Pacific Journal of Mathematics

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Volume 323 No. 1 March 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 1, 2023
https://doi.org/10.2140/pjm.2023.323.129

NONEXISTENCE OF NEGATIVE WEIGHT DERIVATIONS OF THE LOCAL k-TH HESSIAN ALGEBRAS ASSOCIATED TO ISOLATED SINGULARITIES

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A new conjecture about the nonexistence of negative weight derivations of the k-th Hessian algebras for weighted homogeneous isolated hypersurface singularities is proposed. We verify this conjecture up to dimension three.

1. Introduction

A holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is called quasihomogeneous if $f \in J(f)$, where $J(f) := (\partial f/\partial z_0, \partial f/\partial z_1, \dots, \partial f/\partial z_n)$ is the Jacobian ideal. A polynomial $f(z_0, \dots, z_n)$ is called weighted homogeneous of type $(\alpha_0, \dots, \alpha_n; d)$, where $\alpha_0, \dots, \alpha_n$ and d are fixed positive integers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $\alpha_0 i_0 + \cdots + \alpha_n i_n = d$. According to a beautiful theorem of Saito [1971], if $V = V(f)$ has isolated singularities, then f is a weighted homogeneous polynomial after a biholomorphic change of coordinates if and only if f is quasihomogeneous. The order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity, denoted by $\mathrm{mult}(f)$, of the singularity $(V, 0)$.

For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ that is defined by the holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, one has the moduli algebra $A(V) := \mathcal{O}_{n+1}/(f, \partial f/\partial z_0, \dots, \partial f/\partial z_n)$ which is finite dimensional. The well-known Mather–Yau theorem [1982] states that: Let $(V_1, 0)$ and $(V_2, 0)$ be two isolated hypersurface singularities, and let $A(V_1)$ and $A(V_2)$ be their respective moduli algebras, then $(V_1, 0) \cong (V_2, 0) \Longleftrightarrow A(V_1) \cong A(V_2)$. In 1983, Yau introduced the Lie algebra of derivations of $A(V)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. The finite dimensional Lie algebra $L(V)$ is called the Yau algebra, and its dimension $\lambda(V)$ is called the Yau number in ([Khimshiashvili 2006; Yu 1996]). The Yau algebra plays an important role in singularity theory and was used to distinguish complex analytic structures of isolated hypersurface singularities [Seeley and Yau 1990]. Yau and

Zuo was supported by the NSFC (12271280) and the Tsinghua University Initiative Scientific Research Program. Yau was supported by the Tsinghua University Education Foundation (042202008).
MSC2020: 14B05, 32S05.
Keywords: derivations, Hessian algebra, isolated singularity, weighted homogeneous.

his collaborators have been systematically studying the Yau algebras of isolated hypersurface singularities and their generalizations from the eighties (see [Yau 1986; Xu and Yau 1996; Seeley and Yau 1990; Chen et al. 2019; Hussain et al. 2021]). In [Hussain et al. 2021] and [Chen et al. 2020], many new derivation Lie algebras that arise from isolated hypersurface singularities are introduced. These Lie algebras are more subtle invariants of singularities compared with previous Lie algebras. These Lie algebras are defined as follows: For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ defined by the holomorphic function $f(z_0, z_1, \dots, z_n)$, let $\mathrm{Hess}(f)$ be the Hessian matrix (f_{ij}) of the second-order partial derivatives of f and $h(f)$ be the Hessian of f, i.e., the determinant of the matrix $\mathrm{Hess}(f)$. More generally, for each k satisfying $0 \le k \le n+1$, we denote by I_k the ideal in $\mathcal{O}_{n+1}$ generated by all $k \times k$-minors in the matrix $\mathrm{Hess}(f)$. In particular, the ideal $I_{n+1} = (h(f))$ is a principal ideal. For each k as above, consider the graded k-th Hessian algebra of the polynomial f defined by

$$H_k(f) = \mathcal{O}_{n+1}/\big((f) + J(f) + I_k\big).$$

In particular, $H_0(f)$ is exactly the well-known moduli algebra $A(V)$.

It is easy to check that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is contact invariant of f, i.e., it depends only on the isomorphism class of the germ $(V, 0)$ [Dimca and Sticlaru 2015].

In particular, $H_{n+1}(f)$ has a geometric meaning. We recall the following beautiful characterization theorem of zero-dimensional isolated complete intersection singularities:

Theorem 1.1 [Dimca 1984]. *Two zero-dimensional isolated complete intersection singularities X and Y are isomorphic if and only if their singular subspaces* $\mathrm{Sing}(X)$ *and* $\mathrm{Sing}(Y)$ *are isomorphic.*

Remark 1.2. Let $V = V(f)$ be an isolated quasihomogeneous hypersurface singularity. It follows that X, defined by $(\partial f/\partial z_0, \dots, \partial f/\partial z_n)$, is a zero-dimensional isolated complete intersection singularity. In this case, $\mathrm{Sing}(X)$ is defined by

$$\left(f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}, h(f)\right).$$

Theorem 1.1 implies that to study the analytic isomorphism type of the zero-dimensional isolated complete intersection singularity X, we only need to consider the Artinian local algebra $H_{n+1}(f)$, which is the coordinate ring of $\mathrm{Sing}(X)$.

Combining Theorem 1.1 with the Mather–Yau theorem, we know that $H_{n+1}(f)$ is a complete invariant of quasihomogeneous isolated hypersurface singularities (i.e., $H_{n+1}(f)$ determines and is determined by the analytic isomorphism type of the singularity). In [Chen et al. 2020], we also call $H_{n+1}(f)$ the generalized moduli algebra of V. As a generalization of the Yau algebra, it is natural to introduce the following new Lie algebras for isolated hypersurface singularities:

Definition 1.3. Let $V = \{f = 0\}$ be a germ of the isolated hypersurface singularity at the origin of $\mathbb{C}^{n+1}$ defined by $f(z_0, z_1, \ldots, z_n)$, with $n \geq 1$. The series of new derivation Lie algebras arising from the isolated hypersurface singularity $(V, 0)$ is defined as $L_k(V) := \mathrm{Der}(H_k(f), H_k(f))$, where $0 \leq k \leq n+1$, or $\mathrm{Der}(H_k(f))$ for short. The dimension of $L_k(V)$ is denoted by $\lambda_k(V)$.

It is known that the Yau algebra cannot characterize the ADE singularities completely. In fact, Elashvili and Khimshiashvili [2006] proved the following result: If X and Y are two simple singularities except for the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic. However, we have proven that the ADE singularities are characterized completely by the new Lie algebra $L_{n+1}(V)$ as follows. We have reasons to believe that the new Lie algebras $L_k(V)$ and numerical invariants $\lambda_k(V)$, where $1 \leq k \leq n+1$, will also play an important role in the study of singularities.

Theorem 1.4 [Chen et al. 2020]. *If X and Y are two n-dimensional ADE singularities, then $L_{n+1}(X) \cong L_{n+1}(Y)$ as Lie algebras, if and only if X and Y are analytically isomorphic.*

Let A be a weighted zero-dimensional complete intersection, i.e., a commutative algebra of the form

$$A = \mathbb{C}[z_0, z_1, \ldots, z_n]/I,$$

where the ideal I is generated by a regular sequence of length $n+1$, $(f_0, f_1, \ldots, f_n)$. Here, the variables have strictly positive integral weights, denoted by $\mathrm{wt}(z_i) = \alpha_i$, where $0 \leq i \leq n$, and the equations are weighted homogeneous with respect to these weights. Consequently, the algebra A is graded and one may speak about its homogeneous degree k derivations, where k is an integer. Recall that a linear map $D : A \to A$ is a derivation if $D(ab) = D(a)b + aD(b)$, for any $a, b \in A$. The map D belongs to $\mathrm{Der}^k(A)$ if $D : A^* \to A^{*+k}$.

On the one hand, one of the most prominent open problems in rational homotopy theory is related to the vanishing of the above derivations in strictly negative degrees.

Halperin Conjecture [Meier 82; Chen et al. 2019]. *Let*

$$A = \mathbb{C}[z_0, z_1, \ldots, z_n]/I,$$

where the ideal I is generated by a regular sequence of length $n+1$, $(f_0, f_1, \ldots, f_n)$. Here, the variables have strictly positive even integer weights, denoted by $\mathrm{wt}(z_i) = \alpha_i$, $0 \leq i \leq n$, and the equations are weighted homogeneous with respect to these weights. Then $\mathrm{Der}^{<0}(A) = 0$.

The Halperin Conjecture has been verified in several particular cases, see [Papadima and Paunescu 1996; Thomas 1981]. For recent progress, please see [Chen et al. 2019].

Let $(V,0)=\{(z_0,z_1,\dots,z_n)\in\mathbb{C}^{n+1}\colon f(z_0,z_1,\dots,z_n)=0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(z_0,z_1,\dots,z_n)$ of type $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$. Then by a well-known result of Saito [1971], we can always assume, without loss of generality, that $d\geq 2\alpha_i>0$ for all $0\leq i\leq n$. We give the variable z_i weight α_i for $0\leq i\leq n$, thus the moduli algebra $A(V)$ is a graded algebra, i.e., $A(V)=\bigoplus_{i=0}^{\infty}A_i(V)$, and the Lie algebra of derivations $\operatorname{Der}(A(V))$ is also graded. Thus, $L(V)$ is graded.

On the other hand, Yau discovered independently the following conjecture on the nonexistence of the negative weight derivation, which is a special case of the Halperin Conjecture.

Yau Conjecture (see [Chen 1995; Chen et al. 1995]). *Consider the isolated singularity*

$$(V,0)=\left\{(z_0,z_1,\dots,z_n)\in\mathbb{C}^{n+1}\colon f(z_0,z_1,\dots,z_n)=0\right\}$$

defined by the weighted homogeneous polynomial $f(z_0,z_1,\dots,z_n)$ *of weight type* $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$. *Assume that* $d\geq 2\alpha_0\geq 2\alpha_1\geq\cdots\geq 2\alpha_n>0$, *without loss of generality. Then there is no nonzero negative weight derivation on the moduli algebra* (= *Milnor algebra*)

$$A(V)=\mathbb{C}[z_0,z_1,\dots,z_n]\,\Big/\left(\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_n}\right),$$

i.e., $L(V)$ *is nonnegatively graded.*

This conjecture is still open and was only proved in the low-dimensional case $n\leq 3$ by explicit calculations [Chen 1995; Chen et al. 1995]. It was also proved for the high-dimensional singularities under certain conditions [Yau and Zuo 2016] and for homogeneous singularities (see Proposition 2.1).

Theorem 1.5 [Chen 1995, Theorem 2.1]. *Let* $f(z_0,z_1,z_2,z_3)$ *be a weighted homogeneous polynomial of type* $(\alpha_0,\alpha_1,\alpha_2,\alpha_3;d)$ *with an isolated singularity at the origin. Assume that* $d\geq 2\alpha_0\geq 2\alpha_1\geq 2\alpha_2\geq 2\alpha_3>0$, *without loss of generality. Let* D *be a derivation of the moduli algebra*

$$A(V)=\mathbb{C}[z_0,z_1,z_2,z_3]\,\Big/\left(\frac{\partial f}{\partial z_0},\dots,\frac{\partial f}{\partial z_3}\right).$$

Then $D\equiv 0$ *if* D *is negatively weighted.*

Assume that f is a weighted homogeneous polynomial, then the k-th Hessian algebra $H_k(V)$ and $L_k(V)$ are also naturally graded. It is natural to propose the following new conjecture:

Conjecture 1.6. *Let $(V,0)=\{(z_0,z_1,\dots,z_n)\in\mathbb{C}^{n+1}: f(z_0,z_1,\dots,z_n)=0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$. Assume that $d\ge 2\alpha_0\ge 2\alpha_1\ge\cdots\ge 2\alpha_n>0$, without loss of generality. Let $H_k(V)$ be the k-th Hessian algebra. Furthermore, in the case of $1<k\le n$ (respectively, $k=1$), we need to assume that* $\operatorname{mult}(f)\ge 4$ *(respectively, 5). Then for any $0\le k\le n+1$, there is no nonzero, negative weight derivation on the $H_k(V)$, i.e., $L_k(V)$ is nonnegatively graded.*

This Conjecture 1.6 seems very hard to verify in general, in fact, when $k=0$, it is exactly the long-standing Yau Conjecture above. When $k=n+1$, it was also verified for $n\le 3$ as follows:

Theorem 1.7 [Ma et al. 2020]. *Consider the isolated singularity*

$$(V,0)=\left\{(z_0,z_1,\dots,z_n)\in\mathbb{C}^{n+1}: f(z_0,z_1,\dots,z_n)=0\right\}$$

defined by the weighted homogeneous polynomial f of weight type $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$, where $1\le n\le 3$. Assume that $d\ge 2\alpha_0\ge 2\alpha_1\ge\cdots\ge 2\alpha_n>0$, without loss of generality. Let D be a derivation of the algebra

$$\mathbb{C}[z_0,z_1,\dots,z_n]\Big/\left(\frac{\partial f}{\partial z_0},\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n},\det\left(\frac{\partial^2 f}{\partial z_i\partial z_j}\right)_{0\le i,j\le n}\right).$$

Then $D\equiv 0$, if D has negative weight, i.e., $L_{n+1}(V)$ is nonnegatively graded for $1\le n\le 3$.

In this paper, we shall verify Conjecture 1.6 for the case $n=1,2$, with $1\le k\le n$, and $n=3$, with $1<k\le 3$ (the case $n=0$ is trivial). The proof of the case where $n=3$ and $k=1$ is completely different and long. It will appear in our subsequent paper. In this paper, we obtain the following main result which partially verifies the Conjecture 1.6:

Main Theorem. *Let $(V,0)=\{(z_0,z_1,\dots,z_n)\in\mathbb{C}^{n+1}: f(z_0,z_1,\dots,z_n)=0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_0,\alpha_1,\dots,\alpha_n;d)$, where $1\le n\le 3$. Assume, without loss of generality, that $d\ge 2\alpha_0\ge 2\alpha_1\ge\cdots\ge 2\alpha_n>0$. Let $L_k(V)$ be the derivation Lie algebra of the k-th Hessian algebra $H_k(V)$ and $D_k\in L_k(V)$.*

(a) *For $n=1$, if D_1 is of negative weight, then $D_1\equiv 0$.*

(b) *For $n=2$, if D_1 (respectively, D_2) is of negative weight, then $D_1\equiv 0$ (respectively, $D_2\equiv 0$). In this case, we need the assumption* $\operatorname{mult}(f)\ge 4$, *see Example 1.8.*

(c) *For $n=3$, if D_2 (respectively, D_3) is of negative weight, then $D_2\equiv 0$ (respectively, $D_3\equiv 0$). In this case, we need the assumption* $\operatorname{mult}(f)\ge 4$, *see Example 1.9.*

Example 1.8. We need to add the condition $\mathrm{mult}(f) \geq 4$ in Main Theorem (b) due to the following two examples:

(a) Let $f = z_0^3 + z_0 z_1 z_2^2 + z_1^3 z_2 + z_2^5$ with weighted type (5, 4, 3; 15). We have

$$\frac{\partial f}{\partial z_0} = 3z_0^2 + z_1 z_2^2, \quad \frac{\partial f}{\partial z_1} = z_0 z_2^2 + 3z_1^2 z_2, \quad \frac{\partial f}{\partial z_2} = 2z_0 z_1 z_2 + z_1^3 + 5z_2^4$$

and

$$\frac{\partial^2 f}{\partial z_0^2} = 6z_0, \qquad \frac{\partial^2 f}{\partial z_1^2} = 6z_1 z_2, \qquad \frac{\partial f}{\partial z_2^2} = 2z_0 z_1 + 20z_2^3,$$
$$\frac{\partial^2 f}{\partial z_0 \partial z_1} = z_2^2, \qquad \frac{\partial^2 f}{\partial z_0 \partial z_2} = 2z_1 z_2, \qquad \frac{\partial f}{\partial z_1 \partial z_2} = 2z_0 z_2 + 3z_1^2.$$

It is easy to check that $D_1 = z_2(\partial/\partial z_1)$ is a negative weight derivation (weighted degree of D_1 is -1, i.e., $\mathrm{wt}(D_1) = -1$) of

$$\mathbb{C}[z_0, z_1, z_2] \Big/ \left(\frac{\partial^2 f}{\partial z_0^2}, \frac{\partial^2 f}{\partial z_1^2}, \frac{\partial^2 f}{\partial z_2^2}, \frac{\partial^2 f}{\partial z_0 \partial z_1}, \frac{\partial^2 f}{\partial z_0 \partial z_2}, \frac{\partial^2 f}{\partial z_1 \partial z_2} \right),$$

i.e., $D_1 \in L_1(V(f))$.

