We introduce Backström pairs and Backström rings, study their derived categories and construct for them a sort of categorical resolutions. For the latter we define the global dimension, construct a sort of semiorthogonal decomposition of the derived category and deduce that the derived dimension of a Backström ring is at most 2. Using this semiorthogonal decomposition, we define a description of the derived category as the category of elements of a special bimodule. We also construct a partial tilting for a Backström pair to a ring of triangular matrices and define the global dimension of the latter.

Introduction

Backström orders were introduced in [Ringel and Roggenkamp 1979], where it was shown that their representations are in correspondence with those of quivers or species. A special class of Backström orders are nodal orders, which appeared (without this name) in [Drozd 1990] as such pure noetherian algebras that the classification of their finitely generated modules is tame. In [Burban and Drozd 2004] tameness was also proved for the derived categories of nodal orders. Global analogues of nodal algebras, called nodal curves, were considered in [Burban and Drozd 2011; Drozd and Voloshyn 2012; Voloshyn and Drozd 2013]. Namely, in [Burban and Drozd 2011] a sort of tilting theory for such curves was developed, which related them to some quasihereditary finite dimensional algebras. In [Drozd and Voloshyn 2012] a criterion was found for a nodal curve to be tame with respect to the classification of vector bundles, and in [Voloshyn and Drozd 2013] it was proved that the same class of curves has tame derived categories. It was clear that the tilting theory of [Burban and Drozd 2011] can be extended to a general situation, namely, to Backström curves, i.e., noncommutative curves having Backström orders as their localizations. Nodal orders and related gentle algebras appear in studying mirror symmetry, see for instance, [Lekili and Polishchuk 2018]. Finite dimensional
analouges of nodal orders, called *nodal algebras*, were introduced in [Drozd and Zembyk 2013; Zembyk 2014]. In the latter paper their structure was completely described. In [Zembyk 2015] it was shown that certain important classes of algebras, such as gentle and skewed-gentle algebras, are nodal. In [Burban and Drozd 2017] a tilting theory was developed for nodal algebras, which was applied to the study of derived categories of gentle and skewed-gentle algebras.

This paper is devoted to a tilting theory for *Backström rings*, which are a straightforward generalization of Backström orders and algebras.

In Section 1, we propose a variant of partial tilting, which generalizes the technique of minors from [Burban et al. 2017].

In Section 2, we introduce *Backström pairs*, which are pairs of semiperfect rings \( H \supseteq A \) with a common radical; (piecewise) *Backström rings* are likewise introduced as those rings \( A \) that occur in (piecewise) Backström pairs with (piecewise) hereditary \( H \). We construct the *Auslander envelope* \( \tilde{A} \) of a Backström pair and calculate its global dimension. It turns out that this global dimension only depends on the global dimension of \( H \). In particular, Auslander envelopes for Backström rings are of global dimension at most 2.

In Section 3, we apply the tilting technique to show that the derived category of the algebra \( A \) is connected by a recollement with the derived category of its Auslander envelope. This implies that the derived dimension of \( A \) in the sense of [Rouquier 2008] is not greater than that of the Auslander envelope.

In Section 4, we consider a recollement between the derived categories of the algebra \( H \) and of the Auslander envelope. It is used to calculate the derived dimension of the Auslander envelope, thus obtaining an upper bound for the derived dimension of the algebra \( A \). In particular, we prove that the derived dimension of a Backström or piecewise Backström algebra is at most 2. Moreover, if \( A \) is a Backström or piecewise Backström algebra of Dynkin type, then either it is piecewise hereditary of Dynkin type, so der.dim \( A = 0 \), or else der.dim \( A = 1 \).

In Section 5, we establish an equivalence between the category \( \mathcal{D}(\tilde{A}) \) and a bimodule category. This gives a useful instrument for calculations in this derived category. (See, for instance, [Bekkert et al. 2003; Bekkert and Merklen 2003; Burban and Drozd 2004; 2006; 2017; Voloshyn and Drozd 2013].)

In Section 6, we consider another partial tilting for the Auslander envelope \( \tilde{A} \) of a Backström pair, relating its derived category by a recollement to the derived category of an algebra \( B \) of triangular matrices which looks simpler than the Auslander algebra. In this case, we calculate explicitly the global dimension of \( B \) and the kernel of the partial tilting functor

\[ F : \mathcal{D}(B) \to \mathcal{D}(A). \]
1. Partial tilting

Let $\mathcal{T}$ be a triangulated category, $\mathcal{R} \subseteq \text{Ob } \mathcal{T}$. We denote by $\text{Tri}(\mathcal{R})$ the smallest strictly full triangulated subcategory containing $\mathcal{R}$ that is closed under coproducts (this means that if a coproduct of objects from $\text{Tri}(\mathcal{R})$ exists in $\mathcal{T}$, it belongs to $\text{Tri}(\mathcal{R})$). For a DG-category $\mathcal{R}$ we denote by $\mathcal{D}(\mathcal{R})$ its derived category [Keller 1994]. The following result is a generalization of [Lunts 2010, Proposition 2.6]:

**Theorem 1.1.** Let $\mathcal{R}$ be a subset of the set of compact objects of $\text{Ob } \mathcal{D}(\mathcal{A})$, where $\mathcal{A}$ is a Grothendieck category. We consider the DG-category $\mathcal{R}$ with the set of objects $\mathcal{R}$ and the sets of morphisms $\mathcal{R}(T, R) = \mathbb{R}\text{Hom}(T, R)$. Define the functor $F: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{R}^{\text{op}})$ by mapping a complex $C$ to the DG-module $F(C) = \mathbb{R}\text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{R} C, C)$ restricted onto $\mathcal{R}$.

(1) The restriction of $F$ onto $\text{Tri}(\mathcal{R})$ is an equivalence $\text{Tri}(\mathcal{R}) \to \mathcal{D}(\mathcal{R}^{\text{op}})$.

(2) There is a recollement diagram in the sense of [Be˘ılinson et al. 1982, 1.4.3]

$$
\begin{array}{ccc}
\text{Ker } F & \xleftarrow{1^*} & \mathcal{D}(\mathcal{A}) \\
\downarrow{1} & & \downarrow{F} \\
\mathcal{D}(\mathcal{R}) & \xleftarrow{F^*} & \mathcal{D}(\mathcal{R}^{\text{op}})
\end{array}
$$

where $1$ is the embedding.\(^1\)

Recall that this means that the following conditions hold:

(a) $F$ and $I$ are exact.

(b) $FI = 0$.

(c) $F^*$ and $F^!$ are left and right adjoint functors to $F$, respectively.

(d) Both adjunction morphisms $\eta: \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})} \to FF^*$ and $\zeta: FF^! \to \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})}$ are isomorphisms.

(e) The same holds for the triple $(1, 1^*, 1^!)$.

(1-1) Recall that Condition 1.4.3.4 from [Be˘ılinson et al. 1982] is a consequence of the other ones; see [Neeman 2001, 9.2].)

If $\mathcal{R}$ generates $\mathcal{D}(\mathcal{A})$, we obtain an equivalence $\mathcal{D}(\mathcal{A}) \cong \mathcal{D}(\mathcal{R}^{\text{op}})$, as in [Lunts 2010]. If $\mathcal{R}$ consists of one object $R$, we obtain an equivalence $\text{Tri}(R) \cong \mathcal{D}(R^{\text{op}})$, where $R = \mathbb{R}\text{Hom}(R, R)$.

**Proof.** (1) We identify $\mathcal{D}(\mathcal{A})$ with the homotopy category $\mathcal{I}(\mathcal{A})$ of $K$-injective complexes, i.e., complexes $I$ such that $\text{Hom}(C, I)$ is acyclic for every acyclic complex $C$, and suppose that $\mathcal{R} \subseteq \mathcal{I}(\mathcal{A})$. Then, $\mathbb{R}\text{Hom}$ coincides with Hom within the category $\mathcal{I}(\mathcal{A})$; so, for $C \in \mathcal{I}(\mathcal{A})$, $FC = \text{Hom}_{\mathcal{I}(\mathcal{A})}(\mathcal{R}, C)$ restricted onto $\mathcal{R}$. The full subcategory of $\mathcal{I}(\mathcal{A})$ consisting of complexes $C$ such that the natural map $\text{Hom}_{\mathcal{I}(\mathcal{A})}(R, C) \to \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(FR, FC)$ is bijective for all $R \in \mathcal{R}$ contains $\mathcal{R}$, is strictly full, triangulated and closed under coproducts, since all objects from $\mathcal{R}$ are

\(^1\)Note that $\mathcal{R}$ is not necessarily recollement-defining in the sense of [Nicolás and Saorín 2009].
compact. Therefore, it contains Tri( şeklinde). Quite analogously, the full subcategory of complexes C such that the natural map $\text{Hom}_{\mathcal{F}(\mathcal{A})}(C, C') \to \text{Hom}_{\mathcal{A}}(FC, FC')$ is bijective for every $C' \in \text{Tri}( şeklinde)$ also contains Tri( şeklinde). Hence, the restriction of F onto Tri( şeklinde) is fully faithful. Moreover, as the functors $\text{Hom}_{\mathcal{R}}(-, R)$, where R runs through $\mathcal{R}$, generate $\mathcal{D}(\mathcal{R}^{\text{op}})$, the functor F is essentially surjective. Therefore, restricted to Tri( şeklinde), it gives an equivalence $\text{Tri}( şeklinde) \to \mathcal{D}(\mathcal{R})$.

