neq 0$. Thus, $\alpha = 0$.

Now we have $q(z_1, z_2) = z_2 \gamma(z_1, z_2)$ and $\gamma(z_1, z_2)(\partial f/\partial z_0) + c(\partial f/\partial z_1) = u z_2^{l-k}$. So we have

$$\gamma(z_1, z_2)\left(\frac{\partial b_0(z_0, z_1)}{\partial z_0} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_0}\right) + c\left(\frac{\partial b_0(z_0, z_1)}{\partial z_1} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1}\right) = u z_2^{l-k}.$$

It follows that $\gamma(z_1, 0) \neq 0$, otherwise we will have $\partial b_n(z_0, z_1)/\partial z_1 = 0$. So $b_n(z_0, z_1) = z_0^m$ for some n. Therefore, f will be of the form

$$f = z_0^m + c_1(z_1, z_2) z_0^{m-1} + \cdots + c_m(z_1, z_2),$$

which contradicts our assumption.

Now, let $\gamma(z_1, z_2) = v z_1^h + z_2 \bar{\gamma}(z_1, z_2)$, where $h > 0$ and $v \neq 0$. Then we have

$$v z_1^h \frac{\partial b_n(z_0, z_1)}{\partial z_0} + c \frac{\partial b_n(z_0, z_1)}{\partial z_1} = 0,$$

where $v \neq 0$ and $c \neq 0$. Let

$$b_n(z_0, z_1) = d_0 z_0^k z_1^{l_0} + d_1 z_0^{k-1} z_1^{l_1} + \cdots + d_k z_1^{l_k},$$

where $d_0 \neq 0$. Then

$$\frac{\partial b_n(z_0, z_1)}{\partial z_0} = kd_0 z_0^{k-1} z_1^{l_0} + (k-1) d_1 z_0^{k-2} z_1^{l_1} + \cdots + d_{k-1} z_1^{l_{k-1}},$$

$$\frac{\partial b_n(z_0, z_1)}{\partial z_1} = l_0 d_0 z_0^k z_1^{l_0 - 1} + l_1 d_1 z_0^{k-1} z_1^{l_1 - 1} + \cdots + l_k d_k z_1^{l_k - 1}.$$

Clearly, the equation

$$v z_1^h \frac{\partial b_n(z_0, z_1)}{\partial z_0} + c \frac{\partial b_n(z_0, z_1)}{\partial z_1} = 0$$

is true only if $k = 0$ or $l_0 = 0$.

If $k = 0$, then $b_n(z_0, z_1) = d_0 z_1^{l_0}$, which is absurd.

If $l_0 = 0$, then

$$b_n(z_0, z_1) = d_0 z_0^k + d_1 z_0^{k-1} z_1^{l_1} + \cdots + d_k z_1^{l_k}.$$

Hence, f is again of the form

$$f = z_0^m + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_m(z_1, z_2),$$

which is absurd. This completes the proof of Case 2.2. □

Proposition 2.10. *Let $f(z_0, z_1, z_2, z_3)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, d)$ with isolated singularity at the origin and* $\operatorname{mult}(f) > 3$. *Assume that $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2 \geq 2\alpha_3$.*

(a) *Let D_2 be a derivation of the algebra $\mathbb{C}[z_0, z_1, z_2, z_3]/(I, I_2)$. Then $D_2 \equiv 0$, if D_2 is of negative weight.*

(b) *Let D_3 be a derivation of the algebra $\mathbb{C}[z_0, z_1, z_2, z_3]/(I, I_3)$. Then $D_3 \equiv 0$, if D_3 is of negative weight.*

Proof. The derivation D_ν has the following form for $\nu = 2, 3$:

$$D_\nu = p_0(z_1, z_2, z_3) \frac{\partial}{\partial z_0} + p_1(z_2, z_3) \frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2},$$

where c, k are constants and $k \geq 1$.

Lemma 2.11. *The smallest weight of an element in I_2 is greater than or equal to the weight of $\partial f / \partial z_0$ when $m \geq 3$, where m is the exponent of z_0 in Lemma 2.3.*

Proof. It is obvious that

$$I_2 = \left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}, \begin{vmatrix} f_{01} & f_{02} \\ f_{11} & f_{12} \end{vmatrix}, \ldots, \begin{vmatrix} f_{12} & f_{13} \\ f_{23} & f_{33} \end{vmatrix}, \begin{vmatrix} f_{22} & f_{23} \\ f_{23} & f_{33} \end{vmatrix} \right),$$

and

$$\operatorname{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} \right) = 2d - 2\alpha_0 - 2\alpha_1.$$

The element with smallest weight in I_2 is $\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}$. We obtain that

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right) \geq \mathrm{wt}\left(\frac{\partial f}{\partial z_0}\right)$$

if and only if $2d - 2\alpha_0 - 2\alpha_1 \geq d - \alpha_0$ that is $d \geq \alpha_0 + 2\alpha_1$.

Case 1: Let $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2)$. In this case, $d = m\alpha_0$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 3$.

Case 2: Let $f = z_0^m z_1 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$. In this case, $d = m\alpha_0 + \alpha_1$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 + \alpha_1 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 2$.

Case 3: Let $f = z_0^m z_2 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$. In this case, $d = m\alpha_0 + \alpha_2$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 + \alpha_2 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 3$.

Case 4: Let $f = z_0^m z_3 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$. In this case, $d = m\alpha_0 + \alpha_3$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 + \alpha_3 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 3$. □

By Lemma 2.11 and our assumption that when $n = 3$, the multiplicity of f is greater than 3, we obtain that $D_2(\partial f/\partial z_0) = 0$ always holds.

Lemma 2.12. *The smallest weight of an element in I_3 is greater than or equal to the weight of $\partial f/\partial z_0$ when $m \geq 3$, where m is the exponent of z_0 in Lemma 2.3.*

Proof. It is obvious that

$$I_2 = \left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}, \begin{vmatrix} f_{00} & f_{01} & f_{03} \\ f_{01} & f_{12} & f_{13} \\ f_{02} & f_{22} & f_{23} \end{vmatrix}, \ldots, \begin{vmatrix} f_{01} & f_{02} & f_{03} \\ f_{12} & f_{12} & f_{13} \\ f_{13} & f_{23} & f_{33} \end{vmatrix}, \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{vmatrix}\right),$$

and

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right) = 3d - 2\alpha_0 - 2\alpha_1 - 2\alpha_2.$$

The element with the smallest weight in I_3 is

$$\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}.$$

We obtain that

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right) \geq \mathrm{wt}\left(\frac{\partial f}{\partial z_0}\right)$$

if and only if $3d - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 \geq d - \alpha_0$, which is $2d \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$.

Case 1: Let $f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$. In this case, $d = m\alpha_0$. So $2d \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$ if and only if $2m\alpha_0 \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$. This is clearly true when $m \geq 3$.

Case 2: Let

$$f = z_0^m z_1 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3).$$

In this case, $d = m\alpha_0 + \alpha_1$. So $2d \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$ if and only if $2m\alpha_0 + 2\alpha_1 \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$. This is clearly true when $m \geq 2$.

Case 3: Let $f = z_0^m z_2 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$. In this case, $d = m\alpha_0 + \alpha_2$. So $2d \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$ if and only if $2m\alpha_0 + 2\alpha_2 \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$. This is clearly true when $m \geq 2$.

Case 4: Let $f = z_0^m z_3 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$. In this case, $d = m\alpha_0 + \alpha_3$. So $2d \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$ if and only if $2m\alpha_0 + 2\alpha_3 \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$. This is clearly true when $m \geq 3$. □

By Lemma 2.12 and our assumption that when $n = 3$, the multiplicity of f is greater than 3, we obtain that $D_3(\partial f/\partial z_0) = 0$ always holds. The commutator $[\partial/\partial z_i, D_\nu]$ is of the following form by a direct computation:

$$\begin{aligned}
\left[\frac{\partial}{\partial z_0}, D_\nu\right] &= 0,\\
\left[\frac{\partial}{\partial z_1}, D_\nu\right] &= \frac{\partial p_0}{\partial z_1}\frac{\partial}{\partial z_0},\\
\left[\frac{\partial}{\partial z_2}, D_\nu\right] &= \frac{\partial p_0}{\partial z_2}\frac{\partial}{\partial z_0} + \frac{\partial p_1}{\partial z_2}\frac{\partial}{\partial z_1},\\
\left[\frac{\partial}{\partial z_3}, D_\nu\right] &= \frac{\partial p_0}{\partial z_3}\frac{\partial}{\partial z_0} + \frac{\partial p_1}{\partial z_3}\frac{\partial}{\partial z_1} + \frac{\partial (c z_3^k)}{\partial z_3}\frac{\partial}{\partial z_2}.
\end{aligned}$$

By Lemma 2.3, there are also two cases to be considered for f.

Case 1: Let $f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2, z_3)z_0 + a_m(z_1, z_2, z_3)$, with $m \geq 4$.

In the first part, we consider D_2. Firstly, we investigate $D_2(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 - \alpha_1 \leq (2m-2)\alpha_0 - 2\alpha_1 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_2(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 - \alpha_2 \leq (2m-2)\alpha_0 - 2\alpha_1 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

with $m \geq 4$, so $D_2(\partial f/\partial z_1)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D_2(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 - \alpha_3 \leq (2m-2)\alpha_0 - 2\alpha_1 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

with $m \geq 4$, so $D_2(\partial f/\partial z_1)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

In the second part, we consider D_3. Firstly, we investigate $D_3(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 - \alpha_1 \leq (3m-2)\alpha_0 - 2\alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_3(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 - \alpha_2 \leq (3m-2)\alpha_0 - 2\alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 - \alpha_3 \leq (3m-2)\alpha_0 - 2\alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

Case 2: Let

$$f = z_0^m z_i + a_1(z_1,z_2,z_3)z_0^{m-1} + \cdots + a_{m-1}(z_1,z_2,z_3)z_0 + a_m(z_1,z_2,z_3),$$

with $m \geq 3$.

Case 2.1: Assume $i = 1$, i.e.,

$$f = z_0^m z_1 + a_1(z_1,z_2,z_3)z_0^{m-1} + \cdots + a_m(z_1,z_2,z_3).$$

In the first part, we consider D_2. Firstly, we investigate $D_2(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 < (2m-2)\alpha_0 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_2(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 + \alpha_1 - \alpha_2 < (2m-2)\alpha_0 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D_2(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 + \alpha_1 - \alpha_3 < (2m-2)\alpha_0 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

In the second part, we consider D_3. Firstly, we investigate $D_3(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 < (3m-2)\alpha_0 + \alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_3(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 + \alpha_1 - \alpha_2 < (3m-2)\alpha_0 + \alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 + \alpha_1 - \alpha_3 < (3m-2)\alpha_0 + \alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

Case 2.2: Assume $i = 2$, i.e.,

$$f = z_0^m z_2 + a_1(z_1, z_2, z_3) z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3).$$

In the first part, we consider D_2. Firstly, we investigate $D_2(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 - \alpha_1 + \alpha_2 < (2m-2)\alpha_0 - 2\alpha_1 + 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_2(\partial f/\partial z_2)$. When $m \geq 4$, we obtain that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 < (2m-2)\alpha_0 - 2\alpha_1 + 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D_2(\partial f/\partial z_3)$. When $m \geq 4$, we obtain that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 + \alpha_2 - \alpha_3 < (2m-2)\alpha_0 - 2\alpha_1 + 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

In the second part, we consider D_3. Firstly, we investigate $D_3(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 - \alpha_1 + \alpha_2 < (3m-2)\alpha_0 - 2\alpha_1 + \alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_3(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 < (3m-2)\alpha_0 - 2\alpha_1 + \alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 + \alpha_2 - \alpha_3 \le (3m-2)\alpha_0 - 2\alpha_1 + \alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

Case 2.3: Assume $i = 3$, i.e.,

$$f = z_0^m z_3 + a_1(z_1, z_2, z_3) z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3).$$

In the first part, we consider D_2. Firstly, we investigate $D_2(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 - \alpha_1 + \alpha_3 < (2m-2)\alpha_0 - 2\alpha_1 + 2\alpha_3 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_2(\partial f/\partial z_2)$. When $m \ge 4$, we obtain that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 - \alpha_2 + \alpha_3 < (2m-2)\alpha_0 - 2\alpha_1 + 2\alpha_3 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D_2(\partial f/\partial z_3)$. When $m \ge 4$, we obtain that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 < (2m-2)\alpha_0 - 2\alpha_1 + 2\alpha_3 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

In the second part, we consider D_3. Firstly, we investigate $D_3(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 - \alpha_1 + \alpha_3 < (3m-2)\alpha_0 - 2\alpha_1 - 2\alpha_2 + 3\alpha_3 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_3(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right)=m\alpha_0-\alpha_2+\alpha_3<(3m-2)\alpha_0-2\alpha_1-2\alpha_2+3\alpha_3=\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02}\\ f_{01} & f_{11} & f_{12}\\ f_{02} & f_{12} & f_{22}\end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right)=m\alpha_0\le(3m-2)\alpha_0-2\alpha_1-2\alpha_2+3\alpha_3=\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02}\\ f_{01} & f_{11} & f_{12}\\ f_{02} & f_{12} & f_{22}\end{vmatrix}\right),$$

so $D(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

Lemma 2.13. *Let*

$$f(z_0,z_1,z_2,z_3)=z_0^3z_2+a_1(z_1,z_2,z_3)z_0^2+a_2(z_1,z_2,z_3)z_0+a_3(z_1,z_2,z_3)$$

be a weighted homogeneous polynomial of type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *that satisfies* $d\ge 2\alpha_0\ge 2\alpha_1\ge 2\alpha_2\ge 2\alpha_3$ *with isolated singularity at the origin. Assume* $\alpha_0+\alpha_2+\alpha_3<2\alpha_1$, *which is equivalent to*

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right)>\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01}\\ f_{01} & f_{11}\end{vmatrix}\right).$$

Let D_2 *be a derivation of the algebra* $\mathbb{C}[z_0,z_1,z_2,z_3]/(I,I_2)$. *Then* $D_2\equiv 0$, *if* D_2 *is of negative weight.*

Proof. By the assumption, we conclude that $\mathrm{wt}(a_1(z_1,z_2,z_3))=\alpha_0+\alpha_2<2\alpha_1-\alpha_3$. We obtain that $a_1(z_1,z_2,z_3)=z_1f'(z_2,z_3)+f''(z_2,z_3)$. Let

$$D_2=p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2},$$

be a nonzero negative weight derivation.

Firstly, we investigate $D_2(\partial f/\partial z_0)$:

$$\begin{aligned}0=D_2\left(\frac{\partial f}{\partial z_0}\right)&=p_0(z_1,z_2,z_3)\big[6z_0z_2+2a_1(z_1,z_2,z_3)\big]\\&\quad+p_1(z_2,z_3)\left[2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}\right]\\&\quad+cz_3^k\left[3z_0^2+2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_2}+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_2}\right].\end{aligned}$$

So we obtain that $c=0$, i.e., $D_2=p_0(z_1,z_2,z_3)(\partial/\partial z_0)+p_1(z_2,z_3)(\partial/\partial z_1)$.

If $p_1(z_2,z_3)=0$, then $p_0(z_1,z_2,z_3)=0$, which is absurd.

If $p_1(z_2,z_3)\neq 0$, then

$$\begin{cases} 3p_0(z_1,z_2,z_3)z_2+p_1(z_2,z_3)f'(z_2,z_3)=0,\\ 2p_0(z_1,z_2,z_3)\big[z_1f'(z_2,z_3)+f''(z_2,z_3)\big]+p_1(z_2,z_3)\dfrac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}=0.\end{cases} \tag{13}$$

Hence, we obtain that

$$2f'(z_2,z_3)\big[z_1f'(z_2,z_3)+f''(z_2,z_3)\big]=3z_2\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1},$$

which implies $f'(z_2,z_3)$ is divisible by z_2. Let $f'(z_2,z_3)=3dz_2^e+3\sum_i d_iz_2^{e_i}z_3^{s_i}$, with $e>0$ and $e_i>0$ for all i, then we obtain that

$$2\Big(dz_2^{e-1}+\sum_i d_iz_2^{e_i-1}z_3^{s_i}\Big)\Big[3z_1\Big(dz_2^e+\sum_i d_iz_2^{e_i}z_3^{s_i}\Big)+f''(z_2,z_3)\Big] = \frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}. \tag{14}$$

Next, we investigate $D(\partial f/\partial z_1)$:

$$D\Big(\frac{\partial f}{\partial z_1}\Big)=p_0(z_1,z_2,z_3)\Big[2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}\Big] + p_1(z_2,z_3)\Big[z_0\frac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1^2}+\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2}\Big].$$

Since $D(\partial f/\partial z_1)=h(\partial f/\partial z_0)$ and $\partial f/\partial z_0=3z_0^2z_2+\cdots$, then $D(\partial f/\partial z_1)=0$. Hence, we have

$$p_0(z_1,z_2,z_3)\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}+p_1(z_2,z_3)\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2}=0.$$

Let

$$g_1(z_2,z_3):=dz_2^{e-1}+\sum_i d_iz_2^{e_i-1}z_3^{s_i},$$

i.e., $3z_2g_1(z_2,z_3)=f'(z_2,z_3)$. Hence, by (14), we have

$$a_2(z_1,z_2,z_3)=z_1^2g_1(z_2,z_3)f'(z_2,z_3)+2z_1g_1(z_2,z_3)f''(z_2,z_3)+f'''(z_2,z_3),$$

and by (13) and (14), we obtain that

$$a_3(z_1,z_2,z_3)=\tfrac{1}{3}z_1^3[g_1(z_2,z_3)]^2f'(z_2,z_3)+z_1^2[g_1(z_2,z_3)]^2f''(z_2,z_3) + z_1f''''(z_2,z_3)+f'''''(z_2,z_3).$$

By Lemma 2.4, we obtain that one of $z_1^{l_0}$, $z_1^{l_1}z_1$, $z_1^{l_2}z_2$, $z_1^{l_3}z_3$ must be contained in f, which is absurd, i.e., D_2 does not exist. □

Lemma 2.14. *Let*

$$f(z_0,z_1,z_2,z_3)=z_0^3z_3+a_1(z_1,z_2,z_3)z_0^2+a_2(z_1,z_2,z_3)z_0+a_3(z_1,z_2,z_3)$$

be a weighted homogeneous polynomial of type $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *that satisfies* $d\ge 2\alpha_0\ge 2\alpha_1\ge 2\alpha_2\ge 2\alpha_3$ *with isolated singularity at the origin. Assume that* $\alpha_0+2\alpha_3<2\alpha_1$, *which is equivalent to*

$$\operatorname{wt}\left(\frac{\partial f}{\partial z_3}\right)>\operatorname{wt}\left(\begin{vmatrix} f_{00} & f_{01}\\ f_{01} & f_{11}\end{vmatrix}\right).$$

Let D_2 *be a derivation of the algebra* $\mathbb{C}[z_0,z_1,z_2,z_3]/(I,I_2)$. *Then* $D_2\equiv 0$, *if* D_2 *is of negative weight.*

Proof. By the assumption, we have $\operatorname{wt}(a_1(z_1,z_2,z_3))=\alpha_0+\alpha_3<2\alpha_1$. We obtain that $a_1(z_1,z_2,z_3)=z_1f'(z_2,z_3)+f''(z_2,z_3)$. Let

$$D_2=p_0(z_1,z_2,z_3)\frac{\partial}{\partial z_0}+p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2},$$

be a nonzero negative weight derivation.

Firstly, we investigate $D_2(\partial f/\partial z_0)$:

$$\begin{aligned}0=D_2\left(\frac{\partial f}{\partial z_0}\right)&=p_0(z_1,z_2,z_3)\big[6z_0z_3+2a_1(z_1,z_2,z_3)\big]\\&\quad+p_1(z_2,z_3)\left[2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}\right]\\&\quad+cz_3^k\left[2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_2}+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_2}\right].\end{aligned}$$

There are two subcases.

Case 1: Assume $c=0$, i.e., $D_2=p_0(z_1,z_2,z_3)(\partial/\partial z_0)+p_1(z_2,z_3)(\partial/\partial z_1)$.

If $p_1(z_2,z_3)=0$, then $p_0(z_1,z_2,z_3)=0$, which is absurd.

If $p_1(z_2,z_3)\ne 0$, then

$$\begin{cases}3p_0(z_1,z_2,z_3)z_3+p_1(z_2,z_3)f'(z_2,z_3)=0,\\ 2p_0(z_1,z_2,z_3)\big[z_1f'(z_2,z_3)+f''(z_2,z_3)\big]+p_1(z_2,z_3)\dfrac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}=0.\end{cases}$$

Hence, we obtain that

$$2f'(z_2,z_3)\cdot\big[z_1f'(z_2,z_3)+f''(z_2,z_3)\big]=3z_3\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1},$$

which implies $f'(z_2,z_3)$ is divisible by z_3. Let $f'(z_2,z_3)=3dz_3^s+3\sum_i d_iz_2^{e_i}z_3^{s_i}$, with $s>0$ and $s_i>0$ for all i, then we obtain that

$$2\left(dz_3^{s-1}+\sum_i d_iz_2^{e_i}z_3^{s_i-1}\right)\left[3z_1\big(dz_3^s+\sum_i d_iz_2^{e_i}z_3^{s_i}\big)+f''(z_2,z_3)\right]=\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}.$$

Next, we investigate $D(\partial f/\partial z_1)$:

$$D\left(\frac{\partial f}{\partial z_1}\right) = p_0(z_1,z_2,z_3)\left[2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}\right] + p_1(z_2,z_3)\left[z_0\frac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1^2}+\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2}\right].$$

It is obvious that $D(\partial f/\partial z_1)=0$. Hence, we obtain a new relation as follows:

$$p_0(z_1,z_2,z_3)\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}+p_1(z_2,z_3)\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2}=0.$$

Let

$$g_1(z_2,z_3)=dz_3^{s-1}+\sum_i d_i z_2^{e_i} z_3^{s_i-1},$$

i.e., $3z_3 g_1(z_2,z_3)=f'(z_2,z_3)$. Hence,

$$a_2(z_1,z_2,z_3)=z_1^2 g_1(z_2,z_3)f'(z_2,z_3)+2z_1 g_1(z_2,z_3)f''(z_2,z_3)+f_3(z_2,z_3),$$

and

$$a_3(z_1,z_2,z_3)=\tfrac{1}{3}z_1^3[g_1(z_2,z_3)]^2 f'(z_2,z_3)+z_1^2[g_1(z_2,z_3)]^2 f''(z_2,z_3) + z_1 f_4(z_2,z_3)+f_5(z_2,z_3).$$

By Lemma 2.4, we obtain that one of $z_1^{l_0}$, $z_1^{l_1}z_0$, $z_1^{l_2}z_2$, $z_1^{l_3}z_3$ must be contained in f, which is absurd, i.e., D_2 does not exist.

Case 2: Assume $c\neq 0$.

Case 2.1: Let $p_1(z_2,z_3)=0$.

Case 2.1.1: Let $p_0(z_1,z_2,z_3)=0$, i.e., $D=cz_3^k(\partial/\partial z_2)$. Then, we obtain that

$$0=D\left(\frac{\partial f}{\partial z_0}\right)=cz_3^k\left(2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_2}+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_2}\right).$$

This implies that $\partial a_1(z_1,z_2,z_3)/\partial z_2=0$ and $\partial a_2(z_1,z_2,z_3)/\partial z_2=0$. Hence,

$$0=D\left(\frac{\partial f}{\partial z_1}\right)=cz_3^k\left(z_0^2\frac{\partial^2 a_1(z_1,z_2,z_3)}{\partial z_1\partial z_2}+z_0\frac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1\partial z_2}+\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1\partial z_2}\right),$$

and

$$0=D\left(\frac{\partial f}{\partial z_2}\right)=cz_3^k\left(z_0^2\frac{\partial^2 a_1(z_1,z_2,z_3)}{\partial z_2^2}+z_0\frac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_2^2}+\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_2^2}\right).$$

Hence, $\partial^2 a_3(z_1,z_2,z_3)/(\partial z_1\partial z_2)=0$ and $\partial^2 a_3(z_1,z_2,z_3)/\partial z_2^2=0$. By Lemma 2.4, we obtain that one of $z_2^{l_0}$, $z_2^{l_1}z_0$, $z_2^{l_2}z_1$, $z_2^{l_3}z_3$ must be contained in f, which is absurd, i.e., D_2 does not exist.

