THE DEFORMATION SPACE OF
DELAUNAY TRIANGULATIONS OF THE SPHERE

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We determine the topology of the spaces of convex polyhedra inscribed in the unit 2-sphere and the spaces of strictly Delaunay geodesic triangulations of the unit 2-sphere. These spaces can be regarded as discretized groups of diffeomorphisms of the unit 2-sphere. Hence, it is natural to conjecture that these spaces have the same homotopy types as those of their smooth counterparts. The main result of this paper confirms this conjecture for the unit 2-sphere. It follows from an observation on the variational principles on triangulated surfaces developed by I. Rivin.

On the contrary, the similar conjecture does not hold in the cases of flat tori and convex polygons. We will construct simple examples of flat tori and convex polygons such that the corresponding spaces of Delaunay geodesic triangulations are not connected.

1. Introduction

One of the fundamental problems in low dimensional topology is to identify the homotopy types of groups of diffeomorphisms of a smooth manifold. Smale [1959] proved that the group of orientation preserving diffeomorphisms of the 2-sphere is homotopy equivalent to SO(3).

This paper studies two types of finite dimensional spaces which could be considered as discrete analogues of the group of orientation preserving diffeomorphisms of the 2-sphere. They are the deformation spaces of Delaunay triangulations of the unit 2-sphere and the deformation spaces of convex polyhedra inscribed in the unit 2-sphere. The main results of this paper show that these discrete analogues are homotopy equivalent to SO(3).

Theorem 1.1. *The deformation space of Delaunay triangulations of the unit 2-sphere is homeomorphic to* SO(3) $\times \mathbb{R}^k$ *for some $k > 0$.*

Theorem 1.2. *The deformation space of the convex polyhedra inscribed in the unit 2-sphere whose faces are all triangles is homeomorphic to* $\mathrm{SO}(3) \times \mathbb{R}^k$ *for some* $k > 0$.

However, we will construct explicit examples of spaces of Delaunay triangulations of convex polygons and flat tori which have different homotopy types from their smooth counterparts. Specifically, we show the spaces of Delaunay triangulations of some flat tori and spaces of Delaunay triangulations of some convex polygons are not connected.

Let $T = (V, E, F)$ denote a 2-dimensional simplicial complex, where V is the set of vertices, E is the set of edges, and F is the set of triangles. Any edge in E is identified with the closed interval $[0, 1]$, and any triangle in F is identified with a Euclidean equilateral triangle with unit length. Denote $T^{(1)}$ as the 1-skeleton of T, and $|T|$ as the underlying space of T homeomorphic to a surface possibly with boundary.

Delaunay triangulations of the unit sphere. Let $\mathbb{S}^2$ be the unit sphere as a Riemannian surface. Assume $|T|$ is homeomorphic to $\mathbb{S}^2$. An embedding $\varphi : T^{(1)} \to \mathbb{S}^2$ is called a *geodesic triangulation* of $\mathbb{S}^2$ if the restriction of φ on each edge is a geodesic parametrized with constant speed. A geodesic triangulation φ naturally divides $\mathbb{S}^2$ into spherical geodesic triangles. For our convenience, we will only consider the geodesic triangulations where all the spherical triangles are convex. A geodesic triangulation φ of $\mathbb{S}^2$ is called a *convex geodesic triangulation* if any spherical triangle in φ is contained in some open hemisphere. Such a convex geodesic triangulation φ is uniquely determined by the images of the vertices of T.

A convex geodesic triangulation φ is called *Delaunay* if it satisfies the empty circle property, meaning that for any pair of adjacent spherical triangles $\triangle ABC$ and $\triangle ABD$, D is not inside the circumcircle of $\triangle ABC$. This condition is equivalent to the following condition on the angles of a convex geodesic triangulation:

$$(1) \qquad\qquad b + c + b' + c' - a - a' \geq 0,$$

where a, b, c, a', b', c' are the inner angles of two neighbored triangles as in Figure 1. Similarly, a convex geodesic triangulation is called *strictly Delaunay* if for any pair of adjacent spherical triangles $\triangle ABC$ and $\triangle ABD$, D is strictly outside the circumcircle of $\triangle ABC$. This condition is equivalent to the following condition on the angles of a convex geodesic triangulation:

$$(2) \qquad\qquad b + c + b' + c' - a - a' > 0.$$

Delaunay and strictly Delaunay triangulations naturally appear in the study of discrete differential geometry and geometry processing. They are widely investigated
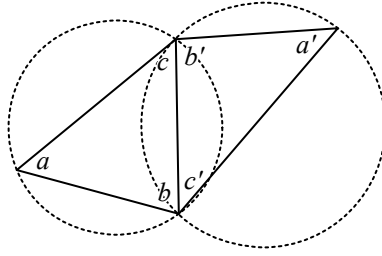
Figure 1. The edge invariant.

and implemented in practice. See [Devadoss and O'Rourke 2011; Edelsbrunner 2001] for example. We will focus on strictly Delaunay triangulations in this paper.