(2) Note that $\mathcal{D}(\mathcal{R}^{\text{op}})$ is cocomplete and compactly generated, hence satisfies the Brown representability theorem [Neeman 2001, Theorem 8.3.3]. Therefore, it is true for Tri( şeklinde) too. Then, [Neeman 2001, Proposition 9.1.19] implies that a Bowsfield localization functor exists for $\text{Tri}( şeklinde) \subseteq \mathcal{D}(\mathcal{A})$ and [Neeman 2001, Proposition 9.1.18] implies that the embedding $E : \text{Tri}( şeklinde) \to \mathcal{D}(\mathcal{A})$ has a right adjoint $\Theta : \mathcal{D}(\mathcal{A}) \to \text{Tri}( şeklinde)$. Let $F' : \mathcal{D}(\mathcal{R}^{\text{op}}) \to \text{Tri}( şeklinde)$ be a quasi-inverse to the restriction of F onto Tri( şeklinde). In particular, $F'$ is a left adjoint to this restriction and the adjunction $FF' \to \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})}$ is an isomorphism. Then,

$$FC = \text{Hom}_{\mathcal{F}(\mathcal{A})}(-, C)|_{\mathcal{R}} \simeq \text{Hom}_{\mathcal{F}(\mathcal{A})}(-, \Theta C)|_{\mathcal{R}} = F\Theta C.$$  

Set $F^* = EF'$. Since $F'M \in \text{Tri}( şeklinde)$ for every $M \in \mathcal{D}(\mathcal{R}^{\text{op}})$,

$$\text{Hom}_{\mathcal{F}(\mathcal{A})}(F^*M, C) \simeq \text{Hom}_{\text{Tri}( şeklinde)}(F'M, \Theta C)$$

$$\simeq \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(M, F\Theta C) \simeq \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(M, FC),$$

for any $M \in \mathcal{D}(\mathcal{R}^{\text{op}})$ and $C \in I(\mathcal{A})$. Hence, $F^*$ is a left adjoint to F. If, moreover, $C \in \text{Tri}( şeklinde)$, we obtain

$$\text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(FF^*M, FC) \simeq \text{Hom}_{\mathcal{F}(\mathcal{A})}(F^*M, C) \simeq \text{Hom}_{\mathcal{D}(\mathcal{R}^{\text{op}})}(M, FC).$$

As F is essentially surjective, this implies that $\eta : FF^* \to \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})}$ is an isomorphism. As all objects from $\mathcal{R}$ are compact, F respects coproducts, hence has a right adjoint $F'$ [Neeman 2001, Theorem 8.4.4]. Now it follows from [Burban et al. 2017, Corollary 2.3] that $\zeta : FF' \to \text{Id}_{\mathcal{D}(\mathcal{R}^{\text{op}})}$ is an isomorphism and there is a recollement diagram (1-1).

Note that $\text{Im} F^* = \text{Tri}( şeklinde)$ by construction, but usually $\text{Im} F' \neq \text{Tri}( şeklinde)$, though it is equivalent to Tri( şeklinde).

**Corollary 1.2.** Under the conditions and notations of the preceding theorem, suppose that $\text{Hom}_{\mathcal{D}(\mathcal{A})}(R, T[m]) = 0$ for $R, T \in \mathcal{R}$ and $m \neq 0$. Then, the functor F induces an equivalence $\text{Tri}( R) \sim \mathcal{D}(\mathcal{R}^{\text{op}})$, where $\mathcal{R}$ is the category with the set of objects $\mathcal{R}$ and the sets of morphisms $\mathcal{R}(A, B) = \text{Hom}_{\mathcal{D}(\mathcal{A})}(A, B)$.

In this situation, we call the functor F a partial tilting functor.
2. Backström pairs

Recall from [Bass 1960; Lambek 1976] that a semiperfect ring is a ring $A$ such that $A/\text{rad } A$ is a semisimple artinian ring and idempotents can be lifted modulo $\text{rad } A$. Equivalently, as a left (or as a right) $A$-module, $A$ decomposes into a direct sum of modules with local endomorphism rings.

**Definition 2.1.** (1) A Backström pair is a pair of semiperfect rings $H \supseteq A$ such that $\text{rad } A = \text{rad } H$. We denote by $C(H, A)$ the conductor of $H$ in $A$: $C(H, A) = \{ \alpha \in A \mid H \alpha \subseteq A \} = \text{ann}(H/A)_A$ (the right subscript $A$ means that we consider $H/A$ as a right $A$-module). Obviously, $C(H, A) \supseteq \text{rad } A$, so both $A/C$ and $H/C$ are semisimple rings.

(2) We call a ring $A$ a (left) Backström ring (resp. piecewise Backström ring) if there is a Backström pair $H \supseteq A$, where the ring $H$ is left hereditary (resp. left piecewise hereditary [Happel 1988], i.e., derived equivalent to a left hereditary ring). If, moreover, both $A$ and $H$ are finite dimensional algebras over a field $k$, we call $A$ a Backström algebra (resp. piecewise Backström algebra).

**Remark 2.2.** If $e$ is an idempotent in $A$, then $\text{rad}(eAe) = e(\text{rad } A)e$, hence, if $H \supseteq A$ is a Backström pair, so is $eHe \supseteq eAe$. This implies that if $P$ is a finitely generated projective $A$-module, $A' = \text{End}_A P$ and $H' = \text{End}_H(H \otimes_A P)$, then $H' \supseteq A'$ is also a Backström pair. Note that if $H$ is left hereditary (or piecewise hereditary), so is $H'$, hence $A'$ is a Backström ring (piecewise Backström ring) whenever $A$ is. In particular, the notion of Backström (or piecewise Backström) ring is Morita invariant. Note also that if $H$ is left hereditary and noetherian, it is also right hereditary, so $A^{\text{op}}$ is also a Backström ring (piecewise Backström ring).

**Examples 2.3.** (1) An important example of Backström algebras are nodal algebras introduced in [Drozd and Zembyk 2013; Zembyk 2014]. By definition, they are finite dimensional algebras such that there is a Backström pair $H \supseteq A$, where $H$ is a hereditary algebra and $\text{length}_A(H \otimes_A U) \leq 2$ for every simple $A$-module $U$. Their structure was completely described in [Zembyk 2014].

(2) Recall that a $k$-algebra $A$ is called gentle [Assem and Skowroński 1987] if $A \cong k\Gamma/J$, where $\Gamma$ is a finite quiver (oriented graph) and $J$ is an ideal in the path algebra $k\Gamma$ such that $(J_+)^2 \supseteq J \supseteq (J_+)^k$ for some $k$, where $J_+$ is the ideal generated by all arrows, and the following conditions hold:

(a) For every vertex $i \in \text{Ver } \Gamma$ there are at most two arrows starting at $i$ and at most two arrows ending at $i$.

(b) If an arrow $a$ starts at $i$ (resp. ends at $i$) and arrows $b_1, b_2$ end at $i$ (resp. start at $i$), then either $ab_1 = 0$ or $ab_2 = 0$ (resp. either $b_1a = 0$ or $b_2a = 0$), but not both.
(c) The ideal $J$ is generated by products of arrows of the sort $ab$.

It is proved in [Zembyk 2015] that such algebras are nodal, hence Backström algebras. The same is true for skewed-gentle algebras [Geißand de la Peña 1999] obtained from gentle algebras by blowing up some vertices.

(3) Backström orders are orders $A$ over a discrete valuation ring such that there is a Backström pair $H \supseteq A$, where $H$ is a hereditary order. They were considered in [Ringel and Roggenkamp 1979].

(4) Let $H = T(n, \mathbb{k})$ be the ring of upper triangular $n \times n$ matrices over a field $\mathbb{k}$ and $A = UT(n, \mathbb{k})$ be its subring of unitriangular matrices $M$, i.e., such that all diagonal elements of $M$ are equal. Then, $H$ is hereditary and $\text{rad } H = \text{rad } A$, hence $A$ is a Backström algebra. In this case, $C(H, A) = \text{rad } A$.

(5) $\Lambda_n = \mathbb{k}[x_1, x_2, \ldots, x_n]/(x_1, x_2, \ldots, x_n)^2$ embeds into $H = \prod_{i=1}^n \mathbb{k}\Gamma_i$, where $\Gamma_i = \cdot x_i \cdot$ ($x_i$ maps to $a_i$). Obviously, under this embedding $\text{rad } \Lambda_n = \text{rad } H$, so $\Lambda_n$ is a Backström algebra.

We consider a fixed Backström pair $H \supseteq A$, set $r = \text{rad } A = \text{rad } H$ and denote by $C$ the conductor $C(H, A)$. Obviously, $C$ is a two-sided $A$-ideal and the biggest left $H$-ideal contained in $A$. Actually, it even turns out to be a two-sided $H$-ideal and its definition is left-right symmetric.

**Lemma 2.4.** Let $R \subseteq S$ be semisimple rings, $I = \{ \alpha \in R \mid S\alpha \subseteq R \}$. Then, $I$ is a two-sided $S$-ideal.