(b) Let $f = z_0^2 z_2 + z_0 z_2^5 + z_1^3$ with weighted type (4, 3, 1; 9). We have

$$\frac{\partial f}{\partial z_0} = 2z_0 z_2 + z_2^5, \quad \frac{\partial f}{\partial z_1} = 3z_1^2, \quad \frac{\partial f}{\partial z_2} = z_0^2 + 5z_0 z_2^4$$

and

$$\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix} = \begin{vmatrix} 2z_2 & 0 & 2z_0 + 5z_2^4 \\ 0 & 6z_1 & 0 \\ 2z_0 + 5z_2^4 & 0 & 20z_0 z_2^3 \end{vmatrix}.$$

Thus $I_2 = \langle f_1, f_2, f_3, f_4 \rangle$, where

$$f_1 = \begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} = 12z_1 z_2, \qquad f_2 = \begin{vmatrix} f_{01} & f_{02} \\ f_{11} & f_{12} \end{vmatrix} = -12z_0 z_1 - 30z_1 z_2^4,$$
$$f_3 = \begin{vmatrix} f_{00} & f_{02} \\ f_{02} & f_{22} \end{vmatrix} = 20z_0 z_2^4 - 4z_0^2 - 25z_2^8, \quad f_4 = \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{12} \end{vmatrix} = 120z_0 z_1 z_2^3.$$

It is easy to check that $D_2 = z_1(\partial/\partial z_0) \in L_2(V(f))$ is a negative weight derivation ($\mathrm{wt}(D_2) = -1$).

Example 1.9. We need to add the condition $\mathrm{mult}(f) \geq 4$ in Main Theorem (c) due to the following example:

Let $f = z_0^2 z_2 + z_2^3 z_0 + z_1^3 + z_3^5$ with weighted type (6, 5, 3, 3; 15). We have

$$\frac{\partial f}{\partial z_0} = 2z_0 z_2 + z_2^3, \quad \frac{\partial f}{\partial z_1} = 3z_1^2, \quad \frac{\partial f}{\partial z_2} = z_0^2 + 3z_2^2 z_0, \quad \frac{\partial f}{\partial z_3} = 5z_3^4$$

and

$$\begin{vmatrix} f_{00} & f_{01} & f_{02} & f_{03} \\ f_{01} & f_{11} & f_{12} & f_{13} \\ f_{02} & f_{12} & f_{22} & f_{23} \\ f_{03} & f_{13} & f_{23} & f_{33} \end{vmatrix} = \begin{vmatrix} 2z_2 & 0 & 2z_0+3z_2^2 & 0 \\ 0 & 6z_1 & 0 & 0 \\ 2z_0+3z_2^2 & 0 & 6z_0z_2 & 0 \\ 0 & 0 & 0 & 20z_3^3 \end{vmatrix}.$$

Thus $I_2 = \langle f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8 \rangle$, where

$$f_1 = \begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} = 12z_1z_2, \qquad f_2 = \begin{vmatrix} f_{01} & f_{02} \\ f_{11} & f_{12} \end{vmatrix} = -12z_0z_1 - 18z_1z_2^2,$$

$$f_3 = \begin{vmatrix} f_{00} & f_{02} \\ f_{02} & f_{22} \end{vmatrix} = -4z_0^2 - 9z_2^4, \qquad f_4 = \begin{vmatrix} f_{00} & f_{03} \\ f_{03} & f_{33} \end{vmatrix} = 40z_2z_3^3,$$

$$f_5 = \begin{vmatrix} f_{02} & f_{23} \\ f_{03} & f_{33} \end{vmatrix} = 40z_0z_3 + 60z_2^2z_3, \qquad f_6 = \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} = 36z_0z_1z_2,$$

$$f_7 = \begin{vmatrix} f_{11} & f_{13} \\ f_{13} & f_{33} \end{vmatrix} = 120z_1z_3^3, \qquad f_8 = \begin{vmatrix} f_{22} & f_{23} \\ f_{23} & f_{33} \end{vmatrix} = 120z_0z_2z_3^3.$$

It is easy to check that $D_2 := z_1(\partial/\partial z_0) \in L_2(V(f))$ is a negative weight derivation ($\mathrm{wt}(D_2) = -1$).

Remark 1.10. Examples 1.8 and 1.9 are interesting because one cannot find such examples when $k = 0$ (see the Yau Conjecture) and $k = n+1$ (see Theorem 1.7).

Xu and Yau [1996] used the property of nonexistence of negative derivations of the moduli algebra $A(V)$ to obtain a characterization of quasihomogeneous singularities (see [Xu and Yau 1996, Theorem 3.2] for details). We believe this characterization can be generalized by using the Lie algebra of derivations of the k-th Hessian algebra. The Main Theorem in this paper provides evidence for the generalization.

2. Proof of the Main Theorem

Firstly, we recall the following known results which will be used in proof of the Main Theorem frequently:

Proposition 2.1 [Xu and Yau 1996, Proposition 2.6]. *Let $A = \bigoplus_{i=0}^k A_i$ be a graded commutative Artinian local algebra with $A_0 = \mathbb{C}$. Suppose the maximal ideal of A is generated by A_j for some $j > 0$. Then $L(A)$ is a graded Lie algebra without negative weight.*

Lemma 2.2 [Yau 1986]. *Let $(A, \mathfrak{m})$ be a commutative local Artinian algebra ($\mathfrak{m}$ is the unique maximal ideal of A and $D \in L(A)$ is the derivation of A). Then D preserves the m-adic filtration of A, i.e., $D(\mathfrak{m}) \subset \mathfrak{m}$.*

Lemma 2.3 [Chen et al. 1995, Lemma 2.1]. *Let f be a weighted homogeneous polynomial with isolated singularity in the variables $z_0, \dots, z_n$ of type $(\alpha_0, \dots, \alpha_n; d)$. Assume $\mathrm{wt}(z_0) = \alpha_0 \geq \mathrm{wt}(z_1) = \alpha_1 \geq \cdots \geq \mathrm{wt}(z_n) = \alpha_n$. Then f must be as in one of the following two cases*:

Case 1: *Let* $f = z_0^m + a_1(z_1, \dots, z_n) z_0^{m-1} + \cdots + a_{m-1}(z_1, \dots, z_n) z_0 + a_m(z_1, \dots, z_n)$.

Case 2: *Let* $f = z_0^m z_i + a_1(z_1, \dots, z_n) z_0^{m-1} + \cdots + a_{m-1}(z_1, \dots, z_n) z_0 + a_m(z_1, \dots, z_n)$.

Lemma 2.4 [Chen 1995, Lemma 1.2]. *Let f be a weighted homogeneous polynomial in $z_0, \dots, z_n$ which defines an isolated singularity at the origin. Then there is a term of the form $z_i^{a_i}$ or $z_i^{a_i} z_j$ in f for any i ($a_i \geq 2$ in the case $z_i^{a_i}$ and $a_i \geq 1$ otherwise).*

Remark 2.5. When we talk about the weight of an element in an ideal, we always assume that the element is nonzero.

Now we begin to prove the Main Theorem.

Proof of the Main Theorem. Let

$$A_\nu := \mathbb{C}[z_0, z_1, \dots, z_n] \Big/ \left(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}, I_\nu \right)$$

and

$$B := \mathbb{C}[z_0, z_1, \dots, z_n] \Big/ \left(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right).$$

It is clear that $A_\nu = B/(I_\nu)$, and A_ν is a commutative Artinian algebra. Let $D_\nu \in L(A_\nu)$ be a derivation of A_ν, and let D_ν be an A_ν-linear combination of $\partial/\partial z_0, \partial/\partial z_1, \dots, \partial f/\partial z_n$. By Lemma 2.2, we know that $D_\nu(\mathfrak{m}) \subset \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal $(z_0, \dots, z_n)$, thus the coefficients of $\partial/\partial z_0, \partial/\partial z_1, \dots, \partial f/\partial z_n$ do not contain the constant term. Moreover, D_ν has negative weight, thus we write

$$D_\nu = p_0(z_1, \dots, z_n) \frac{\partial}{\partial z_0} + p_1(z_2, \dots, z_n) \frac{\partial}{\partial z_1} + \cdots + p_{n-2}(z_{n-1}, z_n) \frac{\partial}{\partial z_{n-2}} + c z_n^k \frac{\partial}{\partial z_{n-1}},$$

where $k \geq 1$ and c is a constant. Observe that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_0} \right) = d - \alpha_0, \ \mathrm{wt}\left(\frac{\partial f}{\partial z_1} \right) = d - \alpha_1, \dots, \ \mathrm{wt}\left(\frac{\partial f}{\partial z_n} \right) = d - \alpha_n,$$

so we have $0 < \mathrm{wt}(\partial/\partial z_0) \leq \mathrm{wt}(\partial/\partial z_1) \leq \cdots \leq \mathrm{wt}(\partial f/\partial z_n)$. Since D_ν is a derivation of A_ν, we have $D_\nu(J_\nu) \subset J_\nu$, where $J_\nu = (\partial f/\partial z_0, \partial f/\partial z_1, \dots, \partial f/\partial z_n, I_\nu)$. Moreover, $\mathrm{wt}(D_\nu(\partial f/\partial z_0)) < \mathrm{wt}(\partial f/\partial z_0)$ implies that $D_\nu(\partial f/\partial z_0)$ does not contain any linear combination of $\partial f/\partial z_0, \partial f/\partial z_1, \dots, \partial f/\partial z_n$.

We divide the proof of the main theorem into four propositions.

Proposition 2.6. *Let $f(z_0, z_1)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1; d)$ with an isolated singularity at the origin. Assume that $d \geq 2\alpha_0 \geq 2\alpha_1$. Let D be a derivation of the algebra*

$$\mathbb{C}[z_0, z_1] \Big/ \left(\frac{\partial^2 f}{\partial z_0^2}, \frac{\partial^2 f}{\partial z_0 \partial z_1}, \frac{\partial^2 f}{\partial z_1^2} \right).$$

Then $D \equiv 0$, if D is of negative weight.

Proof. It is clear that $D(\partial^2 f/\partial z_0^2) = 0$. We have $D = z_1^k(\partial/\partial z_0)$, where $k \geq 1$ and $\mathrm{wt}(D) = k\alpha_1 - \alpha_0 < 0$. Let

$$f(z_0, z_1) = \sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} c_{(n_0, n_1)} z_0^{n_0} z_1^{n_1}.$$

Then we have

$$\begin{aligned} D\left(\frac{\partial^2 f}{\partial z_0^2}\right) &= z_1^k \frac{\partial}{\partial z_0} \left(\sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} n_0(n_0 - 1) c_{(n_0, n_1)} z_0^{n_0 - 2} z_1^{n_1} \right) \\ &= \sum_{\alpha_0 n_0 + \alpha_1 n_1 = d} n_0(n_0 - 1)(n_0 - 2) c_{(n_0, n_1)} z_0^{n_0 - 3} z_1^{n_1 + k} = 0. \end{aligned}$$

So, when $n_0 \geq 3$, $c_{(n_0, n_1)} = 0$, i.e.,

$$f(z_0, z_1) = c_{(2,p)} z_0^2 z_1^r + c_{(1,p)} z_0 z_1^p + c_{(0,q)} z_1^q,$$

where $d = 2\alpha_0 + r\alpha_1 = \alpha_0 + p\alpha_1 = q\alpha_1$.

If $c_{(2,p)} = 0$, then in order for f to have isolated singularity at the origin, we need $p = 1$. So

$$f(z_0, z_1) = c_{(1,p)} z_0 z_1 + c_{(0,q)} z_1^q \quad \text{and} \quad \frac{\partial^2 f}{\partial z_0 \partial z_1} = c_{(1,p)}.$$

So, $D = z_1^k(\partial/\partial z_0)$ is a zero derivation on $\mathbb{C}[z_0, z_1]/(\partial^2 f/\partial z_0^2, \partial^2 f/\partial z_0 \partial z_1, \partial^2 f/\partial z_1^2)$.

If $c_{(2,p)} \neq 0$, then by Lemma 2.4, we obtain that $r = 0$ or $r = 1$. If $r = 0$, then $\partial^2 f/\partial z_0^2 = 2c_{(2,p)}$. If $r = 1$, then $\partial^2 f/\partial z_0^2 = 2c_{(2,p)} z_1$. Hence, $D = z_1^k(\partial/\partial z_0)$ is a zero derivation on $\mathbb{C}[z_0, z_1]/(\partial^2 f/\partial z_0^2, \partial^2 f/\partial z_0 \partial z_1, \partial^2 f/\partial z_1^2)$. □

Proposition 2.7. *Let $(V, 0) = \{(z_0, z_1, z_2) \in \mathbb{C}^3 : f(z_0, z_1, z_2) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial f of weight type $(\alpha_0, \alpha_1, \alpha_2; d)$ with* $\mathrm{mult}(f) \geq 4$. *Assume that $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2 > 0$, without loss of generality. Let $H_1(V)$ be the first Hessian algebra. Let D_1 be a derivation of the algebra $H_1(V)$, i.e., $D_1 \in L_1(V)$, then $D_1 \equiv 0$, if D_1 is of negative weight.*

Proof. For simplicity, we use D to denote D_1. It is clear that $D(\partial^2 f/\partial z_0^2) = 0$. We have $D = p(z_1, z_2)(\partial/\partial z_0) + c z_2^k(\partial/\partial z_1)$, where c is a constant. There are two cases: $c = 0$ or $c \neq 0$.

Case 1: Assume $c = 0$. In this case, $D = p(z_1, z_2)(\partial/\partial z_0)$. By Lemma 2.3, we separate it into two cases.

Case 1.1: Let $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$. Then

$$D\left(\frac{\partial^2 f}{\partial z_0^2}\right) = p(z_1, z_2)\big[m(m-1)(m-2)z_0^{m-3} + (m-1)(m-2)(m-3)a_1(z_1, z_2)z_0^{m-4} + \cdots + 6a_{m-3}(z_1, z_2)\big] = 0,$$

which implies $p(z_1, z_2) = 0$.

Case 1.2: Let $f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2)$. Then

$$D\left(\frac{\partial^2 f}{\partial z_0^2}\right) = p(z_1, z_2)\big[m(m-1)(m-2)z_0^{m-3} z_i + (m-1)(m-2)(m-3)a_1(z_1, z_2)z_0^{m-3} + \cdots + 6a_{m-3}(z_1, z_2)\big],$$

which implies $p(z_1, z_2) = 0$, i.e., $D \equiv 0$.

Case 2: Assume $c \neq 0$. According to Lemma 2.3, we also need to separate it into two cases.