Given an embedding $\psi : T^{(1)} \to \mathbb{S}^2$, we define *the deformation space of Delaunay triangulations of the unit sphere* determined by ψ, denoted by $X(T, \psi)$, as the set of all strictly Delaunay convex geodesic triangulations that are isotopic to ψ in $\mathbb{S}^2$. Then $X(T, \psi)$ is naturally a manifold of dimension $2|V|$ without boundary, if $X(T, \psi)$ is not empty. Notice that $X(T, \psi)$ could be empty for some T since there are 3-connected graphs that cannot be realized as the 1-skeleton of a convex polyhedron with vertices on the unit 2-sphere. See [Steinitz 1928] for noninscribable polytopes.

Theorem 1.1 can be rephrased as:

Theorem 1.3. *Given a strictly Delaunay convex geodesic triangulation ψ, $X(T, \psi)$ is homeomorphic to $\mathbb{R}^{2|V|-3} \times \mathrm{SO}(3)$.*

Notice that by the assumption, $X(T, \psi)$ is not empty in Theorem 1.3.

The topology of spaces of geodesic triangulations of surfaces has been studied since Cairns [1944] first proved the connectivity of the spaces of geodesic triangulations of the 2-sphere. It was conjectured that for constant curvature surfaces they are homotopy equivalent to their smooth counterparts by Connelly et al. [1983]. This conjecture has been confirmed by Bloch, Connelly and Henderson [Bloch et al. 1984] for convex polygons, and a new proof based on Tuttes' embedding theorem was provided by Luo [2022]. Recently, this conjecture was proved for the cases of flat tori and closed surfaces of negative curvature (see the work of Erickson and Lin [2021] and Luo, Wu and Zhu [2021b; 2021a]).

For the case of the unit sphere, Awartani and Henderson [1987] identified the homotopy type of a subspace of the space of geodesic triangulations on the unit 2-sphere, but the general case remains open. Theorem 1.3 provides an affirmative evidence about this conjecture, and we hope that it could be an intermediate step to prove the conjecture for the unit sphere.

Convex polyhedra inscribed in the unit sphere. Assume $|T|$ is homeomorphic to $\mathbb{S}^2$. An embedding $\varphi : |T| \to \mathbb{R}^3$ is called a *polyhedral realization inscribed in the unit sphere* if φ maps any vertex to the unit sphere and maps any face linearly to a Euclidean triangle. Such a polyhedral realization φ is called (strictly) convex if for any triangle $\sigma \in F$, $\varphi(\sigma)$ is a face of the boundary of the convex hull of $\varphi(V)$ in $\mathbb{R}^3$. Given T, denote $Y(T)$ as the set of convex polyhedral realization inscribed in the unit sphere.

We say a point q is *inside* a convex polyhedral surface P if q is in the interior of the convex hull of P. Given a point q in the unit open ball, denote $p_q : \mathbb{R}^3 \backslash \{q\} \to \mathbb{S}^2$ as the radial projection centered at q to the unit sphere. We say two convex polyhedral realizations φ_1, φ_2 in $Y(T)$ *have the same orientation* if and only if $p_{q_1} \circ \varphi_1$ is isotopic to $p_{q_2} \circ \varphi_2$ on $\mathbb{S}^2$, for q_1 inside $\varphi_1(|T|)$ and q_2 inside $\varphi_2(|T|)$. It is straightforward to check that the choice of q_1 and q_2 does not matter.

Given a convex realization polyhedral realization ψ, we define *the deformation space of convex polyhedra inscribed in the sphere* determined by ψ, denoted by $Y(T, \psi) \subset Y(T)$, as the set of all convex realizations φ of $\mathbb{S}^2$ having the same orientation with ψ. Then $Y(T, \psi)$ is naturally a manifold of dimension $2|V|$ without boundary. Theorem 1.2 can be rephrased as

Theorem 1.4. *Given a convex realization ψ, $Y(T, \psi)$ is homeomorphic to $\mathbb{R}^{2|V|-3} \times$* SO(3).

