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MULTIVARIATE CORRELATION INEQUALITIES FOR P-PARTITIONS

SWEE HONG CHAN AND IGOR PAK

Motivated by the Lam–Pylyavskyy inequalities for Schur functions, we give a far reaching multivariate generalization of Fishburn's correlation inequality for the number of linear extensions of posets. We then give a multivariate generalization of the Daykin–Daykin–Paterson inequality proving log-concavity of the order polynomial of a poset. We also prove a multivariate P-partition version of the cross-product inequality by Brightwell, Felsner and Trotter. The proofs are based on a multivariate generalization of the Ahlswede–Daykin inequality.

1. Introduction

Arguably, *linear extensions* play as much a central role in poset theory as *standard Young tableaux* in algebraic combinatorics. While the former combinatorial objects obviously generalize the latter, this connection is yet to be fully explored. In fact, the development in the two areas seem to move along parallel tracks as we explain below.

The story of this paper is an interplay between these two areas of combinatorics, which makes both the motivation and presentation of the results somewhat less accessible. To mitigate this, we include two separate (and almost completely nonoverlapping) versions of the introduction addressing audiences with different background (see also Section 11A).

The results themselves are postponed to later sections and assume fluency in both areas. While the reader may choose to read only the results that are closer to their interests, reading both sides of the story can enhance the experience. To help navigate between the areas, we include detailed notation and some background in Section 2.

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Keywords: poset, linear extension, order polynomial, Schur function, q-analogue, log-concavity, Young diagram, P-partition, correlation inequality, FKG inequality, Ahlswede–Daykin inequality, XYZ inequality, Daykin–Paterson–Paterson inequality, Lam–Pylyavskyy inequality, Fishburn inequality.

Poset theoretic perspective. Our first result (Theorem 3.4) is a self-dual generalization of the remarkable *Fishburn's correlation inequality* (Theorem 3.1) for the numbers of linear extensions of poset order ideals. We further extend it to a correlation inequality for *order polynomials*, and then even further to their q-analogues and multivariate $\boldsymbol{q}$-analogues (Theorems 4.9 and 4.10). To understand the proofs it is worth examining the historical background and motivation behind earlier results.

Following up on the works by Harris (1960) and Kleitman (1966), Fortuin, Kasteleyn and Ginibre introduced the celebrated *FKG inequality* [21]. This correlation inequality was further generalized in a series of papers, most notably by Ahlswede and Daykin [2], who proved a very general *AD inequality* (Theorem 5.1), which is also called the *four functions theorem* [3, Section 6.1]. This result is so general that it has an elementary albeit somewhat involved proof by induction [loc. cit.]. For the many followup investigations of correlation inequalities; see, e.g., [1, Section 15], [33, Section 5], and earlier overviews in [20; 22; 43].

In a direct application to posets, Shepp [36] was able to use the FKG inequality and a clever limit argument to prove the *XYZ inequality* (see, e.g., [3, Section 6.4]), the most remarkable correlation inequality for linear extensions of posets, conjectured earlier by Rival and others. This brings us to Fishburn [18], who established *Fishburn's correlation inequality* (Theorem 3.1) as a tool in his proof of the strict version of the XYZ inequality. We note that Shepp's limit argument does not imply the strict version, so Fishburn's proof uses the AD inequality instead.

Motivated by enumerative applications and Fishburn's work, Björner [5] proved the *q-FKG inequality* generalizing the FKG inequality. Christofides [15] then found the *q-AD inequality*, answering Björner's question. In a joint work with Panova [13], we employed Björner's q-FKG inequality to obtain q-analogues of inequalities for order polynomials of interest in enumerative combinatorics.

In our most recent paper [11], we find several correlation inequalities whose proof required the *combinatorial atlas* technique and does not have a natural q-analogue. Among other results, we proved a series of upper bounds on correlation inequalities (when they are written in the form of a ratio ≥ 1), in some cases serving as a counterpart to the Fishburn's inequality.

The generality of our upper bounds in [11] and the self-dual nature of related results on Young tableaux naturally leads to our self-dual generalization of Fishburn's inequality. Just like the original proofs by Shepp and Fishburn, our proof is via the order polynomial, which naturally arises in this setting. Curiously, to prove our main theorem (Theorem 4.9), we use a multivariate generalization (Theorem 6.1) of Christofides's q-AD inequality.

At this point one would want to compare our results (notably Theorem 4.10), to those by Lam and Pylyavskyy [28], which are closely related and partly inspired this paper. They also prove a multivariate correlation inequality for order preserving maps on posets, which in some cases coincides with ours (see Corollary 4.5 and Remark 8.1). Unfortunately, their meet and join operations on order ideals are noncommutative and are therefore distinct from the more traditional definitions that we use. Thus, while the results in [28] might appear similar and even more general at a first glance (partially because they use the same notation), in full generality the similarity is misleading.

Now, Lam and Pylyavskyy's *cell transfer theorem* [28, Theorem 3.6] has a more general setting given by certain functions on poset's Hasse diagram. When it comes to skew Young diagrams, this allows the authors to recover the same reverse plane partitions results that we do, as well as semistandard Young tableaux results. We also recover their correlation inequality for Schur functions by making additional arguments (Section 8).

To summarize the comparison, neither result implies the other. Our meet and join notions are more standard, leading to a self-dual generalization of Fishburn's inequality. We are also using a more standard tool: the generalized AD inequality. On the other hand, the Lam and Pylyavskyy's ad hoc definitions allow them to recover the same Young tableaux results with an advantage of their proof giving an explicit combinatorial injection (see Section 11B).

We give two applications of the multivariate AD inequality to poset inequalities. First, we prove a multivariate cross-product inequality for order preserving maps on posets (Theorem 10.1), giving a variation on the *cross-product inequality* by Brightwell, Felsner and Trotter [9]. This result is new even for the usual (unweighted) setting. Note that the (original) cross-product inequality remains a conjecture in full generality (Remark 10.2).

Finally, we give a multivariate extension of the *Daykin–Daykin–Paterson* (DDP) *inequality* (Theorem 9.1), which was originally conjectured by Graham in [22], and proved in [17] by an ingenuous direct injection.[1] In fact, Graham originally suggested that the DDP inequality could be proved by the AD inequality (see Remark 9.2). We provide such a proof in Section 9A. Then, motivated by the structure of the multivariate AD inequality, we give a multivariate generalization of the DDP inequality (Theorem 9.3). We conclude with a multivariate log-concavity of the order polynomial (Corollary 9.5), generalizing our recent joint result with Panova [13].

Algebraic combinatorics perspective. Our main result is a generalization of the remarkable *Lam–Pylyavskyy correlation inequality* (Theorem 4.1) for Schur functions and reverse plane partitions to a self-dual (multivariate) correlation inequalities for

[1] This injection eluded us in the first version [13], when we were not aware of [17] and proved an asymptotic version of the DDP inequality which we called *Graham's conjecture*.

general posets (Theorems 4.9 and 4.10). Specializations of our main result give correlation inequalities for q-analogues of the number standard Young tableaux for both straight and skew shapes, which generalize Björner's inequality (Corollary 3.2).

To understand the proofs it is worth examining the historical background and motivation behind earlier results. The study of inequalities for the symmetric functions goes back to Newton (1707), who proved the log-concavity $\boldsymbol{e}_k^2 \geq \boldsymbol{e}_{k+1}\boldsymbol{e}_{k-1}$ of elementary symmetric polynomials $\boldsymbol{e}_k(x_1, \ldots, x_n)$, for all $x_i \in \mathbb{R}$. We refer to [32; 41] for a thorough treatment of symmetric functions.

Over the past century, symmetric functions have received a great deal of attention due to their connections and applications in representation theory, as well as a host of other fields (enumerative algebraic geometry, integrable probability, etc.) With many identities came inequalities, which were often proved by tools from other areas. We refer to [7; 8; 40] for somewhat dated surveys and to [6; 23] for a more recent overviews of positivity results.

Some recent highlights include inequalities for values of Schur functions conjectured by Cuttler, Greene and Skandera [16] and proved by Sra [37], the log-concavity of normalized Schur polynomials by Huh, Matherne, Mészáros and St. Dizier [24], and the *Schur positivity correlation inequality* by Lam, Postnikov and Pylyavskyy [30] (see Remark 4.2). See also [25] for most recent results on correlation inequalities in the context of matroids.

Building on the ideas which go back to MacMahon (1915), Stanley introduced in his thesis [38] the *P-partition theory*, which is closely related to the study of the order polynomial of posets, and to the *major index* statistics on linear extensions [41, Section 3.15]. Motivated by applications to plane partitions, the study of P-partitions became an important subject of its own. The order polynomial of a poset turned out to coincide with the Ehrhart polynomial of the order polytope; see, e.g., [41, Section 4.6.2].

The Lam and Pylyavskyy paper [28] uses Stanley's P-partition theory to obtain inequalities for the numbers of P-partitions with multivariate weights. The authors presented an explicit combinatorial injection called the *cell transfer*, which proves inequalities in a very general setting. As the main application they succeeded in establishing the monomial positivity correlation inequality for Schur functions (Theorem 4.1), which was soon overshadowed by the stronger Schur positivity LPP correlation inequality mentioned above. Their approach also extends to monotonicity of quasisymmetric functions which arise from P-partitions [29].

In this paper, we take the core part of the Lam–Pylyavskyy general inequality and generalize it in the direction which is more natural from the poset theoretic point of view (Theorem 4.10). Since multivariate inequalities are uncommon in poset theory, we give a multivariate extension of the AD inequality, an important tool in the area. We then show that our multivariate extension is strong enough to also imply the above mentioned Lam–Pylyavskyy monomial positivity.

Finally, we show that this multivariate approach can be used to prove new inequalities for general posets. Notably, we prove a new cross-product inequality (Theorem 10.1), and extend DDP and CPP log-concave inequalities for general posets (Theorem 9.3 and Corollary 9.5).

Paper structure. We start with a lengthy Section 2 with the background in both algebraic combinatorics and poset theory. We encourage the reader not to skip this section as we make some minor changes in definitions and standard notation to accommodate partly contradictory traditions in the two areas.

In the next two sections we present both known and new results in the order of increasing generality, pointing out the implications between results along the way. These implications tend to be quick and straightforward, and are included for clarity. In general, we opted for a complete and detailed presentation of all corollaries and special cases as a way to fully explain connections between the results.

In a short Section 3, we present results only about linear extensions and standard Young tableaux. While the results are easy consequences of the P-partition results in Section 4, the idea is to make the linear extension's story completely self-contained. Our most general results (Theorems 4.9 and 4.10) are given at the end of Section 4.

We then proceed to the proofs. In Section 5, we give a self-contained simple proof of the generalized Fishburn's inequality (Theorem 3.4) deducing it from its order polynomial generalization (Theorem 4.8), which is proved via the AD inequality (Theorem 5.1). This proof is based on Fishburn's approach [19], and is included here as a gentle introduction to our multivariate version.

In Section 6, we present the multivariate AD inequality (Theorem 6.1). This is the main tool of the paper, which we use to prove our main results in a short Section 7. In Section 8, we give a new proof of the Lam–Pylyavskyy inequality for Schur functions, also via the multivariate AD inequality.

In Section 9, we give a new proof and then a multivariate generalization of the DPP inequality (Theorem 9.3). We follow this with the cross-product inequality for P-partitions (Theorem 10.1) in Section 10. We conclude with final remarks and open problems in Section 11.

2. Background, definitions and notation

2A. ***Basic notations.*** We use $\mathbb{N}=\{0,1,2,\dots\}$, $\mathbb{N}_{\geq 1}=\{1,2,\dots\}$, $[n]=\{1,2,\dots,n\}$ and $\mathbb{R}_+=\{x\geq 0\}$. To simplify the notation, for an element $a\in X$, we use $X-a$ to denote the subset $X\smallsetminus\{a\}$. Similarly, for a subset $Y\subseteq X$, we write $X-Y$ in place of more general $X\smallsetminus Y$.

For variables $\boldsymbol{q} = (q_1, \ldots, q_n)$ and a vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, we write $\boldsymbol{q}^{\boldsymbol{a}} := q_1^{a_1} \cdots q_n^{a_n}$. For a polynomial $F \in \mathbb{R}[z_1, \ldots, z_n]$, we write that $F \geq 0$ if $F(z_1, \ldots, z_n) \geq 0$ for all $\boldsymbol{z} \in \mathbb{R}^n$. For two polynomials $F, G \in \mathbb{R}[z_1, \ldots, z_n]$, we write $F \geq G$ if $F - G \geq 0$.

For polynomials $F, G \in \mathbb{R}[z]$, we write $F \geqslant_z G$ if $F - G \in \mathbb{R}_+[z]$ is a polynomial with nonnegative coefficients. For multivariate polynomials $F, G \in \mathbb{R}[z_1, \ldots, z_n]$, we define $F \geqslant_{\boldsymbol{z}} G$ analogously. We drop the subscript in $\geqslant$ when the variables are clear. Obviously, $F \geqslant G$ implies $F \geq G$, but not vice versa, e.g., $x^2 + y^2 \geq 2xy$ but $x^2 + y^2 \not\geqslant 2xy$.

2B. ***Posets.*** We refer to [41, Chapter 3] and [42] for standard definitions and notation. Let $\mathcal{P} = (X, \prec)$ be a *partially ordered set* on the ground set X of size $|X| = n$, and with the partial order "$\prec$". A *subposet* is an induced poset $(Y, \prec)$ on the subset $Y \subseteq X$. For an element $x \subseteq X$, we denote by $\mathcal{P} - x$ the subposet of $\mathcal{P}$ on $X - x$.

For a poset $\mathcal{P} = (X, \prec)$, denote by $\mathcal{P}^* = (X, \prec^*)$ the *dual poset* with $x \prec^* y$ if and only if $y \prec x$, for all $x, y \in X$. For posets $\mathcal{P} = (X, \prec_{\mathcal{P}})$ and $\mathcal{Q} = (Y, \prec_{\mathcal{Q}})$, the *parallel sum* $\mathcal{P} + \mathcal{Q} = (Z, \prec)$ is the poset on the disjoint union $Z = X \sqcup Y$, where elements of X retain the partial order of $\mathcal{P}$, elements of Y retain the partial order of $\mathcal{Q}$, and elements $x \in X$ and $y \in Y$ are incomparable. Similarly, the *linear sum* $\mathcal{P} \oplus \mathcal{Q} = (Z, \prec)$, where $x \prec y$ for every two elements $x \in X$ and $y \in Y$ and other relations as in the parallel sum.

We use $\mathcal{C}_n$ and $\mathcal{A}_n$ to denote the n-element *chain* and *antichain*, respectively. Clearly, $\mathcal{C}_n = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_1$ (n times) and $\mathcal{A}_n = \mathcal{C}_1 + \cdots + \mathcal{C}_1$ (n times).

A *lattice* is a poset $\mathbb{L} = (\mathcal{L}, \prec)$ with meet $x \vee y$ (least upper bound) and join $x \wedge y$ (greatest lower bound) well defined, for all $x, y \in \mathcal{L}$. We also use $(\mathcal{L}, \vee, \wedge)$ to denote the lattice and the join and meet operations. The lattice $\mathbb{L} = (\mathcal{L}, \vee, \wedge)$ is *distributive* if it satisfied the distributive law $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Finally, for all $X, Y \subseteq \mathcal{L}$, we denote

$$X \vee Y := \{x \vee y : x \in X, y \in Y\} \quad \text{and} \quad X \wedge Y := \{x \wedge y : x \in X, y \in Y\}.$$

2C. ***Linear extensions and P-partitions.*** A *linear extension* of $\mathcal{P}$ is a bijection $L : X \to [n]$ that is order-preserving: $x \prec y$ implies $L(x) < L(y)$, for all $x, y \in X$. Denote by $\mathcal{E}(\mathcal{P})$ the set of linear extensions of $\mathcal{P}$, and let $e(\mathcal{P}) := |\mathcal{E}(\mathcal{P})|$ be the number of linear extensions. Observe that $e(\mathcal{P}) = e(\mathcal{P}^*)$ and $e(\mathcal{P} \oplus \mathcal{Q}) = e(\mathcal{P}) \cdot e(\mathcal{Q})$.

A subset $A \subseteq X$ is an *upper ideal* if $x \in A$ and $y \succ x$ implies $y \in A$. Similarly, a subset $A \subseteq X$ is a *lower ideal* if $x \in A$ and $y \prec x$ implies $y \in A$. We denote by $e(A)$ the number of linear extensions of the subposet $(A, \prec)$.

Let $\mathcal{P} = (X, \prec)$, where $X = \{x_1, \ldots, x_n\}$. We will always assume that X has a *natural labeling*, i.e., $L : x_i \to i$ is a linear extension. A $\mathcal{P}$-*partition* is an order preserving map $A : X \to \mathbb{N}$, i.e., maps which satisfy $A(x) \le A(y)$ for all $x \prec y$. Denote by $\mathrm{PP}(\mathcal{P})$ the set of P-partitions and let $\mathrm{PP}(\mathcal{P}, t)$ be the set of P-partitions with values at most t.[2]

Let $\Omega(\mathcal{P}, t) := |\mathrm{PP}(\mathcal{P}, t)|$ be the number of $\mathcal{P}$-partitions. This is the *order polynomial* corresponding to the poset $\mathcal{P}$.[3] It is well-known and easy to see that

$$\Omega(\mathcal{P}, t) \sim \frac{e(\mathcal{P})t^n}{n!} \quad \text{as } t \to \infty, \text{ where } |X| = n. \tag{2-1}$$

Denote $|A| := \sum_{x \in X} A(x)$ the sum of the entries in a $\mathcal{P}$-partition. Let

$$\Omega_q(\mathcal{P}, t) := \sum_{A \in \mathrm{PP}(\mathcal{P},t)} q^{|A|}. \tag{2-2}$$

Stanley showed, see [41, Theorem 3.15.7], that there is a statistics $\mathrm{maj} : \mathcal{E}(\mathcal{P}) \to \mathbb{N}$, such that

$$\Omega_q(\mathcal{P}, \infty) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} \sum_{A \in \mathcal{E}(\mathcal{P})} q^{\mathrm{maj}(A)}. \tag{2-3}$$

More generally, let

$$\Omega_{\boldsymbol{q}}(\mathcal{P}, t) := \sum_{A \in \mathrm{PP}(\mathcal{P},t)} q_1^{A(x_1)} \cdots q_n^{A(x_n)}. \tag{2-4}$$

We call this GF the *multivariate order polynomial*. Note that Stanley gave a generalization of (2-3) for $\Omega_{\boldsymbol{q}}(\mathcal{P}, \infty)$ which we will not need; see [41, Theorem 3.15.5]. Finally, for $N \ge 0$, define

$$\mathrm{K}_z(\mathcal{P}, N) := \sum_{A \in \mathrm{PP}(\mathcal{P},N)} z_0^{m_0(A)} \cdots z_N^{m_N(A)}, \tag{2-5}$$

where $m_i(A) := |A^{-1}(i)|$ is the number of values i in the $\mathcal{P}$-partition A.

2D. ***Young diagrams and Young tableaux.*** We refer to [32; 35] and [41, Chapter 7] for standard definitions and notation. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be an *integer partition* of n, write $\lambda \vdash n$, where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell > 0$ and $\lambda_1 + \cdots + \lambda_\ell = n$. Let $\ell(\lambda) := \ell$ denotes the number of parts. A *conjugate partition* $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ is defined by $\lambda'_j = |\{i : \lambda_i \ge j\}|$.

A *Young diagram* is the set of squares $\{(i, j) \in \mathbb{N}^2 : 1 \le j \le \lambda_i, 1 \le i \le \ell\}$. In a mild abuse of notation, we use λ to also denote the corresponding Young diagram,

[2]Stanley [38; 41], uses P-partitions to denote order-reversing rather than order-preserving maps. We adopt this version for clarity and to unify the notation. Displeased readers can always think of dual posets.

[3]A standard definition for order polynomial is $\Omega(\mathcal{P}, t-1)$ as the values in the $\mathcal{P}$-partition are traditionally ≥ 1. We adopt this version to simplify the notation and hope this does not lead to confusion.

and refer to it as the *straight shape*. Let $\mu = (\mu_1, \mu_2, \dots)$ be a partition such that $\mu_i \le \lambda_i$ for all $0 \le i \le \ell$. The difference of Young diagrams is denoted by λ/μ and called the *skew Young diagram* of shape λ/μ, or simply the *skew shape* λ/μ. We use $|\lambda/\mu|$ for the size, i.e., the number of squares in λ/μ.

A *standard Young tableau* of shape λ/μ is a bijection $A : \lambda/\mu \to [n]$ which increases in rows and columns: $A(i, j) < A(i+1, j)$ and $A(i, j) < A(i, j+1)$ whenever these are defined. Denote by $\mathrm{SYT}(\lambda/\mu)$ the set of standard Young tableaux of shape λ/μ. We note that $|\mathrm{SYT}(\lambda)|$ can be computed by the *hook-length formula*, see, e.g., [41, Section 7.21]. Similarly, the number $|\mathrm{SYT}(\lambda/\mu)|$ can be computed by the *Aitken–Feit determinant formula*, see, e.g., [41, Section 7.16].

Let poset $\mathcal{P}_{\lambda/\mu} = (\lambda/\mu, \prec)$ be defined by $(i, j) \preccurlyeq (i', j')$ if $i \le i'$ and $j \le j'$. For example, $P_{31/11} \simeq C_2$ and $\mathcal{P}_{321/21} \simeq A_3$. The set of linear extensions $\mathcal{E}(\mathcal{P}_{\lambda/\mu})$ is in bijection with $\mathrm{SYT}(\lambda/\mu)$, so $e(\mathcal{P}_{\lambda/\mu}) = |\mathrm{SYT}(\lambda/\mu)|$.

2E. ***Schur functions and reverse plane partitions.*** Let $A : \lambda/\mu \to \mathbb{N}$ be a function which increases in rows and columns. In this context, function A is called a *reverse plane partition*.[4] Let $\mathrm{RPP}(\lambda/\mu)$ denote the set of reverse plane partition of shape λ/μ. We think of A as a Young tableau with integers written in squares of λ/μ. If $A \in \mathrm{RPP}(\lambda/\mu)$ is also increasing in columns and has all entries ≥ 1, it is called a *semistandard Young tableau*. The set of such tableaux is denoted $\mathrm{SSYT}(\lambda/\mu)$. We use $\mathrm{RPP}(\lambda/\mu, t)$ and $\mathrm{SSYT}(\lambda/\mu, t)$ to denote reverse plane partitions and semistandard Young tableaux with entries $\le t$.

Schur polynomial is a symmetric polynomial associated with the skew shape λ/μ and can be defined as

$$s_{\lambda/\mu}(z_1, \dots, z_N) = \sum_{A \in \mathrm{SSYT}(\lambda/\mu, N)} z_1^{m_1(A)} \cdots z_N^{m_N(A)}, \tag{2-6}$$

where $m_i(A) = |A^{-1}(i)|$ is the number of i's in A. *Schur functions* are the stable limits of Schur polynomials as $n \to \infty$. They form a linear basis in the space of all symmetric functions.

For reverse plane partitions, observe the connection to the order polynomial

$$\Omega(\lambda/\mu, t) := \Omega(\mathcal{P}_{\lambda/\mu}, t) = \sum_{A \in \mathrm{RPP}(\lambda/\mu, t)} t^{|A|}. \tag{2-7}$$

In similar manner, consider the following multivariate GF for the reverse plane partitions:

$$\mathcal{F}_{\lambda/\mu}(z_0, z_1, \dots, z_N) = \sum_{A \in \mathrm{RPP}(\lambda/\mu, N)} z_0^{m_0(A)} z_1^{m_1(A)} \cdots z_N^{m_N(A)},$$

Note the notation above, we have $\mathcal{F}_{\lambda/\mu}(z_0, z_1, \dots, z_N) = \mathrm{K}_{\mathbf{z}}(\mathcal{P}_{\lambda/\mu}, N)$.

[4] Note that reverse plane partitions for λ/μ are actually $\mathcal{P}_{\lambda/\mu}$ – partitions. This is another notational compromise we make between the areas.

3. Linear extensions

3A. *Fishburn's inequality.* We start with the following fundamental inequality:

Theorem 3.1 (Fishburn's inequality [18]). *Let $\mathcal{P} = (X, \prec)$ be a finite poset, and let $A, B \subset X$ be lower ideals of P. Then*

$$\frac{e(A \cup B) \cdot e(A \cap B)}{e(A) \cdot e(B)} \geq \frac{|A \cup B|! \cdot |A \cap B|!}{|A|! \cdot |B|!}. \tag{3-1}$$

Using the notation

$$f(P) := \frac{e(P)}{|X|!},$$

Fishburn's inequality can be rewritten in a more concise form as a correlation inequality for probabilities:

$$f(A \cup B) \cdot f(A \cap B) \geq f(A) \cdot f(B). \tag{3-2}$$

The original proof of Fishburn's inequality uses the *AD inequality*. Note that it is tight for the antichain $\mathcal{P} = \mathcal{A}_n$.

3B. *Björner's inequality.* For a skew Young diagram $|\lambda/\mu| = n$, we similarly denote

$$f(\lambda/\mu) := f(\mathcal{P}_{\lambda/\mu}) = \frac{|\mathrm{SYT}(\lambda/\mu)|}{n!}.$$

Now (3-2) gives:

Corollary 3.2 (Björner's inequality [5, Section 6]). *Let μ and ν be Young diagrams. Then*

$$f(\mu \vee \nu) \cdot f(\mu \wedge \nu) \geq f(\mu) \cdot f(\nu), \tag{3-3}$$

where $\vee$ and $\wedge$ refer to the union and intersection of the Young diagrams.

Björner's proof used another Fishburn's result combined with the some calculations using the hook-length formula. The following result has an ambiguous status of being nominally new, yet it easily follows from the LP inequality (see Section 4A below).

Corollary 3.3 (generalized Björner's inequality). *Let μ/α and ν/β be skew Young diagrams. Then*

$$f(\mu/\alpha \vee \nu/\beta) \cdot f(\mu/\alpha \wedge \nu/\beta) \geq f(\mu/\alpha) \cdot f(\nu/\beta). \tag{3-4}$$

where $\mu/\alpha \vee \nu/\beta := (\mu \vee \nu)/(\alpha \vee \beta)$ and $\mu/\alpha \wedge \nu/\beta := (\mu \wedge \nu)/(\alpha \wedge \beta)$.

In contrast with Björner's inequality, the generalized Björner inequality does not follow from Fishburn's inequality, at least not directly.