Case 2.1: Let $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$. Then

$$\begin{aligned} D\left(\frac{\partial^2 f}{\partial z_0^2}\right) &= p(z_1, z_2)\big[m(m-1)(m-2)z_0^{m-3} + (m-1)(m-2)(m-3)a_1(z_1, z_2)z_0^{m-4} + \cdots + 6a_{m-3}(z_1, z_2)\big] \\ &\quad + cz_2^k\left[(m-1)(m-2)\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-3} + (m-2)(m-3)\frac{\partial a_2(z_1, z_2)}{\partial z_1}z_0^{m-4} + \cdots + 2\frac{\partial a_{m-2}(z_1, z_2)}{\partial z_1}\right]. \end{aligned}$$

Because $D(\partial^2 f/\partial z_0^2) = 0$ and $m \geq 4$, we have

$$mp(z_1, z_2) = -cz_2^k \frac{\partial a_1(z_1, z_2)}{\partial z_1}.$$

We construct the coordinate transformation

$$\begin{cases} z_0 = z_0' - \dfrac{1}{m}a_1(z_1', z_2'), \\ z_1 = z_1', \\ z_2 = z_2'. \end{cases}$$

Then

$$\begin{aligned} D &= -\frac{1}{m}cz_2^k \frac{\partial a_1(z_1, z_2)}{\partial z_1}\frac{\partial}{\partial z_0} + cz_2^k \frac{\partial}{\partial z_1} \\ &= cz_2^k\left(-\frac{1}{m}\frac{\partial a_1(z_1, z_2)}{\partial z_1}\frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1}\right) = c(z_2')^k \frac{\partial}{\partial z_1'}. \end{aligned}$$

Letting $g(z_0', z_1', z_2') = f(z_0, z_1, z_2)$, we know that g is also a weighted homogeneous polynomial and

$$g = (z_0')^m + b_1(z_1', z_2')(z_0')^{m-1} + \cdots + b_m(z_1', z_2').$$

By the same argument as before, we have

$$D\left(\frac{\partial^2 g}{\partial {z_0'}^2}\right) = 0.$$

So we assume that

$$D\left(\frac{\partial^2 g}{\partial (z_0')^2}\right) = c(z_2')^k \frac{\partial}{\partial z_1'}\left(\frac{\partial^2 g}{\partial (z_0')^2}\right) = 0.$$

So

$$\frac{\partial b_1(z_1', z_2')}{\partial z_1'} = \cdots = \frac{\partial b_{m-2}(z_1', z_2')}{\partial z_1'} = 0.$$

Furthermore,

$$D\left(\frac{\partial^2 g}{\partial z_0' \partial z_1'}\right) = c(z_2')^k \frac{\partial^3 g}{\partial z_0' \partial (z_1')^2} = c(z_2')^k \frac{\partial^2 b_{m-1}(z_1', z_2')}{\partial (z_1')^2}$$

belongs to the principal ideal generated by $\partial^2 g/\partial (z_0')^2$. Hence,

$$c = 0 \quad \text{or} \quad \frac{\partial^2 b_{m-1}(z_1', z_2')}{\partial (z_1')^2} = 0.$$

If $c = 0$, then we have already finished it. In the following, we assume that $\partial^2 b_{m-1}(z_1', z_2')/\partial (z_1')^2 = 0$. So we have

$$\frac{\partial^2 g}{\partial z_0' \partial z_1'} = \frac{\partial b_{m-1}(z_1', z_2')}{\partial z_1'}.$$

Then, it is easy to see that

$$D\left(\frac{\partial^2 g}{\partial (z_1')^2}\right) = c(z_2')^k \frac{\partial^3 g}{\partial (z_1')^3} = c(z_2')^k \frac{\partial^3 b_m(z_1', z_2')}{\partial (z_1')^3}$$

belongs to the ideal generated by $\partial^2 g/\partial (z_0')^2$, $\partial^2 g/(\partial z_0' \partial z_1')$ and $\partial^2 g/(\partial z_0' \partial z_2')$. If one of $b_1, \ldots, b_{m-2}$ is not zero, then $D(\partial^2 g/\partial (z_1')^2)$ belongs to the principal ideal generated by $\partial^2 g/(\partial z_0' \partial z_1')$, i.e., there exists a polynomial $h(z_2, z_3)$ such that

$$D\left(\frac{\partial^2 g}{\partial (z_1')^2}\right) = h \frac{\partial^2 g}{\partial z_0' \partial z_1'},$$

i.e.,

$$c(z_2')^k \frac{\partial^3 b_m(z_1', z_2')}{\partial (z_1')^3} = h(z_1', z_2') \frac{\partial b_{m-1}(z_1', z_2')}{\partial z_1'}. \tag{1}$$

Let $b_{m-1}(z_1', z_2') = pz_1'(z_2')^s + q(z_2')^t$, where $p \neq 0$. By Lemma 2.4, we obtain that at least one of $(z_1')^{l_1}$ and $(z_1')^{l_2} z_2'$ is contained in $b_m(z_1', z_2')$. If $(z_1')^{l_1}$ is contained in $b_m(z_1', z_2')$ as a monomial, then $\partial^3 b_m(z_1', z_2')/\partial(z_1')^3$ is not divisible by z_2'. Hence, $k \geq s$ by (2). Moreover, since $s\alpha_2 + \alpha_1 + \alpha_0 = m\alpha_0$, we easily obtain

$$(m-1)\alpha_0 - \alpha_1 = s\alpha_2 \leq k\alpha_2 < \alpha_1,$$

i.e., $(m-1)\alpha_0 < 2\alpha_0$ which is in contradiction with $m \geq 4$. Hence, $(z_1')^{l_2} z_2'$ must be contained in $b_m(z_1', z_2')$. Then $\partial^3 b_m(z_1', z_2')/\partial(z_1')^3$ is not divisible by $(z_2')^2$. By (1), it is easy to see that $k+1 \geq s$, which implies

$$(m-1)\alpha_0 - \alpha_1 - \alpha_2 = (s-1)\alpha_2 \leq k\alpha_2 < \alpha_1,$$

i.e., $(m-1)\alpha_0 < 2\alpha_1 + \alpha_2$, which is in contradiction with $m \geq 4$.

In the following, we assume that $b_1 = \cdots = b_{m-2} = 0$, then

$$f = (z_0')^m + b_{m-1}(z_1', z_2')z_0' + b_m(z_1', z_2').$$

Hence there exist two polynomial h_1 and h_2 such that

$$(2) \quad c(z_2')^k \frac{\partial^3 b_m(z_1', z_2')}{\partial(z_1')^3} = h_1(z_1', z_2') \frac{\partial b_{m-1}(z_1', z_2')}{\partial z_1'} + h_2(z_1', z_2') \frac{\partial b_{m-1}(z_1', z_2')}{\partial z_2'}.$$

The weight of the left-hand side of (2) is equal to $k\alpha_2 + m\alpha_0 - 3\alpha_1$. The weight of the right-hand side of (2) is equal to $\mathrm{wt}(h_2) + (m-1)\alpha_0 - \alpha_2$. Hence, $\mathrm{wt}(h_2) = k\alpha_2 + \alpha_0 - 3\alpha_1 + \alpha_2 \geq \alpha_2$, which implies that $\alpha_0 \geq 3\alpha_1 - k\alpha_2 > 2\alpha_1$. Let $b_{m-1}(z_1', z_2') = pz_1'(z_2')^s + q(z_2')^t$, where $p \neq 0$. By Lemma 2.4, we obtain that at least one of $(z_1')^{l_1}$ and $(z_1')^{l_2} z_2'$ is contained in $b_m(z_1', z_2')$. If $(z_1')^{l_1}$ is contained in $b_m(z_1', z_2')$ as a monomial, then $\partial^3 b_m(z_1', z_2')/\partial(z_1')^3$ is not divisible by z_2'. Hence, we obtain $k \geq s-1$ by (2). Moreover, since $s\alpha_2 + \alpha_1 + \alpha_0 = m\alpha_0$, we easily obtain

$$(m-1)\alpha_0 - \alpha_1 = s\alpha_2 \leq (k+1)\alpha_2 < \alpha_1 + \alpha_2,$$

i.e., $(m-1)\alpha_0 < 3\alpha_0$, which is in contradiction with $m \geq 4$. Hence, $(z_1')^{l_2} z_2'$ must be contained in $b_m(z_1', z_2')$. Then $\partial^3 b_m(z_1', z_2')/\partial(z_1')^3$ is not divisible by $(z_2')^2$. By (2), it is easy to see that $k+1 \geq s-1$, which implies

$$(m-1)\alpha_0 - \alpha_1 - \alpha_2 = (s-1)\alpha_2 \leq (k+1)\alpha_2 < \alpha_1 + \alpha_2,$$

i.e., $(m-1)\alpha_0 < 2\alpha_1 + 2\alpha_2 < 3\alpha_0$, which is in contradiction with $m \geq 4$.

Case 2.2: Let $f = z_0^m z_i + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$, with $m \geq 3$.

If $i=1$, then we have

$$\begin{aligned}0&=D\Big(\frac{\partial^2 f}{\partial z_0^2}\Big)\\&=p(z_1,z_2)\big[m(m-1)(m-2)z_0^{m-3}z_1+(m-1)(m-2)(m-3)a_1(z_1,z_2)z_0^{m-4}+\cdots\big]\\&\qquad+cz_2^k\Big[m(m-1)z_0^{m-2}+(m-1)(m-2)\frac{\partial a_1(z_1,z_2)}{\partial z_1}z_0^{m-3}+\cdots\Big].\end{aligned}$$

This forces $c=0$ which contradicts our hypothesis. So this case does not occur.

If $i=2$, then we have

$$\begin{aligned}0&=D\Big(\frac{\partial^2 f}{\partial z_0^2}\Big)\\&=p(z_1,z_2)\frac{\partial}{\partial z_0}\Big(\frac{\partial^2 f}{\partial z_0^2}\Big)+cz_2^k\frac{\partial}{\partial z_1}\Big(\frac{\partial^2 f}{\partial z_0^2}\Big)\\&=\frac{\partial^2}{\partial z_0^2}\Big[p(z_1,z_2)\frac{\partial f}{\partial z_0}+cz_2^k\frac{\partial f}{\partial z_1}\Big],\end{aligned}\tag{3}$$

and similarly $D((\partial^2 f)/\partial z_0\partial z_1)$ is a multiple of $\partial^2 f/\partial z_0^2$.

$$\begin{aligned}D\Big(\frac{\partial^2 f}{\partial z_0\partial z_1}\Big)&=p(z_1,z_2)\frac{\partial}{\partial z_0}\Big(\frac{\partial^2 f}{\partial z_0\partial z_1}\Big)+cz_2^k\frac{\partial}{\partial z_1}\Big(\frac{\partial^2 f}{\partial z_0\partial z_1}\Big)\\&=\frac{\partial^2}{\partial z_0\partial z_1}\Big(p(z_1,z_2)\frac{\partial f}{\partial z_0}+cz_2^k\frac{\partial f}{\partial z_1}\Big)-\frac{\partial p(z_1,z_2)}{\partial z_1}\cdot\frac{\partial^2 f}{\partial z_0^2}\\&=h\frac{\partial^2 f}{\partial z_0^2}.\end{aligned}\tag{4}$$

Equation (4) implies that

$$\frac{\partial^2}{\partial z_0\partial z_1}\Big(p(z_1,z_2)\frac{\partial f}{\partial z_0}+cz_2^k\frac{\partial f}{\partial z_1}\Big)=\tilde h\,\frac{\partial^2 f}{\partial z_0^2}.$$

From (3), we know that the left-hand side of this equation is independent of z_0 variable. Since $m\ge 3$, the right-hand side of this equation is independent of z_0 variable only if $\tilde h=0$. Thus we have

$$\frac{\partial^2}{\partial z_0\partial z_1}\Big(p(z_1,z_2)\frac{\partial f}{\partial z_0}+cz_2^k\frac{\partial f}{\partial z_1}\Big)=0.$$

So we have

$$\frac{\partial}{\partial z_0}\Big(p(z_1,z_2)\frac{\partial f}{\partial z_0}+cz_2^k\frac{\partial f}{\partial z_1}\Big)=uz_2^l,$$

where either $u = 0$ or $u \neq 0$ and $l > k$. This is

$$\begin{aligned}(5)\quad &\frac{\partial}{\partial z_0}\Big[p(z_1,z_2)\big(mz_2z_0^{m-1}+(m-1)a_1(z_1,z_2)z_0^{m-2}+\cdots+a_{m-1}(z_1,z_2)\big)\\ &\quad +cz_2^k\Big(\frac{\partial a_1(z_1,z_2)}{\partial z_1}z_0^{m-1}+\frac{\partial a_2(z_1,z_2)}{\partial z_1}z_0^{m-2}+\cdots+\frac{\partial a_m(z_1,z_2)}{\partial z_1}\Big)\Big]=uz_2^l.\end{aligned}$$

As $m \geq 4$, (5) implies

$$mp(z_1,z_2)z_2+cz_2^k\frac{\partial a_1(z_1,z_2)}{\partial z_1}=0.$$

If $cz_2^k(\partial a_1(z_1,z_2)/\partial z_1)=0$, then $p(z_1,z_2)=0$ and (5) becomes

$$\frac{\partial}{\partial z_0}\Big[cz_2^k\Big(\frac{\partial a_1(z_1,z_2)}{\partial z_1}z_0^{m-1}+\frac{\partial a_2(z_1,z_2)}{\partial z_1}z_0^{m-2}+\cdots+\frac{\partial a_m(z_1,z_2)}{\partial z_1}\Big)\Big]=uz_2^l.$$

Since $c \neq 0$ and $m \geq 3$, we have

$$\frac{\partial a_1(z_1,z_2)}{\partial z_1}=\frac{\partial a_2(z_1,z_2)}{\partial z_1}=\cdots=\frac{\partial a_{m-2}(z_1,z_2)}{\partial z_1}=0$$

and

$$\frac{\partial a_{m-1}(z_1,z_2)}{\partial z_1}=ez_2^{l-k},$$

where $e \neq 0$. Hence,

$$a_{m-1}(z_1,z_2)=ez_1z_2^{l-k}+e'z_2^{l'}.$$

Now we consider

$$\frac{\partial^2 f}{\partial z_1^2}=cz_2^k\frac{\partial^3 a_m(z_1,z_2)}{\partial z_1^3}.$$

We can do a similar computation as in Case 2.1 and get a contradiction.

If $cz_2^k(\partial a_1(z_1,z_2)/\partial z_1)\neq 0$. Then

$$p(z_1,z_2)=z_2^{k-1}q(z_1,z_2),$$

where

$$q(z_1,z_2)=-\frac{c}{m}\frac{\partial a_1(z_1,z_2)}{\partial z_1}.$$

So we have

$$\frac{\partial}{\partial z_0}\Big(q(z_1,z_2)\frac{\partial f}{\partial z_0}+cz_2\frac{\partial f}{\partial z_1}\Big)=uz_2^{l-k+1}.$$

We write

$$q(z_1,z_2)=\alpha z_1^s+z_2\gamma(z_1,z_2).$$

We claim that $\alpha = 0$. Suppose on the contrary that $\alpha \neq 0$. If we rewrite f in the form

$$f = b_0(z_0, z_1)z_2^n + b_1(z_0, z_1)z_2^{n-1} + \cdots + b_n(z_0, z_1),$$

then

$$\begin{aligned}(6)\quad \frac{\partial}{\partial z_0}\Bigg[(\alpha z_1^s + z_2\gamma(z_1, z_2))\left(\frac{\partial b_0(z_0, z_1)}{\partial z_0} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_0}\right) \\ + c z_2\left(\frac{\partial b_0(z_0, z_1)}{\partial z_1} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1}\right)\Bigg] \\ = (\alpha z_1^s + z_2\gamma(z_1, z_2))\left(\frac{\partial^2 b_0(z_0, z_1)}{\partial z_0^2} z_2^n + \cdots + \frac{\partial^2 b_n(z_0, z_1)}{\partial z_0^2}\right) \\ + c z_2\left(\frac{\partial^2 b_0(z_0, z_1)}{\partial z_0 \partial z_1} z_2^n + \cdots + \frac{\partial^2 b_n(z_0, z_1)}{\partial z_0 \partial z_1}\right) \\ = u z_2^{l-k+1}.\end{aligned}$$

Considering the coefficient of z_1^s, we know that $\partial^2 b_n(z_0, z_1)/\partial z_0^2 = 0$, and hence from (6) again, we have

$$\begin{aligned}(7)\quad (\alpha z_1^s + z_2\gamma(z_1, z_2))\left(\frac{\partial^2 b_0(z_0, z_1)}{\partial z_0^2} z_2^{n-1} + \cdots + \frac{\partial^2 b_{n-1}(z_0, z_1)}{\partial z_0^2}\right) \\ + c\left(\frac{\partial^2 b_0(z_0, z_1)}{\partial z_0 \partial z_1} z_2^n + \cdots + \frac{\partial^2 b_n(z_0, z_1)}{\partial z_0 \partial z_1}\right) = u z_2^{l-k}.\end{aligned}$$

Recall that either $u = 0$ or $u \neq 0$ and $l > k$. Equation (7) implies

$$\alpha z_1^s \frac{\partial^2 b_{n-1}(z_0, z_1)}{\partial z_0^2} = -c \frac{\partial^2 b_n(z_0, z_1)}{\partial z_0 \partial z_1}.$$

Since $\partial^2 b_n(z_0, z_1)/\partial z_0^2 = 0$, we have

$$\frac{\partial^2 b_{n-1}(z_0, z_1)}{\partial z_0^2} = c' z_1^{s'},$$

where $c' \neq 0$.