Case 2.1.2: Let $p_0(z_1, z_2, z_3) \neq 0$, i.e., $D_2 = p_0(z_1, z_2, z_3)(\partial/\partial z_0) + cz_3^k(\partial/\partial z_2)$. Hence,

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_0}\right) + cz_3^k\frac{\partial}{\partial z_2}\left(\frac{\partial f}{\partial z_0}\right) = \frac{\partial}{\partial z_0}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + cz_3^k\frac{\partial f}{\partial z_2}\right]. \tag{15}$$

Moreover,

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = p_0(z_1, z_2, z_3)\big[6z_0z_3 + 2a_1(z_1, z_2, z_3)\big] + cz_3^k\left[2z_0\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} + \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}\right], \tag{16}$$

and

$$\begin{cases} 3p_0(z_1, z_2, z_3)z_3 + cz_3^k\dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} = 0, \\ 2p_0(z_1, z_2, z_3)\big[z_1f'(z_2, z_3) + f''(z_2, z_3)\big] + cz_3^k\dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_2} = 0. \end{cases}$$

From this, we obtain that

$$2\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2}a_1(z_1, z_2, z_3) = 3z_3\frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}.$$

It is easy to verify that $a_1(z_1, z_2, z_3)$ is divisible by z_3. Let $f'(z_2, z_3) = z_3g'(z_2, z_3)$ and $f''(z_2, z_3) = z_3g''(z_2, z_3)$. So $\partial a_2(z_1, z_2, z_3)/\partial z_2$ is divisible by z_3. Then we consider

$$\begin{aligned} D\left(\frac{\partial f}{\partial z_1}\right) &= p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_1}\right) + cz_3^k\frac{\partial}{\partial z_2}\left(\frac{\partial f}{\partial z_1}\right) \\ &= \frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + cz_3^k\frac{\partial f}{\partial z_2}\right] - \frac{\partial p_0(z_1, z_2, z_3)}{\partial z_1}\frac{\partial f}{\partial z_0} \\ &= h\frac{\partial f}{\partial z_0}. \end{aligned} \tag{17}$$

Equation (17) implies that

$$\frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + cz_3^k\frac{\partial f}{\partial z_2}\right] = \tilde{h}\frac{\partial f}{\partial z_0}.$$

From (15), we know that the left-hand side of this equation is independent of the variable z_0. Since $f = z_0^3z_3 + \cdots$, the right-hand side of this equation is independent of the variable z_0 only if $\tilde{h} = 0$. Thus, we have

$$\frac{\partial}{\partial z_1}\left(p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + cz_3^k\frac{\partial f}{\partial z_2}\right) = 0.$$

Therefore,

$$p_0(z_1,z_2,z_3)\frac{\partial f}{\partial z_0}+cz_3^k\frac{\partial f}{\partial z_2}=g_1(z_2,z_3).$$

This is

$$\begin{aligned}(18)\quad & p_0(z_1,z_2,z_3)\big(3z_3z_0^2+2a_1(z_1,z_2,z_3)z_0+a_2(z_1,z_2,z_3)\big)\\ &+cz_3^k\left[\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_2}z_0^2+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_2}z_0+\frac{\partial a_3(z_1,z_2,z_3)}{\partial z_2}\right]=g_1(z_2,z_3).\end{aligned}$$

Equation (18) implies

$$\begin{cases}3p_0(z_1,z_2,z_3)z_3+cz_3^k\dfrac{\partial a_1(z_1,z_2,z_3)}{\partial z_2}=0,\\ 2p_0(z_1,z_2,z_3)a_1(z_1,z_2,z_3)+cz_3^k\dfrac{\partial a_2(z_1,z_2,z_3)}{\partial z_2}=0,\end{cases}$$

and

$$\begin{aligned}&\frac{\partial\big[p_0(z_1,z_2,z_3)a_2(z_1,z_2,z_3)+cz_3^k(\partial a_3(z_1,z_2,z_3)/\partial z_2)\big]}{\partial z_1}\\ &=\frac{\partial p_0(z_1,z_2,z_3)}{\partial z_1}a_2(z_1,z_2,z_3)+p_0(z_1,z_2,z_3)\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}+cz_3^k\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1\partial z_2}\\ &=0.\end{aligned}$$

It is clear that $a_1(z_1,z_2,z_3)\neq 0$, $\partial a_1(z_1,z_2,z_3)/\partial z_2\neq 0$ and $\partial a_2(z_1,z_2,z_3)/\partial z_2\neq 0$. Hence, there exists $h(z_1,z_2,z_3)$ such that

$$\begin{cases}3z_3h(z_1,z_2,z_3)=2a_1(z_1,z_2,z_3),\\ \dfrac{\partial a_1(z_1,z_2,z_3)}{\partial z_2}h(z_1,z_2,z_3)=\dfrac{\partial a_2(z_1,z_2,z_3)}{\partial z_2}.\end{cases}$$

Thus, $cz_3^k(\partial a_1(z_1,z_2)/\partial z_1)\neq 0$. Then

$$p_0(z_1,z_2,z_3)=z_3^{k-1}q_0(z_1,z_2,z_3),$$

where

$$q_0(z_1,z_2,z_3)=-\frac{c}{3}\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_2}.$$

Hence, $q_0(z_1,z_2,z_3)$ is divisible by z_3, i.e., $p_0(z_1,z_2,z_3)$ is divisible by z_3^k. Thus, the differential equation

$$p_0(z_1,z_2,z_3)+cz_3^k\frac{\partial g(z_1,z_2,z_3)}{\partial z_2}=0$$

has a solution. We also use $g(z_1, z_2, z_3)$ to denote the solution which does not contain the constant term. We construct the following coordinate transformation:

$$\begin{cases} z_0' = z_0 + g(z_1, z_2, z_3), \\ z_1' = z_1, \\ z_2' = z_2, \\ z_3' = z_3. \end{cases}$$

Under this transformation of coordinates, we obtain that

$$\begin{cases} \dfrac{\partial}{\partial z_0} = \dfrac{\partial}{\partial z_0'}, \\ \dfrac{\partial}{\partial z_1} = \dfrac{\partial g}{\partial z_1}\dfrac{\partial}{\partial z_0'} + \dfrac{\partial}{\partial z_1'}, \\ \dfrac{\partial}{\partial z_2} = \dfrac{\partial g}{\partial z_2}\dfrac{\partial}{\partial z_0'} + \dfrac{\partial}{\partial z_2'}. \end{cases}$$

Hence,

$$\begin{aligned} D &= p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + c z_3^k \frac{\partial}{\partial z_2} \\ &= p_0(z_1', z_2', z_3')\frac{\partial}{\partial z_0'} + c(z_3')^k\left(\frac{\partial g}{\partial z_2}\frac{\partial}{\partial z_0'} + \frac{\partial}{\partial z_2'}\right) \\ &= \left(p_0(z_1', z_2', z_3') + c(z_3')^k \frac{\partial g}{\partial z_2}\right)\frac{\partial}{\partial z_0'} + c(z_3')^k\frac{\partial}{\partial z_2'} \\ &= c(z_3')^k \frac{\partial}{\partial z_2'}. \end{aligned}$$

By the discussion of Case 2.1.1, such D does not exist.

Case 2.2: Let $p_1(z_2, z_3) \neq 0$.

Case 2.2.1: Let $p_0(z_1, z_2, z_3) = 0$. Then $D(\partial f/\partial z_0) = 0$ implies that

$$\begin{cases} p_1(z_2, z_3)\dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + c z_3^k \dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} = 0, \\ p_1(z_2, z_3)\dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} + c z_3^k \dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_2} = 0. \end{cases}$$

Hence, $f'(z_2, z_3) = 0$ or a constant multiple of z_3^s.

Next, we investigate $D(\partial f/\partial z_1)$:

$$\begin{aligned} 0 = D\left(\frac{\partial f}{\partial z_1}\right) = p_1(z_2, z_3)&\left[z_0\frac{\partial^2 a_2(z_1, z_2, z_3)}{\partial z_1^2} + \frac{\partial^2 a_3(z_1, z_2, z_3)}{\partial z_1^2}\right] \\ &+ c z_3^k\left[z_0\frac{\partial^2 a_2(z_1, z_2, z_3)}{\partial z_1 \partial z_2} + \frac{\partial^2 a_3(z_1, z_2, z_3)}{\partial z_1 \partial z_2}\right]. \end{aligned}$$

Hence,

$$\begin{cases} p_1(z_2,z_3)\dfrac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1^2}+cz_3^k\dfrac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1\partial z_2}=0,\\ p_1(z_2,z_3)\dfrac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2}+cz_3^k\dfrac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1\partial z_2}=0. \end{cases}$$

If $p_1(z_2,z_3)$ is divisible by z_3^k, then the differential equation

$$p_1(z_2,z_3)+cz_3^k\frac{\partial g(z_2,z_3)}{\partial z_2}=0$$

has a solution. We also use $g(z_2,z_3)$ to denoted the solution which does not contain the constant term. We construct the following coordinate transformation:

$$\begin{cases} z_0'=z_0,\\ z_1'=z_1+g(z_2,z_3),\\ z_2'=z_2,\\ z_3'=z_3. \end{cases}$$

Under this transformation of coordinates, we obtain that

$$\begin{cases} \dfrac{\partial}{\partial z_0}=\dfrac{\partial}{\partial z_0'},\\ \dfrac{\partial}{\partial z_1}=\dfrac{\partial}{\partial z_1'},\\ \dfrac{\partial}{\partial z_2}=\dfrac{\partial g}{\partial z_2}\dfrac{\partial}{\partial z_1'}+\dfrac{\partial}{\partial z_2'}. \end{cases}$$

Hence,

$$\begin{aligned} D&=p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}\\ &=p_1(z_2',z_3')\frac{\partial}{\partial z_1'}+c(z_3')^k\left(\frac{\partial g}{\partial z_2}\frac{\partial}{\partial z_1'}+\frac{\partial}{\partial z_2'}\right)\\ &=\left(p_1(z_2',z_3')+c(z_3')^k\frac{\partial g}{\partial z_2}\right)\frac{\partial}{\partial z_1'}+c(z_3')^k\frac{\partial}{\partial z_2'}. \end{aligned} \tag{19}$$

By Case 2.1.1, such D does not exist. In the following, we assume that $p_1(z_2,z_3)$ is not divisible by z_3^k. Hence, $z_3\mid(\partial a_1/\partial z_1)$, $z_3\mid(\partial a_2/\partial z_1)$, $z_3\mid(\partial^2 a_2/\partial z_1^2)$ and $z_3\mid(\partial^2 a_3/\partial z_1^2)$. By the weight inequality $\alpha_0+2\alpha_3<2\alpha_1$, we have

$$\begin{aligned} f=z_0^3z_3+(z_1f_1+f_0)z_0^2+(z_1^3g_3+z_1^2g_2+z_1g_1+g_0)z_0\\ +z_1^5h_5+z_1^4h_4+z_1^3h_3+z_1^2h_2+z_1h_1+h_0, \end{aligned}$$

where $f_i = f_i(z_2, z_3)$, with $i = 0, 1$; $g_j = g_j(z_2, z_3)$, with $0 \le j \le 3$; and $h_l = h_l(z_2, z_3)$, with $0 \le l \le 5$. Hence, the following equations:

$$\frac{\partial f}{\partial z_0} = 3z_0^2 z_3 + 2z_0(z_1 f_1 + f_0) + (z_1^3 g_3 + z_1^2 g_2 + z_1 g_1 + g_0),$$

$$\frac{\partial f}{\partial z_1} = z_0^2 f_1 + z_0(3z_1^2 g_3 + 2z_1^2 g_2 + g_1) + 5z_1^4 h_5 + 4z_1^3 h_4 + 3z_1^2 h_3 + 2z_1 h_2 + h_1,$$

$$\begin{aligned}\frac{\partial f}{\partial z_2} = z_0^2\left(z_1\frac{\partial f_1}{\partial z_2} + \frac{\partial f_0}{\partial z_2}\right) + z_0\left(z_1^3\frac{\partial g_3}{\partial z_2} + z_1^2\frac{\partial g_2}{\partial z_2} + z_1\frac{\partial g_1}{\partial z_2} + \frac{\partial g_0}{\partial z_2}\right)\\ + \left(z_1^5\frac{\partial h_5}{\partial z_2} + z_1^4\frac{\partial h_4}{\partial z_2} + z_1^3\frac{\partial h_3}{\partial z_2} + z_1^2\frac{\partial h_2}{\partial z_2} + z_1\frac{\partial h_1}{\partial z_2} + \frac{\partial h_0}{\partial z_2}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z_3} = z_0^3 + z_0^2\left(z_1\frac{\partial f_1}{\partial z_3} + \frac{\partial f_0}{\partial z_3}\right) + z_0\left(z_1^3\frac{\partial g_3}{\partial z_3} + z_1^2\frac{\partial g_2}{\partial z_3} + z_1\frac{\partial g_1}{\partial z_3} + \frac{\partial g_0}{\partial z_3}\right)\\ + \left(z_1^5\frac{\partial h_5}{\partial z_3} + z_1^4\frac{\partial h_4}{\partial z_3} + z_1^3\frac{\partial h_3}{\partial z_3} + z_1^2\frac{\partial h_2}{\partial z_3} + z_1\frac{\partial h_1}{\partial z_3} + \frac{\partial h_0}{\partial z_3}\right),\end{aligned}$$

have solution

$$\begin{cases} z_2 = 0,\\ z_3 = 0,\\ \dfrac{\partial f}{\partial z_3} = 0.\end{cases}$$

Hence, f does not have an isolated singularity at the origin.

Case 2.2.2: Let $p_0(z_1, z_2, z_3) \neq 0$. Then

$$\begin{aligned}0 &= D\left(\frac{\partial f}{\partial z_0}\right)\\ &= \frac{\partial}{\partial z_0}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + cz_3^k\frac{\partial f}{\partial z_2}\right]\end{aligned} \tag{20}$$

and

$$\begin{aligned}&D\left(\frac{\partial f}{\partial z_1}\right)\\ &= \frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + cz_3^k\frac{\partial f}{\partial z_2}\right] - \frac{\partial p_0(z_1, z_2, z_3)}{\partial z_1}\frac{\partial f}{\partial z_0}\\ &= h\frac{\partial f}{\partial z_0}.\end{aligned} \tag{21}$$

Equation (21) implies that

$$\frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + cz_3^k\frac{\partial f}{\partial z_2}\right] = \tilde{h}\frac{\partial f}{\partial z_0}.$$

From (20), we know that the left-hand side of this equation is independent of the variable z_0. Since $f = z_0^3 z_3 + \cdots$, the right-hand side of this equation is independent of the variable z_0 only if $\tilde{h} = 0$. Thus, we have

$$\frac{\partial}{\partial z_1}\left(p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c z_3^k \frac{\partial f}{\partial z_2}\right) = 0.$$

Therefore,

$$p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c z_3^k \frac{\partial f}{\partial z_2} = g_1(z_2, z_3).$$

This is

$$\begin{aligned}(22)\quad & p_0(z_1, z_2, z_3)\big(3z_3 z_0^2 + 2a_1(z_1, z_2, z_3) z_0 + a_2(z_1, z_2, z_3)\big) \\ &\quad + p_1(z_2, z_3)\left[z_0^2 \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + z_0 \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} + \frac{\partial a_3(z_1, z_2, z_3)}{\partial z_1}\right] \\ &\quad + c z_3^k \left[\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} z_0^2 + \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2} z_0 + \frac{\partial a_3(z_1, z_2, z_3)}{\partial z_2}\right] \\ & = g_1(z_2, z_3).\end{aligned}$$

Equation (22) implies

$$(23)\quad \begin{cases} 3p_0(z_1, z_2, z_3) z_3 + p_1(z_2, z_3)\dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + c z_3^k \dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} = 0, \\ 2p_0(z_1, z_2, z_3) a_1(z_1, z_2, z_3) + p_1(z_2, z_3)\dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} \\ \qquad\qquad + c z_3^k \dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_2} = 0, \end{cases}$$

and

$$\begin{aligned}(24)\quad & \frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3) a_2(z_1, z_2, z_3) + p_1(z_2, z_3)\frac{\partial a_3(z_1, z_2, z_3)}{\partial z_1} + c z_3^k \frac{\partial a_3(z_1, z_2, z_3)}{\partial z_2}\right] \\ &= \frac{\partial p_0(z_1, z_2, z_3)}{\partial z_1} a_2(z_1, z_2, z_3) + p_0(z_1, z_2, z_3)\frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} \\ &\qquad + p_1(z_2, z_3)\frac{\partial^2 a_3(z_1, z_2, z_3)}{\partial z_1^2} + c z_3^k \frac{\partial^2 a_3(z_1, z_2, z_3)}{\partial z_1 \partial z_2} \\ &= 0.\end{aligned}$$

From (23), we obtain that

$$\begin{aligned}(25)\quad & p_1(z_2, z_3)\left[2a_1(z_1, z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} - 3z_3 \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1}\right] \\ &\qquad + c z_3^k \left[2a_1(z_1, z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} - 3z_3 \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}\right] = 0.\end{aligned}$$

If $p_1(z_2, z_3)$ is divisible by cz_3^k, then the differential equation

$$p_1(z_2, z_3) + cz_3^k \frac{\partial g(z_2, z_3)}{\partial z_2} = 0,$$

has a solution. We also use $g(z_2, z_3)$ to denote the solution which does not contain the constant term. We construct the following coordinate transformation:

$$\begin{cases} z_0' = z_0, \\ z_1' = z_1 + g(z_2, z_3), \\ z_2' = z_2, \\ z_3' = z_3. \end{cases}$$

Under this transformation of coordinates, we obtain that

$$\begin{cases} \dfrac{\partial}{\partial z_0} = \dfrac{\partial}{\partial z_0'}, \\ \dfrac{\partial}{\partial z_1} = \dfrac{\partial}{\partial z_1'}, \\ \dfrac{\partial}{\partial z_2} = \dfrac{\partial g}{\partial z_2}\dfrac{\partial}{\partial z_1'} + \dfrac{\partial}{\partial z_2'}. \end{cases}$$

Hence,

$$\begin{aligned} D &= p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + cz_3^k\frac{\partial}{\partial z_2} \\ &= p_0\big(z_1' - g(z_2', z_3'), z_2', z_3'\big)\frac{\partial}{\partial z_0'} + p_1(z_2', z_3')\frac{\partial}{\partial z_1'} + c(z_3')^k\left(\frac{\partial g}{\partial z_2}\frac{\partial}{\partial z_1'} + \frac{\partial}{\partial z_2'}\right) \\ &= p_0\big(z_1' - g(z_2', z_3'), z_2', z_3'\big)\frac{\partial}{\partial z_0'} + \left(p_1(z_2', z_3') + c(z_3')^k\frac{\partial g}{\partial z_2}\right)\frac{\partial}{\partial z_1'} + c(z_3')^k\frac{\partial}{\partial z_2'}. \end{aligned} \tag{26}$$

By Case 2.1.2, such D does not exist.

In the following, we assume that $p_1(z_2, z_3)$ is not divisible by cz_3^k. According to (25), we obtain that z_3 is divisible by $\partial a_1(z_1, z_2, z_3)/\partial z_1$.

By the weight inequality $\alpha_0 + 2\alpha_3 < 2\alpha_1$, we have

$$\begin{aligned} f = z_0^3 z_3 + (z_1 f_1 + f_0)z_0^2 + (z_1^3 g_3 + z_1^2 g_2 + z_1 g_1 + g_0)z_0 \\ + z_1^5 h_5 + z_1^4 h_4 + z_1^3 h_3 + z_1^2 h_2 + z_1 h_1 + h_0, \end{aligned}$$

where $f_i = f_i(z_2, z_3)$, with $i = 0, 1$; $g_j = g_j(z_2, z_3)$, with $j = 0, 1, 2, 3$; and $h_l = h_l(z_2, z_3)$, with $l = 0, 1, 2, 3, 4, 5$. By (23), we obtain that

$$3p_0 z_3 + p_1 f_1 + cz_3^k\left(z_1\frac{\partial f_1}{\partial z_2} + \frac{\partial f_0}{\partial z_2}\right) = 0, \tag{27}$$

and

$$2p_0(z_1f_1+f_0)+p_1(3z_1^2g_3+2z_1g_2+g_1) + cz_3^k\left(z_1^3\frac{\partial g_3}{\partial z_2}+z_1^2\frac{\partial g_2}{\partial z_2}+z_1\frac{\partial g_1}{\partial z_2}+\frac{\partial g_0}{\partial z_2}\right)=0. \tag{28}$$

If $\partial f_1/\partial z_2=0$, then $p_0(z_1,z_2,z_3)=p_0(z_2,z_3)$ does not depend on z_1. Thus, (27) and (24) become

$$3p_0z_3+p_1f_1+cz_3^k\frac{\partial f_0}{\partial z_2}=0, \tag{29}$$

and

$$p_0(3z_1^2g_3+2z_1g_2+g_1)+p_1(20z_1^3h_5+12z_1^2h_4+6z_1h_3+2h_2) + cz_3^k\left(5z_1^4\frac{\partial h_5}{\partial z_2}+4z_1^3\frac{\partial h_4}{\partial z_2}+3z_1^2\frac{\partial h_3}{\partial z_2}+2z_1\frac{\partial h_2}{\partial z_2}+\frac{\partial h_1}{\partial z_2}\right)=0. \tag{30}$$

By (28), we obtain that

$$\begin{cases} \dfrac{\partial g_3}{\partial z_2}=0,\\ 3p_1g_3+cz_3^k\dfrac{\partial g_2}{\partial z_2}=0,\\ 2p_0f_1+2p_1g_2+cz_3^k\dfrac{\partial g_1}{\partial z_2}=0,\\ 2p_0f_0+p_1g_0+cz_3^k\dfrac{\partial g_0}{\partial z_2}=0. \end{cases} \tag{31}$$

By (30), we obtain that

$$\begin{cases} \dfrac{\partial h_5}{\partial z_2}=0,\\ 20p_1h_5+4cz_3^k\dfrac{\partial h_4}{\partial z_2}=0,\\ 3p_0g_3+12p_1h_4+3cz_3^k\dfrac{\partial h_3}{\partial z_2}=0,\\ 2p_0g_2+6p_1h_3+2cz_3^k\dfrac{\partial h_2}{\partial z_2}=0,\\ p_0g_1+2p_1h_2+cz_3^k\dfrac{\partial h_1}{\partial z_2}=0. \end{cases} \tag{32}$$

If $g_3 = 0$, then $\partial g_2/\partial z_2 = 0$ and $z_3 \mid h_4$ by (31) and (32). By Lemma 2.4 and $z_3 \mid h_4$, we obtain that $h_5 = 0$. Hence, the following equations:

$$\begin{aligned}
\frac{\partial f}{\partial z_0} &= 3z_0^2 z_3 + 2z_0(z_1 f_1 + f_0) + (z_1^3 g_3 + z_1^2 g_2 + z_1 g_1 + g_0),\\
\frac{\partial f}{\partial z_1} &= z_0^2 f_1 + z_0(3z_1^2 g_3 + 2z_1^2 g_2 + g_1) + 5z_1^4 h_5 + 4z_1^3 h_4 + 3z_1^2 h_3 + 2z_1 h_2 + h_1,\\
\frac{\partial f}{\partial z_2} &= z_0^2\left(z_1 \frac{\partial f_1}{\partial z_2} + \frac{\partial f_0}{\partial z_2}\right) + z_0\left(z_1^3 \frac{\partial g_3}{\partial z_2} + z_1^2 \frac{\partial g_2}{\partial z_2} + z_1 \frac{\partial g_1}{\partial z_2} + \frac{\partial g_0}{\partial z_2}\right)\\
&\quad + \left(z_1^5 \frac{\partial h_5}{\partial z_2} + z_1^4 \frac{\partial h_4}{\partial z_2} + z_1^3 \frac{\partial h_3}{\partial z_2} + z_1^2 \frac{\partial h_2}{\partial z_2} + z_1 \frac{\partial h_1}{\partial z_2} + \frac{\partial h_0}{\partial z_2}\right),\\
\frac{\partial f}{\partial z_3} &= z_0^3 + z_0^2\left(z_1 \frac{\partial f_1}{\partial z_3} + \frac{\partial f_0}{\partial z_3}\right) + z_0\left(z_1^3 \frac{\partial g_3}{\partial z_3} + z_1^2 \frac{\partial g_2}{\partial z_3} + z_1 \frac{\partial g_1}{\partial z_3} + \frac{\partial g_0}{\partial z_3}\right)\\
&\quad + \left(z_1^5 \frac{\partial h_5}{\partial z_3} + z_1^4 \frac{\partial h_4}{\partial z_3} + z_1^3 \frac{\partial h_3}{\partial z_3} + z_1^2 \frac{\partial h_2}{\partial z_3} + z_1 \frac{\partial h_1}{\partial z_3} + \frac{\partial h_0}{\partial z_3}\right),
\end{aligned}$$

have solution

$$\begin{cases} z_2 = 0,\\ z_3 = 0,\\ \dfrac{\partial f}{\partial z_3} = 0. \end{cases}$$

Hence f does not have isolated singularity at the origin.

If $g_3 \neq 0$, then $g_3 = z_3^l$, where $l \geq 1$ (due to (32) and omitting a nonzero constant multiple). If $h_5 = 0$, then $\partial h_4/\partial z_2 = 0$ and $h_3 \neq z_2$. Similarly, f does not have isolated singularity at the origin. If $h_5 \neq 0$, then $h_5 = z_3^{l'}$, where $l' \geq 1$ (due to (32) and omitting a nonzero constant multiple). Similarly, f does not have isolated singularity at the origin.