**Proof.** Obviously, $I$ is a left $S$-ideal and a two-sided $R$-ideal. As $R$ is semisimple, $I = Re$ for some central idempotent $e \in R$. Then, $Se \subseteq Re$, so $Se = Re = eR$ and $(1-e)Se = 0$. Hence, $eS(1-e)$ is a left ideal in $S$ and $(eS(1-e))^2 = 0$, so $eS(1-e) = 0$ and $I = Se = eS$ is also a right $S$-ideal. \qed

**Proposition 2.5.** $C$ is a two-sided $H$-ideal. It is the biggest $H$-ideal contained in $A$. Therefore, it coincides with the set $\{ \alpha \in A \mid \alpha H \subseteq A \}$ or with $\text{ann}_A(H/A)$ considered as a left $A$-module.

**Proof.** It follows from the preceding lemma applied to the rings $A/\text{rad } A$ and $H/\text{rad } H$. \qed

In what follows we assume that $A \neq H$, so $C \neq A$. To calculate $C$, we consider a decomposition $A = \bigoplus_{i=1}^m A_i$, where $A_i$ are indecomposable projective left $A$-modules. Arrange them so that $HA_i \neq A_i$ for $1 \leq i \leq r$ and $HA_i = A_i$ for $r < i \leq m$, and set $A^0 = \bigoplus_{i=1}^r A_i$, $H^0 = HA^0$ and $A^1 = \bigoplus_{i=r+1}^m A_i = HA^1$. Then, $A = A^0 \oplus A^1$ and $H = H^0 \oplus A^1$ (possibly, $r = m$, so $A^0 = A$ and $H^0 = H$). Let $A^0 = Ae_0$ and $A^1 = Ae_1$, where $e_0$ and $e_1$ are orthogonal idempotents and $e_0 + e_1 = 1$. Set $A^0_b = e_b Ae_a$ and $H^0_b = e_b H e_a$, where $a, b \in \{0, 1\}$. Note that $A^1_b = H^1_b$ and $A^0_1 = H^0_1$. As $A^0$ and $A^1$ have no isomorphic direct summands,
$A^a_b \subseteq \text{rad } A$ if $a \neq b$. Hence, if we set $\tau^a = \text{rad } A^a (a = 0, 1)$ and consider the Pierce decomposition of the ring $A$

$$A = \begin{pmatrix} A^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix},$$

the Pierce decomposition of the ideal $\tau$ becomes

$$\tau = \begin{pmatrix} \tau^0_0 & A^1_0 \\ A^0_1 & \tau^1_1 \end{pmatrix},$$

where $\tau^a_a = \text{rad } A^a_a, \ a = 0, 1$. This implies that $H^0$ and $H^1$ have no isomorphic direct summands, the Pierce decomposition of $H$ is

$$H = \begin{pmatrix} H^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix},$$

and $\tau^0_0 = \text{rad } H^0_0$. Now, one easily sees that an element $a = (\alpha \beta) \in C$ if and only if $H^0\alpha \subseteq A^0$. We claim that in that case $H^0\alpha \subseteq \text{rad } A^0$. Otherwise $H^0\alpha$ contains an idempotent, hence a direct summand of $A^0$, which is isomorphic to some $A_i$ with $1 \leq i \leq r$. This is impossible, since $HA_i \neq A_i$. Therefore, $\alpha \in \tau^0_0$ and we obtain the following result:

**Proposition 2.6.** The Pierce decomposition of the ideal $C$ is

$$C = \begin{pmatrix} \tau^0_0 & A^1_0 \\ A^0_1 & A^1_1 \end{pmatrix}.$$
We also define $\tilde{H}$ as the ring of $4 \times 4$ matrices of the form

$$\tilde{H} = \begin{pmatrix} H & H \\ C & H \end{pmatrix} \quad \text{or} \quad \tilde{H} = \begin{pmatrix} H_0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \\ v_0^0 & A_1^0 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_0^0 & A_1^1 \end{pmatrix}.$$  

Obviously, $\text{rad} \tilde{H} = \text{rad} \tilde{A}$, so $\tilde{H} \supseteq \tilde{A}$ is also a Backström pair. $\tilde{A}$ is left noetherian if and only if $A$ is left noetherian and $H$ is finitely generated as a left $A$-module.

In the noetherian case one can calculate the global dimensions of $\tilde{A}$ and $\tilde{H}$. It turns out that it only depends on $H$.

**Theorem 2.8.** Suppose that either $A$ (hence also $H$) is left perfect or $A$ is left noetherian and $H$ is finitely generated as a left $A$-module (hence also left noetherian). Then

$$\text{l.gl.dim } \tilde{A} = 1 + \max(1 + \text{pr.dim}_H v^0, \text{pr.dim}_H v^1)$$

$$= \begin{cases} 1 + \text{l.gl.dim } H & \text{if } \text{pr.dim}_H v^0 \geq \text{pr.dim}_H v^1, \\ \text{l.gl.dim } H & \text{if } \text{pr.dim}_H v^0 < \text{pr.dim}_H v^1 \end{cases}$$

and

$$\text{l.gl.dim } \tilde{H} = \text{l.gl.dim } H,$$

where we set $\text{pr.dim} 0 = -1$. In particular, if $A$ is a Backström ring, so is $\tilde{A}$, and if $A$ is not left hereditary, then $\text{l.gl.dim } \tilde{A} = 2$.  

For instance, this is the case for nodal (in particular, gentle or skewed-gentle) algebras (Examples 2.3).

**Proof.** Under these conditions $\tilde{A}$ and $\tilde{H}$ are either left perfect or left noetherian. We recall that if a ring $\Lambda$ is left perfect or left noetherian and semiperfect, then $\text{l.gl.dim } \Lambda = \text{pr.dim}_\Lambda (\Lambda/\text{rad } \Lambda) = 1 + \text{pr.dim}_\Lambda \text{ rad } \Lambda$. The $4 \times 4$ matrix presentation (2-1) of $\tilde{A}$ implies that the corresponding presentation of $\text{rad } \tilde{A}$ is

$$\text{rad } \tilde{A} = \begin{pmatrix} v_0^0 & A_0^1 & H_0^0 & A_0^1 \\ A_1^0 & A_1^1 & A_1^0 & A_1^1 \\ v_0^0 & A_1^0 & v_0^0 & A_1^0 \\ A_1^0 & A_1^1 & A_0^0 & A_1^1 \end{pmatrix}.$$  

An $\tilde{A}$-module $M$ is given by a quadruple $(M', M'', \phi, \psi)$, where $M'$ is an $A$-module, $M''$ is an $H$-module, $\psi : M'' \to M'$ is a homomorphism of $A$-modules and $\phi : C \otimes_A M' \to M''$ is a homomorphism of $H$-modules. Namely, $M' = e'M$, $M'' = e''M$, where $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\psi(m'') = \begin{pmatrix} 0 & 1 \end{pmatrix} m''$ and $\phi(c \otimes m') = \begin{pmatrix} 0 & 0 \end{pmatrix} m'$.

---

2Note that if $\tilde{A}$ is left hereditary, so is $A = e'\tilde{A}e'$ [Sandomierski 1969].
We frequently write \( M = (M')_{M''} \), not mentioning \( \phi \) and \( \psi \). For an \( H \)-module \( N \) we define the \( \tilde{A} \)-module \( N^+ = (N)_{N} \). Then, \( N \mapsto N^+ \) is an exact functor mapping projective modules to projective ones, since \( H^+ = (H)_{H} \) is a projective \( \tilde{A} \)-module.

We denote by \( L^i \) and by \( R^i \) the \( i \)-th column of the presentations (2-1) and (2-2), respectively. Then, \( R^1 = (\epsilon^0)^+ \) and \( R^2 = R^4 = (\epsilon^1)^+ \), where \( \epsilon^a = \epsilon e_a \). If

\[
\cdots \rightarrow F_k \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0
\]

is a minimal projective resolution of an \( H \)-module \( N \),

\[
\cdots \rightarrow F_k^+ \rightarrow \cdots \rightarrow F_1^+ \rightarrow F_0^+ \rightarrow N^+ \rightarrow 0
\]

is a minimal projective resolution of \( N^+ \), so \( \text{pr.dim}_{\tilde{A}} N^+ = \text{pr.dim}_H N \). In particular, \( \text{pr.dim}_{\tilde{A}} R^1 = \text{pr.dim}_H \epsilon^0 \) and \( \text{pr.dim}_{\tilde{A}} R^2 = \text{pr.dim}_H \epsilon^1 \). For the module \( R^3 \) we have an exact sequence

\[
(2-3) \quad 0 \rightarrow (\epsilon^0)^+ \rightarrow R^3 \rightarrow \left( \frac{H^0/\epsilon^0}{0} \right) \rightarrow 0.
\]

Note that \( H^0/\epsilon^0 \) is a semisimple \( A \)-module and \( e_1(H^0/\epsilon^0) = 0 \), hence it contains the same simple direct summands as \( A^0/\epsilon^0 \). The same is true for

\[
\left( \frac{H^0/\epsilon^0}{0} \right) \quad \text{and} \quad \left( \frac{A^0/\epsilon^0}{0} \right) = L^1/R^1.
\]

Hence,

\[
\text{pr.dim}_{\tilde{A}} \left( \frac{H^0/\epsilon^0}{0} \right) = 1 + \text{pr.dim}_{\tilde{A}} R^1 = 1 + \text{pr.dim}_H \epsilon^0.
\]