If $s' = 0$, then $b_{n-1}(z_0, z_1) = c' z_0^2 + \cdots$ and $z_0^2 z_2$ occur in f which is in contradiction with our assumption.

If $s' > 0$, then

$$b_{n-1}(z_0, z_1) = c' z_1^{s'} z_0^2 + c'' z_1^{s''} z_0 + \tilde{u} z_1^{\tau},$$

where $s' > 0$ and $\tau \geq 0$. Now $\partial^2 b_n(z_0, z_1)/\partial z_0^2 = 0$ implies $b_n(z_0, z_1) = w z_0 z_1^t + w' z_1^{t'}$, where $t > 1$. Notice that

$$\frac{\partial f}{\partial z_2} = n b_0(z_0, z_1) z_2^{n-1} + (n-1) b_1(z_0, z_1) z_2^{n-2} + \cdots + b_{n-1}(z_0, z_1).$$

It follows that f is singular along the z_0-axis. The contradiction comes from our hypothesis that $\alpha \neq 0$. Thus, $\alpha = 0$.

Now we have

$$q(z_1, z_2) = z_2\gamma(z_1, z_2) \quad \text{and} \quad \gamma(z_1, z_2)\frac{\partial^2 f}{\partial z_0^2} + c\frac{\partial^2 f}{(\partial z_0 \partial z_1)} = uz_2^{l-k}.$$

So we have

$$\gamma(z_1, z_2)\big[m(m-1)z_0^{m-2}z_2+(m-1)(m-2)a_1(z_1, z_2)z_0^{m-3}+\cdots+2a_{m-2}(z_1, z_2)\big] \\ +c\left[(m-1)\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-2}+\cdots+\frac{\partial a_{m-1}(z_1, z_2)}{\partial z_1}\right] = uz_2^{l-k}.$$

This implies that

$$m\gamma(z_1, z_2)z_2 + c\frac{\partial a_1(z_1, z_2)}{\partial z_1} = 0.$$

Hence, $\partial a_1(z_1, z_2)/\partial z_1$ is divisible by z_2. Let $a_1'(z_1, z_2)$ be a weighted homogeneous polynomial satisfying $\partial a_1'(z_1, z_2)/\partial z_1 = (\partial a_1/\partial z_1)/z_2$. Consider the coordinate transformation

$$\begin{cases} z_0 = z_0' - \dfrac{1}{m}a_1'(z_1', z_2'), \\ z_1 = z_1', \\ z_2 = z_2'. \end{cases}$$

After this coordinate transformation, we have

$$\frac{\partial^2}{\partial (z_0')^2} = \frac{\partial^2}{\partial z_0^2} \quad \text{and} \quad \frac{\partial^2}{\partial z_0' \partial z_1'} = -\frac{1}{m}\frac{\partial a_1'}{\partial z_1'}\frac{\partial^2}{\partial z_0^2} + \frac{\partial^2}{\partial z_0 \partial z_1}.$$

Hence,

$$\begin{aligned} \gamma(z_1, z_2)\frac{\partial^2 f}{\partial z_0^2} + c\frac{\partial^2 f}{\partial z_0 \partial z_1} &= -\frac{c}{m}\frac{\partial a_1'}{\partial z_1}\frac{\partial^2 f}{\partial z_0^2} + c\frac{\partial^2 f}{\partial z_0 \partial z_1} \\ &= c\left(-\frac{1}{m}\frac{\partial a_1'}{\partial z_1}\frac{\partial^2 f}{\partial z_0^2} + \frac{\partial^2 f}{\partial z_0 \partial z_1}\right) \\ &= c\frac{\partial^2 f}{\partial z_0' \partial z_1'} \\ &= u(z_2')^{l-k}. \end{aligned}$$

For simplicity of notation, we still use (z_0, z_1, z_2) to represent the coordinates after coordinate transformation. Without loss of generality, we assume

$$f = z_0^m z_2 + a_1(z_2)z_0^{m-1} + \cdots + a_{m-2}(z_2)z_0^2 + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2).$$

Now we have $cz_2^k(\partial a_1/\partial z_1) = 0$. From a similar discussion as above, we obtain the conclusion. $\square$

Proposition 2.8. *Let $f(z_0, z_1, z_2)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1, \alpha_2; d)$ with isolated singularity at the origin. Assume $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$. Let D_2 be a derivation of the algebra $\mathbb{C}[z_0, z_1, z_2]/(I, I_2)$. Then $D_2 \equiv 0$, if D_2 is of negative weight.*

First, we need the following lemma:

Lemma 2.9. *The smallest weight of an element in I_2 is greater than or equal to the weight of $\partial f/\partial z_0$ when $m \geq 3$, where m is the exponent of z_0 in Lemma 2.3.*

Proof. It is obvious that

$$I_2 = \left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}, \begin{vmatrix} f_{00} & f_{02} \\ f_{01} & f_{12} \end{vmatrix}, \dots, \begin{vmatrix} f_{01} & f_{02} \\ f_{12} & f_{22} \end{vmatrix}, \begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} \right)$$

and

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} \right) = 2d - 2\alpha_0 - 2\alpha_1.$$

Note that $\left|\begin{smallmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{smallmatrix}\right|$ is an element with the smallest weight in I_2. We obtain that

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} \right) \geq \mathrm{wt}\left(\frac{\partial f}{\partial z_0} \right)$$

if and only if $2d - 2\alpha_0 - 2\alpha_1 \geq d - \alpha_0$, which is equivalent to $d \geq \alpha_0 + 2\alpha_1$.

Case 1: Let $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2)$. In this case, $d = m\alpha_0$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 3$.

Case 2: Let $f = z_0^m z_1 + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2)$. In this case, $d = m\alpha_0 + \alpha_1$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 + \alpha_1 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 2$.

Case 3: Let $f = z_0^m z_2 + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_m(z_1, z_2)$. In this case, $d = m\alpha_0 + \alpha_2$. So $d \geq \alpha_0 + 2\alpha_1$ if and only if $m\alpha_0 + \alpha_2 \geq \alpha_0 + 2\alpha_1$. This is clearly true when $m \geq 3$. □

Proof of Proposition 2.8. By Lemma 2.9, we obtain that $D_2(\partial f/\partial z_0) = 0$ when $m \geq 3$. We only need to consider the following two cases:

Case 1: Assume $c = 0$. In this case, $D = p(z_1, z_2)(\partial/\partial z_0)$. By Lemma 2.3, we have to consider two subcases.

Case 1.1: Let $f = z_0^m + a_1(z_1, z_2)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2)z_0 + a_m(z_1, z_2)$. Then

$$D\left(\frac{\partial f}{\partial z_0}\right) = p(z_1, z_2)\big[m(m-1)z_0^{m-2} + (m-1)(m-2)a_1(z_1, z_2)z_0^{m-3} + \cdots + 2a_{m-2}(z_1, z_2)\big] = 0,$$

which implies $p(z_1, z_2) = 0$. If not, the above equation holds only if $m = 1$, which is absurd in view of our assumption. So we must have $p(z_1, z_2) = 0$, i.e., $D \equiv 0$.

Case 1.2: Let $f = z_0^m z_i + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_m(z_1, z_2)$. Then

$$D\left(\frac{\partial f}{\partial z_0}\right) = p(z_1, z_2)\big[m(m-1) z_0^{m-2} z_i + (m-1)(m-2) a_1(z_1, z_2) z_0^{m-3} + \cdots + 2a_{m-2}(z_1, z_2)\big],$$

which implies $p(z_1, z_2) = 0$. If not, the above equation holds only if $m = 1$, which is absurd in view of our assumption. So we must have $p(z_1, z_2) = 0$, i.e., $D \equiv 0$.

Case 2: Assume $c \neq 0$. According to Lemma 2.3, we also need to divide it into two subcases.

Case 2.1: Let $f = z_0^m + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2) z_0 + a_m(z_1, z_2)$. Then

$$\begin{aligned} D\left(\frac{\partial f}{\partial z_0}\right) &= p(z_1, z_2)\big[m(m-1) z_0^{m-2} + (m-1)(m-2) a_1(z_1, z_2) z_0^{m-3} + \cdots + 2a_{m-1}(z_1, z_2)\big] \\ &\quad + c z_2^k \left[(m-1)\frac{\partial a_1(z_1, z_2)}{\partial z_1} z_0^{m-2} + (m-2)\frac{\partial a_2(z_1, z_2)}{\partial z_1} z_0^{m-3} + \cdots + \frac{\partial a_{m-1}(z_1, z_2)}{\partial z_1}\right]. \end{aligned}$$

Because $D(\partial f/\partial z_0) = 0$ and $m \geq 3$, we have

$$m p(z_1, z_2) = -c z_2^k \frac{\partial a_1(z_1, z_2)}{\partial z_1}.$$

We construct the coordinate transformation

$$\begin{cases} z_0 = z_0' - \dfrac{1}{m} a_1(z_1', z_2'), \\ z_1 = z_1', \\ z_2 = z_2'. \end{cases}$$

Then

$$\begin{aligned} D &= -\frac{1}{m} c z_2^k \frac{\partial a_1(z_1, z_2)}{\partial z_1} \frac{\partial}{\partial z_0} + c z_2^k \frac{\partial}{\partial z_1} \\ &= c z_2^k \left(-\frac{1}{m} \frac{\partial a_1(z_1, z_2)}{\partial z_1} \frac{\partial}{\partial z_0} + \frac{\partial}{\partial z_1}\right) \\ &= c (z_2')^k \frac{\partial}{\partial z_1'}. \end{aligned}$$

Letting $g(z_0', z_1', z_2') = f(z_0, z_1, z_2)$, we obtain that g is also a weighted homogeneous polynomial and

$$g = (z_0')^m + b_1(z_1', z_2')(z_0')^{m-1} + \cdots + b_m(z_1', z_2').$$

By the same argument as before, we have $D(\partial g/\partial z_0') = 0$. So

$$D\left(\frac{\partial g}{\partial z_0'}\right) = c(z_2')^k \frac{\partial}{\partial z_1'}\left(\frac{\partial g}{\partial z_0'}\right) = 0.$$

Thus,

$$\frac{\partial b_1(z_1', z_2')}{\partial z_1'} = \cdots = \frac{\partial b_{m-1}(z_1', z_2')}{\partial z_1'} = 0.$$

Consider

$$D\left(\frac{\partial g}{\partial z_1'}\right) = c(z_2')^k \frac{\partial^2 g}{\partial (z_1')^2} = c(z_2')^k \frac{\partial^2 b_m(z_1', z_2')}{\partial (z_1')^2}.$$

Since $\mathrm{wt}(D(\partial g/\partial z_1')) < \mathrm{wt}(\partial g/\partial z_1') = m\alpha_0 - \alpha_1$ and

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right) = (2m-2)\alpha_0 - 2\alpha_1 \geq m\alpha_0 - \alpha_1 = \mathrm{wt}\left(\frac{\partial g}{\partial z_1'}\right),$$

where $m \geq 3$, $D(\partial g/\partial z_1')$ is a multiple of $\partial g/\partial z_0'$:

$$D\left(\frac{\partial g}{\partial z_1'}\right) = c(z_2')^k \frac{\partial^2 g}{\partial (z_1')^2} = c(z_2')^k \frac{\partial^2 b_m(z_1', z_2')}{\partial (z_1')^2},$$
$$\frac{\partial g}{\partial z_0'} = m(z_0')^{m-1} + \cdots + b_{m-1}(z_1', z_2').$$

Further, $D(\partial g/\partial z_1')$ is a multiple of $\partial g/\partial z_0'$, i.e., there exists h such that

$$c(z_2')^k \frac{\partial^2 b_m(z_1', z_2')}{\partial (z_1')^2} = h \frac{\partial g(z_1', z_2')}{\partial z_0'} = h\big[m(z_0')^{m-1} + \cdots + b_{m-1}(z_1', z_2')\big].$$

If $\partial^2 b_m(z_1', z_2')/\partial (z_1')^2 \neq 0$, then we have $m = 1$, which is absurd. So we have $\partial^2 b_m(z_1', z_2')/\partial (z_1')^2 = 0$. This implies

$$b_m(z_1', z_2') = d_1 z_1' (z_2')^{l_1} + d_2 (z_2')^{l_2},$$

where d_1, d_2 are constants. Then g has an isolated singularity at 0 only if $l_1 = 1$. Because $b_m(z_1', z_2')$ is weighted homogeneous, we have $d = \alpha_1 + \alpha_2$. It follows from the assumption $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ that $\alpha_0 = \alpha_1 = \alpha_2$. So g is a homogeneous polynomial. It follows from Proposition 2.1 that $D \equiv 0$.

Case 2.2: Let $f = z_0^m z_i + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2) z_0 + a_m(z_1, z_2)$.

Case 2.2.1: Let $i = 1$. Then we have

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = p(z_1, z_2)\big[m(m-1) z_0^{m-2} z_1 + (m-1)(m-2) a_1(z_1, z_2) z_0^{m-3} + \cdots\big]$$
$$+ c z_2^k \left[m z_0^{m-1} + (m-1) \frac{\partial a_1(z_1, z_2)}{\partial z_1} z_0^{m-2} + \cdots\right].$$

This forces $c = 0$, which is in contradicts with our hypothesis. So this case does not occur.

Case 2.2.2: Let $i = 2$. In this case, $m \geq 3$ by our assumption. Then

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = p(z_1, z_2)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_0}\right) + cz_2^k\frac{\partial}{\partial z_1}\left(\frac{\partial f}{\partial z_0}\right) = \frac{\partial}{\partial z_0}\left[p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right]. \tag{8}$$

Similarly, we obtain that $D(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$:

$$D\left(\frac{\partial f}{\partial z_1}\right) = p(z_1, z_2)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_1}\right) + cz_2^k\frac{\partial}{\partial z_1}\left(\frac{\partial f}{\partial z_1}\right) = \frac{\partial}{\partial z_1}\left(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) - \frac{\partial p(z_1, z_2)}{\partial z_1}\frac{\partial f}{\partial z_0} = h\frac{\partial f}{\partial z_0}. \tag{9}$$

Equation (9) implies that

$$\frac{\partial}{\partial z_1}\left[p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right] = \tilde{h}\frac{\partial f}{\partial z_0}.$$

From (8), we know that the left-hand side of this equation is independent of the variable z_0. Since $m > 1$, the right-hand side of this equation is independent of the variable z_0 only if $\tilde{h} = 0$. Thus, we have

$$\frac{\partial}{\partial z_1}\left(p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1}\right) = 0.$$

So we have

$$p(z_1, z_2)\frac{\partial f}{\partial z_0} + cz_2^k\frac{\partial f}{\partial z_1} = uz_2^l,$$

where either $u = 0$ or $u \neq 0$ and $l > k$. This is

$$p(z_1, z_2)\big(mz_2z_0^{m-1} + (m-1)a_1(z_1, z_2)z_0^{m-2} + \cdots + a_{m-1}(z_1, z_2)\big) + cz_2^k\left[\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-1} + \frac{\partial a_2(z_1, z_2)}{\partial z_1}z_0^{m-2} + \cdots + \frac{\partial z_m(z_1, z_2)}{\partial z_1}\right] = uz_2^l. \tag{10}$$

As $m > 1$, (10) implies

$$mp(z_1, z_2)z_2 + cz_2^k\frac{\partial a_1(z_1, z_2)}{\partial z_1} = 0.$$

If $cz_2^k(\partial a_1(z_1, z_2)/\partial z_1) = 0$, then $p(z_1, z_2) = 0$ and (10) becomes

$$cz_2^k\left(\frac{\partial a_1(z_1, z_2)}{\partial z_1}z_0^{m-1} + \frac{\partial a_2(z_1, z_2)}{\partial z_1}z_0^{m-2} + \cdots + \frac{\partial a_m(z_1, z_2)}{\partial z_1}\right) = uz_2^l.$$

Since $c \neq 0$, we have

$$\frac{\partial a_1(z_1, z_2)}{\partial z_1} = \frac{\partial a_2(z_1, z_2)}{\partial z_1} = \cdots = \frac{\partial a_{m-1}(z_1, z_2)}{\partial z_1} = 0,$$
$$\frac{\partial a_m(z_1, z_2)}{\partial z_1} = e z_2^{l-k},$$

where $e \neq 0$. Hence,

$$a_m(z_1, z_2) = e z_1 z_2^{l-k} + z_2^k.$$

We must have $l - k = 1$ and $e \neq 0$, otherwise f will be singular along the z_1-axis, which is a contradiction. Since the term $z_1 z_2$ appears in f, we conclude that $\alpha_1 + \alpha_2 = d$. The assumption $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2$ implies that $\alpha_0 = \alpha_1 = \alpha_2$. So f is a homogeneous polynomial. By Proposition 2.1, we have that $D \equiv 0$.