In the following, we assume that $\partial f_1/\partial z_2 \neq 0$. Then $p_0(z_1, z_2, z_3)$ depends on z_1. Let $f_1 = z_3 f_1'(z_2, z_3)$, with $f_1' \neq 0$, depend on z_2. By (27), without loss of generality, we can assume that p_0 is divisible by z_3^k. (Otherwise, if

$$p_0 = z_1 z_3^k p_0'(z_2, z_3) + p_0''(z_2, z_3),$$

then we can do the same argument as the condition $\partial f_1/\partial z_2 = 0$.) Moreover, p_1 is not divisible by z_3^k. By (28) and (30), we obtain that

$$z_3 \mid 3z_1^2 g_3 + 2z_1 g_2 + g_1 \quad \text{and} \quad z_3 \mid 20z_1^3 h_5 + 12z_1^2 h_4 + 6z_1 h_3 + 2h_2.$$

This implies that $z_3 \mid g_3$, $z_3 \mid g_2$, $z_3 \mid g_1$, $z_3 \mid h_5$, $z_3 \mid h_4$, $z_3 \mid h_3$, $z_3 \mid h_2$. Moreover we obtain that $z_3 \mid (\partial g_3/\partial z_2)$, $z_3 \mid (\partial g_2/\partial z_2)$, $z_3 \mid (\partial g_1/\partial z_2)$, $z_3 \mid (\partial h_5/\partial z_2)$, $z_3 \mid (\partial h_4/\partial z_2)$, $z_3 \mid (\partial h_3/\partial z_2)$ and $z_3 \mid (\partial h_2/\partial z_2)$. So f does not have isolated singularity at the origin. □

When investigating the derivation D_2, we assume that f has one of the following forms:

(1) $f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$ with $m \geq 4$,

(2) $f = z_0^m z_1 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$ with $m \geq 3$,

(3) $f = z_0^m z_2 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$ with $m \geq 3$,

(4) $f = z_0^m z_3 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$ with $m \geq 3$,

with the following relations:

$$
\begin{aligned}
D_2\left(\frac{\partial f}{\partial z_0}\right) &= 0,\\
D_2\left(\frac{\partial f}{\partial z_1}\right) &= p(z_1, z_2, z_3)\frac{\partial f}{\partial z_0},\\
D_2\left(\frac{\partial f}{\partial z_2}\right) &= q_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + q_1(z_2, z_3)\frac{\partial f}{\partial z_1},\\
D_2\left(\frac{\partial f}{\partial z_3}\right) &= w_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + w_1(z_2, z_3)\frac{\partial f}{\partial z_1} + w_2(z_3)\frac{\partial f}{\partial z_2}.
\end{aligned}
$$

When investigating the derivation D_3, we assume that f has one of the following forms:

(1) $f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$, with $m \geq 3$,

(2) $f = z_0^m z_1 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$, with $m \geq 2$,

(3) $f = z_0^m z_2 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$, with $m \geq 2$,

(4) $f = z_0^m z_3 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$, with $m \geq 3$,

with the following relations:

$$
\begin{aligned}
D_3\left(\frac{\partial f}{\partial z_0}\right) &= 0,\\
D_3\left(\frac{\partial f}{\partial z_1}\right) &= p(z_1, z_2, z_3)\frac{\partial f}{\partial z_0},\\
D_3\left(\frac{\partial f}{\partial z_2}\right) &= q_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + q_1(z_2, z_3)\frac{\partial f}{\partial z_1},\\
D_3\left(\frac{\partial f}{\partial z_3}\right) &= w_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + w_1(z_2, z_3)\frac{\partial f}{\partial z_1} + w_2(z_3)\frac{\partial f}{\partial z_2}.
\end{aligned}
$$

If D_ν is a derivation of A_ν, then D_ν is a derivation of B. By Theorem 1.5, such a derivation is nonnegative. □

In view of Proposition 2.6, Proposition 2.7, Proposition 2.8 and Proposition 2.10, the proof of the Main Theorem is now complete. □

3. Future work

For an isolated hypersurface singularity defined by $f(z_0, \ldots, z_n)$, the moduli algebra is defined by $A(f) = \mathcal{O}_{n+1}/\big((f)+J(f)\big)$ and the k-th Hessian algebra is defined by $H_k(f) = \mathcal{O}_{n+1}/\big((f) + J(f) + I_k\big)$. Suppose that the ideal $\big((f) + J(f) + I_k\big)$ is generated by $g_1, \ldots, g_m$, we use $J_\ell(g_1, \ldots, g_m)$ to denote the ideal generated by all $\ell \times \ell$-minors of the Jacobian matrix of $g_1, \ldots, g_m$, then we introduce a series of local algebras $M_{k,\ell}(f) = \mathcal{O}_{n+1}/\big((f) + J(f) + I_k + J_\ell(g_1, \ldots, g_m)\big)$. We believe Conjecture 1.6 and the Main Theorem can be generalized to these new local algebras. Some progress has been made, and we will include the results in our subsequent papers.

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Received June 9, 2022. Revised September 13, 2022.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.173

THE STRUCTURE OF THE UNRAMIFIED ABELIAN IWASAWA MODULE OF SOME NUMBER FIELDS

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For a given positive integer m, we determine an explicit infinite family of real quadratic number fields F, such that the unramified abelian Iwasawa module over the $\mathbb{Z}_2$-extension of F, is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2^m}$.

1. Introduction

Let p be a prime number and $\mathbb{Z}_p$ be the ring of p-adic integers. We denote by K a number field, K_∞ be the cyclotomic $\mathbb{Z}_p$-extension of K, and for each nonnegative integer n, K_n be the n-th layer of K_∞. For any nonnegative integer n, we denote by $A_n(K)$ the p-class group of K_n. We simply denote by $A(K) := A_0(K)$ the p-class group of K. The unramified abelian Iwasawa module $X_\infty(K)$ of K is defined by

$$X_\infty(K) := \varprojlim A_n(K),$$

where the projective limit is defined with respect to the norm mappings. It is well known, by Iwasawa's results that $X_\infty(K)$ is a finitely generated torsion $\Lambda := \mathbb{Z}_p[[T]]$-module and for large n, we have

$$|A_n(K)| = p^{\lambda_p(K)n+\mu_p(K)p^n+\nu_p(K)},$$

where $\lambda_p(K)$, $\mu_p(K)$ and $\nu_p(K)$ are so called Iwasawa invariants of K_∞/K. In the case where K is abelian over $\mathbb{Q}$, we have $\mu_p(K) = 0$ [3]. It is conjectured that for totally real number fields K, $\lambda_p(K) = \mu_p(K) = 0$ [5]. This conjecture, called Greenberg's conjecture, is considered as one of the fascinating problems in Iwasawa theory of $\mathbb{Z}_p$-extensions. So proving the finiteness of $X_\infty(K)$, leads us to ask the following questions:

- What about the structure of $X_\infty(K)$?
- What is the least nonnegative integer n such that $X_\infty(K) \simeq A_n(K)$?

MSC2020: 11R23, 11R29, 11R32, 11R37.
Keywords: class group, unit group, capitulation problem, Z_2-extension.

We will deal with these questions in a special case of totally real quadratic number fields.

Next, for each group G which is a finitely generated $\mathbb{Z}_p$-module, we denote by $\mathrm{rk}_p(G)$ the p-rank of G, that is, the dimension of the $\mathbb{F}_p$-vectorial space G/G^p.

Note that M. Ozaki [13] constructed a nonexplicit infinite family of cyclic number fields K of degree p, verifying Greenberg's conjecture and such that $\mathrm{rk}_p(X_\infty(K))$ is arbitrarily large.

For $p=2$, several articles tackled the Greenberg's conjecture for some totally real quadratic number fields. Precisely, for the prime numbers ℓ and ℓ', the quadratic number fields $F=\mathbb{Q}(\sqrt{\ell\ell'})$ has been studied intensively, where ℓ and ℓ' are prime numbers such that $\ell\equiv -\ell'\equiv 1 \pmod 4$. In particular, Y. Mizusawa [9] proved that for certain quadratic number fields F, the Galois groups of the maximal unramified pro-2-extensions over the cyclotomic $\mathbb{Z}_2$-extension of F are metacyclic pro-2-groups; he also studied the finiteness of $X_\infty(F)$ in relation with Greenberg's conjecture. Clearly in this case $X_\infty(F)$ is of rank equal to 2. Let us mention the articles [4; 8; 9; 10; 11; 12; 14], where we have found selected explicit totally real quadratic number fields F satisfying Greenberg's conjecture.

The common point in all these articles is that the unramified abelian Iwasawa module $X_\infty(F)$ for the selected number fields F, is of small rank equal to 1 or 2.

Our contribution is to check Greenberg's conjecture for a new family of fields $F=\mathbb{Q}(\sqrt{\ell\ell'})$. Precisely, we give the structure of $X_\infty(F)$ and determine the least positive integer m from which the groups $A_n(F)$ stabilize. The main result of this article is the following theorem.

Theorem 1.1. *Let ℓ and ℓ' be prime numbers such that $\ell\equiv -\ell'\equiv 1 \pmod 4$, $F=\boldsymbol{Q}(\sqrt{\ell\ell'})$. Put $v_2(\ell-1)-2=m$ and $v_2(\ell'+1)-2=m'$. Assume that $(\ell/\ell')=-1$ and $m'\geq m$. Then we have*

$$A_n(F)\simeq(\mathbb{Z}/2\mathbb{Z})^{2^n}\quad \text{for all } n\leq m \text{ and } X_\infty(F)\simeq A_m(F)\simeq(\mathbb{Z}/2\mathbb{Z})^{2^m}$$

2. Totally real quadratic number fields verifying Greenberg's conjecture and the structure of the unramified abelian Iwasawa module

Let p be a prime number, K a number field and K_n the layers of the cyclotomic $\mathbb{Z}_p$-extension of K. For each nonnegative integer n, let L_n be the Hilbert p-class field of K_n and L'_n be the maximal extension of K_n contained in L_n in which all p-adic places of K_n split completely. By class field theory, we have $A_n(K)\simeq\mathrm{Gal}(L_n/K_n)$ and the subgroup $D_n(K)$ of $A_n(K)$ generated by the classes of p-adic primes fixes L'_n, in order that $\mathrm{Gal}(L_n/L'_n)\simeq D_n(K)$. Also, for any nonnegative integer n, we denote by $A'_n(K)$ the group of p-ideal p-classes of K_n, that is, $A_n(K)/D_n(K)$. We simply denote by $A'(K):=A'_0(K)$ the group of p-ideal p-classes of K, that is, $A(K)/D(K)$. We define $L_\infty:=\bigcup L_n$, $L'_\infty=\bigcup L'_n$ and the

Iwasawa module $X'_\infty(K)$ as the projective limit of the groups $A'_n(K)$ with respect to the norm maps

$$X'_\infty(K) = \varprojlim A'_n(K) \simeq \varprojlim \mathrm{Gal}(L'_n/K_n) = \mathrm{Gal}(L'_\infty/K_\infty),$$

where the second projective limit is defined with respect to the restriction maps. Also, we define the group $D_\infty(K)$ as the projective limit of the groups $D_n(K)$, with respect to the norm maps

$$D_\infty(K) := \varprojlim D_n(K).$$

Let γ be a topological generator of $\mathrm{Gal}(K_\infty/K)$, let $w_0 = T = \gamma - 1$, and for each positive integer n, we denote by $w_n = \gamma^{p^n} - 1 = (1+T)^{p^n} - 1$, $\nu_n = w_n/w_0$ and $\Lambda = \mathbb{Z}_p[\![T]\!]$ the ring of formal power series, which is a local ring of maximal ideal (p, T).

Preparation to the proof of the main theorem. We will prove the following general result giving the least layer of the cyclotomic $\mathbb{Z}_p$-extension of K, from which the elementary groups $A'_n(K)/p$ of the layers K_n stabilize.

Proposition 2.1. *Let p be a prime number and K a number field containing a unique p-adic place that is totally ramified in K_∞. Suppose there exists a nonnegative integer m such that $\mathrm{rk}_p(A'_m(K)) < p^m$. Then we have*

$$X'_\infty(K)/p \simeq A'_m(K)/p.$$

Proof. Since K contains a unique p-adic place which is totally ramified in K_∞, then the maximal abelian extension of K_n contained in L'_∞ is $K_\infty L'_n$, and hence $w_n X'_\infty(K)$ fixes $K_\infty L'_n$ [6]. We obtain

$$X'_\infty(K)/w_0 X'_\infty(K) \simeq \mathrm{Gal}(K_\infty L'_0/K_\infty) \simeq \mathrm{Gal}(L'_0/K) \simeq A'(K),$$
$$X'_\infty(K)/w_n X'_\infty(K) \simeq \mathrm{Gal}(K_\infty L'_n/K_\infty) \simeq \mathrm{Gal}(L'_n/K_n) \simeq A'_n(K).$$

Let r be a nonnegative integer such that $\mathrm{rk}_p(A'(K)) = r$:

$$A'(K)/p \simeq (\mathbb{Z}/p\mathbb{Z})^r.$$

Hence from Nakayama's lemma, $X'_\infty(K)$ is a finitely generated Λ-module with r generators. Thus the elementary p-group $X'_\infty(K)/p$ is a $\mathbb{F}_p[\![T]\!]$-module with r generators:

$$X'_\infty(K)/p \simeq \bigoplus_{i=1}^{r} \frac{\mathbb{F}_p[\![T]\!]}{(T^{n_i})},$$

where n_i are positive integers. Clearly we have

$$\mathrm{rk}_p(X'_\infty(K)) = \sum_{i=1}^{r} n_i.$$

As reported above, the groups $A'_n(K)$ are determined by giving quotient of $X'_\infty(K)$ over w_n. Hence we obtain

$$X'_\infty(K)/(p, w_n) \simeq A'_n(K)/p \simeq \bigoplus_{i=1}^{r} \frac{\mathbb{F}_p[\![T]\!]}{(w_n, T^{n_i})}.$$

Hence

$$\mathrm{rk}_p(A'_m(K)) = \sum_{i=1}^{r}\big(\min(\deg(w_m), n_i)\big) = \sum_{i=1}^{r}\big(\min(p^m, n_i)\big).$$

The hypothesis, $\mathrm{rk}_p(A'_m(K)) < p^m$, implies $n_i < p^m$ for each $i = 1, \ldots, r$. We conclude that

$$\mathrm{rk}_p(X'_\infty(K)) = \sum_{i=1}^{r} n_i = \mathrm{rk}_p(A'_m(K). \qquad \square$$

Below we consider the quadratic number field $F = \boldsymbol{Q}(\sqrt{\ell\ell'})$, where ℓ and ℓ' are prime numbers such that $\ell \equiv -\ell' \equiv 1 \pmod 4$. Let $m+2$ and $m'+2$ be respectively the 2-adic valuations of $\ell - 1$ and $\ell' + 1$:

$$v_2(\ell - 1) - 2 = m \quad \text{and} \quad v_2(\ell' + 1) - 2 = m'.$$

Clearly in terms of decomposition in the cyclotomic $\mathbb{Z}_2$-extension of $\boldsymbol{Q}$, we have $\boldsymbol{Q}_m$ and $\boldsymbol{Q}_{m'}$ respectively the decomposition fields of ℓ and ℓ'.

For each positive integer n, denote $\alpha_n = 2\cos(2\pi/2^{n+2})$. The n-th layer of the cyclotomic $\mathbb{Z}_2$-extension of $\boldsymbol{Q}$ is $\boldsymbol{Q}_n = \boldsymbol{Q}(\alpha_n)$. One can verify that $\alpha_{n+1} = \sqrt{2+\alpha_n}$. We have $N_{\boldsymbol{Q}_n/\boldsymbol{Q}}(2+\alpha_n) = 2$ and $(2+\alpha_n)o_{\boldsymbol{Q}_n}$ is the unique prime ideal of $\boldsymbol{Q}_n$ lying over 2, and hence

$$2o_{\boldsymbol{Q}_n} = (2+\alpha_n)^{2^n} o_{\boldsymbol{Q}_n}.$$

Put for each positive integer n, $\beta_n = 2 + \alpha_n$, so

$$\beta_{n+1} = 2 + \alpha_{n+1} = 2 + \sqrt{2+\alpha_n} = 2 + \sqrt{\beta_n}.$$

Then we have

$$\boldsymbol{Q}_n = \boldsymbol{Q}(\beta_n) \quad \text{and} \quad \boldsymbol{Q}_{n+1} = \boldsymbol{Q}_n(\sqrt{\beta_n}).$$

Next, we denote by $E_{\boldsymbol{Q}_n}$ (resp. $E'_{\boldsymbol{Q}_n}$), the group of units (resp. the group of 2-units) of $\boldsymbol{Q}_n$. Clearly, the group $E'_{\boldsymbol{Q}_n}$ is generated by β_n and $E_{\boldsymbol{Q}_n}$.

Proposition 2.2. *Suppose that $m' \geq m$. We have*:

(1) *If $m = 0$, then $A'_n(F) = 0$ for each nonnegative integer n.*

(2) *If $m \geq 1$, then $\frac{1}{2}X'_\infty(F) \simeq \mathbb{Z}/2\mathbb{Z}^{\oplus 2^m - 1}$, precisely we have*

(2-1) $\frac{1}{2}A_n(F) \simeq \frac{1}{2}A'_n(F) \simeq \mathbb{Z}/2\mathbb{Z}^{\oplus 2^n}$ *for all* $n \leq m-1$,

(2-2) $D_n \simeq \mathbb{Z}/2\mathbb{Z}$, $\frac{1}{2}A'_n(F) \simeq \mathbb{Z}/2\mathbb{Z}^{\oplus 2^m - 1}$, $\frac{1}{2}A_n(F) \simeq \mathbb{Z}/2\mathbb{Z}^{\oplus 2^m}$ *for all* $n \geq m$.

Proof. By genus theory, we have $A(F) \simeq \mathbb{Z}/2\mathbb{Z}$. Since F contains a unique 2-adic place, then $X'_\infty(F)/T \simeq A'(F)$ is cyclic (possible trivial). Suppose that $m=0$, then ℓ is inert in $\boldsymbol{Q}_1$, which is equivalent to $(2/\ell)=-1$. Hence, the 2-adic place of F is inert in $\boldsymbol{Q}(\sqrt{\ell}, \sqrt{\ell'})$ the genus field of F, thus $A'(F)$ is trivial. In that case, by Nakayama's lemma $X'_\infty(F)$ is trivial, then we have (1). Next suppose that $m \geq 1$. Then ℓ splits in $\boldsymbol{Q}_1$, so the 2-adic place of F splits in $\boldsymbol{Q}(\sqrt{\ell}, \sqrt{\ell'})$, thus $A'(F)$ is cyclic nontrivial.

On the other hand, since $A(\boldsymbol{Q}_n)$ is trivial, then each class of $A_n(F)$ of order 2 is an ambiguous class relative to the extension $F_n/\boldsymbol{Q}_n$. Hence we obtain

$$\tfrac{1}{2}A_n(F) \simeq A_n(F)^G \quad \text{and} \quad \tfrac{1}{2}A'_n(F) \simeq A'_n(F)^G,$$

where $G = \mathrm{Gal}(F_n/\boldsymbol{Q}_n)$.

From A' version of ambiguous class number formula applied to the extension $F_n/\boldsymbol{Q}_n$ (see, for instance, [2]), we have, for each nonnegative integer n

$$|A'_n(F)^G| = \begin{cases} 2^{2^n+2^n}[E'_{\boldsymbol{Q}_n} : E'_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n}(F_n^*)]^{-1} & \text{for all } n \leq m-1, \\ 2^{2^m+2^n}[E'_{\boldsymbol{Q}_n} : E'_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n}(F_n^*)]^{-1} & \text{for all } m \leq n \leq m', \\ 2^{2^m+2^{m'}}[E'_{\boldsymbol{Q}_n} : E'_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n}(F_n^*)]^{-1} & \text{for all } n \geq m'. \end{cases}$$

Hence to compute the unit index $[E'_{\boldsymbol{Q}_n} : E'_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n} F_n^*]$, it suffices to look to the units of $\boldsymbol{Q}_n$ and β_n whether or not they are norms in the extension $F_n/\boldsymbol{Q}_n$. Clearly, the unit index $[E'_{\boldsymbol{Q}_n} : E'_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n}(F_n^*)]$ is less than or equal to 2^{2^n+1}; we will compute this unit index. It is well known that an element $u \in E'_{\boldsymbol{Q}_n}$ is a norm in the extension $F_n/\boldsymbol{Q}_n$ if and only if the quadratic norm residue symbol $\left(\frac{u,\ell\ell'}{\mathcal{P}}\right)$ relatively to the extension $F_n/\boldsymbol{Q}_n$, is trivial for each prime ideal $\mathcal{P}$ of $\boldsymbol{Q}_n$ ramified in F_n. Note that there is only one 2-adic place $\mathcal{Q}$ of $\boldsymbol{Q}_n$ ramified in F_n. Then from the product formula

$$\prod_{\mathcal{L}|\ell}\left(\frac{u,\ell\ell'}{\mathcal{L}}\right)\prod_{\mathcal{L}'|\ell'}\left(\frac{u,\ell\ell'}{\mathcal{L}'}\right)\left(\frac{u,\ell\ell'}{\mathcal{Q}}\right)=1,$$

u is a norm in the extension $F_n/\boldsymbol{Q}_n$ if and only if $\left(\frac{u,\ell\ell'}{\mathcal{P}}\right)=1$, for each prime ideal $\mathcal{P}$ of $\boldsymbol{Q}_n$ dividing $\ell\ell'$. In particular, since each ℓ-adic (resp. ℓ'-adic) place $\mathcal{L}$ (resp. $\mathcal{L}'$) of $\boldsymbol{Q}_n$ is unramified in $\boldsymbol{Q}_n(\sqrt{\ell'})$ (resp. $\boldsymbol{Q}_n(\sqrt{\ell})$), and by the fact that u is a 2-unit, we obtain

$$\left(\frac{u,\ell}{\mathcal{L}'}\right)=\sqrt{\ell}^{\left(\frac{\mathcal{Q}_m(\sqrt{\beta_m})/\mathcal{Q}_m}{\mathcal{L}'}\right)^{-v_{\mathcal{L}'}((u))}-1}=1, \quad \left(\frac{u,\ell'}{\mathcal{L}}\right)=\sqrt{\ell'}^{\left(\frac{\mathcal{Q}_m(\sqrt{\beta_m})/\mathcal{Q}_m}{\mathcal{L}}\right)^{-v_{\mathcal{L}}((u))}-1}=1,$$

where $\left(\frac{*/*}{*}\right)$ denotes the Artin symbol and $v_{\mathcal{P}}((u))$ is the $\mathcal{P}$-adic valuation of the ideal (u) of $\boldsymbol{Q}_n$ generated by u, so $v_{\mathcal{P}}((u))=0$.

Hence, since for each prime ideal $\mathcal{P}$ dividing $\ell\ell'$, we have $\left(\frac{u,\ell\ell'}{\mathcal{P}}\right)=\left(\frac{u,\ell}{\mathcal{P}}\right)\left(\frac{u,\ell'}{\mathcal{P}}\right)$, then u is a norm in the extension $F_n/\boldsymbol{Q}_n$ if and only if u is a norm in the extensions

$\boldsymbol{Q}_n(\sqrt{\ell})/\boldsymbol{Q}_n$ and $\boldsymbol{Q}_n(\sqrt{\ell'})/\boldsymbol{Q}_n$. Thus, we have the following surjective maps:

$$f : E'_{\boldsymbol{Q}_n}/E'_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n} F_n^* \twoheadrightarrow E'_{\boldsymbol{Q}_n}/E'_{\boldsymbol{Q}_n} \cap N_{\boldsymbol{Q}_n(\sqrt{\ell'})/\boldsymbol{Q}_n} \boldsymbol{Q}_n(\sqrt{\ell'})^*,$$
$$E_{\boldsymbol{Q}_n}/E_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n} F_n^* \twoheadrightarrow E_{\boldsymbol{Q}_n}/E_{\boldsymbol{Q}_n} \cap N_{\boldsymbol{Q}_n(\sqrt{\ell'})/\boldsymbol{Q}_n} \boldsymbol{Q}_n(\sqrt{\ell'})^*.$$

Since $\boldsymbol{Q}(\sqrt{\ell'})$ contains a unique 2-adic place which is totally ramified in the $\mathbb{Z}_2$-extension $(\boldsymbol{Q}(\sqrt{\ell'}))_\infty$, then $X'_\infty(\boldsymbol{Q}(\sqrt{\ell'}))/T \simeq A'_0(\boldsymbol{Q}(\sqrt{\ell'}))$, which is trivial. Hence $A'_n(\boldsymbol{Q}(\sqrt{\ell'}))$ is trivial for each nonnegative integer n. Thus from the ambiguous class number formula applied to the quadratic extension $\boldsymbol{Q}_n(\sqrt{\ell'})/\boldsymbol{Q}_n$, we obtain

$$[E'_{\boldsymbol{Q}_n} : E'_{\boldsymbol{Q}_n} \cap N_{\boldsymbol{Q}_n(\sqrt{\ell'})/\boldsymbol{Q}_n} \boldsymbol{Q}_n(\sqrt{\ell'})^*] = \begin{cases} 2^{2^n} & \text{for all } n \le m', \\ 2^{2^{m'}} & \text{for all } n \ge m'. \end{cases}$$

Similarly, we obtain the maximality of the following unit index for $n \le m'$:

$$[E_{\boldsymbol{Q}_n} : E_{\boldsymbol{Q}_n} \cap N_{\boldsymbol{Q}_n(\sqrt{\ell'})/\boldsymbol{Q}_n} \boldsymbol{Q}_n(\sqrt{\ell'})^*] = \begin{cases} 2^{2^n} & \text{for all } n \le m', \\ 2^{2^{m'}} & \text{for all } n \ge m'. \end{cases}$$

It follows from the above maps that

$$[E'_{\boldsymbol{Q}_n} : E'_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n} F_n^*] \ge \begin{cases} 2^{2^n} & \text{for all } n \le m', \\ 2^{2^{m'}} & \text{for all } n \ge m', \end{cases}$$
$$[E_{\boldsymbol{Q}_n} : E_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n} F_n^*] \ge \begin{cases} 2^{2^n} & \text{for all } n \le m', \\ 2^{2^{m'}} & \text{for all } n \ge m'. \end{cases}$$

Therefore, since $[E_{\boldsymbol{Q}_n} : E_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n} F_n^*] \le 2^{2^n}$, we obtain the maximality of the following unit index:

$$[E_{\boldsymbol{Q}_n} : E_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n} F_n^*] = 2^n \quad \text{for all } n \le m'.$$

For $n \le m-1$, from the hypotheses, the ℓ-adic and ℓ'-adic places of $\boldsymbol{Q}_n$ split in $\boldsymbol{Q}_{n+1} = \boldsymbol{Q}_n(\sqrt{\beta_n})$, then for each prime ideal $\mathcal{P}|\ell\ell'$, by the properties of the norm residue symbol, β_n is a norm in the extension $F_n/\boldsymbol{Q}_n$:

$$\left(\frac{\beta_n, \ell\ell'}{\mathcal{P}}\right) = \left(\frac{\ell\ell', \beta_n}{\mathcal{P}}\right) = \sqrt{\beta_n}^{\left(\frac{\boldsymbol{Q}_n(\sqrt{\beta_n})/\boldsymbol{Q}_n}{\mathcal{P}}\right)^{-v_{\mathcal{P}}((\ell\ell'))} - 1} = \frac{\left(\frac{\boldsymbol{Q}_{n+1}/\boldsymbol{Q}_n}{\mathcal{P}}\right)^{-1}(\sqrt{\beta_n})}{\sqrt{\beta_n}} = 1,$$

where $v_{\mathcal{P}}((\ell\ell')) = 1$ is the $\mathcal{P}$-adic valuation of the ideal $(\ell\ell')$ of $\boldsymbol{Q}_n$ generated by $\ell\ell'$. Hence we obtain

$$[E'_{\boldsymbol{Q}_n} : E'_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n}(F_n^*)] = [E_{\boldsymbol{Q}_n} : E_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n}(F_n^*)] = 2^{2^n}.$$

It follows from the ambiguous class number formula that

$$\left|\tfrac{1}{2}A_n(F)\right| = \left|\tfrac{1}{2}A'_n(F)\right| = |A'_n(F)^G| = 2^{2^n+2^n}[E'_{\boldsymbol{Q}_n} : E'_{\boldsymbol{Q}_n} \cap N_{F_n/\boldsymbol{Q}_n}(F_n^*)]^{-1} = 2^{2^n}.$$

Hence we obtain (2-1) of Proposition 2.2.