3C. ***Generalized Fishburn's inequality.*** Our first new result is a common generalization of both the Fishburn's and the generalized Björner's inequalities.

Theorem 3.4. *Let $\mathcal{P} = (X, \prec)$ be a finite poset. Let $A, B \subseteq X$ be lower ideals, and let $C, D \subseteq X$ be upper ideals of $\mathcal{P}$, such that $A \cap C = B \cap D = \varnothing$. Then*

$$f(X - V) \cdot f(X - W) \geq f(X - A - C) \cdot f(X - B - D), \tag{3-5}$$

where $V := (A \cap B) \cup (C \cup D)$ and $W := (A \cup B) \cup (C \cap D)$.

Note that Fishburn's inequality (Theorem 3.1) is a special case $C = D = \varnothing$, and that Theorem 3.4 is self-dual. We prove the theorem using the AD inequality in Section 5.

Proof of Theorem 3.4 ⇒ Corollary 3.3. Let $\mathcal{P} := \mathcal{P}_\lambda$, where $\lambda := \mu \vee \nu$. In the notation of Theorem 3.4, we have $X = \lambda$. Consider the following four subsets of the Young diagram λ:

$$A := \alpha, \quad B := \beta, \quad C := \lambda/\mu, \quad D := \lambda/\nu. \tag{3-6}$$

Now observe that

$$X - A - C = \mu/\alpha, \qquad X - B - D = \nu/\beta,$$
$$X - V = (\mu \wedge \nu)/(\alpha \wedge \beta), \qquad X - W = (\mu \vee \nu)/(\alpha \vee \beta).$$

Thus, (3-5) implies (3-4), as desired. □

4. *P*-partitions

4A. ***Schur functions.*** The following *LP inequality* is the key result which inspired this paper.

Theorem 4.1 (Lam–Pylyavskyy inequality for Schur polynomials [28, Theorem 4.5]). *Let μ/α and ν/β be skew Young diagrams, and let $z = (z_1, \ldots, z_N)$, where $N \geq \ell(\mu), \ell(\nu)$. Then*

$$s_{\mu\vee\nu}(z) \cdot s_{\mu\wedge\nu}(z) \geqslant_z s_\mu(z) \cdot s_\nu(z). \tag{4-1}$$

More generally, we have

$$s_{\mu/\alpha\vee\nu/\beta}(z) \cdot s_{\mu/\alpha\wedge\nu/\beta}(z) \geqslant_z s_{\mu/\alpha}(z) \cdot s_{\nu/\beta}(z), \tag{4-2}$$

where $\mu/\alpha \vee \nu/\beta := (\mu \vee \nu)/(\alpha \vee \beta)$ and $\mu/\alpha \wedge \nu/\beta := (\mu \wedge \nu)/(\alpha \wedge \beta)$.

The original proof is completely combinatorial and uses an explicit injection. For completeness, we include a short argument showing how the LP inequality implies the Björner's and the generalized Björner's inequality.

Proof of (4-2) $\Rightarrow$ (3-4). Recall the following analogue of (2-3) for skew Schur functions

$$s_{\lambda/\tau}(1, q, q^2, \dots) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^{|\lambda/\tau|})} \sum_{T \in \mathrm{SYT}(\lambda/\tau)} q^{\mathrm{maj}(T)}, \tag{4-3}$$

where maj : $\mathrm{SYT}(\lambda/\tau) \to \mathbb{N}$ is the *major index* of a tableau, see, e.g., [41, Theorem 7.19.11].

Let $n := |\mu/\alpha| + |\nu/\beta|$. Substituting (4-3) into each of the four Schur functions in the LP inequality (4-2), multiplying both sides by $(1-q)^n$ and letting $q \to 1$, gives the generalized Björner's inequality (3-3). □

Remark 4.2. The following truly remarkable *Lam–Postnikov–Pylyavskyy inequality* further extended (4-2) and resolved several open problems in the area

$$s_{\mu/\alpha \vee \nu/\beta} \cdot s_{\mu/\alpha \wedge \nu/\beta} \geqslant_s s_{\mu/\alpha} \cdot s_{\nu/\beta}. \tag{4-4}$$

Here "$\geqslant_s$" stands for *Schur positivity*, which is saying that the difference is a nonnegative sum of Schur functions. Although we will not need this extension, it does give a more conceptual proof of Björner's inequality.

In a different direction, Richards [34] gave an analytic generalization of (4-1) for real $\lambda, \mu \in \mathbb{R}^\ell$ and the determinant definition of Schur polynomials. It would be natural to conjecture that (4-4) also generalizes to this setting.

Proof of (4-4) $\Rightarrow$ (3-3). Recall that for all $\mu \vdash k$, $\nu \vdash n-k$, we have

$$s_\mu \cdot s_\nu = \sum_{\lambda \vdash n} c^\lambda_{\mu\nu} s_\lambda \quad \text{and} \quad \chi^\mu \otimes \chi^\nu \uparrow^{S_n}_{S_k \times S_{n-k}} = \sum_{\lambda \vdash n} c^\lambda_{\mu\nu} \chi^\lambda,$$

where $c^\lambda_{\mu\nu}$ are the *Littlewood–Richardson coefficients*; see, e.g., [35, Section 4.9]. Equating dimensions in the second equality gives

$$f(\mu) \cdot f(\nu) = \sum_{\lambda \vdash n} c^\lambda_{\mu\nu} f(\lambda).$$

Thus $\varphi : s_\lambda \to f(\lambda)$ is a ring homomorphism from the ring of symmetric function to $\mathbb{Q}$ which maps Schur positive symmetric function to $\mathbb{Q}_+$. Applying φ to the inequality (4-4) for $\alpha = \beta = \varnothing$ gives the desired inequality (3-3). □

4B. *RPP variation.* The following RPP variation is an easy corollary of the LP inequality (4-2):

Corollary 4.3. *Let μ and ν be Young diagrams and let $t \geq 0$. Then*

$$\Omega(\mu \vee \nu, t) \cdot \Omega(\mu \wedge \nu, t) \geq \Omega(\mu, t) \cdot \Omega(\nu, t). \tag{4-5}$$

Similarly, for the q-statistics we have

$$\Omega_q(\mu \vee \nu, \infty) \cdot \Omega_q(\mu \wedge \nu, \infty) \geqslant_q \Omega_q(\mu, \infty) \cdot \Omega_q(\nu, \infty). \tag{4-6}$$

More generally, we have

$$\Omega_q(\mu \vee \nu, t) \cdot \Omega_q(\mu \wedge \nu, t) \geqslant_q \Omega_q(\mu, t) \cdot \Omega_q(\nu, t). \tag{4-7}$$

Proof of (4-1) $\Rightarrow$ (4-6). Setting $N \leftarrow \infty$ and $\boldsymbol{z} = (z_1, z_2, \dots) \leftarrow (q, q, \dots)$, we get

$$s_\lambda(q, q, \dots) = \Omega_q(\lambda, \infty) \cdot q^{n(\lambda)}, \quad \text{where } n(\lambda) = \sum_{(i,j)\in\lambda} i. \tag{4-8}$$

Note that $n(\mu \vee \nu) + n(\mu \wedge \nu) = n(\mu) + n(\nu)$. Substituting (4-8) into (4-1) and dividing both sides by $q^{n(\mu)+n(\nu)}$ gives (4-6). $\square$

Corollary 4.4. *Let $\mathcal{P} = (X, \prec)$ be a finite poset, let $t \geq 0$, and let $A, B \subset X$ be lower ideals of P. Then*

$$\Omega(A \cup B, t) \cdot \Omega(A \cap B, t) \geq \Omega(A, t) \cdot \Omega(B, t). \tag{4-9}$$

More generally, we have

$$\Omega_q(A \cup B, t) \cdot \Omega_q(A \cap B, t) \geqslant_q \Omega_q(A, t) \cdot \Omega_q(B, t). \tag{4-10}$$

Proof of (4-9) $\Rightarrow$ (3-2). Let $t \to \infty$ and apply (2-1) to each term in (4-9). $\square$

Corollary 4.4 is a direct generalization of Corollary 4.3, which follows by taking $A \leftarrow \mu$ and $B \leftarrow \nu$. Our next result is a multivariate generalization of Corollary 4.3.

Corollary 4.5. *Let μ and ν be Young diagrams and let $N \geq 0$. Then*

$$\mathcal{F}_{\mu\vee\nu}(\boldsymbol{z}) \cdot \mathcal{F}_{\mu\wedge\nu}(\boldsymbol{z}) \geqslant_{\boldsymbol{z}} \mathcal{F}_\mu(\boldsymbol{z}) \cdot \mathcal{F}_\nu(\boldsymbol{z}), \tag{4-11}$$

where $\boldsymbol{z} = (z_0, z_1, \dots, z_N)$.

Proof of (4-11) $\Rightarrow$ (4-7). Let $N \leftarrow t$, and set $z_i \leftarrow q^i$ for all $0 \leq i \leq N$. $\square$

This result is implicit in [28] and follows from the following general theorem:

Theorem 4.6 (Lam–Pylyavskyy inequality for multivariate order polynomials [28, Proposition 3.7]). *Let $\mathcal{P} = (X, \prec)$ be a finite poset, let $A, B \subset X$ be lower ideals of P, and let $N \geq 0$. Then*

$$\mathrm{K}_{\boldsymbol{z}}(A \cup B, N) \cdot \mathrm{K}_{\boldsymbol{z}}(A \cap B, N) \geqslant_{\boldsymbol{z}} \mathrm{K}_{\boldsymbol{z}}(A, N) \cdot \mathrm{K}_{\boldsymbol{z}}(B, N). \tag{4-12}$$

This is the most general version of the LP inequality that we discuss in this paper. Note that (4-12) $\Rightarrow$ (4-11) by taking $A \leftarrow \mu$ and $B \leftarrow \nu$.

Remark 4.7. As we mention in the introduction, the ultimate Lam–Pylyavskyy generalization uses the meet and join operations which are incompatible with those we employ in this paper. They are in fact, noncommutative and designed to allow the "cell transfer" direct injection.

Notably, (4-2) does not follow from (4-12), but from the proof of this ultimate Lam–Pylyavskyy generalization which happens to apply to skew shapes. We give a more streamlined derivation of (4-2) from our generalization below.

4C. ***Main results.*** We begin with the order polynomial extension of the generalized Fishburn's inequality (Theorem 3.4) and the Lam–Pylyavskyy order polynomial inequality (Corollary 4.4).

Theorem 4.8. *Let $\mathcal{P} = (X, \prec)$ be a finite poset. Let $A, B \subseteq X$ be lower ideals, and let $C, D \subseteq X$ be upper ideals of $\mathcal{P}$, such that $A \cap C = B \cap D = \varnothing$. Then*

$$\Omega(X - V, t) \cdot \Omega(X - W, t) \geq \Omega(X - A - C, t) \cdot \Omega(X - B - D, t), \tag{4-13}$$

where $V := (A \cap B) \cup (C \cup D)$ and $W := (A \cup B) \cup (C \cap D)$. More generally, we have

$$\Omega_q(X - V, t) \cdot \Omega_q(X - W, t) \geqslant_q \Omega_q(X - A - C, t) \cdot \Omega_q(X - B - D, t). \tag{4-14}$$

Corollary 4.4 is a special case of the theorem when $C = D = \varnothing$.

Proof of (4-13) $\Rightarrow$ (3-5). Let $t \to \infty$ and apply (2-1) to each term in (4-13). □

Here is our most general result in this direction, and the ultimate multivariate generalization of Fishburn's inequality (Theorem 3.1).

Theorem 4.9. *Let $\mathcal{P} = (X, \prec)$ be a finite poset. Let $A, B \subseteq X$ be lower ideals, and let $C, D \subseteq X$ be upper ideals of $\mathcal{P}$, such that $A \cap C = B \cap D = \varnothing$. Then*

$$\Omega_{\boldsymbol{q}}(X - V, t) \cdot \Omega_{\boldsymbol{q}}(X - W, t) \geqslant_{\boldsymbol{q}} \Omega_{\boldsymbol{q}}(X - A - C, t) \cdot \Omega_{\boldsymbol{q}}(X - B - D, t), \tag{4-15}$$

where $V := (A \cap B) \cup (C \cup D)$ and $W := (A \cup B) \cup (C \cap D)$.

Proof of (4-15) $\Rightarrow$ (4-14). Take $\boldsymbol{q} \leftarrow (q, \ldots, q)$. □

Finally, we present another generalization of Theorem 4.8 for different choices of rank functions, and furthermore generalizes Lam–Pylyavskyy Theorem 4.6. We prove both theorems in Section 7.

Theorem 4.10. *Let $\mathcal{P} = (X, \prec)$ be a finite poset. Let $A, B \subseteq X$ be lower ideals, and let $C, D \subseteq X$ be upper ideals of $\mathcal{P}$, such that $A \cap C = B \cap D = \varnothing$. Then*

$$\mathrm{K}_{\boldsymbol{z}}(X - V, N) \cdot \mathrm{K}_{\boldsymbol{z}}(X - W, N) \geqslant_{\boldsymbol{z}} \mathrm{K}_{\boldsymbol{z}}(X - A - C, N) \cdot \mathrm{K}_{\boldsymbol{z}}(X - B - D, N), \tag{4-16}$$

where $V := (A \cap B) \cup (C \cup D)$ and $W := (A \cup B) \cup (C \cap D)$.

Proof of (4-16) $\Rightarrow$ (4-14). Take $\boldsymbol{z} \leftarrow (1, q, q^2, \ldots, q^N)$. □

In particular, these two theorems imply the following corollary for skew Young diagrams.

Corollary 4.11. *Let μ/α and ν/β be skew Young diagrams. Then*

$$\Omega_{\boldsymbol{q}}(\mu/\alpha \vee \nu/\beta, t) \cdot \Omega_{\boldsymbol{q}}(\mu/\alpha \wedge \nu/\beta, t) \geqslant_{\boldsymbol{q}} \Omega_{\boldsymbol{q}}(\mu/\alpha, t) \cdot \Omega_{\boldsymbol{q}}(\nu/\beta, t), \tag{4-17}$$

and

$$\mathcal{F}_{\mu/\alpha\vee\nu/\beta}(\boldsymbol{z}) \cdot \mathcal{F}_{\mu/\alpha\wedge\nu/\beta}(\boldsymbol{z}) \geqslant_{\boldsymbol{z}} \mathcal{F}_{\mu/\alpha}(\boldsymbol{z}) \cdot \mathcal{F}_{\nu/\beta}(\boldsymbol{z}), \tag{4-18}$$

where $\boldsymbol{z} = (z_0, z_1, \dots, z_N)$.

Proof. Let $\mathcal{P}$, A, B, C, D be as in (3-6). By applying the same argument as in the proof of the [Theorem 3.4 $\Rightarrow$ Corollary 3.3] implication, the inequality (4-17) now follows from (4-15), while the inequality (4-18) follows from (4-16). $\square$

Remark 4.12. Although the inequalities (4-18) and (4-17) do not appear in [28], they follow from the approach in that paper.

5. The Ahlswede–Daykin inequality

In this section, we prove the first part of Theorem 4.8 by using the Ahlswede–Daykin (AD) inequality. Our approach is based on the proof in [18]. For every $\rho : Z \to \mathbb{R}_+$ and every $X \subseteq Z$, denote

$$\rho(X) := \sum_{x\in X} \rho(x). \tag{5-1}$$

Theorem 5.1 (Ahlswede–Daykin inequality [2]). *Let $\mathbb{L} = (\mathcal{L}, \vee, \wedge)$ be a finite distributive lattice, and let $\alpha, \beta, \gamma, \delta : \mathcal{L} \to \mathbb{R}_+$ be nonnegative functions on $\mathcal{L}$. Suppose we have*

$$\alpha(x) \cdot \beta(y) \le \gamma(x \vee y) \cdot \delta(x \wedge y) \quad \textit{for every } x, y \in \mathcal{L}. \tag{5-2}$$

Then

$$\alpha(X) \cdot \beta(Y) \le \gamma(X \vee Y) \cdot \delta(X \wedge Y) \quad \textit{for every } X, Y \subseteq \mathcal{L}. \tag{5-3}$$

Proof of the first part of Theorem 4.8. Let $\mathcal{P} = (X, \prec)$ be a poset, and let $t \ge 0$. We denote by $\mathbb{L}(\mathcal{P}, t) = (\mathcal{L}, \vee, \wedge)$ the distributive lattice on the set $\mathcal{L} \subseteq \{0, \dots, t\}^X$ given by

$$\mathcal{L} := \mathrm{PP}(\mathcal{P}, t) = \{T : X \to \{0, \dots, t\} : T(x) \le T(y) \text{ for all } x, y \in X \text{ s.t. } x \prec y\}, \tag{5-4}$$

with the join and meet operation given by

$$[S \vee T](x) = \max\{S(x), T(x)\}$$

and

$$[S \wedge T](x) = \min\{S(x), T(x)\}$$

for every $x \in X$. Recall that $\Omega(\mathcal{P}, t) = |\mathcal{L}|$. Let $\alpha, \beta, \gamma, \delta : \mathcal{L} \to \mathbb{R}_+$ be given by

$$\begin{aligned}
\alpha(T) &:= \mathbf{1}\{T(x) = 0 \text{ for all } x \in A,\, T(y) = t \text{ for all } y \in C\},\\
\beta(T) &:= \mathbf{1}\{T(x) = 0 \text{ for all } x \in B,\, T(y) = t \text{ for all } y \in D\},\\
\gamma(T) &:= \mathbf{1}\{T(x) = 0 \text{ for all } x \in A \cap B,\, T(y) = t \text{ for all } y \in C \cup D\},\\
\delta(T) &:= \mathbf{1}\{T(x) = 0 \text{ for all } x \in A \cup B,\, T(y) = t \text{ for all } y \in C \cap D\}.
\end{aligned} \tag{5-5}$$

Note that

$$\begin{aligned}
\alpha(\mathcal{L}) &= \Omega(X - A - C, t), & \beta(\mathcal{L}) &= \Omega(X - B - D, t),\\
\gamma(\mathcal{L}) &= \Omega(X - V, t), & \delta(\mathcal{L}) &= \Omega(X - W, t).
\end{aligned}$$

By the AD inequality (5-3), it thus suffices to verify (5-2), which in this case states

$$\alpha(S) \cdot \beta(T) \le \gamma(S \vee T) \cdot \delta(S \wedge T) \quad \text{for every } S, T \in \mathcal{L}. \tag{5-6}$$

Let $S, T \in \mathcal{L}$ be such that $\alpha(S) = \beta(T) = 1$. Then

$$\begin{aligned}
S(x) &= 0 \ \text{ for } x \in A, & S(y) &= t \ \text{ for } y \in C,\\
T(x) &= 0 \ \text{ for } x \in B, & T(y) &= t \ \text{ for } y \in D.
\end{aligned}$$

This gives

$$\begin{aligned}
\max\{S(x), T(x)\} &= 0 \ \text{ for } x \in A \cap B, & \max\{S(y), T(y)\} &= t \ \text{ for } x \in C \cup D,\\
\min\{S(x), T(x)\} &= 0 \ \text{ for } x \in A \cup B, & \min\{S(y), T(y)\} &= t \ \text{ for } x \in C \cap D.
\end{aligned}$$

The first equation implies $\gamma(S \vee T) = 1$, while the second equation implies $\delta(S \wedge T) = 1$. This implies (5-6) and completes the proof of (4-13). □

Remark 5.2. For the second (more general) part of Theorem 4.8, one can use the same approach with the AD inequality in Theorem 5.1 replaced with q-AD inequality by Christofides [15]. Our proof of Theorem 4.9 given below, extends Theorem 4.8 using the multivariate $\boldsymbol{q}$-AD inequality.

6. Multivariate AD inequality

6A. ***The statement.*** Let $\mathbb{L} := (\mathcal{L}, \wedge, \vee)$ be a finite distributive lattice. Throughout this section, fix variables $q_1, \ldots, q_\ell$, and *modular functions* $r_1, \ldots, r_\ell : \mathcal{L} \to \mathbb{N}$ defined to satisfy

$$r_i(x) + r_i(y) = r_i(x \vee y) + r_i(x \wedge y) \quad \text{for all } x, y \in \mathcal{L} \text{ and } 1 \le i \le \ell.$$

Write $\boldsymbol{q} := (q_1, \ldots, q_\ell)$ and $\boldsymbol{r} := (r_1, \ldots, r_\ell)$. For $x \in \mathcal{L}$, write

$$\boldsymbol{r}(x) := (r_1(x), \ldots, r_\ell(x)) \quad \text{and} \quad \boldsymbol{q}^{\boldsymbol{r}(x)} := q_1^{r_1(x)} \cdots q_\ell^{r_\ell(x)}.$$

For a function $\rho : \mathcal{L} \to \mathbb{R}_+$ and subset $X \subseteq \mathcal{L}$, define

$$\rho_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(X) := \sum_{x \in X} \rho(x) \boldsymbol{q}^{\boldsymbol{r}(x)} \in \mathbb{R}_+[q_1, \dots, q_\ell]. \tag{6-1}$$

Note that (6-1) is a multivariate $\boldsymbol{q}$-analogue of (5-1). We can now state the multivariate $\boldsymbol{q}$-analogue of the Ahlswede–Daykin inequality (Theorem 5.1).

Theorem 6.1 (multivariate AD inequality). *Let $\mathbb{L} = (\mathcal{L}, \wedge, \vee)$ be a finite distributive lattice, and let $\alpha, \beta, \gamma, \delta : \mathcal{L} \to \mathbb{R}_+$ be nonnegative functions on $\mathcal{L}$. Suppose we have*

$$\alpha(x) \cdot \beta(y) \leq \gamma(x \vee y) \cdot \delta(x \wedge y) \quad \textit{for every } x, y \in \mathcal{L}. \tag{6-2}$$

Then

$$\alpha_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(X) \cdot \beta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(Y) \leqslant_{\boldsymbol{q}} \gamma_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(X \vee Y) \cdot \delta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(X \wedge Y) \quad \textit{for every } X, Y \subseteq \mathcal{L}. \tag{6-3}$$

Our proof is strongly inspired by those of Björner [5] and Christofides [15]. We closely follow the presentation from the former while incorporating some ideas from the latter paper.

6B. *The proof.* We start by proving the following special case of Theorem 6.1, which we use to obtain the theorem in the full generality.

Proposition 6.2. *Let $\mathbb{L} = (\mathcal{L}, \wedge, \vee)$, $\alpha, \beta, \gamma, \delta$ be as in Theorem 6.1. Then*

$$\alpha_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) \beta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) \leqslant_{\boldsymbol{q}} \gamma_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) \delta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}). \tag{6-4}$$

Proof of Proposition 6.2 ⇒ Theorem 6.1. Let $\alpha', \beta', \gamma', \delta' : \mathcal{L} \to \mathbb{R}_+$ be functions given by

$$\alpha' := \alpha \circ \mathbf{1}_X, \quad \beta' := \beta \circ \mathbf{1}_Y, \quad \gamma' := \gamma \circ \mathbf{1}_{X \vee Y}, \quad \delta := \delta' \circ \mathbf{1}_{X \wedge Y}.$$

Note that

$$\alpha'(x) \cdot \beta'(y) \leq \gamma'(x \vee y) \cdot \delta'(x \wedge y) \quad \text{for every } x, y \in \mathcal{L}. \tag{6-5}$$

Indeed, the LHS of (6-5) is equal to 0 if $x \notin A$ or $y \notin B$, so suppose that $x \in A$, $y \in B$. Then the inequality reduces to (6-2), which is part of the assumption. The inequality (6-3) then follows from (6-4) by noting that

$$\begin{aligned} \alpha'_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \alpha_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(X), & \beta'_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \beta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(Y), \\ \gamma'_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \gamma_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(X \vee Y), & \delta'_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \delta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(X \wedge Y), \end{aligned}$$

as desired. □

Proof of Proposition 6.2. Let

$$\Phi(\boldsymbol{q},\boldsymbol{r}) := \alpha_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L})\cdot\beta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) - \gamma_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L})\cdot\delta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}).$$

For $x, y \in \mathcal{L}$, we also define

$$\phi(x,y) := \alpha(x)\cdot\beta(y) - \gamma(x)\cdot\delta(y).$$

A simple computation shows that

$$\Phi(\boldsymbol{q},\boldsymbol{r}) = \sum_{(x,y)\in\mathcal{L}^2} \phi(x,y)\boldsymbol{q}^{\boldsymbol{r}(x)+\boldsymbol{r}(y)}.$$

Let $\boldsymbol{d} := (d_1,\ldots,d_\ell) \in \mathbb{N}^\ell$ be an arbitrary integer vector. Denote by

$$\Phi_{\boldsymbol{d}} := [q_1^{d_1}\cdots q_\ell^{d_\ell}]\Phi(q)$$

the coefficient of the monomial $\boldsymbol{q}^{\boldsymbol{d}}$ in $\Phi(q)$. We then have

$$\Phi_{\boldsymbol{d}} = \sum_{\substack{(x,y)\in\mathcal{L}^2,\\ \boldsymbol{r}(x)+\boldsymbol{r}(y)=\boldsymbol{d}}} \phi(x,y).$$

We now consider another, slightly coarser, grouping of terms. For $u, v \in \mathcal{L}$ satisfying $u \prec^\diamond v$, so in particular $u \neq v$, let $C(u,v)$ denote the set of (ordered) pairs (x,y) in the interval $[u,v]$ such that $x \wedge y = v$ and $x \vee y = u$. Let

$$\psi(u,v) := \sum_{(x,y)\in C(u,v)} \phi(x,y).$$

It follows from the modularity of $r_1,\ldots,r_\ell$ that

$$\Phi_{\boldsymbol{d}} = \sum_{\substack{u\prec^\diamond v,\\ \boldsymbol{r}(u)+\boldsymbol{r}(v)=\boldsymbol{d}}} \psi(u,v) + \sum_{\substack{u\in\mathcal{L},\\ 2\boldsymbol{r}(u)=\boldsymbol{d}}} \phi(u,u).$$

Since $\phi(u,u) = \alpha(u)\beta(u) - \gamma(u)\delta(u) \leq 0$ by (6-2), the proposition follows from Claim 6.3 below. □

Claim 6.3. *In notation above, for every $u, v \in \mathcal{L}$ such that $u \prec^\diamond v$, we have $\psi(u,v) \leq 0$.*

Proof of Claim 6.3. Note that $\psi(u,v)$ depends only on elements in the poset interval $[u,v]$, so by restricting to $[u,v]$ if necessary, we can without loss of generality assume that $u = \hat{0}$ is the unique minimal element of $\mathcal{L}$, and $v = \hat{1}$ is the unique maximal element of $\mathcal{L}$.