If $c z_2^k(\partial a_1(z_1, z_2)/\partial z_1) \neq 0$, then

$$p(z_1, z_2) = z_2^{k-1} q(z_1, z_2),$$

where

$$q(z_1, z_2) = -\frac{c}{m} \frac{\partial a_1(z_1, z_2)}{\partial z_1}.$$

So we have

$$q(z_1, z_2) \frac{\partial f}{\partial z_0} + c z_2 \frac{\partial f}{\partial z_1} = u z_2^{l-k+1}.$$

Write

$$q(z_1, z_2) = \alpha z_1^s + z_2 \gamma(z_1, z_2).$$

We claim that $\alpha = 0$; suppose on the contrary that $\alpha \neq 0$. If we rewrite f in the form

$$f = b_0(z_0, z_1) z_2^n + b_1(z_0, z_1) z_2^{n-1} + \cdots + b_n(z_0, z_1),$$

then

$$\begin{aligned}(11)\quad [\alpha z_1^s + z_2\gamma(z_1, z_2)]&\left(\frac{\partial b_0(z_0, z_1)}{\partial z_0} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_0}\right) \\ &+ c z_2\left(\frac{\partial b_0(z_0, z_1)}{\partial z_1} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1}\right) = u z_2^{l-k+1}.\end{aligned}$$

Considering the coefficient of z_1^s, we know that $\partial b_n(z_0, z_1)/\partial z_0 = 0$, and hence from (11) again, we have

$$\begin{aligned}(12)\quad [\alpha z_1^s + z_2\gamma(z_1, z_2)]&\left(\frac{\partial b_0(z_0, z_1)}{\partial z_0} z_2^{n-1} + \cdots + \frac{\partial b_{n-1}(z_0, z_1)}{\partial z_0}\right) \\ &+ c\left(\frac{\partial b_0(z_0, z_1)}{\partial z_1} z_2^n + \cdots + \frac{\partial b_n(z_0, z_1)}{\partial z_1}\right) = u z_2^{l-k}.\end{aligned}$$

Recall that either $u=0$ or $u\neq 0$ and $l>k$. Equation (12) implies

$$\alpha z_1^s \frac{\partial b_{n-1}(z_0,z_1)}{\partial z_0} = -c\frac{\partial b_n(z_0,z_1)}{\partial z_1}.$$

Since $\partial b_n(z_0,z_1)/\partial z_0 = 0$, we have

$$\frac{\partial b_{n-1}(z_0,z_1)}{\partial z_0} = c' z_1^{s'},$$

where $c'\neq 0$.

If $s'=0$, then $b_{n-1}(z_0,z_1)=c'z_0+c''z_1$, and hence z_0z_2 and z_1z_2 occur in f. It follows again that $\alpha_0=\alpha_1=\alpha_2$, and we are finished.

If $s'>0$, then

$$b_{n-1}(z_0,z_1)=c'z_1^{s'}z_0+\tilde{u}z_1^{\tau},$$

where $s'>0$ and $\tau>0$. Now $\partial b_n(z_0,z_1)/\partial z_0=0$ implies $b_n(z_0,z_1)=wz_1^t$, where $t>1$. Notice that

$$\frac{\partial f}{\partial z_2} = nb_0(z_0,z_1)z_2^{n-1}+(n-1)b_1(z_0,z_1)z_2^{n-2}+\cdots+b_{n-1}(z_0,z_1).$$

It follows that f is singular along the z_0-axis. The contradiction comes from our hypothesis that $\alpha\neq 0$. Thus, $\alpha=0$.

Now we have $q(z_1,z_2)=z_2\gamma(z_1,z_2)$ and $\gamma(z_1,z_2)(\partial f/\partial z_0)+c(\partial f/\partial z_1)=uz_2^{l-k}$. So we have

$$\gamma(z_1,z_2)\left(\frac{\partial b_0(z_0,z_1)}{\partial z_0}z_2^n+\cdots+\frac{\partial b_n(z_0,z_1)}{\partial z_0}\right) + c\left(\frac{\partial b_0(z_0,z_1)}{\partial z_1}z_2^n+\cdots+\frac{\partial b_n(z_0,z_1)}{\partial z_1}\right) = uz_2^{l-k}.$$

It follows that $\gamma(z_1,0)\neq 0$, otherwise we will have $\partial b_n(z_0,z_1)/\partial z_1=0$. So $b_n(z_0,z_1)=z_0^m$ for some n. Therefore, f will be of the form

$$f=z_0^m+c_1(z_1,z_2)z_0^{m-1}+\cdots+c_m(z_1,z_2),$$

which contradicts our assumption.

Now, let $\gamma(z_1,z_2)=vz_1^h+z_2\bar{\gamma}(z_1,z_2)$, where $h>0$ and $v\neq 0$. Then we have

$$vz_1^h\frac{\partial b_n(z_0,z_1)}{\partial z_0}+c\frac{\partial b_n(z_0,z_1)}{\partial z_1}=0,$$

where $v\neq 0$ and $c\neq 0$. Let

$$b_n(z_0,z_1)=d_0z_0^kz_1^{l_0}+d_1z_0^{k-1}z_1^{l_1}+\cdots+d_kz_1^{l_k},$$

where $d_0 \neq 0$. Then

$$\frac{\partial b_n(z_0, z_1)}{\partial z_0} = kd_0 z_0^{k-1} z_1^{l_0} + (k-1)d_1 z_0^{k-2} z_1^{l_1} + \cdots + d_{k-1} z_1^{l_{k-1}},$$

$$\frac{\partial b_n(z_0, z_1)}{\partial z_1} = l_0 d_0 z_0^k z_1^{l_0 - 1} + l_1 d_1 z_0^{k-1} z_1^{l_1 - 1} + \cdots + l_k d_k z_1^{l_k - 1}.$$

Clearly, the equation

$$v z_1^h \frac{\partial b_n(z_0, z_1)}{\partial z_0} + c \frac{\partial b_n(z_0, z_1)}{\partial z_1} = 0$$

is true only if $k = 0$ or $l_0 = 0$.

If $k = 0$, then $b_n(z_0, z_1) = d_0 z_1^{l_0}$, which is absurd.

If $l_0 = 0$, then

$$b_n(z_0, z_1) = d_0 z_0^k + d_1 z_0^{k-1} z_1^{l_1} + \cdots + d_k z_1^{l_k}.$$

Hence, f is again of the form

$$f = z_0^m + a_1(z_1, z_2) z_0^{m-1} + \cdots + a_m(z_1, z_2),$$

which is absurd. This completes the proof of Case 2.2. □

Proposition 2.10. *Let $f(z_0, z_1, z_2, z_3)$ be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, d)$ with isolated singularity at the origin and* $\mathrm{mult}(f) > 3$. *Assume that $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2 \geq 2\alpha_3$.*

(a) *Let D_2 be a derivation of the algebra $\mathbb{C}[z_0, z_1, z_2, z_3]/(I, I_2)$. Then $D_2 \equiv 0$, if D_2 is of negative weight.*

(b) *Let D_3 be a derivation of the algebra $\mathbb{C}[z_0, z_1, z_2, z_3]/(I, I_3)$. Then $D_3 \equiv 0$, if D_3 is of negative weight.*

Proof. The derivation D_ν has the following form for $\nu = 2, 3$:

$$D_\nu = p_0(z_1, z_2, z_3) \frac{\partial}{\partial z_0} + p_1(z_2, z_3) \frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2},$$

where c, k are constants and $k \geq 1$.

Lemma 2.11. *The smallest weight of an element in I_2 is greater than or equal to the weight of $\partial f / \partial z_0$ when $m \geq 3$, where m is the exponent of z_0 in Lemma 2.3.*

Proof. It is obvious that

$$I_2 = \left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}, \begin{vmatrix} f_{01} & f_{02} \\ f_{11} & f_{12} \end{vmatrix}, \ldots, \begin{vmatrix} f_{12} & f_{13} \\ f_{23} & f_{33} \end{vmatrix}, \begin{vmatrix} f_{22} & f_{23} \\ f_{23} & f_{33} \end{vmatrix} \right),$$

and

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix} \right) = 2d - 2\alpha_0 - 2\alpha_1.$$

The element with smallest weight in I_2 is $\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}$. We obtain that

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right) \geq \mathrm{wt}\left(\frac{\partial f}{\partial z_0}\right)$$

if and only if $2d-2\alpha_0-2\alpha_1 \geq d-\alpha_0$ that is $d \geq \alpha_0+2\alpha_1$.

Case 1: Let $f = z_0^m + a_1(z_1,z_2)z_0^{m-1} + \cdots + a_m(z_1,z_2)$. In this case, $d = m\alpha_0$. So $d \geq \alpha_0+2\alpha_1$ if and only if $m\alpha_0 \geq \alpha_0+2\alpha_1$. This is clearly true when $m \geq 3$.

Case 2: Let $f = z_0^m z_1 + a_1(z_1,z_2,z_3)z_0^{m-1} + \cdots + a_m(z_1,z_2,z_3)$. In this case, $d = m\alpha_0+\alpha_1$. So $d \geq \alpha_0+2\alpha_1$ if and only if $m\alpha_0+\alpha_1 \geq \alpha_0+2\alpha_1$. This is clearly true when $m \geq 2$.

Case 3: Let $f = z_0^m z_2 + a_1(z_1,z_2,z_3)z_0^{m-1} + \cdots + a_m(z_1,z_2,z_3)$. In this case, $d = m\alpha_0+\alpha_2$. So $d \geq \alpha_0+2\alpha_1$ if and only if $m\alpha_0+\alpha_2 \geq \alpha_0+2\alpha_1$. This is clearly true when $m \geq 3$.

Case 4: Let $f = z_0^m z_3 + a_1(z_1,z_2,z_3)z_0^{m-1} + \cdots + a_m(z_1,z_2,z_3)$. In this case, $d = m\alpha_0+\alpha_3$. So $d \geq \alpha_0+2\alpha_1$ if and only if $m\alpha_0+\alpha_3 \geq \alpha_0+2\alpha_1$. This is clearly true when $m \geq 3$. □

By Lemma 2.11 and our assumption that when $n=3$, the multiplicity of f is greater than 3, we obtain that $D_2(\partial f/\partial z_0) = 0$ always holds.

Lemma 2.12. *The smallest weight of an element in I_3 is greater than or equal to the weight of $\partial f/\partial z_0$ when $m \geq 3$, where m is the exponent of z_0 in Lemma 2.3.*

Proof. It is obvious that

$$I_2 = \left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}, \begin{vmatrix} f_{00} & f_{01} & f_{03} \\ f_{01} & f_{12} & f_{13} \\ f_{02} & f_{22} & f_{23} \end{vmatrix}, \ldots, \begin{vmatrix} f_{01} & f_{02} & f_{03} \\ f_{12} & f_{12} & f_{13} \\ f_{13} & f_{23} & f_{33} \end{vmatrix}, \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{vmatrix}\right),$$

and

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right) = 3d-2\alpha_0-2\alpha_1-2\alpha_2.$$

The element with the smallest weight in I_3 is

$$\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}.$$

We obtain that

$$\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right) \geq \mathrm{wt}\left(\frac{\partial f}{\partial z_0}\right)$$

if and only if $3d-2\alpha_0-2\alpha_1-2\alpha_2 \geq d-\alpha_0$, which is $2d \geq \alpha_0+2\alpha_1+2\alpha_2$.

Case 1: Let $f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$. In this case, $d = m\alpha_0$. So $2d \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$ if and only if $2m\alpha_0 \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$. This is clearly true when $m \geq 3$.

Case 2: Let

$$f = z_0^m z_1 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3).$$

In this case, $d = m\alpha_0 + \alpha_1$. So $2d \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$ if and only if $2m\alpha_0 + 2\alpha_1 \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$. This is clearly true when $m \geq 2$.

Case 3: Let $f = z_0^m z_2 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$. In this case, $d = m\alpha_0 + \alpha_2$. So $2d \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$ if and only if $2m\alpha_0 + 2\alpha_2 \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$. This is clearly true when $m \geq 2$.

Case 4: Let $f = z_0^m z_3 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$. In this case, $d = m\alpha_0 + \alpha_3$. So $2d \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$ if and only if $2m\alpha_0 + 2\alpha_3 \geq \alpha_0 + 2\alpha_1 + 2\alpha_2$. This is clearly true when $m \geq 3$. □

By Lemma 2.12 and our assumption that when $n = 3$, the multiplicity of f is greater than 3, we obtain that $D_3(\partial f/\partial z_0) = 0$ always holds. The commutator $[\partial/\partial z_i, D_\nu]$ is of the following form by a direct computation:

$$\left[\frac{\partial}{\partial z_0}, D_\nu\right] = 0,$$
$$\left[\frac{\partial}{\partial z_1}, D_\nu\right] = \frac{\partial p_0}{\partial z_1}\frac{\partial}{\partial z_0},$$
$$\left[\frac{\partial}{\partial z_2}, D_\nu\right] = \frac{\partial p_0}{\partial z_2}\frac{\partial}{\partial z_0} + \frac{\partial p_1}{\partial z_2}\frac{\partial}{\partial z_1},$$
$$\left[\frac{\partial}{\partial z_3}, D_\nu\right] = \frac{\partial p_0}{\partial z_3}\frac{\partial}{\partial z_0} + \frac{\partial p_1}{\partial z_3}\frac{\partial}{\partial z_1} + \frac{\partial (c z_3^k)}{\partial z_3}\frac{\partial}{\partial z_2}.$$

By Lemma 2.3, there are also two cases to be considered for f.

Case 1: Let $f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2, z_3)z_0 + a_m(z_1, z_2, z_3)$, with $m \geq 4$.

In the first part, we consider D_2. Firstly, we investigate $D_2(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 - \alpha_1 \leq (2m-2)\alpha_0 - 2\alpha_1 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_2(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 - \alpha_2 \leq (2m-2)\alpha_0 - 2\alpha_1 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

with $m \geq 4$, so $D_2(\partial f/\partial z_1)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D_2(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 - \alpha_3 \leq (2m-2)\alpha_0 - 2\alpha_1 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

with $m \geq 4$, so $D_2(\partial f/\partial z_1)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

In the second part, we consider D_3. Firstly, we investigate $D_3(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 - \alpha_1 \leq (3m-2)\alpha_0 - 2\alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_3(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 - \alpha_2 \leq (3m-2)\alpha_0 - 2\alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 - \alpha_3 \leq (3m-2)\alpha_0 - 2\alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

Case 2: Let

$$f = z_0^m z_i + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_{m-1}(z_1, z_2, z_3)z_0 + a_m(z_1, z_2, z_3),$$

with $m \geq 3$.