The space of inscribed convex polyhedra in the unit sphere is closely related to realization spaces of polytopes with a fixed combinatorial type. Steinitz [1922] proved that every planar 3-connected graph is the 1-skeleton of a convex polyhedron in $\mathbb{R}^3$. Moreover, his proof implies that the realization space of polyhedra is a cell after the normalization by affine transformations. See [Richter-Gebert 1996] for a detailed discussion about the realization spaces.

Connections between the two spaces. Denote $Y_0(T)$ as the subset of $Y(T)$ containing all the convex realizations φ such that the origin $O = (0, 0, 0)$ is inside $\varphi(|T|)$. Given a convex realization ψ, denote $Y_0(T, \psi) = Y(T, \psi) \cap Y_0(T)$. If $\varphi \in Y_0$, then the radial projection p_O maps the triangulation structure on $\varphi(|T|)$ to a strictly Delaunay convex geometric triangulation of $\mathbb{S}^2$. This naturally produces a homeomorphism from $Y_0(T, \psi)$ to $X(T, p_O \circ \psi|_{T^{(1)}})$ for any convex realization ψ. Therefore, Theorem 1.3 can be reformulated as

Theorem 1.5. *Given a convex realization $\psi \in Y_0$, $Y_0(T, \psi)$ is homeomorphic to $\mathbb{R}^{2|V|-3} \times$* SO(3).

Organization. In Section 2, we will review the concept of angle structures. In Section 3, we will determine the topology of the spaces of Delaunay triangulations of convex polygons with fixed angles. In Section 4, we will prove Theorem 1.4

and Theorem 1.5. In Section 5, we will provide examples showing the homotopy types of spaces of Delaunay triangulations of flat tori and convex polygons could be disconnected.

2. Angle structures on triangulated surfaces

The tool to study the topology of spaces of Delaunay triangulations on $\mathbb{S}^2$ is the concept of angle structure or angle system on triangulated surfaces. This concept was proposed by Colin de Vedière [1991], and developed by Rivin [1994], Leibon [2002], Luo [2006], Bobenko and Springborn [2004], Springborn [2008], and others. We briefly summarize the theory in the following.

Angle structures on triangulated surfaces. Assume $|T|$ is a 2-dimensional manifold possibly with boundary. A *corner* in T is defined as a vertex-face pair (v, f) in T such that the face f contains v. It represents the inner angle of the face f at the vertex v. A *Euclidean angle structure* θ, or an angle structure in short, on T is a positive function on the set of the corners such that $\theta_1 + \theta_2 + \theta_3 = \pi$ for the three angles in every face f. Every angle structure can be presented as a positive vector in $\mathbb{R}^{3|F|}$. Denote $V_b \subset V$ as the set of boundary vertices, and then the *edge invariant* $\alpha = \alpha(\theta) \in \mathbb{R}^{E \cup V_b}$ is defined as:

(a) $\alpha_e = \theta_1 + \theta_2$, if e is an inner edge, and θ_1 and θ_2 are the two angles opposite to e.

(b) $\alpha_e = \theta_1$, if e is a boundary edge, and θ_1 is the angle opposite to e.

(c) $\alpha_v = \sum_i \theta_i$, if v is a boundary vertex, and θ_i's are the angles at v.

Denote the set of angle structures realizing a prescribed edge invariant $\bar{\alpha} \in \mathbb{R}^{E \cup V_b}$ as $\mathcal{A}(T, \bar{\alpha})$.

Given an edge length function $l \in \mathbb{R}^E$ satisfying the triangle inequalities, we can naturally determine a piecewise Euclidean metric on T and induce an angle structure $\theta(l)$ using the inner angles in this piecewise Euclidean metric. Notice that not every angle structure can be induced from a piecewise Euclidean metric, and there are holonomy conditions on the angle structures so that we can glue the Euclidean triangles determined by the angles to form a Euclidean triangle mesh. We will see that these geometric angle structures can be found by the following variational principles on $\mathcal{A}(T, \bar{\alpha})$.

Variational principles of angle structures. Variational methods are introduced to find piecewise Euclidean surfaces with a prescribed edge invariant. The functionals in these variational principles have elegant geometric interpretations in terms of volumes of polyhedra in the hyperbolic 3-space $\mathbb{H}^3$.