For $x \in \mathcal{L}$, a *complement* of x is an element $y \in \mathcal{L}$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. Note that in a finite distributive lattice every element has at most one complement (see e.g., [4, Theorem 10, page 12]), and we denote this element by x^c

if it exists. Note that $\psi(\hat{0}, \hat{1})$ depends only on elements that have a complement in $\mathcal{L}$, and that the set of complemented elements in a finite distributive lattice form a sublattice of $\mathcal{L}$ (see e.g., [4, page 18]). By restricting to this sublattice if necessary, without loss of generality we can assume that every element $x \in \mathcal{L}$ has a unique complement x^c (i.e., when $\mathcal{L}$ is a Boolean lattice).

Define four new functions $\alpha', \beta', \gamma', \delta' : \mathcal{L} \to \mathbb{R}_+$ as follows:

$$\alpha'(x) := \alpha(x)\beta(x^c). \quad \beta'(x) := \alpha(x^c)\beta(x).$$
$$\gamma'(x) := \gamma(x)\delta(x^c). \quad \delta'(x) := \gamma(x^c)\delta(x).$$

Note that

$$\psi(\hat{0}, \hat{1}) = \sum_{x \in \mathcal{L}} \phi(x, x^c) = \sum_{x \in \mathcal{L}} \alpha(x)\beta(x^c) - \gamma(x)\delta(x^c) = \alpha'(\mathcal{L}) - \gamma'(\mathcal{L}).$$

It thus suffices to show that $\alpha'(\mathcal{L}) \le \gamma'(\mathcal{L})$. Now observe that, for any $x, y \in \mathcal{L}$, we have

$$\begin{aligned} \alpha'(x)\beta'(y) &= (\alpha(x)\beta(y))(\alpha(y^c)\beta(x^c)) \\ &\le_{(6\text{-}2)} (\gamma(x \vee y)\delta(x \wedge y))(\gamma(y^c \vee x^c)\delta(y^c \wedge x^c)) \\ &\le \gamma(x \vee y)\delta((y \vee x)^c)\gamma((y \wedge x)^c)\delta(x \wedge y) \\ &= \gamma'(x \vee y)\delta'(x \wedge y). \end{aligned}$$

It then follows from the (usual) AD inequality (5-3), that

$$\alpha'(\mathcal{L})\beta'(\mathcal{L}) \le \gamma'(\mathcal{L})\delta'(\mathcal{L}). \tag{6-6}$$

On the other hand, note that $\beta'(\mathcal{L}) = \alpha'(\mathcal{L})$ and $\gamma'(\mathcal{L}) = \delta'(\mathcal{L})$ by definition of the functions. Since the functions are nonnegative, (6-6) gives $\alpha'(\mathcal{L}) \le \gamma'(\mathcal{L})$. This completes the proof. □

7. Proof of main results

7A. ***Proof of Theorem 4.9.*** Let $\alpha, \beta, \gamma, \delta : \mathcal{L} \to \mathbb{R}_+$ be as in (5-5). Note that these functions satisfy the assumption (5-6) of the multivariate AD inequality.

Let $\boldsymbol{q} := (q_1, \ldots, q_n)$ be variables, with $n = |X|$. For any $i \in [n]$, let $r_i : \mathcal{L} \to \mathbb{R}_+$ be the modular function given by $r_i(T) := T(x_i)$. For a subset $Y \subseteq X$, denote

$$\boldsymbol{q}^{\mathbf{n}(Y)} := \prod_{x_i \in Y} (q_i)^t.$$

Then

$$\begin{aligned}\alpha_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \Omega_{\boldsymbol{q}}(X-A-C,t)\cdot \boldsymbol{q}^{\mathbf{n}(C)},\\ \beta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \Omega_{\boldsymbol{q}}(X-B-D,t)\cdot \boldsymbol{q}^{\mathbf{n}(D)},\\ \gamma_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \Omega_{\boldsymbol{q}}(X-V,t)\cdot \boldsymbol{q}^{\mathbf{n}(C\cup D)},\\ \delta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \Omega_{\boldsymbol{q}}(X-W,t)\cdot \boldsymbol{q}^{\mathbf{n}(C\cap D)}.\end{aligned}$$

The theorem now follows from the multivariate AD inequality (6-3). □

7B. ***Proof of Theorem 4.10.*** Let $\alpha, \beta, \gamma, \delta : \mathcal{L} \to \mathbb{R}_+$ be as in (5-5), with $t \leftarrow N$. Note that these functions satisfy the assumption of the multivariate AD inequality (see (5-6)). Let $\boldsymbol{q} := (q_0, \dots, q_N)$ be variables. For any $i \in \{0, \dots, N\}$, let $r_i : \mathcal{L} \to \mathbb{R}_+$ be the modular function where $r_i(T) := |\{x \in X : T(x) = i\}|$ is the number of i's in T. Then

$$\begin{aligned}\alpha_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \mathrm{K}_z(X-A-C,N)\cdot q_0^{|A|}q_N^{|C|},\\ \beta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \mathrm{K}_z(X-B-D,M)\cdot q_0^{|B|}q_N^{|D|},\\ \gamma_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \mathrm{K}_z(X-V,N)\cdot q_0^{|A\cap B|}q_N^{|C\cup D|},\\ \delta_{\langle \boldsymbol{q},\boldsymbol{r}\rangle}(\mathcal{L}) &= \mathrm{K}_z(X-W,M)\cdot q_0^{|A\cup B|}q_N^{|C\cap D|}.\end{aligned}$$

The theorem now follows from the multivariate AD inequality (6-3). □

8. Back to Schur polynomials

In this section we give a new proof of the Lam–Pylyavskyy inequality (4-2) for Schur polynomials via the multivariate AD inequality.

Proof of Theorem 4.1. Let $\mathcal{P} := \mathcal{P}_\lambda$ be the poset of the Young diagram of shape λ, where $\lambda := \mu \vee \nu$. Let $\mathbb{L} := (\mathcal{L}', \wedge', \vee')$ be the distributive lattice given by $\mathcal{L}' := \mathrm{RPP}(\lambda, N)$, with the $\vee'$ and $\wedge'$ operation given by

$$(S \vee' T)(i,j) := \max\{S(i,j), T(i,j)\}, \quad (S \wedge' T)(i,j) := \min\{S(i,j), T(i,j)\}.$$

For a skew Young diagram π/τ such that $\pi \subset \lambda$, let $\phi^{\pi/\tau} : \mathcal{L}' \to \mathbb{R}_+$ be the characteristic function of the reverse plane partition $T \in \mathrm{RPP}(\lambda, N)$ satisfying all these properties:

$$\begin{aligned}&T(i.j) \geq 1 \quad \text{for } (i,j) \in \lambda,\\ &T(i,j) = 1 \quad \text{for } (i,j) \in \tau \quad \text{and} \quad T(i,j) = N \quad \text{for } (i,j) \in \lambda/\pi,\\ &T(i,j) < T(i+1,j) \quad \text{if } (i,j), (i+1,j) \in \pi/\tau.\end{aligned}$$

Note that these reverse plane partitions are in bijection with semistandard Young tableau of π/τ in $\mathrm{SSYT}(\pi/\tau, N)$.

We define functions $\zeta, \eta, \xi, \rho : \mathcal{L} \to \mathbb{R}_+$ as follows:

$$\zeta := \phi^{\mu/\alpha}. \quad \eta := \phi^{\nu/\beta}. \quad \xi := \phi^{\mu/\alpha\wedge\nu/\beta}. \quad \rho := \phi^{\mu/\alpha\vee\nu/\beta}.$$

We now show that these functions satisfy the assumption of the multivariate AD inequality, i.e., for any $S, T \in \mathcal{L}$:

$$\zeta(S) \cdot \eta(T) \le \xi(S \vee T) \cdot \rho(S \wedge T).$$

The equation is vacuously true if $\zeta(S) = 0$ or $\eta(T) = 0$, so assume $\zeta(S) = \eta(T) = 1$. We show only the proof that $\xi(S \vee T) = 1$, as the proof of $\rho(S \wedge T) = 1$ is similar. First, for $(i, j) \in \lambda$, we have

$$[S \vee T](i, j) = \max\{S(i, j), T(i, j)\} \ge 1.$$

Second, for $(i, j) \in \alpha \wedge \beta$,

$$[S \vee T](i, j) = \max\{S(i, j), T(i, j)\} = 1.$$

Third, for $(i, j) \in \lambda/(\mu \wedge \nu)$,

$$[S \vee T](i, j) = \max\{S(i, j), T(i, j)\} = N.$$

Fourth, let $(i, j), (i+1, j) \in (\mu \wedge \nu)/(\alpha \wedge \beta)$. We will need to show that

$$[S \vee T](i, j) < [S \vee T](i+1, j). \tag{8-1}$$

Note that we must have either $(i, j) \in (\mu \wedge \nu)/\alpha$ or $(i, j) \in (\mu \wedge \nu)/\beta$. Without loss of generality, we assume the former holds. Then it follows that $(i+1, j) \in (\mu\wedge\nu)/\alpha$. Since $\zeta(S) = 1$, this implies that

$$S(i, j) < S(i+1, j) \le \max\{S(i+1, j), T(i+1, j)\} = [S \vee T](i+1, j).$$

Thus (8-1) follows if $T(i, j) \le S(i, j)$, so suppose instead that $T(i, j) > S(i, j)$. This then implies $T(i, j) > 1$. Since $\eta(T) = 1$, this implies that $(i, j) \in (\mu \wedge \nu)/\beta$, which in turn implies that $(i+1, j) \in (\mu \wedge \nu)/\beta$. Thus we have

$$[S \vee T](i, j) = T(i, j) < T(i+1, j) \le [S \vee T](i+1, j),$$

which completes the proof of (8-1).

Let $\boldsymbol{z} := (z_1, \ldots, z_N)$ be variables, and let $r_i : \mathcal{L} \to \mathbb{N}$, $i \in [N]$, be the modular function defined as follows: $r_i(T) := m_i(T)$ is the number of i's in T. It then follows that

$$A_{\langle \boldsymbol{z},\boldsymbol{r}\rangle} = s_{\mu/\alpha} \cdot q_1^{|\alpha|} q_N^{|\lambda|-|\mu|}, \qquad B_{\langle \boldsymbol{z},\boldsymbol{r}\rangle} = s_{\nu/\beta} \cdot q_1^{|\beta|} q_N^{|\lambda|-|\nu|},$$
$$C_{\langle \boldsymbol{z},\boldsymbol{r}\rangle} = s_{\mu/\alpha\wedge\nu/\beta} \cdot q_1^{|\alpha\wedge\beta|} q_N^{|\lambda|-|\mu\wedge\nu|}, \quad D_{\langle \boldsymbol{z},\boldsymbol{r}\rangle} = s_{\mu/\alpha\wedge\nu/\beta} \cdot q_1^{|\alpha\vee\beta|} q_N^{|\lambda|-|\mu\vee\nu|}.$$

The theorem now follows from the multivariate AD inequality (6-3). $\square$

Remark 8.1. By the arguments analogous to the proofs in this and previous section, specifically the proof of (8-1) to account for strict comparisons, the multivariate AD inequality can be used to prove results analogous to Theorem 4.8 and Theorem 4.10 for both strict and nonstrict $(\mathcal{P}, \omega)$-partitions; see definitions in [41, Section 3.15.1]. Similarly, we can extend out results to the more general $\mathbb{T}$-labeled $(\mathcal{P}, O)$ tableaux defined in [28]. We omit the details for brevity.

9. Multivariate Daykin–Daykin–Paterson inequality

9A. ***The DDP inequality.*** Let $\mathcal{P} = (X, \prec)$ be a *partially ordered set* on $|X| = n$ elements. Fix $t \geq 0$ and an element $z \in X$. For integer $0 \leq k \leq t$, denote by $\mathrm{PP}(\mathcal{P}, t; z, k)$ the set of $\mathcal{P}$-partitions $A \in \mathrm{PP}(\mathcal{P}, t)$ such that $A(z) = k$. Let $\Omega(\mathcal{P}, t; z, k) := |\mathrm{PP}(\mathcal{P}, t; z, k)|$ be the number of such $\mathcal{P}$-partitions. The following inequality was conjectured by Graham [22] and proved by Daykin, Daykin and Paterson [17].

Theorem 9.1 (Daykin–Daykin–Paterson inequality). *Let $\mathcal{P} = (X, \prec)$ be a finite poset, let $t \in \mathbb{N}$, and let $z \in X$. Then, for every $0 \leq k \leq t$, we have*

$$\Omega(\mathcal{P}, t; z, k)^2 \geq \Omega(\mathcal{P}, t; z, k-1) \cdot \Omega(\mathcal{P}, t; z, k+1). \tag{9-1}$$

More generally, for every positive integers $a, b \geq 1$,

$$\Omega(\mathcal{P}, t; z, k+a) \cdot \Omega(\mathcal{P}, t; z, k+b) \geq \Omega(\mathcal{P}, t; z, k) \cdot \Omega(\mathcal{P}, t; z, k+a+b). \tag{9-2}$$

We give a new proof of Theorem 9.1 as an application of the AD inequality (5-3). The proof below sets the stage for the multivariate generalization of the theorem.

Proof of Theorem 9.1. We denote by $\mathbb{L} = (\mathcal{L}, \vee, \wedge)$ the distributive lattice on the set $\mathcal{L}$ given by

$$\mathcal{L} := \{T : X \to \{-b, -b+1, \dots, t\} : T(x) \leq T(y)\ \forall x, y \in X \text{ s.t. } x \prec y\},$$

the set of order-preserving functions such that $-b \leq T(x) \leq t$ for every $x \in X$. The join and meet operation are given by

$$[S \vee T](x) := \max\{S(x), T(x)\} \quad \text{and} \quad [S \wedge T](x) := \min\{S(x), T(x)\},$$

for every $x \in X$. It is straightforward to verify that $\mathbb{L}$ is a distributive lattice.

Let $\alpha, \beta, \gamma, \delta : \mathcal{L} \to \mathbb{R}_+$ be characteristic function of subsets of $\mathcal{L}$ defined as follows:

$$\alpha := \mathbf{1}\{T(z) = k \text{ and } T(x) \geq 0,\ \text{for all } x \in X\}.$$
$$\beta := \mathbf{1}\{T(z) = k+a \text{ and } T(x) \leq t-b,\ \text{for all } x \in X\}.$$
$$\gamma := \mathbf{1}\{T(z) = k+a \text{ and } T(x) \geq 0,\ \text{for all } x \in X\}.$$
$$\delta := \mathbf{1}\{T(z) = k \text{ and } T(x) \leq t-b,\ \text{for all } x \in X\}.$$

We will now verify the assumption of AD inequality, i.e., for every $S, T \in \mathcal{L}$, we have

$$\alpha(S) \cdot \beta(T) \leq \gamma(S \vee T) \cdot \delta(S \wedge T). \tag{9-3}$$

Without loss of generality we can assume that $\alpha(S) = \beta(T) = 1$. Note that

$$[S \vee T](z) = \max\{S(z), T(z)\} = \max\{k, k+a\} = k+a.$$

Also note that, for every $x \in X$,

$$[S \vee T](x) = \max\{S(x), T(x)\} \geq S(x) \geq 0.$$

This shows that $\gamma(S \vee T) = 1$. Similarly, note that

$$[S \wedge T](z) = \min\{S(z), T(z)\} = \min\{k, k+a\} = k.$$

Also note that, for every $x \in X$,

$$[S \wedge T](x) = \min\{S(x), T(x)\} \leq T(x) \leq t-b.$$

This shows that $\delta(S \wedge T) = 1$, and completes the proof of (9-3).

Now note that

$$\alpha(\mathcal{L}) = |\{T \in \mathcal{L} : T(z) = k \text{ and } 0 \leq T(x) \leq t \; \forall x \in X\}| = \Omega(\mathcal{P}, t; z, k), \quad \text{and}$$
$$\gamma(\mathcal{L}) = |\{T \in \mathcal{L} : T(z) = k+a \text{ and } 0 \leq T(x) \leq t \forall x \in X\}| = \Omega(\mathcal{P}, t; z, k+a).$$

Also note that

$$\begin{aligned} \beta(\mathcal{L}) &= |\{T \in \mathcal{L} : T(z) = k+a \text{ and } -b \leq T(x) \leq t-b \; \forall x \in X\}| \\ &= |\{T' \in \mathcal{L} : T'(z) = k+a+b \text{ and } 0 \leq T'(x) \leq t \forall x \in X\}| \\ &= \Omega(\mathcal{P}, t; z, k+a+b), \end{aligned} \tag{9-4}$$

where the second equality is obtained through the substitution $T'(x) \leftarrow T(x)+b$. Similarly, by the same substitution we have

$$\begin{aligned} \delta(\mathcal{L}) &= |\{T \in \mathcal{L} : T(z) = k \text{ and } -b \leq T(x) \leq t-b \; \forall x \in X\}| \\ &= |\{T' \in \mathcal{L} : T'(z) = k+b \text{ and } 0 \leq T'(x) \leq t \forall x \in X\}| \\ &= \Omega(\mathcal{P}, t; z, k+b). \end{aligned} \tag{9-5}$$

Now (9-2) follows from the AD inequality (5-3). □

Remark 9.2. The original proof of the DDP inequality was through an explicit injection [17]. Curiously, Graham believed that there should exist a proof based on the FKG or AD inequalities. He lamented "such a proof has up to now successfully eluded all attempts to find it" [22, page 15]. The proof above validates Graham's supposition.

We should also mention that if the order-preserving functions are replaced with linear extensions, the DPP inequality (9-1) becomes *Stanley's inequality* [39], a major result in the area for which finding a direct combinatorial proof remains a challenging open problem. We refer to [33, Section 6.3] for an extensive discussion and further references.

9B. *Multivariate DDP inequality.* Let $\boldsymbol{q} := (q_1, \dots, q_n)$ be variables, and fix a natural labeling $X = \{x_1, \dots, x_n\}$. Define

$$\Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k) := \sum_{A \in \mathrm{PP}(\mathcal{P}, t; z, k)} q_1^{A(x_1)} \cdots q_n^{A(x_n)}.$$

We now present the multivariate version of DDP inequality (9-1), proved by the multivariate AD inequality (6-3).

Theorem 9.3 (multivariate DDP inequality). *Let $\mathcal{P} = (X, \prec)$ be a finite poset, let $t \in \mathbb{N}$, and let $z \in X$. Then, for every $0 \le k \le t$, we have*

$$\Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k)^2 \geqslant_{\boldsymbol{q}} \Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k-1) \cdot \Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k+1). \tag{9-6}$$

More generally, for every integer $a, b \ge 1$, we have

$$\Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k{+}a) \cdot \Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k{+}b) \geqslant_{\boldsymbol{q}} \Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k) \cdot \Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k{+}a{+}b). \tag{9-7}$$

Note that in contrast with the DPP inequality (9-1), the generalized log-concavity (9-7) does not follow from the (usual) log-concavity (9-6) via telescoping.

Proof. Let $\mathcal{L}, \alpha, \beta, \gamma, \delta$ be as in the proof of Theorem 9.1. Note that these functions satisfy the assumption (6-2) of the multivariate AD inequality (6-3). For all $1 \le i \le n$, let $r_i : \mathcal{L} \to \mathbb{R}_+$ be the modular function given by $r_i(A) := A(x_i)$, where $A \in \mathcal{L}$. Then

$$\begin{aligned} \alpha_{\langle \boldsymbol{q}, \boldsymbol{r} \rangle}(\mathcal{L}) &= \Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k), & \beta_{\langle \boldsymbol{q}, \boldsymbol{r} \rangle}(\mathcal{L}) &= \Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k{+}a{+}b) \cdot (q_1 \cdots q_n)^{-b}, \\ \gamma_{\langle \boldsymbol{q}, \boldsymbol{r} \rangle}(\mathcal{L}) &= \Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k{+}a), & \delta_{\langle \boldsymbol{q}, \boldsymbol{r} \rangle}(\mathcal{L}) &= \Omega_{\boldsymbol{q}}(\mathcal{P}, t; z, k{+}b) \cdot (q_1 \cdots q_n)^{-b}. \end{aligned}$$

The second part of the theorem now follows from the multivariate AD inequality (6-3), and thus also the first part (which is a special case). □

Remark 9.4. In the context of Remark 8.1, Theorem 9.3 holds by the same argument if the order-preserving functions are replaced with the strict order-preserving functions. This approach can be extended to general $\mathbb{T}$-labeled $(\mathcal{P}, O)$ tableaux. However, the analogue of (9-6) does not hold if $\Omega_{\boldsymbol{q}}$ is replaced with K_z. This is because the weight functions for K_z is not invariant under the translation transformation used in (9-4) and (9-5) in the proof of Theorem 9.1.

9C. ***Log-concavity of the multivariate order polynomial.*** The following corollary follows immediately from Theorem 9.3, and can be viewed as a multivariate generalization of [13, Theorem 4.7], and a poset generalization of the first formula in the proof of Lemma 6.13 in [31, page 550].

Corollary 9.5. *Let $\mathcal{P} = (X, \prec)$ be a finite poset, and let $t \in \mathbb{N}_{\geq 1}$ be a positive integer. Then*

$$\Omega_{\boldsymbol{q}}(\mathcal{P}, t)^2 \geqslant_{\boldsymbol{q}} \Omega_{\boldsymbol{q}}(\mathcal{P}, t-1) \cdot \Omega_{\boldsymbol{q}}(\mathcal{P}, t+1).$$

More generally, for every integers $a, b \geq 1$, we have

$$\Omega_{\boldsymbol{q}}(\mathcal{P}, t+a) \cdot \Omega_{\boldsymbol{q}}(\mathcal{P}, t+b) \geqslant_{\boldsymbol{q}} \Omega_{\boldsymbol{q}}(\mathcal{P}, t) \cdot \Omega_{\boldsymbol{q}}(\mathcal{P}, t+a+b).$$

Proof. Let $n := |X|$. Let $\mathcal{P}' := \mathcal{P} \oplus z$ be the linear sum of $\mathcal{P}$ and an extra element z, which is the unique maximal element in $\mathcal{P}'$. Since we use natural labeling, element z corresponds to the variable q_{n+1}.

Note that for every $\ell, t \in \mathbb{N}$, we have

$$\Omega_{\boldsymbol{q}}(\mathcal{P}', t; z, \ell) = \Omega_{\boldsymbol{q}}(\mathcal{P}, \ell) \cdot q_{n+1}^{\ell}. \tag{9-8}$$

On the other hand, it follows from applying Theorem 9.3 to $\mathcal{P}'$ that

$$\Omega_{\boldsymbol{q}}(\mathcal{P}', t; z, k+a) \cdot \Omega_{\boldsymbol{q}}(\mathcal{P}', t; z, k+b) \geqslant_{\boldsymbol{q}} \Omega_{\boldsymbol{q}}(\mathcal{P}', t; z, k) \cdot \Omega_{\boldsymbol{q}}(\mathcal{P}', t; z, k+a+b).$$

The corollary now follows by applying (9-8) to the equation above. □

Remark 9.6. Our proof of the $\boldsymbol{q} = \mathbf{1}$ version in [13, Theorem 4.7] goes along similar lines, but uses the FKG rather than the AD inequality. Note that our [13, Theorem 4.8] gives a strict log-concavity for order polynomials, with a substantially more involved proof.

10. Cross–product inequality for $\mathcal{P}$-partitions

10A. ***The statement.*** Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements. Fix $t \geq 0$ and distinct elements $x, y, z \in X$. For integers $k, \ell \geq 0$, denote by

$$\mathrm{SPP}(\mathcal{P}, t; x, y, z; k, \ell) := \{A \in \mathrm{PP}(\mathcal{P}, t) : A(y) - A(x) = k \text{ and } A(z) - A(y) = \ell\}.$$

Denote

$$\Lambda_q(k, \ell) := \sum_{A \in \mathrm{SPP}(\mathcal{P}, t; x, y, z; k, \ell)} q^{|A|},$$

$$\Lambda_{\boldsymbol{q}}(k, \ell) := \sum_{A \in \mathrm{SPP}(\mathcal{P}, t; x, y, z; k, \ell)} q_1^{A(x_1)} \cdots q_n^{A(x_n)},$$

and let $\mathrm{F}(k, \ell) := \Lambda_1(k, \ell) = |\mathrm{SPP}(\mathcal{P}, t; x, y, z; k, \ell)|$.

Theorem 10.1 (cross-product inequality for $\mathcal{P}$-partitions). *Let $\mathcal{P} = (X, \prec)$ be a finite poset, let $x, y, z \in \mathcal{P}$, and let $t \in \mathbb{N}_{\geq 1}$ be a positive integer. Then, for every $k, \ell \geq 0$, we have*

$$F(k, \ell+1) \cdot F(k+1, \ell) \geq F(k, \ell) \cdot F(k+1, \ell+1). \tag{10-1}$$

More generally

$$\Lambda_q(k, \ell+1) \cdot \Lambda_q(k+1, \ell) \geqslant_q \Lambda_q(k, \ell) \cdot \Lambda_q(k+1, \ell+1). \tag{10-2}$$

Even more generally

$$\Lambda_{\boldsymbol{q}}(k, \ell+1) \cdot \Lambda_{\boldsymbol{q}}(k+1, \ell) \geqslant_{\boldsymbol{q}} \Lambda_{\boldsymbol{q}}(k, \ell) \cdot \Lambda_{\boldsymbol{q}}(k+1, \ell+1). \tag{10-3}$$

Remark 10.2. Note that already the unweighted inequality (10-1) appears to be new. Note also that if the order-preserving functions are replaced with linear extensions, then a version of (10-1) is known as the *cross–product conjecture* [9, Conjecture 3.1], a major open problem in the area. We refer to [12] for an extensive discussion and further references.

10B. ***Proof of Theorem 10.1.*** We denote by $\mathbb{L} = (\mathcal{L}, \vee, \wedge)$ the distributive lattice on the set of order-preserving functions from X to $\{0, 1, \ldots, t\}$:

$$\mathcal{L} := \{T : X \to \{0, 1, \ldots, t\} : T(v) \leq T(w)\, \forall v, w \in X \text{ s.t. } v \prec w\}.$$

The join and meet operation are given by

$$[S \vee T](w) := \max\{S(w) - S(y), T(w) - T(y)\} + \min\{S(y), T(y)\},$$
$$[S \wedge T](w) := \min\{S(w) - S(y), T(w) - T(y)\} + \max\{S(y), T(y)\},$$

for every $w \in X$. This lattice was proved distributive by Shepp [36, Equations 2.4, 2.5], in his proof of the *XYZ inequality*; see also [3, Section 6.4].