Suppose now that $n \geq m$, especially when $n = m$, we have

$$|A'_m(F)^G| = 2^{2^{m+1}}[E'_{\boldsymbol{Q}_m} : E'_{\boldsymbol{Q}_m} \cap N_{F_m/\boldsymbol{Q}_m}(F_m^*)]^{-1}.$$

We will prove that the unit index $[E'_{\boldsymbol{Q}_m} : E'_{\boldsymbol{Q}_m} \cap N_{F_m/\boldsymbol{Q}_m}(F_m^*)]$ is maximal equal to 2^{2^m+1}. If we denote by U a fundamental system of units of $\boldsymbol{Q}_m$, it suffices to look if the system of the classes of units

$$\{\overline{-1}, \bar{\beta}_m, \bar{u} \mid u \in U\}$$

is a base of the $\mathbb{F}_2$-vectorial space $E'_{\boldsymbol{Q}_m}/E'_{\boldsymbol{Q}_m} \cap N_{F_n/\boldsymbol{Q}_m}(F_m^*)$. From the equalities

$$[E'_{\boldsymbol{Q}_m} : E'_{\boldsymbol{Q}_m} \cap N_{\boldsymbol{Q}_m(\sqrt{\ell'})/\boldsymbol{Q}_m}\boldsymbol{Q}_m(\sqrt{\ell'})^*] = [E_{\boldsymbol{Q}_m} : E_{\boldsymbol{Q}_m} \cap N_{\boldsymbol{Q}_m(\sqrt{\ell'})/\boldsymbol{Q}_m}\boldsymbol{Q}_m(\sqrt{\ell'})^*] = 2^m,$$

it is clear that $\{\overline{-1}, \bar{u} \mid u \in U\}$ is a base of the $\mathbb{F}_2$-vectorial space

$$E'_{\boldsymbol{Q}_m}/E'_{\boldsymbol{Q}_m} \cap N_{\boldsymbol{Q}_m(\sqrt{\ell'})/\boldsymbol{Q}_m}\boldsymbol{Q}_m(\sqrt{\ell'})^*.$$

Therefore, $\{\overline{-1}, \bar{u} \mid u \in U\}$, is a free system of the $\mathbb{F}_2$-vectorial space

$$E'_{\boldsymbol{Q}_m}/E'_{\boldsymbol{Q}_m} \cap N_{F_n/\boldsymbol{Q}_m}(F_m^*).$$

On the other hand, from the hypotheses, the ℓ-adic places of $\boldsymbol{Q}_m$ are inert in $\boldsymbol{Q}_{m+1}$. Hence β_m is not norm in the extension $F_m/\boldsymbol{Q}_m$, precisely for each ℓ-adic place $\mathcal{L}$ of $\boldsymbol{Q}_m$, we have

$$\left(\frac{\beta_m, \ell\ell'}{\mathcal{L}}\right) = \left(\frac{\ell\ell', \beta_m}{\mathcal{L}}\right) = \sqrt{\beta_m}^{\left(\frac{\boldsymbol{Q}_m(\sqrt{\beta_m})/\boldsymbol{Q}_m}{\mathcal{L}}\right)^{-v_{\mathcal{L}}((\ell\ell'))}-1} = \sqrt{\beta_m}^{\left(\frac{\boldsymbol{Q}_{m+1}/\boldsymbol{Q}_m}{\mathcal{L}}\right)^{-1}-1} = -1.$$

Hence β_m is not norm in the extension $F_m/\boldsymbol{Q}_m$.

Also, the ℓ'-adic places of $\boldsymbol{Q}_m$ are inert in $\boldsymbol{Q}_{m+1}$ if and only if $m = m'$. Therefore, one of the following two facts can occur:

(i) In the case where $m' \geq m+1$, for each ℓ'-adic place $\mathcal{L}'$ of $\boldsymbol{Q}_m$, we have

$$\left(\frac{\beta_m, \ell'}{\mathcal{L}'}\right) = \left(\frac{\ell', \beta_m}{\mathcal{L}'}\right) = \sqrt{\beta_m}^{\left(\frac{\boldsymbol{Q}_m(\sqrt{\beta_m})/\boldsymbol{Q}_m}{\mathcal{L}'}\right)^{-v_{\mathcal{L}'}((\ell'))}-1} = \sqrt{\beta_m}^{\left(\frac{\boldsymbol{Q}_{m+1}/\boldsymbol{Q}_m}{\mathcal{L}'}\right)^{-1}-1} = 1.$$

Hence, β_m is norm in the extension $\boldsymbol{Q}_m(\sqrt{\ell'})/\boldsymbol{Q}_m$, so the kernel of the previous map f is nontrivial. Thus we obtain

$$\ker(f) = \bar{\beta}_m\mathbb{F}_2.$$

(ii) In the case where $m = m'$, for each ℓ'-adic place $\mathcal{L}'$ of $\boldsymbol{Q}_m$, we have

$$\left(\frac{\beta_m, \ell'}{\mathcal{L}'}\right) = \left(\frac{\ell', \beta_m}{\mathcal{L}'}\right) = \sqrt{\beta_m}^{\left(\frac{\boldsymbol{Q}_m(\sqrt{\beta_m})/\boldsymbol{Q}_m}{\mathcal{L}'}\right)^{-v_{\mathcal{L}'}((\ell'))}-1} = \sqrt{\beta_m}^{\left(\frac{\boldsymbol{Q}_{m+1}/\boldsymbol{Q}_m}{\mathcal{L}'}\right)^{-1}-1} = -1.$$

Thus β_m is not norm in the extension $\boldsymbol{Q}_m(\sqrt{\ell'})/\boldsymbol{Q}_m$, so $\bar{\beta}_m \notin \ker(f)$.

Also, for each ℓ-adic place $\mathcal{L}$ and ℓ'-adic place $\mathcal{L}'$ of $\boldsymbol{Q}_m$, we have

$$\left(\frac{-1,\ell\ell'}{\mathcal{L}}\right)=\left(\frac{-1,\ell}{\mathcal{L}}\right)=\left(\frac{-1}{\ell}\right)=1 \quad \text{and} \quad \left(\frac{-1,\ell'}{\mathcal{L}'}\right)=\left(\frac{-1}{\ell'}\right)=-1.$$

Consequently, in this case, $-\beta_m$ is not norm in the extension $F_m/\boldsymbol{Q}_m$, but norm in the extension $\boldsymbol{Q}_m(\sqrt{\ell'})/\boldsymbol{Q}_m$. Hence the kernel of f is nontrivial:

$$\ker(f)=-\bar{\beta}_m\mathbb{F}_2.$$

Consequently, we conclude that the system $\{\overline{-1},\bar{\beta}_m,\bar{u}\mid u\in U\}$ is free. Thus, we find

$$\left|\tfrac{1}{2}A'_m(F)\right|=|A'_m(F)^G|=2^{2^m+2^m}[E'_{\boldsymbol{Q}_m}:E'_{\boldsymbol{Q}_m}\cap N_{F_m/\boldsymbol{Q}_m}(F_m^*)]^{-1}=2^{2^m-1}.$$

So clearly, $D_m(F)$ is nontrivial. Moreover, since the 2-adic place of F_m is totally ramified in F_∞, then for $n\geq m$, the norm map $D_n(F)\to D_m(F)$ is onto, implies that $D_n(F)$ is nontrivial. Also, since F_n contains a unique 2-adic place and its square is trivial, then we have

$$D_n(F)\simeq D_m(F)\simeq\mathbb{Z}/2\mathbb{Z}.$$

Furthermore, since $\mathrm{rk}_2(A'_m(F))=2^m-1<2^m$, it follows from Proposition 2.1 that

$$\tfrac{1}{2}X'_\infty(F)\simeq\tfrac{1}{2}A'_m(F)\simeq\tfrac{1}{2}A'_n(F)\simeq\mathbb{Z}/2\mathbb{Z}^{\oplus 2^m-1}\quad\text{for all } n\geq m.$$

In addition, by the ambiguous class number formula we conclude that for each $n\geq m$,

$$\mathrm{rk}_2(A_n(F))=\mathrm{rk}_2(A_n(F)^G)=2^m. \qquad \square$$

Corollary 2.3. *We have*

$$X_\infty(F)\simeq X'_\infty(F)\oplus D_\infty(F),$$

where $D_\infty(F)\simeq\mathbb{Z}/2\mathbb{Z}$.

Proof. From Proposition 2.2, for each $n\geq m$, we have

$$D_n(F)\simeq\mathbb{Z}/2\mathbb{Z},\quad \mathrm{rk}_2(A'_n(F))=2^m-1\quad\text{and}\quad\mathrm{rk}_2(A_n(F))=2^m.$$

It follows that $A_n\simeq A'_n\oplus D_n(F)$. Hence, passing to the projective limit with respect to the norm maps, we have the result. $\square$

Proof of the main theorem. From the hypotheses, we have $A(F)=A'(F)\simeq\mathbb{Z}/2\mathbb{Z}$ and generated by the class of the ℓ-adic place. By Proposition 2.2, we have $\mathrm{rank}(A'_m(F))<2^m$, then $A'(F)$ capitulates in F_m [15, Lemma 7]. Consider the commutative diagram [6, Theorems 6 and 7]:

$$\begin{array}{ccc} A'(F) & \xrightarrow{\sim} & X'_\infty(F)/w_0X'_\infty(F) \\ \downarrow & & \downarrow \nu_m \\ A'_m(F) & \xrightarrow{\sim} & X'_\infty(F)/w_mX'_\infty(F) \end{array}$$

Since $A'(F)$ capitulates in F_m, then the left vertical map is trivial, thus

$$\nu_m X'_\infty(F) \subset w_m X'_\infty(F).$$

Hence we obtain

$$w_m X'_\infty(F) = \nu_m X'_\infty(F) = w_0(\nu_m X'_\infty(F)).$$

On the other hand, since $\nu_m X'_\infty(F)$ is a finitely generated Λ-module and w_0 is contained in (p, T), then by Nakayama's lemma we obtain $w_m X'_\infty(F) = \nu_m X'_\infty(F) = 0$; hence $X'_\infty(F) \simeq A'_m(F)$. Consequently, from Corollary 2.3, we have

$$X_\infty(F) \simeq X'_\infty(F) \oplus D_\infty(F) \simeq A_m(F) \simeq A'_m(F) \oplus \mathbb{Z}/2\mathbb{Z}.$$

Also, from Proposition 2.2, we have $\mathrm{rk}_2(A_{m-1}(F)) = 2^{m-1} < \mathrm{rk}_2(A_m(F)) = 2^m$, then $X_\infty(F) \not\simeq A_{m-1}(F)$.

Now, we will prove that $X_\infty(F)$ is an elementary abelian 2-group. We will use other notations. For each nonnegative integer $n \leq m'$, let S_n be the set of ℓ'-adic places of F_n, and D_{S_n} the subgroup of $A_n(F)$ generated by the classes of places in S_n. Let $A_n^{S_n}$ be the group of S_n-classes, that is, $A_n^{S_n} := A_n(F)/D_{S_n}$. Let M_n be the maximal abelian unramified 2-extension over F_n, in which all places of S_n split completely. By class field theory, we have

$$\mathrm{Gal}(M_n/F_n) \simeq A_n^{S_n}.$$

Since F contains a unique 2-adic place which is totally ramified in F_∞ and the ℓ'-adic place of F splits completely in $F_{m'}$, then the maximal abelian unramified extension of F contained in $M_{m'}$ is $F_{m'}M_0$. On the other hand, $A_{m'}^{S_{m'}}$ is a finitely generated $\Lambda = \mathbb{Z}_2[[T]]$-module and $A_{m'}^{S_{m'}}/T \simeq A_0^{S_0}$. By the hypotheses, we have $(\ell/\ell') = -1$, then $A_0^{S_0} = 0$ and by Nakayama's lemma, $A_{m'}^{S_{m'}} = 0$. It follows that for each nonnegative integers $n \leq m'$, we have $A_n(F) \simeq D_{S_n}$. But, all classes of places in S_n are trivial or of order 2, then $A_n(F)$ is an elementary 2-group, thus $X_\infty(F)$ is an elementary group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2^m}$. □

Application to the $\mathbb{Z}_2$-torsion of $X_\infty(K)$, for some imaginary biquadratic number fields K. It is well known from the results of Ferrero and Kida [2; 7] that the $\mathbb{Z}_2$-torsion part $X^0_\infty(K)$ of the unramified abelian Iwasawa module $X_\infty(K)$ of any imaginary quadratic number field K is trivial or cyclic of order 2. As an application of the main theorem, we will determine an infinite family of imaginary biquadratic number fields K, in which the $\mathbb{Z}_2$-torsion part of the Iwasawa module $X_\infty(K)$ is an elementary group of arbitrary large rank.

M. Atsuta [1] studied the minus quotient $X_\infty^-(K)$ of the Iwasawa module $X_\infty(K)$ for CM number fields K, that is,

$$X_\infty^-(K) = X_\infty(K)/(1+J)X_\infty(K),$$

where J is the complex conjugation. He determined the maximal finite submodule of X_∞^- under some mild assumptions. Precisely for a CM number field K such that its totally real maximal subfield K^+ is unramified at 2 and contains a unique 2-adic place, then $X_\infty^-(K)$ has no nontrivial finite Λ-submodule [1, Example 2.8]. So from the exact sequence

$$0 \to X_\infty(K^+) \to X_\infty(K) \to X_\infty^-(K) \to 0,$$

we have the maximal finite Λ-submodule of $X_\infty(K)$ which coincides with the maximal finite submodule of $X_\infty(K^+)$:

$$X_\infty^0(K) = X_\infty^0(K^+).$$

We reconsider now, the quadratic number field $F = \boldsymbol{Q}(\sqrt{\ell\ell'})$ of the main Theorem 1.1. Recall that ℓ and ℓ' are two prime numbers such that

$$\ell \equiv -\ell' \equiv 1 \pmod 4 \quad \text{and} \quad (\ell/\ell') = -1.$$

The positive integers m and m' are defined as

$$v_2(\ell-1)-2 = m \quad \text{and} \quad v_2(\ell'+1)-2 = m' \qquad (m' \geq m).$$

Then we have:

Proposition 2.4. *For the imaginary biquadratic number field $K = F(i)$, we have the structure of the unramified abelian Iwasawa module $X_\infty(K)$ of K:*

$$X_\infty(K) \simeq \mathbb{Z}_2^{\lambda_2(K)} \oplus X_\infty^0(K),$$

where $\lambda_2(K) = 2^m + 2^{m'} - 1$ and $X_\infty^0(K) \simeq X_\infty(F) \simeq (\mathbb{Z}/2\mathbb{Z})^{2^m}$.

Proof. From Kida's formula [7, Theorem 3], we see immediately that

$$\lambda(K) = 2^m + 2^{m'} - 1.$$

On the other hand, since the quadratic extension K/K^+ (here $K^+ = F$) is unramified at 2-adic primes, then $X_\infty^-(K)$ has no nontrivial Λ-submodule [1, Corollary 1.4]. Hence, the $\mathbb{Z}_2$-torsion $X_\infty^0(K)$ of the Iwasawa module $X_\infty(K)$ coincides with the Iwasawa module $X_\infty(F)$:

$$X_\infty^0(K) = X_\infty(F).$$

Consequently from Theorem 1.1, we obtain

$$X_\infty(K) \simeq \mathbb{Z}_2^{2^m+2^{m'}-1} \oplus (\mathbb{Z}/2\mathbb{Z})^{2^m}. \qquad \square$$

Acknowledgement

The author is grateful to the referee for careful reading of the manuscript and for valuable comments and suggestions for the improvement of this article.

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Received October 25, 2022. Revised February 28, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.185

CONJUGACY CLASSES OF π-ELEMENTS AND NILPOTENT/ABELIAN HALL π-SUBGROUPS

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Let G be a finite group and π be a set of primes. We study finite groups with a large number of conjugacy classes of π-elements. In particular, we obtain precise lower bounds for this number in terms of the π-part of the order of G to ensure the existence of a nilpotent or abelian Hall π-subgroup in G.

1. Introduction

Let G be a finite group. The number $k(G)$ of conjugacy classes of G is an important and much investigated invariant in group theory. It is equal to the number of complex irreducible representations of G. The probability $\Pr(G)$ that two uniformly and randomly chosen elements from G commute is given by $k(G)/|G|$ where $|G|$ denotes the order of G. This is called the commuting probability or the commutativity degree of G and it has a large literature; see [Gustafson 1973; Neumann 1989; Lescot 2001; Guralnick and Robinson 2006; Eberhard 2015]. The commuting probability has also been studied for infinite groups; see [Tointon 2020].

A starting point of our work is a much cited theorem of Gustafson [1973] stating that $\Pr(G) > \frac{5}{8}$ for a finite group G if and only if it is abelian. Let p be the smallest prime divisor of the order of a finite group G. It was observed by Guralnick and Robinson [2006, Lemma 2] that if $\Pr(G) > 1/p$, then G is nilpotent. Moreover, Burness, Guralnick, Moretó and Navarro [Burness et al. 2021, Lemma 4.2] recently showed that if $\Pr(G) > (p^2+p-1)/p^3$, then G is abelian. An aim of this paper is to give a generalization of all three of these results.

Hung and Martínez would like to thank Alexander Moretó for helpful conversations. We are grateful to the referee for several suggestions that have greatly improved the exposition in the paper. Part of this work was done while Hung was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM), and he thanks the VIASM for financial support and hospitality. Maróti would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge for its support and hospitality during the programme Groups, representations and applications: new perspectives when work on this paper was undertaken. The work of Maróti has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 741420); he was also supported by the National Research, Development and Innovation Office (NKFIH) Grant No. K138596, No. K132951 and Grant No. K138828.

MSC2020: primary 20E45; secondary 20D10, 20D20.

Keywords: finite groups, conjugacy classes, π-elements, Hall subgroups.

Let π be a set of primes. A positive integer is called a π-number if it is not divisible by any prime outside π. The π-part n_π of a positive integer n is the largest π-number which divides n. An element of a finite group is called a π-element if its order is a π-number. The set of all π-elements in a finite group is a union of conjugacy classes of the group. Let $k_\pi(G)$ be the number of conjugacy classes of π-elements in a finite group G and let

$$d_\pi(G) := k_\pi(G)/|G|_\pi.$$

This invariant is always at most 1 by an old result of Robinson; see [Malle et al. 2021, Lemma 3.5]. The main theorem of the paper [Maróti and Nguyen 2014] is that if $d_\pi(G) > \frac{5}{8}$ for a finite group G and a set of primes π, then G possesses an abelian Hall π-subgroup. The following result is a far reaching generalization of this statement.

Theorem 1.1. *Let G be a finite group and let π be a set of primes. Let p be the smallest member of π. If $\mathrm{d}_\pi(G) > 1/p$, then G has a nilpotent Hall π-subgroup, whose derived subgroup has size at most p. Moreover, if $\mathrm{d}_\pi(G) > (p^2+p-1)/p^3$, then G has an abelian Hall π-subgroup.*

A well-known theorem of Wielandt [1954] states that if a finite group G contains a nilpotent Hall π-subgroup for some set of primes π then all Hall π-subgroups of G are conjugate and every π-subgroup of G is contained in a Hall π-subgroup. Therefore, the π-subgroups of a group satisfying the hypothesis of Theorem 1.1 behave like Sylow subgroups.

There are several results in the literature on the existence of abelian or nilpotent Hall subgroups in finite groups. For example [Beltrán et al. 2016, Theorem B] states that if G is a finite group and π a set of primes, then G has nilpotent Hall π-subgroups if and only if for every pair of distinct primes p, q in π the class sizes of the p-elements of G are not divisible by q.

For certain sets π, Tong-Viet [2020] proved some nice results on the existence of normal π-complements in finite groups G under the condition that $\mathrm{d}_\pi(G)$ is large. For example, [Tong-Viet 2020, Theorem E] states that if $p > 2$ is the smallest prime in π and $d_\pi(G) > (p+1)/2p$, then G contains not only an abelian Hall π-subgroup but also a normal π-complement. Another is [loc. cit., Theorem A], which states that if $d_2(G) > \frac{1}{2}$ then G has a normal 2-complement. We in fact make use of this result to prove Theorem 1.1 in the case $2 \in \pi$. As a consequence, the proof for this case does not depend on the classification of finite simple groups. The other case $2 \notin \pi$, however, is more challenging and our proof has to rely on the classification.

The paper is organized as follows. In Section 2 we prove some preliminary results on the commuting probability $\Pr(G)$. In Section 3 we prove some basic properties of the π-class invariant $\mathrm{d}_\pi(G)$ and, in particular, we show in Theorem 3.4 that in order to prove the main result, it suffices to show the existence of a nilpotent Hall π-subgroup under the hypothesis $\mathrm{d}_\pi(G) > \frac{1}{p}$. We then establish this statement in Section 4, modulo a result on finite simple groups (Theorem 4.9) that will be proved in Section 5. Finally, in Section 6, we present examples showing that the converse of Theorem 1.1 is false and that the obtained bounds are sharp in general.

2. Commuting probability

In this section we recall and prove some results about the commuting probability $\Pr(G)$ that will be needed later.

The first lemma is a generalization of Gustafson's result [1973] mentioned earlier. The inequality part is due to Burness, Guralnick, Moretó, and Navarro [2021].

Lemma 2.1. *Let G be a finite group and p the smallest prime dividing $|G|$. If G is not abelian, then $\Pr(G) \le (p^2 + p - 1)/p^3$ with equality if and only if $G/\boldsymbol{Z}(G) = C_p \times C_p$.*

Proof. The first part of the lemma is [Burness et al. 2021, Lemma 4.2]. Following its proof, we see that the equality $\Pr(G) = (p^2 + p - 1)/p^3$ holds if and only if $G/\boldsymbol{Z}(G) = C_p \times C_p$ and $|x^G| = p$ for every $x \in G \setminus \boldsymbol{Z}(G)$. It suffices to prove that if $G/\boldsymbol{Z}(G) = C_p \times C_p$, then $|x^G| = p$ for every $x \in G \setminus \boldsymbol{Z}(G)$.

Assume that $G/\boldsymbol{Z}(G) = C_p \times C_p$ and let $x \in G \setminus \boldsymbol{Z}(G)$. Since $x \in \boldsymbol{C}_G(x) \setminus \boldsymbol{Z}(G)$, we have that $\boldsymbol{Z}(G) < \boldsymbol{C}_G(x)$. Therefore, $|x^G| = |G|/|\boldsymbol{C}_G(x)|$ is a proper divisor of $|G|/|\boldsymbol{Z}(G)| = p^2$. On the other hand, since x is not central, $|x^G| > 1$. Thus, $|x^G| = p$, and the claim follows. □

Note that if G is an extra-special p-group of order p^3 with p odd or if G is a dihedral group when $p = 2$, then $G/\boldsymbol{Z}(G) = C_p \times C_p$. Therefore, the bound in Lemma 2.1 is sharp for all p.

We next give a bound for $\Pr(G)$ in terms of the smallest prime factor of the order of G and the order of its derived subgroup G'.

