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BACKSTRÖM ALGEBRAS

YURIY DROZD

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We introduce *Backström pairs* and *Backström rings*, study their derived categories and construct for them a sort of *categorical resolutions*. For the latter we define the global dimension, construct a sort of semiorthogonal decomposition of the derived category and deduce that the derived dimension of a Backström ring is at most 2. Using this semiorthogonal decomposition, we define a description of the derived category as the category of elements of a special bimodule. We also construct a partial tilting for a Backström pair to a ring of triangular matrices and define the global dimension of the latter.

Introduction

Backström orders were introduced in [Ringel and Roggenkamp 1979], where it was shown that their representations are in correspondence with those of quivers or species. A special class of Backström orders are *nodal orders*, which appeared (without this name) in [Drozd 1990] as such pure noetherian algebras that the classification of their finitely generated modules is tame. In [Burban and Drozd 2004] tameness was also proved for the derived categories of nodal orders. Global analogues of nodal algebras, called *nodal curves*, were considered in [Burban and Drozd 2011; Drozd and Voloshyn 2012; Voloshyn and Drozd 2013]. Namely, in [Burban and Drozd 2011] a sort of tilting theory for such curves was developed, which related them to some quasihereditary finite dimensional algebras. In [Drozd and Voloshyn 2012] a criterion was found for a nodal curve to be tame with respect to the classification of vector bundles, and in [Voloshyn and Drozd 2013] it was proved that the same class of curves has tame derived categories. It was clear that the tilting theory of [Burban and Drozd 2011] can be extended to a general situation, namely, to *Backström curves*, i.e., noncommutative curves having Backström orders as their localizations. Nodal orders and related gentle algebras appear in studying mirror symmetry, see for instance, [Lekili and Polishchuk 2018]. Finite dimensional

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analogues of nodal orders, called *nodal algebras*, were introduced in [Drozd and Zembyk 2013; Zembyk 2014]. In the latter paper their structure was completely described. In [Zembyk 2015] it was shown that certain important classes of algebras, such as gentle and skewed-gentle algebras, are nodal. In [Burban and Drozd 2017] a tilting theory was developed for nodal algebras, which was applied to the study of derived categories of gentle and skewed-gentle algebras.

This paper is devoted to a tilting theory for *Backström rings*, which are a straightforward generalization of Backström orders and algebras.

In Section 1, we propose a variant of partial tilting, which generalizes the technique of minors from [Burban et al. 2017].

In Section 2, we introduce *Backström pairs*, which are pairs of semiperfect rings $\mathbf{H} \supseteq \mathbf{A}$ with a common radical; (piecewise) *Backström rings* are likewise introduced as those rings \mathbf{A} that occur in (piecewise) Backström pairs with (piecewise) hereditary \mathbf{H} . We construct the *Auslander envelope* $\tilde{\mathbf{A}}$ of a Backström pair and calculate its global dimension. It turns out that this global dimension only depends on the global dimension of \mathbf{H} . In particular, Auslander envelopes for Backström rings are of global dimension at most 2.

In Section 3, we apply the tilting technique to show that the derived category of the algebra \mathbf{A} is connected by a recollement with the derived category of its Auslander envelope. This implies that the derived dimension of \mathbf{A} in the sense of [Rouquier 2008] is not greater than that of the Auslander envelope.

In Section 4, we consider a recollement between the derived categories of the algebra \mathbf{H} and of the Auslander envelope. It is used to calculate the derived dimension of the Auslander envelope, thus obtaining an upper bound for the derived dimension of the algebra \mathbf{A} . In particular, we prove that the derived dimension of a Backström or piecewise Backström algebra is at most 2. Moreover, if \mathbf{A} is a Backström or piecewise Backström algebra of Dynkin type, then either it is piecewise hereditary of Dynkin type, so $\text{der.dim } \mathbf{A} = 0$, or else $\text{der.dim } \mathbf{A} = 1$.

In Section 5, we establish an equivalence between the category $\mathcal{D}(\tilde{\mathbf{A}})$ and a bimodule category. This gives a useful instrument for calculations in this derived category. (See, for instance, [Bekkert et al. 2003; Bekkert and Merklen 2003; Burban and Drozd 2004; 2006; 2017; Voloshyn and Drozd 2013].)

In Section 6, we consider another partial tilting for the Auslander envelope $\tilde{\mathbf{A}}$ of a Backström pair, relating its derived category by a recollement to the derived category of an algebra \mathbf{B} of triangular matrices which looks simpler than the Auslander algebra. In this case, we calculate explicitly the global dimension of \mathbf{B} and the kernel of the partial tilting functor

$$F : \mathcal{D}(\mathbf{B}) \rightarrow \mathcal{D}(\mathbf{A}).$$

1. Partial tilting

Let \mathcal{T} be a triangulated category, $\mathfrak{X} \subseteq \text{Ob } \mathcal{T}$. We denote by $\text{Tri}(\mathfrak{X})$ the smallest strictly full triangulated subcategory containing \mathfrak{X} that is closed under coproducts (this means that if a coproduct of objects from $\text{Tri}(\mathfrak{X})$ exists in \mathcal{T} , it belongs to $\text{Tri}(\mathfrak{X})$). For a DG-category \mathcal{R} we denote by $\mathcal{D}(\mathcal{R})$ its derived category [Keller 1994]. The following result is a generalization of [Lunts 2010, Proposition 2.6]:

Theorem 1.1. *Let \mathfrak{X} be a subset of the set of compact objects of $\text{Ob } \mathcal{D}(\mathcal{A})$, where \mathcal{A} is a Grothendieck category. We consider the DG-category \mathcal{R} with the set of objects \mathfrak{X} and the sets of morphisms $\mathcal{R}(T, R) = \mathbb{R}\text{Hom}(T, R)$. Define the functor $F : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{R}^{\text{op}})$ by mapping a complex C to the DG-module $FC = \mathbb{R}\text{Hom}_{\mathcal{D}(\mathcal{A})}(-, C)$ restricted onto \mathfrak{X} .*

- (1) *The restriction of F onto $\text{Tri}(\mathfrak{X})$ is an equivalence $\text{Tri}(\mathfrak{X}) \rightarrow \mathcal{D}(\mathcal{R}^{\text{op}})$.*
- (2) *There is a recollement diagram in the sense of [Beilinson et al. 1982, 1.4.3]*

$$(1-1) \quad \text{Ker } F \begin{array}{c} \xleftarrow{l^*} \\ \xrightarrow{l} \\ \xleftarrow{l^!} \end{array} \mathcal{D}(\mathcal{A}) \begin{array}{c} \xleftarrow{F^*} \\ \xrightarrow{F} \\ \xleftarrow{F^!} \end{array} \mathcal{D}(\mathcal{R}^{\text{op}}),$$

where l is the embedding.¹

Recall that this means that the following conditions hold:

- (a) F and l are exact.
- (b) $Fl = 0$.
- (c) F^* and $F^!$ are left and right adjoint functors to F , respectively.
- (d) Both adjunction morphisms $\eta : \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})} \rightarrow FF^*$ and $\zeta : FF^! \rightarrow \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})}$ are isomorphisms.
- (e) The same holds for the triple $(l, l^*, l^!)$.

(Note that Condition 1.4.3.4 from [Beilinson et al. 1982] is a consequence of the other ones; see [Neeman 2001, 9.2].)

If \mathfrak{X} generates $\mathcal{D}(\mathcal{A})$, we obtain an equivalence $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{R}^{\text{op}})$, as in [Lunts 2010]. If \mathfrak{X} consists of one object R , we obtain an equivalence $\text{Tri}(R) \simeq \mathcal{D}(\mathbf{R}^{\text{op}})$, where $\mathbf{R} = \mathbb{R}\text{Hom}(R, R)$.

Proof. (1) We identify $\mathcal{D}(\mathcal{A})$ with the homotopy category $\mathcal{I}(\mathcal{A})$ of K -injective complexes, i.e., complexes I such that $\text{Hom}(C, I)$ is acyclic for every acyclic complex C , and suppose that $\mathfrak{X} \subseteq \mathcal{I}(\mathcal{A})$. Then, $\mathbb{R}\text{Hom}$ coincides with Hom within the category $\mathcal{I}(\mathcal{A})$; so, for $C \in \mathcal{I}(\mathcal{A})$, $FC = \text{Hom}_{\mathcal{I}(\mathcal{A})}(-, C)$ restricted onto \mathfrak{X} . The full subcategory of $\mathcal{I}(\mathcal{A})$ consisting of complexes C such that the natural map $\text{Hom}_{\mathcal{I}(\mathcal{A})}(R, C) \rightarrow \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(FR, FC)$ is bijective for all $R \in \mathfrak{X}$ contains \mathfrak{X} , is strictly full, triangulated and closed under coproducts, since all objects from \mathcal{R} are

¹Note that \mathfrak{X} is not necessarily *recollement-defining* in the sense of [Nicolás and Saorín 2009].

compact. Therefore, it contains $\text{Tri}(\mathfrak{X})$. Quite analogously, the full subcategory of complexes C such that the natural map $\text{Hom}_{\mathcal{J}(\mathcal{A})}(C, C') \rightarrow \text{Hom}_{\mathcal{J}(\mathcal{A})}(FC, FC')$ is bijective for every $C' \in \text{Tri}(\mathfrak{X})$ also contains $\text{Tri}(\mathfrak{X})$. Hence, the restriction of F onto $\text{Tri}(\mathfrak{X})$ is fully faithful. Moreover, as the functors $\text{Hom}_{\mathcal{D}}(-, R)$, where R runs through \mathfrak{X} , generate $\mathcal{D}(\mathcal{R}^{\text{op}})$, the functor F is essentially surjective. Therefore, restricted to $\text{Tri}(\mathfrak{X})$, it gives an equivalence $\text{Tri}(\mathfrak{X}) \rightarrow \mathcal{D}(\mathcal{R})$.

