Pacific Journal of Mathematics

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Volume 323 No. 2 April 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 323, No. 2, 2023
https://doi.org/10.2140/pjm.2023.323.307

ON THE THEORY OF GENERALIZED ULRICH MODULES

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Dedicated with gratitude to the memory of Professor Shiro Goto

We further develop the theory of generalized Ulrich modules introduced in 2014 by Goto et al. Our main goal is to address the problem as to when the operations of taking the Hom functor and horizontal linkage preserve the Ulrich property. One of the applications is a new characterization of quadratic hypersurface rings. Moreover, in the Gorenstein case, we deduce that applying linkage to sufficiently high syzygy modules of Ulrich ideals yields Ulrich modules. Finally, we explore connections to the theory of modules with minimal multiplicity, and as a byproduct we determine the Chern number of an Ulrich module as well as the Castelnuovo–Mumford regularity of its Rees module.

1. Introduction

This work is concerned with the theory of generalized Ulrich modules (over Cohen–Macaulay local rings) by Goto et al. [2014], which widely extended the classical study of maximally generated maximal Cohen–Macaulay modules — or *Ulrich modules*, as coined in [Herzog and Kühl 1987] — initiated in the 1980s by B. Ulrich [1984]. The term *generalized* refers to the fact that Ulrich modules are taken relatively to a zero-dimensional ideal which is not necessarily the maximal ideal, the latter situation corresponding to the classical theory; despite the apparent naivety of the idea, this passage adds considerable depth to the theory and enlarges its horizon of applications.

Motivated by the remarkable advances in [Goto et al. 2014], our purpose here is to present further progress which includes generalizations of several known results on Ulrich modules, from the above paper as well as [Kobayashi and Takahashi 2019; Ooishi 1991; Wiebe 2003], and connections to some other important classes such as that of modules with minimal multiplicity; for the latter task, we employ suitable numerical invariants such as the Castelnuovo–Mumford regularity of blowup modules.

MSC2020: primary 13C05, 13C14, 13H10; secondary 13A30, 13C13, 13C40, 13D07.

Keywords: Ulrich module, maximal Cohen–Macaulay module, horizontal linkage, module of minimal multiplicity, blowup module.

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It is worth recalling that the original notion of an Ulrich module (together with the classical existence problem; see, however, Yhee's [2021] construction of local domains which do not admit Ulrich modules or (weakly) lim Ulrich sequences) has been extensively explored since its inception, in both commutative algebra and algebraic geometry. Echoing and complementing the second paragraph of the Introduction of [Yhee 2021], the applications include criteria for the Gorenstein property [Hanes and Huneke 2005; Ulrich 1984], the investigation of maximal Cohen–Macaulay modules over Gorenstein local rings and factoriality of certain rings [Herzog and Kühl 1987], the development of the theory of almost Gorenstein rings [Goto et al. 2015], strategies to tackle certain resistant conjectures in multiplicity theory — e.g., Ma's [2023] resolution of Lech's conjecture in the graded case by introducing and using the notion of (weakly) lim Ulrich sequences, which gives yet another way to generalize the classical Ulrich property — and methods for constructing resultants and Chow forms of projective algebraic varieties (see [Eisenbud and Schreyer 2003], where the concepts of Ulrich sheaf and Ulrich bundle were introduced).

In essence, the general approach suggested in [Goto et al. 2014] extended the definition of an Ulrich module M over a (commutative, Noetherian) Cohen–Macaulay local ring $(R, \mathscr{M})$ to a relative setting that takes into account an $\mathscr{M}$-primary ideal $\mathscr{I}$ containing a parameter ideal as a reduction, so that the case $\mathscr{I} = \mathscr{M}$ retrieves the standard theory. For instance, the condition of the freeness of $M/\mathscr{I}M$ over $R/\mathscr{I}$, which was hidden in the classical setting as $M/\mathscr{M}M$ is simply a vector space, is now required. Following this line of investigation, other works have appeared in the literature, including [Goto et al. 2016a; 2016b; 2019; Numata 2017].

We will briefly comment on our main results, section by section. Preliminary definitions and some known auxiliary results, which are used throughout the paper, are given in Section 2. The main goal of Section 3 is to investigate the Ulrich property under the Hom functor. In this regard, our main result is Theorem 3.2, which can be viewed as a generalization of [Goto et al. 2014, Theorem 5.1] and of [Kobayashi and Takahashi 2019, Proposition 4.1]. Moreover, Corollary 3.5 generalizes [Goto et al. 2014, Corollary 5.2], and Corollary 3.6 is a far-reaching extension of [Brennan et al. 1987, Lemma 2.2]. We also study a connection to the theory of semidualizing modules (see Corollary 3.8) and use it to derive a new characterization of when R is regular (see Corollary 3.9). In addition, in the last subsection, we provide some freeness criteria for $M/\mathscr{I}M$ over the Artinian local ring $R/\mathscr{I}$, which is one of the requirements for Ulrichness with respect to $\mathscr{I}$.

In Section 4 we are essentially interested in the behavior of the Ulrich property under the operation of horizontal linkage over Gorenstein local rings. The main result here is Theorem 4.1 (see also Corollary 4.4), from which we derive a curious characterization of quadratic hypersurface local rings (see Corollary 4.3). In

Corollary 4.7, we record the special case of sufficiently high syzygy modules of a nonparameter Ulrich ideal, in case R is Gorenstein.

In Section 5 we consider the class of modules with minimal multiplicity (in the sense of [Puthenpurakal 2003]) and then connect this concept to the Ulrich property, both taken with respect to $\mathscr{I}$. The basic relation is that Ulrich R-modules have minimal multiplicity (see Proposition 5.6), and as a consequence we use the Chern number — the first Hilbert coefficient — as an ingredient to obtain a characterization of Ulrichness (see Corollary 5.9) which generalizes [Ooishi 1991, Corollary 1.3(1)]. Under this perspective, modules with trivial Chern number are provided in Corollary 5.10, and considerations about the structure of the Hilbert–Samuel polynomial of an Ulrich module are given in Remarks 5.11. Our main technical result in this section is Theorem 5.14, which curiously does not contain Ulrich-like properties in its statement and, more precisely, characterizes modules with minimal multiplicity as follows:

Theorem 5.14. *Let $(R, \mathscr{M})$ be a Noetherian local ring with infinite residue field, M a Cohen–Macaulay R-module of dimension $t > 0$ and I an $\mathscr{M}$-primary ideal of R. Let $J = (z_1, \dots, z_t)$ be a minimal M-reduction of I. The following assertions are equivalent*:

(i) *M has minimal multiplicity with respect to I.*

(ii) $\operatorname{reg} \mathcal{R}(I, M) = \operatorname{reg} \mathcal{G}(I, M) = r_J(I, M) \leq 1$.

(iii) $r_J(I, M) \leq 1$.

Here, $\operatorname{reg}(-)$ denotes (Castelnuovo–Mumford) regularity, and $\mathcal{R}(I, M)$ and $\mathcal{G}(I, M)$ stand respectively for the Rees module and the associated graded module of I relative to M. Also, $r_J(I, M)$ is the reduction number of I with respect to J relative to M. We emphasize that Theorem 5.14 answers affirmatively the module-theoretic analogue of Sally's [1983] question about independence of reduction numbers for the class of modules with minimal multiplicity. Additionally, from this theorem we derive Corollary 5.15, which determines the regularity of the Rees and associated graded modules of $\mathscr{I}$ relative to an Ulrich module (this result partially generalizes [Ooishi 1991, Proposition 1.1]), and also Corollary 5.16, where we deal once again with high syzygy modules of Ulrich ideals.

Finally, Section 6 provides a detailed example to illustrate some of our main corollaries.

2. Conventions, preliminaries, and some auxiliary results

Throughout this paper, all rings are assumed to be commutative and Noetherian with 1, and by *finite* module we mean a finitely generated module.

In this section, we recall some of the basic notions and tools that will play an important role throughout the paper. Other auxiliary notions will be introduced as they become necessary.

2A. *Ulrich ideals and modules.* Let $(R, \mathscr{M})$ be a local ring, M a finite R-module, and $I \neq R$ an ideal of definition of M, i.e., $\mathscr{M}^n M \subset IM$ for some $n > 0$. Let us establish some notations. We denote by $\nu(M)$ and $e_I^0(M)$, respectively, the minimal number of generators of M and the multiplicity of M with respect to I. When $I = \mathscr{M}$, we simply write $e(M)$ in place of $e^0_{\mathscr{M}}(M)$.

Definition 2.1. Let $(R, \mathscr{M})$ be a local ring. A finite R-module M is *Cohen–Macaulay* if $\operatorname{depth}_R M = \dim M$, and *maximal Cohen–Macaulay* if $\operatorname{depth}_R M = \dim R$. Note the zero module is not maximal Cohen–Macaulay as its depth is set to be $+\infty$. Moreover, M is called *Ulrich* if M is a maximal Cohen–Macaulay R-module satisfying $\nu(M) = e(M)$.

For instance, if $(R, \mathscr{M})$ is a 1-dimensional Cohen–Macaulay local ring, then the power $\mathscr{M}^{e(R)-1}$ is an Ulrich module. Several other classes of examples can be found in [Brennan et al. 1987].

Ulrich modules are also dubbed *maximally generated maximal Cohen–Macaulay modules*. This is due to the fact that there is an inequality $\nu(M) \leq e(M)$ whenever the local ring R is Cohen–Macaulay and M is maximal Cohen–Macaulay; see [Brennan et al. 1987, Proposition 1.1].

Convention 2.2. Henceforth, in the entire paper, we adopt the following convention and notations. Whenever $(R, \mathscr{M})$ is a d-dimensional Cohen–Macaulay local ring, we will let $\mathscr{I}$ (to be distinguished from the notation I) stand for an $\mathscr{M}$-primary ideal that contains a parameter ideal

$$Q = (\boldsymbol{x}) = (x_1, \ldots, x_d)$$

as a reduction, i.e., $Q\mathscr{I}^r = \mathscr{I}^{r+1}$ for some integer $r \geq 0$. As is well known, any $\mathscr{M}$-primary ideal of R has this property provided that the residue class field $R/\mathscr{M}$ is infinite, or that R is analytically irreducible with $d = 1$.

Definition 2.3. Let R be a Cohen–Macaulay local ring. We say that the ideal $\mathscr{I}$ is *Gorenstein* if the quotient ring $R/\mathscr{I}$ is Gorenstein.

Next, we recall the general notions of Ulrich ideal and Ulrich module as introduced in [Goto et al. 2014], where in addition several explicit examples are given. As will be made clear, the latter Definition 2.7 below generalizes Definition 2.1.

Definition 2.4 [Goto et al. 2014]. Let R be a Cohen–Macaulay local ring. We say that the ideal $\mathscr{I}$ is *Ulrich* if $\mathscr{I}^2 = Q\mathscr{I}$ (the reduction number of $\mathscr{I}$ with respect to Q is at most 1) and $\mathscr{I}/\mathscr{I}^2$ is a free $R/\mathscr{I}$-module.

Examples 2.5. (i) [Kumashiro 2023, Proposition 3.10] Let $S = K[\![x, y, z]\!]$ be a formal power series ring over an infinite field K, and fix any regular sequence $\{f, g, h\} \subset (x, y, z)$. Then, $R = S/(f^2 - gh, g^2 - hf, h^2 - fg)$ is a 1-dimensional Cohen–Macaulay local ring and $\mathscr{I} = (f, g, h)R$ is an Ulrich ideal.

(ii) [Goto et al. 2014, Example 2.7(2)] One way to produce examples in arbitrary positive dimension is as follows. Given a field K and integers $d, s \geq 1$, consider the d-dimensional local hypersurface ring $R = K[\![z_1, \dots, z_{d+1}]\!]/(z_1^2 + \cdots + z_d^2 + z_{d+1}^{2s})$, where $z_1, \dots, z_{d+1}$ are formal indeterminates over k. Then, the ideal

$$\mathscr{I} = (z_1, \dots, z_d, z_{d+1}^s)R$$

is Ulrich and contains the parameter ideal $Q = (z_1, \dots, z_d)R$ as a reduction.