For each face f in F, an energy functional is defined in terms of three angles at the corners of the face in an angle structure. For a face in a Euclidean angle structure
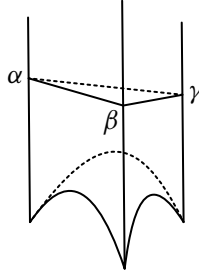
Figure 2. The volume of an ideal tetrahedron.

with three angles (α, β, γ), the energy functional is the volume of ideal hyperbolic tetrahedron whose horospherical section is similar to a Euclidean triangle with three angles (α, β, γ). See Figure 2. The volume is given by

$$V(\alpha, \beta, \gamma) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where Λ is the *Lobachevsky function*

$$\Lambda(x) = -\int_0^x \log(2\sin\theta)\,d\theta.$$

The total energy for a given angle structure is defined as the sum of functionals on each face

$$\mathcal{E}(\theta) = \sum_{f_i \in F} V_i(\alpha_i, \beta_i, \gamma_i).$$

The variational principles for these energy functionals can be summarized as follows.

Theorem 2.1 [Rivin 1994]. *Assume $\bar{\alpha} \in (0, \pi]^{E \cup V_b}$ and $\mathcal{A}(T, \bar{\alpha})$ is nonempty, then*:

(a) *The energy functional $\mathcal{E}$ is strictly concave down on $\mathcal{A}(T, \bar{\alpha})$.*

(b) *There exists a unique critical point $\theta = \Theta(\bar{\alpha})$ of $\mathcal{E}$ in $\mathcal{A}(T, \bar{\alpha})$.*

(c) *$\Theta(\bar{\alpha})$ is the unique angle structure in $\mathcal{A}(T, \bar{\alpha})$ that could be induced from a piecewise Euclidean metric on T.*

Denote $\mathcal{A}_0(T)$ as the set of angle structures θ such that $\alpha(\theta) \in (0, \pi)^{E \cup V_b}$ and the angle sum $\sum_i \theta_i$ around any interior vertex is 2π. Denote $\mathcal{A}_E(T)$ as the set of angle structures θ in $\mathcal{A}_0(T)$ that can be induced from a piecewise Euclidean metric on T. Notice that the angle structure induced from a Delaunay triangulation of a convex polygon in the plane belongs to $\mathcal{A}_E(T)$. Then by Theorem 2.1, we have the following.

Lemma 2.2. *If $\mathcal{A}_E(T)$ is nonempty, then $\mathcal{A}_E(T)$ is homeomorphic to $\mathbb{R}^k$ for some $k \geq 0$.*

Proof. If $\mathcal{A}_E(T)$ is nonempty, then $\mathcal{A}_0(T)$ is nonempty. From the definition we can see that $\mathcal{A}_0(T)$ is an open convex subset in an affine subspace of $\mathbb{R}^{3|F|}$. Then its image $\alpha(\mathcal{A}_0(T))$ under the edge invariant map α, which is a linear map, is an open convex subset of an affine subspace of $\mathbb{R}^{E \cup V_b}$. Hence, $\alpha(\mathcal{A}_0(T))$ is homeomorphic to $\mathbb{R}^k$ for some $k \geq 0$.

It remains to show that $\bar{\alpha} \mapsto \Theta(\bar{\alpha})$ is a homeomorphism from $\alpha(\mathcal{A}_0(T))$ to $\mathcal{A}_E(T)$. It is straightforward to show that such a map is continuous from $\alpha(\mathcal{A}_0(T))$ to $\mathbb{R}^{3|F|}$. Moreover, $\bar{\alpha} \mapsto \Theta(\bar{\alpha}) \mapsto \alpha(\Theta(\bar{\alpha}))$ is the identity map on $\alpha(\mathcal{A}_0(T))$. By Theorem 2.1, $\theta \mapsto \alpha(\theta) \mapsto \Theta(\alpha(\theta))$ is the identity map on $\mathcal{A}_E(T)$. Then we only need to show that the image $\Theta(\bar{\alpha})$ is in $\mathcal{A}_E(T)$. By the definition we only need to verify that for any interior vertex v, the angle sum around v in $\Theta(\bar{\alpha})$ is equal to the angle sum around v in θ. This is because the angle sum of an angle structure θ around an interior vertex v is determined by the edge invariant $\alpha(\theta)$ as the following.

$$\sum_{f \in F: f \ni v} \theta_{v,f} = \sum_{f \in F: f \ni v} \pi - \sum_{e \in E: e \ni v} \alpha_e. \qquad \square$$

The dimension of the space $\mathcal{A}_E(T)$ can be explicitly computed in the next section.