Case 2.1: Assume $i = 1$, i.e.,

$$f = z_0^m z_1 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3).$$

In the first part, we consider D_2. Firstly, we investigate $D_2(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 < (2m-2)\alpha_0 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_2(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 + \alpha_1 - \alpha_2 < (2m-2)\alpha_0 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D_2(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 + \alpha_1 - \alpha_3 < (2m-2)\alpha_0 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

In the second part, we consider D_3. Firstly, we investigate $D_3(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 < (3m-2)\alpha_0 + \alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_3(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 + \alpha_1 - \alpha_2 < (3m-2)\alpha_0 + \alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 + \alpha_1 - \alpha_3 < (3m-2)\alpha_0 + \alpha_1 - 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

Case 2.2: Assume $i = 2$, i.e.,

$$f = z_0^m z_2 + a_1(z_1, z_2, z_3) z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3).$$

In the first part, we consider D_2. Firstly, we investigate $D_2(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right) = m\alpha_0 - \alpha_1 + \alpha_2 < (2m-2)\alpha_0 - 2\alpha_1 + 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_2(\partial f/\partial z_2)$. When $m \geq 4$, we obtain that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 < (2m-2)\alpha_0 - 2\alpha_1 + 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D_2(\partial f/\partial z_3)$. When $m \geq 4$, we obtain that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 + \alpha_2 - \alpha_3 < (2m-2)\alpha_0 - 2\alpha_1 + 2\alpha_2 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

In the second part, we consider D_3. Firstly, we investigate $D_3(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right)=m\alpha_0-\alpha_1+\alpha_2<(3m-2)\alpha_0-2\alpha_1+\alpha_2=\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02}\\ f_{01} & f_{11} & f_{12}\\ f_{02} & f_{12} & f_{22}\end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_3(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right)=m\alpha_0<(3m-2)\alpha_0-2\alpha_1+\alpha_2=\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02}\\ f_{01} & f_{11} & f_{12}\\ f_{02} & f_{12} & f_{22}\end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right)=m\alpha_0+\alpha_2-\alpha_3\leq(3m-2)\alpha_0-2\alpha_1+\alpha_2=\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02}\\ f_{01} & f_{11} & f_{12}\\ f_{02} & f_{12} & f_{22}\end{vmatrix}\right),$$

so $D(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

Case 2.3: Assume $i=3$, i.e.,

$$f=z_0^m z_3+a_1(z_1,z_2,z_3)z_0^{m-1}+\cdots+a_m(z_1,z_2,z_3).$$

In the first part, we consider D_2. Firstly, we investigate $D_2(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right)=m\alpha_0-\alpha_1+\alpha_3<(2m-2)\alpha_0-2\alpha_1+2\alpha_3=\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01}\\ f_{01} & f_{11}\end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_2(\partial f/\partial z_2)$. When $m\geq 4$, we obtain that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right)=m\alpha_0-\alpha_2+\alpha_3<(2m-2)\alpha_0-2\alpha_1+2\alpha_3=\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01}\\ f_{01} & f_{11}\end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D_2(\partial f/\partial z_3)$. When $m\geq 4$, we obtain that

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right)=m\alpha_0<(2m-2)\alpha_0-2\alpha_1+2\alpha_3=\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01}\\ f_{01} & f_{11}\end{vmatrix}\right),$$

so $D_2(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

In the second part, we consider D_3. Firstly, we investigate $D_3(\partial f/\partial z_1)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_1}\right)=m\alpha_0-\alpha_1+\alpha_3<(3m-2)\alpha_0-2\alpha_1-2\alpha_2+3\alpha_3=\mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02}\\ f_{01} & f_{11} & f_{12}\\ f_{02} & f_{12} & f_{22}\end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_1)$ is a multiple of $\partial f/\partial z_0$.

Secondly, we investigate $D_3(\partial f/\partial z_2)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_2}\right) = m\alpha_0 - \alpha_2 + \alpha_3 < (3m-2)\alpha_0 - 2\alpha_1 - 2\alpha_2 + 3\alpha_3 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D_3(\partial f/\partial z_2)$ is a linear combination of $\partial f/\partial z_0$ and $\partial f/\partial z_1$.

Thirdly, we investigate $D(\partial f/\partial z_3)$:

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) = m\alpha_0 \le (3m-2)\alpha_0 - 2\alpha_1 - 2\alpha_2 + 3\alpha_3 = \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} & f_{02} \\ f_{01} & f_{11} & f_{12} \\ f_{02} & f_{12} & f_{22} \end{vmatrix}\right),$$

so $D(\partial f/\partial z_3)$ is a linear combination of $\partial f/\partial z_0$, $\partial f/\partial z_1$ and $\partial f/\partial z_2$.

Lemma 2.13. *Let*

$$f(z_0, z_1, z_2, z_3) = z_0^3 z_2 + a_1(z_1, z_2, z_3) z_0^2 + a_2(z_1, z_2, z_3) z_0 + a_3(z_1, z_2, z_3)$$

be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *that satisfies* $d \ge 2\alpha_0 \ge 2\alpha_1 \ge 2\alpha_2 \ge 2\alpha_3$ *with isolated singularity at the origin. Assume* $\alpha_0 + \alpha_2 + \alpha_3 < 2\alpha_1$, *which is equivalent to*

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) > \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right).$$

Let D_2 *be a derivation of the algebra* $\mathbb{C}[z_0, z_1, z_2, z_3]/(I, I_2)$. *Then* $D_2 \equiv 0$, *if* D_2 *is of negative weight.*

Proof. By the assumption, we conclude that $\mathrm{wt}(a_1(z_1, z_2, z_3)) = \alpha_0 + \alpha_2 < 2\alpha_1 - \alpha_3$. We obtain that $a_1(z_1, z_2, z_3) = z_1 f'(z_2, z_3) + f''(z_2, z_3)$. Let

$$D_2 = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2},$$

be a nonzero negative weight derivation.

Firstly, we investigate $D_2(\partial f/\partial z_0)$:

$$\begin{aligned} 0 = D_2\left(\frac{\partial f}{\partial z_0}\right) &= p_0(z_1, z_2, z_3)\big[6z_0 z_2 + 2a_1(z_1, z_2, z_3)\big] \\ &\quad + p_1(z_2, z_3)\left[2z_0 \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1}\right] \\ &\quad + c z_3^k\left[3z_0^2 + 2z_0 \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} + \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}\right]. \end{aligned}$$

So we obtain that $c = 0$, i.e., $D_2 = p_0(z_1, z_2, z_3)(\partial/\partial z_0) + p_1(z_2, z_3)(\partial/\partial z_1)$.

If $p_1(z_2, z_3) = 0$, then $p_0(z_1, z_2, z_3) = 0$, which is absurd.

If $p_1(z_2,z_3)\neq 0$, then

$$(13)\quad \begin{cases} 3p_0(z_1,z_2,z_3)z_2+p_1(z_2,z_3)f'(z_2,z_3)=0,\\ 2p_0(z_1,z_2,z_3)\big[z_1f'(z_2,z_3)+f''(z_2,z_3)\big]+p_1(z_2,z_3)\dfrac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}=0. \end{cases}$$

Hence, we obtain that

$$2f'(z_2,z_3)\big[z_1f'(z_2,z_3)+f''(z_2,z_3)\big]=3z_2\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1},$$

which implies $f'(z_2,z_3)$ is divisible by z_2. Let $f'(z_2,z_3)=3dz_2^e+3\sum_i d_iz_2^{e_i}z_3^{s_i}$, with $e>0$ and $e_i>0$ for all i, then we obtain that

$$(14)\quad 2\Big(dz_2^{e-1}+\sum_i d_iz_2^{e_i-1}z_3^{s_i}\Big)\Big[3z_1\Big(dz_2^e+\sum_i d_iz_2^{e_i}z_3^{s_i}\Big)+f''(z_2,z_3)\Big] = \frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}.$$

Next, we investigate $D(\partial f/\partial z_1)$:

$$D\Big(\frac{\partial f}{\partial z_1}\Big)=p_0(z_1,z_2,z_3)\Big[2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1}+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}\Big] +p_1(z_2,z_3)\Big[z_0\frac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1^2}+\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2}\Big].$$

Since $D(\partial f/\partial z_1)=h(\partial f/\partial z_0)$ and $\partial f/\partial z_0=3z_0^2z_2+\cdots$, then $D(\partial f/\partial z_1)=0$. Hence, we have

$$p_0(z_1,z_2,z_3)\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}+p_1(z_2,z_3)\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2}=0.$$

Let

$$g_1(z_2,z_3):=dz_2^{e-1}+\sum_i d_iz_2^{e_i-1}z_3^{s_i},$$

i.e., $3z_2g_1(z_2,z_3)=f'(z_2,z_3)$. Hence, by (14), we have

$$a_2(z_1,z_2,z_3)=z_1^2g_1(z_2,z_3)f'(z_2,z_3)+2z_1g_1(z_2,z_3)f''(z_2,z_3)+f'''(z_2,z_3),$$

and by (13) and (14), we obtain that

$$a_3(z_1,z_2,z_3)=\tfrac{1}{3}z_1^3[g_1(z_2,z_3)]^2f'(z_2,z_3)+z_1^2[g_1(z_2,z_3)]^2f''(z_2,z_3) +z_1f''''(z_2,z_3)+f'''''(z_2,z_3).$$

By Lemma 2.4, we obtain that one of $z_1^{l_0}$, $z_1^{l_1}z_1$, $z_1^{l_2}z_2$, $z_1^{l_3}z_3$ must be contained in f, which is absurd, i.e., D_2 does not exist. □

Lemma 2.14. *Let*

$$f(z_0, z_1, z_2, z_3) = z_0^3 z_3 + a_1(z_1, z_2, z_3) z_0^2 + a_2(z_1, z_2, z_3) z_0 + a_3(z_1, z_2, z_3)$$

be a weighted homogeneous polynomial of type $(\alpha_0, \alpha_1, \alpha_2, \alpha_3; d)$ *that satisfies* $d \geq 2\alpha_0 \geq 2\alpha_1 \geq 2\alpha_2 \geq 2\alpha_3$ *with isolated singularity at the origin. Assume that* $\alpha_0 + 2\alpha_3 < 2\alpha_1$, *which is equivalent to*

$$\mathrm{wt}\left(\frac{\partial f}{\partial z_3}\right) > \mathrm{wt}\left(\begin{vmatrix} f_{00} & f_{01} \\ f_{01} & f_{11} \end{vmatrix}\right).$$

Let D_2 be a derivation of the algebra $\mathbb{C}[z_0, z_1, z_2, z_3]/(I, I_2)$. Then $D_2 \equiv 0$, if D_2 is of negative weight.

Proof. By the assumption, we have $\mathrm{wt}(a_1(z_1, z_2, z_3)) = \alpha_0 + \alpha_3 < 2\alpha_1$. We obtain that $a_1(z_1, z_2, z_3) = z_1 f'(z_2, z_3) + f''(z_2, z_3)$. Let

$$D_2 = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + p_1(z_2, z_3)\frac{\partial}{\partial z_1} + c z_3^k \frac{\partial}{\partial z_2},$$

be a nonzero negative weight derivation.

Firstly, we investigate $D_2(\partial f/\partial z_0)$:

$$\begin{aligned} 0 = D_2\left(\frac{\partial f}{\partial z_0}\right) &= p_0(z_1, z_2, z_3)\big[6 z_0 z_3 + 2 a_1(z_1, z_2, z_3)\big] \\ &\quad + p_1(z_2, z_3)\left[2 z_0 \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1}\right] \\ &\quad + c z_3^k \left[2 z_0 \frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} + \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}\right]. \end{aligned}$$

There are two subcases.

Case 1: Assume $c = 0$, i.e., $D_2 = p_0(z_1, z_2, z_3)(\partial/\partial z_0) + p_1(z_2, z_3)(\partial/\partial z_1)$.

If $p_1(z_2, z_3) = 0$, then $p_0(z_1, z_2, z_3) = 0$, which is absurd.

If $p_1(z_2, z_3) \neq 0$, then

$$\begin{cases} 3 p_0(z_1, z_2, z_3) z_3 + p_1(z_2, z_3) f'(z_2, z_3) = 0, \\ 2 p_0(z_1, z_2, z_3)\big[z_1 f'(z_2, z_3) + f''(z_2, z_3)\big] + p_1(z_2, z_3)\dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} = 0. \end{cases}$$

Hence, we obtain that

$$2 f'(z_2, z_3) \cdot \big[z_1 f'(z_2, z_3) + f''(z_2, z_3)\big] = 3 z_3 \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1},$$

which implies $f'(z_2, z_3)$ is divisible by z_3. Let $f'(z_2, z_3) = 3 d z_3^s + 3 \sum_i d_i z_2^{e_i} z_3^{s_i}$, with $s > 0$ and $s_i > 0$ for all i, then we obtain that

$$2\left(d z_3^{s-1} + \sum_i d_i z_2^{e_i} z_3^{s_i - 1}\right)\left[3 z_1 (d z_3^s + \sum_i d_i z_2^{e_i} z_3^{s_i}) + f''(z_2, z_3)\right] = \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1}.$$

Next, we investigate $D(\partial f/\partial z_1)$:

$$D\left(\frac{\partial f}{\partial z_1}\right) = p_0(z_1,z_2,z_3)\left[2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_1} + \frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}\right] + p_1(z_2,z_3)\left[z_0\frac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1^2} + \frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2}\right].$$

It is obvious that $D(\partial f/\partial z_1)=0$. Hence, we obtain a new relation as follows:

$$p_0(z_1,z_2,z_3)\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1} + p_1(z_2,z_3)\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2} = 0.$$

Let

$$g_1(z_2,z_3) = dz_3^{s-1} + \sum_i d_i z_2^{e_i} z_3^{s_i-1},$$

i.e., $3z_3 g_1(z_2,z_3) = f'(z_2,z_3)$. Hence,

$$a_2(z_1,z_2,z_3) = z_1^2 g_1(z_2,z_3) f'(z_2,z_3) + 2z_1 g_1(z_2,z_3) f''(z_2,z_3) + f_3(z_2,z_3),$$

and

$$a_3(z_1,z_2,z_3) = \tfrac{1}{3}z_1^3[g_1(z_2,z_3)]^2 f'(z_2,z_3) + z_1^2[g_1(z_2,z_3)]^2 f''(z_2,z_3) + z_1 f_4(z_2,z_3) + f_5(z_2,z_3).$$

By Lemma 2.4, we obtain that one of $z_1^{l_0}$, $z_1^{l_1}z_0$, $z_1^{l_2}z_2$, $z_1^{l_3}z_3$ must be contained in f, which is absurd, i.e., D_2 does not exist.

Case 2: Assume $c \neq 0$.

Case 2.1: Let $p_1(z_2,z_3)=0$.

Case 2.1.1: Let $p_0(z_1,z_2,z_3)=0$, i.e., $D = cz_3^k(\partial/\partial z_2)$. Then, we obtain that

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = cz_3^k\left(2z_0\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_2} + \frac{\partial a_2(z_1,z_2,z_3)}{\partial z_2}\right).$$

This implies that $\partial a_1(z_1,z_2,z_3)/\partial z_2 = 0$ and $\partial a_2(z_1,z_2,z_3)/\partial z_2 = 0$. Hence,

$$0 = D\left(\frac{\partial f}{\partial z_1}\right) = cz_3^k\left(z_0^2\frac{\partial^2 a_1(z_1,z_2,z_3)}{\partial z_1\partial z_2} + z_0\frac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1\partial z_2} + \frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1\partial z_2}\right),$$

and

$$0 = D\left(\frac{\partial f}{\partial z_2}\right) = cz_3^k\left(z_0^2\frac{\partial^2 a_1(z_1,z_2,z_3)}{\partial z_2^2} + z_0\frac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_2^2} + \frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_2^2}\right).$$

Hence, $\partial^2 a_3(z_1,z_2,z_3)/(\partial z_1\partial z_2)=0$ and $\partial^2 a_3(z_1,z_2,z_3)/\partial z_2^2 = 0$. By Lemma 2.4, we obtain that one of $z_2^{l_0}$, $z_2^{l_1}z_0$, $z_2^{l_2}z_1$, $z_2^{l_3}z_3$ must be contained in f, which is absurd, i.e., D_2 does not exist.