Remark 2.6. In a Gorenstein local ring, every Ulrich ideal is Gorenstein; see [Goto et al. 2014, Corollary 2.6].

Definition 2.7 [Goto et al. 2014]. Let R be a Cohen–Macaulay local ring and let M be a finite R-module. We say that M is *Ulrich with respect to* $\mathscr{I}$ if the following conditions hold:

(i) M is a maximal Cohen–Macaulay R-module.

(ii) $\mathscr{I}M = QM$.

(iii) $M/\mathscr{I}M$ is a free $R/\mathscr{I}$-module.

Remarks 2.8. (i) Let us recall the discussion in the paragraph after Definition 1.2 in [Goto et al. 2014]. Denote the length of R-modules by $\ell_R(-)$. If R is a Cohen–Macaulay local ring and M is a maximal Cohen–Macaulay R-module, then

$$e^0_{\mathscr{I}}(M) = e^0_Q(M) = \ell_R(M/QM) \geq \ell_R(M/\mathscr{I}M),$$

so that condition (ii) of Definition 2.7 is equivalent to saying that the equality $e^0_{\mathscr{I}}(M) = \ell_R(M/\mathscr{I}M)$ takes place. In particular, if $\mathscr{I} = \mathscr{M}$, condition (ii) is the same as $e(M) = \nu(M)$. Therefore, M is an Ulrich module with respect to $\mathscr{M}$ if and only if M is an Ulrich module in the sense of Definition 2.1.

(ii) Clearly, if $d = 1$ and $\mathscr{I}$ is an Ulrich ideal of R, then $\mathscr{I}$ is an Ulrich R-module with respect to $\mathscr{I}$.

(iii) Let us recall the following more general recipe to obtain Ulrich modules from Ulrich ideals (in the setting of Convention 2.2). If $\mathscr{I}$ is an Ulrich ideal of R which is not a parameter ideal, then for any $i \geq d$ the i-th syzygy module (see Section 2B below) of $R/\mathscr{I}$ is an Ulrich R-module with respect to $\mathscr{I}$, and conversely (we refer to [Goto et al. 2014, Theorem 4.1]). This is a very helpful property and will be explored in some of our results and examples.

2B. *Linkage.* The concepts recalled in this subsection can be described in the general context of semiperfect rings, but in this paper we focus on the special case of (finite modules over) a local ring R, since this is the setup where our results will be proved.

Given a finite R-module M, we write $M^* = \operatorname{Hom}_R(M, R)$. The (*Auslander*) *transpose* $\operatorname{Tr} M$ of M is defined as the cokernel of the dual $\partial_1^* = \operatorname{Hom}_R(\partial_1, R)$ of the first differential map ∂_1 in a minimal free resolution of M over R. Hence there is an exact sequence

$$0 \longrightarrow M^* \longrightarrow F_0^* \xrightarrow{\partial_1^*} F_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0$$

for suitable finite free R-modules F_0, F_1. The (*first*) *syzygy* module $\Omega^1 M = \Omega M$ of M is the image of ∂_1, hence a submodule of F_0. We recursively put $\Omega^k M = \Omega(\Omega^{k-1} M)$, the k-th syzygy module of M, for any $k \geq 2$.

Note that the modules $\operatorname{Tr} M$ and ΩM are uniquely determined up to isomorphism, since the same is true of a minimal free resolution of M. By [Auslander 1966, Proposition 6.3], we have an exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^1(\operatorname{Tr} M, R) \longrightarrow M \xrightarrow{e_M} M^{**} \longrightarrow \operatorname{Ext}_R^2(\operatorname{Tr} M, R) \longrightarrow 0, \tag{1}$$

where e_M is the evaluation map.

Martsinkovsky and Strooker [2004] generalized the classical theory of linkage for ideals to the context of modules by means of the operator $\lambda = \Omega \operatorname{Tr}$, i.e., a finite R-module M is sent to the composite $\Omega \operatorname{Tr} M$ defined from a minimal free presentation of M.

Definition 2.9 [Martsinkovsky and Strooker 2004]. Two finite R-modules M and N are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$. In the case where M and λM are horizontally linked, $M \cong \lambda^2 M$, we simply say that the module M is horizontally linked.

We also recall that a *stable* module is a finite module with no nonzero free direct summand. A finite R-module M is called a *syzygy module* if it is embedded in a finite free R-module, that is if $M \cong \Omega N$ for some finite R-module N. Here is a well-known characterization of horizontally linked modules.

Lemma 2.10 [Martsinkovsky and Strooker 2004, Theorem 2 and Corollary 6]. *A finite R-module M is horizontally linked if and only if it is stable and* $\operatorname{Ext}_R^1(\operatorname{Tr} M, R) = 0$, *if and only if M is a stable syzygy module.*

Lemma 2.11 [Martsinkovsky and Strooker 2004, Proposition 4]. *Suppose M is horizontally linked. Then, λM is also horizontally linked and, in particular, λM is stable.*

2C. *Canonical modules.* In the sequel we collect basic facts about canonical modules.

Lemma 2.12 [Bruns and Herzog 1993]. *Let R be a Cohen–Macaulay local ring with canonical module ω_R. Let M be a maximal Cohen–Macaulay R-module. Then the following statements hold*:

(i) $\operatorname{Hom}_R(M, \omega_R)$ *is a maximal Cohen–Macaulay R-module.*

(ii) $\operatorname{Ext}^i_R(M, \omega_R) = 0$ *for all $i > 0$.*

(iii) $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$.

(iv) *If $\boldsymbol{y}$ is an R-sequence, then $R/(\boldsymbol{y})$ has a canonical module $\omega_{R/(\boldsymbol{y})} \cong \omega_R/\boldsymbol{y}\omega_R$.*

(v) *Let $\varphi : R \to S$ be a local homomorphism of Cohen–Macaulay local rings such that S is a finite R-module. Then S has a canonical module $\omega_S \cong \operatorname{Ext}^t_R(S, \omega_R)$, where $t = \dim R - \dim S$.*

3. Hom functor and the Ulrich property

In this section we investigate, in essence, the behavior of the Ulrich property under the Hom functor.

3A. *Key lemma, main result, and corollaries.* We start with the following basic lemma, which will be a key ingredient in the proof of the main result of this section.

Lemma 3.1. *Let R be a Cohen–Macaulay local ring, M, N be maximal Cohen–Macaulay R-modules, and $\boldsymbol{y} = y_1, \dots, y_n$ be an R-sequence for some $n \geq 1$.*

(i) *If either $n = 1$ or $\operatorname{Ext}^i_R(M, N) = 0$ for all $i = 1, \dots, n-1$, there is an injection*

$$\operatorname{Hom}_R(M, N)/\boldsymbol{y}\operatorname{Hom}_R(M, N) \hookrightarrow \operatorname{Hom}_{R/(\boldsymbol{y})}(M/\boldsymbol{y}M, N/\boldsymbol{y}N).$$

(ii) *If $\operatorname{Ext}^i_R(M, N) = 0$ for all $i = 1, \dots, n$, there is an isomorphism*

$$\operatorname{Hom}_R(M, N)/\boldsymbol{y}\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{R/(\boldsymbol{y})}(M/\boldsymbol{y}M, N/\boldsymbol{y}N).$$

Proof. We shall prove the assertion (i), which from the arguments below (essentially from (2)) will be easily seen to imply (ii). Set $R' = R/(y_1)$, $M' = M/y_1M$, and $N' = N/y_1N$. We will proceed by induction on n. Consider first the case $n = 1$, which is standard, but we supply the proof for convenience. Since M and N are maximal Cohen–Macaulay R-modules and $y_1 \in \mathscr{M}$ is R-regular, where $\mathscr{M}$ is the maximal ideal of R, it follows that y_1 is both M-regular and N-regular. In particular, we have the short exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow M' \longrightarrow 0,$$

which induces the exact sequence

$$(2)\quad 0 \to \operatorname{Hom}_R(M', N) \to \operatorname{Hom}_R(M, N) \xrightarrow{y_1} \operatorname{Hom}_R(M, N) \to \operatorname{Ext}^1_R(M', N) \\ \to \cdots \to \operatorname{Ext}^i_R(M, N) \to \operatorname{Ext}^{i+1}_R(M', N) \to \operatorname{Ext}^{i+1}_R(M, N) \to \cdots .$$

It follows an injection

$$(3)\qquad \operatorname{Hom}_R(M, N)/y_1 \operatorname{Hom}_R(M, N) \hookrightarrow \operatorname{Ext}^1_R(M', N).$$

Because y_1 is N-regular and $y_1 M' = 0$, there are isomorphisms

$$(4)\qquad \operatorname{Ext}^i_{R'}(M', N') \cong \operatorname{Ext}^{i+1}_R(M', N) \quad \text{for all } i \geq 0,$$

see [Bruns and Herzog 1993, Lemma 3.1.16]. In particular,

$$(5)\qquad \operatorname{Hom}_{R'}(M', N') \cong \operatorname{Ext}^1_R(M', N),$$

and the result follows by (3) and (5).

Now let $n \geq 2$. Clearly, R' is a Cohen–Macaulay ring and M', N' are maximal Cohen–Macaulay R'-modules. By assumption, $\operatorname{Ext}^i_R(M, N) = 0$ for all $i = 1, \ldots, n-1$. Thus, using (2) and (4), we obtain isomorphisms

$$(6)\quad \operatorname{Ext}^i_{R'}(M', N') \cong \begin{cases} \operatorname{Hom}_R(M, N)/y_1 \operatorname{Hom}_R(M, N) & \text{if } i = 0, \\ 0 & \text{if } i = 1, \ldots, n-2. \end{cases}$$

Since $\boldsymbol{y}' = y_2, \ldots, y_n$ is an R'-sequence, the induction hypothesis yields an injection

$$\operatorname{Hom}_{R'}(M', N')/\boldsymbol{y}' \operatorname{Hom}_{R'}(M', N') \hookrightarrow \operatorname{Hom}_{R'/\boldsymbol{y}'R'}(M'/\boldsymbol{y}'M', N'/\boldsymbol{y}'N'),$$

where the latter module is clearly isomorphic to $\operatorname{Hom}_{R/(\boldsymbol{y})}(M/\boldsymbol{y}M, N/\boldsymbol{y}N)$. Now the conclusion follows by (6) with $i = 0$. $\square$

The theorem below is our main result in this section.

Theorem 3.2. *Let R be a Cohen–Macaulay local ring of dimension d. Let M and N be maximal Cohen–Macaulay R-modules such that* $\operatorname{Hom}_R(M, N) \neq 0$ *and* $\operatorname{Ext}^i_R(M, N) = 0$ *for all $i = 1, \ldots, n$, where either $n = d - 1$ or $n = d$. Let $\mathscr{I}$ and Q be as in Convention 2.2. Assume that M, resp. N, is an Ulrich R-module with respect to $\mathscr{I}$, and consider the following conditions*:

(i) $\operatorname{Hom}_R(M, N)$ *is an Ulrich R-module with respect to $\mathscr{I}$.*

(ii) $\operatorname{Hom}_R(M, N)/\mathscr{I} \operatorname{Hom}_R(M, N)$ *is a free $R/\mathscr{I}$-module.*

(iii) $\operatorname{Hom}_{R/Q}(R/\mathscr{I}, N/QN)$, *resp.* $\operatorname{Hom}_{R/Q}(M/QM, R/\mathscr{I})$, *is a free $R/\mathscr{I}$-module.*

Then the following statements hold:

(a) *If $n = d - 1$ then (i) $\Longleftrightarrow$ (ii).*

(b) *If $n = d$ then (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).*

Proof. (a) Applying the functor $\mathrm{Hom}_R(-, N)$ to a free resolution

$$\cdots \longrightarrow F_{d+1} \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of the R-module M, and using the hypothesis that $\mathrm{Ext}^i_R(M, N) = 0$ for $i = 1, \dots, d-1$, we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F_0, N) \to \cdots \to \mathrm{Hom}_R(F_{d-1}, N) \to \mathrm{Hom}_R(F_d, N).$$

Now set $X_0 := \mathrm{Hom}_R(M, N)$ and $X_i := \mathrm{Im}(\mathrm{Hom}_R(F_{i-1}, N) \to \mathrm{Hom}_R(F_i, N))$ for $i = 1, \dots, d$. Since N is maximal Cohen–Macaulay, then $\mathrm{depth}_R \mathrm{Hom}_R(F_i, N) = d$ for all $i = 0, \dots, d$. Thus, by the short exact sequence

$$0 \longrightarrow X_i \longrightarrow \mathrm{Hom}_R(F_i, N) \longrightarrow X_{i+1} \longrightarrow 0,$$

we get $\mathrm{depth}_R X_i \geq \min\{d, \mathrm{depth}_R X_{i+1} + 1\}$; see, e.g., [Bruns and Herzog 1993, Proposition 1.2.9]. Therefore,

$$\mathrm{depth}_R \mathrm{Hom}_R(M, N) \geq \min\{d, \mathrm{depth}_R X_d + d\} = d,$$

i.e., $\mathrm{Hom}_R(M, N)$ is a maximal Cohen–Macaulay R-module.