Lemma 2.2. *If p is the smallest prime dividing the order of a finite group G, then*

$$\Pr(G) \le \frac{1 + (p^2 - 1)/|G'|}{p^2}.$$

Proof. Let $\mathrm{Irr}(G)$ denote the set of all irreducible complex characters of G. We have

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 \ge |G/G'| + p^2(k(G) - |G/G'|),$$

since $\chi(1)$ divides $|G|$ for every $\chi \in \mathrm{Irr}(G)$. After dividing both sides of the previous inequality by $|G|$, we obtain $1 \geq 1/|G'| + p^2(\mathrm{Pr}(G) - 1/|G'|)$. This yields $\mathrm{Pr}(G) \leq (1 + (p^2-1)/|G'|)/p^2$, as we claimed. □

Lemma 2.3. *Let G be a finite group and p the smallest prime dividing $|G|$. Suppose that $|G'| \leq p$. Then $G' \leq \mathbf{Z}(G)$, and thus $G/\mathbf{Z}(G)$ is abelian. In particular, G is nilpotent.*

Proof. The case $|G'| = 1$ is obvious, so we assume $|G'| = p$. Since G' is normal and its order is the smallest prime dividing $|G|$, we deduce that G' is central in G, and the result follows. □

Next we refine Lemma 2.1. It follows from [Guralnick and Robinson 2006, Lemma 2(xiii)] that if $\mathrm{Pr}(G) > 1/p$, where p is the smallest prime dividing $|G|$, then G is nilpotent.

Theorem 2.4. *Let G be a finite group and p the smallest prime dividing $|G|$. Then $1/p < \mathrm{Pr}(G) \leq (p^2 + p - 1)/p^3$ if and only if $|G'| = p$. Moreover, in such case,*

$$\mathrm{Pr}(G) = \frac{1}{p} + \frac{p-1}{p|G : \mathbf{Z}(G)|}.$$

Proof. By Lemma 2.1 we may assume that G is nonabelian. Assume that $|G'| > p$. Then $|G'| \geq p + 1$ and hence, applying Lemma 2.2, we have $\mathrm{Pr}(G) \leq 1/p$. The only if part is therefore done.

Conversely, assume that $|G'| = p$. Then $G' \leq \mathbf{Z}(G)$ by Lemma 2.3. By [Isaacs 1976, Proble 2.13], we have $\chi(1)^2 = |G : \mathbf{Z}(G)|$ for every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$. We deduce that

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|/p + |G : \mathbf{Z}(G)|(k(G) - |G|/p),$$

and it follows that

$$\mathrm{Pr}(G) = \frac{1}{p} + \frac{p-1}{p|G : \mathbf{Z}(G)|} > \frac{1}{p},$$

as stated. □

Remark 2.5. It is worth noting that if $G/\mathbf{Z}(G) \cong C_p \times C_p$, then, by Lemma 2.1, we have $\mathrm{Pr}(G) = (p^2 + p - 1)/p^3 > 1/p$, and hence $|G'| = p$ by Theorem 2.4.

Let us denote

$$g_p(x) := \frac{1 + (p^2 - 1)/x}{p^2}.$$

We note that the function $g_p(x)$ is decreasing in terms of x. Also, $g_p(1) = 1$, $g_p(p) = (p^2 + p - 1)/p^3$, and $g_p(p+1) = 1/p$. These values of g_p, that appear in our main result, explain the relevance of g_p.

The next theorem could be compared with a result of Lescot [2001] stating that $\Pr(G) = \frac{1}{2}$ if and only if G is isoclinic to the symmetric group Σ_3.

Theorem 2.6. *Let G be a finite group and p the smallest prime dividing $|G|$. If $|G'| > p$, then*

$$\Pr(G) \le \frac{n(p) + p^2 - 1}{p^2 n(p)} \le \frac{1}{p},$$

where $n(p)$ denotes the smallest prime larger than p. Moreover, $\Pr(G) = 1/p$ *if and only if $p = 2$ and $G/\mathbf{Z}(G) \cong \Sigma_3$.*

Proof. By Bertrand's postulate, we know that $n(p) < 2p \le p^2$. Therefore, if $|G'| > p$ then $|G'| \ge n(p)$ and hence, applying Lemma 2.2, we have

$$\Pr(G) \le g_p(n(p)) = \frac{n(p) + p^2 - 1}{p^2 n(p)}.$$

The second inequality holds because $g_p(n(p)) \le g_p(p+1) = 1/p$.

Suppose that $\Pr(G) = 1/p$. This forces $n(p) = p + 1$, which implies that $p = 2$ and $|G'| = 3$. We claim that $\Pr(G) = \frac{1}{2}$ if and only if $G/\mathbf{Z}(G) = \Sigma_3$. Assume first that $G/\mathbf{Z}(G) = \Sigma_3$. Let q be a prime dividing $|G|$ and let $Q \in \mathrm{Syl}_q(G)$. Since $G/\mathbf{Z}(G) = \Sigma_3$, we deduce that $|Q : \mathbf{Z}(Q)| \le q$ and hence Q is abelian. It follows that G possesses an abelian Sylow q-subgroup for every prime q dividing $|G|$. Thus, by [Guralnick and Robinson 2006, Lemma 2(xiii)], we have

$$\Pr(G) = \Pr(G/\mathbf{Z}(G)) = \Pr(\Sigma_3) = \tfrac{1}{2}.$$

The other direction of the claim follows from the above-mentioned theorem of Lescot [2001] since if G is isoclinic to Σ_3, then $G/\mathbf{Z}(G) = \Sigma_3$. □

3. Hall π-subgroups

In this section we prove that the second statement of Theorem 1.1 follows from the first.

Let $\mathcal{D}_\pi$ be the collection of all finite groups G such that G has a Hall π-subgroup, any two Hall π-subgroups of G are conjugate, and any π-subgroup of G is contained in a Hall π-subgroup. Of course $\mathcal{D}_\pi$ is everything when π is a single prime by Sylow's theorems. Also, $\mathcal{D}_\pi$ contains all π-separable groups. The following easy observation is useful to bound $\mathrm{d}_\pi(G)$ in the case $G \in \mathcal{D}_\pi$.

Lemma 3.1. *Let G be a finite group in $\mathcal{D}_\pi$. If H is a Hall π-subgroup of G, then*

$$\mathrm{d}_\pi(G) \le \Pr(H).$$

Proof. Since $|H| = |G|_\pi$, it suffices to see that $k_\pi(G) \le k(H)$. If $x, y \in H$ are not conjugate in G, then they cannot be conjugate in H. Since $G \in \mathcal{D}_\pi$, every G-class of π-elements has a representative in H. □

From this, we can easily prove Theorem 1.1 in case $G \in \mathcal{D}_\pi$.

Theorem 3.2. *Let π be a set of primes and G a finite group in $\mathcal{D}_\pi$. Then Theorem 1.1 holds for G.*

Proof. By hypothesis, G has a Hall π-subgroup H and all the Hall π-subgroups of G are G-conjugates of H. Thus, by Lemma 3.1, we have $\mathrm{d}_\pi(G) \leq \mathrm{Pr}(H)$. Let p be the smallest prime in π. Assume that $\mathrm{d}_\pi(G) > 1/p$. We then have

$$\mathrm{Pr}(H) > \frac{1}{p}.$$

Theorem 2.4 and Lemma 2.3 then imply that $|H'| \leq p$ and H is nilpotent, as claimed. Moreover, if $\mathrm{d}_\pi(G) > (p^2+p-1)/p^3$ then H is abelian by Lemma 2.1. □

As a consequence of Theorem 3.2, we have that Theorem 1.1 holds if $\pi = \{p\}$ or if G is π-separable.

We also recall some facts on the groups in $\mathcal{D}_\pi$. The first one is a result of Wielandt [1954] mentioned in the Introduction and the second one is due to Hall [1956, Theorem D5].

Lemma 3.3. *Let G be a finite group and π a set of primes*:

(i) *If G possesses a nilpotent Hall π-subgroup, then $G \in \mathcal{D}_\pi$.*

(ii) *If N possesses nilpotent Hall π-subgroups, G/N possesses solvable Hall π-subgroups, and $G/N \in \mathcal{D}_\pi$, then $G \in \mathcal{D}_\pi$.*

Theorem 3.4. *The second statement of Theorem 1.1 follows from the first.*

Proof. Let G be a group with $\mathrm{d}_\pi(G) > (p^2+p-1)/p^3 > 1/p$. By hypothesis, G possesses a nilpotent Hall π-subgroup. It then follows that $G \in \mathcal{D}_\pi$ by Lemma 3.3. The result follows by Theorem 3.2. □

The rest of the paper is therefore devoted to prove that G has a nilpotent Hall π-subgroup under the condition $\mathrm{d}_\pi(G) > 1/p$.

4. Reducing to a problem on simple groups

In this section we prove Theorem 1.1, assuming a result on finite simple groups.

Reducing to simple groups. We begin by recalling two properties of $\mathrm{d}_\pi(G)$. The first one is [Maróti and Nguyen 2014, Proposition 5], essentially due to Robinson. The second is due to Fulman and Guralnick [2012, Lemma 2.3].

Lemma 4.1. *Let G be a finite group and π a set of primes.*

(i) *Let $\mu \subseteq \pi$. Then* $\mathrm{d}_\pi(G) \leq \mathrm{d}_\mu(G)$.

(ii) $\mathrm{d}_\pi(G) \leq \mathrm{d}_\pi(N)\,\mathrm{d}_\pi(G/N)$ *for any normal subgroup N of G.*

Lemma 4.2. *Let G be a finite group, π a set of primes, and p the smallest prime in π. Let $q \in \pi$ and $Q \in \mathrm{Syl}_q(G)$. Suppose $\mathrm{d}_\pi(G) > 1/p$. We have:*

(i) *$Q/\mathbf{Z}(Q)$ is abelian and $|Q'| \le q$.*

(ii) *If $q \in \pi \setminus \{p\}$, then Q is abelian.*

Proof. By Sylow's theorems and Lemma 3.1 we have $\mathrm{d}_q(G) \le \Pr(Q)$. On the other hand, by Lemma 4.1(i), we have $\mathrm{d}_\pi(G) \le \mathrm{d}_q(G)$. We deduce that

$$\frac{1}{q} \le \frac{1}{p} < \Pr(Q).$$

Theorem 2.4 and Lemma 2.3 now imply that $Q/\mathbf{Z}(Q)$ is abelian and $|Q'| \le q$.

Suppose $q > p$. Then $q \ge p+1$, and one can easily check that $(q^2+q-1)/q^3 < 1/p$. Now $\Pr(Q) > (q^2+q-1)/q^3$, and thus Q must be abelian by Lemma 2.1. □

The next lemma is [Moretó 2013, Lemma 3.1], which allows us to work with a set of two primes instead of an arbitrary set.

Lemma 4.3 (Moretó). *Let G be a finite group and let π a set of primes. If G possesses a nilpotent Hall τ-subgroup for every $\tau \subseteq \pi$ with $|\tau| = 2$, then G possesses a nilpotent Hall π-subgroup.*

Proposition 4.4. *Suppose that Theorem 1.1 is false for a group G. Then there exists $\pi = \{p, q\}$, where $p < q$ are two primes, such that G does not possess nilpotent Hall π-subgroups and for all $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, $P/\mathbf{Z}(P)$ is abelian, $|P'| \le p$, and Q is abelian.*

Proof. By Theorem 3.4, we may assume that there exists π, a set of primes, such that $\mathrm{d}_\pi(G) > 1/p$, but G does not possess nilpotent Hall π-subgroups, where p is the smallest member of π.

If G has a nilpotent Hall τ-subgroup for every $\tau \subseteq \pi$ with $|\tau| = 2$, then by Lemma 4.3, G has nilpotent Hall π-subgroups. Thus, there exists $\{q, r\} \subseteq \pi$ with $q < r$ such that G does not possess a nilpotent Hall $\{q, r\}$-subgroup. By Lemma 4.1(i), we also have $\mathrm{d}_\pi(G) \le \mathrm{d}_{\{q,r\}}(G)$, and it follows that

$$\frac{1}{q} \le \frac{1}{p} < \mathrm{d}_\pi(G) \le \mathrm{d}_{\{q,r\}}(G).$$

Therefore, Theorem 1.1 fails for G and the set $\{q, r\}$, and hence we may assume that $|\pi| = 2$, that is $\pi = \{p, q\}$ with $p < q$.

Finally, the assertion on the Sylow subgroups follows from Lemma 4.2. □

Proposition 4.5. *Let π be a set of primes and p the smallest member in π. Let G be a finite group with minimal order subject to the conditions that $\mathrm{d}_\pi(G) > 1/p$ and G does not possess nilpotent Hall π-subgroups. Then G is nonabelian simple.*

Proof. We may assume that G is nonabelian and not simple. Let N be a nontrivial proper normal subgroup in G. By Lemma 4.1(ii), we have

$$\frac{1}{p} < \mathrm{d}_\pi(G) \leq \mathrm{d}_\pi(G/N)\,\mathrm{d}_\pi(N).$$

It follows that $1/p < \mathrm{d}_\pi(G/N)$ and $1/p < \mathrm{d}_\pi(N)$, as both $d_\pi(N)$ and $d_\pi(G/N)$ are at most one; see [Malle et al. 2021, Lemma 3.5]. By the minimality of G, N and G/N possess nilpotent Hall π-subgroups. Applying Lemma 3.3, we then deduce that both N and G/N are members of $\mathcal{D}_\pi$. It follows that $G/N \in \mathcal{D}_\pi$, G/N possesses solvable Hall π-subgroups and N possesses nilpotent Hall π-subgroups. By Lemma 3.3(ii), we have $G \in \mathcal{D}_\pi$. Therefore, by Theorem 3.2, we have that G possesses nilpotent Hall π-subgroups, which is a contradiction. We conclude that G is nonabelian simple. □

Reducing to a question on simple groups. The following is a consequence of a result of Tong-Viet, which asserts that if $\mathrm{d}_2(G) > \frac{1}{2}$ then G possesses a normal 2-complement.

Lemma 4.6. *Let S be a nonabelian simple group and π be a set of primes containing 2. Then* $\mathrm{d}_\pi(S) \leq \frac{1}{2}$.

Proof. Suppose that $\mathrm{d}_\pi(S) > \frac{1}{2}$. Then $\frac{1}{2} < \mathrm{d}_\pi(S) \leq \mathrm{d}_2(S)$. By [Tong-Viet 2020, Theorem A], S possesses a normal 2-complement, which is impossible. □

Proposition 4.7. *Let G be a group and π a set of primes such that* $\mathrm{d}_\pi(G) > 1/p$, *where p is the smallest prime in π. Let $q \in \pi$ but $q \neq p$. Then q does not divide $|\boldsymbol{N}_G(P) : \boldsymbol{C}_G(P)|$ where $P \in \mathrm{Syl}_p(G)$.*

Proof. Assume by contradiction that q divides $|\boldsymbol{N}_G(P)/\boldsymbol{C}_G(P)|$. Let x be an element of order q in $\boldsymbol{N}_G(P)/\boldsymbol{C}_G(P)$ where $P \in \mathrm{Syl}_p(G)$. Consider the action of $X = \langle x \rangle$ on P. Let r be the number of elements of P fixed by X.

We claim that $r > |P|/p^2$. Assume to the contrary that $r \leq |P|/p^2$. We have $|P| = r + t \cdot q$, implying that $t = (|P| - r)/q$. Since each X-orbit on P is contained in a conjugacy class of p-elements it is easy to see that $k_p(G) \leq r + t$. Now we have

$$\frac{1}{p} < \mathrm{d}_\pi(G) \leq \mathrm{d}_p(G) = \frac{k_p(G)}{|P|} \leq \frac{r+t}{|P|} = \frac{1}{q}\left((q-1)\frac{r}{|P|} + 1\right) \leq \frac{1}{q}\left((q-1)\frac{1}{p^2} + 1\right).$$

It is not hard to see that this implies $q \leq p$, which is a contradiction. We have shown that $r > |P|/p^2$.

Since r divides $|P|$, it follows that

$$r \in \{|P|, |P|/p\}.$$

If $r = |P|$ then X centralizes P, which is impossible. Thus $r = |P|/p$ and hence there exists a subgroup H of order $|P|/p$ that is centralized by X. That is,

$$H = \boldsymbol{C}_P(X) = \{z \in P \mid z^x = z \text{ for all } x \in X\}.$$

Let $L := P : X$ be the semidirect product of the relevant action of X on P. Then $L/H \cong C : X$ for some $C \cong C_p$. Since H is maximal in P, the subgroup H is normal in P, and it is X-invariant, applying [Isaacs 2008, Corollary 3.28], we have

$$\boldsymbol{C}_{P/H}(X) = \boldsymbol{C}_P(X)H/H = H/H,$$

and hence X acts nontrivially on C. Let $\mathcal{O}$ be a nontrivial orbit of the action of X on C. We now have $q = |X| = |\mathcal{O}| \leq |C| = p$, which is a contradiction. □

Corollary 4.8. *Let G be a group and $\pi = \{p, q\}$ a set of primes with $p < q$ such that $\mathrm{d}_\pi(G) > 1/p$. Let $P \in \mathrm{Syl}_p(G)$. Then q divides $|\mathrm{Syl}_p(G)| = |G : \boldsymbol{N}_G(P)|$ or G possesses a nilpotent Hall π-subgroup.*

Proof. We know that $|G|_q$ divides

$$|G| = |G : \boldsymbol{N}_G(P)||\boldsymbol{N}_G(P) : \boldsymbol{C}_G(P)||\boldsymbol{C}_G(P)|$$

but q cannot divide $|\boldsymbol{N}_G(P) : \boldsymbol{C}_G(P)|$ by Proposition 4.7. Assume that q does not divide $|G : \boldsymbol{N}_G(P)|$. Then $|G|_q$ divides $|\boldsymbol{C}_G(P)|$. Therefore, there exists $Q \in \mathrm{Syl}_q(G)$ with $Q \leq \boldsymbol{C}_G(P)$. Now PQ is a nilpotent Hall π-subgroup of G. □

Now we can prove Theorem 1.1, modulo the following statement about simple groups whose proof is deferred to the next section.

Theorem 4.9. *Let G be a nonabelian simple group and $\pi = \{p, q\}$ be a set of two odd primes with $p < q$. Assume that there exist $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $P/\boldsymbol{Z}(P)$ is abelian, $|P'| \leq p$, Q is abelian, and q divides $|G : \boldsymbol{N}_G(P)|$. Then $\mathrm{d}_\pi(G) \leq 1/p$.*

Observe that in Theorem 4.9 we may assume that both p and q divide the order of G.

Theorem 4.10. *Let G be a finite group, π be a set of primes, and p be the smallest prime in π. Assume Theorem 4.9. If $\mathrm{d}_\pi(G) > 1/p$ then G has a nilpotent Hall π-subgroup.*

Proof. Assume that the theorem is false and let G be a minimal counterexample. In particular, $\mathrm{d}_\pi(G) > 1/p$ but G has no nilpotent Hall π-subgroups. By Proposition 4.5, we know that G is nonabelian simple. Using Lemma 4.6, we know furthermore that $p \neq 2$.

By Proposition 4.4, there exists $\pi = \{p, q\}$ with (odd) $p < q$ such that $\mathrm{d}_\pi(G) > 1/p$, $P/\boldsymbol{Z}(P)$ is abelian, $|P'| \leq p$, and Q is abelian, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. We also have that q divides $|G : \boldsymbol{N}_G(P)|$, by Corollary 4.8. We now have all the hypotheses of Theorem 4.9, and therefore deduce that $\mathrm{d}_\pi(G) \le 1/p$. This is a contradiction. □

We remark that we have indeed proved Theorem 1.1 when the set π contains the prime 2, and this result does not rely on the classification of finite simple groups.

5. Simple groups

In this section we prove Theorem 4.9, by using the classification. We begin with the alternating groups.

Lemma 5.1. *Let p be an odd prime, $n \ge 5$ be an integer and $P \in \mathrm{Syl}_p(\mathsf{A}_n)$:*

(i) *If $n \ge p^2$, then $P/\mathbf{Z}(P)$ is not abelian.*

(ii) *If $n < p^2$, then P is elementary abelian.*

Proof. For (i) it is sufficient to exhibit a subgroup H of P such that $H/\mathbf{Z}(H)$ is not abelian. If $n \ge p^2$, then $H = C_p \wr C_p$ is such a subgroup of P. Statement (ii) follows from the description of the Sylow p-subgroups of A_n found in [Huppert 1967, Satz III.15.3]. □

Theorem 5.2. *Let $n \ge 5$, $\pi = \{p, q\}$ be a set of two odd primes with $p < q$, and $P \in \mathrm{Syl}_p(\mathsf{A}_n)$. Assume that both p and q divide the order of A_n. If $P/\mathbf{Z}(P)$ is abelian, then $\mathrm{d}_\pi(\mathsf{A}_n) \le 1/p$. In particular, Theorem 4.9 holds for alternating groups.*

Proof. Let $P \in \mathrm{Syl}_p(\mathsf{A}_n)$ and $Q \in \mathrm{Syl}_q(\mathsf{A}_n)$. Since $P/\mathbf{Z}(P)$ is abelian, $n < p^2$ by Lemma 5.1. Let $n = rp + s = lq + t$, where $r, s \in \{0, 1, \dots, p-1\}$ and $l, t \in \{0, 1, \dots, q-1\}$. Then $P = (C_p)^r$ and $Q = (C_q)^l$ with both r and l at least 1.

It is easy to see that every π-element of A_n can be expressed as a product of the form $xy = yx$, where x is a product of cycles of length p and y is a product of cycles of length q. Since $n < p^2$, the supports of x and y are disjoint.

Assume first that $n \ge p + q + 2$. In this case we have that $k_p(\mathsf{A}_n) = 1 + r \le p$, $k_q(\mathsf{A}_n) = 1 + l \le q$ and $|\mathsf{A}_n|_\pi = p^r q^l$. Thus we have

$$\mathrm{d}_\pi(\mathsf{A}_n) = \frac{k_\pi(\mathsf{A}_n)}{|\mathsf{A}_n|_\pi} \le \frac{pq}{p^r q^l}.$$

If $(r, l) \ne (1, 1)$, then $\mathrm{d}_\pi(\mathsf{A}_n) \le 1/p$. Assume now that $r = l = 1$. Then $k_\pi(\mathsf{A}_n) \le k_p(\mathsf{A}_n) k_q(\mathsf{A}_n) = 4$ and hence $\mathrm{d}_\pi(\mathsf{A}_n) \le (4/q)(1/p) < 1/p$, where the last inequality holds because $q \ge 5$.

Assume now that $n \le p + q + 1$ and so $l = 1$. In this case it may happen that a Σ_n-conjugacy class of π-elements splits in two different A_n-conjugacy classes. We thus have $k_\pi(\mathsf{A}_n) \le (1+r)(1+l) + 1 = 2(1+r) + 1 = 2r + 3$. It follows that $\mathrm{d}_\pi(\mathsf{A}_n) \le (2r+1)/(p^r q)$. If $r \ge 2$, then $(2r+3)/(p^r q) < 1/q < 1/p$. If $r = 1$, then $2r + 3 = 5 \le q$ and so once again $\mathrm{d}_\pi(\mathsf{A}_n) \le 1/p$. □

For convenience, we will consider the Tits group ${}^2F_4(2)'$ as a sporadic simple group.

Theorem 5.3. *Let S be a sporadic simple group and $\pi = \{p, q\}$ where $p < q$ are odd primes dividing $|S|$. If $(S, \pi) \neq (J_1, \{3, 5\})$ then $\mathrm{d}_\pi(S) \leq 1/p$. In particular, Theorem 4.9 holds for S.*

Proof. In what follows we use information in [Conway et al. 1985] without further notice. We may assume that π is a set of primes such that $k_\pi(S) \geq 6$, for otherwise

$$\mathrm{d}_\pi(S) = \frac{k_\pi(S)}{|S|_\pi} \leq \frac{5}{pq} \leq \frac{1}{p}$$

There is no such π for the four smallest Mathieu groups. For each of the groups M_{24}, HS, J_2 there are two possibilities for π. In each of the six cases $k_\pi(S)$ is at most $|S|_p$ or $|S|_q$ and this is sufficient to obtain the bound $\mathrm{d}_\pi(S) \leq 1/p$.

So we assume that S is not one of the groups already analyzed. If S is different from Fi_{23}, Fi'_{24} and J_1, then we count the total number of conjugacy classes of S of elements of odd order. These numbers are usually less that $|S|_r$ for a given prime divisor r of $|S|$. If this is the case for a prime r, then we can assume that r does not lie in π (otherwise we would be done). This gives strong restrictions on the set π. In fact, given that $k_\pi(S) \geq 6$, we find this way that S must be J_4 and π is either $\{3, 7\}$ or $\{5, 7\}$. In each of these two cases we count the number of π-classes in S to obtain our bound of $1/p$ for $\mathrm{d}_\pi(S)$.

If S is Fi_{23} or Fi'_{24}, then we again count the number of conjugacy classes of S of elements of odd order. This allows us to conclude that 3 cannot lie in π. We then count the number of conjugacy classes of S whose elements have orders divisible neither by 2 nor 3. This number is 8 in the first case and 14 in the second. By looking at the prime factorization of $|S|$, the only case to consider is $S = Fi'_{24}$ and $\pi = \{11, 13\}$. But it turns out that $k_\pi(S) = 3$ in this case.

The only group remaining is $S = J_1$. The number of conjugacy classes of S of elements of odd order is 11 forcing π to be a subset of $\{3, 5, 7\}$. Then $k_\pi(S) = 3$ or $\pi = \{3, 5\}$ and $k_\pi(S) = 6$, giving $\mathrm{d}_\pi(S) = \frac{2}{5}$.

The last assertion follows from the fact that if $P \in \mathrm{Syl}_3(J_1)$, then 5 does not divide $|J_1 : N_{J_1}(P)|$. □

We are left with the case of simple groups of Lie type $S \neq {}^2F_4(2)'$. For the sake of convenience, we rename the prime q in Theorem 4.9 to s in order to reserve q for the size of the underlying field of S.