Let $\alpha, \beta, \gamma, \delta : \mathcal{L} \to \mathbb{R}_+$ be characteristic function of subsets of $\mathcal{L}$ defined as follows:

$$\alpha := \mathbf{1}\{T(y) - T(x) = k \text{ and } T(z) - T(y) = \ell\}.$$
$$\beta := \mathbf{1}\{T(y) - T(x) = k+1 \text{ and } T(z) - T(y) = \ell+1\}.$$
$$\gamma := \mathbf{1}\{T(y) - T(x) = k \text{ and } T(z) - T(y) = \ell+1\}.$$
$$\delta := \mathbf{1}\{T(y) - T(x) = k+1 \text{ and } T(z) - T(y) = \ell\}.$$

We will now verify the assumption (6-2) of the multivariate AD inequality

$$\alpha(S) \cdot \beta(T) \leq \gamma(S \vee T) \cdot \delta(S \wedge T), \tag{10-4}$$

for every $S, T \in \mathcal{L}$. Without loss of generality we can assume that $\alpha(S) = \beta(T) = 1$. We have

$$\begin{aligned}
[S \vee T](x) - [S \vee T](y) &= \max\{S(x) - S(y), T(x) - T(y)\} \\
&= \max\{-k, -k-1\} = -k, \\
[S \vee T](z) - [S \vee T](y) &= \max\{S(z) - S(y), T(z) - T(y)\} \\
&= \max\{\ell, \ell+1\} = \ell+1, \\
[S \wedge T](x) - [S \wedge T](y) &= \min\{S(x) - S(y), T(x) - T(y)\} \\
&= \min\{-k, -k-1\} = -k-1, \\
[S \wedge T](z) - [S \wedge T](y) &= \min\{S(z) - S(y), T(z) - T(y)\} \\
&= \min\{\ell, \ell+1\} = \ell.
\end{aligned}$$

This shows that $\gamma(S \vee T) = \delta(S \wedge T) = 1$ and proves (10-4).

Finally, consider modular functions $r_i : \mathcal{L} \to \mathbb{R}_+$, for all $1 \le i \le n$, given by $r_i(T) := T(x_i)$. Then we have

$$\begin{aligned}
\alpha_{\langle \boldsymbol{q}, \boldsymbol{r} \rangle}(\mathcal{L}) &= \Lambda_{\boldsymbol{q}}(k, \ell), & \beta_{\langle \boldsymbol{q}, \boldsymbol{r} \rangle}(\mathcal{L}) &= \Lambda_{\boldsymbol{q}}(k+1, \ell+1), \\
\gamma_{\langle \boldsymbol{q}, \boldsymbol{r} \rangle}(\mathcal{L}) &= \Lambda_{\boldsymbol{q}}(k, \ell+1), & \delta_{\langle \boldsymbol{q}, \boldsymbol{r} \rangle}(\mathcal{L}) &= \Lambda_{\boldsymbol{q}}(k+1, \ell).
\end{aligned}$$

The theorem now follows from the multivariate AD inequality (6-3). □

Remark 10.3. Let us also mention that the proof in [12, Section 3.1] shows that Theorem 10.1 implies a (multivariate) $\mathcal{P}$-partition version of the *Kahn–Saks inequality* [27, Theorem 2.5]. On the other hand, while the KS inequality easily implies Stanley's inequality discussed earlier in Remark 9.2 (see e.g., [14, Section 1.2]), the multivariate DPP inequality (Theorem 9.3) does not similarly follow from cross–product inequality for $\mathcal{P}$-partitions (Theorem 10.1). This is also demonstrated by the fact that different lattices are used in the proofs of the two theorems.

11. Final remarks and open problems

11A. This paper grew out of [11, Section 4.1] where we obtained superficially similar correlation inequalities which appear to have a very different nature and whose only known proof uses the combinatorial atlas technology. Our investigation was also partly motivated by the desire to bridge the gap between the two areas of combinatorics. Notably, we would like to emphasize the importance of the AD inequality to algebraic combinatorics, and the multivariate weighting to poset theory.

Note that there is a weighted version of $e(\mathcal{P})$ introduced in [10, Section 1.16]. While the results in [11] translate verbatim to the weighted setting, these weights seem incompatible with $\boldsymbol{q}$-weights in this paper. Similarly, the q-weight on $e(\mathcal{P})$ in

[12] is also of different nature. On the other hand, the q-weighted order polynomial in [13] is exactly $\Omega_q(\mathcal{P}, t)$.

11B. One distinguishing feature of poset inequalities is the difficulty of getting the equality conditions, see, e.g., [13, Section 9.9] for an overview. We are not aware of any equality conditions for the inequalities in this paper, proved or conjectured.

Another difficulty is finding a combinatorial interpretation for the difference of two sides. This was a major motivation for our investigation in [10]. We show in [26, Section 7.4] that the AD inequality (5-3) does not have a combinatorial interpretation in full generality, in a sense of being in #P. Of course, the Lam–Postnikov–Pylyavskyy deep algebraic approach in [30] (see Remark 4.2) is even less likely to give a combinatorial interpretation. We refer to [33, Section 6] for an extensive survey.

Now, the Lam–Pylyavskyy's injective approach in [28] shows that the difference of coefficients on both sides in (4-12) has a combinatorial interpretation. By contrast, the limit arguments we use throughout this paper do not give a combinatorial interpretation for Fishburn's inequality (3-1). It would be interesting to see if (3-1) and the generalized Fishburn inequality (3-5) can be proved by a direct combinatorial argument giving a combinatorial interpretation.

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COMPATIBILITY IN OZSVÁTH–SZABÓ'S BORDERED HFK VIA HIGHER REPRESENTATIONS

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We equip the basic local crossing bimodules in Ozsváth–Szabó's theory of bordered knot Floer homology with the structure of 1-morphisms of 2-representations, categorifying the $U_q(\mathfrak{gl}(1|1)^+)$-intertwining property of the corresponding maps between ordinary representations. Besides yielding a new connection between bordered knot Floer homology and higher representation theory in line with work of Rouquier and Manion, this structure gives an algebraic reformulation of a "compatibility between summands" property for Ozsváth and Szabó's bimodules that is important when building their theory up from local crossings to more global tangles and knots.

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1. Introduction

Ozsváth and Szabó's theory [2018; 2019a; 2019b; 2020] of bordered knot Floer homology, or bordered HFK, has proven to be highly efficient for computations (see [Ozsváth and Szabó 2023] for a fast computer program based on the theory). It works by assigning certain dg algebras to sets of n tangle endpoints (oriented up or down) and certain A_∞ bimodules to tangles; one recovers HFK for closed knots by taking appropriate tensor products of these bimodules.

Manion [2019] showed that the dg algebras of bordered HFK categorify representations of the quantum supergroup $U_q(\mathfrak{gl}(1|1))$ and that the tangle bimodules categorify intertwining maps between these representations. While Manion [2019]

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did not consider a categorified action of the quantum group on the bordered HFK algebras, such an action (for Khovanov's categorification $\mathcal{U}$ [2014] of the positive half $U_q(\mathfrak{gl}(1|1)^+) = \mathbb{C}(q)[E]/(E^2)$) was defined in [Lauda and Manion 2021], compatibly (via [Lekili and Polishchuk 2020; Manion et al. 2020]) with a more general family of higher actions defined in [Manion and Rouquier 2020].

Since Ozsváth and Szabó's tangle bimodules categorify intertwining maps between representations, it is natural to ask whether the bimodules themselves intertwine the higher actions of $\mathcal{U}$ on the bordered HFK algebras. Since a higher action of $\mathcal{U}$ on a dg algebra $\mathcal{A}$ amounts to a dg bimodule $\mathcal{E}$ over $\mathcal{A}$ together with some extra data, one (roughly) asks whether tangle bimodules X satisfy $X \otimes_{\mathcal{A}} \mathcal{E} \cong \mathcal{E} \otimes_{\mathcal{A}} X$. A structured way to require such commutativity is to equip X with the data of a 1-morphism between 2-representations of $\mathcal{U}$.

The main result of this paper is that one can naturally equip Ozsváth and Szabó's local crossing bimodules with this 1-morphism structure.

Theorem 1.1. *Ozsváth and Szabó's local bimodules $\mathcal{P}$ and $\mathcal{N}$, for a positive and negative crossing between two strands, can be equipped with the structure of* 1-*morphisms of* 2-*representations over* $\mathcal{U}$, *encoding the commutativity of $\mathcal{P}$ and $\mathcal{N}$ with the* 2-*action bimodule* $\mathcal{E}$.

In fact, the algebra over which $\mathcal{P}$ and $\mathcal{N}$ are defined has two natural 2-actions of $\mathcal{U}$, and we prove Theorem 1.1 for both 2-actions. Below we comment a bit more on the motivation and potential applications for Theorem 1.1, as well as future directions for study.

Remark 1.2. Theorem 1.1 is an algebraic expression of an important "compatibility between summands" property of the bordered HFK bimodules. Indeed, like the general strands algebras $\mathcal{A}(\mathcal{Z})$ of bordered Heegaard Floer homology, Ozsváth–Szabó's bordered HFK algebras have a direct sum decomposition indexed by $\mathbb{Z}$ (in Heegaard diagram terms this index describes occupancy number, while representation-theoretically it encodes a $\mathfrak{gl}(1|1)$ weight space decomposition). The A_∞ bimodules for tangles respect this decomposition, and there is a certain compatibility between the bimodule summands for different k. In [Ozsváth and Szabó 2018], this compatibility is encoded in a graph from which one can define all summands of the bimodules. Because of how the 2-action bimodules $\mathcal{E}$ interact with the index of the direct sum decomposition, Theorem 1.1 is a more algebraic way to formulate this compatibility.

In [Ozsváth and Szabó 2018], this compatibility is the key ingredient in the "global extension" of the two-strand crossing bimodules to bimodules, over larger algebras, for n strands with one crossing between two adjacent strands (this extension is necessary when using the theory of Ozsváth and Szabó [2018] to compute HFK for knots). The global extension is one of the most technical parts of [Ozsváth and

Szabó 2018]; the main hoped-for application of the results of this paper is a more algebraic treatment of the global extension, based on higher representation theory.

Remark 1.3. The 1-morphism structure of Theorem 1.1 can be interpreted as an instance of an extra layer of the connection between higher representation theory and cornered Heegaard Floer homology, beyond what was explored in [Manion and Rouquier 2020]. This extra layer involves 3-manifolds, not just 1- and 2-manifolds, and begins to relate to the parts of cornered Heegaard Floer homology that use holomorphic disk counts and domains in Heegaard diagrams with corners. Generalizing from Theorem 1.1, there should be a general family of Heegaard diagrams (with the diagrams underlying the bordered HFK bimodules as special cases) whose bimodules can be upgraded to 1-morphisms of 2-representations, and the data needed for this upgrade should come from counting holomorphic disks whose domains have positive multiplicities at the corners of the Heegaard diagram.

Remark 1.4. This paper is focused on the local two-strand aspects of bordered HFK, since these are the elementary building blocks to which one wants to apply a global extension procedure to obtain n-strand tangle invariants. One could also ask whether the globally extended n-strand tangle bimodules of bordered HFK give 1-morphisms of 2-representations of $\mathcal{U}$; we expect this to be true. Furthermore, the local bimodules considered here are adapted to two strands pointing in the same direction (downwards, in the conventions of [Ozsváth and Szabó 2018]). For strands with other orientations, one has a choice of more elaborate theories from [Ozsváth and Szabó 2018; 2019b; 2019a], some involving curved dg algebras. We expect that the bimodules of these more elaborate theories also give 1-morphisms of 2-representations of $\mathcal{U}$, once, e.g., 2-representations are appropriately defined on the curved dg algebras.

Remark 1.5. Since it follows from [Lekili and Polishchuk 2020; Manion et al. 2020] that the local Ozsváth–Szabó algebras appearing in this paper are quasiisomorphic to certain (larger) dg strands algebras $\mathcal{A}(\mathcal{Z})$, it is natural to ask whether there are bimodules corresponding to $\mathcal{P}$ and $\mathcal{N}$ over the larger algebras, and if so, whether these bimodules give 1-morphisms between the 2-representation structures on $\mathcal{A}(\mathcal{Z})$ defined directly in [Manion and Rouquier 2020]. The answer in both cases appears to be "yes;" the authors of [Manion et al. 2020] hope to address this question in work in preparation.

Remark 1.6. Along with E, there is another odd generator F of $U_q(\mathfrak{gl}(1|1))$; since we are discussing actions of E here, it is natural to ask about F as well. While the framework of [Manion and Rouquier 2020] is based on a categorification of $U_q(\mathfrak{gl}(1|1)^+)$ and fundamentally gives us E but not F, one can categorify at least a relative F' of F by taking homomorphisms of left $\mathcal{A}$-modules from the $\mathcal{E}$ bimodule into $\mathcal{A}$ (as discussed e.g., in [Lauda and Manion 2021, Theorem 1.3] with slightly different conventions, as well as in [Manion and Rouquier 2020]). If we take $\mathcal{E}$ to be projective on the left ("type DA") as in this paper, then the bimodule $\mathcal{F}' := \mathrm{Hom}_{\mathcal{A} \text{ on left}}(\mathcal{E}, \mathcal{A})$ will be projective on the right ("type AD"), so since X is type DA and has higher A_∞ actions on the right, it's more natural to look at the bimodules $\mathcal{E} \otimes_{\mathcal{A}} X$ and $X \otimes_{\mathcal{A}} \mathcal{E}$ than the bimodules $\mathcal{F}' \otimes_{\mathcal{A}} X$ and $X \otimes_{\mathcal{A}} \mathcal{F}'$.

If we did define $\mathcal{F}' \otimes_{\mathcal{A}} -$ and $- \otimes_{\mathcal{A}} \mathcal{F}'$ appropriately, then we would expect adjunctions in the homotopy category $(\mathcal{E} \otimes_{\mathcal{A}} -) \dashv (\mathcal{F}' \otimes_{\mathcal{A}} -)$ and $(- \otimes_{\mathcal{A}} \mathcal{F}') \dashv (- \otimes_{\mathcal{A}} \mathcal{E})$. Specifying maps $X \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} X$ and $X \otimes_{\mathcal{A}} \mathcal{F}' \to \mathcal{F}' \otimes_{\mathcal{A}} X$ would be equivalent, up to homotopy, to specifying maps $X \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} X$ and $\mathcal{E} \otimes_{\mathcal{A}} X \to X \otimes_{\mathcal{A}} \mathcal{E}$.

In our case, we will show that $\mathcal{E} \otimes_{\mathcal{A}} X$ and $X \otimes_{\mathcal{A}} \mathcal{E}$ are literally the same up to a renaming of basis elements, so that neither direction is singled out and we have maps both ways giving an isomorphism. Based on the above, after making the right definitions one would get a map $X \otimes_{\mathcal{A}} \mathcal{F}' \to \mathcal{F}' \otimes_{\mathcal{A}} X$ up to homotopy; since we only have an adjunction one way, it's not immediate that this map would be an isomorphism in the homotopy category, although it seems likely that $X \otimes_{\mathcal{A}} \mathcal{F}' \cong \mathcal{F}' \otimes_{\mathcal{A}} X$ is still true here. We will not investigate further, though; work in preparation of the second author at the decategorified level suggests that in some settings, but not the one under consideration, one should legitimately have actions of both odd generators E and F of $\mathfrak{gl}(1|1)$, whereas here we only have E along with whatever modifications we want to do to it algebraically.

Organization. In Section 2 we review algebraic definitions from bordered Heegaard Floer homology, including a matrix-based notation from [Manion 2020] that will be useful here. In Section 3 we review what we need from Ozsváth and Szabó's theory of bordered HFK. In Section 4 we review the relevant input from higher representation theory and define 2-actions of $\mathcal{U}$ on the local bordered HFK algebras. In Section 5 we show that Theorem 1.1 holds for Ozsváth–Szabó's local positive-crossing bimodule $\mathcal{P}$, and in Section 6 we do the same for the local negative-crossing bimodule $\mathcal{N}$.

2. Bordered algebra

2A. *DA bimodules.* We will work with DA bimodules, as defined by Lipshitz, Ozsváth and Thurston [Lipshitz et al. 2015, Section 2.2.4], over associative algebras with no differentials. We will assume that these associative algebras $\mathcal{A}$ are defined over a field k of characteristic 2 and come equipped with a finite collection of orthogonal idempotents $\{I_1, \dots, I_n\}$ such that $I_1 + \cdots + I_n = 1$. We will refer to the I_j as distinguished idempotents.

Remark 2.1. An equivalent perspective is to view $\mathcal{A}$ as a k-linear category with objects $\{I_1, \dots, I_n\}$.

For such an algebra $\mathcal{A}$, we will let $\mathcal{I}_{\mathcal{A}}$ denote the ring of idempotents of $\mathcal{A}$, i.e., a finite direct product of copies of k (one for each idempotent I_j), viewed as a subalgebra of $\mathcal{A}$.

We will also assume that $\mathcal{A}$ is equipped with two $\mathbb{Z}$-gradings which we will call the intrinsic and homological gradings; we let $[1]$ denote an upward shift by 1 in the homological grading (we use upward rather than downward shifts because, following the conventions of [Lipshitz et al. 2015; Ozsváth and Szabó 2018], we use differentials that decrease the homological grading by 1).

Definition 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be graded associative algebras over a field k of characteristic 2. A DA bimodule over $(\mathcal{A}, \mathcal{B})$ is given by the data $(X, (\delta_i^1)_{i=1}^{\infty})$ where X is a $\mathbb{Z} \oplus \mathbb{Z}$-graded bimodule over $(\mathcal{I}_{\mathcal{A}}, \mathcal{I}_{\mathcal{B}})$ and, for $i \geq 1$,

$$\delta_i^1 : X \otimes \mathcal{B}[1]^{\otimes(i-1)} \to \mathcal{A}[1] \otimes X$$

(tensor products are over $\mathcal{I}_{\mathcal{A}}$ or $\mathcal{I}_{\mathcal{B}}$ as appropriate) is a bidegree-preserving morphism of bimodules over $(\mathcal{I}_{\mathcal{A}}, \mathcal{I}_{\mathcal{B}})$ such that the DA bimodule relations are satisfied, i.e., such that

$$\sum_{j_1+j_2=i+1} (\mu_{\mathcal{A}} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{\mathcal{A}} \otimes \delta_{j_1}^1) \circ (\delta_{j_2}^1 \otimes \mathrm{id}_{\mathcal{B}^{\otimes(j_1-1)}}) + \sum_{j=1}^{i-2} \delta_{i-1}^1 \circ (\mathrm{id}_{\mathcal{B}^{\otimes(j-1)}} \otimes \mu_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{B}^{\otimes(i-j-2)}}) = 0$$

for all $i \geq 1$, where $\mu_{\mathcal{A}}$ and $\mu_{\mathcal{B}}$ are the multiplication operations on $\mathcal{A}$ and $\mathcal{B}$.

We will often refer to $(X, (\delta_i^1)_{i=1}^{\infty})$ simply as X. We say that X is strictly unital if $\delta_2^1(x, 1) = 1 \otimes x$ for all $x \in X$ and $\delta_i^1(x, b_1, \ldots, b_{i-1}) = 0$ if $i > 2$ and any b_j is in the idempotent ring $\mathcal{I}_{\mathcal{B}}$.

If we have a k-basis for X and x, x' are basis elements with $a \otimes x'$ appearing as a nonzero term of $\delta_i^1(x \otimes b_1 \otimes \cdots \otimes b_{i-1})$ (where $a \in \mathcal{A}$ and $b_1, \ldots, b_{i-1} \in \mathcal{B}$), we will sometimes depict the situation using a "DA module operation graph" as in [Lipshitz et al. 2015, Definition 2.2.45]. See Figure 1 for an example. In this notation, the DA bimodule relations are shown in Figure 2.

Remark 2.3. For all DA bimodules $(X, (\delta_i^1)_{i=1}^{\infty})$ considered in this paper, X will be finite-dimensional over k, as well as left and right bounded in the sense of [Lipshitz et al. 2015, Definition 2.2.46].

Remark 2.4. If X is a DA bimodule over $(\mathcal{A}, \mathcal{B})$, then $\mathcal{A} \otimes_{\mathcal{I}_{\mathcal{A}}} X$ is an A_∞ bimodule over $(\mathcal{A}, \mathcal{B})$ such that the left action of $\mathcal{A}$ has no higher A_∞ terms and such that, as a left $\mathcal{A}$-module, X is a direct sum of projective modules $\mathcal{A} \cdot I$ for distinguished idempotents I of $\mathcal{A}$ (disregarding the differential). One can think of the definition of DA bimodule as a convenient way of specifying and reasoning about such A_∞ bimodules.

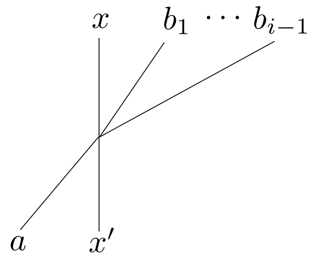

Figure 1. A DA module operation graph showing a term $a \otimes x'$ of the action of δ^1_i on $x \otimes b_1 \otimes \cdots \otimes b_{i-1}$; this notation allows $i = 1$ in which case there are no edges to the right of the vertical line from x to x'.

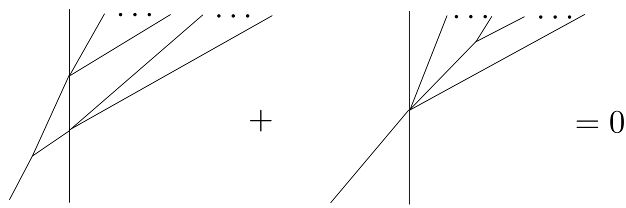

Figure 2. The DA bimodule relations in Definition 2.2.

2B. ***The box tensor product.*** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be associative algebras as in Section 2A and let X and Y be DA bimodules over $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{C})$ respectively. Assuming X is left bounded or Y is right bounded, Lipshitz–Ozsváth–Thurston define a DA bimodule $X \boxtimes Y$ in [Lipshitz et al. 2015, Section 2.3.2].

Definition 2.5. As a bimodule over $(\mathcal{I}_{\mathcal{A}}, \mathcal{I}_{\mathcal{C}})$, $X \boxtimes Y$ is defined to be $X \otimes_{\mathcal{I}_{\mathcal{B}}} Y$. For $i \geq 1$, the DA bimodule operation $\delta^{\boxtimes,1}_i$ on $X \boxtimes Y$ is defined in terms of the operations $\delta^{X,1}_*$ on X and $\delta^{Y,1}_*$ on Y by

$$\begin{aligned}\delta^{\boxtimes,1}_i = \sum_{j \geq 0} \sum_{i_1+\cdots+i_j = i+j-1} (\delta^{X,1}_j \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \mathrm{id}_{\mathcal{B}^{\otimes(j-1)}} \otimes \delta^{Y,1}_{i_j}) \\ \circ (\mathrm{id}_X \otimes \mathrm{id}_{\mathcal{B}^{\otimes(j-2)}} \otimes \delta^{Y,1}_{i_{j-1}} \otimes \mathrm{id}_{\mathcal{A}^{\otimes(i_j-1)}}) \\ \circ \cdots \circ (\mathrm{id}_X \otimes \delta^{Y,1}_{i_1} \otimes \mathrm{id}_{\mathcal{A}^{\otimes(i_2+\cdots+i_j-j+1)}}).\end{aligned}$$

In terms of DA module operation graphs, the general pattern for the operation $\delta^{\boxtimes,1}_i$ on $X \boxtimes Y$ is shown in Figure 3.

Remark 2.6. By [Lipshitz et al. 2015, Proposition 2.3.10], if X and Y are both left bounded then so is $X \boxtimes Y$.

Remark 2.7. Assuming suitable boundedness, the box tensor product $X \boxtimes Y$ is a convenient way of working with the derived tensor product $(\mathcal{A} \otimes_{\mathcal{I}_{\mathcal{A}}} X) \tilde{\otimes}_{\mathcal{B}} (\mathcal{B} \otimes_{\mathcal{I}_{\mathcal{B}}} Y)$;

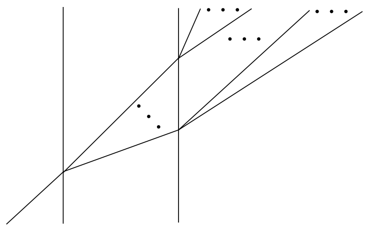

Figure 3. The general pattern for the operation $\delta_i^{\boxtimes,1}$ on $X \boxtimes Y$.

indeed, by [Lipshitz et al. 2015, Proposition 2.3.18] we have

$$\mathcal{A} \otimes_{\mathcal{I}_{\mathcal{A}}} (X \boxtimes Y) \simeq (\mathcal{A} \otimes_{\mathcal{I}_{\mathcal{A}}} X) \tilde{\otimes}_{\mathcal{B}} (\mathcal{B} \otimes_{\mathcal{I}_{\mathcal{B}}} Y)$$

where $\simeq$ denotes homotopy equivalence of DA bimodules; see [Lipshitz et al. 2015, Section 2.2.4].

2C. *Matrix notation.* We will describe DA bimodules using the matrix-based notation of [Manion 2020, Section 2.2]; we recall this notation here. When using this notation to describe a DA bimodule over $(\mathcal{A}, \mathcal{B})$, it is assumed that $\mathcal{B}$ comes equipped with a choice of k-basis such that:

- Distinguished idempotents of $\mathcal{B}$ are basis elements.
- Each basis element b satisfies $I \cdot b \cdot I' = b$ for unique distinguished idempotents I of $\mathcal{A}$ and I' of $\mathcal{B}$ (called the left and right idempotents of b respectively) with $\tilde{I} \cdot b \cdot \tilde{I}' = 0$ whenever $\tilde{I}, \tilde{I}'$ are distinguished idempotents of $\mathcal{A}$ and $\mathcal{B}$ with $\tilde{I} \neq I$ or $\tilde{I}' \neq I'$.
- Each basis element of $\mathcal{B}$ is homogeneous with respect to the bigrading.

Definition 2.8. To specify a DA bimodule $(X, (\delta_i^1)_{i=1}^{\infty})$ over $(\mathcal{A}, \mathcal{B})$ (finite-dimensional over k), we specify two matrices, a *primary matrix* and a *secondary matrix*:

- The primary matrix is a set-valued matrix (each entry is a finite set with a $\mathbb{Z} \oplus \mathbb{Z}$-bidegree specified for each element) with columns indexed by the distinguished idempotents of $\mathcal{B}$ and rows indexed by the distinguished idempotents of $\mathcal{A}$. Given such a matrix, the bimodule X over $(\mathcal{I}_{\mathcal{A}}, \mathcal{I}_{\mathcal{B}})$ is taken to have a k-basis given by the union of the sets in each entry (with each basis element given its specified bidegree). More specifically, the left-action of $\mathcal{I}_{\mathcal{A}}$ and right-action of $\mathcal{I}_{\mathcal{B}}$ are fixed by saying that, for distinguished idempotents I of $\mathcal{A}$ and I' of $\mathcal{B}$, the vector space $I \cdot X \cdot I'$ has a basis given by the set in row I and column I'. For an element x of this set, we say that I is the left idempotent of x and I' is the right idempotent of x.