Now, as in Convention 2.2, let $\boldsymbol{x} = x_1, \dots, x_d$ be a generating set of the parameter ideal Q. Then $\boldsymbol{x}$ is an R-sequence (see [Bruns and Herzog 1993, Theorem 2.1.2(d)]), and so by Lemma 3.1(i) there is an injection

$$\mathrm{Hom}_R(M, N)/Q\,\mathrm{Hom}_R(M, N) \hookrightarrow \mathrm{Hom}_{R/Q}(M/QM, N/QN). \tag{7}$$

Because M (resp. N) is assumed to be Ulrich with respect to $\mathscr{I}$, the module M/QM (resp. N/QN) is annihilated by $\mathscr{I}$, and hence so is $\mathrm{Hom}_{R/Q}(M/QM, N/QN)$. In either case, it follows from (7) that the quotient $\mathrm{Hom}_R(M, N)/Q\,\mathrm{Hom}_R(M, N)$ is annihilated by $\mathscr{I}$. Thus,

$$\mathscr{I}\,\mathrm{Hom}_R(M, N) = Q\,\mathrm{Hom}_R(M, N).$$

Therefore, $\mathrm{Hom}_R(M, N)$ is Ulrich with respect to $\mathscr{I}$ if and only if the quotient module $\mathrm{Hom}_R(M, N)/\mathscr{I}\,\mathrm{Hom}_R(M, N)$ is $R/\mathscr{I}$-free, so (i) $\Longleftrightarrow$ (ii).

(b) As seen above, there is an equality $\mathscr{I}\,\mathrm{Hom}_R(M, N) = Q\,\mathrm{Hom}_R(M, N)$. Notice that, furthermore, Lemma 3.1(ii) yields an isomorphism

$$\mathrm{Hom}_R(M, N)/Q\,\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{R/Q}(M/QM, N/QN). \tag{8}$$

Now suppose that M is Ulrich with respect to $\mathscr{I}$. From $M/QM = M/\mathscr{I}M \cong (R/\mathscr{I})^m$ for some integer $m > 0$, we deduce that

$$\mathrm{Hom}_{R/Q}(M/QM, N/QN) \cong (\mathrm{Hom}_{R/Q}(R/\mathscr{I}, N/QN))^m. \tag{9}$$

By (8) and (9), we get

$$\operatorname{Hom}_R(M, N)/\mathscr{I} \operatorname{Hom}_R(M, N) \cong (\operatorname{Hom}_{R/Q}(R/\mathscr{I}, N/QN))^m.$$

Therefore, the quotient $\operatorname{Hom}_R(M, N)/\mathscr{I} \operatorname{Hom}_R(M, N)$ is $R/\mathscr{I}$-free if and only if the module $\operatorname{Hom}_{R/Q}(R/\mathscr{I}, N/QN)$ is $R/\mathscr{I}$-free. The case where N is Ulrich with respect to $\mathscr{I}$ is completely similar. This shows (ii) $\Longleftrightarrow$ (iii) and concludes the proof of the theorem. □

Remark 3.3. It is worth observing that the condition $\operatorname{Hom}_R(M, N) = 0$ can hold even if M and N are both Ulrich. For instance, over the local ring $R = K[\![x, y]\!]/(xy)$, where x, y are formal variables over a field K, we have

$$\operatorname{Hom}_R(R/xR, R/yR) = 0.$$

We point out that Theorem 3.2 generalizes [Kobayashi and Takahashi 2019, Proposition 4.1] (see Corollary 3.7, to be given shortly) and, in addition, recovers the following result from [Goto et al. 2014]:

Corollary 3.4 [Goto et al. 2014, Theorem 5.1]. *Let R be a Cohen–Macaulay local ring with canonical module ω_R, and let M be an Ulrich R-module with respect to $\mathscr{I}$. Then the following assertions are equivalent*:

(i) $\operatorname{Hom}_R(M, \omega_R)$ *is an Ulrich R-module with respect to $\mathscr{I}$.*

(ii) $\mathscr{I}$ *is a Gorenstein ideal.*

Proof. By Lemma 2.12(ii), we have $\operatorname{Ext}^i_R(M, \omega_R) = 0$ for all $i > 0$. Since R/Q and $R/\mathscr{I}$ are zero-dimensional local rings and the ideal Q is generated by an R-sequence, there are isomorphisms

$$\omega_{R/\mathscr{I}} \cong \operatorname{Hom}_{R/Q}(R/\mathscr{I}, \omega_{R/Q}) \cong \operatorname{Hom}_{R/Q}(R/\mathscr{I}, \omega_R/Q\omega_R)$$

according to standard facts; see parts (iv) and (v) of Lemma 2.12. Now, applying Theorem 3.2(b) with $N = \omega_R$, we derive that $\operatorname{Hom}_R(M, \omega_R)$ is Ulrich with respect to $\mathscr{I}$ if and only if $\omega_{R/\mathscr{I}}$ is $R/\mathscr{I}$-free, or equivalently, $R/\mathscr{I}$ is a Gorenstein ring. □

Taking Remark 2.6 into account, the corollary below is readily seen to generalize [Goto et al. 2014, Corollary 5.2].

Corollary 3.5. *Let R be a Cohen–Macaulay local ring with canonical module ω_R, and let M be a maximal Cohen–Macaulay R-module. Assume that the ideal $\mathscr{I}$ is Gorenstein. Then the following assertions are equivalent*:

(i) *M is an Ulrich R-module with respect to $\mathscr{I}$.*

(ii) $\operatorname{Hom}_R(M, \omega_R)$ *is an Ulrich R-module with respect to $\mathscr{I}$.*

Proof. There is an isomorphism $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$ by Lemma 2.12(iii). The conclusion follows by Corollary 3.4. □

Our next result is a far-reaching extension of [Brennan et al. 1987, Lemma 2.2]; see also Corollary 3.9.

Corollary 3.6. *Let R be a Cohen–Macaulay local ring with canonical module ω_R. Assume that the ideal $\mathscr{I}$ is Gorenstein. Then the following assertions are equivalent*:

(i) *$\mathscr{I}$ is a parameter ideal.*

(ii) *R is an Ulrich R-module with respect to $\mathscr{I}$.*

(iii) *ω_R is an Ulrich R-module with respect to $\mathscr{I}$.*

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is immediate from Definition 2.7 and holds regardless of $\mathscr{I}$ being Gorenstein. Now, by virtue of the isomorphisms $\omega_R \cong \operatorname{Hom}_R(R, \omega_R)$ and $\operatorname{Hom}_R(\omega_R, \omega_R) \cong R$, our Corollary 3.5 yields (ii) $\Longleftrightarrow$ (iii). $\square$

As yet another byproduct of Theorem 3.2, we retrieve [Kobayashi and Takahashi 2019, Proposition 4.1], which in turn generalizes the local version of [Wiebe 2003, Proposition 3.5].

Corollary 3.7 [Kobayashi and Takahashi 2019, Proposition 4.1]. *Let R be a Cohen–Macaulay local ring of dimension d. Let M, N be maximal Cohen–Macaulay R-modules such that* $\operatorname{Hom}_R(M, N) \neq 0$ *and* $\operatorname{Ext}^i_R(M, N) = 0$ *for all $i = 1, \dots, d-1$. If either M or N is an Ulrich R-module*, *then so is* $\operatorname{Hom}_R(M, N)$.

Proof. As observed in Remarks 2.8(i), M is an Ulrich R-module if and only if M is an Ulrich R-module with respect to the maximal ideal $\mathscr{M}$ of R. Now, being a (finite-dimensional) vector space over the residue field $k = R/\mathscr{M}$, the module $\operatorname{Hom}_R(M, N)/\mathscr{M} \operatorname{Hom}_R(M, N)$ is k-free. Thus, $\operatorname{Hom}_R(M, N)$ is Ulrich by Theorem 3.2(a). $\square$

3B. ***Hom with values in a semidualizing module.*** Let us recall that a finite module $\mathscr{C}$ over a ring R is called *semidualizing* if the morphism $R \to \operatorname{Hom}_R(\mathscr{C}, \mathscr{C})$ given by homothety is an isomorphism and $\operatorname{Ext}^i_R(\mathscr{C}, \mathscr{C}) = 0$ for all $i > 0$. In this case, a finite R-module M is said to be *totally $\mathscr{C}$-reflexive* if the biduality map $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, \mathscr{C}), \mathscr{C})$ is an isomorphism and, in addition, $\operatorname{Ext}^i_R(M, \mathscr{C}) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(M, \mathscr{C}), \mathscr{C})$ for all $i > 0$. Note every totally $\mathscr{C}$-reflexive module is maximal Cohen–Macaulay by virtue of the relative Auslander–Bridger formula; see [Sather-Wagstaff 2010, Proposition 6.4.2]. A detailed account about the theory of semidualizing modules is given in [Sather-Wagstaff 2010].

As a matter of illustration, R is semidualizing as a module over itself, and, for any semidualizing R-module $\mathscr{C}$, both R and $\mathscr{C}$ are totally $\mathscr{C}$-reflexive. More interestingly, if R is a Cohen–Macaulay local ring possessing a canonical module ω_R, then ω_R is semidualizing and, in addition, every maximal Cohen–Macaulay R-module is totally ω_R-reflexive (to see this, use Lemma 2.12). It should also be pointed out, based on the existence of several examples in the literature, that not every semidualizing R-module must be isomorphic to R or ω_R; see for example [Araya and Iima 2018, Section 5; Sather-Wagstaff 2010, Section 2.3].

Corollary 3.8. *Let R be a Cohen–Macaulay local ring with a semidualizing module $\mathscr{C}$, and let M be a totally $\mathscr{C}$-reflexive R-module. Then, M is an Ulrich R-module if and only if* $\operatorname{Hom}_R(M, \mathscr{C})$ *is an Ulrich R-module.*

Proof. We have $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \mathscr{C}), \mathscr{C})$, which in particular forces the module $\operatorname{Hom}_R(M, \mathscr{C})$ to be nontrivial, and in addition

$$\operatorname{Ext}^i_R(M, \mathscr{C}) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(M, \mathscr{C}), \mathscr{C}) \quad \text{for all } i > 0.$$

Since $\mathscr{C}$ is semidualizing, $\operatorname{depth}_R \mathscr{C} = \operatorname{depth} R$ (see [Sather-Wagstaff 2010, Theorem 2.2.6(c)]) and hence $\mathscr{C}$ is maximal Cohen–Macaulay. The result is clear by Corollary 3.7. □

Note that Corollary 3.8 gives a different proof of the case $\mathscr{I} = \mathscr{M}$ of Corollary 3.5 by taking $\mathscr{C} = \omega_R$. Another byproduct of Corollary 3.8 is the following curious characterization of regular local rings.