(2) Note that $\mathcal{D}(\mathcal{R}^{\text{op}})$ is cocomplete and compactly generated, hence satisfies the Brown representability theorem [Neeman 2001, Theorem 8.3.3]. Therefore, it is true for $\text{Tri}(\mathfrak{X})$ too. Then, [Neeman 2001, Proposition 9.1.19] implies that a Bousfield localization functor exists for $\text{Tri}(\mathfrak{X}) \subseteq \mathcal{D}(\mathcal{A})$ and [Neeman 2001, Proposition 9.1.18] implies that the embedding $E : \text{Tri}(\mathfrak{X}) \rightarrow \mathcal{D}(\mathcal{A})$ has a right adjoint $\Theta : \mathcal{D}(\mathcal{A}) \rightarrow \text{Tri}(\mathfrak{X})$. Let $F' : \mathcal{D}(\mathcal{R}^{\text{op}}) \rightarrow \text{Tri}(\mathfrak{X})$ be a quasi-inverse to the restriction of F onto $\text{Tri}(\mathfrak{X})$. In particular, F' is a left adjoint to this restriction and the adjunction $FF' \rightarrow \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})}$ is an isomorphism. Then,

$$FC = \text{Hom}_{\mathcal{J}(\mathcal{A})}(-, C)|_{\mathfrak{X}} \simeq \text{Hom}_{\mathcal{J}(\mathcal{A})}(-, \Theta C)|_{\mathfrak{X}} = F\Theta C.$$

Set $F^* = EF'$. Since $F'M \in \text{Tri}(\mathfrak{X})$ for every $M \in \mathcal{D}(\mathcal{R}^{\text{op}})$,

$$\begin{aligned} \text{Hom}_{\mathcal{J}(\mathcal{A})}(F^*M, C) &\simeq \text{Hom}_{\text{Tri}(\mathfrak{X})}(F'M, \Theta C) \\ &\simeq \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(M, F\Theta C) \simeq \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(M, FC), \end{aligned}$$

for any $M \in \mathcal{D}(\mathcal{R}^{\text{op}})$ and $C \in \mathcal{I}(\mathcal{A})$. Hence, F^* is a left adjoint to F . If, moreover, $C \in \text{Tri}(\mathfrak{X})$, we obtain

$$\text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(FF^*M, FC) \simeq \text{Hom}_{\mathcal{J}(\mathcal{A})}(F^*M, C) \simeq \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(M, FC).$$

As F is essentially surjective, this implies that $\eta : FF^* \rightarrow \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})}$ is an isomorphism. As all objects from \mathfrak{X} are compact, F respects coproducts, hence has a right adjoint $F^!$ [Neeman 2001, Theorem 8.4.4]. Now it follows from [Burban et al. 2017, Corollary 2.3] that $\zeta : FF^! \rightarrow \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})}$ is an isomorphism and there is a recollement diagram (1-1). \square

Note that $\text{Im } F^* = \text{Tri}(\mathfrak{X})$ by construction, but usually $\text{Im } F^! \neq \text{Tri}(\mathfrak{X})$, though it is equivalent to $\text{Tri}(\mathfrak{X})$.

Corollary 1.2. *Under the conditions and notations of the preceding theorem, suppose that $\text{Hom}_{\mathcal{D}(\mathcal{A})}(R, T[m]) = 0$ for $R, T \in \mathfrak{X}$ and $m \neq 0$. Then, the functor F induces an equivalence $\text{Tri}(R) \xrightarrow{\sim} \mathcal{D}(\mathcal{R}^{\text{op}})$, where \mathcal{R} is the category with the set of objects \mathfrak{X} and the sets of morphisms $\mathcal{R}(A, B) = \text{Hom}_{\mathcal{D}(\mathcal{A})}(A, B)$.*

In this situation, we call the functor F a partial tilting functor.

2. Backström pairs

Recall from [Bass 1960; Lambek 1976] that a *semiperfect ring* is a ring A such that $A/\text{rad } A$ is a semisimple artinian ring and idempotents can be lifted modulo $\text{rad } A$. Equivalently, as a left (or as a right) A -module, A decomposes into a direct sum of modules with local endomorphism rings.

Definition 2.1. (1) A *Backström pair* is a pair of semiperfect rings $H \supseteq A$ such that $\text{rad } A = \text{rad } H$. We denote by $C(H, A)$ the *conductor* of H in A :

$$C(H, A) = \{\alpha \in A \mid H\alpha \subseteq A\} = \text{ann}(H/A)_A$$

(the right subscript $_A$ means that we consider H/A as a right A -module). Obviously, $C(H, A) \supseteq \text{rad } A$, so both A/C and H/C are semisimple rings.

(2) We call a ring A a (left) *Backström ring* (resp. *piecewise Backström ring*) if there is a Backström pair $H \supseteq A$, where the ring H is left hereditary (resp. *left piecewise hereditary* [Happel 1988], i.e., derived equivalent to a left hereditary ring). If, moreover, both A and H are finite dimensional algebras over a field \mathbb{k} , we call A a *Backström algebra* (resp. *piecewise Backström algebra*).

Remark 2.2. If e is an idempotent in A , then $\text{rad}(eAe) = e(\text{rad } A)e$, hence, if $H \supseteq A$ is a Backström pair, so is $eHe \supseteq eAe$. This implies that if P is a finitely generated projective A -module, $A' = \text{End}_A P$ and $H' = \text{End}_H(H \otimes_A P)$, then $H' \supseteq A'$ is also a Backström pair. Note that if H is left hereditary (or piecewise hereditary), so is H' , hence A' is a Backström ring (piecewise Backström ring) whenever A is. In particular, the notion of Backström (or piecewise Backström) ring is Morita invariant. Note also that if H is left hereditary and noetherian, it is also right hereditary, so A^{op} is also a Backström ring (piecewise Backström ring).

Examples 2.3. (1) An important example of Backström algebras are *nodal algebras* introduced in [Drozd and Zembyk 2013; Zembyk 2014]. By definition, they are finite dimensional algebras such that there is a Backström pair $H \supseteq A$, where H is a hereditary algebra and $\text{length}_A(H \otimes_A U) \leq 2$ for every simple A -module U . Their structure was completely described in [Zembyk 2014].

(2) Recall that a \mathbb{k} -algebra A is called *gentle* [Assem and Skowroński 1987] if $A \simeq \mathbb{k}\Gamma/J$, where Γ is a finite quiver (oriented graph) and J is an ideal in the path algebra $\mathbb{k}\Gamma$ such that $(J_+)^2 \supseteq J \supseteq (J_+)^k$ for some k , where J_+ is the ideal generated by all arrows, and the following conditions hold:

- (a) For every vertex $i \in \text{Ver } \Gamma$ there are at most two arrows starting at i and at most two arrows ending at i .
- (b) If an arrow a starts at i (resp. ends at i) and arrows b_1, b_2 end at i (resp. start at i), then either $ab_1 = 0$ or $ab_2 = 0$ (resp. either $b_1a = 0$ or $b_2a = 0$), but not both.

(c) The ideal J is generated by products of arrows of the sort ab .

It is proved in [Zembyk 2015] that such algebras are nodal, hence Backström algebras. The same is true for skewed-gentle algebras [Geißband de la Peña 1999] obtained from gentle algebras by blowing up some vertices.

(3) *Backström orders* are orders A over a discrete valuation ring such that there is a Backström pair $H \supseteq A$, where H is a hereditary order. They were considered in [Ringel and Roggenkamp 1979].

(4) Let $H = T(n, \mathbb{k})$ be the ring of upper triangular $n \times n$ matrices over a field \mathbb{k} and $A = \text{UT}(n, \mathbb{k})$ be its subring of unitriangular matrices M , i.e., such that all diagonal elements of M are equal. Then, H is hereditary and $\text{rad } H = \text{rad } A$, hence A is a Backström algebra. In this case, $C(H, A) = \text{rad } A$.

(5) $\Lambda_n = \mathbb{k}[x_1, x_2, \dots, x_n]/(x_1, x_2, \dots, x_n)^2$ embeds into $H = \prod_{i=1}^n \mathbb{k}\Gamma_i$, where $\Gamma_i = \cdot \xrightarrow{a_i} \cdot$ (x_i maps to a_i). Obviously, under this embedding $\text{rad } \Lambda_n = \text{rad } H$, so Λ_n is a Backström algebra.

We consider a fixed Backström pair $H \supseteq A$, set $\tau = \text{rad } A = \text{rad } H$ and denote by C the conductor $C(H, A)$. Obviously, C is a two-sided A -ideal and the biggest left H -ideal contained in A . Actually, it even turns out to be a two-sided H -ideal and its definition is left-right symmetric.

Lemma 2.4. *Let $R \subseteq S$ be semisimple rings, $I = \{\alpha \in R \mid S\alpha \subseteq R\}$. Then, I is a two-sided S -ideal.*

Proof. Obviously, I is a left S -ideal and a two-sided R -ideal. As R is semisimple, $I = Re$ for some central idempotent $e \in R$. Then, $Se \subseteq Re$, so $Se = Re = eR$ and $(1 - e)Se = 0$. Hence, $eS(1 - e)$ is a left ideal in S and $(eS(1 - e))^2 = 0$, so $eS(1 - e) = 0$ and $I = Se = eS$ is also a right S -ideal. \square

Proposition 2.5. *C is a two-sided H -ideal. It is the biggest H -ideal contained in A . Therefore, it coincides with the set $\{\alpha \in A \mid \alpha H \subseteq A\}$ or with $\text{ann}_A(H/A)$ considered as a left A -module.*

Proof. It follows from the preceding lemma applied to the rings $A/\text{rad } A$ and $H/\text{rad } H$. \square

In what follows we assume that $A \neq H$, so $C \neq A$. To calculate C , we consider a decomposition $A = \bigoplus_{i=1}^m A_i$, where A_i are indecomposable projective left A -modules. Arrange them so that $HA_i \neq A_i$ for $1 \leq i \leq r$ and $HA_i = A_i$ for $r < i \leq m$, and set $A^0 = \bigoplus_{i=1}^r A_i$, $H^0 = HA^0$ and $A^1 = \bigoplus_{i=r+1}^m A_i = HA^1$. Then, $A = A^0 \oplus A^1$ and $H = H^0 \oplus A^1$ (possibly, $r = m$, so $A^0 = A$ and $H^0 = H$). Let $A^0 = Ae_0$ and $A^1 = Ae_1$, where e_0 and e_1 are orthogonal idempotents and $e_0 + e_1 = 1$. Set $A_b^a = e_b A e_a$ and $H_b^a = e_b H e_a$, where $a, b \in \{0, 1\}$. Note that $A_b^1 = H_b^1$ and $A_1^0 = H_1^0$. As A^0 and A^1 have no isomorphic direct summands,