Therefore, the exact sequence (2-3) shows that \( \text{pr.dim}_{\tilde{A}} R^3 = 1 + \text{pr.dim}_H \epsilon^0 \) and 

\[
\text{pr.dim}_{\tilde{A}} \text{rad } \tilde{A} = \max(1 + \text{pr.dim}_H \epsilon^0, \text{pr.dim}_H \epsilon^1),
\]

which gives the necessary result for \( \tilde{A} \). On the other hand, \( R^3 \) is a projective \( \tilde{H} \)-module, whence \( \text{l.gl.dim } \tilde{H} = \text{l.gl.dim } H \).

\[\square\]

3. The structure of derived categories

In what follows we denote by \( \mathcal{D}(A) \) the derived category \( \mathcal{D}(A-\text{Mod}) \). We denote by \( \mathcal{D}_f(A) \) the full subcategory of \( \mathcal{D}(A) \) consisting of complexes quasi-isomorphic to complexes of finitely generated projective modules. If \( A \) is left noetherian, it coincides with the derived category of the category \( A\-\text{mod} \) of finitely generated \( A \)-modules. We also use the usual superscripts \( +, -, b \). By Perf(\( A \)) we denote the full subcategory of perfect complexes from \( \mathcal{D}(A) \), i.e., complexes quasi-isomorphic to finite complexes of finitely generated projective modules. It coincides with the full subcategory of compact objects in \( \mathcal{D}(A) \) [Rouquier 2008]. If \( A \) is left
noetherian, an $A$-module $M$ belongs to $\text{Perf}(A)$ if and only if it is finitely generated and of finite projective dimension.

There are close relations between the categories $\mathcal{D}(A)$, $\mathcal{D}(H)$ and $\mathcal{D}(\tilde{A})$ based on the following construction [Burban et al. 2017]:

Let $P = (\frac{A}{C})$. It is a projective $\tilde{A}$-module and $\text{End} \; P \simeq A^{\text{op}}$, so it can be considered as a right $A$-module. Consider the functors

$$F = \text{Hom}_A(P, -) \simeq P^\vee \otimes_A - : \tilde{A}\text{-Mod} \to A\text{-Mod},$$

$$F^* = P \otimes_A - : A\text{-Mod} \to \tilde{A}\text{-Mod},$$

$$F'_* = \text{Hom}_A(P^\vee, -) : A\text{-Mod} \to \tilde{A}\text{-Mod},$$

where $P^\vee = \text{Hom}_A(P, \tilde{A}) \simeq (A \; H)$ is the dual right projective $\tilde{A}$-module, the functor $F$ is exact, $F^*$ is its left adjoint and $F'_*$ is its right adjoint. Moreover, the adjunction morphisms $F F^* \to \text{Id}_{A\text{-Mod}}$ and $\text{Id}_{A\text{-Mod}} \to F F'_*$ are isomorphisms [Burban et al. 2017, Theorem 4.3]. The functors $F^*$ and $F'_*$ are fully faithful and $F$ is essentially surjective, i.e., every $A$-module is isomorphic to $FM$ for some $\tilde{A}$-module $M$. $\text{Ker} \; F$ is a Serre subcategory of $\tilde{A}\text{-Mod}$ equivalent to $\overline{H}\text{-Mod}$, where $\overline{H} = H/C \simeq \tilde{A}/(\frac{A \; H}{C \; C})$. The embedding functor $l : \text{Ker} \; F \to \tilde{A}\text{-Mod}$ has a left adjoint $l^*$ and a right adjoint $l'$ and we obtain a recollement diagram

$$\text{Ker} \; F \xleftarrow{l^*} \tilde{A}\text{-Mod} \xrightarrow{F^*} A\text{-Mod}. $$

As the functor $F$ is exact, it extends to the functor between the derived categories $DF : \mathcal{D}(\tilde{A}) \to \mathcal{D}(A)$ acting on complexes componentwise. The derived functors $LF^*$ and $RF_!$ are its left and right adjoints, respectively, the adjunction morphisms $\text{Id}_{\mathcal{D}(A)} \to DF \cdot LF^*$ and $DF \cdot LF^* \to \text{Id}_{\mathcal{D}(A)}$ are again isomorphisms and we have a recollement diagram

$$\text{Ker} \; DF \xleftarrow{DF_!} \mathcal{D}(\tilde{A}) \xrightarrow{DF^*} \mathcal{D}(A).$$

(It also follows from Corollary 1.2.) Here $\text{Ker} \; DF = \mathcal{D}(\overline{H})(\tilde{A})$, the full subcategory of complexes whose cohomologies are $\overline{H}$-modules, i.e., are annihilated by the ideal $(\frac{A \; H}{C \; C})$. Note that, as a rule, it is not equivalent to $\mathcal{D}(\overline{H})$. From the definition of $F$ it follows that

$$\text{Ker} \; DF = P^\perp = \{ C \in \mathcal{D}(\tilde{A}) \mid \text{Hom}_{\mathcal{D}(\tilde{A})}(P, C[k]) = 0 \text{ for all } k \}. $$

Obviously, $DF$ maps $\mathcal{D}^\sigma(\tilde{A})$ to $D^\sigma(A)$ for $\sigma \in \{+, -, b\}$, $LF^*$ maps $\mathcal{D}^-(A)$ to $\mathcal{D}^-(\tilde{A})$ and $RF_!$ maps $\mathcal{D}^+(A)$ to $\mathcal{D}^+(\tilde{A})$. If $\tilde{A}$ is left noetherian, $DF$ maps $\mathcal{D}_f(\tilde{A})$ to $\mathcal{D}_f(A)$ and $LF^*$ maps $\mathcal{D}_f(A)$ to $\mathcal{D}_f(\tilde{A})$. Finally, both $DF$ and $LF^*$ have right adjoints, hence map compact objects (i.e., perfect complexes) to compact ones.
On the contrary, usually $\mathcal{L}F^*$ does not map $\mathcal{D}^b(A)$ to $\mathcal{D}^b(\tilde{A})$. For instance, it is definitely so if $\text{l.gl.dim} \, \tilde{A} < \infty$ while $\text{l.gl.dim} \, A = \infty$ as in Examples 2.3 (4, 5). If $\text{l.gl.dim} \, H$ is finite, so is $\text{l.gl.dim} \, \tilde{A}$, thus this recollement can be considered as a sort of categorical resolution of the category $\mathcal{D}(A)$. In any case, it is useful for studying the categories $A\text{-Mod}$ and $\mathcal{D}(A)$ if we know the structure of the categories $\tilde{A}\text{-Mod}$ and $\mathcal{D}(\tilde{A})$. For instance, it is so if we are interested in the derived dimension, i.e., the dimension of the category $\mathcal{D}^b_f(A)$ in the sense of [Rouquier 2008].

**Definition 3.1.** Let $\mathcal{T}$ be a triangular category and $\mathcal{M}$ be a set of objects from $\mathcal{T}$.

1. We denote by $\langle \mathcal{M} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{M}$ and closed under direct sums, direct summands and shifts (not closed under cones, so not a triangulated subcategory).
2. If $\mathcal{N}$ is another subset of $\mathcal{T}$, we denote by $\mathcal{M} \uparrow \mathcal{N}$ the set of objects $C$ from $\mathcal{T}$ such that there is an exact triangle $A \to B \to C \to$, where $A \in \mathcal{M}$, $B \in \mathcal{N}$.
3. We define $\langle \mathcal{M} \rangle_k$ recursively, setting $\langle \mathcal{M} \rangle_1 = \langle \mathcal{M} \rangle$ and $\langle \mathcal{M} \rangle_{k+1} = \langle \langle \mathcal{M} \rangle \uparrow \langle \mathcal{M} \rangle_k \rangle$.
4. The dimension $\text{dim} \, \mathcal{T}$ of $\mathcal{T}$ is the smallest $k$ such that there is a finite set of objects $\mathcal{M}$ such that $\langle \mathcal{M} \rangle_{k+1} = \mathcal{T}$ (if it exists). We call the dimension $\text{dim} \, \mathcal{D}^b_f(A)$ the derived dimension of the ring $A$ and denote it by $\text{der.dim} \, A$.

As the functor $F$ is exact and essentially surjective, the next result is evident:

**Proposition 3.2.** We have $\text{der.dim} \, A \leq \text{der.dim} \, \tilde{A}$. Namely, if $\mathcal{D}^b_f(\tilde{A}) = \langle \mathcal{M} \rangle_{k+1}$, then $\mathcal{D}^b_f(A) = \langle \mathcal{D}(\mathcal{M}) \rangle_{k+1}$.