The proof of Theorem 4.9 for groups of Lie type is divided into two fundamentally different cases: π contains the defining characteristic of S and π does not. The former case is fairly straightforward.

Theorem 5.4. *Let S be a finite simple group of Lie type in characteristic $p > 2$ and $\pi = \{p, s\}$, where s is an odd prime dividing $|S|$. Then,*

$$\mathrm{d}_\pi(S) \leq \frac{1}{s}.$$

In particular, Theorem 4.9 holds for simple groups of Lie type when π contains the defining characteristic of S.

Proof. First we observe that the desired inequality is satisfied if $k_\pi(S) \leq |S|_p$. We shall make use of well-known bounds of Fulman and Guralnick [2012] for the numbers of conjugacy classes of finite Chevalley groups to show that, when S has high enough rank, even the stronger inequality $k(S) \leq |S|_p$ holds true.

Let $S = \mathrm{PSL}(n, q)$. Then $k(S) \leq \min\{2.5q^{n-1}, q^{n-1} + 3q^{n-2}\}$ by [Fulman and Guralnick 2012, Proposition 3.6]. This is certainly smaller than $|S|_p = q^{n(n-1)/2}$ if $n \geq 4$. Therefore, we just need to verify the theorem for $n = 2$ or 3. The theorem is in fact straightforward to verify for these low rank cases, using the known information on conjugacy classes of the group (see [Dornhoff 1971, Chapter 38] for $n = 2$ and [Simpson and Frame 1973] for $n = 3$). The case $S = \mathrm{PSU}(n, q)$ is entirely similar.

Next, we consider $\mathrm{PSp}(2n, q)$ with $n \geq 3$. Then $k(S) \leq 10.8q^n$ for odd q, and it easily follows that $k(S) \leq |S|_p = q^{n^2}$. The case of orthogonal groups is similar, with a remark that $k(\Omega(2n+1, q)) \leq 7.3q^n$ for $n \geq 2$ and $k(\mathrm{P}\Omega^{\pm}(2n, q)) \leq 6.8q^n$ for $n \geq 4$.

Now we turn to exceptional groups. Recall that the defining characteristic p of S is odd, so we will exclude the types ${}^2B_2$ and ${}^2F_4$. By [Fulman and Guralnick 2012, Table 1] (or [Lübeck $\geq$ 2023] for more details), we observe that $k(S)$ is bounded above by a polynomial with positive coefficients, say g_S, evaluated at q. Suppose S is one of ${}^3D_4(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, or $E_8(q)$. We then have

$$k(S) \leq g_S(1)q^{\deg(g_S)} \leq 252q^{\deg(g_S)} \text{ and } \frac{q^{\deg(g_S)}}{|S|_p} \leq \frac{1}{q^8}.$$

Therefore,

$$\mathrm{d}_\pi(S) \leq \frac{k(S)}{|S|_p|S|_s} \leq \frac{252}{sq^8} < \frac{1}{s},$$

as wanted. The remaining cases of the types G_2 and ${}^2G_2$ are even easier, using the more refined bounds $k(G_2(q)) \leq q^2 + 2q + 9$ and $k({}^2G_2(q)) \leq q + 8$. □

Lemma 5.5. *Let G be a finite group and let π be a set of primes such that $|\mathbf{Z}(G)|_\pi = 1$. Then, $k_\pi(G) = k_\pi(G/\mathbf{Z}(G))$.*

Proof. Let $Z := \mathbf{Z}(G)$. Every coset gZ of Z in G contains at most one π-element of G since $|Z|_\pi = 1$. The π-elements of G/Z are gZ where g runs through the π-elements of G. If g is a π-element, then the conjugacy class of gZ in G/Z

consists of hZ where $h \in g^G$. Thus, there is a bijection between the π-conjugacy classes of G and the π-conjugacy classes of G/Z. □

In the case when π does not contain the defining characteristic of S, the conjugacy classes of π-elements of S will be semisimple classes, which can be conveniently described via an ambient algebraic group of S and its Weyl group.

It is well-known that every simple group of Lie type $S \neq {}^2F_4(2)'$ is of the form $S = \boldsymbol{G}^F/\boldsymbol{Z}(\boldsymbol{G}^F)$ for some simple algebraic group $\boldsymbol{G}$ of simply connected type and a suitable Steinberg endomorphism F on $\boldsymbol{G}$; see [Malle and Testerman 2011, Theorem 24.17] for instance.

Theorem 5.6. *Let S be a finite simple group of Lie type and $\boldsymbol{G}$, F as above. Let $\pi = \{p, s\}$ with $p < s$ be a set of primes not containing the defining characteristic of S. Suppose that s divides $|\mathrm{Syl}_p(S)|$. Then*

$$\mathrm{d}_\pi(\boldsymbol{G}^F) \leq \frac{1}{p}.$$

In particular, if $|\boldsymbol{Z}(\boldsymbol{G}^F)|_\pi = 1$, then $\mathrm{d}_\pi(S) \leq 1/p$.

Proof. Let $G := \boldsymbol{G}^F$. We first claim that a Hall π-subgroup of G, if exists, cannot be abelian. Assume by contradiction that G does have such subgroup, say H. Then $\overline{H} := H\boldsymbol{Z}(G)/\boldsymbol{Z}(G)$ would be an abelian Hall π-subgroup of S, implying that $\boldsymbol{N}_S(P)$ contains $\overline{H}$, where P is a Sylow p-subgroup of S that is contained in $\overline{H}$. It follows that s does not divide $|S : \boldsymbol{N}_S(P)|$, violating the hypothesis.

Let $\boldsymbol{T}$ be an F-stable maximal torus of $\boldsymbol{G}$, and let $W = N_{\boldsymbol{G}}(\boldsymbol{T})/\boldsymbol{T}$ be the Weyl group of $\boldsymbol{G}$. Since π does not contain the defining characteristic of S, the conjugacy classes of π-elements of G are semisimple classes. According to [Carter 1985, Proposition 3.7.3] and its proof, there is a well-defined bijection

$$\tau : \mathrm{Cl}_{ss}(G) \to (\boldsymbol{T}/W)^F$$

between the set $\mathrm{Cl}_{ss}(G)$ of semisimple conjugacy classes of G and the set $(\boldsymbol{T}/W)^F$ of F-stable orbits of W on $\boldsymbol{T}$. Malle, Navarro, and Robinson showed [Malle et al. 2021, Theorem 3.15] that this bijection τ preserves element orders, and therefore the counting formula (and its proof) for the number of F-stable orbits of W on $\boldsymbol{T}$ in [Carter 1985, Proposition 3.7.4] implies that

$$k_\pi(G) = \frac{1}{|W|}\sum_{w \in W} |\boldsymbol{T}^{w^{-1}F}|_\pi.$$

It follows that

$$\mathrm{d}_\pi(G) = \frac{1}{|W|}\sum_{w \in W} \frac{|\boldsymbol{T}^{w^{-1}F}|_\pi}{|G|_\pi}.$$

Now, if $|\boldsymbol{T}^{w^{-1}F}|_\pi = |G|_\pi$ for some $w \in W$ then a Hall π-subgroup of $\boldsymbol{T}^{w^{-1}F}$, which is abelian, would be a Hall π-subgroup of G, and this contradicts the above claim. Thus

$$\frac{|\boldsymbol{T}^{w^{-1}F}|_\pi}{|G|_\pi} \le \frac{1}{p}$$

for every $w \in W$. It then follows that

$$\mathrm{d}_\pi(G) \le \frac{1}{p},$$

proving the first part of the theorem.

For the second part, assume that $|\boldsymbol{Z}(G)|_\pi = 1$. By Lemma 5.5, we then have

$$\mathrm{d}_\pi(S) = \mathrm{d}_\pi(G/\boldsymbol{Z}(G)) = \mathrm{d}_\pi(G) \le \frac{1}{p},$$

as stated. □

Theorem 5.6 already proves Theorem 4.9 in several cases, as seen in the next result. In what follows, to unify the notation, we use GL^ϵ, SL^ϵ and PSL^ϵ for linear groups when $\epsilon = +$ and for unitary groups when $\epsilon = -$. We also use E_6^+ for E_6 and E_6^- for ${}^2E_6$.

Theorem 5.7. *Let S be a simple group of Lie type, π be a set of two odd primes not containing the defining characteristic of S, and p be the smaller prime in π. Assume that we are not in one of the following situations*:

(i) $S = E_6^\epsilon(q)$ *and* $3 \in \pi$.

(ii) $S = \mathrm{PSL}^\epsilon(n, q)$ *with* $n \ge 3$ *and* $\gcd(n, q-\epsilon)_\pi \ne 1$.

Then $\mathrm{d}_\pi(S) \le 1/p$.

Proof. Let $\boldsymbol{G}$ and F be as in Theorem 5.6. According to [Malle and Testerman 2011, Table 24.12], if we are not in one of the stated situations, then $|\boldsymbol{Z}(\boldsymbol{G}^F)|_\pi = 1$. The result then follows from Theorem 5.6. □

Next we prove Theorem 4.9 for case (i) in Theorem 5.7.

Proposition 5.8. *Let $S = E_6^\epsilon(q)$ with $(3, q) = 1$ and $P \in \mathrm{Syl}_3(S)$. Then $|P'| > 3$. In particular, Theorem 4.9 holds in the case $S = E_6^\epsilon(q)$ and $3 \in \pi$.*

Proof. Let $\boldsymbol{G}$ be a simple algebraic group of simply connected type and $F : \boldsymbol{G} \to \boldsymbol{G}$ a Frobenius map such that $S = \boldsymbol{G}^F/\boldsymbol{Z}(\boldsymbol{G}^F)$. By [Malle and Testerman 2011, Theorem 25.17], we know that every Sylow 3-subgroup of $\boldsymbol{G}^F$ lies in $\boldsymbol{N}_{\boldsymbol{G}^F}(\boldsymbol{T})$ for some maximal F-stable torus $\boldsymbol{T}$ of $\boldsymbol{G}$. Therefore Sylow 3-subgroups of $\boldsymbol{N}_{\boldsymbol{G}^F}(\boldsymbol{T})/\boldsymbol{T}^F = \mathrm{SO}(5, 3)$ (the Weyl group of E_6) are homomorphic images of Sylow 3-subgroups of $S = \boldsymbol{G}^F/\boldsymbol{Z}(\boldsymbol{G}^F)$. Since the size of the derived subgroup of Sylow 3-subgroups of $\mathrm{SO}(5, 3)$ is 9, we deduce that $|P'| > 3$. □

For the rest of this section, we will prove Theorem 4.9 for case (ii) in Theorem 5.7.

Lemma 5.9. *Let p be an odd prime and $S = \mathrm{PSL}^\epsilon(n, q)$. Assume that p divides* $\gcd(n, q-\epsilon)$ *and Sylow p-subgroups of S are abelian. Then $n = p = 3$. Furthermore, $q-\epsilon$ is divisible by* 3 *but not* 9.

Proof. It is argued in Lemma 2.8 of [Koshitani and Sakurai 2021] that if Sylow p-subgroups of S are abelian and $p \geq 5$ then p cannot divide $|\boldsymbol{Z}(\mathrm{SL}^\epsilon(n, q))|$. Therefore our hypotheses imply that $p = 3$.

We first prove that $n = 3$. The condition $p = 3$ divides $\gcd(n, q-\epsilon)$, implies that $n \geq 3$. Assume by contradiction that $n > 3$. Let w be the (unique) element of order 3 of $\mathbb{F}_{q^2}^\times$, and consider the element $g := \mathrm{diag}(I_{n-2}, w, w^{-1})$. We have

$$\boldsymbol{C}_{\mathrm{GL}^\epsilon(n,q)}(g) = \mathrm{GL}^\epsilon(n-2, q) \times \mathrm{GL}^\epsilon(1, q)^2,$$

and so

$$|\mathrm{GL}^\epsilon(n, q) : \boldsymbol{C}_{\mathrm{GL}^\epsilon(n,q)}(g)| = q^{2n-1}\frac{(q^n - \epsilon^n)(q^{n-1} - \epsilon^{n-1})}{(q-\epsilon)^2}.$$

Since 3 divides $\gcd(n, q-\epsilon)$, we have that 3 must divide $|\mathrm{GL}^\epsilon(n, q) : \boldsymbol{C}_{\mathrm{GL}^\epsilon(n,q)}(g)|$. In fact, we also have 3 divides $|\mathrm{SL}^\epsilon(n, q) : \boldsymbol{C}_{\mathrm{SL}^\epsilon(n,q)}(g)|$. On the other hand, as 1 is the only eigenvalue of g with multiplicity larger than 1 (recall that $n > 3$), it is easy to see that $\boldsymbol{C}_{\mathrm{SL}^\epsilon(n,q)}(g)$ is the full preimage of $\boldsymbol{C}_{\mathrm{PSL}^\epsilon(n,q)}(\bar{g})$ under the natural projection from SL^ϵ to PSL^ϵ, where $\bar{g}$ is the image of g in $\mathrm{PSL}^\epsilon(n, q)$. In particular, $|\mathrm{SL}^\epsilon(n, q) : \boldsymbol{C}_{\mathrm{SL}^\epsilon(n,q)}(g)| = |\mathrm{PSL}^\epsilon(n, q) : \boldsymbol{C}_{\mathrm{PSL}^\epsilon(n,q)}(\bar{g})|$, and hence 3 divides $|\mathrm{PSL}^\epsilon(n, q) : \boldsymbol{C}_{\mathrm{PSL}^\epsilon(n,q)}(\bar{g})|$, implying that Sylow 3-subgroups of $S = \mathrm{PSL}^\epsilon(n, q)$ are not abelian. We have shown that $n = 3$.

Finally, assume that 9 divides $q-\epsilon$. Let λ be the element of order 9 in $\mathbb{F}_{q^2}^\times$ and consider $h := \mathrm{diag}(\lambda, \lambda^3, \lambda^5) \in \mathrm{SL}^\epsilon(3, q)$, also of order 9. We then have $\boldsymbol{C}_{\mathrm{GL}^\epsilon(3,q)}(h) = \mathrm{GL}^\epsilon(1, q)^3$, so that $|\boldsymbol{C}_{\mathrm{SL}^\epsilon(3,q)}(h)| = (q-\epsilon)^2$. Moreover, as $\{\lambda, \lambda^3, \lambda^5\} = \{a\lambda, a\lambda^3, a\lambda^5\}$ if and only if $a = 1$, $\boldsymbol{C}_{\mathrm{SL}^\epsilon(3,q)}(h)$ is the full preimage of $\boldsymbol{C}_{\mathrm{PSL}^\epsilon(3,q)}(\bar{h})$. We deduce that $|\boldsymbol{C}_{\mathrm{PSL}^\epsilon(3,q)}(\bar{h})| = (q-\epsilon)^2/3$. This is smaller than the 3-part of $|\mathrm{PSL}^\epsilon(3, q)|$, and thus Sylow 3-subgroups of $\mathrm{PSL}^\epsilon(3, q)$ are not abelian, violating the hypothesis. So 9 cannot divide $q-\epsilon$, as stated. □

Theorem 5.10. *Let p be an odd prime, $n \geq 4$, and $(n, p) \neq (6, 3)$. Let $G := \mathrm{SL}^\epsilon(n, q)$ defined in characteristic not equal to p, $S := G/\boldsymbol{Z}(G) = \mathrm{PSL}^\epsilon(n, q)$, and $P \in \mathrm{Syl}_p(S)$. Suppose that $P/\boldsymbol{Z}(P)$ is abelian. Then p does not divide $|\boldsymbol{Z}(G)|$.*

Proof. Assume by contradiction that $p \mid |\boldsymbol{Z}(G)| = \gcd(n, q-\epsilon)$. Lemma 5.9 already shows that P is nonabelian, but we need to work harder to achieve that $P/\boldsymbol{Z}(P)$ is nonabelian. Let $\lambda \in \mathbb{F}_{q^2}^\times$ be an element of order p and consider the p-element

$$x := \mathrm{diag}(\lambda, \lambda^{-1}, I_{n-2}) \in G.$$

Let $V = \mathbb{F}_q^n$, respectively $\mathbb{F}_{q^2}^n$, denote the natural G-module for $\epsilon = +$, respectively $\epsilon = -$. Fix a basis $B = \{v_1, v_2, \ldots, v_n\}$ of V, and consider the permutation y on B defined by

$$y := \{v_1 \mapsto v_2, v_2 \mapsto v_3, \ldots, v_{p-1} \mapsto v_p, v_p \mapsto v_1, v_i \mapsto v_i \text{ for } p < i \le n\},$$

which is well-defined as $p \le n$. Note that, as $p > 2$, we have $y \in G$ and $\mathrm{ord}(y) = p$. Direct calculation shows that

$$[x, y] = \mathrm{diag}(\lambda^{-1}, \lambda^2, \lambda^{-1}, I_{n-3}) =: s.$$

Suppose that the p-part of $q - \epsilon$ is p^a and let C be the (unique) cyclic subgroup of order p^a of $\mathbb{F}_{q^2}^\times$. As y permutes the diagonal matrices in G with diagonal entries in C, one can form the corresponding semidirect product that is then a p-group. It follows that x and y both belong to a Sylow p-subgroup, say $\widehat{P}$, of G. We deduce that $s = [x, y] \in \widehat{P}'$, which implies that $s\boldsymbol{Z}(G) \in P'$, where $P \in \mathrm{Syl}_p(S)$ is the image of $\widehat{P}$ under the natural projection $\mathrm{SL}^\epsilon \to \mathrm{PSL}^\epsilon$.

We will show that $s\boldsymbol{Z}(G)$ does not belong to $\boldsymbol{Z}(P)$, which is enough to conclude that $P/\boldsymbol{Z}(P)$ is not abelian.

Let $\widetilde{G} := \mathrm{GL}^\epsilon(n, q)$. We have

$$\boldsymbol{C}_{\widetilde{G}}(s) = \begin{cases} \mathrm{GL}^\epsilon(3, q) \times \mathrm{GL}^\epsilon(n-3, q) & \text{if } p = 3, \\ \mathrm{GL}^\epsilon(1, q) \times \mathrm{GL}^\epsilon(2, q) \times \mathrm{GL}^\epsilon(n-3, q) & \text{if } p > 3. \end{cases}$$

It is easy to see that $|S : \boldsymbol{C}_S(s\boldsymbol{Z}(G))| = |G : \boldsymbol{C}_G(s)| = |\widetilde{G} : \boldsymbol{C}_{\widetilde{G}}(s)|$. Hence,

$$|S : \boldsymbol{C}_S(s\boldsymbol{Z}(G))| = \begin{cases} \frac{|\mathrm{GL}^\epsilon(n,q)|}{|\mathrm{GL}^\epsilon(3,q)||\mathrm{GL}^\epsilon(n-3,q)|} & \text{if } p = 3, \\ \frac{|\mathrm{GL}^\epsilon(n,q)|}{|\mathrm{GL}^\epsilon(1,q)||\mathrm{GL}^\epsilon(2,q)||\mathrm{GL}^\epsilon(n-3,q)|} & \text{if } p > 3. \end{cases}$$

It follows that, if ℓ is the defining characteristic of S, then

$$|S : \boldsymbol{C}_S(s\boldsymbol{Z}(G))|_{\ell'} = \begin{cases} \frac{(q^n-\epsilon^n)(q^{n-1}-\epsilon^{n-1})(q^{n-2}-\epsilon^{n-2})}{(q-\epsilon)(q^2-1)(q^3-\epsilon^3)} & \text{if } p = 3, \\ \frac{(q^n-\epsilon^n)(q^{n-1}-\epsilon^{n-1})(q^{n-2}-\epsilon^{n-2})}{(q-\epsilon)^2(q^2-1)} & \text{if } p > 3. \end{cases}$$

Using the condition $p \mid \gcd(n, q - \epsilon)$ and the assumption $(n, p) \ne (6, 3)$, we see that this is divisible by p. It follows that $s\boldsymbol{Z}(G)$ does not belong to $\boldsymbol{Z}(P)$, and this finishes the proof. □

Lemma 5.11. *Let* $S = \mathrm{PSL}^\epsilon(n, q)$ *with* $n \ge 4$. *If* 3 *divides* $q - \epsilon$, *then* $d_3(S) \le \frac{1}{3}$. *In particular, if* 3 *divides* $q - \epsilon$ *and* $3 \in \pi$, *then* $d_\pi(S) \le \frac{1}{3}$.

Proof. Assume, to the contrary, that $d_3(S) > \frac{1}{3}$. Then $d_3(P) > \frac{1}{3}$, and thus $|P'| \le 3$ by Theorem 2.4. The proof of Theorem 5.10 shows that P' contains two elements $s\boldsymbol{Z}(G)$ and $t\boldsymbol{Z}(G)$, where $s = \mathrm{diag}(\lambda^{-1}, \lambda^2, \lambda^{-1}, I_{n-3})$ and $t = \mathrm{diag}(1, \lambda^{-1}, \lambda^2, \lambda^{-1}, I_{n-4})$. Obviously these elements generate a group of order greater than 3, a contradiction. □

Lemma 5.12. *Let $S = \mathrm{PSL}^{\epsilon}(3, q)$ and π a set of odd primes with $3 \in \pi$. Then* $\mathrm{d}_{\pi}(S) \leq \frac{1}{3}$.

Proof. If 3 does not divide $q - \epsilon$, then the result follows by Theorem 5.6. We therefore assume that 3 divides $q - \epsilon$. In particular, 3 divides $q^2 + \epsilon q + 1$. Denote $t := (q - \epsilon)_3/3$. We have

$$|S|_3 = \frac{((q-\epsilon)^2(q+\epsilon)(q^2+\epsilon q+1))_3}{3} \geq (q-\epsilon)_3^2 = 9t^2.$$

On the other hand, counting the number of conjugacy classes of 3-elements in $\mathrm{PSL}^{\epsilon}(3, q)$ (see for example [Simpson and Frame 1973]) we have $k_3(S) = (t^2 + t + 2)/2 \leq 2t^2$. Therefore,

$$\mathrm{d}_{\pi}(S) \leq \mathrm{d}_3(S) = \frac{k_3(S)}{|S|_3} \leq \frac{2t^2}{9t^2} < \frac{1}{3},$$

as wanted. □

Proposition 5.13. *Theorem 4.9 holds for $S = \mathrm{PSL}^{\epsilon}(n, q)$ with $n \geq 3$ and $\pi = \{p, s\}$ with $p < s$ be odd primes such that q is not divisible by neither p nor s.*

Proof. The result follows by Theorem 5.6 in the case $\gcd(n, q-\epsilon)_{\pi} = 1$. So assume that $\gcd(n, q-\epsilon)_{\pi} > 1$, so that there exists $r \in \pi$ such that r divides $\gcd(n, q-\epsilon)$. The case $n = 3$ is then done by Lemma 5.12. So we assume furthermore that $n \geq 4$.

Let $R \in \mathrm{Syl}_r(S)$. We have that $R/\mathbf{Z}(R)$ is abelian by hypothesis. This and the condition r divides $\gcd(n, q-\epsilon)$ contradict Theorem 5.10 if $r \geq 5$. The remaining case $r = 3$ is handled by Lemma 5.11. □

We have completed the proof of Theorem 4.9, by combining Theorems 5.2, 5.3, 5.4, 5.7 and Propositions 5.8 and 5.13.

As mentioned before, Theorem 1.1 follows from Theorems 4.9 and 4.10 together with Theorem 3.4.

6. Examples and further discussion

In this section, we present examples showing that the converses of both statements of Theorem 1.1 are false and the bounds are generically sharp.

Consider the converse of the first part of Theorem 1.1. Assume first that $2 \in \pi$ and $3 \notin \pi$. If G is the direct product of Σ_4 and an abelian group, then $\mathrm{d}_{\pi}(G) = \frac{1}{6}$. Now, let π have size at least 2 and $p > 2$. Let P be a finite p-group with $|P'| = p$. Let C be the cyclic group which is the direct product of the groups C_q where q runs over all primes in π except for p. Let T be the elementary abelian 2-group of rank $|\pi|-1$. Let $G = P \times (C:T)$ where $C:T = \prod_{p\neq q\in\pi}(C_q : C_2)$. In this case

$$\mathrm{d}_\pi(G) \le \left(\frac{p^2+p-1}{p^3}\right)\left(\prod_{p\neq q\in\pi}\frac{q+1}{2q}\right) \le \left(\frac{p^2+p-1}{p^3}\right)\cdot\left(\frac{p+1}{2p}\right)^{|\pi|-1}$$
$$\le \left(\frac{p^2+p-1}{p^3}\right)\cdot\left(\frac{p+1}{2p}\right).$$

Since $p \ge 3$, this is less than $\frac{5}{6p}$, so the converse of the first statement is false.

Consider the converse of the second statement. Assume first that $2 \in \pi$ and $3 \notin \pi$. If G is the direct product of A_4 and an abelian group, then $\mathrm{d}_\pi(G) = \frac{1}{6}$. Now, let $p \neq 2$ and let $|\pi| \ge 3$. Let $C = \prod_{q\in\pi} C_q$. Let $T = C_{p-1} \times (C_2)^{|\pi|-1}$ and set $G = C:T$. Then

$$\mathrm{d}_\pi(G) = \frac{2}{p}\cdot\prod_{p\neq q\in\pi}\frac{q+1}{2q}.$$

Since $|\pi| \ge 3$, $q \ge p+2$ and all primes q in π are odd, we get

$$\mathrm{d}_\pi(G) \le \left(\frac{2}{p}\right)\cdot\left(\frac{(p+2)+1}{2(p+2)}\right)\cdot\left(\frac{(p+4)+1}{2(p+4)}\right) \le \frac{24}{35p}.$$

Thus the converse of the second statement of Theorem 1.1 is also false.