Corollary 3.9. *Let R be a Cohen–Macaulay local ring with a semidualizing module $\mathscr{C}$. Then, R is regular if and only if $\mathscr{C}$ is an Ulrich R-module.*

Proof. According to [Sather-Wagstaff 2010, Proposition 2.1.12], saying that $\mathscr{C}$ is semidualizing is tantamount to R being a totally $\mathscr{C}$-reflexive R-module. Now, Corollary 3.8 yields that R is Ulrich over itself if and only if $\mathscr{C}$ is an Ulrich R-module. The former situation, as observed in [Brennan et al. 1987, Lemma 2.2], is equivalent to the regularity of R. □

We raise the following question and a related remark.

Question 3.10. Does Corollary 3.4 hold with $\mathscr{C}$ (a given semidualizing R-module) in place of ω_R?

Remark 3.11. An affirmative answer to Question 3.10 would imply the validity of Corollary 3.5 with $\mathscr{C}$ in place of ω_R as well, provided that R is a normal domain. Indeed, it suffices to note that in this case the maximal Cohen–Macaulay R-module M is necessarily reflexive in the usual sense, and thus by [Sather-Wagstaff 2010, Corollary 5.4.7], which also requires R to be normal, we have

$$M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \mathscr{C}), \mathscr{C})$$

via the natural biduality map.

3C. ***Freeness criteria for $M/\mathscr{I}M$ via (co)homology vanishing.*** We close the section providing some criteria for the freeness of the $R/\mathscr{I}$-module $M/\mathscr{I}M$, which is of interest since this is one of the requirements for M to be Ulrich with respect to $\mathscr{I}$; see Definition 2.7.

As we have been investigating how Ulrichness behaves under the Hom ($=\mathrm{Ext}^0$) functor, it seems natural to wonder about the relevance of higher Ext modules in the theory, and in fact we shall see that the vanishing of finitely many "diagonal" Ext modules $\mathrm{Ext}^i_{R/\mathscr{I}}(M/\mathscr{I}M, M/\mathscr{I}M)$, under suitable hypotheses, can detect freeness over the Artinian local ring $R/\mathscr{I}$, which we will assume to be Gorenstein. Vanishing of homology modules, namely "diagonal" Tor modules $\mathrm{Tor}^{R/\mathscr{I}}_j(M/\mathscr{I}M, M/\mathscr{I}M)$, will also play a role. Essentially, our criteria will consist of adaptations of some results from [Huneke et al. 2004] and one from [Şega 2011].

In the proposition below, and as before, $(R, \mathscr{M})$ and $\mathscr{I}$ (also Q, which appears in the proof) are as in Convention 2.2, and $\ell_R(-)$ stands for length of R-modules.

Proposition 3.12. *Suppose $R/\mathscr{I}$ is Gorenstein (e.g., R is Gorenstein and $\mathscr{I}$ is Ulrich; see Remark 2.6) and let M be a finite R-module. Assume any one of the following situations:*

(i) $\mathscr{M}^2M \subset \mathscr{I}M$ *and* $\mathrm{Ext}^i_{R/\mathscr{I}}(M/\mathscr{I}M, M/\mathscr{I}M) = 0$ *for all i satisfying* $1 \leq i \leq \max\{3, \nu(M), \ell_R(M/\mathscr{I}M) - \nu(M)\}$.

(ii) $\mathscr{M}^3 \subset \mathscr{I}$ *and* $\mathrm{Ext}^i_{R/\mathscr{I}}(M/\mathscr{I}M, M/\mathscr{I}M) = 0$ *for some $i > 0$.*

(iii) *($R/\mathscr{I}$ need not be Gorenstein.) $R/\mathscr{M}$ is infinite, $\mathscr{I}$ is not a parameter ideal,* $\mathscr{M}^3 \subset \mathscr{I}$, $e^0_{\mathscr{I}}(R) \leq 2\ell_R(\mathscr{M}/(\mathscr{M}^2 + \mathscr{I}))$, *and* $\mathrm{Tor}^{R/\mathscr{I}}_j(M/\mathscr{I}M, M/\mathscr{I}M) = 0$ *for three consecutive values of $j \geq 2$.*

(iv) $\mathscr{M}^4 \subset \mathscr{I}$, *there exists $x \in \mathscr{M} \setminus \mathscr{I}$ such that the ideal $(\mathscr{I} : x)/\mathscr{I}$ is principal, and* $\mathrm{Tor}^{R/\mathscr{I}}_j(M/\mathscr{I}M, M/\mathscr{I}M) = 0$ *for all $j \gg 0$.*

Then, $M/\mathscr{I}M$ is $R/\mathscr{I}$-free.

Proof. For simplicity, set $\overline{R} = R/\mathscr{I}$, $\overline{\mathscr{M}} = \mathscr{M}/\mathscr{I}$, and $\overline{M} = M/\mathscr{I}M$. Let us assume (i). By assumption $\overline{\mathscr{M}}^2\overline{M} = 0$, hence

$$\nu(\overline{\mathscr{M}}\,\overline{M}) = \ell_R(\overline{\mathscr{M}}\,\overline{M}) = \ell_R(\mathscr{M}M/\mathscr{I}M).$$

On the other hand, by the short exact sequence

$$0 \longrightarrow \mathscr{M}M/\mathscr{I}M \longrightarrow M/\mathscr{I}M \longrightarrow M/\mathscr{M}M \longrightarrow 0,$$

we have $\ell_R(\mathscr{M}M/\mathscr{I}M) = \ell_R(M/\mathscr{I}M) - \ell_R(M/\mathscr{M}M)$. Therefore we obtain $\nu(\overline{\mathscr{M}}\,\overline{M}) = \ell_R(M/\mathscr{I}M) - \nu(M)$. In addition it is clear that $\nu(\overline{M}) = \nu(M)$. Now we can apply [Huneke et al. 2004, Proposition 4.4(1)], which ensures that the $R/\mathscr{I}$-module $M/\mathscr{I}M$ is either free or injective. Since $R/\mathscr{I}$ is Gorenstein, $M/\mathscr{I}M$ is necessarily free, as needed.

Assume that (ii) holds. Notice that $\bar{\mathscr{M}}^3 = 0$ by hypothesis. Now, since $R/\mathscr{I}$ is Gorenstein, the freeness of $M/\mathscr{I}M$ follows readily by [Huneke et al. 2004, Theorem 4.1(2)].

Now suppose (iii). Let $\ell\ell(\bar{R})$ denote the *Loewy length* of $\bar{R}$, which is the smallest integer n such that $\bar{\mathscr{M}}^n = 0$, i.e., $\mathscr{M}^n \subset \mathscr{I}$. Thus, by assumption, $\ell\ell(\bar{R}) \leq 3$. If $\ell\ell(\bar{R}) = 1$ ($\mathscr{I} = \mathscr{M}$), there is nothing to prove. If $\ell\ell(\bar{R}) = 2$, then $M/\mathscr{I}M$ is free by [Huneke et al. 2004, Remark 2.1]. So we can assume $\ell\ell(\bar{R}) = 3$. Using Remarks 2.8(i) and the hypothesis that $\mathscr{I}$ is not a parameter ideal (so that the inclusion $Q \subset \mathscr{I}$ is strict), we get $e^0_{\mathscr{I}}(R) = \ell_R(R/Q) \geq \ell_R(R/\mathscr{I}) + 1$. Therefore,

$$2\nu(\bar{\mathscr{M}}) = 2\ell_R(\mathscr{M}/(\mathscr{M}^2 + \mathscr{I})) \geq e^0_{\mathscr{I}}(R) \geq \ell_R(\bar{R}) + 1 = \ell_R(\bar{R}) - \ell\ell(\bar{R}) + 4.$$

We are now in a position to apply [Huneke et al. 2004, Theorem 3.1(2)] to conclude that $M/\mathscr{I}M$ is free.

Finally, suppose (iv). So $R/\mathscr{I}$ is Gorenstein and $\bar{\mathscr{M}}^4 = 0$, and in addition note that $(\mathscr{I} : x)/\mathscr{I}$ is the annihilator of $x\bar{R}$. Then $M/\mathscr{I}M$ is free by [Şega 2011, Theorem 3.3]. □

Remark 3.13. From the proof in the situation (iii) it is clear that, for general $\mathscr{I}$ (possibly a parameter ideal), the hypothesis on the multiplicity must be replaced with $e^0_{\mathscr{I}}(R) \leq 2\ell_R(\mathscr{M}/(\mathscr{M}^2 + \mathscr{I})) - 1$.

4. Horizontal linkage and the Ulrich property

We begin this section by pointing out the warming-up fact that, if the local ring R is Gorenstein, then it follows from [Martsinkovsky and Strooker 2004, Theorem 1] that every stable Ulrich R-module with respect to $\mathscr{I}$, where $\mathscr{I}$ is as in Convention 2.2, is horizontally linked (note that maximal Cohen–Macaulay modules are precisely the totally reflexive modules, since R is Gorenstein). See Section 2B for terminology.

In essence, our goal herein is to develop a further study of linkage of Ulrich modules with respect to $\mathscr{I}$, also assumed to be Ulrich but not a parameter ideal, the main result being the theorem below, which in particular shows that the operation of horizontal linkage over a Gorenstein local ring preserves the Ulrich property with respect to $\mathscr{I}$ for horizontally linked modules.

Theorem 4.1. *Let $(R, \mathscr{M})$ be a Cohen–Macaulay local ring of dimension d, and suppose the ideal $\mathscr{I}$ is Ulrich but not a parameter ideal. Consider the following assertions*:

(i) *R is Gorenstein.*

(ii) *M is Ulrich with respect to $\mathscr{I}$ if and only if λM is Ulrich with respect to $\mathscr{I}$, whenever M is a horizontally linked R-module.*

(iii) *λM is maximal Cohen–Macaulay, whenever M is a horizontally linked R-module which is Ulrich with respect to $\mathscr{I}$.*

(iv) $\operatorname{Ext}_R^{d+2}(R/\mathscr{I}, R) = 0$.

Then the following statements hold:

(a) *(i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii)*.

(b) *If* $d \geq 2$, *then (iii)* $\Longrightarrow$ *(iv)*.

(c) *If* $d \geq 2$ *and* $\mathscr{I} = \mathscr{M}$, *then all the four conditions above are equivalent.*

Proof. (a) (i) $\Longrightarrow$ (ii). Let M be a horizontally linked R-module. By Lemma 2.10, M is a stable R-module. Assume that M is an Ulrich R-module with respect to $\mathscr{I}$. By [Goto et al. 2014, Corollary 5.3], the Auslander transpose $\operatorname{Tr} M$ is Ulrich with respect to $\mathscr{I}$. Moreover, since M is stable, we obtain by [Anderson and Fuller 1992, Theorem 32.13] that $\operatorname{Tr} M$ is stable as well. Applying [Goto et al. 2014, Corollary 5.3] we conclude that the syzygy module $\Omega \operatorname{Tr} M = \lambda M$ is Ulrich with respect to $\mathscr{I}$. Now, to see the converse, it suffices to apply Lemma 2.11 to the module λM and to use that $M \cong \lambda^2 M$. Notice that (ii) $\Longrightarrow$ (iii) is obvious. This concludes the proof of (a).