### 4. Semiorthogonal decomposition

There is another recollement diagram for $\mathcal{D}(\tilde{A})$ related to the projective module $Q = (\begin{smallmatrix} H \\ H \end{smallmatrix})$ with $\text{End} \, Q \simeq H^{\text{op}}$. Namely, we set

$$G = \text{Hom}_{\tilde{A}}(Q, -) \simeq Q^\vee \otimes_{\tilde{A}} - : \tilde{A}\text{-Mod} \to H\text{-Mod},$$

$$G^* = Q \otimes_H - : H\text{-Mod} \to \tilde{A}\text{-Mod},$$

$$G^! = \text{Hom}_H(Q^\vee, -) : H\text{-Mod} \to \tilde{A}\text{-Mod},$$

where $Q^\vee = \text{Hom}_{\tilde{A}}(Q, \tilde{A}) \simeq (\begin{smallmatrix} C & H \end{smallmatrix})$,

$$\mathcal{DG} : \mathcal{D}(\tilde{A}) \to \mathcal{D}(H) \text{ is } G \text{ applied componentwise},$$

$$\mathcal{LG}^* : \mathcal{D}(A) \to \mathcal{D}(\tilde{A}) \text{ is the left adjoint of } \mathcal{DG},$$

$$\mathcal{RG}^! : \mathcal{D}(A) \to \mathcal{D}(\tilde{A}) \text{ is the right adjoint of } \mathcal{DG}.$$
We also set \( \widetilde{A} = A/C \simeq \bar{A}/(C^H_H) \). Then, we have recollement diagrams

\[
\begin{array}{ccc}
\text{Ker } G & \xrightarrow{J^*} & \text{A-Mod} \\
\xleftarrow{J'} & & \xleftarrow{G^*} \text{H-Mod}
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Ker } DG & \xrightarrow{DJ^*} & \mathcal{D}(\bar{A}) \\
\xleftarrow{RJ^*} & & \xleftarrow{LG^*} \mathcal{D}(H)
\end{array}
\]

where Ker \( G \simeq \text{A-Mod} \). Since the \( \bar{A} \)-ideal \( (C^H_H) \) is projective as a right \( \bar{A} \)-module, \cite[Theorem 4.6]{burban2017} implies that Ker \( DG \simeq \mathcal{D}(\bar{A}) \).

As usual, this recollement diagram gives semiorthogonal decompositions \cite[Corollary 2.6]{burban2017}

\[ (4-1) \quad \mathcal{D}(\bar{A}) = (\text{Ker } DG, \text{Im } LG^*) = (\text{Im } RG^1, \text{Ker } DG) \]

with Ker \( DG \simeq \mathcal{D}(\bar{A}) \) and \( \text{Im } LG^* \simeq \text{Im } RG^1 \simeq \mathcal{D}(H) \) (though usually \( \text{Im } LG^* \neq \text{Im } RG^1 \)).

Recall from \cite{KuznetsovLunts2015} that a \textit{semiorthogonal decomposition} \( \mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2) \), where \( \mathcal{I}_1, \mathcal{I}_2 \) are full triangulated subcategories of \( \mathcal{I} \), means that \( \text{Hom}(\mathcal{I}, T_2, T_1) = 0 \) for all \( T_1 \in \mathcal{I}_1 \) and \( T_2 \in \mathcal{I}_2 \), and for every object \( T \in \mathcal{I} \) there is an exact triangle \( T_1 \rightarrow T_2 \rightarrow T \rightarrow \)

\textbf{Lemma 4.1.} If \( \mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2) \) is a semiorthogonal decomposition of a triangulated category \( \mathcal{I} \), then

\[ \text{dim } \mathcal{I} \leq \text{dim } \mathcal{I}_1 + \text{dim } \mathcal{I}_2 + 1. \]

\textbf{Proof.} First we show that for any subsets \( M, N \) of objects of the category \( \mathcal{I} \)

\[ (4-2) \quad (M)_{k+1} \uparrow M \subseteq (M) \uparrow ((M)_{k} \uparrow N) \subseteq (M) \uparrow ((M) \uparrow ((M) \uparrow \cdots ((M) \uparrow N) \cdots)). \]

Indeed, let \( C \in (M)_{k+1} \uparrow M \), i.e., there is an exact triangle \( A \rightarrow B \rightarrow C \rightarrow \), where \( A \in (M)_{k+1}, B \in M \). There is also an exact triangle \( A_1 \rightarrow A \rightarrow A_2 \rightarrow \), where \( A_1 \in (M)_k, A_2 \in (M) \). The octahedron axiom implies that there are exact triangles \( A_1 \rightarrow B \rightarrow B' \rightarrow \) and \( A_2 \rightarrow B' \rightarrow C \rightarrow \). Therefore, \( B' \in (M)_k \uparrow M \) and \( C \in (M) \uparrow ((M) \uparrow N) \).

Now, let \( (M)_{k+1} = \mathcal{I}_1 \) and \( (M)_{l+1} = \mathcal{I}_2 \). Then, for every \( T \in \mathcal{I} \) there is an exact triangle \( T_1 \rightarrow T_2 \rightarrow T \rightarrow \), where \( T_1 \in (M)_{k+1}, T_2 \in (M)_{l+1} \). But, according to \((4-2), (M)_{k+1} \uparrow (M)_{l+1} \subseteq (M \cup M)_{k+l+2}, \) so \( \mathcal{I} = (M \cup M)_{k+l+2} \) and \( \text{dim } \mathcal{I} \leq k+l+1 \). \( \Box \)

\textsuperscript{3}In \cite[Theorem 7.4]{Psaroudakis2014} this result is proved in the case when this decomposition arises from a recollement.
Since $\bar{A}$ is semisimple, any indecomposable object from $\mathcal{D}(\bar{A})$ is just a shifted simple module, so $\mathcal{D}_f^b(\bar{A}) = \langle \bar{A} \rangle$ and $\text{der.dim} \bar{A} = 0$. If $H$ is hereditary, every indecomposable object from $\mathcal{D}_f^b(H)$ is a shift of a module. For every module $M$ there is an exact sequence $0 \to P' \to P \to M \to 0$ with projective modules $P, P'$ and, since $H$ is semiperfect, every indecomposable projective $H$-module is a direct summand of $H$. Hence, $\mathcal{D}_f^b(H) = \langle H \rangle_2$ and $\text{der.dim} H \leq 1$.

**Corollary 4.2.** We have $\text{der.dim} A \leq \text{der.dim} H + 1$. In particular, if $A$ is a Backström (or piecewise Backström) ring, $\text{der.dim} A \leq 2$.

A finite dimensional hereditary algebra is said to be of Dynkin type if it has finitely many isomorphism classes of indecomposable modules. Such algebras, up to Morita equivalence, correspond to Dynkin diagrams [Dlab and Ringel 1976; Gabriel 1972]. If the derived category of an algebra $H$ is equivalent to the derived category of a hereditary algebra of Dynkin type, we say that $H$ is piecewise hereditary of Dynkin type. We say that a Backström (or piecewise Backström) algebra $A$ is of Dynkin type if there is a Backström pair $H \supset A$, where $H$ is a hereditary (piecewise hereditary) algebra of Dynkin type. For instance, it is so if $A$ is a gentle or skewed-gentle algebra [Zembyk 2015], or the algebra $\text{UT}(n \kappa)$ of unitriangular matrices over a field (Examples 2.3 (4)), or the algebra $\Lambda_n$ from Examples 2.3 (5). In this case, $\mathcal{D}_f^b(H) = \langle M_1, M_2, \ldots, M_m \rangle_1$, where $M_1, M_2, \ldots, M_m$ are all pairwise nonisomorphic indecomposable $H$-modules, so $\text{der.dim} H = 0$.

In [Chen et al. 2008] it was proved that $\text{der.dim} A = 0$ for a finite dimensional algebra $A$ if and only if $A$ is a piecewise hereditary algebra of Dynkin type.

**Corollary 4.3.** If $A$ is a Backström (or piecewise Backström) algebra of Dynkin type (for instance, gentle or skewed-gentle), but is not piecewise hereditary of Dynkin type, then $\text{der.dim} A = 1$.

**Example 4.4.** The path algebra of the commutative quiver

\[
\begin{array}{c}
1 \\
\alpha_0 \\
\beta_0 \\
\beta_1 \\
\gamma \\
\gamma' \\
3 \\
\alpha_1 \\
2 \\
2' \\
4 \\
\alpha_1 \alpha_0 = \beta_1 \beta_0
\end{array}
\]

is a tilted (hence piecewise hereditary) algebra of type $\tilde{D}_5$. At the same time it is a Backström algebra of type $A_4$. Namely, it is a skewed-gentle algebra obtained from the path algebra of the quiver $1 \to 2 \to 3 \to 4$ by blowing up vertices 2 and 4.\footnote{It is proved in [Happel 1988] that piecewise hereditary algebras of Dynkin type are just iterated tilted algebras of Dynkin type.}

\footnote{See [Zembyk 2014] for the construction of blowing up and its relation to nodal algebras.}
5. Relation to bimodule categories

In this section, we explain how a semiorthogonal decomposition allows us to apply to calculations in a triangulated category the technique of matrix problems, namely, of bimodule categories, as in [Drozd 2010].

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, $\mathcal{U}$ be an $\mathcal{A} \rightarrow \mathcal{B}$-bimodule, i.e., a biadditive functor $\mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \text{Ab}$. Recall from [Drozd 2010] that the bimodule category or the category of elements of the bimodule $\mathcal{U}$ is the category $\text{El}(\mathcal{U})$ whose set of objects is $\bigcup_{A \in \mathcal{A}, B \in \mathcal{B}} \mathcal{U}(A, B)$ and whose morphisms from $u \in \mathcal{U}(A, B)$ to $v \in \mathcal{U}(A', B')$ are the pairs $(\alpha, \beta)$ such that $u \alpha = \beta v$, where $\alpha : A' \rightarrow A$, $\beta : B \rightarrow B'$. Here, as usual, we wrote $u \alpha$ and $\beta v$ instead of $\mathcal{U}(\alpha, 1_B) u$ and $\mathcal{U}(1_A, \beta) v$. Bimodule categories appear when there is a semiorthogonal decomposition of a triangulated category.