3. Delaunay Triangulations of Convex Polygons

Assume that $|T|$ is homeomorphic to a closed disk, an embedding $\varphi : |T| \to \mathbb{R}^2$ is called a *triangulation of a polygon* if φ is linear on any triangle of T. Further such φ is called a *triangulation of a convex polygon* if the inner angle of the polygon $\varphi(|T|)$ at $\varphi(v_i)$ is less than π for any boundary vertex v_i of T. Such φ is called *strictly Delaunay* if for any pair of adjacent triangles $\triangle ABC$ and $\triangle ABD$ in $\varphi(T)$, D is strictly outside the circumcircle of $\triangle ABC$. This condition is equivalent to that $a + a' < \pi$, where a, a' are the inner angles of two neighbored triangles as in Figure 1.

Denote $\theta(\varphi)$ as the angle structure induced from the triangulation φ, and $Z(T) = \{\varphi : \theta(\varphi) \in \mathcal{A}_E(T)\}$ as the set of strictly Delaunay triangulations of a convex polygon. We say two embeddings φ, ψ from $|T|$ to $\mathbb{R}^2$ have the same orientation if $\psi \circ \varphi^{-1}$ is an orientation preserving map on $\varphi(|T|)$. Given a triangulation ψ of a polygon, denote $Z(T, \psi)$ as the set of strictly Delaunay triangulations φ of a convex polygon that have the same orientation with ψ.

Furthermore, if we are given a directed edge e_{ij} of T, denote $Z(T, \psi, e_{ij})$ as the set of strictly Delaunay triangulations $\varphi \in Z(T, \psi)$ satisfying that $\varphi(j) - \varphi(i) = (\lambda, 0)$ for some $\lambda > 0$. Then it is elementary to see that a triangulation in $Z(T, \psi, e_{ij})$ is uniquely determined by the induced angle structure $\theta(\varphi)$, $\varphi(i)$ and $\varphi(j) - \varphi(i)$. Therefore, $\varphi \mapsto (\theta(\varphi), \varphi(i), \varphi(j) - \varphi(i))$ gives a homeomorphism from $Z(T, \psi, e_{ij})$ to $\mathcal{A}_E(T) \times \mathbb{R}^2 \times \mathbb{R}_+$. On the other hand, the space $Z(T, \psi, e_{ij})$ is a $(2|V| - 1)$-dimensional manifold if not empty, then we have the following from Lemma 2.2.
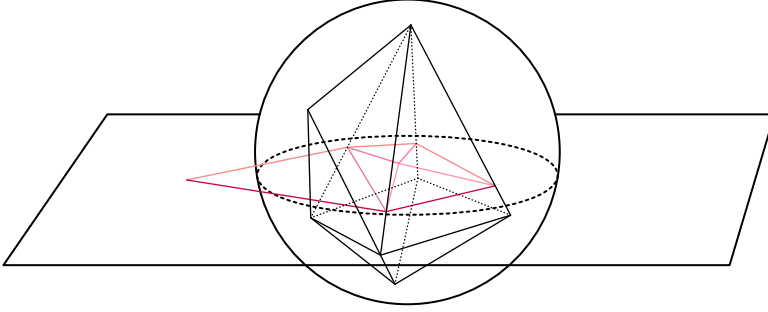
Figure 3. The stereographic projection of an inscribed convex polyhedron.

Corollary 3.1. *Given any Delaunay triangulation of a convex polygon ψ, and a directed edge e_{ij}, $Z(T, \psi, e_{ij})$ is homeomorphic to $\mathbb{R}^{2|V|-1}$.*

In the next section, we will reduce the spaces of Delaunay triangulations on the sphere and the spaces of convex polyhedra inscribed in the sphere to the space $Z(T, \psi, e_{ij})$.

4. Proof of the main theorems

We will prove Theorems 1.4 and 1.5 in this section using the stereographic projection. It is well known that the stereographic projection

$$\pi : (x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

gives an angle-preserving diffeomorphism from $\mathbb{S}^2 \backslash \{(0, 0, 1)\}$ to $\mathbb{R}^2$. For a circle Γ on $\mathbb{S}^2$, the stereographic projection maps Γ to a circle on $\mathbb{R}^2$ if Γ does not contain $(0, 0, 1)$, and maps $\Gamma \backslash \{(0, 0, 1)\}$ to a straight line in $\mathbb{R}^2$ if Γ contains $(0, 0, 1)$.