- The secondary matrix is a matrix whose entries are formal sums of expressions a (for $a \in \mathcal{A}$) and $a \otimes (b_1, \ldots, b_{i-1})$ (for $a \in \mathcal{A}$ and each b_j a basis element for $\mathcal{B}$). The sums are allowed to be infinite, but there should be finitely many terms of the form a (without the $\otimes$ symbol) and finitely many terms for each given sequence $(b_1, \ldots, b_{i-1})$. The rows and columns of the secondary matrix are each indexed by the union of all entries of the primary matrix, in some fixed order. Given such a matrix, the operations δ^1_i on X are defined as follows for a basis element x of X (a column label of the secondary matrix):
  - $\delta^1_1(x)$ is the sum of all elements $a \otimes y$ where a is a term (without the $\otimes$ symbol) of a secondary matrix entry in column x and y is the row label of the entry containing this term.
  - For $i > 1$ and a sequence $(b_1, \ldots, b_{i-1})$ of basis elements of $\mathcal{B}$, $\delta^1_i(x \otimes b_1 \otimes \cdots \otimes b_{i-1})$ is the sum of all elements $a \otimes y$ where $a \otimes (b_1, \ldots, b_{i-1})$ is a term of a secondary matrix entry in column x and y is the row label of the entry containing this term.

An example of a DA bimodule specified by primary and secondary matrices can be found in Definition 3.3 below. We use the following conventions:

Convention 2.9. If indices such as k or l appear in entries of the secondary matrix, we take an infinite sum over all $k \geq 0$ or $l \geq 0$ unless otherwise specified.

Convention 2.10. When using matrix notation to specify a strictly unital DA bimodule, the above rules would say that in each diagonal entry of the secondary matrix (corresponding to an entry x of the primary matrix), there is a term $I \otimes I'$ where I and I' are the left and right idempotents of x respectively (it should also be the case that no basis element b_j appearing in an entry $a \otimes (b_1, \ldots, b_{i-1})$ is a distinguished idempotent). However, we will omit the terms $I \otimes I'$ when we write the secondary matrix.

If the primary or secondary matrix has block form, we will often give each block separately.

Remark 2.11. One advantage of this matrix-based notation is that the DA bimodule relations can be checked using linear-algebraic manipulations. Indeed, to check the DA bimodule relations, one forms two new matrices from the secondary matrix. The first matrix, which we will call the "squared secondary matrix," is obtained by multiplying the secondary matrix by itself. When doing so, one will need to take products of secondary matrix entries; these products are defined by:

- $a \cdot a' = a'a$.
- $a \cdot (a' \otimes (b'_1, \ldots, b'_{i-1})) = a'a \otimes (b'_1, \ldots, b'_{i-1})$.
- $(a \otimes (b_1, \ldots, b_{i-1})) \cdot a' = a'a \otimes (b_1, \ldots, b_{i-1})$.
- $(a \otimes (b_1, \ldots, b_{i-1})) \cdot (a' \otimes (b'_1, \ldots, b'_{j-1})) = a'a \otimes (b'_1, \ldots, b'_{j-1}, b_1, \ldots, b_{i-1})$.

The second matrix, which we will call the "multiplication matrix," is obtained by, for each b_j in an entry $a \otimes (b_1, \dots, b_{i-1})$ and each pair of $\mathcal{B}$-basis elements (b', b'') (neither a distinguished idempotent in the strictly unital case) such that Cb_j is a term of the basis expansion of $b'b''$ for some nonzero element $C \in k$, adding the term $Ca \otimes (b_1, \dots, b_{j-1}, b', b'', b_{j+1}, \dots, b_{i-1})$ to the corresponding entry of the multiplication matrix.

Once these two matrices are formed, the DA bimodule relations amount to saying that the squared secondary matrix and the multiplication matrix sum to zero.

2D. *Box tensor products in matrix notation.* Suppose we have DA bimodules X over $(\mathcal{A}, \mathcal{B})$ and Y over $(\mathcal{B}, \mathcal{C})$ as in Section 2B. To specify $X \boxtimes Y$ in matrix notation, one can do the following manipulations:

- The primary matrix for $X \boxtimes Y$ is the matrix product of the primary matrix for X (on the left) and the primary matrix for Y (on the right). When multiplying two entries of these primary matrices, one uses the Cartesian product of sets, and when adding these products together, one uses the disjoint union.
- Let (x, y) and (x', y') be two elements of the primary matrix for $X \boxtimes Y$. To obtain the secondary matrix element in row (x', y') and column (x, y), there are two cases to consider:
  - For entries a (with no $\otimes$ symbol) in row x' and column x of the secondary matrix for X, if $y = y'$ then add an entry a to the secondary matrix for $X \boxtimes Y$ in row (x', y') and column (x, y). If $y \neq y'$, do not add such an entry.
  - For entries $a \otimes (b_1, \dots, b_{i-1})$ in row x' and column x of the secondary matrix for X, look for all sequences $(y = y_1, y_2, \dots, y_i = y')$ of primary matrix entries for Y such that, for $1 \leq j \leq i-1$, there is a term $b \otimes (c_1^j, \dots, c_{m_j-1}^j)$ in row y_{j+1} and column y_j of the secondary matrix for Y such that $C_j b_j$ is a term of the basis expansion of b for some nonzero $C_j \in k$. For all such sequences $(y_1, \dots, y_i)$ and all such choices of terms $b \otimes (c_1^j, \dots, c_{m_j-1}^j)$, add an entry
  $$C_1 \cdots C_{i-1} a \otimes (c_1^1, \dots, c_{m_1-1}^1, \dots, c_1^{i-1}, \dots, c_{m_{i-1}-1}^{i-1})$$
  to the secondary matrix of $X \boxtimes Y$ in row (x', y') and column (x, y).

3. Bordered HFK

3A. *Algebras.* We now review Ozsváth and Szabó's algebra $\mathcal{B}(2) = \bigoplus_{k=0}^{3} \mathcal{B}(2, k)$ from [Ozsváth and Szabó 2018, Section 3.2], which is an algebra over $\mathbb{F}_2$.

Definition 3.1. The algebra $\mathcal{B}(2, 0)$ is $\mathbb{F}_2$. The algebra $\mathcal{B}(2, 1)$ is the path algebra of the quiver shown in Figure 4 modulo the relations $[R_i, U_j] = 0$, $[L_i, U_j] = 0$,

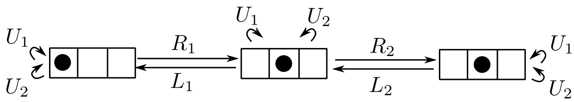

Figure 4. The quiver for $\mathcal{B}(2, 1)$.

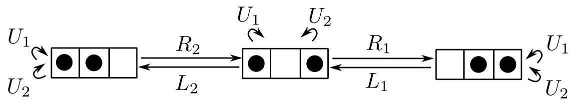

Figure 5. The quiver for $\mathcal{B}(2, 2)$.

$R_i L_i = U_i$, $L_i R_i = U_i$, $R_1 R_2 = 0$, $L_2 L_1 = 0$, $U_2 = 0$ at the leftmost node, and $U_1 = 0$ at the rightmost node.

The algebra $\mathcal{B}(2, 2)$ is the path algebra of the quiver shown in Figure 5 modulo the relations $[R_i U_j] = 0$, $[L_i, U_j] = 0$, $R_i L_i = U_i$, and $L_i R_i = U_i$. The algebra $\mathcal{B}(2, 3)$ is $\mathbb{F}_2[U_1, U_2]$. We set $\mathcal{B}(2) = \bigoplus_{k=0}^{3} \mathcal{B}(2, k)$.

Our definition matches Ozsváth and Szabó's by [Manion et al. 2021, Theorem 1.1]; also see [Ozsváth and Szabó 2018, Figure 10] for $\mathcal{B}(2, 1)$, although in this figure Ozsváth and Szabó leave out some of the relations. We define an intrinsic grading on $\mathcal{B}(2)$ by setting $\deg(R_i) = \deg(L_i) = 1$ and $\deg(U_i) = 2$; this grading is twice Ozsváth and Szabó's single Alexander grading (the doubling is related to the expression $t = q^2$ when obtaining the Alexander polynomial from representations of $U_q(\mathfrak{gl}(1|1))$). We define the homological grading to be identically zero on the generators of $\mathcal{B}(2)$.

The algebras $\mathcal{B}(2, 1)$ and $\mathcal{B}(2, 2)$ each have three distinguished idempotents given by the length-zero paths at each node. Ordering the nodes from left to right and following Ozsváth and Szabó's notation, for $\mathcal{B}(2, 1)$ we can call these idempotents I_0, I_1, and I_2. For $\mathcal{B}(2, 2)$ we can call them I_{01}, I_{02}, and I_{12}. The unique nonzero element of $\mathcal{B}(2, 0)$ is its distinguished idempotent and we can call it $I_\varnothing$; for $\mathcal{B}(2, 3)$ the distinguished idempotent is $1 \in \mathbb{F}_2[U_1, U_2]$ and we can call it I_{012}.

To avoid subscripts as much as possible, we will relabel these idempotents as follows:

$$\varnothing := I_\varnothing,$$
$$A := I_0, \quad B := I_1, \quad C := I_2,$$
$$AB := I_{01}, \quad AC := I_{02}, \quad BC := I_{12},$$
$$ABC := I_{012}.$$

To clarify the conventions: in Figure 4 the left and right idempotents of R_1 are A and B respectively, while in Figure 5 the left and right idempotents of R_1 are AC and BC respectively.

The following proposition can be deduced from the definition of $\mathcal{B}(2)$.

Proposition 3.2. *A $\mathbb{F}_2$-basis for $\mathcal{B}(2,1)$ is given by*

$$\{U_1^k(A), U_1^k(B), U_2^k(B), U_2^k(C), R_1U_1^k, L_1U_1^k, R_2U_2^k, L_2U_2^k\}$$

(k *runs over all integers* ≥ 0). *A $\mathbb{F}_2$-basis for $\mathcal{B}(2,2)$ is given by*

$$\{U_1^kU_2^l(AB), U_1^kU_2^l(AC), U_1^kU_2^l(BC), R_1U_1^kU_2^l, L_1U_1^kU_2^l, R_2U_1^kU_2^l, L_2U_1^kU_2^l\}$$

(k *and* l *run over all integers* ≥ 0).

The algebra $\mathcal{B}(2,0) = \mathbb{F}_2$ has a unique $\mathbb{F}_2$-basis, and for $\mathcal{B}(2,3)$ we use the basis of monomials $U_1^kU_2^l$ for $k, l \geq 0$.

3B. ***Bimodules.*** Next we review, in matrix notation, Ozsváth and Szabó's DA bimodules $\mathcal{P}$ and $\mathcal{N}$ over $\mathcal{B}(2)$. One thinks of these bimodules as being associated to two-strand tangles consisting of a single positive crossing and a single negative crossing respectively and containing the minimal amount of data necessary to build the bimodules for n-strand single-crossing tangles. They can be obtained by counting holomorphic disks in the Heegaard diagrams shown in Section 3B3 below.

3B1. *The bimodule $\mathcal{P}$.* This bimodule is defined in [Ozsváth and Szabó 2018, Section 5.1]; here we translate Ozsváth and Szabó's definition into matrix notation.

Definition 3.3. The primary matrix for $\mathcal{P}$ has rows and columns indexed by the distinguished idempotents

$$\varnothing, A, B, C, AB, AC, BC, ABC$$

of $\mathcal{B}(2)$. The matrix has block-diagonal form with blocks specified by the following matrices:

$$\begin{array}{c|c} & \varnothing \\ \hline \varnothing & \begin{bmatrix} {}_{\varnothing}S_{\varnothing} \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} A & \quad B & \quad C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} \{{}_AS_A\} & \varnothing & \varnothing \\ \{{}_BW_A\} & \{{}_BN_B\} & \{{}_BE_C\} \\ \varnothing & \varnothing & \{{}_CS_C\} \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} AB & \quad AC & \quad BC \end{matrix} \\ \begin{matrix} AB \\ AC \\ BC \end{matrix} & \begin{bmatrix} \{{}_{AB}N_{AB}\} & \{{}_{AB}E_{AC}\} & \varnothing \\ \varnothing & \{{}_{AC}S_{AC}\} & \varnothing \\ \varnothing & \{{}_{BC}W_{AC}\} & \{{}_{BC}N_{BC}\} \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & ABC \\ ABC & \begin{bmatrix} \{{}_{ABC}N_{ABC}\} \end{bmatrix} \end{array}$$

Below we will abuse notation slightly and omit the braces $\{\}$, writing e.g., ${}_AS_A$ instead of $\{{}_AS_A\}$. The secondary matrix for $\mathcal{P}$ has a corresponding block-diagonal form; the blocks are:

$$\begin{array}{cc} & {}_{\varnothing}S_{\varnothing} \\ {}_{\varnothing}S_{\varnothing} & \left[\begin{array}{c} 0 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccccc} {}_AS_A & {}_BW_A & {}_BN_B & {}_BE_C & {}_CS_C \end{array} \\ \begin{array}{c} {}_AS_A \\ {}_BW_A \\ {}_BN_B \\ {}_BE_C \\ {}_CS_C \end{array} & \left[\begin{array}{ccccc} 0 & L_1 & 0 & 0 & 0 \\ 0 & U_2^{k+1}\otimes U_1^{k+1} & U_2^{k+1}\otimes L_1U_1^k & 0 & L_2U_2^k\otimes(L_2, L_1U_1^k) \\ R_1U_1^k\otimes(R_1, U_2^{k+1}) & U_2^k\otimes R_1U_1^k & U_2^{k+1}\otimes U_1^{k+1}+U_1^{k+1}\otimes U_2^{k+1} & U_1^k\otimes L_2U_2^k & L_2U_2^k\otimes(L_2, U_1^{k+1}) \\ R_1U_1^k\otimes(R_1, R_2U_2^k) & 0 & U_1^{k+1}\otimes R_2U_2^k & U_1^{k+1}\otimes U_2^{k+1} & 0 \\ 0 & 0 & 0 & R_2 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccccc} {}_{AB}N_{AB} & {}_{AB}E_{AC} & {}_{AC}S_{AC} & {}_{BC}W_{AC} & {}_{BC}N_{BC} \end{array} \\ \begin{array}{c} {}_{AB}N_{AB} \\ {}_{AB}E_{AC} \\ {}_{AC}S_{AC} \\ {}_{BC}W_{AC} \\ {}_{BC}N_{BC} \end{array} & \left[\begin{array}{ccccc} U_1^lU_2^k\otimes U_1^kU_2^l & U_1^k\otimes L_2U_2^k & *_1 & L_1L_2U_2^k\otimes L_2U_1^{k+1} & L_1L_2U_1^lU_2^k\otimes L_1L_2U_1^kU_2^l \\ U_1^{l+1}U_2^k\otimes R_2U_1^kU_2^l & U_1^{k+1}\otimes U_2^{k+1} & *_2 & L_1L_2U_2^k\otimes U_1^{k+1} & L_1L_2U_1^lU_2^k\otimes L_1U_1^kU_2^l \\ 0 & R_2 & 0 & L_1 & 0 \\ R_2R_1U_1^lU_2^k\otimes R_2U_1^kU_2^l & R_2R_1U_1^k\otimes U_2^{k+1} & *_3 & U_2^{k+1}\otimes U_1^{k+1} & U_1^lU_2^{k+1}\otimes L_1U_1^kU_2^l \\ R_2R_1U_1^lU_2^k\otimes R_2R_1U_1^kU_2^l & R_2R_1U_1^k\otimes R_1U_2^{k+1} & *_4 & U_2^k\otimes R_1U_1^k & U_1^lU_2^k\otimes U_1^kU_2^l \end{array}\right] \end{array}$$

$$\begin{array}{cc} & {}_{ABC}N_{ABC} \\ {}_{ABC}N_{ABC} & \left[\begin{array}{c} U_1^lU_2^k\otimes U_1^kU_2^l \end{array}\right]. \end{array}$$

The entries $*_i$ for $1 \le i \le 4$ are specified below; also, in any entry of the form $U_1^lU_2^k \otimes U_1^kU_2^l$, we disallow $(k, l) = (0, 0)$ to match Convention 2.10. The entry $*_1$ in column ${}_{AC}S_{AC}$ and row ${}_{AB}N_{AB}$ is

$$\begin{aligned} & L_2U_1^tU_2^n \otimes (U_1^{n+1}, L_2U_2^t) && (0 \le n < t) \\ &\quad + L_2U_1^tU_2^n \otimes (R_1U_1^n, L_1L_2U_2^t) && (0 \le n < t) \\ &\quad + L_2U_1^tU_2^n \otimes (L_2U_1^{n+1}, U_2^t) && (0 \le n < t) \\ &\quad + L_2U_1^tU_2^n \otimes (L_2U_2^t, U_1^{n+1}) && (1 \le t \le n) \\ &\quad + L_2U_1^tU_2^n \otimes (U_2^t, L_2U_1^{n+1}) && (1 \le t \le n) \\ &\quad + L_2U_1^tU_2^n \otimes (R_1U_2^t, L_1L_2U_1^n) && (1 \le t \le n) \\ &\quad + L_2U_2^n \otimes (L_2, U_1^{n+1}) && (0 \le n). \end{aligned}$$

The entry $*_2$ in column ${}_{AC}S_{AC}$ and row ${}_{AB}E_{AC}$ is

$$\begin{aligned} & L_2U_1^tU_2^n \otimes (U_1^{n+1}, U_2^t) && (0 \le n < t) \\ &\quad + L_2U_1^tU_2^n \otimes (R_1U_1^n, L_1U_2^t) && (0 \le n < t) \\ &\quad + L_2U_1^tU_2^n \otimes (L_2U_1^{n+1}, R_2U_2^{t-1}) && (0 \le n < t) \\ &\quad + L_2U_1^tU_2^n \otimes (U_2^t, U_1^{n+1}) && (1 \le t \le n) \\ &\quad + L_2U_1^tU_2^n \otimes (R_1U_2^t, L_1U_1^n) && (1 \le t \le n) \\ &\quad + L_2U_1^tU_2^n \otimes (L_2U_2^{t-1}, R_2U_1^{n+1}) && (1 \le t \le n). \end{aligned}$$

The entry $*_3$ in column ${}_{AC}S_{AC}$ and row ${}_{BC}W_{AC}$ is

$$\begin{aligned}
&R_1U_1^tU_2^n \otimes (U_2^{t+1}, U_1^n) && (0 \le t < n)\\
&\quad + R_1U_1^tU_2^n \otimes (L_2U_2^t, R_2U_1^n) && (0 \le t < n)\\
&\quad + R_1U_1^tU_2^n \otimes (R_1U_2^{t+1}, L_1U_1^{n-1}) && (0 \le t < n)\\
&\quad + R_1U_1^tU_2^n \otimes (U_1^n, U_2^{t+1}) && (1 \le n \le t)\\
&\quad + R_1U_1^tU_2^n \otimes (L_2U_1^n, R_2U_2^t) && (1 \le n \le t)\\
&\quad + R_1U_1^tU_2^n \otimes (R_1U_1^{n-1}, L_1U_2^{t+1}) && (1 \le n \le t)
\end{aligned}$$

The entry $*_4$ in column ${}_{AC}S_{AC}$ and row ${}_{BC}N_{BC}$ is

$$\begin{aligned}
&R_1U_1^tU_2^n \otimes (U_2^{t+1}, R_1U_1^n) && (0 \le t < n)\\
&\quad + R_1U_1^tU_2^n \otimes (L_2U_2^t, R_2R_1U_1^n) && (0 \le t < n)\\
&\quad + R_1U_1^tU_2^n \otimes (R_1U_2^{t+1}, U_1^n) && (0 \le t < n)\\
&\quad + R_1U_1^tU_2^n \otimes (R_1U_1^n, U_2^{t+1}) && (1 \le n \le t)\\
&\quad + R_1U_1^tU_2^n \otimes (U_1^n, R_1U_2^{t+1}) && (1 \le n \le t)\\
&\quad + R_1U_1^tU_2^n \otimes (L_2U_1^n, R_2R_1U_2^t) && (1 \le n \le t)\\
&\quad + R_1U_1^t \otimes (R_1, U_2^{t+1}) && (0 \le t).
\end{aligned}$$

3B2. *The bimodule $\mathcal{N}$.* The bimodule $\mathcal{N}$ is defined in [Ozsváth and Szabó 2018, Section 5.5] using a symmetry relationship with $\mathcal{P}$. Explicitly, $\mathcal{N}$ has the same primary matrix as $\mathcal{P}$. The blocks of the secondary matrix of $\mathcal{N}$ are:

$$\begin{array}{cc}
 & {}_{\varnothing}S_{\varnothing}\\
{}_{\varnothing}S_{\varnothing} & \left[\begin{array}{c} 0 \end{array}\right]
\end{array}$$

$$\begin{array}{cc}
 & \begin{array}{ccccc} {}_AS_A & {}_BW_A & {}_BN_B & {}_BE_C & {}_CS_C \end{array}\\
\begin{array}{c} {}_AS_A\\ {}_BW_A\\ {}_BN_B\\ {}_BE_C\\ {}_CS_C \end{array} &
\left[\begin{array}{ccccc}
0 & 0 & L_1U_1^k\otimes(U_2^{k+1}, L_1) & L_1U_1^k\otimes(L_2U_2^k, L_1) & 0\\
R_1 & U_2^{k+1}\otimes U_1^{k+1} & U_2^k\otimes L_1U_1^k & 0 & 0\\
0 & U_2^{k+1}\otimes R_1U_1^k & U_2^{k+1}\otimes U_1^{k+1}+U_1^{k+1}\otimes U_2^{k+1} & U_1^{k+1}\otimes L_2U_2^k & 0\\
0 & 0 & U_1^k\otimes R_2U_2^k & U_1^{k+1}\otimes U_2^{k+1} & L_2\\
0 & R_2U_2^k\otimes(R_1U_1^k, R_2) & R_2U_2^k\otimes(U_1^{k+1}, R_2) & 0 & 0
\end{array}\right]
\end{array}$$

$$\begin{array}{cc}
 & \begin{array}{ccccc} {}_{AB}N_{AB} & {}_{AB}E_{AC} & {}_{AC}S_{AC} & {}_{BC}W_{AC} & {}_{BC}N_{BC} \end{array}\\
\begin{array}{c} {}_{AB}N_{AB}\\ {}_{AB}E_{AC}\\ {}_{AC}S_{AC}\\ {}_{BC}W_{AC}\\ {}_{BC}N_{BC} \end{array} &
\left[\begin{array}{ccccc}
U_1^lU_2^k\otimes U_1^kU_2^l & U_1^{l+1}U_2^k\otimes L_2U_1^kU_2^l & 0 & L_1L_2U_1^lU_2^k\otimes L_2U_1^kU_2^l & L_1L_2U_1^lU_2^k\otimes L_1L_2U_1^kU_2^l\\
U_1^k\otimes R_2U_2^k & U_1^{k+1}\otimes U_2^{k+1} & L_2 & L_1L_2U_1^k\otimes U_2^{k+1} & L_1L_2U_1^k\otimes L_1U_2^{k+1}\\
*_1' & *_2' & 0 & *_3' & *_4'\\
R_2R_1U_2^k\otimes R_2U_1^{k+1} & R_2R_1U_2^k\otimes U_1^{k+1} & R_1 & U_2^{k+1}\otimes U_1^{k+1} & U_2^k\otimes L_1U_1^k\\
R_2R_1U_1^lU_2^k\otimes R_2R_1U_1^kU_2^l & R_2R_1U_1^lU_2^k\otimes R_1U_1^kU_2^l & 0 & U_1^lU_2^{k+1}\otimes R_1U_1^kU_2^l & U_1^lU_2^k\otimes U_1^kU_2^l
\end{array}\right]
\end{array}$$

$$\begin{array}{cc}
 & {}_{ABC}N_{ABC}\\
{}_{ABC}N_{ABC} & \left[\begin{array}{c} U_1^lU_2^k\otimes U_1^kU_2^l \end{array}\right]
\end{array}$$

where in any entry of the specific form $U_1^l U_2^k \otimes U_1^k U_2^l$ we disallow $(k,l)=(0,0)$ to match Convention 2.10. The entry $*_1'$ in column ${}_{AB}N_{AB}$ and row ${}_{AC}S_{AC}$ is:

$$\begin{aligned} & R_2U_1^tU_2^n \otimes (R_2U_2^t, U_1^{n+1}) && (0 \le n < t) \\ &\quad + R_2U_1^tU_2^n \otimes (R_2R_1U_2^t, L_1U_1^n) && (0 \le n < t) \\ &\quad + R_2U_1^tU_2^n \otimes (U_2^t, R_2U_1^{n+1}) && (0 \le n < t) \\ &\quad + R_2U_1^tU_2^n \otimes (U_1^{n+1}, R_2U_2^t) && (1 \le t \le n) \\ &\quad + R_2U_1^tU_2^n \otimes (R_2U_1^{n+1}, U_2^t) && (1 \le t \le n) \\ &\quad + R_2U_1^tU_2^n \otimes (R_2R_1U_1^n, L_1U_2^t) && (1 \le t \le n) \\ &\quad + R_2U_2^n \otimes (U_1^{n+1}, R_2) && (0 \le n). \end{aligned}$$

The entry $*_2'$ in column ${}_{AB}E_{AC}$ and row ${}_{AC}S_{AC}$ is

$$\begin{aligned} & R_2U_1^tU_2^n \otimes (U_2^t, U_1^{n+1}) && (0 \le n < t) \\ & R_2U_1^tU_2^n \otimes (R_1U_2^t, L_1U_1^n) && (0 \le n < t) \\ & R_2U_1^tU_2^n \otimes (L_2U_2^{t-1}, R_2U_1^{n+1}) && (0 \le n < t) \\ & R_2U_1^tU_2^n \otimes (U_1^{n+1}, U_2^t) && (1 \le t \le n) \\ & R_2U_1^tU_2^n \otimes (R_1U_1^n, L_1U_2^t) && (1 \le t \le n) \\ & R_2U_1^tU_2^n \otimes (L_2U_1^{n+1}, R_2U_2^{t-1}) && (1 \le t \le n). \end{aligned}$$

The entry $*_3'$ in column ${}_{BC}W_{AC}$ and row ${}_{AC}S_{AC}$ is

$$\begin{aligned} & L_1U_1^tU_2^n \otimes (U_1^n, U_2^{t+1}) && (0 \le t < n) \\ & L_1U_1^tU_2^n \otimes (L_2U_1^n R_2U_2^t) && (0 \le t < n) \\ & L_1U_1^tU_2^n \otimes (R_1U_1^{n-1}, L_1U_2^{t+1}) && (0 \le t < n) \\ & L_1U_1^tU_2^n \otimes (U_2^{t+1}, U_1^n) && (1 \le n \le t) \\ & L_1U_1^tU_2^n \otimes (L_2U_2^t, R_2U_1^n) && (1 \le n \le t) \\ & L_1U_1^tU_2^n \otimes (R_1U_2^{t+1}, L_1U_1^{n-1}) && (1 \le n \le t). \end{aligned}$$

The entry $*_4'$ in column ${}_{BC}N_{BC}$ and row ${}_{AC}S_{AC}$ is

$$\begin{array}{ll}
L_1U_1^tU_2^n \otimes (L_1U_1^n, U_2^{t+1}) & (0 \le t < n) \\
L_1U_1^tU_2^n \otimes (L_1L_2U_1^n, R_2U_2^t) & (0 \le t < n) \\
L_1U_1^tU_2^n \otimes (U_1^n, L_1U_2^{t+1}) & (0 \le t < n) \\
L_1U_1^tU_2^n \otimes (U_2^{t+1}, L_1U_1^n) & (1 \le n \le t) \\
L_1U_1^tU_2^n \otimes (L_1U_2^{t+1}, U_1^n) & (1 \le n \le t) \\
L_1U_1^tU_2^n \otimes (L_1L_2U_2^t, R_2U_1^n) & (1 \le n \le t) \\
L_1U_1^t \otimes (U_2^{t+1}, L_1) & (0 \le t).
\end{array}$$

The starred terms in row ${}_{AC}S_{AC}$ of middle block of the secondary matrix for $\mathcal{N}$, as well as in the column ${}_{AC}S_{AC}$ of the middle block of the secondary matrix for $\mathcal{P}$, encode the A_∞ terms of the right algebra actions on (the middle summands of) the bimodules; see [Lipshitz et al. 2015, Section 2.2.4] for more context on these A_∞ structures in general.