**Theorem 5.1.** Let $(\mathcal{A}, \mathcal{B})$ be a semiorthogonal decomposition of a triangulated category $\mathcal{C}$. Consider the $\mathcal{A} \rightarrow \mathcal{B}$-bimodule $\mathcal{U}$ such that $\mathcal{U}(A, B) = \text{Hom}_\mathcal{E}(A, B)$, $A \in \mathcal{A}$, $B \in \mathcal{B}$. For every $f : A \rightarrow B$ fix a cone $Cf$, that is, an exact triangle $A \rightarrow B \rightarrow Cf \rightarrow A[1]$. The map $f \mapsto Cf$ induces an equivalence of categories $\mathcal{C} : \text{El}(\mathcal{U}) \rightarrow \mathcal{C} / \mathcal{J}$, where $\mathcal{J}$ is the ideal of $\mathcal{C}$ consisting of morphisms $\eta$ such that there are factorizations $\eta = \eta' \xi = \xi \eta''$, where the source of $\eta'$ is in $\mathcal{A}$ and the target of $\eta''$ is in $\mathcal{B}$. Moreover, $\mathcal{J}^2 = 0$, so $\mathcal{C}$ induces a bijection between isomorphism classes of objects from $\text{El}(\mathcal{U})$ and from $\mathcal{C}$.

**Proof.** As $(\mathcal{A}, \mathcal{B})$ is a semiorthogonal decomposition of $\mathcal{C}$, every object from $\mathcal{C}$ occurs in an exact triangle $A \rightarrow B \rightarrow C \rightarrow$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$, so $f$ is an object from $\text{El}(\mathcal{U})$ and $C \simeq Cf$. Let $f' : A' \rightarrow B'$ be another object of $\text{El}(\mathcal{U})$ and $(\alpha, \beta) : f \rightarrow f'$ be a morphism from $\text{El}(\mathcal{U})$. Fix a commutative diagram

$$
\begin{array}{cccc}
A & \xrightarrow{f} & B & \xrightarrow{f_1} & Cf & \xrightarrow{f_2} & A[1] \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \alpha[1] \\
A' & \xrightarrow{f'} & B' & \xrightarrow{f'_1} & Cf' & \xrightarrow{f'_2} & A'[1]
\end{array}
$$

It exists, though is not unique. Let $\gamma'$ be another morphism making the diagram (5-1) commutative and set $\eta = \gamma - \gamma'$. Then, $\eta f_1 = 0$, hence $\eta$ factors through $f_2$, and $f'_2 \eta = 0$, hence $\eta$ factors through $f'_1$. Thus, $\eta \in \mathcal{J}$. On the other hand, if $\eta : Cf \rightarrow Cf'$ is in $\mathcal{J}$, the decomposition $\eta = \eta' \xi$ implies that $\eta f_1 = \eta' \xi f_1 = 0$ and the decomposition $\eta = \xi \eta''$ implies that $f'_2 \eta = f'_2 \xi \eta'' = 0$, hence the morphism $\gamma' = \gamma + \eta$ makes the diagram (5-1) commutative. Therefore, the class $\mathcal{C}(\alpha, \beta)$ of $\gamma$ modulo $\mathcal{J}$ is uniquely defined, so the maps $f \mapsto Cf$ and $(\alpha, \beta) \mapsto \mathcal{C}(\alpha, \beta)$ define a functor $\mathcal{C} : \text{El}(\mathcal{U}) \rightarrow \mathcal{C} / \mathcal{J}$.

---

6This theorem is a partial case of [Drozd 2010, Theorem 1.1].
Let now $\gamma : Cf \to Cf'$ be any morphism. Then, $f_2' \gamma f_1 = 0$, so $\gamma f_1 = f_1' \beta$ for some $\beta : B \to B'$. Hence, there is a morphism $\alpha : A \to A'$ making the diagram (5-1) commutative, i.e., defining a morphism $(\alpha, \beta) : f \to f'$ such that $\gamma \equiv C(\alpha, \beta) \mod \mathcal{J}$. If $(\alpha', \beta')$ is another such morphism, $f_1'(\beta - \beta') = 0$, so $\beta - \beta' = f_1' \xi$ for some $\xi : B \to A$. But $\xi = 0$, so $\beta = \beta'$. In the same way $\alpha = \alpha'$. Hence, the functor $C$ is fully faithful. As we have already noticed, it is essentially surjective, and therefore defines an equivalence $\text{El}(\mathcal{U}) \sim \mathcal{C}/\mathcal{J}$. The equality $\mathcal{J}^2 = 0$ follows immediately from the definition and the conditions of the theorem. 

We apply Theorem 5.1 to Backström pairs $H \subseteq A$ such that $A$ is left noetherian and $H$ is left hereditary and finitely generated as a left $A$-module. For instance, it is so in the case of Backström algebras or Backström orders. Then, the ring $\bar{A}$ is also noetherian and $C$ is projective as a left $H$-module. According to (4-1), $(\text{Ker} DG, \text{Im} LG^*)$ is a semiorthogonal decomposition of $\mathcal{D}(\bar{A})$. Moreover, both $G$ and $G^*$ map finitely generated modules to finitely generated modules, so the same is valid if we consider their restrictions onto $\mathcal{D}_f(\bar{A})$ and $\mathcal{D}_f(H)$. Note also that $G^*$ is exact, so $G^*$ can be applied to complexes componentwise. The $\bar{A}$-module $G^* M$ can be identified with the module of columns $M^+ = (\begin{array}{c} M \\ M \end{array})$ with the action of $\bar{A}$ given by matrix multiplication. It gives an equivalence of $\mathcal{D}(H)$ with $\text{Im} LG^*$. As $H$ is left hereditary, every complex from $\mathcal{D}(H)$ is equivalent to a direct sum of shifted modules (see [Keller 2007, Section 2.5]). On the other hand, $\text{Ker} DG \simeq \mathcal{D}(\bar{A})$ and $\bar{A}$ is semisimple, since $C \supset r$. Hence, every complex from $\mathcal{D}(\bar{A})$ is isomorphic to a direct sum of shifted simple $\bar{A}$-modules, which are direct summands of $\bar{A}$. So, to calculate the bimodule $\mathcal{U}$, we only have to calculate $\text{Ext}^1_{\bar{A}}(\bar{A}, M^+)$, where $M$ is an $H$-module. Note also that $C^+$ is a projective $\bar{A}$-module, since $C$ is a projective $H$-module. Therefore, a projective resolution of $\bar{A}$ is $0 \to C^+ \xrightarrow{\varepsilon} P \to \bar{A} \to 0$ and $\text{pr.dim}_{\bar{A}} \bar{A} = 1$. Hence, we only have to calculate $\text{Hom}_{\bar{A}}(\bar{A}, M^+)$ and $\text{Ext}^1_{\bar{A}}(\bar{A}, M^+)$. 

**Theorem 5.2.** (1) $\text{Hom}_{\bar{A}}(\bar{A}, M^+) \simeq \text{ann}_M C = \{ u \in M \mid Cu = 0 \}$. 

(2) $\text{Ext}^1_{\bar{A}}(\bar{A}, M^+) \simeq \text{Hom}_H(C, M)/(M/\text{ann}_M C)$, where the quotient $M/\text{ann}_M C$ embeds into $\text{Hom}_H(C, M)$ if we map an element $u \in M$ to the homomorphism $\mu_u : c \mapsto cu$.

**Proof.** (1) $\text{Hom}_{\bar{A}}(\bar{A}, M^+)$ is identified with the set of homomorphisms $\phi : P \to M^+$ such that $\phi \varepsilon = 0$. A homomorphism $\phi : P \to M^+$ is uniquely defined by an element $u \in M$ such that $\phi(\begin{array}{c} 1 \\ 0 \end{array}) = (\begin{array}{c} u \\ 0 \end{array})$. Namely, $\phi(\begin{array}{c} a \\ c \end{array}) = (\begin{array}{c} au \\ cu \end{array})$. Obviously, $\phi \varepsilon = 0$ if and only if $Cu = 0$, i.e., $u \in \text{ann}_M C$.