(b) (iii) $\Longrightarrow$ (iv). Let $\bar{R} = R/\mathscr{I}$, and assume on the contrary that $\operatorname{Ext}_R^{d+2}(\bar{R}, R) \neq 0$. First notice that $\Omega^{d+1}\bar{R}$ is stable, otherwise R would be a direct summand of $\Omega^{d+1}\bar{R}$ and then, by [Avramov 1998, Corollary 1.2.5],

$$d + 1 \leq \max\{0, \operatorname{depth} R - \operatorname{depth}_R \bar{R}\} = d - \operatorname{depth}_R \bar{R},$$

which is absurd. Now, by Lemma 2.10, $\Omega^{d+1}\bar{R}$ is a horizontally linked R-module. By [Goto et al. 2014, Theorem 3.2], $\Omega^{d+1}\bar{R}$ is an Ulrich R-module with respect to $\mathscr{I}$. It follows from the assumption of (iii) that $\lambda\Omega^{d+1}\bar{R}$ is a maximal Cohen–Macaulay R-module, which in turn fits into a short exact sequence

$$0 \longrightarrow \lambda\Omega^{d+1}\bar{R} \longrightarrow F \longrightarrow \operatorname{Tr} \Omega^{d+1}\bar{R} \longrightarrow 0$$

for some free R-module F. By [Bruns and Herzog 1993, Proposition 1.2.9],

$$\operatorname{depth}_R \operatorname{Tr} \Omega^{d+1}\bar{R} \geq \min\{\operatorname{depth}_R F, \operatorname{depth}_R \lambda\Omega^{d+1}\bar{R} - 1\} = d - 1 > 0. \tag{10}$$

Using (1), there is an exact sequence

$$0 \to \operatorname{Ext}_R^1(\operatorname{Tr}\operatorname{Tr}\Omega^{d+1}\bar{R}, R) \to \operatorname{Tr}\Omega^{d+1}\bar{R} \to (\operatorname{Tr}\Omega^{d+1}\bar{R})^{**} \to \operatorname{Ext}_R^2(\operatorname{Tr}\operatorname{Tr}\Omega^{d+1}\bar{R}, R) \to 0,$$

and since $\Omega^{d+1}\bar{R}$ is stable, we have $\operatorname{Tr}\operatorname{Tr}\Omega^{d+1}\bar{R} \cong \Omega^{d+1}\bar{R}$ by [Anderson and Fuller 1992, Corollary 32.14(4)]. Thus, we obtain the exact sequence

$$0 \to \operatorname{Ext}_R^{d+2}(\bar{R}, R) \to \operatorname{Tr}\Omega^{d+1}\bar{R} \to (\operatorname{Tr}\Omega^{d+1}\bar{R})^{**} \to \operatorname{Ext}_R^{d+3}(\bar{R}, R) \to 0. \tag{11}$$

As $\mathscr{I}$ is $\mathscr{M}$-primary, the nonzero module $\operatorname{Ext}_R^{d+2}(\bar{R}, R)$ must have finite length, which in particular implies $\operatorname{depth}_R \operatorname{Ext}_R^{d+2}(\bar{R}, R) = 0$. On the other hand, by virtue of (10) and (11), we get $\operatorname{depth}_R \operatorname{Ext}_R^{d+2}(\bar{R}, R) > 0$, a contradiction.

(c) (iv) $\Longrightarrow$ (i). If $\operatorname{Ext}_R^{d+2}(R/\mathscr{M}, R) = 0$ then, by [Matsumura 1986, Theorem 18.1], the local ring R is Gorenstein. $\square$

In order to provide the first application of our theorem, we invoke the following classical concept:

Definition 4.2. A d-dimensional Cohen–Macaulay local ring R is said to have *minimal multiplicity* if its multiplicity and embedding dimension are related by $e(R) = \operatorname{edim} R - d + 1$. As is well known, there is in general an inequality $e(R) \geq \operatorname{edim} R - d + 1$, which originates the terminology.

Now recall that a local ring R is a *hypersurface* ring if $R \cong S/(f)$, where $(S, \mathscr{N})$ is a regular local ring and $f \in \mathscr{N}$. Such a ring is said to be a *quadratic* hypersurface ring if $f \in \mathscr{N}^2 \setminus \mathscr{N}^3$. Clearly, a hypersurface ring $R \cong S/(f)$ with $f \in \mathscr{N}^2$ is quadratic if and only if R has minimal multiplicity (equal to 2).

Our Theorem 4.1 yields a characterization of quadratic hypersurface rings in terms of linkage of Ulrich modules in the classical sense, in the case $\mathscr{I} = \mathscr{M}$. It is worth recalling an interesting connection, which we shall use in the proof of Corollary 4.5, between quadratic hypersurface rings and the Ulrich property. To wit, every nonfree maximal Cohen–Macaulay module over such a ring is a direct sum of an Ulrich module and a free module (see [Herzog and Kühl 1987, Corollary 1.4]); in particular, any such ring admits an Ulrich module.

Corollary 4.3. *Let R be a nonregular Cohen–Macaulay local ring of minimal multiplicity with dimension $d \geq 2$ and infinite residue field k. The following assertions are equivalent*:

(i) *R is a (quadratic) hypersurface ring.*

(ii) *M is Ulrich if and only if λM is Ulrich, whenever M is a horizontally linked R-module.*

(iii) *λM is maximal Cohen–Macaulay, whenever M is a horizontally linked Ulrich R-module.*

(iv) $\operatorname{Ext}_R^{d+2}(k, R) = 0$.

Proof. As before let $\mathscr{M}$ be the maximal ideal of R. Since $R/\mathscr{M}$ is infinite, it is well known that R has minimal multiplicity if and only if

$$\mathscr{M}^2 = (\boldsymbol{x})\mathscr{M}$$

with $\boldsymbol{x}$ an R-sequence (see [Bruns and Herzog 1993, Exercise 4.6.14]), which in turn means that $\mathscr{M}$ is an Ulrich ideal in the sense of Definition 2.4. Since R is nonregular,

$\mathscr{M}$ is not a parameter ideal. Therefore, as every hypersurface ring is Gorenstein, the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) follow readily by Theorem 4.1 with $\mathscr{I} = \mathscr{M}$. Now, as recalled in the proof of the theorem, condition (iv) forces R to be Gorenstein. But it is well known that a Gorenstein local ring having minimal multiplicity is just a quadratic hypersurface ring, as needed. □

Connections between a more general notion of minimal multiplicity and the Ulrich property with respect to $\mathscr{I}$ will be given in Section 5.

Before establishing another consequence of Theorem 4.1 over Gorenstein local rings, we invoke an auxiliary invariant which will be used in the proof, namely, the *Gorenstein dimension* of a finite R-module M, which is denoted by $\text{G-dim}_R M$ (for the definition, see, e.g., [Christensen 2000, Definition 1.2.3]). Recall that if R is Gorenstein then $\text{G-dim}_R M < \infty$ for every finite R-module M. If R is local and M is a finite R-module with $\text{G-dim}_R M < \infty$ then the so-called Auslander–Bridger formula states that $\text{G-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$. In particular, if R is Gorenstein, then $\text{G-dim}_R M = 0$ if and only if M is maximal Cohen–Macaulay. For details, see [Auslander and Bridger 1969; Christensen 2000].

Corollary 4.4. *Let R be a Gorenstein local ring, and suppose the ideal $\mathscr{I}$ is Ulrich but not a parameter ideal. Let M be a stable maximal Cohen–Macaulay R-module. Then, M is an Ulrich R-module with respect to $\mathscr{I}$ if and only if λM is an Ulrich R-module with respect to $\mathscr{I}$.*

Proof. Since R is Gorenstein and M is maximal Cohen–Macaulay, then as observed above we have $\text{G-dim}_R M = 0$. By [Martsinkovsky and Strooker 2004, Theorem 1], M is horizontally linked. Now the result follows from Theorem 4.1(a). □

Corollary 4.5. *Let R be a quadratic hypersurface local ring with infinite residue field, and let M be a stable maximal Cohen–Macaulay R-module. Then, λM is an Ulrich R-module.*

Proof. Over such a ring, any maximal Cohen–Macaulay module M is either free or satisfies

$$M \cong U \oplus F,$$

for some Ulrich module U and free module F, according to [Herzog and Kühl 1987, Corollary 1.4]. Thus, if in addition M is stable (in particular, nonfree), then it must be Ulrich. Also note the maximal ideal $\mathscr{M}$ of R is Ulrich but not a parameter ideal. Now we can apply Corollary 4.4 with $\mathscr{I} = \mathscr{M}$ to get the result. □

Before giving more consequences of Corollary 4.4, we recall a useful lemma.

Lemma 4.6 [Herzog and Kühl 1987, Lemma 1.2]. *Let R be a Gorenstein local ring. If M is a maximal Cohen–Macaulay R-module, then ΩM is a stable R-module.*

Corollary 4.7. *Let R be a Gorenstein local ring of dimension d, and suppose the ideal $\mathscr{I}$ is Ulrich but not a parameter ideal. Then, $\lambda(\Omega^k \mathscr{I})$ is an Ulrich R-module with respect to $\mathscr{I}$ for all $k \geq d$.*

Proof. First, as recalled in Remarks 2.8(iii), the R-module $\Omega^k(R/\mathscr{I})$ is Ulrich with respect to $\mathscr{I}$ (in particular, maximal Cohen–Macaulay) for all $k \geq d$. It follows by Lemma 4.6 that the R-module $\Omega^{k+1}(R/\mathscr{I}) = \Omega^k \mathscr{I}$ is stable for all $k \geq d$, and thus Corollary 4.4 concludes the proof. □

Corollary 4.8. *Let R be a 1-dimensional Gorenstein local ring. If $\mathscr{I}$ is an Ulrich ideal of R which is not a parameter ideal, then $\lambda\mathscr{I}$ is an Ulrich R-module with respect to $\mathscr{I}$.*

Proof. By Remarks 2.8(ii), $\mathscr{I}$ is an Ulrich R-module with respect to $\mathscr{I}$. Note $\mathscr{I}$ is stable as it is a nonprincipal ideal, hence a nonfree R-module. Now, apply Corollary 4.4. □

5. Minimal multiplicity and Ulrich properties

We start the section presenting a few preparatory definitions (Rees and associated graded modules, and relative reduction numbers) as well as some auxiliary facts.

Let I be a proper ideal of a ring R. Recall that the Rees algebra of I is the graded ring $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n$ (as usual, we put $I^0 = R$), which can be realized as the standard graded subalgebra $R[Iu] \subset R[u]$, where u is an indeterminate over R. The associated graded ring of I is given by $\mathcal{G}(I) = \bigoplus_{n\geq 0} I^n/I^{n+1} = \mathcal{R}(I) \otimes_R R/I$, which is standard graded over R/I.

Definition 5.1. If M is a finite R-module, the *Rees module* and the *associated graded module* of I relative to M are, respectively, given by

$$\mathcal{R}(I, M) = \bigoplus_{n\geq 0} I^n M, \quad \mathcal{G}(I, M) = \bigoplus_{n\geq 0} \frac{I^n M}{I^{n+1} M} = \mathcal{R}(I, M) \otimes_R R/I,$$

which are finite graded modules over $\mathcal{R}(I)$ and $\mathcal{G}(I)$, respectively.

Now consider a local ring $(R, \mathscr{M})$ with residue field k. For a proper ideal I of R, recall that the fiber cone of I is the special fiber ring of $\mathcal{R}(I)$, i.e., the standard graded k-algebra $\mathcal{F}(I) = \bigoplus_{n\geq 0} I^n/\mathscr{M} I^n = \mathcal{R}(I) \otimes_R k$. We can also consider the finite graded $\mathcal{F}(I)$-module $\mathcal{F}(I, M) = \bigoplus_{n\geq 0} I^n M/\mathscr{M} I^n M = \mathcal{R}(I, M) \otimes_R k$, whose Krull dimension (called *analytic spread* of I relative to M) is denoted by

$$s_M(I) = \dim \mathcal{F}(I, M).$$

Definition 5.2. Let I be a proper ideal of a ring R and let M be a nonzero finite R-module. An ideal $J \subset I$ is called an *M-reduction of I* if $J I^n M = I^{n+1} M$ for some integer $n \geq 0$. Such an M-reduction J is said to be *minimal* if it is minimal with respect to inclusion. If J is an M-reduction of I, we define the *reduction number of I with respect to J relative to M* as

$$r_J(I, M) = \min\{m \in \mathbb{N} \mid JI^m M = I^{m+1}M\}.$$

The lemma below detects a useful connection between minimal M-reductions and the so-called (*maximal*) *M-superficial sequences* of a given $\mathscr{M}$-primary ideal in a local ring $(R, \mathscr{M})$. For the definition and details about the latter concept, we refer to [Rossi and Valla 2010, Sections 1.2 and 1.3]; also see [Conti 2006].