The inequality $\mathrm{d}_\pi(G) > (p^2+p-1)/p^3$ in the second statement of Theorem 1.1 is sharp for every set of primes π. Take G to be the direct product of a finite nonabelian p-group P such that $P/\mathbf{Z}(P)$ is isomorphic to $C_p \times C_p$ with an abelian group. In this case $\mathrm{d}_\pi(G) = (p^2+p-1)/p^3$ and G does not contain an abelian Hall π-subgroup.

Let us consider now the inequality $\mathrm{d}_\pi(G) > 1/p$ of the first part. This condition is best possible when $p = 2$ and $3 \in \pi$, for if G is a direct product of Σ_3 and an abelian group, then $\mathrm{d}_\pi(G) = \frac{1}{2}$ and G does not contain a nilpotent Hall π-subgroup. However the bound is certainly not best possible when p is odd. In fact, following our proofs closely, it can be seen that in such case, the group G still possesses a nilpotent Hall π-subgroup even when $d_\pi(G) = 1/p$.

Now let p be odd. We will show that in certain cases the inequality $\mathrm{d}_\pi(G) > 1/2p$ does not imply that G has a nilpotent Hall π-subgroup. To see this let $\pi = \{p, q\}$ where $q = 2p+1$; that is, p is a Sophie Germain prime. Let G be the direct product of $C_q : C_p$ and an abelian group. Elementary character theory gives $k_\pi(C_q : C_p) = p + (q-1)/p$. Thus

$$\mathrm{d}_\pi(G) = \mathrm{d}_\pi(C_q : C_p) = \frac{1}{2p+1}\left(1+\frac{2}{p}\right),$$

which is strictly larger than $1/2p$.

The last example naturally raises the following question: for π a set of *odd* primes, what is the exact (lower) bound for $d_\pi(G)$ to ensure the existence of a nilpotent Hall π-subgroup in G? This seems nontrivial to us at the time of this writing.

Let G be a finite group and let p be the smallest prime dividing $|G|$. If $n(p)$ denotes the smallest prime larger than p and

$$\Pr(G) > \frac{n(p) + p^2 - 1}{p^2 n(p)} =: f(p),$$

then $|G'| \le p$ and thus G is nilpotent by Theorem 2.6 and Lemma 2.3. Note that $f(p) \le 1/p$ and equality occurs if and only if $p = 2$.

Now let π be a set of primes and p be the smallest member in π. It is perhaps true that if $d_\pi(G) > f(p)$ then G possesses a nilpotent Hall π-subgroup, but this would require significant more effort, especially on the part of simple groups of Lie type in characteristic not belong to π. We have decided to work with the bound $1/p$ instead in order to make our arguments flowing smoothly. We certainly do not claim that $f(p)$ is the (conjectural) best possible bound for $d_\pi(G)$ to ensure the existence of a nilpotent Hall π-subgroup in G, and thus the question we just raised above remains open.

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Received July 18, 2022.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.205

THE CLASSIFICATION OF NONDEGENERATE UNICONNECTED CYCLE SETS

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Dedicated to B. V. M.

It is known that the set-theoretic solutions to the Yang–Baxter equation studied by Etingof et al. (1999) are equivalent to a class of sets with a binary operation, called nondegenerate cycle sets. There is a covering theory for cycle sets which associates a universal covering to any indecomposable cycle set. The cycle sets arising as universal covers are said to be uniconnected. In this paper, the category of nondegenerate uniconnected cycle sets is determined, and it is proved that up to isomorphism, a nondegenerate uniconnected cycle set is given by a brace A with a transitive cycle base (an adjoint orbit which generates the additive group of A). The theorem is applied to braces with cyclic additive or adjoint group, where a more explicit classification is obtained.

Introduction

Set-theoretic solutions to the Yang–Baxter equation [2; 30] are self-maps $S : X \times X \to X \times X$ which satisfy the equation

$$(S \times 1_X)(1_X \times S)(S \times 1_X) = (1_X \times S)(S \times 1_X)(1_X \times S)$$

in $X \times X \times X$. A solution $S(x, y) = ({}^x y, x^y)$ is said to be *nondegenerate* if the maps $y \mapsto {}^x y$ and $y \mapsto y^x$ are bijective for all $x \in X$. Suggested by Drinfeld [10], set-theoretic solutions were found on the symplectic leaves of a Poisson Lie group [29] and in connection with semigroups of I-type [14; 28]. A systematic study of nondegenerate *involutive* ($S^2 = 1_{X\times X}$) solutions was first given by Etingof, Schedler, and Soloviev [11]. By [18, Propositions 1 and 2], nondegenerate involutive solutions on X are equivalent to *nondegenerate cycle sets* $(X; \cdot)$, that is, sets with a binary operation such that the maps $\sigma(x) : X \to X$ with $\sigma(x)(y) := x \cdot y$ and $x \mapsto x \cdot x$ are bijective, and the equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$

MSC2020: 05E18, 16T25, 81R50.
Keywords: braces, coverings, cycle sets, Yang–Baxter equation.

holds in X. The correspondence is given as follows. For a nondegenerate involutive solution S on X, the inverse maps $y \mapsto x \cdot y$ of $y \mapsto y^x$ make X into a nondegenerate cycle set, and every nondegenerate cycle set X gives rise to a nondegenerate involutive solution $S(x, y) = (x^y \cdot y, x^y)$. If X is finite, the bijectivity of $x \mapsto x \cdot x$ is redundant.

For any cycle set X, the $\sigma(x)$ generate a permutation group $G(X) = (G(X); \circ)$ on X. If X is nondegenerate, $G(X)$ admits a unique cycle set structure such that

$$\sigma : X \to G(X)$$

is a morphism of cycle sets. In other words, a nondegenerate involutive solution S on X lifts to a solution on $G(X)$. The cycle set structure on $G(X)$ gives rise to a left action of $G(X)$ on the underlying set: $(a \circ b) \cdot c = a \cdot (b \cdot c)$, and the operation

$$a + b := (a \cdot b) \circ a$$

is commutative. Such a structure $(G; \circ, +, \cdot)$ is called a *brace* [19]. For any brace A, the underlying cycle set is nondegenerate, and $(A; +)$ is an abelian group. The group $A^\circ := (A; \circ)$ is called the *adjoint group* of A. Its action on A makes A into a left A°-module. Every abelian group A can be regarded as a *trivial brace* with $a \circ b = a + b$ and $a \cdot b = b$ for all $a, b \in A$. For a trivial brace A, we write $A^\times$ for the set of its generators as a group.

While cycle sets give rise to solutions to the Yang–Baxter equation, braces form an efficient tool for the study of cycle sets. Besides this, braces arise in affine geometry [3; 4; 5], the theory of solvable groups [6; 7; 23], Hopf–Galois structures [1; 8; 13; 15], and other topics [21].

For a nondegenerate cycle set X, let $A(X)$ denote the associated brace with adjoint group $G(X)$. Any surjective morphism $f : X \twoheadrightarrow Y$ of nondegenerate cycle sets lifts along the morphism $\sigma : X \to A(X)$ to a brace morphism $A(f) : A(X) \twoheadrightarrow A(Y)$. If X is *indecomposable* [12], that is, $G(X)$ acts transitively on X, then Y is indecomposable. If, in addition, $A(f)$ is invertible, the morphism f is said to be a *covering* [24]. A strategy to classify indecomposable cycle sets was outlined in the latter reference: any indecomposable cycle set X admits a *universal covering* $\widetilde{p} : \widetilde{X} \twoheadrightarrow X$ so that no noninvertible covering of $\widetilde{X}$ is possible. The cycle set $\widetilde{X}$ is indecomposable in a strong sense: the permutation group $G(\widetilde{X})$ acts freely and transitively on $\widetilde{X}$. If $\widetilde{p} : \widetilde{X} \twoheadrightarrow X$ is invertible, X is said to be *uniconnected*. The descent from $\widetilde{X}$ to a cycle set X is given by the *fundamental group* $\pi_1(X)$ of X and is described in [24, Theorem 3.3 and Theorem 4.3].

So the complete classification of indecomposable cycle sets hinges decisively on the determination of the uniconnected cycle sets. By the choice of a base point, the free transitive action of the permutation group allows to identify a uniconnected cycle set X with $G(X)$. If X is nondegenerate, the cycle set structure of X can be determined explicitly in terms of the brace $A(X)$. For cycle sets X with cyclic permutation group $G(X)$, this has been applied in [26, Theorem 1], but ignoring the dependence of a base point, it was falsely assumed that the isomorphism class of X is determined by the brace $A(X)$. The fact that the base point matters was observed recently by Jedlička et al. [17], who give a classification of uniconnected cycle sets with a finite cyclic permutation group. Note that these cycle sets are nondegenerate [18].

In this paper, we determine the category of all nondegenerate uniconnected cycle sets (Theorem 2). Its objects are braces A with a distinguished element $e \in A$ such that the adjoint orbit X of e generates the additive group of A. Such a subcycle set X is said to be a *transitive cycle base* [24] of A. Morphisms can be viewed as affine extensions of brace morphisms with a translational part in the adjoint group. As a corollary, it follows that a complete set of invariants of a nondegenerate uniconnected cycle set X consists in the brace $A(X)$ together with the transitive cycle base $\sigma(X)$. Conversely, every brace A with a transitive cycle base corresponds to a unique nondegenerate uniconnected cycle set X, up to isomorphism. If the permutation group $G(X)$ is cyclic, the result of [17] is obtained in a more conceptional form: the isomorphism classes of nondegenerate uniconnected cycle sets X with $A(X) = A$ correspond bijectively to the set $(A/(\mathrm{Soc}(A) + A^2))^\times$ of generators of the trivial brace $A/(\mathrm{Soc}(A) + A^2)$ (Theorem 17). For finite A, the ideals A^2 and $\mathrm{Soc}(A)$ — the *socle* [19] — are complementary in the sense that $|A| = |A^2| \cdot |\mathrm{Soc}(A)|$.

For general braces A, a certain duality between A^2 and $\mathrm{Soc}(A)$ remains true: A^2 is the smallest ideal I such that A/I is a trivial brace, and $\mathrm{Soc}(A)$ is the largest ideal I such that the adjoint group I° acts trivially on A.

The correspondence between nondegenerate uniconnected cycle sets and braces with a transitive cycle base leads us to the question of which braces A have a nonempty set $\mathscr{T}(A)$ of transitive cycle bases. As a necessary condition, we show that the group A/A^2 is cyclic if $\mathscr{T}(A) \neq \varnothing$ (Proposition 7). For *abelian* braces A (i.e., with an abelian adjoint group A°) with $\mathscr{T}(A) \neq \varnothing$, we prove that $\mathscr{T}(A) = (A/A^2)^\times$ (Proposition 12). Abelian braces A are equivalent to commutative radical rings, with $a \circ b = ab + a + b$. If A is nilpotent, the necessary condition of Proposition 7 for $\mathscr{T}(A) \neq \varnothing$ is sufficient (Corollary 13).

It is well known that a nontrivial brace A with A° cyclic must be finite. As a commutative radical ring, A is a direct product of its primary components A_p. For odd primes p, the brace A_p is *cyclic*, which means that its additive group is cyclic, and A_2 is cyclic unless $(A_2; +)$ is the Klein four-group. Thus, with a trivial exception, braces A with A° cyclic are contained in the class of cyclic braces, which have been classified in [22]. By [20, Section 7], there are six infinite classes of exceptional cyclic braces of order 2^n, and all these braces are determined by their adjoint group. We refine this result by showing that up to brace automorphisms, these braces A admit a unique transitive cycle base, hence a unique nondegenerate uniconnected cycle set associated with A (Theorem 19).

1. Uniconnected cycle sets

Recall [23] that an *affine structure* on a group $(A;\circ)$ is given by a binary operation $(A;\cdot)$ satisfying the equations

$$(a\circ b)\cdot c = a\cdot(b\cdot c), \tag{1}$$

$$(a\cdot b)\circ a = (b\cdot a)\circ b. \tag{2}$$

By [23, Theorem 2.1], a group with an affine structure is equivalent to a *brace* [19], that is,

$$a+b := (a\cdot b)\circ a \tag{3}$$

defines an abelian group structure on A with the same unit element as $(A,\circ)$. Equivalently (see [15]), a brace can be described as an abelian group $(A;+)$ with a second group structure $(A,\circ)$ such that

$$(a+b)\circ c + c = a\circ c + b\circ c \tag{4}$$

holds for $a,b,c\in A$. Equation (4) shows that $a+b=c+d$ implies that $a\circ e+b\circ e = c\circ e+d\circ e$. Thus each $b\in A$ defines an affine map $a\mapsto a\circ b$ on $(A;+)$. The group $A^\circ := (A;\circ)$ is said to be the *adjoint group* of the brace A. For example, any radical ring [16] is a brace with $a\circ b := ab+a+b$, which explains the terminology. Accordingly, we write 0 for the common unit element of $(A;+)$ and A°, and a' for the inverse of a in A°.

The adjoint group acts on $(A;+)$ via $b\mapsto a\cdot b$ which makes A into a left A°-module:

$$a\cdot(b+c) = (a\cdot b)+(a\cdot c). \tag{5}$$

If $a\mapsto a^b$ denotes the inverse of $a\mapsto b\cdot a$, then (3) can be rewritten as

$$a\circ b = a^b + b. \tag{6}$$

The right action $a\mapsto a^b$ makes A into a right A°-module:

$$a^{b\circ c} = (a^b)^c, \quad (a+b)^c = a^c + b^c.$$

Equation (6) shows that $a\mapsto a^b$ is the linear part of the affine map $a\mapsto a\circ b$, with translational part $a\mapsto a+b$. Recall that a set $(X;\cdot)$ with a binary operation is said to be a *cycle set* [18] if the left multiplications $\sigma(x): X\to X$ with $\sigma(x)(y) := x\cdot y$ are bijective and the equation

$$(x\cdot y)\cdot(x\cdot z) = (y\cdot x)\cdot(y\cdot z) \tag{7}$$

holds in X. A cycle set X is said to be *nondegenerate* [18] if the square map $x \mapsto x \cdot x$ is bijective. Every finite cycle set is nondegenerate [18, Theorem 2]. By [18, Proposition 1], nondegenerate cycle sets are equivalent to nondegenerate involutive set-theoretic solutions to the Yang–Baxter equation [11]. By (1) and (3), every brace A is a cycle set with

$$(a + b) \cdot c = (a \cdot b) \cdot (a \cdot c). \tag{8}$$

For $b = -a$, (3) gives $0 = a - a = (a \cdot (-a)) \circ a$. Hence

$$a' = a \cdot (-a) = -(a \cdot a),$$

which shows that every brace is nondegenerate as a cycle set. Equations (6), (5), (3), and (7) show that a brace satisfies $a \cdot (b \circ c) = a \cdot (b^c + c) = (a \cdot b^c) + (a \cdot c) = ((a \cdot c) \cdot (a \cdot b^c)) \circ (a \cdot c) = ((c \cdot a) \cdot (c \cdot b^c)) \circ (a \cdot c)$. Thus

$$a \cdot (b \circ c) = ((c \cdot a) \cdot b) \circ (a \cdot c). \tag{9}$$

A *morphism* of cycle sets X, Y is a map $f : X \to Y$ with $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. A *morphism* of braces is a group homomorphism for the additive and adjoint group:

$$f(a + b) = f(a) + f(b), \quad f(a \circ b) = f(a) \circ f(b).$$

By (3), this implies that f is a morphism of cycle sets. The category of braces will be denoted by **Bra**.

For a cycle set X, the group $G(X)$ generated by all $\sigma(x)$, $x \in X$, is called the *permutation group* of X. So there is a natural map $\sigma : X \to G(X)$, and $G(X)$ acts from the left on X. If this action is transitive, X is said to be *indecomposable*. For a nondegenerate cycle set X, the permutation group $G(X)$ is the adjoint group of a brace $A(X)$ such that

$$\sigma : X \to A(X) \tag{10}$$

is a morphism of cycle sets; see [19, Section 1]. Equations (1) and (9) show that the brace $A(X)$ is uniquely determined by these properties. The image σX is of the map (10) together with the cycle set morphism $X \twoheadrightarrow \sigma X$ is called the *retraction* of X.

By [24, Section 2], every surjective morphism $f : X \twoheadrightarrow Y$ of cycle sets extends to a unique group homomorphism $G(f)$ so that the diagram

$$\begin{CD} X @>{f}>> Y \\ @V{\sigma}VV @VV{\sigma}V \\ G(X) @>{G(f)}>> G(Y) \end{CD}$$

commutes. If X is indecomposable, then $Y = f(X)$ is indecomposable, too. If X is indecomposable and $G(f)$ invertible, f is said to be a *covering* [24]. Then f is equivariant under the action of $G(X)$. By [24, Corollary 3.7], every indecomposable cycle set X has a universal covering $\widetilde{p} : \widetilde{X} \twoheadrightarrow X$ so that every covering of $\widetilde{X}$ is invertible. If $\widetilde{p}$ is invertible, X is said to be *uniconnected*, in analogy to simply connected spaces in topology. By [24, Corollary 3.9], X is uniconnected if and only if $G(X)$ acts freely and transitively on X. With the choice of a base point $e \in X$, the group $G(X)$ can then be identified with X.

Recall that a subset X of a brace A is said to be a *cycle base* [19] if X is invariant under the action of A° and X generates the additive group of A. If A° acts transitively on X, then X is said to be a *transitive cycle base* [24]. By $\mathscr{T}(A)$ we denote the set of transitive cycle bases of A. Equation (6) shows that a cycle base X of A also generates the adjoint group A°.

The following characterization was proved in [27]:

Theorem 1. *Let A be a brace with a transitive cycle base X and $e \in X$. Then*

$$a \odot b := b \circ (e^a)' \tag{11}$$

makes A into a nondegenerate uniconnected cycle set. Every nondegenerate uniconnected cycle set arises in this way.

Remark. Theorem 1 is proved, but not correctly stated, in [27] where the condition $e \in X \in \mathscr{T}(A)$ is replaced by the stronger assumption that e generates A. By [25, Theorem 3], both conditions are equivalent if the multipermutation level of X is finite. In general, this is not true even if X is finite; see [25, Example 2].

Equation (11) shows that $e^{a \odot b} = e^{b \circ (e^a)'} = e^a \cdot e^b$. So the map

$$\exp : (A; \odot) \twoheadrightarrow X \tag{12}$$

with $\exp(a) := e^a$ is a cycle set morphism onto $X = e^A := \{e^a \mid a \in A\}$. For a, b in $(A; \odot)$, we have $\sigma(a) = \sigma(b) \iff e^a = e^b$. Thus, up to isomorphism, (12) is the retraction of $(A; \odot)$. Moreover, Equation (11) shows that A° is isomorphic to the permutation group of $(A; \odot)$. Hence $A \cong A(A; \odot)$.

2. Coaffine brace morphisms

In this section, we determine the category of nondegenerate uniconnected cycle sets by means of Theorem 1. To this end, we need a weak type of brace morphism.

Definition. We define a *coaffine* map $A \to B$ between braces to be a pair (b, f) with a brace morphism $f : A \to B$ and a constant $b \in B$ such that $(b, f)(a) := b \circ f(a)$ for all $a \in A$. We write $\mathrm{Hom}^\sharp(A, B)$ for the set of coaffine maps $f : A \to B$.

The composition of coaffine maps is given by

$$(c, g)(b, f) = (c \circ g(b), gf). \tag{13}$$

It is easily checked that the composition is associative with $(0, 1_A) : A \to A$ as unit morphisms. So **Bra** is a subcategory of the category $\mathbf{Bra}^\sharp$ of braces with coaffine maps as morphisms. A morphism (b, f) in $\mathbf{Bra}^\sharp$ is invertible if and only if f is bijective. Then

$$(b, f)^{-1} = (f^{-1}(b)', f^{-1}).$$

Every coaffine map $(b, f) : A \to B$ has a *translational part* $b = (b, f)(0)$, so that

$$(b, f) = (b, 1_B)(0, f).$$

The translations $(b, 1_B)$ are left translations of the adjoint group B°. Therefore, we speak of "coaffine" rather than "affine" maps. The invertible coaffine maps $A \to A$ form a group $\mathrm{Aut}^\sharp(A)$ with the brace automorphisms as a subgroup $\mathrm{Aut}^\flat(A)$. The translations in $\mathrm{Aut}^\sharp(A)$ form a normal subgroup isomorphic to A°. Since $A^\circ \cap \mathrm{Aut}^\flat(A) = \{(0, 1_A)\}$, we have a semidirect product

$$\mathrm{Aut}^\sharp(A) = A^\circ \rtimes \mathrm{Aut}^\flat(A).$$

Let $\mathbf{Bra}_\sharp$ be the category of braces with morphisms $[b, f] : A \to B$ given by a brace morphism $f : A \to B$ and an element $b \in B$ such that

$$[b, f](a) := b \cdot f(a).$$

Note that in contrast to $\mathbf{Bra}^\sharp$, the morphisms in $\mathbf{Bra}_\sharp$ are additive maps. The composition is given by $[c, g][b, f](a) = c \cdot g(b \cdot f(a)) = c \cdot (g(b) \cdot gf(a))$. By (1), this gives

$$[c, g][b, f] = [c \circ g(b), gf],$$

similarly to (13). We write $\mathrm{Hom}_\sharp(A, B)$ for the morphisms $A \to B$ in $\mathbf{Bra}_\sharp$. So the maps $(b, f) \mapsto [b, f]$ provide surjections $\mathrm{Hom}^\sharp(A, B) \twoheadrightarrow \mathrm{Hom}_\sharp(A, B)$ which are compatible with compositions.

Theorem 1 shows that every nondegenerate uniconnected cycle set can be represented by a brace A together with an element $e \in X \in \mathscr{T}(A)$. In what follows, we write $(A; e)$ for the uniconnected cycle set $(A; \odot)$ of Theorem 1. By (11),

$$e = (0 \odot 0)'.$$

Theorem 2. *Let $(A; e)$ and $(B; u)$ be nondegenerate uniconnected cycle sets. The cycle set morphisms $(A; e) \to (B; u)$ coincide with the coaffine morphisms $(c, f) : A \to B$ with $u = [c, f](e)$.*

Proof. Assume first that $(c, f): A \to B$ is a coaffine morphism with $u = [c, f](e) = c \cdot f(e)$. For $a, b \in A$, we have $(c, f)(a \odot b) = c \circ f(b \circ (e^a)') = c \circ f(b) \circ (f(e)^{f(a)})' = c \circ f(b) \circ \big((c \cdot f(e))^{c \circ f(a)}\big)' = (c, f)(b) \circ (u^{(c,f)(a)})' = (c, f)(a) \odot (c, f)(b)$. Thus (f, c) is a morphism $(A; e) \to (B; u)$.

Conversely, let $g : (A; e) \to (B; u)$ be a cycle set morphism. Then $g(b \circ (e^a)') = g(a \odot b) = g(a) \odot g(b) = g(b) \circ (u^{g(a)})'$ for all $a, b \in A$. Replacing b by $b \circ e^a$ gives $g(b) = g(b \circ e^a) \circ (u^{g(a)})'$. So we have

$$g(b \circ e^a) = g(b) \circ u^{g(a)}, \tag{14}$$

$$g(b \circ (e^a)') = g(b) \circ (u^{g(a)})'. \tag{15}$$

Recursively, we define $a^{\circ n}$ by $a^{\circ 1} := a$ and $a^{\circ(n+1)} = a^{\circ n} \circ a$. By induction, (14)–(15) give

$$g(b \circ (e^a)^{\circ n}) = g(b) \circ (u^{g(a)})^{\circ n}$$

for all $n \in \mathbb{Z}$, and a further induction yields

$$g(b \circ (e^{a_1})^{\circ n_1} \circ \cdots \circ (e^{a_r})^{\circ n_r}) = g(b) \circ (u^{g(a_1)})^{\circ n_1} \circ \cdots \circ (u^{g(a_r)})^{\circ n_r}. \tag{16}$$

Since e^A is a cycle base of A, each element $a \in A$ is of the form

$$a = (e^{a_1})^{\circ n_1} \circ \cdots \circ (e^{a_r})^{\circ n_r}$$

for some $a_1, \dots, a_r \in A$ and $n_1, \dots, n_r \in \mathbb{Z}$. For $b = 0$, (16) turns into $g(a) = g(0) \circ (u^{g(a_1)})^{\circ n_1} \circ \cdots \circ (u^{g(a_r)})^{\circ n_r}$. Hence

$$g(b \circ a) = g(b) \circ g(0)' \circ g(a) \tag{17}$$

holds for all $a, b \in A$. So the map $f : A \to B$ with $f(a) := g(0)' \circ g(a)$ satisfies

$$f(a \circ b) = f(a) \circ f(b).$$

Now we show that f is a brace morphism. For $b = a'$, (17) gives $g(0) = g(a') \circ g(0)' \circ g(a)$. Hence

$$g(a') = g(0) \circ g(a)' \circ g(0). \tag{18}$$

With $b = 0$, (14) gives $g(e^a) = g(0) \circ u^{g(a)}$. Hence

$$f(e^a) = u^{g(a)}. \tag{19}$$

Thus (3), (19), (17), and (18) yield

$$\begin{aligned}f(a+e^b) &= f((a\cdot e^b)\circ a) = f(e^{b\circ a'}\circ a)\\ &= f(e^{b\circ a'})\circ f(a) = u^{g(b\circ a')}\circ f(a) = u^{g(b)\circ g(0)'\circ g(a')}\circ f(a)\\ &= u^{g(b)\circ g(a)'\circ g(0)}\circ f(a) = ((g(0)'\circ g(a))\cdot u^{g(b)})\circ f(a)\\ &= (f(a)\cdot u^{g(b)})\circ f(a) = f(a)+u^{g(b)} = f(a)+f(e^b).\end{aligned}$$

Since e^A is a cycle base of A, we obtain $f(a+b)=f(a)+f(b)$ for all $a,b\in A$, by induction. Hence f is a brace morphism, and $g=(g(0),f)$. For $a=0$, (19) gives $f(e)=u^{g(0)}$. Thus $u=g(0)\cdot f(e)=[g(0),f](e)$. □

Corollary 3. *Let A be a brace with $e,u\in X\in\mathscr{T}(A)$. Then $(A;e)\cong(A;u)$.*

Proof. There is an element $b\in A$ with $b\cdot e=u$. Hence $[b,1_A](e)=u$, and thus $(A;e)\cong(A;u)$. □

Now let **Uni** be the category of braces A with a distinguished transitive cycle base $X\in\mathscr{T}(A)$. We write $(A;X)$ for the objects of **Uni**. Morphisms $(A;X)\to(B;Y)$ are brace morphisms $f:A\to B$ with $f(X)\subset Y$. Recall that a functor F is said to be *conservative* if it reflects isomorphisms: if $F(g)$ is an isomorphism, then g is an isomorphism. We define a *factor category* of a category $\mathscr{C}$ to be a category $\mathscr{D}$ with a full conservative functor $F:\mathscr{C}\to\mathscr{D}$ such that each object of $\mathscr{D}$ is isomorphic to an object $F(C)$. So the existence of such a functor implies that up to isomorphism, $\mathscr{C}$ and $\mathscr{D}$ have the same objects.