(2) $\text{Ext}^1_{\bar{A}}(\bar{A}, M^+) \simeq \text{Hom}_{\bar{A}}(C^+, M^+)/\text{Hom}_{\bar{A}}(P, M^+) \varepsilon$. As the functor $G^*$ is fully faithful, $\text{Hom}_{\bar{A}}(C^+, M^+) \simeq \text{Hom}_H(C, M)$. Namely, $\psi : C \to M$ induces
\( \psi^+: C^+ \to M^+ \) mapping \((a \ b)_b\) to \((\psi(a) \ \psi(b))\). Let \( \phi: P \to M^+ \) correspond, as above, to an element \( u \in M \). Then,

\[
\phi \varepsilon \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} au \\ cu \end{pmatrix},
\]

so it equals \( \mu_u \), and \( \text{Hom}_{\tilde{A}}(P, M^+) \varepsilon \) is identified with \( M/\text{ann}_M C \) embedded into \( \text{Hom}_{H}(C, M) \) as above. \( \square \)

Actually, in our case an object \( E \) from the category \( \text{El}(\mathcal{U}) \) (therefore, also an object from \( \mathcal{D}^b(\tilde{A}) \)) is given by the vertices and solid arrows of a diagram

\[
\begin{array}{cccccc}
A_n & \xrightarrow{\mu_n} & M_n & \xrightarrow{\beta_n} & M_{n+1} & \xrightarrow{\gamma_n} & A_{n+1} \\
\downarrow{\alpha_n} & & \downarrow{\mu_{n+1}} & & \downarrow{\mu_{n+2}} & & \downarrow{\mu_{n+3}} \\
A_{n+1} & \xrightarrow{\eta_{n+1}} & M_{n+1} & \xrightarrow{\beta_{n+1}} & M_{n+2} & \xrightarrow{\gamma_{n+1}} & A_{n+2} \\
\downarrow{\alpha_{n+2}} & & \downarrow{\mu_{n+3}} & & \downarrow{\beta_{n+2}} & & \downarrow{\gamma_{n+3}} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\end{array}
\]

(of arbitrary length), where \( A_i \) are \( \tilde{A} \)-modules, \( M_i \) are \( H \)-modules, \( \mu_i \) belongs to \( \text{Hom}_{\tilde{A}}(A_i, M_i^+) \) and \( \eta_i \) belongs to \( \text{Ext}^1_{\tilde{A}}(A_i, M_{i-1}^+) \). A morphism between \( E \) and \( E' \) is given by the dotted arrows, where

\[
\begin{align*}
\alpha_i &\in \text{Hom}_{\tilde{A}}(A_i, A'_i) \simeq \text{Hom}_{\tilde{A}}(A_i, A_i), \\
\gamma_i &\in \text{Hom}_{H}(M_i, M'_i) \simeq \text{Hom}_{\tilde{A}}(M_i^+, (M_i')^+), \\
\beta_i &\in \text{Ext}^1_{H}(M_i, M'_{i+1}) \simeq \text{Ext}^1_{\tilde{A}}(M_i^+, (M_{i+1})^+).
\end{align*}
\]

These morphisms must satisfy the relations

\[
\mu'_i \alpha_i = \gamma_i \mu_i, \quad \eta'_i \alpha_i = \gamma_{i+1} \eta_i + \beta_i \mu_i.
\]

6. Partial tilting for Backström pairs

Let \( H \subseteq A \) be a Backström pair. Consider the ring \( B \) of triangular matrices of the form

\[
B = \begin{pmatrix} \tilde{A} & \tilde{H} \\ 0 & H \end{pmatrix}.
\]

Let \( e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and \( e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \), and let \( B_1 = Be_1 \) and \( B_2 = Be_2 \) be projective \( B \)-modules given by the first and the second column of \( B \), i.e.,

\[
B_1 = \begin{pmatrix} \tilde{A} \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \tilde{H} \\ H \end{pmatrix}.
\]
A \textit{B}-module $M$ is defined by a triple $\left( \begin{array}{c} M_1 \\ M_2 \end{array} \right)x_M$, where $M_1 = e_1 M$ is an \textit{A}-module, $M_2 = e_2 M$ is an \textit{H}-module and $x_M : M_2 \to M_1$ is an \textit{A}-homomorphism such that $\text{Ker } x_M \supseteq C M_2$ (it is necessary since $C M_1 = 0$). Namely, $x_M$ is multiplication by $\left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$. We write an element $u \in M$ as a column $\left( \begin{array}{c} u_1 \\ u_2 \end{array} \right)$, where $u_1 = e_1 u$, $u_2 = e_2 u$. Then,

$$
\left( \begin{array}{cc} a & b \\ 0 & c \end{array} \right) \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) = \left( \begin{array}{c} au_1 + x_M(bu_2) \\ cu_2 \end{array} \right).
$$

A homomorphism $\alpha : M \to N$ is defined by two homomorphisms $\alpha_1 : M_1 \to N_1$ and $\alpha_2 : M_2 \to N_2$ such that $\alpha_1 x_M = x_N \alpha_2$. We write $\alpha = \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right)$.

**Proposition 6.1.** We have $\text{l.gl.dim } B = \max(\text{l.gl.dim } H, \text{w.dim } \overline{H}_H + 1)$.

In particular, if $H$ is left hereditary and $\overline{H}$ is not flat as a right $H$-module, then $\text{l.gl.dim } B = 2$.

**Proof.** [Palmér and Roos 1973, Theorem 5] shows that $\text{l.gl.dim } B \leq n$ if and only if $\text{l.gl.dim } H \leq n$ and $\mathbb{R}^n \text{Hom}_A(\overline{H} \otimes_H -, -) = 0$.

As the ring $\overline{A}$ is semisimple, $\mathbb{R}^n \text{Hom}_A(\overline{H} \otimes_H -, -) = \text{Hom}_\overline{A}(\text{Tor}_n^H(\overline{H}, -), -)$. This implies the first assertion. The second is obvious, since $\text{Tor}_1^H(\overline{H}, -) = 0$ if and only if $\overline{H}$ is flat as a right $H$-module. \hfill \Box

We denote by $R$ the $B$-module given by the triple $\left( \begin{array}{c} H/A \\ \pi \end{array} \right)$, where $\pi : H \to H/A$ is the natural surjection.

**Proposition 6.2.**

(1) $\text{End}_B R \simeq A^{\text{op}}$.

(2) $\text{pr.dim}_B R = 1$.

(3) $\text{Ext}_B^1(R, R) = 0$.

Recall that conditions (2) and (3) mean that $R$ is a partial tilting $B$-module.

**Proof.** The minimal projective resolution of $R$ is

$$
0 \to B_1 \xrightarrow{\varepsilon} B_2 \to R \to 0,
$$

where $\varepsilon$ is the embedding, which gives (2). Any endomorphism $\gamma$ of $R$ induces a commutative diagram:

$\begin{array}{ccc}
B_1 & \xrightarrow{\varepsilon} & B_2 \\
\gamma_1 \downarrow & & \downarrow \gamma_2 \\
B_1 & \xrightarrow{\varepsilon} & B_2
\end{array}$

As $\text{End}_B B_2 \simeq H^{\text{op}}$, $\gamma_2$ is given by multiplication with an element $h \in H$ on the right. If there is a commutative diagram as above, necessarily $h \in A$, which proves (1).
Finally, a homomorphism $\alpha : B_1 \to R$ maps the generator $(1_0)$ of $B_1$ to an element $(\tilde{h}_0) \in R$. If $h$ is a preimage of $\tilde{h}$ in $H$, then $\alpha$ extends to the homomorphism $B_2 \to R$ that maps the generator $(0_1)$ of $B_2$ to $(0_\tilde{h}) \in R$. This implies (3).

Now Theorem 1.1 applied to the module $R$ gives the following result:

**Theorem 6.3.** (1) The functor $F = \mathbb{R}\text{Hom}(R, -)$ induces an equivalence

$$\text{Tri}(R) \xrightarrow{\sim} \mathcal{D}(\mathcal{A}).$$

(2) $\text{Ker } F$ consists of complexes $C$ such that the map $\chi_{H^k(C)}$ is bijective for all $k$.

(3) There is a recollement diagram

$$\text{Ker } F \xleftarrow{\iota} \mathcal{D}(\mathcal{B}) \xrightarrow{F} \mathcal{D}(\mathcal{A}) \xleftarrow{\iota'}.$$

Actually, claim (2) means that a complex $C$ is in $\text{Ker } F$ if and only if its cohomologies are direct sums of $\mathcal{B}$-modules of the form $(U_1 1_U)$, where $U$ is a simple $\bar{H}$-module.

$F$ is a partial tilting functor in the sense of Corollary 1.2.

**Proof.** (1) and (3) follow from Proposition 6.2 and Theorem 1.1, since the complex $P : 0 \to B_1 \xrightarrow{\varepsilon} B_2 \to 0$ is perfect, hence compact, and isomorphic to $R$ in $\mathcal{D}(\mathcal{B})$.

To find $\text{Ker } F$, consider a complex

$$C : \cdot \to C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \to \cdots,$$

where $C^k$ is defined by a triple $(C^i, C^j, C^k)$ and $d^k = (d^1_2, d^2_2)$, where $d^1_2 \chi_k = \chi_{k+1} d^2_2$ for all $k$. Note that $C_i = (C^i, d^i)$ $(i = 1, 2)$ are complexes, $(\chi_k)$ is a homomorphism of complexes and $H^k(C) = \left(\begin{array}{c} H^k(C_i) \\ H^k(C_j) \end{array} \right) \tilde{\chi}_k$, where $\tilde{\chi}_k = \chi_{H^k(C)}$ is induced by $\chi_k$. A homomorphism $P \to C[k]$ is a pair of homomorphisms $\alpha : B_2 \to C^1$, $\beta : B_1 \to C^{k-1}$ such that $\alpha_1 \pi = \chi_k \alpha_2$, $\beta_2 = 0$, $d^1_2 \alpha_1 = 0$ $(i = 1, 2)$ and $d^{k-1} \beta_1 = \alpha_1 \varepsilon$. Let $\alpha_2(1) = x \in C^2_2$ and $\beta_1(1) = y \in C^{k-1}_1$. These values completely define $\alpha$ and $\beta$.

The conditions for $\alpha$ and $\beta$ mean that $d^2_2 x = 0$ and $d^{k-1} y = \chi_k x$.