Lemma 5.3 [Conti 2006, corollario 3.14]. *Let $(R, \mathscr{M})$ be a local ring with infinite residue field and let I be an $\mathscr{M}$-primary ideal. Let M be a finite R-module of positive dimension. Then, every minimal M-reduction of I can be generated by a maximal M-superficial sequence of I. Conversely, an ideal generated by a maximal M-superficial sequence of I is necessarily a minimal M-reduction of I.*

Next we invoke a central notion in this section, and a helpful lemma. As in Section 2A, if I is an ideal of definition of a finite R-module M then $e_I^0(M)$ denotes the multiplicity of M with respect to I. Moreover, we let $e_I^1(M)$ stand for the first Hilbert coefficient — the so-called *Chern number* — of M with respect to I.

Definition 5.4 [Puthenpurakal 2003, Definition 15]. Let $(R, \mathscr{M})$ be a local ring, M a Cohen–Macaulay R-module of dimension t and I a proper ideal of R such that $\mathscr{M}^n M \subset IM$ for some $n > 0$. Then M has *minimal multiplicity with respect to I* if

$$e_I^0(M) = (1-t)\ell_R(M/IM) + \ell_R(IM/I^2M).$$

Notice that by taking $M = R$ and $I = \mathscr{M}$ we recover Definition 4.2.

Lemma 5.5 [Puthenpurakal 2003, Theorem 16]. *Let $(R, \mathscr{M})$ be a local ring, M a Cohen–Macaulay R-module of dimension t and I a proper ideal of R such that $\mathscr{M}^n M \subset IM$ for some $n > 0$. The following conditions are equivalent*:

(i) *M has minimal multiplicity with respect to I.*

(ii) $(z_1, \dots, z_t)IM = I^2M$, *for every maximal M-superficial sequence $z_1, \dots, z_t$.*

(iii) $(z_1, \dots, z_t)IM = I^2M$, *for some maximal M-superficial sequence $z_1, \dots, z_t$.*

(iv) $e_I^1(M) = e_I^0(M) - \ell_R(M/IM)$.

Here we observe that item (iii) above is not present in [Puthenpurakal 2003], but a simple inspection of the proof easily shows that this assertion is also equivalent to the ones given in Theorem 16 of that paper.

Our first result in this part is the following. As in the previous sections, we let $Q = (x_1, \dots, x_d) \subset \mathscr{I}$ be as in Convention 2.2.

Proposition 5.6. *Suppose R is a Cohen–Macaulay local ring with infinite residue field. Then, every Ulrich R-module with respect to $\mathscr{I}$ has minimal multiplicity with respect to $\mathscr{I}$.*

Proof. Let M be an Ulrich module with respect to $\mathscr{I}$. In particular, M is maximal Cohen–Macaulay. Let $\operatorname{grade}(\mathscr{I}, M)$ denote the maximal length of an M-sequence contained in $\mathscr{I}$. By [Kadu 2011, Lemma 1.3 and Lemma 1.6], we have

$$\operatorname{grade}(\mathscr{I}, M) \leq s_M(\mathscr{I}) \leq \dim M.$$

As $\mathscr{I}$ is $\mathscr{M}$-primary, $\operatorname{grade}(\mathscr{I}, M) = \operatorname{depth} M = d$, where as before $d = \dim R$. Hence $s_M(\mathscr{I}) = d = \nu(Q)$, where $\nu(-)$ stands for minimal number of generators. As is well known (see, e.g., [Conti 2006, corollario 3.22]), this implies that Q is a minimal M-reduction of $\mathscr{I}$, and therefore Lemma 5.3 gives that $x_1, \dots, x_d$ is in fact a maximal M-superficial sequence of $\mathscr{I}$. On the other hand, because M is Ulrich, we have $QM = \mathscr{I}M$ and so

$$Q\mathscr{I}M = \mathscr{I}^2 M.$$

We conclude, by Lemma 5.5, that M has minimal multiplicity with respect to $\mathscr{I}$. □

Remark 5.7. The converse of Proposition 5.6 fails even in the classical case $\mathscr{I} = \mathscr{M}$; see [Puthenpurakal 2005, Example 4.12].

Combining Proposition 5.6 and [Puthenpurakal 2003, Theorem 16], we immediately obtain the following property.

Corollary 5.8. *Suppose R is a Cohen–Macaulay local ring with infinite residue field. If M is an Ulrich R-module with respect to $\mathscr{I}$, then the associated graded $\mathcal{G}(\mathscr{I})$-module $\mathcal{G}(\mathscr{I}, M)$ is Cohen–Macaulay.*

The next consequence deals with the Chern number and gives a generalization of [Ooishi 1991, Corollary 1.3(1)].

Corollary 5.9. *Let $(R, \mathscr{M})$ be a Cohen–Macaulay local ring with infinite residue field and positive dimension, and let M be a maximal Cohen–Macaulay R-module. Then $e^1_{\mathscr{I}}(M) \geq 0$, and the following assertions are equivalent:*

(i) *M is an Ulrich R-module with respect to $\mathscr{I}$.*

(ii) *$M/\mathscr{I}M$ is a free $R/\mathscr{I}$-module and $e^1_{\mathscr{I}}(M) = 0$.*

Proof. Applying [Puthenpurakal 2003, Proposition 12] and Remarks 2.8(i), we get

$$e^1_{\mathscr{I}}(M) \geq e^0_{\mathscr{I}}(M) - \ell_R(M/\mathscr{I}M) \geq 0.$$

If M is Ulrich with respect to $\mathscr{I}$ then, by definition, the $R/\mathscr{I}$-module $M/\mathscr{I}M$ is free and in addition $e^0_{\mathscr{I}}(M) = \ell_R(M/\mathscr{I}M)$ (use again Remarks 2.8(i)). On the other hand, Proposition 5.6 ensures that M has minimal multiplicity with respect to $\mathscr{I}$, and therefore Lemma 5.5 gives $e^1_{\mathscr{I}}(M) = e^0_{\mathscr{I}}(M) - \ell_R(M/\mathscr{I}M) = 0$.

Conversely, suppose (ii). Since M is already assumed to be maximal Cohen–Macaulay, it remains to show that $\mathscr{I}M = QM$, which as we know is equivalent to the equality $e^0_{\mathscr{I}}(M) = \ell_R(M/\mathscr{I}M)$. But this follows from $0 \leq e^0_{\mathscr{I}}(M) - \ell_R(M/\mathscr{I}M) \leq e^1_{\mathscr{I}}(M) = 0$. This concludes the proof. □

Corollary 5.10. *Let $(R, \mathscr{M})$ be a Cohen–Macaulay local ring with infinite residue field and dimension $d \geq 1$. If $\mathscr{I}$ is an Ulrich ideal of R which is not a parameter ideal, then $e^1_{\mathscr{I}}(\Omega^k \mathscr{I}) = 0$ for all $k \geq d-1$. If in addition R is Gorenstein, then*

$$e^1_{\mathscr{I}}(\lambda(\Omega^k \mathscr{I})) = 0 \quad \textit{for all } k \geq d.$$

Proof. Recall that the R-module $\Omega^{k+1}(R/\mathscr{I}) = \Omega^k \mathscr{I}$ is Ulrich with respect to $\mathscr{I}$ (in particular, maximal Cohen–Macaulay) for all $k \geq d-1$; see Remarks 2.8(iii). Then the vanishing of $e^1_{\mathscr{I}}(\Omega^k \mathscr{I})$ follows by Corollary 5.9. Now if R is Gorenstein then, by Corollary 4.7, the module $\lambda(\Omega^k \mathscr{I})$ is Ulrich with respect to $\mathscr{I}$ for all $k \geq d$, and we again apply Corollary 5.9. □

Remarks 5.11. (i) Let M be a d-dimensional Cohen–Macaulay R-module (assume the setting of Convention 2.2, with $d > 0$ and $R/\mathscr{M}$ infinite). Recall that, for $k \gg 0$, the Hilbert–Samuel function $H^M_{\mathscr{I}}(k) = \ell_R(M/\mathscr{I}^k M)$ coincides with a degree d polynomial $P^M_{\mathscr{I}}(k)$, the Hilbert–Samuel polynomial of M with respect to $\mathscr{I}$, which can be expressed as

$$P^M_{\mathscr{I}}(k) = \sum_{i=0}^{d} (-1)^i e^i_{\mathscr{I}}(M) \binom{k+d-i-1}{d-i}.$$

Now if M is Ulrich with respect to $\mathscr{I}$, then in particular $M/\mathscr{I}M \cong (R/\mathscr{I})^{\nu(M)}$ and therefore, by Corollary 5.9, we get $e^0_{\mathscr{I}}(M) = \ell_R(M/\mathscr{I}M) = \nu(M)\ell_R(R/\mathscr{I})$ and $e^1_{\mathscr{I}}(M) = 0$. Thus, if for instance $d = 1$ then $P^M_{\mathscr{I}}(k) = \nu(M)\ell_R(R/\mathscr{I})k$. If $d = 2$, we have

$$P^M_{\mathscr{I}}(k) = \nu(M)\ell_R(R/\mathscr{I})\binom{k+1}{2} + e^2_{\mathscr{I}}(M),$$

which raises the problem of finding $e^2_{\mathscr{I}}(M)$. Of course, in case we know an integer k_0 satisfying $P^M_{\mathscr{I}}(k) = H^M_{\mathscr{I}}(k)$ for all $k \geq k_0$, then $e^2_{\mathscr{I}}(M)$ can be computed from the expression above by evaluating $k = k_0$.

(ii) If $d \geq 1$ and $\mathscr{I}$ is an Ulrich ideal of R then, as we know, the j-th syzygy module of $\mathscr{I}$ is Ulrich with respect to $\mathscr{I}$ for all $j \geq d-1$. Now assume $d = 1$. Applying the preceding part to the module $\Omega^j \mathscr{I}$ for any $j \geq 0$, and noticing that $\nu(\Omega^j \mathscr{I})$ is precisely the j-th Betti number $\beta_j(\mathscr{I})$ of $\mathscr{I}$, we obtain the simple formula

$$P^{\Omega^j \mathscr{I}}_{\mathscr{I}}(k) = \beta_j(\mathscr{I})\ell_R(R/\mathscr{I})k.$$

In addition, considering linkage and assuming that R is Gorenstein, our Corollary 4.7 yields that $\lambda(\Omega^j \mathscr{I})$ is also Ulrich with respect to $\mathscr{I}$ for any $j \geq 1$, and observe that $\nu(\lambda(\Omega^j \mathscr{I})) = \beta_j(\mathscr{I})$ as well. It follows that $P_{\mathscr{I}}^{\Omega^j \mathscr{I}}(k) = P_{\mathscr{I}}^{\lambda(\Omega^j \mathscr{I})}(k)$.

Our next result, Theorem 5.14 below, provides a characterization of modules of minimal multiplicity in terms of reduction number and Castelnuovo–Mumford regularity (of blowup modules). For completeness, we recall the definition of the latter, which is of great importance in commutative algebra and algebraic geometry, for instance in the study of degrees of syzygies over polynomial rings; we refer to [Brodmann and Sharp 1998, Chapter 15].

Let $S = \bigoplus_{n \geq 0} S_n$ be a finitely generated standard graded algebra over a ring S_0. As usual, we write $S_+ = \bigoplus_{n \geq 1} S_n$. For a graded S-module $A = \bigoplus_{n \in \mathbb{Z}} A_n$ satisfying $A_n = 0$ for all $n \gg 0$, we set

$$\operatorname{end} A = \begin{cases} \max\{n \mid A_n \neq 0\} & \text{if } A \neq 0, \\ -\infty & \text{if } A = 0. \end{cases}$$

Now fix a finite graded S-module $N \neq 0$. Given $j \geq 0$, let

$$H^j_{S_+}(N) = \varinjlim_k \operatorname{Ext}^j_S(S/S^k_+, N)$$

be the j-th local cohomology module of N. Recall $H^j_{S_+}(N)$ is a graded module such that $H^j_{S_+}(N)_n = 0$ for all $n \gg 0$; see [Brodmann and Sharp 1998, Proposition 15.1.5(ii)]. Thus, $\operatorname{end} H^j_{S_+}(N) < \infty$.

Definition 5.12. The *Castelnuovo–Mumford regularity* of the graded S-module N is given by

$$\operatorname{reg} N = \max\{\operatorname{end} H^j_{S_+}(N) + j \mid j \geq 0\}.$$

The following lemma will be very useful to the proof of Theorem 5.14, since it interprets the regularity of Rees modules as a relative reduction number in a suitable setting. It was originally stated in more generality (involving d-sequences) but here the special case of regular sequences suffices for our purposes.