The symmetry relationship between $\mathcal{P}$ and $\mathcal{N}$ described in [Ozsváth and Szabó 2018, Section 5.5] can be summarized by saying the secondary matrix of $\mathcal{N}$ is obtained from that of $\mathcal{P}$ by performing the following operations:

- Take the transpose of the secondary matrix of $\mathcal{P}$.
- In each entry, replace L_i with R_i and vice versa, while reversing the order of multiplication when relevant (so e.g., L_1L_2 becomes R_2R_1).
- For any entry $a \otimes (b_1, b_2)$, reverse the order of b_1 and b_2.

3B3. *Heegaard diagram origins.* We comment briefly here on the Heegaard diagram origins of the DA bimodules $\mathcal{P}$ and $\mathcal{N}$. Roughly, they can be thought of as DA bimodules associated to the bordered sutured Heegaard diagrams shown in Figure 6 and Figure 7 respectively. A detailed study of the relationship of the algebraically defined bimodules $\mathcal{P}$ and $\mathcal{N}$ to the holomorphic geometry associated with these diagrams can be found in [Ozsváth and Szabó 2019a], although in that paper Ozsváth and Szabó do not use the language of bordered sutured Heegaard Floer homology.

Remark 3.4. The diagrams in Figures 6 and 7 do not satisfy all the hypotheses necessary to be covered by Lipshitz, Ozsváth and Thurston's results [2015] or Zarev's results [2011]; Ozsváth and Szabó [2019a] show that they can still be analyzed using a generalization of the analytic setup of bordered or bordered sutured Heegaard Floer homology. However, a more literal generalization of these theories would yield bimodules over the larger dg algebras of [Lekili and Polishchuk 2020; Manion et al. 2020] rather than over the associative algebra $\mathcal{B}(2)$. The second

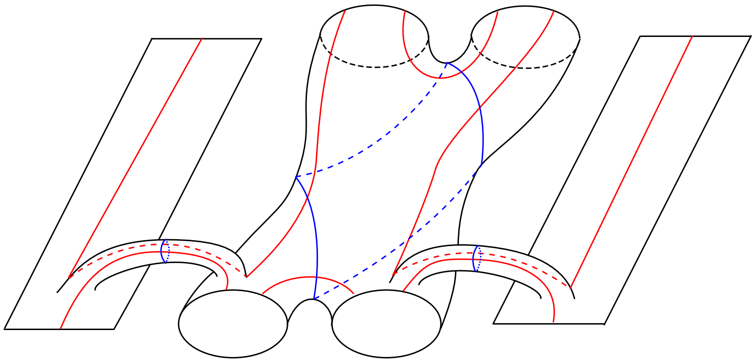

Figure 6. The bordered sutured Heegaard diagram for $\mathcal{P}$.

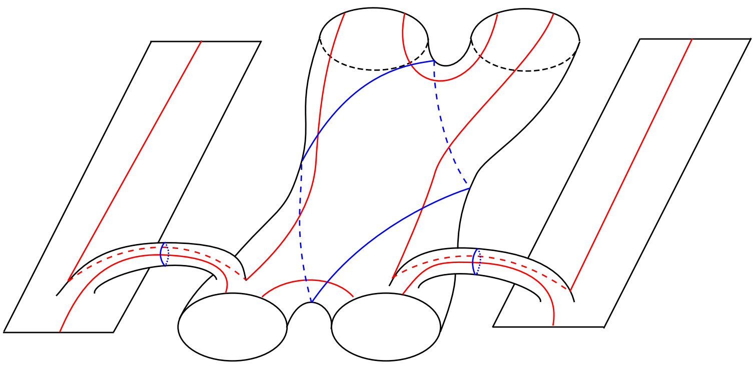

Figure 7. The bordered sutured Heegaard diagram for $\mathcal{N}$.

author, with Marengon and Willis, hope to address this difference in future work, defining DA bimodules over the larger dg algebras and relating them to $\mathcal{P}$ and $\mathcal{N}$.

4. Higher representations

4A. *General setup.* We now briefly review how higher representation theory interacts with bordered Heegaard Floer homology, as discussed in more generality in [Manion and Rouquier 2020].

4A1. *Monoidal category.* The following differential monoidal category $\mathcal{U}$ was defined in [Khovanov 2014], and 2-actions of $\mathcal{U}$ are a main subject of [Manion and Rouquier 2020]; see also [Douglas and Manolescu 2014; Douglas et al. 2019].

Definition 4.1. Let $\mathcal{U}$ denote the strict differential monoidal category with objects generated under $\otimes$ by a single object e and with morphisms generated under $\otimes$ and composition by an endomorphism τ of $e \otimes e$, subject to the relations $\tau^2 = 0$ and

$$(\mathrm{id}_e \otimes \tau) \circ (\tau \otimes \mathrm{id}_e) \circ (\mathrm{id}_e \otimes \tau) = (\tau \otimes \mathrm{id}_e) \otimes (\mathrm{id}_e \otimes \tau) \otimes (\tau \otimes \mathrm{id}_e),$$

and with differential determined by $d(\tau) = \mathrm{id}_{e\otimes e}$.

Remark 4.2. A grading on $\mathcal{U}$ is defined in [Khovanov 2014], making it into a dg category. Here we will not need to work with this grading; indeed, in the 2-actions of $\mathcal{U}$ we consider below, τ will act as zero.

The endomorphism algebra in $\mathcal{U}$ of $e^{\otimes m}$ is the nil-Coxeter dg algebra denoted by $\mathfrak{N}_m$ in [Douglas and Manolescu 2014].

4A2. *2-representations.* We will be especially concerned with 2-representations of $\mathcal{U}$ on associative algebras in the setting of DA bimodules; we give a concrete definition of this notion below.

Definition 4.3. Let $\mathcal{A}$ be an associative algebra (we make the same assumptions on $\mathcal{A}$ as in Section 2A). A (DA bimodule) 2-representation of $\mathcal{U}$ on $\mathcal{A}$ is the data of a DA bimodule $\mathcal{E}$ over $\mathcal{A}$ and a (typically nonclosed) DA bimodule morphism τ from $\mathcal{E} \boxtimes \mathcal{E}$ to itself satisfying $\tau^2 = 0$,

$$(\mathrm{id}_{\mathcal{E}} \boxtimes \tau) \circ (\tau \boxtimes \mathrm{id}_{\mathcal{E}}) \circ (\mathrm{id}_{\mathcal{E}} \boxtimes \tau) = (\tau \boxtimes \mathrm{id}_{\mathcal{E}}) \circ (\mathrm{id}_{\mathcal{E}} \boxtimes \tau) \circ (\tau \boxtimes \mathrm{id}_{\mathcal{E}}),$$

and $d(\tau) = 1$. We also assume that $\mathcal{E}$ is left bounded in the sense of [Lipshitz et al. 2015, Definition 2.2.46].

We will write the above data as $(\mathcal{A}, \mathcal{E}, \tau)$.

Remark 4.4. The definitions of DA bimodule morphisms, their tensor products, and their differentials can be found in [Lipshitz et al. 2015, Section 2.2.4 and Section 2.3.2], but we will refrain from spelling out these definitions here because in the examples we will consider, $\mathcal{E} \boxtimes \mathcal{E}$ will be the zero DA bimodule and τ will be the zero morphism.

4A3. *1-morphisms of 2-representations.* We will also work with a DA bimodule version of 1-morphisms between 2-representations of $\mathcal{U}$.

Definition 4.5. Let $(\mathcal{A}, \mathcal{E}, \tau)$ and $(\mathcal{A}', \mathcal{E}', \tau')$ be (DA bimodule) 2-representations of $\mathcal{U}$ on associative algebras $\mathcal{A}$ and $\mathcal{A}'$. A (DA bimodule) 1-morphism of 2-representations from $(\mathcal{A}, \mathcal{E}, \tau)$ to $(\mathcal{A}', \mathcal{E}', \tau')$ consists of a left bounded DA bimodule X over $(\mathcal{A}', \mathcal{A})$ together with a homotopy equivalence

$$\alpha : X \boxtimes \mathcal{E} \to \mathcal{E}' \boxtimes X,$$

satisfying

$$(\tau' \boxtimes \mathrm{id}_X) \circ (\mathrm{id}_{\mathcal{E}'} \boxtimes \alpha) \circ (\alpha \boxtimes \mathrm{id}_{\mathcal{E}}) = (\mathrm{id}_{\mathcal{E}'} \boxtimes \alpha) \circ (\alpha \boxtimes \mathrm{id}_{\mathcal{E}}) \circ (\mathrm{id}_X \boxtimes \tau)$$

as morphisms from $X \boxtimes \mathcal{E} \boxtimes \mathcal{E}$ to $\mathcal{E}' \boxtimes \mathcal{E}' \boxtimes X$.

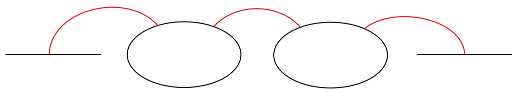

Figure 8. The arc diagram $\mathcal{Z}$ such that $\mathcal{A}(\mathcal{Z})$ is quasiisomorphic to $\mathcal{B}(2)$; the 2-to-1 matching is indicated by the arcs (red), and by symmetry one may take any orientation on the circles and intervals.

Remark 4.6. We will not elaborate on the definition of homotopy equivalence of DA bimodules here (it can be found in [Lipshitz et al. 2015, Section 2.2.4]); in this paper the homotopy equivalences α will be isomorphisms given by bijections between primary matrix entries such that the corresponding secondary matrices agree.

4B. ***Actions on bordered HFK algebras.*** In [Manion and Rouquier 2020], 2-representations of $\mathcal{U}$ are defined on the algebras $\mathcal{A}(\mathcal{Z})$ appearing in bordered sutured Heegaard Floer homology. Here $\mathcal{Z}$ denotes an arc diagram, i.e., a finite collection of oriented intervals and circles equipped with a 2-to-1 matching of finitely many points in the interiors of the intervals and circles, and there is a 2-representation of $\mathcal{U}$ on $\mathcal{A}(\mathcal{Z})$ for each interval in $\mathcal{Z}$.

The algebra $\mathcal{B}(2)$ was shown in [Manion et al. 2020; Lekili and Polishchuk 2020] to be quasiisomorphic to $\mathcal{A}(\mathcal{Z})$ where $\mathcal{Z}$ is the arc diagram shown in Figure 8. Since $\mathcal{Z}$ has two intervals, we should expect two 2-actions of $\mathcal{U}$ on $\mathcal{B}(2)$; we define these 2-actions below; see [Lauda and Manion 2021] for a related 2-representation of $\mathcal{U}$ on an n-strand Ozsváth–Szabó algebra from [Ozsváth and Szabó 2018]. In more detail, we will define DA bimodules $\mathcal{E}_1$ and $\mathcal{E}_2$ over $\mathcal{B}(2)$; these bimodules will satisfy $\mathcal{E}_i \boxtimes \mathcal{E}_i = 0$, so that $(\mathcal{A}, \mathcal{E}_i, 0)$ is a 2-representation of $\mathcal{U}$.

Remark 4.7. The arc diagram shown in Figure 8 can also be seen on the front and back edges of the Heegaard diagrams in Figure 6 and Figure 7, with the red arcs in Figure 8 determined by the matching pattern of the red arcs in the Heegaard diagrams.

Definition 4.8. The primary matrix for $\mathcal{E}_1$ has block form with the following blocks (we write e.g., X_1 for the singleton set $\{X_1\}$):

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \varnothing & \begin{bmatrix} X_1 & \varnothing & \varnothing \end{bmatrix} \end{array} \quad \begin{array}{cc} & \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} \varnothing & \varnothing & \varnothing \\ X_2 & \varnothing & \varnothing \\ \varnothing & X_3 & \varnothing \end{bmatrix} \end{array} \quad \begin{array}{cc} & ABC \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \begin{bmatrix} \varnothing \\ \varnothing \\ X_4 \end{bmatrix} \end{array}$$

The secondary matrix for $\mathcal{E}_1$ has a corresponding block form with blocks:

$$\begin{array}{cc} & X_1 \\ X_1 & \left[\begin{array}{c} 0 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} X_2 & X_3 \end{array} \\ \begin{array}{c} X_2 \\ X_3 \end{array} & \left[\begin{array}{cc} U_1^{k+1}\otimes U_1^{k+1}+U_2^{k+1}\otimes U_2^{k+1} & L_2U_2^k\otimes L_2U_2^k \\ R_2U_2^k\otimes R_2U_2^k & U_2^{k+1}\otimes U_2^{k+1} \end{array}\right] \end{array}$$

$$\begin{array}{cc} & X_4 \\ X_4 & \left[\begin{array}{c} U_1^kU_2^l\otimes U_1^kU_2^l \end{array}\right] \end{array}$$

In the final block we disallow $(k,l)=(0,0)$ to match Convention 2.10.

Definition 4.9. The primary matrix for $\mathcal{E}_2$ has block form with the following blocks (again we write e.g., Y_1 for the singleton set $\{Y_1\}$):

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \varnothing & \left[\begin{array}{ccc} \varnothing & \varnothing & Y_1 \end{array}\right] \end{array} \quad \begin{array}{cc} & \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \left[\begin{array}{ccc} \varnothing & Y_2 & \varnothing \\ \varnothing & \varnothing & Y_3 \\ \varnothing & \varnothing & \varnothing \end{array}\right] \end{array} \quad \begin{array}{cc} & ABC \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \left[\begin{array}{c} Y_4 \\ \varnothing \\ \varnothing \end{array}\right] \end{array}$$

The secondary matrix for $\mathcal{E}_2$ has a corresponding block form with blocks:

$$\begin{array}{cc} & Y_1 \\ Y_1 & \left[\begin{array}{c} 0 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} Y_2 & Y_3 \end{array} \\ \begin{array}{c} Y_2 \\ Y_3 \end{array} & \left[\begin{array}{cc} U_1^{k+1}\otimes U_1^{k+1} & L_1U_1^k\otimes L_1U_1^k \\ R_1U_1^k\otimes R_1U_1^k & U_1^{k+1}\otimes U_1^{k+1}+U_2^{k+1}\otimes U_2^{k+1} \end{array}\right] \end{array}$$

$$\begin{array}{cc} & Y_4 \\ Y_4 & \left[\begin{array}{c} U_1^kU_2^l\otimes U_1U_2^l \end{array}\right] \end{array}$$

In the final block we disallow $(k,l)=(0,0)$ to match Convention 2.10.

By multiplying the primary matrix for $\mathcal{E}_i$ by itself ($i=1,2$), one can see that $\mathcal{E}_i\boxtimes\mathcal{E}_i$ has a primary matrix with each entry the empty set; in other words, $\mathcal{E}_i\boxtimes\mathcal{E}_i$ is zero as claimed above.

5. 1-morphism structure for $\mathcal{P}$

5A. *Commutativity with $\mathcal{E}_1$.*

5A1. *The bimodule $\mathcal{E}_1\boxtimes\mathcal{P}$.* We give a matrix description for $\mathcal{E}_1\boxtimes\mathcal{P}$ following Section 2D. To get the primary matrix for $\mathcal{E}_1\boxtimes\mathcal{P}$, we multiply the primary matrices for $\mathcal{E}_1$ and $\mathcal{P}$. We can do this block-by-block, so the primary matrix for $\mathcal{E}_1 \boxtimes \mathcal{P}$ has block form with blocks given by

$$\begin{array}{c} \\ \varnothing \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} X_1 & \varnothing & \varnothing \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} S_A & \varnothing & \varnothing \\ W & N & E \\ \varnothing & \varnothing & S_C \end{bmatrix} \end{array} = \begin{array}{c} \\ \varnothing \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} X_1 S_C & \varnothing & \varnothing \end{bmatrix} \end{array},$$

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{bmatrix} \varnothing & \varnothing & \varnothing \\ X_2 & \varnothing & \varnothing \\ \varnothing & X_3 & \varnothing \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ AB \\ AC \\ BC \end{array}\begin{array}{c} \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{bmatrix} N_{AB} & E & \varnothing \\ \varnothing & S & \varnothing \\ \varnothing & W & N_{BC} \end{bmatrix} \end{array} = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{bmatrix} \varnothing & \varnothing & \varnothing \\ X_2 N_{AB} & X_2 E & \varnothing \\ \varnothing & X_3 X & \varnothing \end{bmatrix} \end{array},$$

$$\begin{array}{c} \\ AB \\ AC \\ BC \end{array}\begin{array}{c} ABC \\ \begin{bmatrix} \varnothing \\ \varnothing \\ X_4 \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ ABC \end{array}\begin{array}{c} ABC \\ \begin{bmatrix} N \end{bmatrix} \end{array} = \begin{array}{c} \\ AB \\ AC \\ BC \end{array}\begin{array}{c} ABC \\ \begin{bmatrix} \varnothing \\ \varnothing \\ X_4 N \end{bmatrix} \end{array}.$$

In these matrices, we indicate idempotents only when necessary to distinguish primary matrix entries in the same block (so, for example, in the block with rows and columns A, B, C, we distinguish between two types of S generators, but the only N generator in this block is ${}_B N_B$ so we omit the idempotents and just write N).

The secondary matrix for $\mathcal{E}_1 \boxtimes \mathcal{P}$ also has block form with blocks given by:

$$\begin{array}{c} \\ X_1 S_C \end{array}\begin{array}{c} X_1 S_C \\ \begin{bmatrix} 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ X_2 N_{AB} \\ X_2 E \\ X_3 S \end{array}\begin{array}{c} \begin{array}{ccc} X_2 N_{AB} & X_2 E & X_3 S \end{array} \\ \begin{bmatrix} U_2^{k+1} \otimes U_1^{k+1} + U_1^{k+1} \otimes U_2^{k+1} & U_1^k \otimes L_2 U_2^k & L_2 U_2^k \otimes (L_2, U_1^{k+1}) \\ U_1^{k+1} \otimes R_2 U_2^k & U_1^{k+1} \otimes U_2^{k+1} & 0 \\ 0 & R_2 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ X_4 N \end{array}\begin{array}{c} X_4 N \\ \begin{bmatrix} U_1^l U_2^k \otimes U_1^k U_2^l \end{bmatrix} \end{array}$$

In the final block we disallow $(k, l) = (0, 0)$. An explanation for the terms in the secondary matrix is given in Figure 9, which uses the operation graph depictions of Figure 3.

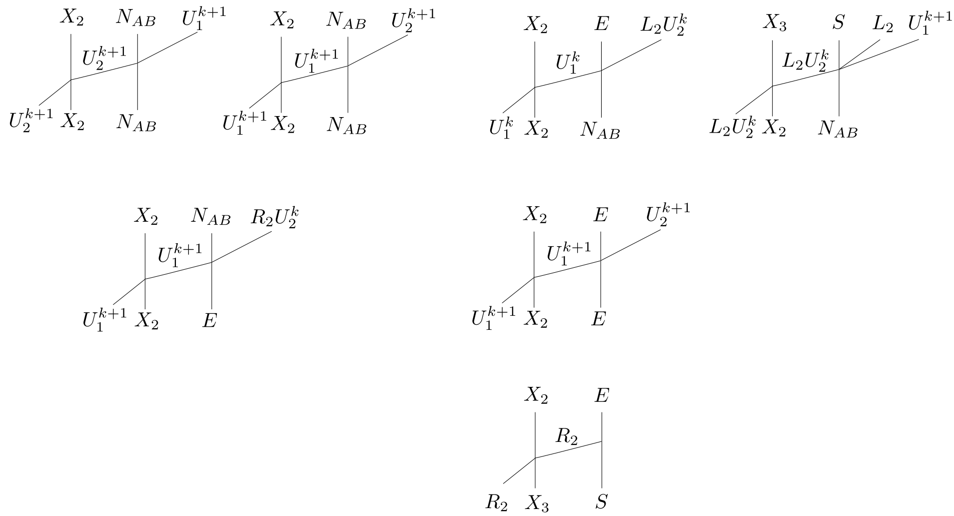

Figure 9. Operation graphs for the terms in the secondary matrix of $\mathcal{E}_1 \boxtimes \mathcal{P}$.

5A2. *The bimodule* $\mathcal{P} \boxtimes \mathcal{E}_1$. Similarly, we give a matrix description for $\mathcal{P} \boxtimes \mathcal{E}_1$. The primary matrix has block form with blocks

$$\begin{array}{cc} & \varnothing \\ \varnothing & \left[\begin{array}{c} S \end{array}\right] \end{array} \cdot \begin{array}{cccc} & A & B & C \\ \varnothing & \left[\begin{array}{c} X_1 \end{array}\right. & \varnothing & \left.\varnothing\right] \end{array} = \begin{array}{cccc} & A & B & C \\ \varnothing & \left[\begin{array}{c} SX_1 \end{array}\right. & \varnothing & \left.\varnothing\right] \end{array},$$

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & S_A & \varnothing & \varnothing \\ B & W & N & E \\ C & \varnothing & \varnothing & S_C \end{array} \cdot \begin{array}{c|ccc} & AB & AC & BC \\ \hline A & \varnothing & \varnothing & \varnothing \\ B & X_2 & \varnothing & \varnothing \\ C & \varnothing & X_3 & \varnothing \end{array} = \begin{array}{c|ccc} & AB & AC & BC \\ \hline A & \varnothing & \varnothing & \varnothing \\ B & NX_2 & EX_3 & \varnothing \\ C & \varnothing & S_CX_3 & \varnothing \end{array},$$

$$\begin{array}{c|ccc} & AB & AC & BC \\ \hline AB & N_{AB} & E & \varnothing \\ AC & \varnothing & S & \varnothing \\ BC & \varnothing & W & N_{BC} \end{array} \cdot \begin{array}{c|c} & ABC \\ \hline AB & \varnothing \\ AC & \varnothing \\ BC & X_4 \end{array} = \begin{array}{c|c} & ABC \\ \hline AB & \varnothing \\ AC & \varnothing \\ BC & N_{BC}X_4 \end{array}.$$

The secondary matrix for $\mathcal{P} \boxtimes \mathcal{E}_1$ also has block form with blocks:

$$\begin{array}{cc} & SX_1 \\ SX_1 & [\,0\,] \end{array}$$

$$\begin{array}{c|ccc} & NX_2 & EX_3 & S_CX_3 \\ \hline NX_2 & U_2^{k+1} \otimes U_1^{k+1} + U_1^{k+1} \otimes U_2^{k+1} & U_1^k \otimes L_2U_2^k & L_2U_2^k \otimes (L_2, U_1^{k+1}) \\ EX_3 & U_1^{k+1} \otimes R_2U_2^k & U_1^{k+1} \otimes U_2^{k+1} & 0 \\ S_CX_3 & 0 & R_2 & 0 \end{array}$$

$$\begin{array}{cc} & N_{BC}X_4 \\ N_{BC}X_4 & \left[\, U_1^lU_2^k \otimes U_1^kU_2^l \,\right] \end{array}$$

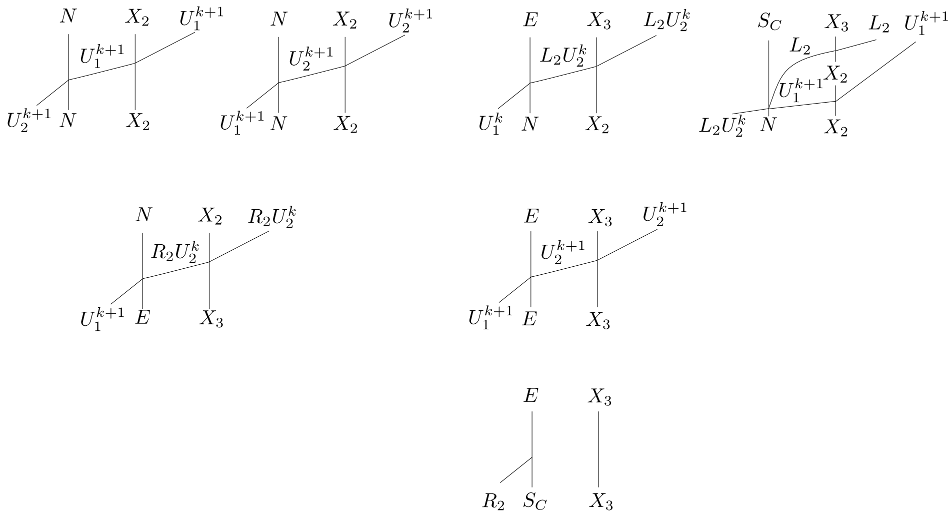

Figure 10. Operation graphs for the terms in the secondary matrix of $\mathcal{P} \boxtimes \mathcal{E}_1$.