This morphism is homotopic to zero if and only if there are maps $\sigma : B_2 \to C^{k-1}$ and $\tau : B_1 \to C^{k-2}$ such that $\alpha = d^{k-1} \sigma$ and $\beta = \sigma \varepsilon + d^{k-2} \tau$. Again, $\sigma$ is defined by the element $z = \sigma_2(1) \in C^{k-1}_2$ and $\tau$ is defined by the element $t = \tau_1(1) \in C^{k-2}_1$. Then, the conditions for $\alpha$ and $\beta$ mean that $x = d^{k-1} z$ and $y = \chi_{k-1} z + d^{k-2} t$.

Suppose that any homomorphism $P \to C[k]$ is homotopic to zero. Let $\tilde{x}$ in $H^k(C^2)$ be such that $\tilde{\chi}_k(\tilde{x}) = 0$ and $x \in \text{Ker } d^2_2$ be a representative of $\tilde{x}$. Then, $\chi_k(x) = d^{k-1} y$ for some $y \in C^{k-1}$, so the pair $(x, y)$ defines a homomorphism $P \to C[k]$. Therefore, there must be $z \in C^{k-1}_2$ such that $x = d^{k-1} z$; thus $\tilde{x} = 0$ and $\tilde{\chi}_k$ is injective. Let now $\tilde{y} \in H^{k-1}(C_2)$ and $y \in C^{k-1}_2$ be its representative. Then, the pair $(0, y)$ defines a homomorphism $P \to C[k]$, so there must be elements $z \in C^{k-1}_2$. 


and \( t \in C^{k-2}_1 \) such that \( d^{k-1}_1z = 0 \) and \( y = \chi_{k-1}z + d^{k-2}_1t \). Hence, \( \bar{y} = \bar{\chi}_{k-1}(\bar{z}) \), so \( \bar{\chi}_{k-1} \) is surjective. As this holds for all \( k \), we have that all maps \( \bar{\chi}_k \) are bijective.

On the contrary, suppose that not all \( \bar{\chi}_k \) are bijective. If a pair \((x, y)\) defines a homomorphism \( P \rightarrow C[k] \), then \( \chi_k(x) = d^{k-1}_1y \), so \( \bar{\chi}_k(x) = 0 \). Therefore, \( \bar{x} = 0 \), i.e., \( x = d^{k-1}_2z \) for some \( z \in C^{k-1}_2 \) and \( \chi_kx = d^{k-1}_1\chi_{k-1}z \). Then, \( d^{k-1}_1(y - \chi_kz) = 0 \), hence there is an element \( z' \in C^{k-1}_2 \) such that \( d^{k-1}_1z' = 0 \) and the cohomology class of \( y - \chi_kz \) equals \( \bar{\chi}_{k-1}(\bar{z}') \), i.e., \( y - \chi_kz = \chi_{k-1}z' + d^{k-2}_1t \) for some \( t \). Then, \( x = d^{k-1}_2(z + z') \) and \( y = \chi_{k-1}(z + z') + d^{k-2}_1t \), so this homomorphism is homotopic to zero. \( \square \)

As usual, we identify the category \( \mathcal{A}\text{-Mod} \) with the full subcategory of \( \mathcal{D}(\mathcal{A}) \) consisting of the complexes \( C \) concentrated in degree 0. The following result shows how the partial titling functor \( \mathcal{F} \) behaves with respect to modules:

**Corollary 6.4.** Let a \( \mathcal{B}\text{-module} M \) be given by the triple \( (\chi_M, M_1, M_2) \).

1. \( FM \in \mathcal{A}\text{-Mod} \) if and only if \( \chi_M \) is surjective. Namely, then \( FM \simeq \ker \chi_M \).
2. \( FM \in \mathcal{A}\text{-Mod}[1] \) if and only if \( \chi_M \) is injective. Namely, then \( FM \simeq \coker \chi_M[1] \).

**Proof.** Note that \( \Hom_B(B_1, M) \simeq M_1 \), \( \Hom_B(B_2, M) \simeq M_2 \) and if \( \phi : B_2 \rightarrow M \) maps \((0,0)\) to \((0,0)\), then \( \phi e \) maps \((1,0)\) to \((\chi_M(x),0)\). Therefore, \( \mathbb{R}\Hom_B(R, M) \) is the complex

\[
0 \rightarrow M_2 \xrightarrow{\chi_M} M_1 \rightarrow 0,
\]

which proves the claim. \( \square \)

**Remark 6.5.** There are several derived equivalences related to \( \tilde{\mathcal{A}} \).

1. If \( \mathcal{A} \) is a Backström order, it is known (see [Burban et al. 2017]) that the complex \( T = B_1[1] \oplus H^+ \), where \( B_1 = \left( \begin{smallmatrix} \tilde{A} \\ 0 \end{smallmatrix} \right) \), is a tilting complex for \( \tilde{\mathcal{A}} \) and \((\End_{\mathcal{D}(\tilde{\mathcal{A}})})^{\text{op}} T \simeq B \), hence \( \tilde{\mathcal{A}} \) is derived equivalent to \( B \). Nevertheless, in the general situation of Backström rings (even of Backström algebras) this is not so. First of all, \( \Hom_{\tilde{\mathcal{A}}}(B_1, H^+) \cong \text{ann}_H C \), so it can happen that \( \Hom_{\mathcal{D}(\tilde{\mathcal{A}})}(T, T[1]) \neq 0 \). This is so, for instance, for the pair \((T(n, k), UT(n, k))\) from Equation (2-3) (4), since in this case the matrix unit \( e_{nn} \) belongs to \( \text{ann}_H C \). This is also so for Equation (2-3) (5). Moreover, even if \( \text{ann}_H C = 0 \), one can see that \( \overline{H} = \text{Ext}^1_{\tilde{\mathcal{A}}}(B_1, H^+) \simeq C^{-1}/cH \), where \( C^{-1} = \Hom_H(C, H) \) and \( cH = H/\text{ann}_H C \) is naturally embedded into \( C^{-1} \). Therefore, in this case,

\[
(\End_{\mathcal{D}(\tilde{\mathcal{A}})} T)^{\text{op}} \simeq B' = \begin{pmatrix} \tilde{A} & \overline{H} \\ 0 & H \end{pmatrix},
\]

which need not coincide with \( B \) (see Example 6.6 below). If \( H \) is a hereditary order, then \( \text{ann}_H C = 0 \) and \( \overline{H} \simeq \overline{H} \), hence \( B' \simeq B \), in accordance with [Burban et al. 2017].
(2) On the other hand, set \( T' = (A^H/A)_C \) considered as a left \( \tilde{A} \)-module. One can check it is a tilting module for \( \tilde{A} \) and
\[
(\text{End}_{\mathcal{D}(\tilde{A})} T')^{\text{op}} \simeq \tilde{B} = \begin{pmatrix} A & H/A \\ 0 & A \end{pmatrix},
\]
hence \( \tilde{A} \) is derived equivalent to \( \tilde{B} \). Unfortunately, this ring can be not so good from the homological point of view. At least, it is not better than \( \tilde{A} \) itself. Namely, as one can easily check,
\[
\text{l.gl.dim} \tilde{B} = \max(\text{l.gl.dim} A, 1 + \text{pr.dim}_A(H/A)),
\]
which is either \( \text{l.gl.dim} A \) or (more often) \( \text{l.gl.dim} A + 1 \).

(3) One more observation: Consider the right \( \tilde{A} \)-modules \( (\tilde{A} 0) \) and \( (C H) \). One can check that \( T'' = (\tilde{A} 0)[1] \oplus (C H) \) is a tilting complex for \( \mathcal{D}(\tilde{A}^{\text{op}}) \) and \( \text{End}_{\mathcal{D}(\tilde{A}^{\text{op}})} T'' \simeq B'' = \begin{pmatrix} \tilde{A} & 0 \\ H & H \end{pmatrix} \), hence \( \tilde{A}^{\text{op}} \) is derived equivalent to \( (B'')^{\text{op}} \).

Note that the functor \( P \mapsto \text{Hom}_R(P, R) \) induces an exact duality
\[
\text{Perf}(R) \rightarrow \text{Perf}(R^{\text{op}})
\]
for any ring \( R \). Hence, \( \text{Perf}(\tilde{A}) \simeq \text{Perf}(B'') \).

**Example 6.6.** Let \( H = T(3, \mathbb{k}) \) and \( A = \{(a_{ij}) \in H \mid a_{11} = a_{22}\} \). Set \( H_i = H e_{ii} \) and \( U_i = H_i / \text{rad} H_i \). Then, \( C = \{(a_{ij}) \in H \mid a_{11} = a_{22} = 0\} \), hence \( \overline{H} = U_1 \oplus U_2 \).

On the other hand, \( C = \text{rad} H_2 \oplus H_3 \simeq H_1 \oplus H_3 \), so \( C^{-1} = \text{Hom}_H(C, H) \) can be identified with the set of \( 3 \times 2 \) matrices \( (b_{ij}) \) such that \( b_{12} = b_{22} = 0 \). One can check that \( C H \) is identified with the subset \( \{(b_{ij}) \mid b_{11} = 0\} \subset C^{-1} \) and \( \overline{H} ' \simeq U_2 \not\simeq \overline{H} \) (even \( \dim_{\mathbb{k}} \overline{H} ' \neq \dim_{\mathbb{k}} \overline{H} \)).

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**References**


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