Lemma 5.13 [Giral and Planas-Vilanova 2008, Theorem 5.3]. *Let R be a ring, I an ideal of R and M a finite R-module. Let $z_1, \dots, z_s$ be an M-sequence such that the ideal $J = (z_1, \dots, z_s)$ is an M-reduction of I. Let $r_J(I, M) = r$. Suppose either $s = 1$, or else $s \geq 2$ and*

$$(z_1, \dots, z_i)M \cap I^{r+1}M = (z_1, \dots, z_i)I^r M \quad \textit{for all } i = 1, \dots, s-1.$$

Then, $\operatorname{reg} \mathcal{R}(I, M) = r_J(I, M)$.

We are now ready for the main technical result of this section, which in particular will lead us to a byproduct on Ulrich modules. Note this theorem also gives a generalization of [Ooishi 1991, Proposition 1.2], where the situation $\mathscr{I} = \mathscr{M}$ was treated; more precisely, the condition $g_{\Delta}(M) = 0$ in that paper is equivalent to Puthenpurakal's notion of minimal multiplicity when $\mathscr{I} = \mathscr{M}$.

Theorem 5.14. *Let $(R, \mathscr{M})$ be a local ring with infinite residue field, M a Cohen–Macaulay R-module of dimension $t > 0$ and I an $\mathscr{M}$-primary ideal of R. Let $J = (z_1, \dots, z_t)$ be a minimal M-reduction of I. The following assertions are equivalent*:

(i) *M has minimal multiplicity with respect to I.*

(ii) $\operatorname{reg} \mathcal{R}(I, M) = \operatorname{reg} \mathcal{G}(I, M) = r_J(I, M) \leq 1$.

(iii) $r_J(I, M) \leq 1$.

Proof. First, notice that $z_1, \dots, z_t$ is a (maximal) M-superficial sequence of I by Lemma 5.3. As a consequence, since M is Cohen–Macaulay and I is $\mathscr{M}$-primary, $z_1, \dots, z_t$ must be in fact an M-sequence according to [Rossi and Valla 2010, Lemma 1.2]. Now, the core of the proof is the implication (i) $\Longrightarrow$ (ii), so assume first that (i) holds. In general, we have $\operatorname{reg} \mathcal{R}(I, M) = \operatorname{reg} \mathcal{G}(I, M)$, see [Zamani 2014, Corollary 3], and so it remains to prove that $\operatorname{reg} \mathcal{R}(I, M) = r_J(I, M)$, which we shall accomplish by means of Lemma 5.13.

Moreover, since $z_1, \dots, z_t$ is maximal M-superficial, Lemma 5.5 yields $JIM = I^2M$, i.e., $r_J(I, M) \leq 1$. Now, to simplify notation, set $\mathbf{z}_i = z_1, \dots, z_i$ for $i = 1, \dots, t-1$ (note we can assume $t > 1$ by Lemma 5.13). Since clearly $(\mathbf{z}_i)M \cap IM = (\mathbf{z}_i)M$ for all $i = 1, \dots, t-1$, the case $r_J(I, M) = 0$ is trivial by virtue of Lemma 5.13. Now suppose $r_J(I, M) = 1$. Again in view of Lemma 5.13, all we need to prove is that

$$(\mathbf{z}_i)M \cap I^2M = (\mathbf{z}_i)IM \quad \text{for all } i = 1, \dots, t-1.$$

First, it is clear that $(\mathbf{z}_i)IM \subset (\mathbf{z}_i)M \cap I^2M$. To show the other inclusion, take an arbitrary $f \in (\mathbf{z}_i)M \cap I^2M$. Because $JIM = I^2M$, we have

$$f = z_1m_1 + \cdots + z_im_i = z_1a_1m'_1 + \cdots + z_ta_tm'_t$$

with $m_j, m'_k \in M$ and $a_k \in I$. Hence

$$\overline{z_ta_tm'_t} = \bar{0} \in M/(\mathbf{z}_{t-1})M,$$

and since the sequence is regular on M, we have $\overline{a_tm'_t} = \bar{0} \in M/(\mathbf{z}_{t-1})M$, that is, $a_tm'_t = z_1w_{t,1} + \cdots + z_{t-1}w_{t,t-1}$ with $w_{t,j} \in M$. Therefore, f can be expressed as

$$z_1m_1 + \cdots + z_im_i = z_1(a_1m'_1 + z_tw_{t,1}) + \cdots + z_{t-1}(a_{t-1}m'_{t-1} + z_tw_{t,t-1}), \tag{12}$$

whose right-hand side shows $f \in (\mathbf{z}_{t-1})IM$, thus settling the case $i = t-1$. Next, for $i < t-1$, we reduce (12) modulo $(\mathbf{z}_{t-2})M$ and apply an analogous argument to the term $z_{t-1}(a_{t-1}m'_{t-1}+z_t w_{t,t-1})$ in order to obtain

$$a_{t-1}m'_{t-1}+z_t w_{t,t-1}=z_1 w_{t-1,1}+\cdots+z_{t-2}w_{t-1,t-2} \tag{13}$$

with $w_{t-1,j}\in M$. Thus, by (12) and (13),

$$f=z_1(a_1m'_1+z_t w_{t,1}+z_{t-1}w_{t-1,1})+\cdots+z_{t-2}(a_{t-2}m'_{t-2}+z_t w_{t,t-2}+z_{t-1}w_{t-1,t-2}).$$

Continuing with the argument, we get an equality

$$f=z_1(a_1m'_1+z_t w_{t,1}+\cdots+z_{i+1}w_{i+1,1})+\cdots+z_i(a_im'_i+z_t w_{t,i}+\cdots+z_{i+1}w_{i+1,i}).$$

Since $a_1,\ldots,a_i,z_{i+1},\ldots,z_t\in I$, it follows that $f\in(z_i)IM$, as needed.

The implication (ii) $\Longrightarrow$ (iii) is obvious. Finally, suppose (iii) holds. Then $JIM=I^2M$, and we have seen that $z_1,\ldots,z_t$ is a maximal M-superficial sequence. By Lemma 5.5, we conclude that M has minimal multiplicity with respect to I. □

As a consequence of Theorem 5.14, we determine the regularity of blowup modules of $\mathscr{I}$ relative to an Ulrich module. Also, taking $\mathscr{I}=\mathscr{M}$ the result retrieves part of [Ooishi 1991, Proposition 1.1].

Corollary 5.15. *Let $(R,\mathscr{M})$ be a Cohen–Macaulay local ring with infinite residue field and positive dimension, and let Q be as in Convention 2.2. If M is an Ulrich R-module with respect to $\mathscr{I}$, then*

$$\operatorname{reg}\mathcal{R}(\mathscr{I},M)=\operatorname{reg}\mathcal{G}(\mathscr{I},M)=r_Q(\mathscr{I},M)=0.$$

The converse holds in case M is maximal Cohen–Macaulay and $M/\mathscr{I}M$ is $R/\mathscr{I}$-free.

Proof. First, notice that Q is an M-reduction of $\mathscr{I}$, so the number $r_Q(\mathscr{I},M)$ makes sense. Now, because M is Ulrich with respect to $\mathscr{I}$, we have $QM=\mathscr{I}M$, which means $r_Q(\mathscr{I},M)=0$. On the other hand, Proposition 5.6 and its proof ensure that M has minimal multiplicity with respect to $\mathscr{I}$ and that Q is in fact a minimal M-reduction of $\mathscr{I}$, and so we can apply Theorem 5.14 to obtain $\operatorname{reg}\mathcal{R}(\mathscr{I},M)=\operatorname{reg}\mathcal{G}(\mathscr{I},M)=r_Q(\mathscr{I},M)$. The converse is clear. □

Corollary 5.16. *Let $(R,\mathscr{M})$ be a Cohen–Macaulay local ring with infinite residue field and positive dimension. Suppose $\mathscr{I}$ is an Ulrich ideal of R but not a parameter ideal. Then, $\operatorname{reg}\mathcal{R}(\mathscr{I},\Omega^k\mathscr{I})=0$ for all $k\geq d-1$. If in addition R is Gorenstein, then*

$$\operatorname{reg}\mathcal{R}(\mathscr{I},\lambda(\Omega^k\mathscr{I}))=0 \quad \textit{for all } k\geq d.$$

Proof. As we know, the R-module $\Omega^{k+1}(R/\mathscr{I})=\Omega^k\mathscr{I}$ is Ulrich with respect to $\mathscr{I}$ for all $k\geq d-1$. Thus the first part follows from Corollary 5.15. If R is Gorenstein then by Corollary 4.7 the R-module $\lambda(\Omega^k\mathscr{I})$ is Ulrich with respect to $\mathscr{I}$ for all $k\geq d$. Now we again apply Corollary 5.15. □

Corollary 5.17. *Let $(R, \mathscr{M})$ be a 1-dimensional Cohen–Macaulay local ring with infinite residue field. If $\mathscr{I}$ is an Ulrich ideal, then*

$$\operatorname{reg} \mathcal{R}(\mathscr{I})_+ = 0.$$

Proof. Using Remarks 2.8(ii) and Corollary 5.15, we obtain $\operatorname{reg} \mathcal{R}(\mathscr{I}, \mathscr{I}) = 0$. On the other hand, we clearly have $\mathcal{R}(\mathscr{I}, \mathscr{I}) = \bigoplus_{i\geq 0} \mathscr{I}^{i+1} = \mathcal{R}(\mathscr{I})_+$. □

Example 5.18. Consider the local ring $R = K[\![x, y]\!]/(x^2 + y^4)$, where K is an infinite field. The ideal $\mathscr{I} = (x, y^2)R$ is Ulrich (this is the case $d = 1$ and $s = 2$ of Examples 2.5(ii)). Then, Corollary 5.17 gives $\operatorname{reg} \mathcal{R}(\mathscr{I})_+ = 0$. To write this graded ideal explicitly, we can use (degree 1) variables T, U over R in order to determine a presentation of the Rees algebra

$$\mathcal{R}(\mathscr{I}) = R[T, U]/\mathscr{K}, \quad \mathscr{K} = (xT + y^2U, y^2T - xU, T^2 + U^2), \quad \mathcal{R}(\mathscr{I})_0 = R,$$

so that $\mathcal{R}(\mathscr{I})_+ = (T, U)R[T, U]/\mathscr{K}$.

Now let us use the same example to illustrate the determination of the Hilbert–Samuel polynomial $P^{\mathscr{I}}_{\mathscr{I}}(k)$. Notice that $\ell_R(R/\mathscr{I}) = \dim_K(K[\![y]\!]/(y^2)) = 2$ and $\nu(\mathscr{I}) = 2$. By Remarks 5.11(i), we have $P^{\mathscr{I}}_{\mathscr{I}}(k) = \nu(\mathscr{I})\ell_R(R/\mathscr{I})k = 4k$, i.e.,

$$\ell_R(\mathscr{I}/\mathscr{I}^{k+1}) = 4k \quad \text{for all } k \gg 0.$$

6. A detailed example

In this last section, we fix formal indeterminates x, y, z over an infinite field K as well as the 2-dimensional local hypersurface ring $R = K[\![x, y, z]\!]/(x^2 + y^2 + z^4)$. The ideal

$$\mathscr{I} = (x, y, z^2)R$$

is Ulrich — this is the case $d = s = 2$ of Examples 2.5(ii) — and not a parameter ideal. Our goal here is to find (explicit) Ulrich R-modules with respect to $\mathscr{I}$ and study their multiplicities, Chern numbers, and the regularity of the associated blowup modules.