Assume $|T|$ is homeomorphic to $\mathbb{S}^2$, and v_0 is a vertex of T, and $\psi \in Y(T)$ is a convex realization inscribed in the unit sphere, then denote $Y(T, \psi, v_0)$ (resp. $Y(T, v_0)$, $Y_0(T, \psi, v_0)$, $Y_0(T, v_0)$) as the set of $\varphi \in Y(T, \psi)$ (resp. $\varphi \in Y(T)$, $Y_0(T, \psi)$, $Y_0(T)$) with $\varphi(v_0) = (0, 0, 1)$.

Lemma 4.1. *Assume $|T|$ is homeomorphic to $\mathbb{S}^2$, v_0 is a vertex of T, T_0 denotes the subcomplex of T obtained by removing the open 1-ring neighborhood of v_0, and e_{ij} is a directed edge in T_0:*

(a) *There exists a map $\tilde{\pi} : Y(T, v_0) \to Z(T_0)$ induced by π such that $\phi = \tilde{\pi}(\varphi)$ is the strictly Delaunay triangulation of a convex polygon determined by $\phi(v) = \pi(\varphi(v))$ for any vertex v of T_0; see Figure 3.*

(b) *There exists a map $\tilde{\eta} : Z(T_0) \to Y(T, v_0)$ induced by π^{-1} such that $\varphi = \tilde{\eta}(\phi)$ is the convex realization determined by $\varphi(v) = \pi^{-1}(\phi(v))$ for any vertex v of T_0.*

(c) $\tilde{\pi}$ and $\tilde{\eta}$ are inverse to each other and then $\tilde{\pi}$ is a homeomorphism from $Y(T, v_0)$ to $Z(T_0)$.

(d) Given a convex realization $\psi \in Y(T, v_0)$, $\tilde{\pi}$ gives a homeomorphism from $Y(T, \psi, v_0)$ to $Z(T_0, \tilde{\pi}(\psi))$.

(e) If $\phi \in \tilde{\pi}(Y_0(T, v_0))$:
  (i) The origin $(0, 0)$ is in the interior of $\phi(|T_0|)$.
  (ii) $\lambda\phi$ is also in $\tilde{\pi}(Y_0(T, v_0))$ for any $\lambda \in (0, 1)$.

(f) For any $\varphi \in Y(T, \psi)$, there exists a unique $\varphi_0 \in Y(T, \psi, v_0)$ and $g \in \mathrm{SO}(3)$, such that $\varphi = g \circ \varphi_0$ and $\tilde{\pi}(\varphi_0) \in Z(T_0, \tilde{\pi}(\psi), e_{ij})$. Then $Y(T, \psi)$ is homeomorphic to $Z(T_0, \tilde{\pi}(\psi), e_{ij}) \times \mathrm{SO}(3)$.

(g) For any $\varphi \in Y_0(T, \psi)$, there exists a unique $\varphi_0 \in Y_0(T, \psi, v_0)$ and $g \in \mathrm{SO}(3)$, such that $\varphi = g \circ \varphi_0$ and $\tilde{\pi}(\varphi_0) \in Z(T_0, \tilde{\pi}(\psi), e_{ij})$. Then $Y_0(T, \psi)$ is homeomorphic to $(\tilde{\pi}(Y_0(T, v_0)) \cap Z(T_0, \tilde{\pi}(\psi), e_{ij})) \times \mathrm{SO}(3)$.

Proof. (a) and (b) are true by the empty circle property of the (strict) Delaunay triangulations and the fact that the stereographic projection preserves circles.

(c) This is a direct consequence from the definition.

(d) Given a convex realization $\varphi \in Y(T, v_0)$ and q_1 inside $\psi(|T|)$ and q_2 inside $\varphi(|T|)$, the following elementary facts related to orientations are equivalent by the definition and properties of stereographic projections:

  (i) $\varphi \in Y(T, \psi, v_0)$.

 (ii) ψ and φ have the same orientation.

(iii) $\pi_{q_1} \circ \psi$ is isotopic to $\pi_{q_2} \circ \varphi$ in $\mathbb{S}^2$.

(iv) $\pi_{q_1} \circ \psi$ and $\pi_{q_2} \circ \varphi$ have the same orientation.