$A_b^a \subseteq \text{rad } A$ if $a \neq b$. Hence, if we set $\tau^a = \text{rad } A^a$ ($a = 0, 1$) and consider the Pierce decomposition of the ring A

$$A = \begin{pmatrix} A_0^0 & A_0^1 \\ A_1^0 & A_1^1 \end{pmatrix},$$

the Pierce decomposition of the ideal τ becomes

$$\tau = \begin{pmatrix} \tau_0^0 & A_0^1 \\ A_1^0 & \tau_1^1 \end{pmatrix},$$

where $\tau_a^a = \text{rad } A_a^a$, $a = 0, 1$. This implies that H^0 and H^1 have no isomorphic direct summands, the Pierce decomposition of \mathbf{H} is

$$\mathbf{H} = \begin{pmatrix} H_0^0 & A_0^1 \\ A_1^0 & A_1^1 \end{pmatrix}$$

and $\tau_0^0 = \text{rad } H_0^0$. Now, one easily sees that an element $a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ belongs to \mathbf{C} if and only if $H^0\alpha \subseteq A^0$. We claim that in that case $H^0\alpha \subseteq \text{rad } A^0$. Otherwise $H^0\alpha$ contains an idempotent, hence a direct summand of A^0 , which is isomorphic to some A_i with $1 \leq i \leq r$. This is impossible, since $\mathbf{H}A_i \neq A_i$. Therefore, $\alpha \in \tau_0^0$ and we obtain the following result:

Proposition 2.6. *The Pierce decomposition of the ideal \mathbf{C} is*

$$\mathbf{C} = \begin{pmatrix} \tau_0^0 & A_0^1 \\ A_1^0 & A_1^1 \end{pmatrix}.$$

Definition 2.7. Analogously to [Burban and Drozd 2011], we define the *Auslander envelope* of the Backström pair $\mathbf{H} \supseteq A$ as the ring $\tilde{\mathbf{A}}$ of 2×2 matrices of the form

$$\tilde{\mathbf{A}} = \begin{pmatrix} A & \mathbf{H} \\ \mathbf{C} & \mathbf{H} \end{pmatrix}$$

with the usual matrix multiplication.

Using Pierce decompositions of A , \mathbf{H} and \mathbf{C} , we also present $\tilde{\mathbf{A}}$ as the ring of 4×4 matrices

$$(2-1) \quad \tilde{\mathbf{A}} = \begin{pmatrix} A_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \\ \tau_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \end{pmatrix}.$$

We also define $\tilde{\mathbf{H}}$ as the ring of 4×4 matrices of the form

$$\tilde{\mathbf{H}} = \begin{pmatrix} \mathbf{H} & \mathbf{H} \\ \mathbf{C} & \mathbf{H} \end{pmatrix} \quad \text{or} \quad \tilde{\mathbf{H}} = \begin{pmatrix} H_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \\ \tau_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \end{pmatrix}.$$

Obviously, $\text{rad } \tilde{\mathbf{H}} = \text{rad } \tilde{\mathbf{A}}$, so $\tilde{\mathbf{H}} \supseteq \tilde{\mathbf{A}}$ is also a Backström pair. $\tilde{\mathbf{A}}$ is left noetherian if and only if \mathbf{A} is left noetherian and \mathbf{H} is finitely generated as a left \mathbf{A} -module.

In the noetherian case one can calculate the global dimensions of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{H}}$. It turns out that it only depends on \mathbf{H} .

Theorem 2.8. *Suppose that either \mathbf{A} (hence also \mathbf{H}) is left perfect or \mathbf{A} is left noetherian and \mathbf{H} is finitely generated as a left \mathbf{A} -module (hence also left noetherian).*

Then

$$\begin{aligned} \text{l.gl.dim } \tilde{\mathbf{A}} &= 1 + \max(1 + \text{pr.dim}_{\mathbf{H}} \tau^0, \text{pr.dim}_{\mathbf{H}} \tau^1) \\ &= \begin{cases} 1 + \text{l.gl.dim } \mathbf{H} & \text{if } \text{pr.dim}_{\mathbf{H}} \tau^0 \geq \text{pr.dim}_{\mathbf{H}} \tau^1, \\ \text{l.gl.dim } \mathbf{H} & \text{if } \text{pr.dim}_{\mathbf{H}} \tau^0 < \text{pr.dim}_{\mathbf{H}} \tau^1 \end{cases} \end{aligned}$$

and

$$\text{l.gl.dim } \tilde{\mathbf{H}} = \text{l.gl.dim } \mathbf{H},$$

where we set $\text{pr.dim } 0 = -1$. In particular, if \mathbf{A} is a Backström ring, so is $\tilde{\mathbf{A}}$, and if \mathbf{A} is not left hereditary, then $\text{l.gl.dim } \tilde{\mathbf{A}} = 2$.²

For instance, this is the case for nodal (in particular, gentle or skewed-gentle) algebras (Examples 2.3).

Proof. Under these conditions $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{H}}$ are either left perfect or left noetherian. We recall that if a ring Λ is left perfect or left noetherian and semiperfect, then $\text{l.gl.dim } \Lambda = \text{pr.dim}_{\Lambda} (\Lambda/\text{rad } \Lambda) = 1 + \text{pr.dim}_{\Lambda} \text{rad } \Lambda$. The 4×4 matrix presentation (2-1) of $\tilde{\mathbf{A}}$ implies that the corresponding presentation of $\text{rad } \tilde{\mathbf{A}}$ is

$$(2-2) \quad \text{rad } \tilde{\mathbf{A}} = \begin{pmatrix} \tau_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & \tau_1^1 & A_1^0 & \tau_1^1 \\ \tau_0^0 & A_0^1 & \tau_0^0 & A_0^1 \\ A_1^0 & \tau_1^1 & A_1^0 & \tau_1^1 \end{pmatrix}.$$

An $\tilde{\mathbf{A}}$ -module M is given by a quadruple (M', M'', ϕ, ψ) , where M' is an \mathbf{A} -module, M'' is an \mathbf{H} -module, $\psi : M'' \rightarrow M'$ is a homomorphism of \mathbf{A} -modules and $\phi : \mathbf{C} \otimes_{\mathbf{A}} M' \rightarrow M''$ is a homomorphism of \mathbf{H} -modules. Namely, $M' = e'M$, $M'' = e''M$, where $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\psi(m'') = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} m''$ and $\phi(c \otimes m') = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} m'$.

²Note that if $\tilde{\mathbf{A}}$ is left hereditary, so is $\mathbf{A} = e' \tilde{\mathbf{A}} e'$ [Sandomierski 1969].

We frequently write $M = \begin{pmatrix} M' \\ M'' \end{pmatrix}$, not mentioning ϕ and ψ . For an \mathbf{H} -module N we define the $\tilde{\mathbf{A}}$ -module $N^+ = \begin{pmatrix} N \\ N \end{pmatrix}$. Then, $N \mapsto N^+$ is an exact functor mapping projective modules to projective ones, since $\mathbf{H}^+ = \begin{pmatrix} \mathbf{H} \\ \mathbf{H} \end{pmatrix}$ is a projective $\tilde{\mathbf{A}}$ -module.

We denote by L^i and by R^i the i -th column of the presentations (2-1) and (2-2), respectively. Then, $R^1 = (\tau^0)^+$ and $R^2 = R^4 = (\tau^1)^+$, where $\tau^a = \tau e_a$. If

$$\cdots \rightarrow F_k \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0$$

is a minimal projective resolution of an \mathbf{H} -module N ,

$$\cdots \rightarrow F_k^+ \rightarrow \cdots \rightarrow F_1^+ \rightarrow F_0^+ \rightarrow N^+ \rightarrow 0$$

is a minimal projective resolution of N^+ , so $\text{pr.dim}_{\tilde{\mathbf{A}}} N^+ = \text{pr.dim}_{\mathbf{H}} N$. In particular, $\text{pr.dim}_{\tilde{\mathbf{A}}} R^1 = \text{pr.dim}_{\mathbf{H}} \tau^0$ and $\text{pr.dim}_{\tilde{\mathbf{A}}} R^2 = \text{pr.dim}_{\mathbf{H}} \tau^1$. For the module R^3 we have an exact sequence

$$(2-3) \quad 0 \rightarrow (\tau^0)^+ \rightarrow R^3 \rightarrow \begin{pmatrix} H^0/\tau^0 \\ 0 \end{pmatrix} \rightarrow 0.$$

Note that H^0/τ^0 is a semisimple \mathbf{A} -module and $e_1(H^0/\tau^0) = 0$, hence it contains the same simple direct summands as A^0/τ^0 . The same is true for

$$\begin{pmatrix} H^0/\tau^0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A^0/\tau^0 \\ 0 \end{pmatrix} = L^1/R^1.$$

Hence,

$$\text{pr.dim}_{\tilde{\mathbf{A}}} \begin{pmatrix} H^0/\tau^0 \\ 0 \end{pmatrix} = 1 + \text{pr.dim}_{\tilde{\mathbf{A}}} R^1 = 1 + \text{pr.dim}_{\mathbf{H}} \tau^0.$$

Therefore, the exact sequence (2-3) shows that $\text{pr.dim}_{\tilde{\mathbf{A}}} R^3 = 1 + \text{pr.dim}_{\mathbf{H}} \tau^0$ and

$$\text{pr.dim}_{\tilde{\mathbf{A}}} \text{rad } \tilde{\mathbf{A}} = \max(1 + \text{pr.dim}_{\mathbf{H}} \tau^0, \text{pr.dim}_{\mathbf{H}} \tau^1),$$

which gives the necessary result for $\tilde{\mathbf{A}}$. On the other hand, R^3 is a projective $\tilde{\mathbf{H}}$ -module, whence $\text{l.gl.dim } \tilde{\mathbf{H}} = \text{l.gl.dim } \mathbf{H}$. \square

3. The structure of derived categories

In what follows we denote by $\mathcal{D}(\mathbf{A})$ the derived category $\mathcal{D}(\mathbf{A}\text{-Mod})$. We denote by $\mathcal{D}_f(\mathbf{A})$ the full subcategory of $\mathcal{D}(\mathbf{A})$ consisting of complexes quasi-isomorphic to complexes of finitely generated projective modules. If \mathbf{A} is left noetherian, it coincides with the derived category of the category $\mathbf{A}\text{-mod}$ of finitely generated \mathbf{A} -modules. We also use the usual superscripts $+$, $-$, b . By $\text{Perf}(\mathbf{A})$ we denote the full subcategory of perfect complexes from $\mathcal{D}(\mathbf{A})$, i.e., complexes quasi-isomorphic to finite complexes of finitely generated projective modules. It coincides with the full subcategory of compact objects in $\mathcal{D}(\mathbf{A})$ [Rouquier 2008]. If \mathbf{A} is left

noetherian, an \mathbf{A} -module M belongs to $\text{Perf}(\mathbf{A})$ if and only if it is finitely generated and of finite projective dimension.