First, $\mathscr{I}$ has an infinite (in fact, periodic) minimal R-free resolution

$$\cdots \longrightarrow R^4 \xrightarrow{\Phi} R^4 \xrightarrow{\Phi} R^4 \xrightarrow{\Phi} R^4 \xrightarrow{\Psi} R^3 \longrightarrow \mathscr{I} \longrightarrow 0, \tag{14}$$

where

$$\Phi = \begin{pmatrix} -z^2 & 0 & -y & x \\ 0 & -z^2 & x & y \\ -y & x & z^2 & 0 \\ x & y & 0 & z^2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} -z^2 & 0 & -y & x \\ 0 & -z^2 & x & y \\ x & y & 0 & z^2 \end{pmatrix}.$$

In what follows, as a matter of standard notation, whenever φ is a $p \times q$ matrix with entries in R, we let $\operatorname{Im}\varphi$ denote the R-submodule of R^p generated by the column vectors of φ. Below we observe a few facts.

- We claim that the R-submodules $\operatorname{Im}\Phi \subset R^4$ and $\operatorname{Im}\Psi \subset R^3$ are Ulrich with respect to $\mathscr{I}$. To see this, using Remarks 2.8(iii) we get that $\Omega^k\mathscr{I}$ is Ulrich with respect to $\mathscr{I}$ whenever $k \geq 1$. But in the present case, by (14), these modules are

$$\Omega\mathscr{I} = \operatorname{Im}\Psi, \quad \Omega^k\mathscr{I} = \operatorname{Im}\Phi, \quad \text{for all } k \geq 2,$$

  thus showing the claim. Also notice (by the symmetry of Φ) that $\lambda(\Omega^k\mathscr{I}) = \lambda(\operatorname{Im}\Phi) = \operatorname{Im}\Phi^* = \operatorname{Im}\Phi$ for all $k \geq 2$. In particular, $\operatorname{Im}\Phi$ is horizontally linked.

- Let us compute multiplicities and Chern numbers. First, since $\operatorname{Im}\Psi$ is Ulrich with respect to $\mathscr{I}$, we must have $\operatorname{Im}\Psi/\mathscr{I}\operatorname{Im}\Psi \cong (R/\mathscr{I})^{\nu(\operatorname{Im}\Psi)}$. Note $\ell_R(R/\mathscr{I}) = \dim_K(K[\![z]\!]/(z^2)) = 2$. Thus, by Remarks 2.8(i),

$$e^0_{\mathscr{I}}(\operatorname{Im}\Psi) = \ell_R(\operatorname{Im}\Psi/\mathscr{I}\operatorname{Im}\Psi) = \nu(\operatorname{Im}\Psi)\ell_R(R/\mathscr{I}) = 4\cdot 2 = 8.$$

  Since $\nu(\operatorname{Im}\Phi) = 4$ as well, we have $e^0_{\mathscr{I}}(\operatorname{Im}\Phi) = 8$. As to the Chern numbers, Corollary 5.10 gives $e^1_{\mathscr{I}}(\Omega^k\mathscr{I}) = 0$ for all $k \geq 1$. Hence,

$$e^1_{\mathscr{I}}(\operatorname{Im}\Psi) = e^1_{\mathscr{I}}(\operatorname{Im}\Phi) = 0.$$

- For the Castelnuovo–Mumford regularity of blowup modules, Corollary 5.16 yields $\operatorname{reg}\mathcal{R}(\mathscr{I}, \Omega^k\mathscr{I}) = 0$ for all $k \geq 1$, and therefore

$$\operatorname{reg}\mathcal{R}(\mathscr{I}, \operatorname{Im}\Psi) = \operatorname{reg}\mathcal{R}(\mathscr{I}, \operatorname{Im}\Phi) = 0.$$

  Finally, the associated graded $\mathcal{G}(\mathscr{I})$-modules $\mathcal{G}(\mathscr{I}, \operatorname{Im}\Psi)$ and $\mathcal{G}(\mathscr{I}, \operatorname{Im}\Phi)$ have regularity zero as well (see Corollary 5.15), and notice they are Cohen–Macaulay by Corollary 5.8.

Acknowledgments

Miranda-Neto was partially supported by the CNPq-Brazil grants 301029/2019-9 and 406377/2021-9. Queiroz was supported by a CAPES Doctoral Scholarship. The authors are indebted to the referee for careful inspection of the paper, including corrections, questions, and interesting suggestions which substantially improved it. Last but not least, the authors are also grateful to Shiro Goto (who sadly passed away on July 26, 2022, and to whom this article is dedicated) and to Naoki Endo for some correspondence following the first version of the manuscript.

References

[Anderson and Fuller 1992] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, 2nd ed., Graduate Texts in Mathematics **13**, Springer, New York, 1992. MR Zbl

[Araya and Iima 2018] T. Araya and K.-i. Iima, "Remarks on torsionfreeness and its applications", *Comm. Algebra* **46**:1 (2018), 191–200. MR Zbl

[Auslander 1966] M. Auslander, "Coherent functors", pp. 189–231 in *Proceedings of the Conference on Categorical Algebra* (La Jolla, CA, 1965), edited by S. Eilenberg et al., Springer, New York, 1966. MR Zbl

[Auslander and Bridger 1969] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society **94**, Providence, RI, 1969. MR Zbl

[Avramov 1998] L. L. Avramov, "Infinite free resolutions", pp. 1–118 in *Six lectures on commutative algebra* (Bellaterra, 1996), edited by J. Elías et al., Progr. Math. **166**, Birkhäuser, Basel, 1998. MR Zbl

[Brennan et al. 1987] J. P. Brennan, J. Herzog, and B. Ulrich, "Maximally generated Cohen–Macaulay modules", *Math. Scand.* **61**:2 (1987), 181–203. MR Zbl

[Brodmann and Sharp 1998] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics **60**, Cambridge University Press, 1998. MR Zbl

[Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1993. MR Zbl

[Christensen 2000] L. W. Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics **1747**, Springer, Berlin, 2000. MR Zbl

[Conti 2006] D. Conti, *Successioni superficiali e riduzioni per filtrazioni stabili di moduli*, Tesi di Laurea, Università di Pisa, 2006, available at https://etd.adm.unipi.it/t/etd-11072006-004214/.

[Eisenbud and Schreyer 2003] D. Eisenbud and F.-O. Schreyer, "Resultants and Chow forms via exterior syzygies", *J. Amer. Math. Soc.* **16**:3 (2003), 537–579. MR Zbl

[Giral and Planas-Vilanova 2008] J. M. Giral and F. Planas-Vilanova, "Integral degree of a ring, reduction numbers and uniform Artin–Rees numbers", *J. Algebra* **319**:8 (2008), 3398–3418. MR Zbl

[Goto et al. 2014] S. Goto, K. Ozeki, R. Takahashi, K.-I. Watanabe, and K.-I. Yoshida, "Ulrich ideals and modules", *Math. Proc. Cambridge Philos. Soc.* **156**:1 (2014), 137–166. MR Zbl

[Goto et al. 2015] S. Goto, R. Takahashi, and N. Taniguchi, "Almost Gorenstein rings — towards a theory of higher dimension", *J. Pure Appl. Algebra* **219**:7 (2015), 2666–2712. MR Zbl

[Goto et al. 2016a] S. Goto, K. Ozeki, R. Takahashi, K.-I. Watanabe, and K.-I. Yoshida, "Ulrich ideals and modules over two-dimensional rational singularities", *Nagoya Math. J.* **221**:1 (2016), 69–110. MR Zbl

[Goto et al. 2016b] S. Goto, R. Takahashi, and N. Taniguchi, "Ulrich ideals and almost Gorenstein rings", *Proc. Amer. Math. Soc.* **144**:7 (2016), 2811–2823. MR Zbl

[Goto et al. 2019] S. Goto, R. Isobe, and S. Kumashiro, "The structure of chains of Ulrich ideals in Cohen–Macaulay local rings of dimension one", *Acta Math. Vietnam.* **44**:1 (2019), 65–82. MR Zbl

[Hanes and Huneke 2005] D. Hanes and C. Huneke, "Some criteria for the Gorenstein property", *J. Pure Appl. Algebra* **201**:1-3 (2005), 4–16. MR Zbl

[Herzog and Kühl 1987] J. Herzog and M. Kühl, "Maximal Cohen–Macaulay modules over Gorenstein rings and Bourbaki-sequences", pp. 65–92 in *Commutative algebra and combinatorics* (Kyoto, 1985), edited by M. Nagata and H. Matsumura, Adv. Stud. Pure Math. **11**, North-Holland, Amsterdam, 1987. MR Zbl

[Huneke et al. 2004] C. Huneke, L. M. Şega, and A. N. Vraciu, "Vanishing of Ext and Tor over some Cohen–Macaulay local rings", *Illinois J. Math.* **48**:1 (2004), 295–317. MR Zbl

[Kadu 2011] G. S. Kadu, "Ideals of analytic deviation one with respect to a Cohen–Macaulay module", *Indian J. Pure Appl. Math.* **42**:2 (2011), 73–97. MR Zbl

[Kobayashi and Takahashi 2019] T. Kobayashi and R. Takahashi, "Ulrich modules over Cohen–Macaulay local rings with minimal multiplicity", *Q. J. Math.* **70**:2 (2019), 487–507. MR Zbl

[Kumashiro 2023] S. Kumashiro, "The Auslander–Reiten conjecture for certain non-Gorenstein Cohen–Macaulay rings", *J. Pure Appl. Algebra* **227**:11 (2023), art. id. 107420.

[Ma 2023] L. Ma, "Lim Ulrich sequences and Lech's conjecture", *Invent. Math.* **231**:1 (2023), 407–429. MR Zbl

[Martsinkovsky and Strooker 2004] A. Martsinkovsky and J. R. Strooker, "Linkage of modules", *J. Algebra* **271**:2 (2004), 587–626. MR Zbl

[Matsumura 1986] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1986. MR Zbl

[Numata 2017] T. Numata, "Ulrich ideals of Gorenstein numerical semigroup rings with embedding dimension 3", *J. Commut. Algebra* **9**:1 (2017), 143–156. MR Zbl

[Ooishi 1991] A. Ooishi, "On the associated graded modules of canonical modules", *J. Algebra* **141**:1 (1991), 143–157. MR Zbl

[Puthenpurakal 2003] T. J. Puthenpurakal, "Hilbert-coefficients of a Cohen–Macaulay module", *J. Algebra* **264**:1 (2003), 82–97. MR Zbl

[Puthenpurakal 2005] T. J. Puthenpurakal, "The Hilbert function of a maximal Cohen–Macaulay module", *Math. Z.* **251**:3 (2005), 551–573. MR Zbl

[Rossi and Valla 2010] M. E. Rossi and G. Valla, *Hilbert functions of filtered modules*, Lecture Notes of the Unione Matematica Italiana **9**, Springer, Berlin, 2010. MR Zbl

[Sally 1983] J. D. Sally, "Reductions, local cohomology and Hilbert functions of local rings", pp. 231–241 in *Commutative algebra* (Durham, 1981), edited by R. Y. Sharp, London Math. Soc. Lecture Note Ser. **72**, Cambridge Univ. Press, 1983. MR Zbl

[Sather-Wagstaff 2010] K. Sather-Wagstaff, "Semidualizing modules", lecture notes, 2010, available at http://ssather.people.clemson.edu/DOCS/sdm.pdf.

[Şega 2011] L. M. Şega, "Self-tests for freeness over commutative Artinian rings", *J. Pure Appl. Algebra* **215**:6 (2011), 1263–1269. MR Zbl

[Ulrich 1984] B. Ulrich, "Gorenstein rings and modules with high numbers of generators", *Math. Z.* **188**:1 (1984), 23–32. MR Zbl

[Wiebe 2003] A. Wiebe, "Ulrich modules of rank one over determinantal rings", Sem. Ser. Math. Algebra, Universitatea "Ovidius" din Constanța, 2003, available at https://math.univ-ovidius.ro/sna/down/e12/17.pdf.

[Yhee 2021] F. C. Yhee, "Ulrich modules and weakly lim Ulrich sequences do not always exist", preprint, 2021. arXiv 2104.05766

[Zamani 2014] N. Zamani, "Regularity of the Rees and associated graded modules", *Eur. J. Pure Appl. Math.* **7**:4 (2014), 429–436. MR Zbl

Received August 18, 2022. Revised March 15, 2023.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 323 No. 2 April 2023