 (v) $\tilde{\pi}(\psi)$ and $\tilde{\pi}(\varphi)$ have the same orientation.

(vi) $\tilde{\pi}(\varphi) \in Z(T_0, \tilde{\pi}(\psi))$.

(e) If $\varphi \in Y_0(T, v_0)$, then the origin is inside $\varphi(|T|)$. Then the ray starting from the north pole passing through the origin intersects with $\varphi(|T|)$ at a unique point q in the interior of $\varphi(|T_0|)$. So part (i) is true. We prove part (ii) by contradiction. If $\lambda\phi$ is not in $\tilde{\pi}(Y_0(T, v_0))$ for some $\lambda \in (0, 1)$, then the origin $(0, 0, 0)$ is not inside $\tilde{\eta}(\lambda\phi)$ and there is an open hemisphere H on $\mathbb{S}^2$ not intersecting $\tilde{\eta}(\lambda\phi)(V)$. Notice that H does not contain the north pole so $\pi(H)$ is well-defined. Then $\pi(H)$ is an open disk containing $(0, 0)$ or an open half plane with $(0, 0)$ on its boundary, and $\pi(H)$ does not intersect $(\lambda\phi)(V)$. So $\pi(H)$ does not intersect $\phi(V)$, meaning that H does not intersect $\tilde{\eta}(\phi)(V)$. So $(0, 0, 0)$ is not inside $\tilde{\eta}(\phi)(|T|)$, but this contradicts with that $\phi \in \tilde{\pi}(Y_0(T, v_0))$.
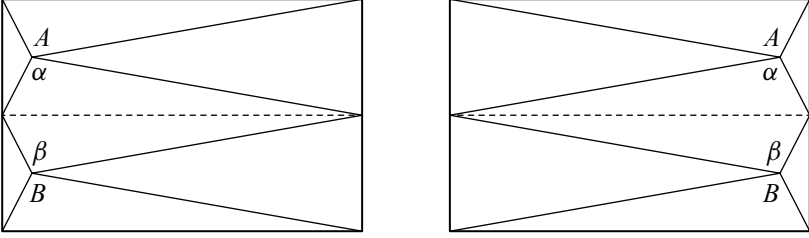
Figure 4. Counterexample, a convex polygon.

(f) and (g) Follow from the fact that the rotation along the z-axis (or the origin in the xy-plane) is invariant under the stereographic projection. □

Proof of Theorem 1.4. This is an immediate consequence of Corollary 3.1 and part (f) of Lemma 4.1. □

Proof of Theorem 1.5. By part (g) of Lemma 4.1, we only need to show that $\tilde{\pi}(Y_0(T, v_0)) \cap Z(T_0, \tilde{\pi}(\psi), e_{ij})$ is homeomorphic to $\mathbb{R}^{2|V|-3}$. By part (e) of Lemma 4.1 it is elementary to verify that

$$\varphi \in \tilde{\pi}(Y_0(T, v_0)) \cap Z(T_0, \tilde{\pi}(\psi), e_{ij})$$

is uniquely determined by $\theta(\varphi)$, $\varphi^{-1}(0, 0)$ and $d(\varphi)$, where $d(\varphi)$ is the Euclidean diameter of $\varphi(|T|)$ and describes the scaling transformation needed to determine φ. So $\varphi \mapsto (\theta(\varphi), \varphi^{-1}(0, 0), d(\varphi))$ gives a continuous injective map from $\tilde{\pi}(Y_0(T, v_0)) \cap Z(T_0, \tilde{\pi}(\psi), e_{ij})$ to $\mathcal{A}_E(T_0) \times \text{int}(|T_0|) \times (0, \infty)$, where $\text{int}(|T_0|) = |T_0| \backslash \partial(|T_0|)$ is homeomorphic to $\mathbb{R}^2$. Then by Lemma 2.2 and a dimension counting, we complete the proof. □

5. Delaunay triangulations of other surfaces

In this section, we will discuss the space of Delaunay geodesic triangulations of convex polygons and flat tori.

Convex polygons. A convex polygon P in the plane is determined by the position of a sequence of cyclically ordered vertices. The following simple example in Figure 4 shows that for a fixed convex polygon P in the plane with a triangulation $\psi : T \to P$, denote the space of Delaunay triangulations of P which are isotopic to ψ and have the same orientation with ψ as $X(T, \psi)$. Notice that $X(T, \psi)$ is different from the space $Z(T, \psi)$ in Section 3, since the positions of boundary vertices of T for elements in X are fixed.