Case 2.1.2: Let $p_0(z_1, z_2, z_3) \neq 0$, i.e., $D_2 = p_0(z_1, z_2, z_3)(\partial/\partial z_0) + cz_3^k(\partial/\partial z_2)$. Hence,

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_0}\right) + cz_3^k\frac{\partial}{\partial z_2}\left(\frac{\partial f}{\partial z_0}\right) = \frac{\partial}{\partial z_0}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + cz_3^k\frac{\partial f}{\partial z_2}\right]. \tag{15}$$

Moreover,

$$0 = D\left(\frac{\partial f}{\partial z_0}\right) = p_0(z_1, z_2, z_3)\big[6z_0z_3 + 2a_1(z_1, z_2, z_3)\big] + cz_3^k\left[2z_0\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} + \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}\right], \tag{16}$$

and

$$\begin{cases} 3p_0(z_1, z_2, z_3)z_3 + cz_3^k\dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} = 0, \\ 2p_0(z_1, z_2, z_3)\big[z_1f'(z_2, z_3) + f''(z_2, z_3)\big] + cz_3^k\dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_2} = 0. \end{cases}$$

From this, we obtain that

$$2\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2}a_1(z_1, z_2, z_3) = 3z_3\frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}.$$

It is easy to verify that $a_1(z_1, z_2, z_3)$ is divisible by z_3. Let $f'(z_2, z_3) = z_3g'(z_2, z_3)$ and $f''(z_2, z_3) = z_3g''(z_2, z_3)$. So $\partial a_2(z_1, z_2, z_3)/\partial z_2$ is divisible by z_3. Then we consider

$$\begin{aligned} D\left(\frac{\partial f}{\partial z_1}\right) &= p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0}\left(\frac{\partial f}{\partial z_1}\right) + cz_3^k\frac{\partial}{\partial z_2}\left(\frac{\partial f}{\partial z_1}\right) \\ &= \frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + cz_3^k\frac{\partial f}{\partial z_2}\right] - \frac{\partial p_0(z_1, z_2, z_3)}{\partial z_1}\frac{\partial f}{\partial z_0} \\ &= h\frac{\partial f}{\partial z_0}. \end{aligned} \tag{17}$$

Equation (17) implies that

$$\frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + cz_3^k\frac{\partial f}{\partial z_2}\right] = \tilde{h}\frac{\partial f}{\partial z_0}.$$

From (15), we know that the left-hand side of this equation is independent of the variable z_0. Since $f = z_0^3z_3 + \cdots$, the right-hand side of this equation is independent of the variable z_0 only if $\tilde{h} = 0$. Thus, we have

$$\frac{\partial}{\partial z_1}\left(p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + cz_3^k\frac{\partial f}{\partial z_2}\right) = 0.$$

Therefore,

$$p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + cz_3^k\frac{\partial f}{\partial z_2} = g_1(z_2, z_3).$$

This is

$$\begin{aligned} p_0(z_1,z_2,z_3)\big(3z_3z_0^2&+2a_1(z_1,z_2,z_3)z_0+a_2(z_1,z_2,z_3)\big)\\ &+cz_3^k\bigg[\frac{\partial a_1(z_1,z_2,z_3)}{\partial z_2}z_0^2+\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_2}z_0+\frac{\partial a_3(z_1,z_2,z_3)}{\partial z_2}\bigg]=g_1(z_2,z_3). \end{aligned} \tag{18}$$

Equation (18) implies

$$\begin{cases} 3p_0(z_1, z_2, z_3)z_3 + cz_3^k\dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} = 0, \\ 2p_0(z_1, z_2, z_3)a_1(z_1, z_2, z_3) + cz_3^k\dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_2} = 0, \end{cases}$$

and

$$\begin{aligned} &\frac{\partial\big[p_0(z_1,z_2,z_3)a_2(z_1,z_2,z_3)+cz_3^k(\partial a_3(z_1,z_2,z_3)/\partial z_2)\big]}{\partial z_1}\\ &=\frac{\partial p_0(z_1,z_2,z_3)}{\partial z_1}a_2(z_1,z_2,z_3)+p_0(z_1,z_2,z_3)\frac{\partial a_2(z_1,z_2,z_3)}{\partial z_1}+cz_3^k\frac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1\partial z_2}\\ &=0. \end{aligned}$$

It is clear that $a_1(z_1, z_2, z_3) \neq 0$, $\partial a_1(z_1, z_2, z_3)/\partial z_2 \neq 0$ and $\partial a_2(z_1, z_2, z_3)/\partial z_2 \neq 0$. Hence, there exists $h(z_1, z_2, z_3)$ such that

$$\begin{cases} 3z_3h(z_1, z_2, z_3) = 2a_1(z_1, z_2, z_3), \\ \dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_2}h(z_1, z_2, z_3) = \dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}. \end{cases}$$

Thus, $cz_3^k(\partial a_1(z_1, z_2)/\partial z_1) \neq 0$. Then

$$p_0(z_1, z_2, z_3) = z_3^{k-1}q_0(z_1, z_2, z_3),$$

where

$$q_0(z_1, z_2, z_3) = -\frac{c}{3}\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2}.$$

Hence, $q_0(z_1, z_2, z_3)$ is divisible by z_3, i.e., $p_0(z_1, z_2, z_3)$ is divisible by z_3^k. Thus, the differential equation

$$p_0(z_1, z_2, z_3) + cz_3^k\frac{\partial g(z_1, z_2, z_3)}{\partial z_2} = 0$$

has a solution. We also use $g(z_1, z_2, z_3)$ to denote the solution which does not contain the constant term. We construct the following coordinate transformation:

$$\begin{cases} z_0' = z_0 + g(z_1, z_2, z_3), \\ z_1' = z_1, \\ z_2' = z_2, \\ z_3' = z_3. \end{cases}$$

Under this transformation of coordinates, we obtain that

$$\begin{cases} \dfrac{\partial}{\partial z_0} = \dfrac{\partial}{\partial z_0'}, \\ \dfrac{\partial}{\partial z_1} = \dfrac{\partial g}{\partial z_1}\dfrac{\partial}{\partial z_0'} + \dfrac{\partial}{\partial z_1'}, \\ \dfrac{\partial}{\partial z_2} = \dfrac{\partial g}{\partial z_2}\dfrac{\partial}{\partial z_0'} + \dfrac{\partial}{\partial z_2'}. \end{cases}$$

Hence,

$$\begin{aligned} D &= p_0(z_1, z_2, z_3)\frac{\partial}{\partial z_0} + cz_3^k\frac{\partial}{\partial z_2} \\ &= p_0(z_1', z_2', z_3')\frac{\partial}{\partial z_0'} + c(z_3')^k\left(\frac{\partial g}{\partial z_2}\frac{\partial}{\partial z_0'} + \frac{\partial}{\partial z_2'}\right) \\ &= \left(p_0(z_1', z_2', z_3') + c(z_3')^k\frac{\partial g}{\partial z_2}\right)\frac{\partial}{\partial z_0'} + c(z_3')^k\frac{\partial}{\partial z_2'} \\ &= c(z_3')^k\frac{\partial}{\partial z_2'}. \end{aligned}$$

By the discussion of Case 2.1.1, such D does not exist.

Case 2.2: Let $p_1(z_2, z_3) \neq 0$.

Case 2.2.1: Let $p_0(z_1, z_2, z_3) = 0$. Then $D(\partial f/\partial z_0) = 0$ implies that

$$\begin{cases} p_1(z_2, z_3)\dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + cz_3^k\dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} = 0, \\ p_1(z_2, z_3)\dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} + cz_3^k\dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_2} = 0. \end{cases}$$

Hence, $f'(z_2, z_3) = 0$ or a constant multiple of z_3^s.

Next, we investigate $D(\partial f/\partial z_1)$:

$$\begin{aligned} 0 = D\left(\frac{\partial f}{\partial z_1}\right) = p_1(z_2, z_3)&\left[z_0\frac{\partial^2 a_2(z_1, z_2, z_3)}{\partial z_1^2} + \frac{\partial^2 a_3(z_1, z_2, z_3)}{\partial z_1^2}\right] \\ &+ cz_3^k\left[z_0\frac{\partial^2 a_2(z_1, z_2, z_3)}{\partial z_1 \partial z_2} + \frac{\partial^2 a_3(z_1, z_2, z_3)}{\partial z_1 \partial z_2}\right]. \end{aligned}$$

Hence,

$$\begin{cases} p_1(z_2,z_3)\dfrac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1^2}+cz_3^k\dfrac{\partial^2 a_2(z_1,z_2,z_3)}{\partial z_1\partial z_2}=0,\\ p_1(z_2,z_3)\dfrac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1^2}+cz_3^k\dfrac{\partial^2 a_3(z_1,z_2,z_3)}{\partial z_1\partial z_2}=0.\end{cases}$$

If $p_1(z_2,z_3)$ is divisible by z_3^k, then the differential equation

$$p_1(z_2,z_3)+cz_3^k\frac{\partial g(z_2,z_3)}{\partial z_2}=0$$

has a solution. We also use $g(z_2,z_3)$ to denoted the solution which does not contain the constant term. We construct the following coordinate transformation:

$$\begin{cases} z_0'=z_0,\\ z_1'=z_1+g(z_2,z_3),\\ z_2'=z_2,\\ z_3'=z_3.\end{cases}$$

Under this transformation of coordinates, we obtain that

$$\begin{cases} \dfrac{\partial}{\partial z_0}=\dfrac{\partial}{\partial z_0'},\\ \dfrac{\partial}{\partial z_1}=\dfrac{\partial}{\partial z_1'},\\ \dfrac{\partial}{\partial z_2}=\dfrac{\partial g}{\partial z_2}\dfrac{\partial}{\partial z_1'}+\dfrac{\partial}{\partial z_2'}.\end{cases}$$

Hence,

$$\begin{aligned} D&=p_1(z_2,z_3)\frac{\partial}{\partial z_1}+cz_3^k\frac{\partial}{\partial z_2}\\ &=p_1(z_2',z_3')\frac{\partial}{\partial z_1'}+c(z_3')^k\left(\frac{\partial g}{\partial z_2}\frac{\partial}{\partial z_1'}+\frac{\partial}{\partial z_2'}\right)\\ &=\left(p_1(z_2',z_3')+c(z_3')^k\frac{\partial g}{\partial z_2}\right)\frac{\partial}{\partial z_1'}+c(z_3')^k\frac{\partial}{\partial z_2'}.\end{aligned}\tag{19}$$

By Case 2.1.1, such D does not exist. In the following, we assume that $p_1(z_2,z_3)$ is not divisible by z_3^k. Hence, $z_3\mid(\partial a_1/\partial z_1)$, $z_3\mid(\partial a_2/\partial z_1)$, $z_3\mid(\partial^2 a_2/\partial z_1^2)$ and $z_3\mid(\partial^2 a_3/\partial z_1^2)$. By the weight inequality $\alpha_0+2\alpha_3<2\alpha_1$, we have

$$\begin{aligned} f=z_0^3z_3+(z_1f_1+f_0)z_0^2+(z_1^3g_3+z_1^2g_2+z_1g_1+g_0)z_0\\ +z_1^5h_5+z_1^4h_4+z_1^3h_3+z_1^2h_2+z_1h_1+h_0,\end{aligned}$$

where $f_i = f_i(z_2, z_3)$, with $i = 0, 1$; $g_j = g_j(z_2, z_3)$, with $0 \le j \le 3$; and $h_l = h_l(z_2, z_3)$, with $0 \le l \le 5$. Hence, the following equations:

$$\frac{\partial f}{\partial z_0} = 3z_0^2 z_3 + 2z_0(z_1 f_1 + f_0) + (z_1^3 g_3 + z_1^2 g_2 + z_1 g_1 + g_0),$$

$$\frac{\partial f}{\partial z_1} = z_0^2 f_1 + z_0(3z_1^2 g_3 + 2z_1^2 g_2 + g_1) + 5z_1^4 h_5 + 4z_1^3 h_4 + 3z_1^2 h_3 + 2z_1 h_2 + h_1,$$

$$\begin{aligned}\frac{\partial f}{\partial z_2} &= z_0^2\left(z_1\frac{\partial f_1}{\partial z_2} + \frac{\partial f_0}{\partial z_2}\right) + z_0\left(z_1^3\frac{\partial g_3}{\partial z_2} + z_1^2\frac{\partial g_2}{\partial z_2} + z_1\frac{\partial g_1}{\partial z_2} + \frac{\partial g_0}{\partial z_2}\right)\\ &\quad + \left(z_1^5\frac{\partial h_5}{\partial z_2} + z_1^4\frac{\partial h_4}{\partial z_2} + z_1^3\frac{\partial h_3}{\partial z_2} + z_1^2\frac{\partial h_2}{\partial z_2} + z_1\frac{\partial h_1}{\partial z_2} + \frac{\partial h_0}{\partial z_2}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z_3} &= z_0^3 + z_0^2\left(z_1\frac{\partial f_1}{\partial z_3} + \frac{\partial f_0}{\partial z_3}\right) + z_0\left(z_1^3\frac{\partial g_3}{\partial z_3} + z_1^2\frac{\partial g_2}{\partial z_3} + z_1\frac{\partial g_1}{\partial z_3} + \frac{\partial g_0}{\partial z_3}\right)\\ &\quad + \left(z_1^5\frac{\partial h_5}{\partial z_3} + z_1^4\frac{\partial h_4}{\partial z_3} + z_1^3\frac{\partial h_3}{\partial z_3} + z_1^2\frac{\partial h_2}{\partial z_3} + z_1\frac{\partial h_1}{\partial z_3} + \frac{\partial h_0}{\partial z_3}\right),\end{aligned}$$

have solution

$$\begin{cases} z_2 = 0,\\ z_3 = 0,\\ \dfrac{\partial f}{\partial z_3} = 0.\end{cases}$$

Hence, f does not have an isolated singularity at the origin.