Corollary 4. *The object map $(A;e)\mapsto(A;e^A)$ makes* **Uni** *into a factor category of the category of nondegenerate uniconnected cycle sets.*

Proof. Every morphism $(A;e)\to(B;u)$ is given by a coaffine morphism $(c,f):A\to B$ with $u=c\cdot f(e)$. Hence $f:(A;e^A)\to(B;u^B)$ is a morphism in **Uni**. Conversely, let $f:(A;e^A)\to(B;u^B)$ be a morphism in **Uni**. Then $f(e)=c\cdot u$ for some $c\in B$. Hence (c,f) is a morphism $(A;e)\to(B;u)$ which is mapped to $f:(A;e^A)\to(B;u^B)$. If f is invertible, (c,f) is invertible, too. □

This implies two characterizations of nondegenerate uniconnected cycle sets:

Corollary 5. *Let A be a brace. The automorphism group $\mathrm{Aut}^\flat(A)$ acts on the set $\mathscr{T}(A)$ of transitive cycle bases, and there is a bijection between the isomorphism classes of nondegenerate uniconnected cycle sets X with $A(X)\cong A$, and the set $\overline{\mathscr{T}}(A)$ of $\mathrm{Aut}^\flat(A)$-orbits of $\mathscr{T}(A)$.*

Corollary 6. *Two nondegenerate uniconnected cycle sets X and Y are isomorphic if and only if there is a brace isomorphism $f:A(X)\xrightarrow{\sim}A(Y)$ which maps the retraction σX onto σY.*

To analyse $\mathscr{T}(A)$ for a brace A, we have to recall the concept of brace ideal. To stress the analogy to ring theory, consider the operation $ab := a^b - a$ in A, which satisfies

$$a \circ b = ab + a + b$$

for all $a, b \in A$. An additive subgroup I of a brace A is said to be an *ideal* [19] if $ab \in I$ and $ba \in I$ whenever $a \in I$ and $b \in A$. If only $ab \in I$ is required, I is said to be a *right ideal* of A. The residue classes $I + b = \{a + b \mid a \in I\}$ of an ideal form a brace A/I with the induced operations in analogy with ring-theoretic ideals. In particular, $b \mapsto I + b$ is a brace morphism $A \twoheadrightarrow A/I$. For example, the *socle* [19] of any brace A is an ideal

$$\mathrm{Soc}(A) := \{a \in A \mid \forall b \in A : a \cdot b = b\},$$

such that $A \twoheadrightarrow A/\mathrm{Soc}(A)$ is the retraction of A as a cycle set. Furthermore, the finite sums $a_1b_1 + \cdots + a_nb_n$ with $a_i, b_i \in A$ form an ideal A^2 of A. The brace A/A^2 is *trivial* in the sense that all products ab are zero, or equivalently, $a \circ b = a + b$ for all $a, b \in A/A^2$. For an abelian group A and its corresponding trivial brace, we write $A^\times$ for the set of its generators. Thus $A^\times$ is empty if the group $(A; +) = A^\circ$ is not cyclic.

Proposition 7. *Let A be a brace with a transitive cycle base. Then A/A^2 is a cyclic group.*

Proof. Two elements x, y of a transitive cycle base satisfy $x = y^a$ for some $a \in A$. Hence $x - y = y^a - y = ya \in A^2$. Thus $A \twoheadrightarrow A/A^2$ maps a transitive cycle base X to a single element $g \in A/A^2$. Since X generates the additive group of A, the element g generates A/A^2. □

Proposition 8. *Let $f : A \twoheadrightarrow B$ be a surjective morphism of braces. Any transitive cycle base X of A is mapped to a transitive cycle base $f(X) \in \mathscr{T}(B)$.*

Proof. Since X generates the additive group of A, the image $f(X)$ generates $(B; +)$. For any pair $x, y \in X$, there is an element $a \in A$ with $y = a \cdot x$. Hence $f(a) \cdot f(x) = f(a \cdot x) = f(y)$, which shows that $f(X) \in \mathscr{T}(B)$. □

3. Abelian braces

Recall that a brace A is said to be *abelian* [20] if its adjoint group A° is commutative. Such braces are radical rings, so that no ambiguity with respect to the powers A^n is possible; see [19, Section 3]. For a finite brace A, the additive group is the direct sum of its primary components $A_p := \{a \in A \mid \exists n \in \mathbb{N} : p^n a = 0\}$, which are right ideals of A. In what follows, we study the set $\bar{\mathscr{T}}(A)$ of $\mathrm{Aut}^\flat(A)$-orbits of $\mathscr{T}(A)$. By Corollary 5, the elements of $\bar{\mathscr{T}}(A)$ correspond to the isomorphism classes of nondegenerate uniconnected cycle sets X with $A(X) \cong A$.

Proposition 9. *Let A be a finite abelian brace. Then $\mathscr{T}(A) \cong \prod_p \mathscr{T}(A_p)$ and $\bar{\mathscr{T}}(A) \cong \prod_p \overline{\mathscr{T}(A_p)}$.*

Proof. By [20, Proposition 3], the brace A is a product of its primary components A_p. Thus, Proposition 8 implies that the projection X_p of a cycle base $X \in \mathscr{T}(A)$ into A_p is a transitive cycle base, and $X = \prod_p X_p$. Conversely, let (X_p) be a collection of cycle bases $X_p \in \mathscr{T}(A_p)$ for each prime p. Then $X := \prod_p X_p$ is a transitive cycle base of A. Hence $\mathscr{T}(A) \cong \prod_p \mathscr{T}(A_p)$. The second statement follows since $\mathrm{Aut}^\flat(A) \cong \prod_p \mathrm{Aut}^\flat(A_p)$. □

By Proposition 9, the classification of finite uniconnected cycle sets X with $G(X)$ abelian reduces to the case that $G(X)$ is a finite p-group. Since any transitive action of an abelian group on a set is free, we have the following:

Proposition 10. *Every indecomposable cycle set with an abelian permutation group is uniconnected.*

The following example shows that uniconnected cycle sets with an abelian permutation group need not be nondegenerate.

Example 11. Let $\mathscr{C}(X)$ be the ring of continuous real functions on a nonempty topological space X. With respect to the partial order $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in X$, the additive group of $\mathscr{C}(X)$ is an *abelian ℓ-group* [9], that is, an abelian group with a lattice structure satisfying $(f \vee g) + h = (f + h) \vee (g + h)$. With

$$f \cdot g := g - (f \vee 0),$$

$\mathscr{C}(X)$ satisfies

$$\begin{aligned}(f \cdot g) \cdot (f \cdot h) &= (g - (f \vee 0)) \cdot (h - (f \vee 0)) \\ &= h - (f \vee 0) - ((g - (f \vee 0)) \vee 0) \\ &= h - \big(((g - (f \vee 0)) \vee 0) + (f \vee 0)\big) \\ &= h - (g \vee f \vee 0).\end{aligned}$$

Thus, by symmetry, $\mathscr{C}(X)$ is a cycle set. Since every continuous function is of the form $f - g$ with $f, g \geqslant 0$, the permutation group of $\mathscr{C}(X)$ is the additive group of $\mathscr{C}(X)$. Its action on $\mathscr{C}(X)$ is transitive. Hence $\mathscr{C}(X)$ is uniconnected. However, $f \cdot f = f - (f \vee 0) \leqslant 0$ shows that $\mathscr{C}(X)$ is degenerate.

Proposition 12. *Let A be an abelian brace with a transitive cycle base X and $e \in X$. Then $A = \mathbb{Z}e + eA$ and $\mathscr{T}(A) = (A/A^2)^\times$. If $e^n \in A^{n+1}$, then $e^n = 0$.*

Proof. By [20, Proposition 3], A is a commutative radical ring. Since $X \in \mathscr{T}(A)$, we have $X = e^A = e + eA$. Hence $A = \mathbb{Z}e + eA$, which yields $A^2 = (\mathbb{Z}e + eA)A = eA$. Thus $\mathscr{T}(A) = (A/A^2)^\times$. Now assume that $e^n \in A^{n+1}$. By induction, we have $A^{n+1} = e^n A$. Indeed, $A^2 = eA$, and if $A^{n+1} = e^n A$ holds for some $n \geqslant 1$, then

$A^{n+2} = (e^n A)A = e^n(eA) = e^{n+1}A$. Hence $e^n \in e^n A \subset A^{n+2} = e^{n+1}A$, and thus $e^n = e^{n+1}a$ for some $a \in A$. So we obtain $e^n = e^n ea = e^{n+2}a^2 = \cdots = e^{2n}a^n$. Hence $i := e^n a^n$ is idempotent. Thus $i^{-i} = i(-i) + i = -i + i = 0$, which yields $i = 0$. Therefore, we get $e^n = e^n i = 0$. □

Corollary 13. *A nilpotent abelian brace admits a transitive cycle base if and only if A/A^2 is cyclic.*

Proof. By Proposition 7, $\mathscr{T} \neq \varnothing$ implies that A/A^2 is cyclic. Conversely, let A/A^2 be cyclic. Choose $e \in A$ such that $e + A^2$ generates A/A^2. Then $A = \mathbb{Z}e + A^2$. Hence $A^2 = eA + A^3$. Assume that $A^n \subset eA + A^{n+1}$ holds for some $n \geqslant 2$. Then $A^{n+1} \subset eA^2 + A^{n+2} \subset eA + A^{n+2}$. By induction, this yields $A^2 = eA$. Thus $A = \mathbb{Z}e + eA$, which shows that $e + eA = \{e^a \mid a \in A\}$ is a transitive cycle base. □

Corollary 14. *Let A be an abelian brace with $\mathscr{T}(A) \neq \varnothing$. Then there is an element $e \in A$ such that for all $n \in \mathbb{N}$, we have $A^{n+1} = e^n A$ and*

$$A = \mathbb{Z}e + \mathbb{Z}e^2 + \cdots + \mathbb{Z}e^n + e^n A. \tag{20}$$

Proof. By Proposition 12 and its proof, $A = \mathbb{Z}e + eA$ and $A^{n+1} = e^n A$. Assume that (20) holds for some $n \geqslant 1$. Then $A = \mathbb{Z}e + eA = \mathbb{Z}e + \mathbb{Z}e^2 + \cdots + Ze^{n+1} + e^{n+1}A$. By induction, this proves the claim. □

Let A be an abelian brace with $\mathscr{T}(A) \neq \varnothing$. For positive integers n, we define

$$I_n := \{m \in \mathbb{Z} \mid mA^n \subset A^{n+1}\}.$$

This gives an increasing chain of ideals $I_1 \subset I_2 \subset \cdots$ in $\mathbb{Z}$. So there are unique integers $r_n \in \mathbb{N}$ with $I_n = \mathbb{Z}r_n$ and divisibility relations

$$\cdots \mid r_4 \mid r_3 \mid r_2 \mid r_1.$$

Definition. Let A be an abelian brace with $\mathscr{T}(A) \neq \varnothing$. If n is the smallest integer with $r_n = r_{n+1}$, we call $(r_1, \ldots, r_n)$ the *characteristic sequence* of A.

If $e \in X \in \mathscr{T}(A)$, Corollary 14 implies that $A^{n+1} = e^n A$. So the characteristic sequence is given by

$$r_n \mid m \iff me^n \in e^n A.$$

By Proposition 12, $r_n = 1$ implies that $e^n = 0$. Thus A is nilpotent if and only if the last entry of the characteristic sequence is 1.

Let $\widehat{A} := \varprojlim A/A^n$ be the inverse limit of the sequence of radical rings

$$\cdots \twoheadrightarrow A/A^4 \twoheadrightarrow A/A^3 \twoheadrightarrow A/A^2.$$

By Corollary 14, each element $a \in A$ can be developed into a power series $a = m_1 e + m_2 e^2 + m_3 e^3 + \cdots$ with unique coefficients $m_i \in \mathbb{Z}$ satisfying $0 \leqslant m_i < r_i$

for $r_i > 0$. So there is an exact sequence

$$\bigcap_{n=1}^{\infty} A^n \hookrightarrow A \twoheadrightarrow \hat{A}.$$

Example 15. Let p be a prime and $\mathbb{Z}_p$ the ring of p-adic rational numbers. Consider the local ring $\mathbb{Z}_p \oplus \mathbb{Q}$ of pairs $(a, x) \in \mathbb{Z}_p \oplus \mathbb{Q}$ with $(a, x)(b, y) := (ab, ay + bx)$. The Jacobson radical of $\mathbb{Z}_p \oplus \mathbb{Q}$ is $A := p\mathbb{Z}_p \oplus \mathbb{Q}$. Hence $A^n = p^n\mathbb{Z}_p \oplus \mathbb{Q}$ and $\bigcap_{n=1}^{\infty} A^n = \mathbb{Q}$. The cycle bases are $X_r = (rp + p^2\mathbb{Z}_p) \oplus \mathbb{Q}$ with $r \in \{1, \dots, p-1\}$. As there are no nonzero additive maps $\mathbb{Q} \to p\mathbb{Z}_p$, a brace automorphism Φ of $A = p\mathbb{Z}_p \oplus \mathbb{Q}$ is given by a matrix

$$\begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}.$$

Applying Φ to $(p, 0)(0, 1) = (0, p)$, we get $(\alpha(p), \beta(p))(0, \gamma(1)) = (0, \gamma(p))$. So $\alpha(p)\gamma(1) = \gamma(p) = p\gamma(1)$, which gives $\alpha(p) = p$. Since α is a ring automorphism, this implies that $\alpha = 1$. So the A°-orbits of $\mathscr{T}(A)$ are trivial, which shows that $|\bar{\mathscr{T}}(A)| = p - 1$.

Example 16. The Jacobson radical of the power series ring $\mathbb{Z}[\![e]\!]$ is $A := e\mathbb{Z}[\![e]\!]$. Its characteristic sequence is (0). There are two cycle bases $e + eA$ and $-e + eA$, and $e \mapsto -e$ induces a brace automorphism. Thus $|\bar{\mathscr{T}}(A)| = 1$.

4. Cyclic and cocyclic braces

Recall that a brace A is said to be *cyclic* [20] if its additive group is cyclic. If the adjoint group A° is cyclic, A is said to be *cocyclic* [22]. Note that in contrast to cocyclic braces, cyclic braces need not be abelian. In this section, we apply Corollary 5 to cyclic and cocyclic braces.

Theorem 17. *Let A be a cocyclic brace. There is a one-to-one correspondence between the isomorphism classes of nondegenerate uniconnected cycle sets X with $A(X) \cong A$ and the set $(A/(\mathrm{Soc}(A) + A^2))^\times$.*

Proof. By Proposition 12, we have $\mathscr{T}(A) = (A/A^2)^\times$. Since A/A^2 is cyclic, the epimorphism $A/A^2 \twoheadrightarrow A/(A^2 + \mathrm{Soc}(A))$ restricts to a surjection

$$p : (A/A^2)^\times \twoheadrightarrow (A/(\mathrm{Soc}(A) + A^2))^\times.$$

The embedding $\mathrm{Aut}^\flat(A) \hookrightarrow \mathrm{Aut}(A^\circ) = (A^\circ)^\times$ shows that every brace automorphism of A is given by a map $a \mapsto a^{\circ k}$ for some $k \in \mathbb{Z}$. By [22, Proposition 12], $a \mapsto a^{\circ k}$ is a brace automorphism if and only if $a^{\circ(k-1)} \in \mathrm{Soc}(A)$ for all $a \in A$. So we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Aut}^\flat(A) & \hookrightarrow & (A^\circ)^\times & \twoheadrightarrow & (A^\circ/\operatorname{Soc}(A)^\circ)^\times \\ & & \downarrow & & \downarrow \\ & & (A/A^2)^\times & \overset{p}{\twoheadrightarrow} & (A/(\operatorname{Soc}(A)+A^2))^\times \end{array}$$

with an exact first row. Now $\mathrm{Aut}^\flat(A)$ acts on $\mathscr{T}(A) = (A/A^2)^\times$, and the orbit of a cycle base $e + A^2 = A^2 \circ e$ is $A^2 \circ e \circ \operatorname{Soc}(A) = (A^2 \circ e) + \operatorname{Soc}(A) = e + A^2 + \operatorname{Soc}(A)$. So the $\mathrm{Aut}^\flat(A)$-orbits of $\mathscr{T}(A)$ correspond to the elements of $(A/(\operatorname{Soc}(A) + A^2))^\times$. By Corollary 5, this completes the proof. □

Remark. Theorem 17 corrects [26, Theorem 1], and its corollary, where it is falsely assumed that $\mathrm{Aut}^\flat(A)$ acts transitively on $\mathscr{T}(A)$ for a cocyclic brace. The inaccuracy was observed recently by Jedlička et al. [17] who gave a correct classification by using different methods.

Example 18. Up to isomorphism, there is a single infinite cocyclic brace A, and A is trivial; see [22, Section 3]. The brace $A = \langle u \rangle$ has two transitive cycle bases, $\{u\}$ and $\{u^{-1}\}$, and there are two brace automorphisms. So the only uniconnected cycle set X with $A(X) \cong A$ is given by $a \odot b = b \circ u'$, that is, $a \odot u^{\circ n} = u^{\circ(n-1)}$.

We turn our attention to cyclic braces A. It is convenient to identify the additive group of A with $\mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$. Assume first that $n = 0$. Besides the trivial infinite cyclic brace, there is a nonabelian one, given by $k^\ell := k(-1)^\ell$; see [20, proof of Proposition 6]. So there is a single transitive cycle base $\{1, -1\}$, which shows that $|\overline{\mathscr{T}}(A)| = 1$. The corresponding uniconnected cycle set is given by

$$a \odot b = (-1)^a - b.$$

Now let A be finite. For simplicity, we restrict ourselves to the primary case: $|A| = p^n$ for some prime p. (For a classification of all cyclic braces, see [22].) Let d denote the order of the socle $\operatorname{Soc}(A)$. Then A is said to be *exceptional* [20; 22] if A is nontrivial (i.e., $d < p^n$) and either $d = p^{n-1} \neq 1$ or $A/\operatorname{Soc}(A)$ is not cocyclic.

By [20, Theorem 3], a cyclic brace A with $|A| = p^n$ is cocyclic or exceptional. If A is exceptional, then $p = 2$, and the adjoint group A° admits a cyclic subgroup of order 2^{n-1}. Moreover, the isomorphism class of A is uniquely determined by the adjoint group A°. The following result refines this fact by showing that there is a single isomorphism class of uniconnected cycle sets X with $G(X) = A^\circ$.

Theorem 19. *Let A be an exceptional cyclic brace with $|A| = 2^n$. Then $|\overline{\mathscr{T}}(A)| = 1$.*

Proof. By [20, Section 7], there are six classes of exceptional cyclic braces A with $|A| = 2^n$, $n \geqslant 2$, according to their adjoint group; see [20, Proposition 11; 22, Section 6]. Let C_m denote the cyclic group of order m. The first two classes contain abelian braces:

(1a) A° is cyclic with $|A| \geqslant 8$ and $\mathrm{Soc}(A) = 2A$. There are 2^{n-2} cycle bases with two elements each. By [22, Proposition 1], every automorphism of $(A;+)$ is a brace automorphism. Hence $|\bar{\mathscr{T}}(A)| = 1$.

(1b) $A^\circ = \langle -1 \rangle \times \langle 1 \rangle \cong C_2 \times C_{2^{n-1}}$ is abelian, not cyclic. The socle of A is $\{0, 2^{n-1}\}$. By [22, Proposition 1], the cycle set structure of A is given by

$$a^b = 2ab + a.$$

Hence there is a single cycle base $\{1, 3, 5, \ldots\}$. Thus $|\bar{\mathscr{T}}(A)| = 1$.

For the next three cases, the adjoint group A° is one of the following groups:

$$\begin{aligned} D_{2^m} &= \{a, b \mid a^{2^m} = b^2 = 1,\ bab^{-1} = a^{-1}\}, && m \geqslant 2, \\ Q_{2^m} &= \{a, b \mid a^{2^{m+1}} = 1,\ b^2 = a^{2^m},\ bab^{-1} = a^{-1}\}, && m \geqslant 1, \\ SD_{2^m} &= \{a, b \mid a^{2^m} = b^2 = 1,\ bab^{-1} = a^{-1+2^{m-1}}\}, && m \geqslant 3. \end{aligned}$$

These groups are the dihedral group D_{2^m} of order 2^{m+1} (type 2a), the generalized quaternion group Q_{2^m} of order 2^{m+2} (type 2b), and the semidihedral group SD_{2^m} of order 2^{m+1} (type 3a). For the groups D_{2^m} and Q_{2^m}, the brace structure is given by $a = 2$ and $b = 1$, and

$$x \cdot y = y^x = \begin{cases} (-1)^x y & \text{for type (2a)}, \\ (-1 + 2^{m+1})^x y & \text{for type (2b)}. \end{cases}$$

and $\mathrm{Soc}(A) = \langle a \rangle = 2A$. Hence, as in case (1a), $\mathrm{Aut}^\flat(A) = \mathrm{Aut}(A;+)$, which yields $|\bar{\mathscr{T}}(A)| = 1$.

(3a) $A^\circ = SD_{2^m}$. The brace structure is given by

$$x \cdot y = y^x = \begin{cases} y & \text{for } x \equiv 0 \pmod 4, \\ (-1 + 2^m) y & \text{for } x \equiv 1 \pmod 4, \\ (1 + 2^m) y & \text{for } x \equiv 2 \pmod 4, \\ (-1) y & \text{for } x \equiv 3 \pmod 4, \end{cases}$$

with $a = 2$ and $b = -1$, and $\mathrm{Soc}(A) = 4A$. The transitive cycle bases are

$$\{k, -k + 2^m, k + 2^m, -k\}$$

for $k \equiv 1 \pmod 4$, and the brace automorphisms are $x \mapsto xk$ with $k \equiv 1 \pmod 4$. Hence $|\bar{\mathscr{T}}(A)| = 1$.

(3b) Here A° is the group

$$M_{2^m} := \{a, b \mid bab^{-1} = a^{1+2^{m-1}}\}, \quad m \geqslant 3,$$

of order 2^{m+1}. By [20, Proposition 10], the brace structure is given by $a = 1$ and $b = -1$, and

$$c^1 = c(3+2^m), \quad c^{-1} = -c \tag{21}$$

for $c \in A = \mathbb{Z}/2^{m+1}\mathbb{Z}$. We show that

$$2^{m+1} \mid (3+2^m)^k - 1 \iff 2^{m-1} \mid k \tag{22}$$

holds for $m \geqslant 3$ and $k \in \mathbb{N}$. Modulo 8, we have $(3+2^m)^k \equiv 3$ for odd k, and $(3+2^m)^k \equiv 1$ for even k. Thus, to verify (22), we can assume that $k = 2\ell$. Modulo 2^{m+1}, the binomial formula gives $(3+2^m)^k \equiv 3^k \equiv 9^\ell$. By [20, Lemma 4], we have

$$2^{m-2} \mid \tfrac{1}{8}((1+8)^\ell - 1) \iff 2^{m-2} \mid \ell.$$

Hence $2^{m+1} \mid (3+2^m)^k - 1 \iff 2^{m-2} \mid \frac{1}{8}((1+8)^\ell - 1) \iff 2^{m-2} \mid \ell \iff 2^{m-1} \mid k$. This proves (22). Thus $3 + 2^m$ is of order 2^{m-1} in the ring $A = \mathbb{Z}/2^{m+1}\mathbb{Z}$, and $(3+2^m)^k \equiv 1$ or $\equiv 3 \pmod 8$ for all k. By (21), it follows that there is a single cycle base $A \setminus 2A$ of A. Whence $|\bar{\mathscr{T}}(A)| = 1$. □

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Received March 23, 2022. Revised October 8, 2022.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 323 No. 1 March 2023