The following example shows that $X(T, \psi)$ may not be connected.

In Figure 4, there are nine interior edges in the triangulation, eight of which are Delaunay. The dashed edge might not be Delaunay. In Figure 4, if the vertices A
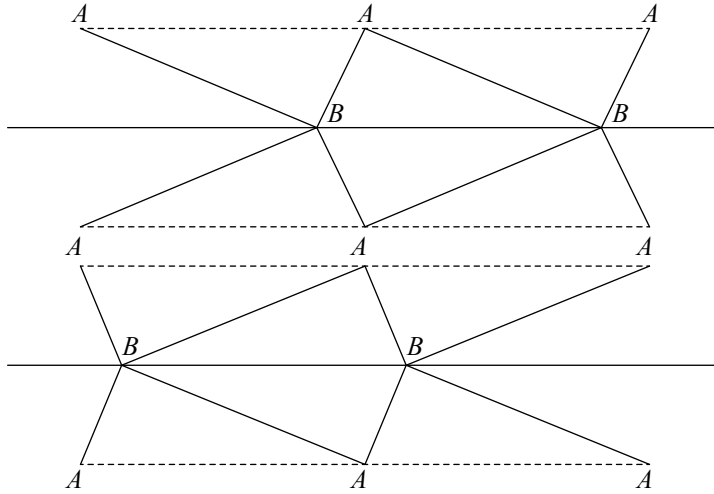
Figure 5. Counterexample, a flat torus.

and B are close to the vertical boundaries, then α and β are both acute, so we can construct two Delaunay triangulations τ_1 and τ_2 on the left and right. If there is a family of Delaunay triangulations connecting τ_1 and τ_2, the vertex A or B will pass the perpendicular bisector of the horizontal boundary of this rectangle. If the rectangle is flat enough, the angle sum $\alpha + \beta > \pi$ when one of A and B lies on the perpendicular bisector. This shows that $X(T, \psi)$ for this rectangle P is not connected.

Delaunay triangulations on flat tori. Assume $|T|$ is homeomorphic to the torus $\mathbb{T}^2$ with a marking homeomorphism whose restriction on $T^{(1)}$ is denoted as ψ. An embedding $\varphi : T^{(1)} \rightarrow \mathbb{T}^2$ is a *Delaunay geodesic triangulation* with the combinatorial type (T, ψ) satisfying:

(a) The restriction φ_{ij} of φ on each edge e_{ij}, identified with a unit interval $[0, 1]$, is a geodesic parametrized with constant speed.

(b) φ is homotopic to ψ.

(c) Equation (2) is satisfied for all edges in T.

Let $X = X(T, \psi)$ denote the set of all such geodesic triangulations, which is called the *deformation space of Delaunay geodesic triangulations of $\mathbb{T}^2$* of combinatorial type (T, ψ).

The following example shows that the space of Delaunay geodesic triangulations $X = X(T, \psi)$ may not be connected.

In Figure 5, we draw two geodesic triangulations τ_1 and τ_2 on a flat torus. For each geodesic triangulation, we draw two fundamental domains of this torus. The

triangulation has two vertices and six edges. Fixing the vertex A at a point in the universal covering, we can see that the position of the vertex B determines a geodesic triangulation of this flat torus. Notice that τ_1 and τ_2 are both Delaunay, since all the angles in these two triangulations are acute when B is sufficiently close to the vertical line connecting two adjacent copies of A in the universal covering.

We can choose the shape of the fundamental domain of the flat torus as shown in the picture. Then τ_1 and τ_2 are in two different connected components of the space of Delaunay triangulations of this flat torus. This observation is based on the following fact: any path connecting τ_1 and τ_2 needs to move the vertex B from the right to the left. However, we can choose a flat enough fundamental domain such that when B passes the perpendicular bisector of the dashed edge, the dashed edge is never Delaunay. This implies that the space $X = X(T, \psi)$ for this flat torus is not connected.

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YANWEN LUO
DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
PISCATAWAY, NJ
UNITED STATES

yl1594@rutgers.edu

TIANQI WU
DEPARTMENT OF MATHEMATICS
CLARK UNIVERSITY
WORCESTER, MA
UNITED STATES

tianwu@clarku.edu

XIAOPING ZHU
DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ
UNITED STATES

xiaoping.zhu@rutgers.edu

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