In the final block we disallow $(k, l) = (0, 0)$. An explanation for the terms in the secondary matrix is given in Figure 10.

Corollary 5.1. *The DA bimodules $\mathcal{E}_1 \boxtimes \mathcal{P}$ and $\mathcal{P} \boxtimes \mathcal{E}_1$ are isomorphic to each other.*

Proof. The primary and secondary matrices for $\mathcal{E}_1 \boxtimes \mathcal{P}$ and $\mathcal{P} \boxtimes \mathcal{E}_1$ agree up to a relabeling of primary matrix entries. $\square$

5B. *Commutativity with $\mathcal{E}_2$.*

5B1. *The bimodule $\mathcal{E}_2 \boxtimes \mathcal{P}$.* Next we give a matrix description of $\mathcal{E}_2 \boxtimes \mathcal{P}$. The primary matrix has block form with blocks

$$\begin{array}{c} \\ \varnothing \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} \varnothing & \varnothing & Y_1 \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} S_A & \varnothing & \varnothing \\ W & N & E \\ \varnothing & \varnothing & S_C \end{bmatrix} \end{array} = \begin{array}{c} \\ \varnothing \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} \varnothing & \varnothing & Y_1 S_C \end{bmatrix} \end{array},$$

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{bmatrix} \varnothing & Y_2 & \varnothing \\ \varnothing & \varnothing & Y_3 \\ \varnothing & \varnothing & \varnothing \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ AB \\ AC \\ BC \end{array}\begin{array}{c} \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{bmatrix} N_{AB} & E & \varnothing \\ \varnothing & S & \varnothing \\ \varnothing & W & N_{BC} \end{bmatrix} \end{array} = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{bmatrix} \varnothing & Y_2 S & \varnothing \\ \varnothing & Y_3 W & Y_3 N_{BC} \\ \varnothing & \varnothing & \varnothing \end{bmatrix} \end{array},$$

$$\begin{array}{c} \\ AB \\ AC \\ BC \end{array}\begin{array}{c} ABC \\ \begin{bmatrix} Y_4 \\ \varnothing \\ \varnothing \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ ABC \end{array}\begin{array}{c} ABC \\ \begin{bmatrix} N \end{bmatrix} \end{array} = \begin{array}{c} \\ AB \\ AC \\ BC \end{array}\begin{array}{c} ABC \\ \begin{bmatrix} Y_4 N \\ \varnothing \\ \varnothing \end{bmatrix} \end{array}.$$

The secondary matrix for $\mathcal{E}_2 \boxtimes \mathcal{P}$ also has block form with blocks:

$$\begin{array}{cc} & Y_1S_C \\ Y_1S_C & \left[\begin{array}{c} 0 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} Y_2S & Y_3W & Y_3N_{BC} \end{array} \\ \begin{array}{c} Y_2S \\ Y_3W \\ Y_3N_{BC} \end{array} & \left[\begin{array}{ccc} 0 & L_1 & 0 \\ 0 & U_2^{k+1} \otimes U_1^{k+1} & U_2^{k+1} \otimes L_1U_1^k \\ R_1U_1^k \otimes (R_1, U_2^{k+1}) & U_2^k \otimes R_1U_1^k & U_1^{k+1} \otimes U_2^{k+1} + U_2^{k+1} \otimes U_1^{k+1} \end{array}\right] \end{array}$$

$$\begin{array}{cc} & Y_4N \\ Y_4N & \left[\begin{array}{c} U_1^kU_2^l \otimes U_1^lU_2^k \end{array}\right] \end{array}$$

In the final block we disallow $(k, l) = (0, 0)$. One can draw operation graphs for the secondary matrix entries as we did above in Figures 9 and 10, but we will omit the graphs here.

5B2. *The bimodule $\mathcal{P} \boxtimes \mathcal{E}_2$.* The primary matrix for $\mathcal{P} \boxtimes \mathcal{E}_2$ has block form with blocks

$$\begin{array}{cc} & \varnothing \\ \varnothing & \left[\begin{array}{c} S \end{array}\right] \end{array} \cdot \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \varnothing & \left[\begin{array}{ccc} \varnothing & \varnothing & Y_1 \end{array}\right] \end{array} = \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \varnothing & \left[\begin{array}{ccc} \varnothing & \varnothing & SY_1 \end{array}\right] \end{array},$$

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \left[\begin{array}{ccc} S_A & \varnothing & \varnothing \\ W & N & E \\ \varnothing & \varnothing & S_C \end{array}\right] \end{array} \cdot \begin{array}{cc} & \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \left[\begin{array}{ccc} \varnothing & Y_2 & \varnothing \\ \varnothing & \varnothing & Y_3 \\ \varnothing & \varnothing & \varnothing \end{array}\right] \end{array} = \begin{array}{cc} & \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \left[\begin{array}{ccc} \varnothing & S_AY_2 & \varnothing \\ \varnothing & WY_2 & NY_3 \\ \varnothing & \varnothing & \varnothing \end{array}\right] \end{array},$$

$$\begin{array}{cc} & \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \left[\begin{array}{ccc} N_{AB} & E & \varnothing \\ \varnothing & S & \varnothing \\ \varnothing & W & N_{BC} \end{array}\right] \end{array} \cdot \begin{array}{cc} & ABC \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \left[\begin{array}{c} Y_4 \\ \varnothing \\ \varnothing \end{array}\right] \end{array} = \begin{array}{cc} & ABC \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \left[\begin{array}{c} N_{AB}Y_4 \\ \varnothing \\ \varnothing \end{array}\right] \end{array}.$$

The secondary matrix for $\mathcal{P} \boxtimes \mathcal{E}_2$ also has block form with blocks:

$$\begin{array}{cc} & SY_1 \\ SY_1 & \left[\begin{array}{c} 0 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} S_AY_2 & WY_2 & NY_3 \end{array} \\ \begin{array}{c} S_AY_2 \\ WY_2 \\ NY_3 \end{array} & \left[\begin{array}{ccc} 0 & L_1 & 0 \\ 0 & U_2^{k+1} \otimes U_1^{k+1} & U_2^{k+1} \otimes L_1U_1^k \\ R_1U_1^k \otimes (R_1, U_2^{k+1}) & U_2^k \otimes R_1U_1^k & U_1^{k+1} \otimes U_2^{k+1} + U_2^{k+1} \otimes U_1^{k+1} \end{array}\right] \end{array}$$

$$\begin{array}{cc} & N_{AB}Y_4 \\ N_{AB}Y_4 & \left[\begin{array}{c} U_1^kU_2^l \otimes U_1^lU_2^k \end{array}\right] \end{array}$$

In the final block we disallow $(k,l)=(0,0)$. As with $\mathcal{E}_2 \boxtimes \mathcal{P}$, we will omit drawing the operation graphs.

Corollary 5.2. *The DA bimodules $\mathcal{E}_2 \boxtimes \mathcal{P}$ and $\mathcal{P} \boxtimes \mathcal{E}_2$ are isomorphic to each other.*

Proof. The primary and secondary matrices for $\mathcal{E}_2 \boxtimes \mathcal{P}$ and $\mathcal{P} \boxtimes \mathcal{E}_2$ agree up to a relabeling of primary matrix entries. □

6. 1-morphism structure for $\mathcal{N}$

Here we summarize, with fewer details, the computations for $\mathcal{N}$ that are analogous to those for $\mathcal{P}$ in Section 5.

6A. *Commutativity with $\mathcal{E}_1$.*

6A1. *The bimodule $\mathcal{E}_1 \boxtimes \mathcal{N}$.* The primary matrix for $\mathcal{E}_1 \boxtimes \mathcal{N}$ has block form with the same blocks as for $\mathcal{E}_1 \boxtimes \mathcal{P}$, namely

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \varnothing & \begin{bmatrix} \varnothing & \varnothing & X_1 S_C \end{bmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} \varnothing & \varnothing & \varnothing \\ X_2 N_{AB} & X_2 E & \varnothing \\ \varnothing & X_3 X & \varnothing \end{bmatrix} \end{array}, \quad \begin{array}{cc} & ABC \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \begin{bmatrix} \varnothing \\ \varnothing \\ X_4 N \end{bmatrix} \end{array}.$$

The secondary matrix for $\mathcal{E}_1 \boxtimes \mathcal{N}$ has block form with blocks given by:

$$\begin{array}{cc} & X_1 S_C \\ X_1 S_C & \begin{bmatrix} 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} X_2 N_{AB} & X_2 E & X_3 S \end{array} \\ \begin{array}{c} X_2 N_{AB} \\ X_2 E \\ X_3 S \end{array} & \begin{bmatrix} U_2^{k+1} \otimes U_1^{k+1} + U_1^{k+1} \otimes U_2^{k+1} & U_1^{k+1} \otimes L_2 U_2^k & 0 \\ U_1^k \otimes R_2 U_2^k & U_1^{k+1} \otimes U_2^{k+1} & L_2 \\ R_2 U_2^k \otimes (U_1^{k+1}, R_2) & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & X_4 N \\ X_4 N & \begin{bmatrix} U_1^l U_2^k \otimes U_1^k U_2^l \end{bmatrix} \end{array}$$

In the final block we disallow $(k,l)=(0,0)$.

6A2. *The bimodule $\mathcal{N} \boxtimes \mathcal{E}_1$.* The primary matrix for $\mathcal{N} \boxtimes \mathcal{E}_1$ has block form with the same blocks as for $\mathcal{P} \boxtimes \mathcal{E}_1$, namely

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \varnothing & \begin{bmatrix} S X_1 & \varnothing & \varnothing \end{bmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} \varnothing & \varnothing & \varnothing \\ N X_2 & E X_3 & \varnothing \\ \varnothing & S_C X_3 & \varnothing \end{bmatrix} \end{array}, \quad \begin{array}{cc} & ABC \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \begin{bmatrix} \varnothing \\ \varnothing \\ N_{BC} X_4 \end{bmatrix} \end{array}.$$

The secondary matrix for $\mathcal{N} \boxtimes \mathcal{E}_1$ has block form with blocks given by

$$\begin{array}{c|c} & SX_1 \\ \hline SX_1 & \begin{bmatrix} 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} & \begin{array}{ccc} NX_2 & EX_3 & S_C X_3 \end{array} \\ \begin{array}{c} NX_2 \\ EX_3 \\ S_C X_3 \end{array} & \begin{bmatrix} U_2^{k+1} \otimes U_1^{k+1} + U_1^{k+1} \otimes U_2^{k+1} & U_1^{k+1} \otimes L_2 U_2^k & 0 \\ U_1^k \otimes R_2 U_2^k & U_1^{k+1} \otimes U_2^{k+1} & L_2 \\ R_2 U_2^k \otimes (U_1^{k+1}, R_2) & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} & N_{BC}X_4 \\ \hline N_{BC}X_4 & \begin{bmatrix} U_1^l U_2^k \otimes U_1^k U_2^l \end{bmatrix} \end{array}$$

In the final block we disallow $(k, l) = (0, 0)$.

Corollary 6.1. *The DA bimodules $\mathcal{E}_1 \boxtimes \mathcal{N}$ and $\mathcal{N} \boxtimes \mathcal{E}_1$ are isomorphic to each other.*

6B. *Commutativity with $\mathcal{E}_2$.*

6B1. *The bimodule $\mathcal{E}_2 \boxtimes \mathcal{N}$.* The primary matrix for $\mathcal{E}_2 \boxtimes \mathcal{N}$ has block form with the same blocks as for $\mathcal{E}_2 \boxtimes \mathcal{P}$, namely

$$\begin{array}{c|c} & \begin{array}{ccc} A & B & C \end{array} \\ \varnothing & \begin{bmatrix} \varnothing & \varnothing & Y_1 S_C \end{bmatrix} \end{array}, \quad \begin{array}{c|c} & \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} \varnothing & Y_2 S & \varnothing \\ \varnothing & Y_3 W & Y_3 N_{BC} \\ \varnothing & \varnothing & \varnothing \end{bmatrix} \end{array}, \quad \begin{array}{c|c} & ABC \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \begin{bmatrix} Y_4 N \\ \varnothing \\ \varnothing \end{bmatrix} \end{array}.$$

The secondary matrix for $\mathcal{E}_2 \boxtimes \mathcal{N}$ has block form with blocks given by

$$\begin{array}{c|c} & Y_1 S_C \\ \hline Y_1 S_C & \begin{bmatrix} 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} & \begin{array}{ccc} Y_2 S & Y_3 W & Y_3 N_{BC} \end{array} \\ \begin{array}{c} Y_2 S \\ Y_3 W \\ Y_3 N_{BC} \end{array} & \begin{bmatrix} 0 & 0 & L_1 U_1^k \otimes (U_2^{k+1}, L_1) \\ R_1 & U_2^{k+1} \otimes U_1^{k+1} & U_2^k \otimes L_1 U_1^k \\ 0 & U_2^{k+1} \otimes R_1 U_1^k & U_1^{k+1} \otimes U_2^{k+1} + U_2^{k+1} \otimes U_1^{k+1} \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} & Y_4 N \\ \hline Y_4 N & \begin{bmatrix} U_1^k U_2^l \otimes U_1^l U_2^k \end{bmatrix} \end{array}$$

In the final block we disallow $(k, l) = (0, 0)$.

6B2. *The bimodule $\mathcal{N} \boxtimes \mathcal{E}_2$.* The primary matrix for $\mathcal{N} \boxtimes \mathcal{E}_2$ has block form with the same blocks as for $\mathcal{P} \boxtimes \mathcal{E}_2$, namely

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \varnothing & \left[\begin{array}{ccc} \varnothing & \varnothing & SY_1 \end{array}\right] \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} AB & AC & BC \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \left[\begin{array}{ccc} \varnothing & S_A Y_2 & \varnothing \\ \varnothing & W Y_2 & N Y_3 \\ \varnothing & \varnothing & \varnothing \end{array}\right] \end{array}, \quad \begin{array}{cc} & ABC \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \left[\begin{array}{c} N_{AB} Y_4 \\ \varnothing \\ \varnothing \end{array}\right] \end{array}.$$

The secondary matrix for $\mathcal{N} \boxtimes \mathcal{E}_2$ has block form with blocks given by

$$\begin{array}{cc} & SY_1 \\ SY_1 & \left[\begin{array}{c} 0 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} S_A Y_2 & W Y_2 & N Y_3 \end{array} \\ \begin{array}{c} S_A Y_2 \\ W Y_2 \\ N Y_3 \end{array} & \left[\begin{array}{ccc} 0 & 0 & L_1 U_1^k \otimes (U_2^{k+1}, L_1) \\ R_1 & U_2^{k+1} \otimes U_1^{k+1} & U_2^k \otimes L_1 U_1^k \\ 0 & U_2^{k+1} \otimes R_1 U_1^k & U_1^{k+1} \otimes U_2^{k+1} + U_2^{k+1} \otimes U_1^{k+1} \end{array}\right] \end{array}$$

$$\begin{array}{cc} & N_{AB} Y_4 \\ N_{AB} Y_4 & \left[\begin{array}{c} U_1^k U_2^l \otimes U_1^l U_2^k \end{array}\right] \end{array}$$

In the final block we disallow $(k, l) = (0, 0)$.

Corollary 6.2. *The DA bimodules $\mathcal{E}_2 \boxtimes \mathcal{N}$ and $\mathcal{N} \boxtimes \mathcal{E}_2$ are isomorphic to each other.*

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THE FOX–HATCHER CYCLE AND A VASSILIEV INVARIANT OF ORDER THREE

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We show that the integration of a 1-cocycle $I(X)$ of the space of long knots in $\mathbb{R}^3$ over the *Fox–Hatcher* 1-*cycles* gives rise to a Vassiliev invariant of order exactly three. This result can be seen as a continuation of the previous work of the Sakai (2011), proving that the integration of $I(X)$ over the *Gramain* 1-*cycles* is the *Casson invariant*, the unique nontrivial Vassiliev invariant of order two (up to scalar multiplications). The result in the present paper is also analogous to part of Mortier's result (2015). Our result differs from, but is motivated by, Mortier's one in that the 1-cocycle $I(X)$ is given by the *configuration space integrals* associated with graphs while Mortier's cocycle is obtained in a combinatorial way.

1. Introduction

Spaces of smooth embeddings of manifolds are receiving a lot of attention in topology, on the ground that various important methods in algebraic and geometric topology are being applied to the spaces. In this paper we study the space of (*framed*) *long knots* in $\mathbb{R}^3$.

Definition 1.1. A *long knot* is an embedding $f\colon \mathbb{R}^1 \hookrightarrow \mathbb{R}^3$ satisfying $f(x) = (x, 0, 0)$ for any $x \in \mathbb{R}^1$ with $|x| \geq 1$. A *framed long knot* is a smooth map $\tilde{f} = (f, w)\colon \mathbb{R}^1 \to \mathbb{R}^3 \times SO(3)$ such that f is a long knot, the first column of $w(x) \in SO(3)$ is equal to $f'(x)/|f'(x)|$ and $w(x)$ is the identity matrix for any $x \in \mathbb{R}^1$ with $|x| \geq 1$. The space of all long knots (respectively framed long knots) is denoted by $\mathcal{K}$ (respectively $\widetilde{\mathcal{K}}$).

The recent studies of $\mathcal{K}$ (and its high dimensional analogues) are revealing relations between the topological nature of $\mathcal{K}$ and the *Vassiliev invariants* (see for example [12]) for knots and links. In [17] Sakai has constructed a de Rham 1-cocycle $I(X)$ of $\mathcal{K}$ (see Section 3), by means of the integrations over configuration spaces associated with a *graph cocycle* X (see Figure 6), and has shown that the integration

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of $I(X)$ over the *Gramain cycles* of $\mathcal{K}$ gives rise to the *Casson invariant* v_2, the Vassiliev invariant of order two uniquely characterized by $v_2(\text{trivial knot}) = 0$ and $v_2(\text{trefoil knot}) = 1$. This may be seen as a real valued version of [19, Theorem 2]. After that Mortier has given another 1-cocycle α_3^1 of $\mathcal{K}$ in a combinatorial way and has shown that its evaluations over the Gramain cycles and the *Fox–Hatcher cycles FH* are Vassiliev invariants of orders respectively two and three [14, Theorem 4.1]. In [7; 8; 10] 1-cocycles on $\mathcal{K}$ are also studied in detail from a combinatorial viewpoint.

The main result in the present paper is analogous to the order three part of Mortier's result.

Theorem 1.2. *The integration of $I(X)$ over the Fox–Hatcher cycles gives rise to a Vassiliev invariant of order three for framed long knots. More precisely we have*

$$\int_{p_* FH_{\tilde{f}}} I(X) = 6v_3(f) - \mathrm{lk}(\tilde{f})v_2(f), \tag{1-1}$$

where

- $p\colon \widetilde{\mathcal{K}} \to \mathcal{K}$ *is the first projection and* $f = p(\tilde{f})$,
- v_2 *is the Casson invariant, and* v_3 *is the Vassiliev invariant of order three characterized by the conditions*

$$v_3(\textit{trivial knot}) = 0, \quad v_3(3_1^+) = 1, \quad v_3(3_1^-) = -1 \tag{1-2}$$

  (3_1^+ *and* 3_1^- *are respectively the right-handed and the left-handed trefoil knots*), *and*
- $\mathrm{lk}(\tilde{f}) \in \mathbb{Z}$ *is the* ***framing number*** *of* $\tilde{f}$ (*see Remark 1.3 below*).

Remark 1.3. The framing number $\mathrm{lk}(\tilde{f})$ is the linking number of $f = p(\tilde{f})$ and f', where f' is the long knot obtained by moving f slightly into the direction of the second column of w. In fact the map $p \times \mathrm{lk}\colon \widetilde{\mathcal{K}} \to \mathcal{K} \times \mathbb{Z}$ is a homotopy equivalence [5, Proposition 9], and the framing number uniquely determines the framing w up to homotopy. Thus we may regard a framed long knot as a pair (f, w) of $f \in \mathcal{K}$ and $w \in \mathbb{Z}$.

The 1-cocycle $I(X)$ is constructed by means of the *configuration space integral associated with graphs*, that was developed in [1; 4; 13] to describe Vassiliev invariants and was generalized in [6] to obtain a cochain map from a graph complex to $\Omega^*_{DR}(\mathcal{K})$ (up to some correction terms, that vanish in the cases of the spaces of long knots in high dimensional spaces). Vassiliev invariants (which are examples of 0-cocycles of $\mathcal{K}$) are obtained from trivalent graphs, while our 1-cocycle $I(X)$ comes from nontrivalent graphs (see Figure 6). It is very interesting, although not strange, that nontrivalent graphs may also have information of Vassiliev invariants.

We note that the right hand side of (1-1) coincides with the formula for Mortier's invariant of order three. We thus expect that the 1-cocycle $I(X)$ is cohomologous to Mortier's α_3^1. This is true on the connected components of torus and hyperbolic knots, since $I(X)$ agrees with α_3^1 on the Gramain and the Fox–Hatcher cycles by Theorem 1.2, [17, Theorem 3.1] and [14, Theorem 4.1], and these cycles generate π_1 of the components of torus and hyperbolic knots [11, page 2].

This paper is organized as follows: In Section 2 the Fox–Hatcher cycle is introduced, and in Section 3 the construction of the 1-cocycle $I(X)$ is reviewed. Our invariant v, the left hand side of (1-1), is shown to be of order three in Corollary 4.2. The key ingredient is Theorem 4.1 and is proved in Section 4B. The formula (1-1) is verified in Section 4C.

2. The Fox–Hatcher cycle

2A. ***The Fox–Hatcher cycle.*** The Fox–Hatcher cycle was introduced in [9], and was later studied in [11] from the viewpoint of the space of knots. If $f = p(\tilde{f})$ is not trivial, it then gives a nonzero element of $\pi_1(\widetilde{\mathcal{K}}_{\tilde{f}})$, where $\widetilde{\mathcal{K}}_{\tilde{f}}$ is the path component of $\widetilde{\mathcal{K}}$ containing $\tilde{f}$.

The Fox–Hatcher cycle is defined as follows. A framed long knot can be seen as a based embedding $f: S^1 \hookrightarrow S^3$ (we see S^3 as in $\mathbb{R}^4 \approx \mathbb{C}^2$) together with a framing w, with a prescribed behavior near the basepoint. For $t \in S^1$, $w(t)$ is an orthonormal basis of $T_{f(t)}S^3$ whose first vector is $f'(t)/|f'(t)|$. There exists an S^1-action on the space of such embeddings defined by $(\theta \cdot (f, w))(t) := (A(\theta)^{-1} f(t-\theta), A(\theta)^{-1} w(t-\theta))$, where $A(\theta) \in \mathrm{SO}(4)$ is the matrix given by $A(\theta) = (w(\theta), f(\theta))$. For any $\tilde{f} \in \widetilde{\mathcal{K}}$, this action determines a 1-cycle $FH_{\tilde{f}}: S^1 \to \widetilde{\mathcal{K}}_{\tilde{f}}$ and we call it the *Fox–Hatcher cycle*. We notice that the S^1-action looks very similar to the natural S^1-action on free loop spaces by the reparametrization, and in fact this action defines a *BV-operation* on $H_*(\widetilde{\mathcal{K}})$ [18].

Practically it is convenient to describe *FH* on knot diagrams. In this paper a framed long knot is drawn in a usual knot diagram with so-called *blackboard framing*.

Definition 2.1. Let D be a knot diagram of $\tilde{f}$ with blackboard framing and c the "left-most" crossing, namely the crossing that we meet first when traveling from $f(-1)$ along the natural orientation of f. We call the transformation shown in Figure 1 the *Fox–Hatcher move* (FH-move for short) *on* c.

The left-most crossing c disappears after the FH-move on c and the right-most crossing c' is created. If the arc that moves in the FH-move is the over-arc (resp. under-arc) at c, then after the FH-move it becomes the over-arc (resp. under-arc) at c'. We arrive the original diagram D after performing the FH-moves for all

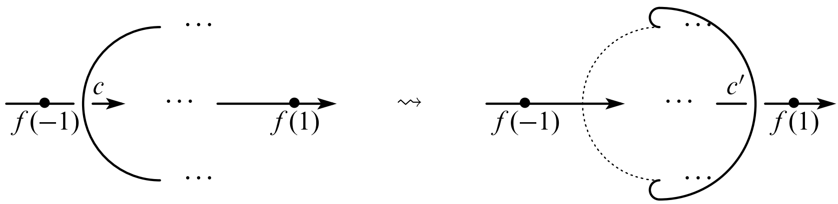

Figure 1. The Fox–Hatcher move on c.

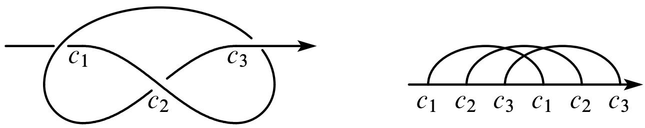

Figure 2. A knot diagram and its Gauss diagram.

the other crossings c of D and the newborn crossings c'. The sequence of these FH-moves realizes $FH_{\tilde{f}}$.

2B. ***FH moves and Gauss diagrams.*** The configuration of crossings of a knot diagram is encoded by (*linear*) *Gauss diagrams*. Here we see how the FH-move on the left-most crossing changes the Gauss diagram.

Definition 2.2. A (*linear*) *Gauss diagram* is a partition of $\{1, 2, \dots, 2n\}$ for some natural number n into a union $\bigcup_{1 \le k \le n} \{i_k, j_k\}$ of n subsets of cardinality 2.

A Gauss diagram can be seen as a graph on $\mathbb{R}^1$ with an even number of vertices all of which are on $\mathbb{R}^1$ and with each vertex joined by exactly one edge with another vertex. Here segments in $\mathbb{R}^1$ interposed between two vertices are not regarded as edges. See Figure 2 for example.

Definition 2.3 [17, Definition 3.3]. Let $c_1, \dots, c_n$ be (part of the) crossings of a knot diagram of $f \in \mathcal{K}$ such that each c_i corresponds to $f(p_i)$ and $f(q_i)$, with $-1 < p_1 < \cdots < p_n < 1$ and $p_i < q_i$ for any $i = 1, \dots, n$. We say that the crossings $c_1, \dots, c_n$ *respect* a Gauss diagram G if G is isomorphic to the Gauss diagram $G_{c_1,\dots,c_n}$ obtained by joining p_i and q_i for $i = 1, \dots, n$. See Figure 2.