There are close relations between the categories $\mathcal{D}(\mathbf{A})$, $\mathcal{D}(\mathbf{H})$ and $\mathcal{D}(\tilde{\mathbf{A}})$ based on the following construction [Burban et al. 2017]:

Let $\mathbf{P} = \begin{pmatrix} \mathbf{A} \\ \mathbf{C} \end{pmatrix}$. It is a projective $\tilde{\mathbf{A}}$ -module and $\text{End } \mathbf{P} \simeq \mathbf{A}^{\text{op}}$, so it can be considered as a right \mathbf{A} -module. Consider the functors

$$\begin{aligned} \mathbf{F} &= \text{Hom}_{\tilde{\mathbf{A}}}(\mathbf{P}, -) \simeq \mathbf{P}^\vee \otimes_{\tilde{\mathbf{A}}} - : \tilde{\mathbf{A}}\text{-Mod} \rightarrow \mathbf{A}\text{-Mod}, \\ \mathbf{F}^* &= \mathbf{P} \otimes_{\mathbf{A}} - : \mathbf{A}\text{-Mod} \rightarrow \tilde{\mathbf{A}}\text{-Mod}, \\ \mathbf{F}^\dagger &= \text{Hom}_{\mathbf{A}}(\mathbf{P}^\vee, -) : \mathbf{A}\text{-Mod} \rightarrow \tilde{\mathbf{A}}\text{-Mod}, \end{aligned}$$

where $\mathbf{P}^\vee = \text{Hom}_{\tilde{\mathbf{A}}}(\mathbf{P}, \tilde{\mathbf{A}}) \simeq (\mathbf{A} \ \mathbf{H})$ is the dual right projective $\tilde{\mathbf{A}}$ -module, the functor \mathbf{F} is exact, \mathbf{F}^* is its left adjoint and \mathbf{F}^\dagger is its right adjoint. Moreover, the adjunction morphisms $\mathbf{F}\mathbf{F}^* \rightarrow \text{Id}_{\mathbf{A}\text{-Mod}}$ and $\text{Id}_{\mathbf{A}\text{-Mod}} \rightarrow \mathbf{F}\mathbf{F}^\dagger$ are isomorphisms [Burban et al. 2017, Theorem 4.3]. The functors \mathbf{F}^* and \mathbf{F}^\dagger are fully faithful and \mathbf{F} is essentially surjective, i.e., every \mathbf{A} -module is isomorphic to $\mathbf{F}M$ for some $\tilde{\mathbf{A}}$ -module M . $\text{Ker } \mathbf{F}$ is a Serre subcategory of $\tilde{\mathbf{A}}\text{-Mod}$ equivalent to $\overline{\mathbf{H}}\text{-Mod}$, where $\overline{\mathbf{H}} = \mathbf{H}/\mathbf{C} \simeq \tilde{\mathbf{A}}/\begin{pmatrix} \mathbf{A} & \mathbf{H} \\ \mathbf{C} & \mathbf{C} \end{pmatrix}$. The embedding functor $\mathbf{l} : \text{Ker } \mathbf{F} \rightarrow \tilde{\mathbf{A}}\text{-Mod}$ has a left adjoint \mathbf{l}^* and a right adjoint \mathbf{l}^\dagger and we obtain a recollement diagram

$$\begin{array}{ccccc} & \longleftarrow \mathbf{l}^* & & \longleftarrow \mathbf{F}^* & \\ \text{Ker } \mathbf{F} & \xrightarrow{\mathbf{l}} & \tilde{\mathbf{A}}\text{-Mod} & \xrightarrow{\mathbf{F}} & \mathbf{A}\text{-Mod} \\ & \longleftarrow \mathbf{l}^\dagger & & \longleftarrow \mathbf{F}^\dagger & \end{array}$$

As the functor \mathbf{F} is exact, it extends to the functor between the derived categories $\text{DF} : \mathcal{D}(\tilde{\mathbf{A}}) \rightarrow \mathcal{D}(\mathbf{A})$ acting on complexes componentwise. The derived functors LF^* and RF^\dagger are its left and right adjoints, respectively, the adjunction morphisms $\text{Id}_{\mathcal{D}(\mathbf{A})} \rightarrow \text{DF} \cdot \text{LF}^*$ and $\text{DF} \cdot \text{RF}^\dagger \rightarrow \text{Id}_{\mathcal{D}(\tilde{\mathbf{A}})}$ are again isomorphisms and we have a recollement diagram

$$\begin{array}{ccccc} & \longleftarrow \text{LI}^* & & \longleftarrow \text{LF}^* & \\ \text{Ker } \text{DF} & \xrightarrow{\text{DI}} & \mathcal{D}(\tilde{\mathbf{A}}) & \xrightarrow{\text{DF}} & \mathcal{D}(\mathbf{A}) \\ & \longleftarrow \text{RI}^\dagger & & \longleftarrow \text{RF}^\dagger & \end{array}$$

(It also follows from Corollary 1.2.) Here $\text{Ker } \text{DF} = \mathcal{D}_{\overline{\mathbf{H}}}(\tilde{\mathbf{A}})$, the full subcategory of complexes whose cohomologies are $\overline{\mathbf{H}}$ -modules, i.e., are annihilated by the ideal $\begin{pmatrix} \mathbf{A} & \mathbf{H} \\ \mathbf{C} & \mathbf{C} \end{pmatrix}$. Note that, as a rule, it is not equivalent to $\mathcal{D}(\overline{\mathbf{H}})$. From the definition of \mathbf{F} it follows that

$$\text{Ker } \text{DF} = \mathbf{P}^\perp = \{C \in \mathcal{D}(\tilde{\mathbf{A}}) \mid \text{Hom}_{\mathcal{D}(\tilde{\mathbf{A}})}(\mathbf{P}, C[k]) = 0 \text{ for all } k\}.$$

Obviously, DF maps $\mathcal{D}^\sigma(\tilde{\mathbf{A}})$ to $D^\sigma(\mathbf{A})$ for $\sigma \in \{+, -, b\}$, LF^* maps $\mathcal{D}^-(\mathbf{A})$ to $\mathcal{D}^-(\tilde{\mathbf{A}})$ and RF^\dagger maps $\mathcal{D}^+(\mathbf{A})$ to $\mathcal{D}^+(\tilde{\mathbf{A}})$. If $\tilde{\mathbf{A}}$ is left noetherian, DF maps $\mathcal{D}_f(\tilde{\mathbf{A}})$ to $\mathcal{D}_f(\mathbf{A})$ and LF^* maps $\mathcal{D}_f(\mathbf{A})$ to $\mathcal{D}_f(\tilde{\mathbf{A}})$. Finally, both DF and LF^* have right adjoints, hence map compact objects (i.e., perfect complexes) to compact ones.

On the contrary, usually LF^* does not map $\mathcal{D}^b(\mathbf{A})$ to $\mathcal{D}^b(\tilde{\mathbf{A}})$. For instance, it is definitely so if $\text{l.gl.dim } \tilde{\mathbf{A}} < \infty$ while $\text{l.gl.dim } \mathbf{A} = \infty$ as in Examples 2.3 (4, 5). If $\text{l.gl.dim } \mathbf{H}$ is finite, so is $\text{l.gl.dim } \tilde{\mathbf{A}}$, thus this recollement can be considered as a sort of categorical resolution of the category $\mathcal{D}(\mathbf{A})$. In any case, it is useful for studying the categories $\mathbf{A}\text{-Mod}$ and $\mathcal{D}(\mathbf{A})$ if we know the structure of the categories $\tilde{\mathbf{A}}\text{-Mod}$ and $\mathcal{D}(\tilde{\mathbf{A}})$. For instance, it is so if we are interested in the *derived dimension*, i.e., the dimension of the category $\mathcal{D}_f^b(\mathbf{A})$ in the sense of [Rouquier 2008].

Definition 3.1. Let \mathcal{T} be a triangulated category and \mathfrak{M} be a set of objects from \mathcal{T} .

- (1) We denote by $\langle \mathfrak{M} \rangle$ the smallest full subcategory of \mathcal{T} containing \mathfrak{M} and closed under direct sums, direct summands and shifts (not closed under cones, so not a triangulated subcategory).
- (2) If \mathfrak{N} is another subset of \mathcal{T} , we denote by $\mathfrak{M} \dagger \mathfrak{N}$ the set of objects C from \mathcal{T} such that there is an exact triangle $A \rightarrow B \rightarrow C \xrightarrow{\pm}$, where $A \in \mathfrak{M}$, $B \in \mathfrak{N}$.
- (3) We define $\langle \mathfrak{M} \rangle_k$ recursively, setting $\langle \mathfrak{M} \rangle_1 = \langle \mathfrak{M} \rangle$ and $\langle \mathfrak{M} \rangle_{k+1} = \langle \langle \mathfrak{M} \rangle \dagger \langle \mathfrak{M} \rangle_k \rangle$.
- (4) The *dimension* $\dim \mathcal{T}$ of \mathcal{T} is the smallest k such that there is a finite set of objects \mathfrak{M} such that $\langle \mathfrak{M} \rangle_{k+1} = \mathcal{T}$ (if it exists). We call the dimension $\dim \mathcal{D}_f^b(\mathbf{A})$ the *derived dimension* of the ring \mathbf{A} and denote it by $\text{der.dim } \mathbf{A}$.