Case 2.2.2: Let $p_0(z_1, z_2, z_3) \neq 0$. Then

$$\begin{aligned}0 &= D\left(\frac{\partial f}{\partial z_0}\right)\\ &= \frac{\partial}{\partial z_0}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + cz_3^k\frac{\partial f}{\partial z_2}\right]\end{aligned} \tag{20}$$

and

$$\begin{aligned}&D\left(\frac{\partial f}{\partial z_1}\right)\\ &= \frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + cz_3^k\frac{\partial f}{\partial z_2}\right] - \frac{\partial p_0(z_1, z_2, z_3)}{\partial z_1}\frac{\partial f}{\partial z_0}\\ &= h\frac{\partial f}{\partial z_0}.\end{aligned} \tag{21}$$

Equation (21) implies that

$$\frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + cz_3^k\frac{\partial f}{\partial z_2}\right] = \tilde{h}\frac{\partial f}{\partial z_0}.$$

From (20), we know that the left-hand side of this equation is independent of the variable z_0. Since $f = z_0^3 z_3 + \cdots$, the right-hand side of this equation is independent of the variable z_0 only if $\tilde{h} = 0$. Thus, we have

$$\frac{\partial}{\partial z_1}\left(p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c z_3^k \frac{\partial f}{\partial z_2}\right) = 0.$$

Therefore,

$$p_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + p_1(z_2, z_3)\frac{\partial f}{\partial z_1} + c z_3^k \frac{\partial f}{\partial z_2} = g_1(z_2, z_3).$$

This is

$$\begin{aligned}(22)\quad & p_0(z_1, z_2, z_3)\big(3z_3 z_0^2 + 2a_1(z_1, z_2, z_3)z_0 + a_2(z_1, z_2, z_3)\big) \\ &\qquad + p_1(z_2, z_3)\left[z_0^2\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + z_0\frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} + \frac{\partial a_3(z_1, z_2, z_3)}{\partial z_1}\right] \\ &\qquad + c z_3^k\left[\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2}z_0^2 + \frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}z_0 + \frac{\partial a_3(z_1, z_2, z_3)}{\partial z_2}\right] \\ &= g_1(z_2, z_3).\end{aligned}$$

Equation (22) implies

$$(23)\quad \begin{cases} 3p_0(z_1, z_2, z_3)z_3 + p_1(z_2, z_3)\dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} + c z_3^k \dfrac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} = 0, \\ 2p_0(z_1, z_2, z_3)a_1(z_1, z_2, z_3) + p_1(z_2, z_3)\dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} \\ \qquad\qquad + c z_3^k \dfrac{\partial a_2(z_1, z_2, z_3)}{\partial z_2} = 0, \end{cases}$$

and

$$\begin{aligned}(24)\quad & \frac{\partial}{\partial z_1}\left[p_0(z_1, z_2, z_3)a_2(z_1, z_2, z_3) + p_1(z_2, z_3)\frac{\partial a_3(z_1, z_2, z_3)}{\partial z_1} + c z_3^k \frac{\partial a_3(z_1, z_2, z_3)}{\partial z_2}\right] \\ &= \frac{\partial p_0(z_1, z_2, z_3)}{\partial z_1}a_2(z_1, z_2, z_3) + p_0(z_1, z_2, z_3)\frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1} \\ &\qquad + p_1(z_2, z_3)\frac{\partial^2 a_3(z_1, z_2, z_3)}{\partial z_1^2} + c z_3^k \frac{\partial^2 a_3(z_1, z_2, z_3)}{\partial z_1 \partial z_2} \\ &= 0.\end{aligned}$$

From (23), we obtain that

$$\begin{aligned}(25)\quad & p_1(z_2, z_3)\left[2a_1(z_1, z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_1} - 3z_3\frac{\partial a_2(z_1, z_2, z_3)}{\partial z_1}\right] \\ &\qquad + c z_3^k\left[2a_1(z_1, z_2, z_3)\frac{\partial a_1(z_1, z_2, z_3)}{\partial z_2} - 3z_3\frac{\partial a_2(z_1, z_2, z_3)}{\partial z_2}\right] = 0.\end{aligned}$$

If $p_1(z_2, z_3)$ is divisible by cz_3^k, then the differential equation

$$p_1(z_2, z_3) + cz_3^k \frac{\partial g(z_2, z_3)}{\partial z_2} = 0,$$

has a solution. We also use $g(z_2, z_3)$ to denote the solution which does not contain the constant term. We construct the following coordinate transformation:

$$\begin{cases} z_0' = z_0, \\ z_1' = z_1 + g(z_2, z_3), \\ z_2' = z_2, \\ z_3' = z_3. \end{cases}$$

Under this transformation of coordinates, we obtain that

$$\begin{cases} \dfrac{\partial}{\partial z_0} = \dfrac{\partial}{\partial z_0'}, \\ \dfrac{\partial}{\partial z_1} = \dfrac{\partial}{\partial z_1'}, \\ \dfrac{\partial}{\partial z_2} = \dfrac{\partial g}{\partial z_2} \dfrac{\partial}{\partial z_1'} + \dfrac{\partial}{\partial z_2'}. \end{cases}$$

Hence,

$$\begin{aligned} D &= p_0(z_1, z_2, z_3) \frac{\partial}{\partial z_0} + p_1(z_2, z_3) \frac{\partial}{\partial z_1} + cz_3^k \frac{\partial}{\partial z_2} \\ &= p_0\bigl(z_1' - g(z_2', z_3'), z_2', z_3'\bigr) \frac{\partial}{\partial z_0'} + p_1(z_2', z_3') \frac{\partial}{\partial z_1'} + c(z_3')^k \left(\frac{\partial g}{\partial z_2} \frac{\partial}{\partial z_1'} + \frac{\partial}{\partial z_2'} \right) \\ &= p_0\bigl(z_1' - g(z_2', z_3'), z_2', z_3'\bigr) \frac{\partial}{\partial z_0'} + \left(p_1(z_2', z_3') + c(z_3')^k \frac{\partial g}{\partial z_2} \right) \frac{\partial}{\partial z_1'} + c(z_3')^k \frac{\partial}{\partial z_2'}. \end{aligned} \tag{26}$$

By Case 2.1.2, such D does not exist.

In the following, we assume that $p_1(z_2, z_3)$ is not divisible by cz_3^k. According to (25), we obtain that z_3 is divisible by $\partial a_1(z_1, z_2, z_3)/\partial z_1$.

By the weight inequality $\alpha_0 + 2\alpha_3 < 2\alpha_1$, we have

$$\begin{aligned} f = z_0^3 z_3 + (z_1 f_1 + f_0) z_0^2 + (z_1^3 g_3 + z_1^2 g_2 + z_1 g_1 + g_0) z_0 \\ + z_1^5 h_5 + z_1^4 h_4 + z_1^3 h_3 + z_1^2 h_2 + z_1 h_1 + h_0, \end{aligned}$$

where $f_i = f_i(z_2, z_3)$, with $i = 0, 1$; $g_j = g_j(z_2, z_3)$, with $j = 0, 1, 2, 3$; and $h_l = h_l(z_2, z_3)$, with $l = 0, 1, 2, 3, 4, 5$. By (23), we obtain that

$$3p_0 z_3 + p_1 f_1 + cz_3^k \left(z_1 \frac{\partial f_1}{\partial z_2} + \frac{\partial f_0}{\partial z_2} \right) = 0, \tag{27}$$

and

$$2p_0(z_1 f_1 + f_0) + p_1(3z_1^2 g_3 + 2z_1 g_2 + g_1) + cz_3^k\left(z_1^3 \frac{\partial g_3}{\partial z_2} + z_1^2 \frac{\partial g_2}{\partial z_2} + z_1 \frac{\partial g_1}{\partial z_2} + \frac{\partial g_0}{\partial z_2}\right) = 0. \tag{28}$$

If $\partial f_1/\partial z_2 = 0$, then $p_0(z_1, z_2, z_3) = p_0(z_2, z_3)$ does not depend on z_1. Thus, (27) and (24) become

$$3p_0 z_3 + p_1 f_1 + cz_3^k \frac{\partial f_0}{\partial z_2} = 0, \tag{29}$$

and

$$p_0(3z_1^2 g_3 + 2z_1 g_2 + g_1) + p_1(20z_1^3 h_5 + 12z_1^2 h_4 + 6z_1 h_3 + 2h_2) + cz_3^k\left(5z_1^4 \frac{\partial h_5}{\partial z_2} + 4z_1^3 \frac{\partial h_4}{\partial z_2} + 3z_1^2 \frac{\partial h_3}{\partial z_2} + 2z_1 \frac{\partial h_2}{\partial z_2} + \frac{\partial h_1}{\partial z_2}\right) = 0. \tag{30}$$

By (28), we obtain that

$$\begin{cases} \dfrac{\partial g_3}{\partial z_2} = 0, \\ 3p_1 g_3 + cz_3^k \dfrac{\partial g_2}{\partial z_2} = 0, \\ 2p_0 f_1 + 2p_1 g_2 + cz_3^k \dfrac{\partial g_1}{\partial z_2} = 0, \\ 2p_0 f_0 + p_1 g_0 + cz_3^k \dfrac{\partial g_0}{\partial z_2} = 0. \end{cases} \tag{31}$$

By (30), we obtain that

$$\begin{cases} \dfrac{\partial h_5}{\partial z_2} = 0, \\ 20p_1 h_5 + 4cz_3^k \dfrac{\partial h_4}{\partial z_2} = 0, \\ 3p_0 g_3 + 12p_1 h_4 + 3cz_3^k \dfrac{\partial h_3}{\partial z_2} = 0, \\ 2p_0 g_2 + 6p_1 h_3 + 2cz_3^k \dfrac{\partial h_2}{\partial z_2} = 0, \\ p_0 g_1 + 2p_1 h_2 + cz_3^k \dfrac{\partial h_1}{\partial z_2} = 0. \end{cases} \tag{32}$$

If $g_3 = 0$, then $\partial g_2/\partial z_2 = 0$ and $z_3 \mid h_4$ by (31) and (32). By Lemma 2.4 and $z_3 \mid h_4$, we obtain that $h_5 = 0$. Hence, the following equations:

$$\frac{\partial f}{\partial z_0} = 3z_0^2 z_3 + 2z_0(z_1 f_1 + f_0) + (z_1^3 g_3 + z_1^2 g_2 + z_1 g_1 + g_0),$$

$$\frac{\partial f}{\partial z_1} = z_0^2 f_1 + z_0(3z_1^2 g_3 + 2z_1^2 g_2 + g_1) + 5z_1^4 h_5 + 4z_1^3 h_4 + 3z_1^2 h_3 + 2z_1 h_2 + h_1,$$

$$\begin{aligned}\frac{\partial f}{\partial z_2} = z_0^2\left(z_1\frac{\partial f_1}{\partial z_2} + \frac{\partial f_0}{\partial z_2}\right) + z_0\left(z_1^3\frac{\partial g_3}{\partial z_2} + z_1^2\frac{\partial g_2}{\partial z_2} + z_1\frac{\partial g_1}{\partial z_2} + \frac{\partial g_0}{\partial z_2}\right)\\ + \left(z_1^5\frac{\partial h_5}{\partial z_2} + z_1^4\frac{\partial h_4}{\partial z_2} + z_1^3\frac{\partial h_3}{\partial z_2} + z_1^2\frac{\partial h_2}{\partial z_2} + z_1\frac{\partial h_1}{\partial z_2} + \frac{\partial h_0}{\partial z_2}\right),\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z_3} = z_0^3 + z_0^2\left(z_1\frac{\partial f_1}{\partial z_3} + \frac{\partial f_0}{\partial z_3}\right) + z_0\left(z_1^3\frac{\partial g_3}{\partial z_3} + z_1^2\frac{\partial g_2}{\partial z_3} + z_1\frac{\partial g_1}{\partial z_3} + \frac{\partial g_0}{\partial z_3}\right)\\ + \left(z_1^5\frac{\partial h_5}{\partial z_3} + z_1^4\frac{\partial h_4}{\partial z_3} + z_1^3\frac{\partial h_3}{\partial z_3} + z_1^2\frac{\partial h_2}{\partial z_3} + z_1\frac{\partial h_1}{\partial z_3} + \frac{\partial h_0}{\partial z_3}\right),\end{aligned}$$

have solution

$$\begin{cases} z_2 = 0, \\ z_3 = 0, \\ \dfrac{\partial f}{\partial z_3} = 0. \end{cases}$$

Hence f does not have isolated singularity at the origin.

If $g_3 \neq 0$, then $g_3 = z_3^l$, where $l \geq 1$ (due to (32) and omitting a nonzero constant multiple). If $h_5 = 0$, then $\partial h_4/\partial z_2 = 0$ and $h_3 \neq z_2$. Similarly, f does not have isolated singularity at the origin. If $h_5 \neq 0$, then $h_5 = z_3^{l'}$, where $l' \geq 1$ (due to (32) and omitting a nonzero constant multiple). Similarly, f does not have isolated singularity at the origin.

In the following, we assume that $\partial f_1/\partial z_2 \neq 0$. Then $p_0(z_1, z_2, z_3)$ depends on z_1. Let $f_1 = z_3 f_1'(z_2, z_3)$, with $f_1' \neq 0$, depend on z_2. By (27), without loss of generality, we can assume that p_0 is divisible by z_3^k. (Otherwise, if

$$p_0 = z_1 z_3^k p_0'(z_2, z_3) + p_0''(z_2, z_3),$$

then we can do the same argument as the condition $\partial f_1/\partial z_2 = 0$.) Moreover, p_1 is not divisible by z_3^k. By (28) and (30), we obtain that

$$z_3 \mid 3z_1^2 g_3 + 2z_1 g_2 + g_1 \quad \text{and} \quad z_3 \mid 20z_1^3 h_5 + 12z_1^2 h_4 + 6z_1 h_3 + 2h_2.$$

This implies that $z_3 \mid g_3$, $z_3 \mid g_2$, $z_3 \mid g_1$, $z_3 \mid h_5$, $z_3 \mid h_4$, $z_3 \mid h_3$, $z_3 \mid h_2$. Moreover we obtain that $z_3 \mid (\partial g_3/\partial z_2)$, $z_3 \mid (\partial g_2/\partial z_2)$, $z_3 \mid (\partial g_1/\partial z_2)$, $z_3 \mid (\partial h_5/\partial z_2)$, $z_3 \mid (\partial h_4/\partial z_2)$, $z_3 \mid (\partial h_3/\partial z_2)$ and $z_3 \mid (\partial h_2/\partial z_2)$. So f does not have isolated singularity at the origin. $\square$

When investigating the derivation D_2, we assume that f has one of the following forms:

(1) $f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$ with $m \geq 4$,

(2) $f = z_0^m z_1 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$ with $m \geq 3$,

(3) $f = z_0^m z_2 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$ with $m \geq 3$,

(4) $f = z_0^m z_3 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$ with $m \geq 3$,

with the following relations:

$$
\begin{aligned}
D_2\left(\frac{\partial f}{\partial z_0}\right) &= 0, \\
D_2\left(\frac{\partial f}{\partial z_1}\right) &= p(z_1, z_2, z_3)\frac{\partial f}{\partial z_0}, \\
D_2\left(\frac{\partial f}{\partial z_2}\right) &= q_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + q_1(z_2, z_3)\frac{\partial f}{\partial z_1}, \\
D_2\left(\frac{\partial f}{\partial z_3}\right) &= w_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + w_1(z_2, z_3)\frac{\partial f}{\partial z_1} + w_2(z_3)\frac{\partial f}{\partial z_2}.
\end{aligned}
$$

When investigating the derivation D_3, we assume that f has one of the following forms:

(1) $f = z_0^m + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$, with $m \geq 3$,

(2) $f = z_0^m z_1 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$, with $m \geq 2$,

(3) $f = z_0^m z_2 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$, with $m \geq 2$,

(4) $f = z_0^m z_3 + a_1(z_1, z_2, z_3)z_0^{m-1} + \cdots + a_m(z_1, z_2, z_3)$, with $m \geq 3$,

with the following relations:

$$
\begin{aligned}
D_3\left(\frac{\partial f}{\partial z_0}\right) &= 0, \\
D_3\left(\frac{\partial f}{\partial z_1}\right) &= p(z_1, z_2, z_3)\frac{\partial f}{\partial z_0}, \\
D_3\left(\frac{\partial f}{\partial z_2}\right) &= q_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + q_1(z_2, z_3)\frac{\partial f}{\partial z_1}, \\
D_3\left(\frac{\partial f}{\partial z_3}\right) &= w_0(z_1, z_2, z_3)\frac{\partial f}{\partial z_0} + w_1(z_2, z_3)\frac{\partial f}{\partial z_1} + w_2(z_3)\frac{\partial f}{\partial z_2}.
\end{aligned}
$$

If D_ν is a derivation of A_ν, then D_ν is a derivation of B. By Theorem 1.5, such a derivation is nonnegative. $\square$

In view of Proposition 2.6, Proposition 2.7, Proposition 2.8 and Proposition 2.10, the proof of the Main Theorem is now complete. $\square$

3. Future work

For an isolated hypersurface singularity defined by $f(z_0, \dots, z_n)$, the moduli algebra is defined by $A(f) = \mathcal{O}_{n+1}/\big((f)+J(f)\big)$ and the k-th Hessian algebra is defined by $H_k(f) = \mathcal{O}_{n+1}/\big((f)+J(f)+I_k\big)$. Suppose that the ideal $\big((f)+J(f)+I_k\big)$ is generated by $g_1, \dots, g_m$, we use $J_\ell(g_1, \dots, g_m)$ to denote the ideal generated by all $\ell \times \ell$-minors of the Jacobian matrix of $g_1, \dots, g_m$, then we introduce a series of local algebras $M_{k,\ell}(f) = \mathcal{O}_{n+1}/\big((f)+J(f)+I_k+J_\ell(g_1, \dots, g_m)\big)$. We believe Conjecture 1.6 and the Main Theorem can be generalized to these new local algebras. Some progress has been made, and we will include the results in our subsequent papers.

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Received June 9, 2022. Revised September 13, 2022.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 323 No. 1 March 2023