As the functor F is exact and essentially surjective, the next result is evident:

Proposition 3.2. *We have $\text{der.dim } \mathbf{A} \leq \text{der.dim } \tilde{\mathbf{A}}$. Namely, if $\mathcal{D}_f^b(\tilde{\mathbf{A}}) = \langle \mathfrak{M} \rangle_{k+1}$, then $\mathcal{D}_f^b(\mathbf{A}) = \langle \text{DF}(\mathfrak{M}) \rangle_{k+1}$.*

4. Semiorthogonal decomposition

There is another recollement diagram for $\mathcal{D}(\tilde{\mathbf{A}})$ related to the projective module $\mathbf{Q} = \begin{pmatrix} \mathbf{H} \\ \mathbf{H} \end{pmatrix}$ with $\text{End } \mathbf{Q} \simeq \mathbf{H}^{\text{op}}$. Namely, we set

$$\begin{aligned} \mathbf{G} &= \text{Hom}_{\tilde{\mathbf{A}}}(\mathbf{Q}, -) \simeq \mathbf{Q}^\vee \otimes_{\tilde{\mathbf{A}}} - : \tilde{\mathbf{A}}\text{-Mod} \rightarrow \mathbf{H}\text{-Mod}, \\ \mathbf{G}^* &= \mathbf{Q} \otimes_{\mathbf{H}} - : \mathbf{H}\text{-Mod} \rightarrow \tilde{\mathbf{A}}\text{-Mod}, \\ \mathbf{G}^\dagger &= \text{Hom}_{\mathbf{H}}(\mathbf{Q}^\vee, -) : \mathbf{H}\text{-Mod} \rightarrow \tilde{\mathbf{A}}\text{-Mod}, \end{aligned}$$

where $\mathbf{Q}^\vee = \text{Hom}_{\tilde{\mathbf{A}}}(\mathbf{Q}, \tilde{\mathbf{A}}) \simeq (\mathbf{C} \ \mathbf{H})$,

$$\begin{aligned} \text{DG} &: \mathcal{D}(\tilde{\mathbf{A}}) \rightarrow \mathcal{D}(\mathbf{H}) \text{ is } \mathbf{G} \text{ applied componentwise,} \\ \text{LG}^* &: \mathcal{D}(\mathbf{A}) \rightarrow \mathcal{D}(\tilde{\mathbf{A}}) \text{ is the left adjoint of } \text{DG}, \\ \text{RG}^\dagger &: \mathcal{D}(\mathbf{A}) \rightarrow \mathcal{D}(\tilde{\mathbf{A}}) \text{ is the right adjoint of } \text{DG}. \end{aligned}$$

We also set $\bar{A} = A/C \simeq \tilde{A}/\begin{pmatrix} C & H \\ C & H \end{pmatrix}$. Then, we have recollement diagrams

$$\begin{array}{ccccc} \text{Ker } G & \begin{array}{c} \longleftarrow J^* \\ \xrightarrow{J} \\ \longleftarrow J' \end{array} & \tilde{A}\text{-Mod} & \begin{array}{c} \longleftarrow G^* \\ \xrightarrow{G} \\ \longleftarrow G' \end{array} & \mathbf{H}\text{-Mod} \end{array}$$

and

$$\begin{array}{ccccc} \text{Ker } DG & \begin{array}{c} \longleftarrow LJ^* \\ \xrightarrow{DJ} \\ \longleftarrow RJ' \end{array} & \mathcal{D}(\tilde{A}) & \begin{array}{c} \longleftarrow LG^* \\ \xrightarrow{DG} \\ \longleftarrow RG' \end{array} & \mathcal{D}(\mathbf{H}), \end{array}$$

where $\text{Ker } G \simeq \bar{A}\text{-Mod}$. Since the \tilde{A} -ideal $\begin{pmatrix} C & H \\ C & H \end{pmatrix}$ is projective as a right \tilde{A} -module, [Burban et al. 2017, Theorem 4.6] implies that $\text{Ker } DG \simeq \mathcal{D}(\bar{A})$.

As usual, this recollement diagram gives semiorthogonal decompositions [Burban et al. 2017, Corollary 2.6]

$$(4-1) \quad \mathcal{D}(\tilde{A}) = (\text{Ker } DG, \text{Im } LG^*) = (\text{Im } RG^!, \text{Ker } DG)$$

with $\text{Ker } DG \simeq \mathcal{D}(\bar{A})$ and $\text{Im } LG^* \simeq \text{Im } RG^! \simeq \mathcal{D}(\mathbf{H})$ (though usually $\text{Im } LG^* \neq \text{Im } RG^!$).

Recall from [Kuznetsov and Lunts 2015] that a *semiorthogonal decomposition* $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$, where $\mathcal{T}_1, \mathcal{T}_2$ are full triangulated subcategories of \mathcal{T} , means that

$$\text{Hom}_{\mathcal{T}}(T_2, T_1) = 0 \quad \text{for all } T_1 \in \mathcal{T}_1 \text{ and } T_2 \in \mathcal{T}_2,$$

and for every object $T \in \mathcal{T}$ there is an exact triangle $T_1 \rightarrow T_2 \rightarrow T \xrightarrow{\pm}$, where $T_i \in \mathcal{T}_i$.

Lemma 4.1.³ *If $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2)$ is a semiorthogonal decomposition of a triangulated category \mathcal{T} , then*

$$\dim \mathcal{T} \leq \dim \mathcal{T}_1 + \dim \mathcal{T}_2 + 1.$$

Proof. First we show that for any subsets $\mathfrak{M}, \mathfrak{N}$ of objects of the category \mathcal{T}

$$(4-2) \quad \langle \mathfrak{M} \rangle_{k+1} \dagger \mathfrak{N} \subseteq \langle \mathfrak{M} \rangle \dagger \langle \langle \mathfrak{M} \rangle_k \dagger \mathfrak{N} \rangle \subseteq \underbrace{\langle \mathfrak{M} \rangle \dagger \langle \langle \mathfrak{M} \rangle \dagger \langle \langle \mathfrak{M} \rangle \dagger \cdots \langle \langle \mathfrak{M} \rangle \dagger \mathfrak{N} \rangle \cdots \rangle}_{k+1}.$$

Indeed, let $C \in \langle \mathfrak{M} \rangle_{k+1} \dagger \mathfrak{N}$, i.e., there is an exact triangle $A \rightarrow B \rightarrow C \xrightarrow{\pm}$, where $A \in \langle \mathfrak{M} \rangle_{k+1}$, $B \in \mathfrak{N}$. There is also an exact triangle $A_1 \rightarrow A \rightarrow A_2 \xrightarrow{\pm}$, where $A_1 \in \langle \mathfrak{M} \rangle_k$, $A_2 \in \langle \mathfrak{M} \rangle$. The octahedron axiom implies that there are exact triangles $A_1 \rightarrow B \rightarrow B' \xrightarrow{\pm}$ and $A_2 \rightarrow B' \rightarrow C \xrightarrow{\pm}$. Therefore, $B' \in \langle \mathfrak{M} \rangle_k \dagger \mathfrak{N}$ and $C \in \langle \mathfrak{M} \rangle \dagger \langle \langle \mathfrak{M} \rangle_k \dagger \mathfrak{N} \rangle$.

Now, let $\langle \mathfrak{M} \rangle_{k+1} = \mathcal{T}_1$ and $\langle \mathfrak{N} \rangle_{l+1} = \mathcal{T}_2$. Then, for every $T \in \mathcal{T}$ there is an exact triangle $T_1 \rightarrow T_2 \rightarrow T \xrightarrow{\pm}$, where $T_1 \in \langle \mathfrak{M} \rangle_{k+1}$, $T_2 \in \langle \mathfrak{N} \rangle_{l+1}$. But, according to (4-2), $\langle \mathfrak{M} \rangle_{k+1} \dagger \langle \mathfrak{N} \rangle_{l+1} \subseteq \langle \mathfrak{M} \cup \mathfrak{N} \rangle_{k+l+2}$, so $\mathcal{T} = \langle \mathfrak{M} \cup \mathfrak{N} \rangle_{k+l+2}$ and $\dim \mathcal{T} \leq k+l+1$. \square

³In [Psaroudakis 2014, Theorem 7.4] this result is proved in the case when this decomposition arises from a recollement.

Since \bar{A} is semisimple, any indecomposable object from $\mathcal{D}(\bar{A})$ is just a shifted simple module, so $\mathcal{D}_f^b(\bar{A}) = \langle \bar{A} \rangle$ and $\text{der.dim } \bar{A} = 0$. If \mathbf{H} is hereditary, every indecomposable object from $\mathcal{D}_f^b(\mathbf{H})$ is a shift of a module. For every module M there is an exact sequence $0 \rightarrow P' \rightarrow P \rightarrow M \rightarrow 0$ with projective modules P, P' and, since \mathbf{H} is semiperfect, every indecomposable projective \mathbf{H} -module is a direct summand of \mathbf{H} . Hence, $\mathcal{D}_f^b(\mathbf{H}) = \langle \mathbf{H} \rangle_2$ and $\text{der.dim } \mathbf{H} \leq 1$.

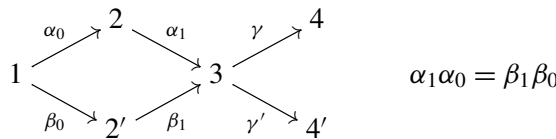
Corollary 4.2. *We have $\text{der.dim } A \leq \text{der.dim } \mathbf{H} + 1$. In particular, if A is a Backström (or piecewise Backström) ring, $\text{der.dim } A \leq 2$.*

A finite dimensional hereditary algebra is said to be of *Dynkin type* if it has finitely many isomorphism classes of indecomposable modules. Such algebras, up to Morita equivalence, correspond to Dynkin diagrams [Dlab and Ringel 1976; Gabriel 1972]. If the derived category of an algebra \mathbf{H} is equivalent to the derived category of a hereditary algebra of Dynkin type, we say that \mathbf{H} is *piecewise hereditary of Dynkin type*.⁴ We say that a Backström (or piecewise Backström) algebra A is of *Dynkin type* if there is a Backström pair $\mathbf{H} \supseteq A$, where \mathbf{H} is a hereditary (piecewise hereditary) algebra of Dynkin type. For instance, it is so if A is a gentle or skewed-gentle algebra [Zembyk 2015], or the algebra $\text{UT}(n\mathbb{k})$ of unitriangular matrices over a field (Examples 2.3 (4)), or the algebra Λ_n from Examples 2.3 (5). In this case, $\mathcal{D}_f^b(\mathbf{H}) = \langle M_1, M_2, \dots, M_m \rangle_1$, where M_1, M_2, \dots, M_m are all pairwise nonisomorphic indecomposable \mathbf{H} -modules, so $\text{der.dim } \mathbf{H} = 0$.

In [Chen et al. 2008] it was proved that $\text{der.dim } A = 0$ for a finite dimensional algebra A if and only if A is a piecewise hereditary algebra of Dynkin type.

Corollary 4.3. *If A is a Backström (or piecewise Backström) algebra of Dynkin type (for instance, gentle or skewed-gentle), but is not piecewise hereditary of Dynkin type, then $\text{der.dim } A = 1$.*

Example 4.4. The path algebra of the commutative quiver



is a tilted (hence piecewise hereditary) algebra of type \tilde{D}_5 . At the same time it is a Backström algebra of type A_4 . Namely, it is a skewed-gentle algebra obtained from the path algebra of the quiver $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ by blowing up vertices 2 and 4.⁵

⁴It is proved in [Happel 1988] that piecewise hereditary algebras of Dynkin type are just iterated tilted algebras of Dynkin type.

⁵See [Zembyk 2014] for the construction of blowing up and its relation to nodal algebras.

5. Relation to bimodule categories

In this section, we explain how a semiorthogonal decomposition allows us to apply to calculations in a triangulated category the technique of matrix problems, namely, of bimodule categories, as in [Drozd 2010].

Let \mathcal{A} and \mathcal{B} be additive categories, \mathcal{U} be an \mathcal{A} - \mathcal{B} -bimodule, i.e., a biadditive functor $\mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \mathbf{Ab}$. Recall from [Drozd 2010] that the *bimodule category* or the *category of elements* of the bimodule \mathcal{U} is the category $\text{El}(\mathcal{U})$ whose set of objects is $\bigcup_{A \in \mathcal{A}, B \in \mathcal{B}} \mathcal{U}(A, B)$ and whose morphisms from $u \in \mathcal{U}(A, B)$ to $v \in \mathcal{U}(A', B')$ are the pairs (α, β) such that $u\alpha = \beta v$, where $\alpha : A' \rightarrow A$, $\beta : B \rightarrow B'$. Here, as usual, we wrote $u\alpha$ and βv instead of $\mathcal{U}(\alpha, 1_B)u$ and $\mathcal{U}(1_{A'}, \beta)v$. Bimodule categories appear when there is a semiorthogonal decomposition of a triangulated category.

Theorem 5.1. *Let $(\mathcal{A}, \mathcal{B})$ be a semiorthogonal decomposition of a triangulated category \mathcal{C} . Consider the \mathcal{A} - \mathcal{B} -bimodule \mathcal{U} such that $\mathcal{U}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$, $A \in \mathcal{A}$, $B \in \mathcal{B}$. For every $f : A \rightarrow B$ fix a cone Cf , that is, an exact triangle $A \xrightarrow{f} B \xrightarrow{f_1} Cf \xrightarrow{f_2} A[1]$. The map $f \mapsto Cf$ induces an equivalence of categories $C : \text{El}(\mathcal{U}) \xrightarrow{\sim} \mathcal{C} / \mathcal{J}$, where \mathcal{J} is the ideal of \mathcal{C} consisting of morphisms η such that there are factorizations $\eta = \eta'\xi = \zeta\eta''$, where the source of η' is in \mathcal{A} and the target of η'' is in \mathcal{B} . Moreover, $\mathcal{J}^2 = 0$, so C induces a bijection between isomorphism classes of objects from $\text{El}(\mathcal{U})$ and from \mathcal{C} .⁶*

Proof. As $(\mathcal{A}, \mathcal{B})$ is a semiorthogonal decomposition of \mathcal{C} , every object from \mathcal{C} occurs in an exact triangle $A \xrightarrow{f} B \rightarrow C \xrightarrow{\pm}$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$, so f is an object from $\text{El}(\mathcal{U})$ and $C \simeq Cf$. Let $f' : A' \rightarrow B'$ be another object of $\text{El}(\mathcal{U})$ and $(\alpha, \beta) : f \rightarrow f'$ be a morphism from $\text{El}(\mathcal{U})$. Fix a commutative diagram

$$(5-1) \quad \begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{f_1} & Cf & \xrightarrow{f_2} & A[1] \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \alpha[1] \downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{f'_1} & Cf' & \xrightarrow{f'_2} & A'[1] \end{array}$$

It exists, though is not unique. Let γ' be another morphism making the diagram (5-1) commutative and set $\eta = \gamma - \gamma'$. Then, $\eta f_1 = 0$, hence η factors through f_2 , and $f'_2 \eta = 0$, hence η factors through f'_1 . Thus, $\eta \in \mathcal{J}$. On the other hand, if $\eta : Cf \rightarrow Cf'$ is in \mathcal{J} , the decomposition $\eta = \eta'\xi$ implies that $\eta f_1 = \eta'\xi f_1 = 0$ and the decomposition $\eta = \zeta\eta''$ implies that $f'_2 \eta = f'_2 \zeta \eta'' = 0$, hence the morphism $\gamma' = \gamma + \eta$ makes the diagram (5-1) commutative. Therefore, the class $C(\alpha, \beta)$ of γ modulo \mathcal{J} is uniquely defined, so the maps $f \mapsto Cf$ and $(\alpha, \beta) \mapsto C(\alpha, \beta)$ define a functor $C : \text{El}(\mathcal{U}) \rightarrow \mathcal{C} / \mathcal{J}$.

⁶This theorem is a partial case of [Drozd 2010, Theorem 1.1].

Let now $\gamma : Cf \rightarrow Cf'$ be any morphism. Then, $f_2'\gamma f_1 = 0$, so $\gamma f_1 = f_1'\beta$ for some $\beta : B \rightarrow B'$. Hence, there is a morphism $\alpha : A \rightarrow A'$ making the diagram (5-1) commutative, i.e., defining a morphism $(\alpha, \beta) : f \rightarrow f'$ such that $\gamma \equiv C(\alpha, \beta) \pmod{\mathcal{J}}$. If (α', β') is another such morphism, $f_1'(\beta - \beta') = 0$, so $\beta - \beta' = f'\xi$ for some $\xi : B \rightarrow A$. But $\xi = 0$, so $\beta = \beta'$. In the same way $\alpha = \alpha'$. Hence, the functor C is fully faithful. As we have already noticed, it is essentially surjective, and therefore defines an equivalence $\text{El}(\mathcal{U}) \xrightarrow{\sim} \mathcal{C}/\mathcal{J}$. The equality $\mathcal{J}^2 = 0$ follows immediately from the definition and the conditions of the theorem. \square

We apply Theorem 5.1 to Backström pairs $\mathbf{H} \subseteq \mathbf{A}$ such that \mathbf{A} is left noetherian and \mathbf{H} is left hereditary and finitely generated as a left \mathbf{A} -module. For instance, it is so in the case of Backström algebras or Backström orders. Then, the ring $\tilde{\mathbf{A}}$ is also noetherian and \mathbf{C} is projective as a left \mathbf{H} -module. According to (4-1), $(\text{Ker DG}, \text{Im LG}^*)$ is a semiorthogonal decomposition of $\mathcal{D}(\tilde{\mathbf{A}})$. Moreover, both G and G^* map finitely generated modules to finitely generated modules, so the same is valid if we consider their restrictions onto $\mathcal{D}_f(\tilde{\mathbf{A}})$ and $\mathcal{D}_f(\mathbf{H})$. Note also that G^* is exact, so G^* can be applied to complexes componentwise. The $\tilde{\mathbf{A}}$ -module G^*M can be identified with the module of columns $M^+ = \begin{pmatrix} M \\ M \end{pmatrix}$ with the action of $\tilde{\mathbf{A}}$ given by matrix multiplication. It gives an equivalence of $\mathcal{D}(\mathbf{H})$ with Im LG^* . As \mathbf{H} is left hereditary, every complex from $\mathcal{D}(\mathbf{H})$ is equivalent to a direct sum of shifted modules (see [Keller 2007, Section 2.5]). On the other hand, $\text{Ker DG} \simeq \mathcal{D}(\bar{\mathbf{A}})$ and $\bar{\mathbf{A}}$ is semisimple, since $\mathbf{C} \supseteq \mathbf{r}$. Hence, every complex from $\mathcal{D}(\bar{\mathbf{A}})$ is isomorphic to a direct sum of shifted simple $\bar{\mathbf{A}}$ -modules, which are direct summands of $\bar{\mathbf{A}}$. So, to calculate the bimodule \mathcal{U} , we only have to calculate $\text{Ext}_{\tilde{\mathbf{A}}}^i(\bar{\mathbf{A}}, M^+)$, where M is an \mathbf{H} -module. Note also that \mathbf{C}^+ is a projective $\tilde{\mathbf{A}}$ -module, since \mathbf{C} is a projective \mathbf{H} -module. Therefore, a projective resolution of $\bar{\mathbf{A}}$ is $0 \rightarrow \mathbf{C}^+ \xrightarrow{\varepsilon} \mathbf{P} \rightarrow \bar{\mathbf{A}} \rightarrow 0$ and $\text{pr.dim}_{\tilde{\mathbf{A}}} \bar{\mathbf{A}} = 1$. Hence, we only have to calculate $\text{Hom}_{\tilde{\mathbf{A}}}(\bar{\mathbf{A}}, M^+)$ and $\text{Ext}_{\tilde{\mathbf{A}}}^1(\bar{\mathbf{A}}, M^+)$.

Theorem 5.2. (1) $\text{Hom}_{\tilde{\mathbf{A}}}(\bar{\mathbf{A}}, M^+) \simeq \text{ann}_M \mathbf{C} = \{u \in M \mid \mathbf{C}u = 0\}$.

(2) $\text{Ext}_{\tilde{\mathbf{A}}}^1(\bar{\mathbf{A}}, M^+) \simeq \text{Hom}_{\mathbf{H}}(\mathbf{C}, M) / (M / \text{ann}_M \mathbf{C})$, where the quotient $M / \text{ann}_M \mathbf{C}$ embeds into $\text{Hom}_{\mathbf{H}}(\mathbf{C}, M)$ if we map an element $u \in M$ to the homomorphism $\mu_u : c \mapsto cu$.

Proof. (1) $\text{Hom}_{\tilde{\mathbf{A}}}(\bar{\mathbf{A}}, M^+)$ is identified with the set of homomorphisms $\phi : \mathbf{P} \rightarrow M^+$ such that $\phi\varepsilon = 0$. A homomorphism $\phi : \mathbf{P} \rightarrow M^+$ is uniquely defined by an element $u \in M$ such that $\phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}$. Namely, $\phi \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} au \\ cu \end{pmatrix}$. Obviously, $\phi\varepsilon = 0$ if and only if $\mathbf{C}u = 0$, i.e., $u \in \text{ann}_M \mathbf{C}$.

(2) $\text{Ext}_{\tilde{\mathbf{A}}}^1(\bar{\mathbf{A}}, M^+) \simeq \text{Hom}_{\tilde{\mathbf{A}}}(\mathbf{C}^+, M^+) / \text{Hom}_{\tilde{\mathbf{A}}}(\mathbf{P}, M^+)\varepsilon$. As the functor G^* is fully faithful, $\text{Hom}_{\tilde{\mathbf{A}}}(\mathbf{C}^+, M^+) \simeq \text{Hom}_{\mathbf{H}}(\mathbf{C}, M)$. Namely, $\psi : \mathbf{C} \rightarrow M$ induces

$\psi^+ : \mathbf{C}^+ \rightarrow M^+$ mapping $\begin{pmatrix} a \\ b \end{pmatrix}$ to $\begin{pmatrix} \psi(a) \\ \psi(b) \end{pmatrix}$. Let $\phi : \mathbf{P} \rightarrow M^+$ correspond, as above, to an element $u \in M$. Then,

$$\phi \varepsilon \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} au \\ cu \end{pmatrix},$$

so it equals μ_u , and $\text{Hom}_{\bar{A}}(\mathbf{P}, M^+) \varepsilon$ is identified with $M / \text{ann}_M \mathbf{C}$ embedded into $\text{Hom}_{\mathbf{H}}(\mathbf{C}, M)$ as above. \square

Actually, in our case an object E from the category $\text{El}^{(0)}(\mathcal{U})$ (therefore, also an object from $\mathcal{D}^b(\bar{A})$) is given by the vertices and solid arrows of a diagram

$$\begin{array}{ccccccc}
 & \alpha_n & & \alpha_{n+1} & & \alpha_{n+2} & & \alpha_{n+3} & & \dots \\
 & \curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright & & \\
 A_n & & & A_{n+1} & & A_{n+2} & & A_{n+3} & & \dots \\
 \downarrow \mu_n & \searrow \eta_n & & \downarrow \mu_{n+1} & \searrow \eta_{n+1} & & \downarrow \mu_{n+2} & \searrow \eta_{n+3} & & \downarrow \mu_{n+3} \\
 M_n & \xrightarrow{\beta_n} & M_{n+1} & \xrightarrow{\beta_{n+1}} & M_{n+2} & \xrightarrow{\beta_{n+2}} & M_{n+3} & \xrightarrow{\beta_{n+3}} & \dots & \\
 \curvearrowleft \gamma_n & & \curvearrowleft \gamma_{n+1} & & \curvearrowleft \gamma_{n+2} & & \curvearrowleft \gamma_{n+3} & & &
 \end{array}$$

(of arbitrary length), where A_i are \bar{A} -modules, M_i are \mathbf{H} -modules, μ_i belongs to $\text{Hom}_{\bar{A}}(A_i, M_i^+)$ and η_i belongs to $\text{Ext}_{\bar{A}}^1(A_i, M_{i-1}^+)$. A morphism between E and E' is given by the dotted arrows, where

$$\begin{aligned}
 \alpha_i &\in \text{Hom}_{\bar{A}}(A_i, A'_i) \simeq \text{Hom}_{\bar{A}}(A_i, A'_i), \\
 \gamma_i &\in \text{Hom}_{\mathbf{H}}(M_i, M'_i) \simeq \text{Hom}_{\bar{A}}(M_i^+, (M'_i)^+), \\
 \beta_i &\in \text{Ext}_{\mathbf{H}}^1(M_i, M'_{i+1}) \simeq \text{Ext}_{\bar{A}}^1(M_i^+, (M'_{i+1})^+).
 \end{aligned}$$

These morphisms must satisfy the relations

$$\mu'_i \alpha_i = \gamma_i \mu_i, \quad \eta'_i \alpha_i = \gamma_{i+1} \eta_i + \beta_i \mu_i.$$

6. Partial tilting for Backström pairs

Let $\mathbf{H} \subseteq \mathbf{A}$ be a Backström pair. Consider the ring \mathbf{B} of triangular matrices of the form

$$\mathbf{B} = \begin{pmatrix} \bar{A} & \bar{H} \\ 0 & \mathbf{H} \end{pmatrix}.$$

Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and let $B_1 = \mathbf{B}e_1$ and $B_2 = \mathbf{B}e_2$ be projective \mathbf{B} -modules given by the first and the second column of \mathbf{B} , i.e.,

$$B_1 = \begin{pmatrix} \bar{A} \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \bar{H} \\ \mathbf{H} \end{pmatrix}.$$

A \mathbf{B} -module M is defined by a triple $(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix} \chi_M)$, where $M_1 = e_1 M$ is an $\bar{\mathbf{A}}$ -module, $M_2 = e_2 M$ is an \mathbf{H} -module and $\chi_M : M_2 \rightarrow M_1$ is an \mathbf{A} -homomorphism such that $\text{Ker } \chi_M \supseteq \mathbf{C}M_2$ (it is necessary since $\mathbf{C}M_1 = 0$). Namely, χ_M is multiplication by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We write an element $u \in M$ as a column $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where $u_1 = e_1 u$, $u_2 = e_2 u$. Then,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} au_1 + \chi_M(bu_2) \\ cu_2 \end{pmatrix}.$$

A homomorphism $\alpha : M \rightarrow N$ is defined by two homomorphisms $\alpha_1 : M_1 \rightarrow N_1$ and $\alpha_2 : M_2 \rightarrow N_2$ such that $\alpha_1 \chi_M = \chi_N \alpha_2$. We write $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$.

Proposition 6.1. *We have $\text{l.gl.dim } \mathbf{B} = \max(\text{l.gl.dim } \mathbf{H}, \text{w.dim } \bar{\mathbf{H}}_{\mathbf{H}} + 1)$.*

In particular, if \mathbf{H} is left hereditary and $\bar{\mathbf{H}}$ is not flat as a right \mathbf{H} -module, then $\text{l.gl.dim } \mathbf{B} = 2$.

Proof. [Palmér and Roos 1973, Theorem 5] shows that $\text{l.gl.dim } \mathbf{B} \leq n$ if and only if

$$\text{l.gl.dim } \mathbf{H} \leq n \text{ and } \mathbb{R}^n \text{Hom}_{\bar{\mathbf{A}}}(\bar{\mathbf{H}} \otimes_{\mathbf{H}} -, -) = 0.$$

As the ring $\bar{\mathbf{A}}$ is semisimple,

$$\mathbb{R}^n \text{Hom}_{\bar{\mathbf{A}}}(\bar{\mathbf{H}} \otimes_{\mathbf{H}} -, -) = \text{Hom}_{\bar{\mathbf{A}}}(\text{Tor}_n^{\mathbf{H}}(\bar{\mathbf{H}}, -), -).$$

This implies the first assertion. The second is obvious, since $\text{Tor}_1^{\mathbf{H}}(\bar{\mathbf{H}}, -) = 0$ if and only if $\bar{\mathbf{H}}$ is flat as a right \mathbf{H} -module. \square

We denote by R the \mathbf{B} -module given by the triple $(\begin{smallmatrix} \mathbf{H}/\mathbf{A} \\ \mathbf{H} \end{smallmatrix} \pi)$, where $\pi : \mathbf{H} \rightarrow \mathbf{H}/\mathbf{A}$ is the natural surjection.

Proposition 6.2. (1) $\text{End}_{\mathbf{B}} R \simeq \mathbf{A}^{\text{op}}$.

(2) $\text{pr.dim}_{\mathbf{B}} R = 1$.

(3) $\text{Ext}_{\mathbf{B}}^1(R, R) = 0$.

Recall that conditions (2) and (3) mean that R is a partial tilting \mathbf{B} -module.

Proof. The minimal projective resolution of R is

$$0 \rightarrow B_1 \xrightarrow{\varepsilon} B_2 \rightarrow R \rightarrow 0,$$

where ε is the embedding, which gives (2). Any endomorphism γ of R induces a commutative diagram:

$$\begin{array}{ccc} B_1 & \xrightarrow{\varepsilon} & B_2 \\ \gamma_1 \downarrow & & \downarrow \gamma_2 \\ B_1 & \xrightarrow{\varepsilon} & B_2 \end{array}$$

As $\text{End}_{\mathbf{B}} B_2 \simeq \mathbf{H}^{\text{op}}$, γ_2 is given by multiplication with an element $h \in \mathbf{H}$ on the right. If there is a commutative diagram as above, necessarily $h \in \mathbf{A}$, which proves (1).

Finally, a homomorphism $\alpha : B_1 \rightarrow R$ maps the generator $\binom{1}{0}$ of B_1 to an element $\binom{\bar{h}}{0} \in R$. If h is a preimage of \bar{h} in \mathbf{H} , then α extends to the homomorphism $B_2 \rightarrow R$ that maps the generator $\binom{0}{1}$ of B_2 to $\binom{0}{h} \in R$. This implies (3). \square

Now Theorem 1.1 applied to the module R gives the following result:

Theorem 6.3. (1) *The functor $F = \mathbb{R}\text{Hom}(R, -)$ induces an equivalence*

$$\text{Tri}(R) \xrightarrow{\sim} \mathcal{D}(\mathbf{A}).$$

(2) *Ker F consists of complexes C such that the map $\chi_{H^k(C)}$ is bijective for all k .*

(3) *There is a recollement diagram*

$$\text{Ker } F \begin{array}{c} \xleftarrow{I^*} \\ \xrightarrow{I} \\ \xleftarrow{I'} \end{array} \mathcal{D}(\mathbf{B}) \begin{array}{c} \xleftarrow{F^*} \\ \xrightarrow{F} \\ \xleftarrow{F'} \end{array} \mathcal{D}(\mathbf{A}).$$

Actually, claim (2) means that a complex C is in $\text{Ker } F$ if and only if its cohomologies are direct sums of \mathbf{B} -modules of the form $\binom{U}{U} 1_U$, where U is a simple $\bar{\mathbf{H}}$ -module.

F is a partial tilting functor in the sense of Corollary 1.2.

Proof. (1) and (3) follow from Proposition 6.2 and Theorem 1.1, since the complex $P : 0 \rightarrow B_1 \xrightarrow{\varepsilon} B_2 \rightarrow 0$ is perfect, hence compact, and isomorphic to R in $\mathcal{D}(\mathcal{B})$. To find $\text{Ker } F$, consider a complex

$$C : \cdots \rightarrow C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \rightarrow \cdots,$$

where C^k is defined by a triple $\binom{C_1^k}{C_2^k} \chi_k$ and $d^k = \binom{d_1^k}{d_2^k}$, where $d_1^k \chi_k = \chi_{k+1} d_2^k$ for all k . Note that $C_i = (C_i^k, d_i^k)$ ($i = 1, 2$) are complexes, (χ_k) is a homomorphism of complexes and $H^k(C) = \binom{H^k(C_1)}{H^k(C_2)} \bar{\chi}_k$, where $\bar{\chi}_k = \chi_{H^k(C)}$ is induced by χ_k . A homomorphism $P \rightarrow C[k]$ is a pair of homomorphisms $\alpha : B_2 \rightarrow C^k$, $\beta : B_1 \rightarrow C^{k-1}$ such that $\alpha_1 \pi = \chi_k \alpha_2$, $\beta_2 = 0$, $d_i^k \alpha_i = 0$ ($i = 1, 2$) and $d^{k-1} \beta_1 = \alpha_1|_{\bar{\mathbf{A}}}$. Let $\alpha_2(1) = x \in C_2^k$ and $\beta_1(1) = y \in C_1^{k-1}$. These values completely define α and β . The conditions for α and β mean that $d_2^k x = 0$ and $d^{k-1} y = \chi_k x$.

This morphism is homotopic to zero if and only if there are maps $\sigma : B_2 \rightarrow C^{k-1}$ and $\tau : B_1 \rightarrow C^{k-2}$ such that $\alpha = d^{k-1} \sigma$ and $\beta = \sigma \varepsilon + d^{k-2} \tau$. Again, σ is defined by the element $z = \sigma_2(1) \in C_2^{k-1}$ and τ is defined by the element $t = \tau_1(1) \in C_1^{k-2}$. Then, the conditions for α and β mean that $x = d_2^{k-1} z$ and $y = \chi_{k-1} z + d_1^{k-2} t$.

Suppose that any homomorphism $P \rightarrow C[k]$ is homotopic to zero. Let \bar{x} in $H^k(C^2)$ be such that $\bar{\chi}_k(\bar{x}) = 0$ and $x \in \text{Ker } d_2^k$ be a representative of \bar{x} . Then, $\chi_k(x) = d_1^{k-1} y$ for some $y \in C^{k-1}$, so the pair (x, y) defines a homomorphism $P \rightarrow C[k]$. Therefore, there must be $z \in C_2^{k-1}$ such that $x = d^{k-1} z$; thus $\bar{x} = 0$ and $\bar{\chi}_k$ is injective. Let now $\bar{y} \in H^{k-1}(C_2)$ and $y \in C_2^{k-1}$ be its representative. Then, the pair $(0, y)$ defines a homomorphism $P \rightarrow C[k]$, so there must be elements $z \in C_2^{k-1}$

and $t \in C_1^{k-2}$ such that $d_1^{k-1}z = 0$ and $y = \chi_{k-1}z + d_1^{k-2}t$. Hence, $\bar{y} = \bar{\chi}_{k-1}(\bar{z})$, so $\bar{\chi}_{k-1}$ is surjective. As this holds for all k , we have that all maps $\bar{\chi}_k$ are bijective.

On the contrary, suppose that all $\bar{\chi}_k$ are bijective. If a pair (x, y) defines a homomorphism $P \rightarrow C[k]$, then $\chi_k(x) = d_1^{k-1}y$, so $\bar{\chi}_k(x) = 0$. Therefore, $\bar{x} = 0$, i.e., $x = d_2^{k-1}z$ for some $z \in C_2^{k-1}$ and $\chi_k x = d_1^{k-1}\chi_{k-1}z$. Then, $d_1^{k-1}(y - \chi_{k-1}z) = 0$, hence there is an element $z' \in C_2^{k-1}$ such that $d_2^{k-1}z' = 0$ and the cohomology class of $y - \chi_{k-1}z$ equals $\bar{\chi}_{k-1}\bar{z}'$, i.e., $y - \chi_{k-1}z = \chi_{k-1}z' + d_1^{k-2}t$ for some t . Then, $x = d_2^{k-1}(z + z')$ and $y = \chi_{k-1}(z + z') + d_1^{k-2}t$, so this homomorphism is homotopic to zero. \square

As usual, we identify the category $\mathbf{A}\text{-Mod}$ with the full subcategory of $\mathcal{D}(\mathbf{A})$ consisting of the complexes C concentrated in degree 0. The following result shows how the partial tilting functor F behaves with respect to modules:

Corollary 6.4. *Let a \mathbf{B} -module M be given by the triple $(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix} \chi_M)$.*

- (1) $FM \in \mathbf{A}\text{-Mod}$ if and only if χ_M is surjective. Namely, then $FM \simeq \text{Ker } \chi_M$.
- (2) $FM \in \mathbf{A}\text{-Mod}[1]$ if and only if χ_M is injective. Namely, then $FM \simeq \text{Cok } \chi_M[1]$.

Proof. Note that $\text{Hom}_{\mathbf{B}}(B_1, M) \simeq M_1$, $\text{Hom}_{\mathbf{B}}(B_2, M) \simeq M_2$ and if $\phi : B_2 \rightarrow M$ maps $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ x \end{pmatrix}$, then $\phi\varepsilon$ maps $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \chi_M(x) \\ 0 \end{pmatrix}$. Therefore, $\mathbb{R}\text{Hom}_{\mathbf{B}}(R, M)$ is the complex

$$0 \rightarrow M_2 \xrightarrow{\chi_M} M_1 \rightarrow 0,$$

which proves the claim. \square

Remark 6.5. There are several derived equivalences related to $\tilde{\mathbf{A}}$.

(1) If \mathbf{A} is a Backström order, it is known (see [Burban et al. 2017]) that the complex $T = B_1[1] \oplus \mathbf{H}^+$, where $B_1 = \begin{pmatrix} \bar{\mathbf{A}} \\ 0 \end{pmatrix}$, is a tilting complex for $\tilde{\mathbf{A}}$ and $(\text{End}_{\mathcal{D}(\tilde{\mathbf{A}})} T)^{\text{op}} \simeq \mathbf{B}$, hence $\tilde{\mathbf{A}}$ is derived equivalent to \mathbf{B} . Nevertheless, in the general situation of Backström rings (even of Backström algebras) this is not so. First of all, $\text{Hom}_{\tilde{\mathbf{A}}}(B_1, \mathbf{H}^+) \simeq \text{ann}_{\mathbf{H}} \mathbf{C}$, so it can happen that $\text{Hom}_{\mathcal{D}(\tilde{\mathbf{A}})}(T, T[1]) \neq 0$. This is so, for instance, for the pair $(T(n, \mathbb{k}), \text{UT}(n, \mathbb{k}))$ from Equation (2-3) (4), since in this case the matrix unit e_{nn} belongs to $\text{ann}_{\mathbf{H}} \mathbf{C}$. This is also so for Equation (2-3) (5). Moreover, even if $\text{ann}_{\mathbf{H}} \mathbf{C} = 0$, one can see that $\bar{\mathbf{H}}' = \text{Ext}_{\tilde{\mathbf{A}}}^1(B_1, \mathbf{H}^+) \simeq C^{-1}/{}_C\mathbf{H}$, where $C^{-1} = \text{Hom}_{\mathbf{H}}(\mathbf{C}, \mathbf{H})$ and ${}_C\mathbf{H} = \mathbf{H}/\text{ann}_{\mathbf{H}} \mathbf{C}$ is naturally embedded into C^{-1} . Therefore, in this case,

$$(\text{End}_{\mathcal{D}(\tilde{\mathbf{A}})} T)^{\text{op}} \simeq \mathbf{B}' = \begin{pmatrix} \bar{\mathbf{A}} & \bar{\mathbf{H}}' \\ 0 & \mathbf{H} \end{pmatrix},$$

which need not coincide with \mathbf{B} (see Example 6.6 below). If \mathbf{H} is a hereditary order, then $\text{ann}_{\mathbf{H}} \mathbf{C} = 0$ and $\bar{\mathbf{H}}' \simeq \bar{\mathbf{H}}$, hence $\mathbf{B}' \simeq \mathbf{B}$, in accordance with [Burban et al. 2017].

(2) On the other hand, set $T' = \begin{pmatrix} A & H/A \\ C & \bar{H} \end{pmatrix}$ considered as a left \tilde{A} -module. One can check it is a tilting module for \tilde{A} and

$$(\text{End}_{\mathcal{D}(\tilde{A})} T')^{\text{op}} \simeq \tilde{B} = \begin{pmatrix} A & H/A \\ 0 & \bar{A} \end{pmatrix},$$

hence \tilde{A} is derived equivalent to \tilde{B} . Unfortunately, this ring can be not so good from the homological point of view. At least, it is not better than A itself. Namely, as one can easily check,

$$\text{l.gl.dim } \tilde{B} = \max(\text{l.gl.dim } A, 1 + \text{pr.dim}_A(H/A)),$$

which is either $\text{l.gl.dim } A$ or (more often) $\text{l.gl.dim } A + 1$.

(3) One more observation: Consider the right \tilde{A} -modules $(\bar{A} \ 0)$ and $(C \ H)$. One can check that $T'' = (\bar{A} \ 0)[1] \oplus (C \ H)$ is a tilting complex for $\mathcal{D}(\tilde{A}^{\text{op}})$ and

$$\text{End}_{\mathcal{D}(\tilde{A}^{\text{op}})} T'' \simeq B'' = \begin{pmatrix} \bar{A} & 0 \\ \bar{H} & H \end{pmatrix},$$

hence \tilde{A}^{op} is derived equivalent to $(B'')^{\text{op}}$.

Note that the functor $P \mapsto \text{Hom}_{\mathbf{R}}(P, \mathbf{R})$ induces an exact duality

$$\text{Perf}(\mathbf{R}) \rightarrow \text{Perf}(\mathbf{R}^{\text{op}})$$

for any ring \mathbf{R} . Hence, $\text{Perf}(\tilde{A}) \simeq \text{Perf}(B')$.

Example 6.6. Let $H = \mathbb{T}(3, \mathbb{k})$ and $A = \{(a_{ij}) \in H \mid a_{11} = a_{22}\}$. Set $H_i = H e_{ii}$ and $U_i = H_i / \text{rad } H_i$. Then, $C = \{(a_{ij}) \in H \mid a_{11} = a_{22} = 0\}$, hence $\bar{H} = U_1 \oplus U_2$. On the other hand, $C = \text{rad } H_2 \oplus H_3 \simeq H_1 \oplus H_3$, so $C^{-1} = \text{Hom}_H(C, H)$ can be identified with the set of 3×2 matrices (b_{ij}) such that $b_{12} = b_{22} = 0$. One can check that ${}_C H$ is identified with the subset $\{(b_{ij}) \mid b_{11} = 0\} \subset C^{-1}$ and $\bar{H}' \simeq U_2 \not\simeq \bar{H}$ (even $\dim_{\mathbb{k}} \bar{H}' \neq \dim_{\mathbb{k}} \bar{H}$).

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YURIY DROZD
INSTITUTE OF MATHEMATICS
NATIONAL ACADEMY OF SCIENCES OF UKRAINE
KYIV
UKRAINE
y.a.drozd@gmail.com
drozd@imath.kiev.ua

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EDITORS

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University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Matthias Aschenbrenner
Fakultät für Mathematik
Universität Wien
Vienna, Austria
matthias.aschenbrenner@univie.ac.at

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Robert Lipshitz
Department of Mathematics
University of Oregon
Eugene, OR 97403
lipshitz@uoregon.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

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
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