Under the setting of Definition 2.3, the left-most crossing is c_1. Let G be the Gauss diagram that $c_1, \dots, c_n$ respect. Then the new knot diagram obtained by performing the FH-move on c_1 has crossings $c_2, \dots, c_n, c_1'$ that respect the Gauss diagram G' obtained by moving the left-most vertex (corresponding to c_1) to the right-most one. See Figure 3.

We eventually arrive the original Gauss diagram after performing the FH-moves on all the crossings c of the original diagram and the newborn crossings c'. This sequence produces a cycle of Gauss diagrams (see Figures 7, 8, 9). In this way the set of all the Gauss diagrams is decomposed into the disjoint cycles.

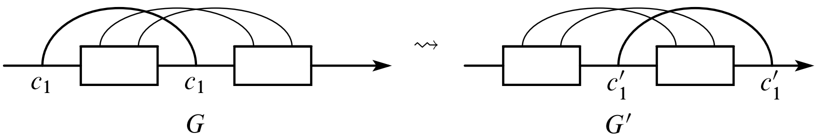

Figure 3. The FH-move on c_1 on the Gauss diagram.

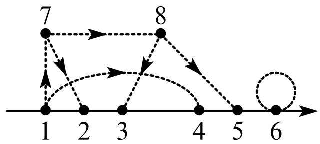

Figure 4. An example of graphs; the i-vertices are those labeled by $1, \dots, 6$ and the f-vertices are those labeled by 7, 8, and there is a loop at the i-vertex labeled by 6.

3. The cocycle $I(X)$

In this section we give a quick review of the construction of differential forms on $\mathcal{K}$ associated with graphs. See also [1; 4; 6; 13; 20] for details.

By a *graph* we mean the oriented real line $\mathbb{R}^1$ together with two kind of vertices, one is called *interval* and the other *free*, and *oriented edges* connecting them (see Figure 4).

The interval vertices (or i-vertices for short) are placed on the oriented line while the free vertices (or f-vertices for short) are not on the line. The i-vertices and the f-vertices of a graph X are labeled by respectively the numbers $1, \dots, \mathrm{v_i}$ and $\mathrm{v_i}+1, \dots, \mathrm{v_i}+\mathrm{v_f}$, where $\mathrm{v_i}$ and $\mathrm{v_f}$ are respectively the numbers of the i-vertices and the f-vertices of X, so that the labels of the i-vertices respect the orientation of the real line. We allow graphs to have loops, where a *loop* is an edge that has exactly one i-vertex as its endpoint (see Figure 4).

For a graph X, let E_X be the configuration space

$$E_X := \big\{(f, (y_1, \dots, y_{\mathrm{v_i}+\mathrm{v_f}})) \in \mathcal{K} \times \mathrm{Conf}_{\mathrm{v_i}+\mathrm{v_f}}(\mathbb{R}^3) \\ \mid y_i = f(x_i) \text{ for some } x_i \in \mathbb{R}^1 \text{ for } i = 1, \dots, \mathrm{v_i}\big\}, \tag{3-1}$$

where

$$\mathrm{Conf}_k(M) := \{(x_1, \dots, x_k) \in M^{\times k} \mid x_i \neq x_j \text{ if } i \neq j\} \tag{3-2}$$

is the space of k-point configurations on a space M.

To an oriented edge α of X from the i-th vertex to the j-th vertex ($i \neq j$), we assign a map

$$\varphi_\alpha \colon E_X \to S^2, \quad \varphi_\alpha(f, (y_1, \dots, y_{v_i+v_f})) := \frac{y_j - y_i}{|y_j - y_i|}. \tag{3-3}$$

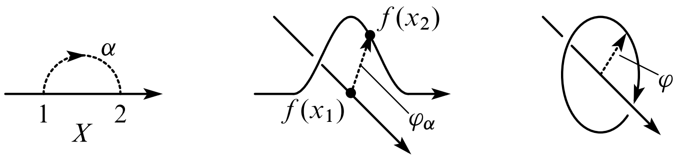

Figure 5. The graph X in Example 3.2 (the left), configurations where the image of φ_α is contained in supp(vol) (the center), the Hopf link (the right).

To a loop α at k-th i-vertex ($1 \le i \le \mathrm{v_i}$) we assign

$$\varphi_\alpha \colon E_X \to S^2, \quad \varphi_\alpha(f, (y_1, \dots, y_{v_i+v_f})) := \frac{f'(x_k)}{|f'(x_k)|}, \tag{3-4}$$

where $x_k \in \mathbb{R}^1$ satisfies $y_k = f(x_k)$.

Let $\mathrm{vol} \in \Omega^2_{DR}(S^2)$ be a unit volume form of S^2 that is antisymmetric, meaning that $i^* \mathrm{vol} = -\mathrm{vol}$ for the antipodal map $i \colon S^2 \to S^2$. Define $\omega_X \in \Omega^{2e}_{DR}(E_X)$ by

$$\omega_X := \bigwedge_{\text{edges } \alpha \text{ of } X} \varphi^*_\alpha(\mathrm{vol}), \tag{3-5}$$

where e is the number of edges of X. The order of the edges is not important because $\deg \mathrm{vol} = 2$ is even.

Let $\pi_X \colon E_X \to \mathcal{K}$ be the first projection. This is a fiber bundle with fiber

$$\pi_X^{-1}(f) = \{y \in \mathrm{Conf}_{\mathrm{v_i}+\mathrm{v_f}}(\mathbb{R}^3) \mid y_i = f(x_i) \text{ for some } x_i \in \mathbb{R}^1 \text{ for } i = 1, \dots, \mathrm{v_i}\} \tag{3-6}$$

of dimension $\mathrm{v_i} + 3\mathrm{v_f}$. Integrating ω_X along the fiber, we get

$$I(X) := \pi_{X*}(\omega_X) \in \Omega^{2e-\mathrm{v_i}-3\mathrm{v_f}}_{DR}(\mathcal{K}). \tag{3-7}$$

Remark 3.1. The integration (3-7) converges since we can compactify all the fibers of π_X by adding the boundary faces to (3-6) so that the maps φ_α are smoothly extended to the compactification. See [3; 4; 6; 13].

Example 3.2. Let X be the graph that has only one edge α joining two i-vertices (Figure 5, the left).

Then $E_X \approx \mathcal{K} \times \mathrm{Conf}_2(\mathbb{R}^1)$ and $I(X) \in \Omega^0_{DR}(\mathcal{K})$ is a function on $\mathcal{K}$, but is not a locally constant function (i.e., not an isotopy invariant), as we see below.

In this paper we use an antisymmetric unit volume form vol whose support is contained in (small) neighborhoods $U_\pm$ of the poles $(0, 0, \pm 1) \in S^2$. Suppose $f \in \mathcal{K}$ is "almost planer," meaning that

- the image of f coincides with a knot diagram D on $\mathbb{R}^2 \times \{0\}$ except for neighborhoods of crossings of D,
- near the crossings the image of f is contained in $\mathbb{R}^2 \times (-\epsilon, \epsilon)$, and
- the unit tangent vectors $f'(x)/|f'(x)|$ are not contained in $U_\pm$.

Then $\varphi_\alpha\colon \{f\} \times \mathrm{Conf}_2(\mathbb{R}^1) \to S^2$ has its image in $U_\pm$ only on the subspace of (x_1, x_2) such that $f(x_1)$ and $f(x_2)$ are on the over- and under-arcs of a crossing of D, one on each arc (Figure 5, the center). Each crossing contributes to the value $I(X)(f)$ by half of its sign; because this contribution is the half of the linking number of the Hopf link (Figure 5, the right), which is equal to the sign of the crossing.

By the generalized Stokes' theorem for fiber integrations, we have

$$dI(X) = \pi_{X*}(d\omega_X) \pm \pi^{\partial}_{X*}(\omega) = \pm\pi^{\partial}_{X*}(\omega), \tag{3-8}$$

where π^{∂}_X is the restriction of π_X to the fiberwise boundary. There exists "almost" 1-1 correspondence between

- the codimension 1 faces of the boundary that nontrivially contribute to $dI(X)$, and
- the graphs obtained from X by contracting one of its edges and *arcs* (segments in $\mathbb{R}^1$ interposed between two i-vertices).

Here we in fact need the antisymmetry of vol. We thus have

$$dI(X) = I(\partial X) + (\text{correction terms}), \tag{3-9}$$

where ∂X is a formal sum of graphs obtained from X by contracting one of its edges and arcs. The above correspondence is not rigorously 1-1 and we need "correction terms," that are conjectured to vanish. We can therefore get a closed form of $\mathcal{K}$ if we have a *graph cocycle*, a formal sum X of graphs with $\partial X = 0$ (and if we have appropriate correction terms). It is known that any $\mathbb{R}$-valued Vassiliev invariant can be produced from a *trivalent* graph cocycle.

In [16; 17] Sakai has given an example of *nontrivalent* graph cocycle

$$X = \sum_{1\le k\le 9} a_k X_k, \quad (a_1, \dots, a_9) = (-2, 1, 2, -2, 2, -1, 1, -1, 1) \tag{3-10}$$

(see Figure 6), and has proved that $I(X) \in H^1_{DR}(\mathcal{K})$ is not zero.[1] This follows from:

Theorem 3.3 [17]. *The differential form $I(X) \in \Omega^1_{DR}(\mathcal{K})$ is closed, and its integration over the* ***Gramain cycle*** *G_f (see Remark 3.4 below) is equal to the* ***Casson invariant*** *$v_2(f)$.*

[1]The coefficients a_7, a_8, a_9 in [16; 17] are wrong and those in (3-10) are correct. The main results in [16; 17] still hold since the graphs X_7, X_8, X_9 are not essential in the integration of $I(X)$ over the Gramain cycles. See [16, Lemma 4.2].

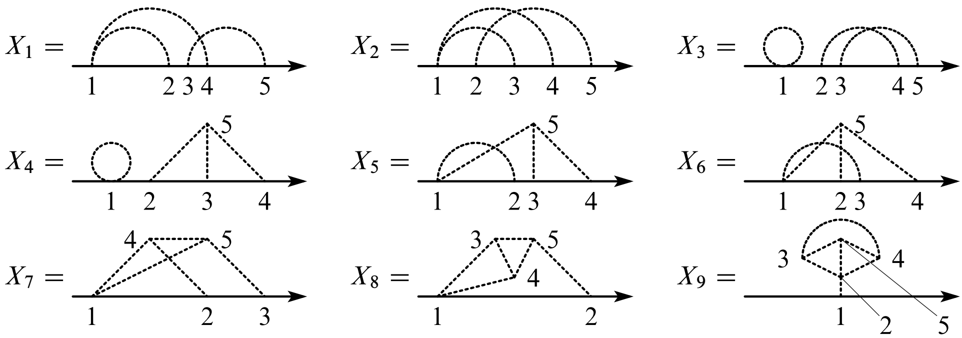

Figure 6. The graphs $X_1, \ldots, X_9$ that give a graph cocycle $\sum_i a_i X_i$; the edges are oriented from the vertex with the smaller labels.

Remark 3.4. The *Gramain* 1-*cycle* $G_f \colon S^1 \to \mathcal{K}$ for $f \in \mathcal{K}$ is a cycle that rotates f around the "long axis" $\mathbb{R}^1 \times \{(0, 0)\}$. Explicitly G_f is given by

$$G_f(\theta)(x) := \begin{pmatrix} 1 & & \\ & \cos\theta & \\ & & \sin\theta \end{pmatrix} f(x) \quad \text{for } \theta \in S^1, x \in \mathbb{R}^1. \tag{3-11}$$

Mortier [14, Theorem 4.1] has given a 1-cocycle α_3^1 of $\mathcal{K}$ in a combinatorial way and has proved that

$$\langle \alpha_3^1, G_f \rangle = v_2(f) \quad \text{and} \quad \langle \alpha_3^1, p_* FH_{(f,w)} \rangle = 6v_3(f) - w \cdot v_2(f) \tag{3-12}$$

for $(f, w) \in \mathcal{K} \times \mathbb{Z} \simeq \widetilde{\mathcal{K}}$. This result motivates us to compute the integration of $I(X)$ over the FH-cycles. We will give another proof of $\langle I(X), G_f \rangle = v_2(f)$; see Corollary 4.9 (actually this corrects the proof of [17, Theorem 3.1], see Remark 4.11).

4. Integration of $I(X)$ over the Fox–Hatcher cycle

Recall that $p \colon \widetilde{\mathcal{K}} \to \mathcal{K}$ is the map forgetting the framing of $\tilde{f}$. For any $\tilde{f} \in \widetilde{\mathcal{K}}$ we define

$$v(\tilde{f}) := \int_{p_* FH_{\tilde{f}}} I(X) = \sum_{1 \le k \le 9} a_k \int_{p_* FH_{\tilde{f}}} I(X_k). \tag{4-1}$$

This gives an isotopy invariant v for framed long knots. Our goal is to describe v as a linear combination of the Vassiliev invariants of order less or equal to three.

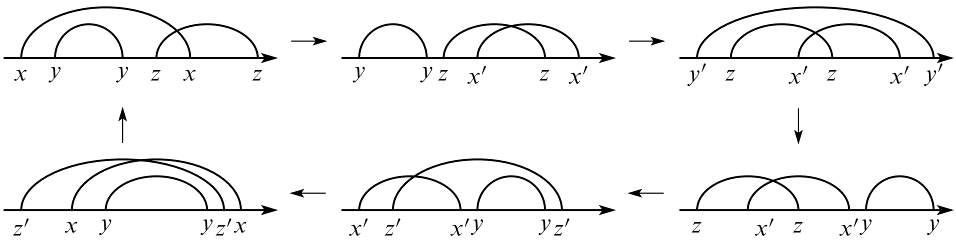

Figure 7. Type I cycle of the Gauss diagrams respecting three crossings under consideration; $\{x, y, z\} = \{c_1, c_2, c_3\}$.

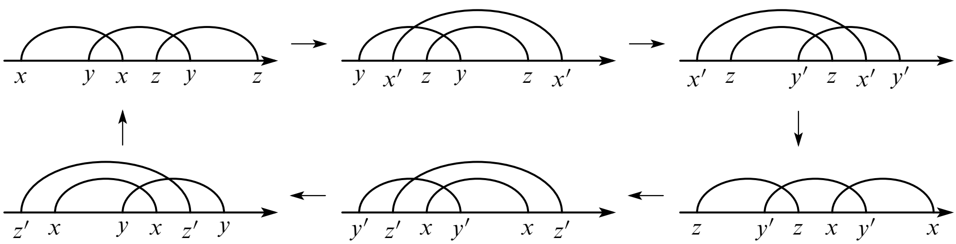

Figure 8. Type II cycle of the Gauss diagrams respecting three crossings under consideration; $\{x, y, z\} = \{c_1, c_2, c_3\}$.

4A. ***The invariant v is of order three.*** For any $\tilde{f} \in \widetilde{\mathcal{K}}$ and crossings $c_1, \dots, c_n$ of its diagram, define

$$D^n v(\tilde{f}) := \sum_{\epsilon_1, \dots, \epsilon_n \in \{+1, -1\}} \epsilon_1 \cdots \epsilon_n v(\tilde{f}_{\epsilon_1, \dots, \epsilon_n}), \tag{4-2}$$

where $\tilde{f}_{\epsilon_1, \dots, \epsilon_n}$ is a framed long knot obtained by changing, if necessary, the crossings c_i so that its sign is equal to ϵ_i. It should be noticed that $D^n v$ depends on the choice of crossings $c_1, \dots, c_n$, although it is not explicit in the notation. What we want to show is $D^4 v(\tilde{f}) = 0$ for any choice of $\tilde{f}$ and $c_1, \dots, c_4$.

Let c_1, c_2, c_3 be (part of the) crossings of a diagram D of $\tilde{f} \in \widetilde{\mathcal{K}}$ respecting the Gauss diagram G (Definition 2.3). Let us perform the FH-moves (described in Section 2) on all the crossings c of D and the corresponding newborn crossings c'. The Gauss diagram that the three crossings under consideration respect changes as in Figure 3 when the FH-move is performed on one of c_i and c'_i $(i = 1, 2, 3)$, and in the sequence of the FH-moves realizing the FH-cycle, six Gauss diagrams (some of which may be equal to each other) respected by the three crossings under consideration form a cycle. Figures 7, 8 and 9 show three such cycles.

There are 15 Gauss diagrams with three edges, and only 10 of them are included in these three cycles. The remaining five Gauss diagrams form the other two cycles, that we omit since in fact they do not contribute to our computation in Section 4B.

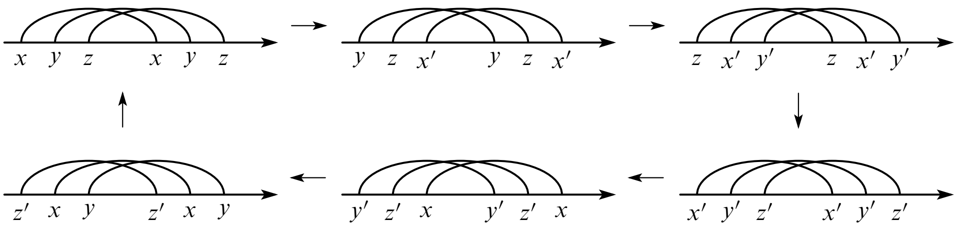

Figure 9. Type III cycle of the Gauss diagrams respecting three crossings under consideration; $\{x, y, z\} = \{c_1, c_2, c_3\}$.

Theorem 4.1. *For any crossings c_1, c_2, c_3, $D^3 v(\tilde{f})$ is given by*

$$D^3 v(\tilde{f}) = \begin{cases} -2 & \textit{if } c_1, c_2 \textit{ and } c_3 \textit{ respect one of the Gauss diagrams in type I cycle}, \\ 2 & \textit{if } c_1, c_2 \textit{ and } c_3 \textit{ respect one of the Gauss diagrams in type II cycle}, \\ 6 & \textit{if } c_1, c_2 \textit{ and } c_3 \textit{ respect the unique Gauss diagram in type III cycle}, \\ 0 & \textit{otherwise}. \end{cases} \tag{4-3}$$

Corollary 4.2. *The invariant v is a Vassiliev invariant for framed long knots of order exactly three.*

Proof. Let $c_1, \dots, c_4$ be crossings of a diagram of $\tilde{f} \in \widetilde{\mathcal{K}}$. Let $\tilde{f}_\pm$ be knots obtained by changing c_4 so that its sign is respectively ± 1. Then by definition

$$D^4 v(f) = D^3 v(f_+) - D^3 v(f_-). \tag{4-4}$$

Moreover c_1, c_2 and c_3 of f_+ and f_- respect the same Gauss diagram. Thus we have $D^3 v(\tilde{f}_+) = D^3 v(\tilde{f}_-)$ by Theorem 4.1, concluding $D^4 v(\tilde{f}) = 0$.

Theorem 4.1 also says that $D^3 v(\tilde{f})$ can be nonzero, and v is not of order two or less. □

The next subsection is devoted to the proof of Theorem 4.1.

4B. ***Computation of $D^3 v$.*** We again remind that $D^3 v$ depends on the choice of crossings c_1, c_2, c_3. As in Example 3.2, we assume that

- $\mathrm{vol} \in \Omega^2_{DR}(S^2)$ is an antisymmetric unit volume form of S^2 whose support is contained in small neighborhoods of poles $(0, 0, \pm 1) \in S^2$, and
- we compute $D^3 v(\tilde{f})$ after transforming $\tilde{f}$ to be "almost planar."

We moreover assume, just for simplicity, that

- $\tilde{f}$ runs parallel to the x- and y-axes at each crossings (see Figure 15).

For $k = 1, \dots, 9$, consider the pullback square:

$$\begin{tikzcd}
(p \circ FH_{\tilde f})^* E_{X_k} \arrow[rr, "\overline{p \circ FH_{\tilde f}}"] \arrow[d, "\pi'_{X_k}"'] & & E_{X_k} \arrow[d, "\pi_{X_k}"] \\
S^1 \arrow[r, "FH_{\tilde f}"] & \widetilde{\mathcal{K}} \arrow[r, "p"] & \mathcal{K}
\end{tikzcd} \tag{4-5}$$

Then

$$\begin{aligned}
\int_{p_* FH_{\tilde f}} I(X_k) &= \int_{S^1} (p \circ FH_{\tilde f})^* \pi_{X_k *} \omega_{X_k} \\
&= \int_{S^1} \pi'_{X_k *} \overline{p \circ FH_{\tilde f}}^* \omega_{X_k} \\
&= \int_{(p \circ FH_{\tilde f})^* E_{X_k}} \overline{p \circ FH_{\tilde f}}^* \omega_{X_k}.
\end{aligned} \tag{4-6}$$

Note that $(p \circ FH_{\tilde f})^* E_{X_k}$ is explicitly given by

$$\begin{aligned}
&(p \circ FH_{\tilde f})^* E_{X_k} \\
&= \left\{ (p(FH_{\tilde f}(\theta)), y) \in \mathcal{K} \times \mathrm{Conf}_{\mathrm{v_i}+\mathrm{v_f}}(\mathbb{R}^3) \;\middle|\; \begin{array}{l} \theta \in S^1,\ y_i = p(FH_{\tilde f}(\theta))(x_i) \\ \text{for some } x_i \in \mathbb{R}^1,\ 1 \le i \le \mathrm{v_i} \end{array} \right\} \\
&\subset E_{X_k}.
\end{aligned} \tag{4-7}$$

Suppose a diagram D of $\tilde f$ has n crossings. Then $FH_{\tilde f}$ can be realized on knot diagram by the sequence of $2n$ FH-moves on c or c', where c is one of the crossings of D and c' is a newly created crossing after the FH-move on c. We can decompose S^1 into $2n$ intervals

$$S^1 = \bigcup_c (I_c \cup I_{c'}) \tag{4-8}$$

such that $FH_{\tilde f}$ restricted on I_c (resp. $I_{c'}$) realizes the FH-move on c (resp. c').

Definition 4.3. Under the above setting, define

$$E_{k;c,c'} := \{(p_*(FH_{\tilde f}(\theta)), y) \in (p \circ H_{\tilde f})^* E_{X_k} \mid \theta \in I_c \cup I_{c'}\}. \tag{4-9}$$

By definition we have

$$(p \circ FH_{\tilde f})^* E_{X_k} = \bigcup_c E_{k;c,c'} \tag{4-10}$$

and hence

$$\int_{(p \circ FH_{\tilde f})^* E_X} \omega_{X_k} = \sum_c \int_{E_{k;c,c'}} \overline{p \circ FH_{\tilde f}}^* \omega_{X_k}. \tag{4-11}$$

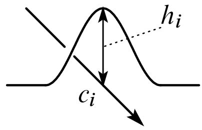

Figure 10. h_i $(i = 1, 2, 3)$.

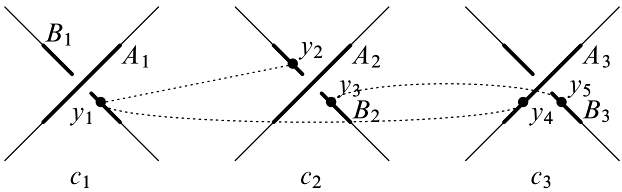

Figure 11. An element of $E_{1;c_1,c_1',A_1}$.

Combining (4-1), (4-2), (4-6) and (4-11), we have

$$D^3 v(\tilde{f}) = \sum_{1\le k\le 9} a_k \sum_{c} \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \epsilon_1\epsilon_2\epsilon_3 \int_{E_{k;c,c'}} \overline{p\circ FH_{\tilde{f}_{\epsilon_1,\epsilon_2,\epsilon_3}}}^{\,*}\omega_{X_k}. \tag{4-12}$$

4B1. *Eliminating $X_3, \dots, X_9$.* Let h_i $(i = 1, 2, 3)$ be the distance between two arcs at c_i, $i = 1, 2, 3$ (Figure 10).

We may compute $D^3 v(\tilde{f})$ in the limit $h_i \to 0$ $(i = 1, 2, 3)$ since v is an invariant. In this limit, only the graphs X_1 and X_2 essentially contribute to $D^3 v(\tilde{f})$;

Proposition 4.4. (1) *For $k = 1, \dots, 9$ and any crossing c other than c_1, c_2, c_3, we have*

$$\lim_{h_1,h_2,h_3\to 0} \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \epsilon_1\epsilon_2\epsilon_3 \int_{E_{k;c,c'}} \overline{p\circ FH_{\tilde{f}_{\epsilon_1,\epsilon_2,\epsilon_3}}}^{\,*}\omega_{X_k} = 0. \tag{4-13}$$

(2) *If $k = 3, \dots, 9$, then* (4-13) *also holds for $c \in \{c_1, c_2, c_3\}$.*

Consequently

$$D^3 v(\tilde{f}) = \lim_{h_1,h_2,h_3\to 0} \sum_{k=1,2} a_k \times \sum_{c\in\{c_1,c_2,c_3\}} \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \epsilon_1\epsilon_2\epsilon_3 \int_{E_{k;c,c'}} \overline{p\circ FH_{\tilde{f}_{\epsilon_1,\epsilon_2,\epsilon_3}}}^{\,*}\omega_{X_k}. \tag{4-14}$$

Proof of Proposition 4.4 (1). Let $-1 < p_i < q_i < 1$ $(i = 1, 2, 3)$ with $p_1 < p_2 < p_3$ be the real numbers such that $f(p_i)$ and $f(q_i)$ correspond to c_i, and let A_i, B_i be small open intervals that include respectively p_i and q_i (see Figure 11). Let $E_{k;c,c',A_1} \subset E_{k;c,c'}$ be the subspace consisting of (θ, y) with no y_j $(1 \le j \le \mathrm{v_i})$ being in A_1.

Then even if we set $h_1 = 0$, any two points y_j and $y_{j'}$ corresponding to endpoints of a single edge of X_k do not collide in $E_{k;c,c',A_1}$, and the maps φ_α and hence the integrand ω_{X_k} can be defined on $E_{k;c,c',A_1}$. This implies

$$\lim_{h_1\to 0}\left(\int_{E_{k;c,c',A_1}} \overline{p\circ FH_{\tilde f_{+1,\epsilon_2,\epsilon_3}}}^*\omega_{X_k} - \int_{E_{k;c,c',A_1}} \overline{p\circ FH_{\tilde f_{-1,\epsilon_2,\epsilon_3}}}^*\omega_{X_k}\right) = 0. \tag{4-15}$$

If we analogously define $E_{k;c,c',A_m}$ and $E_{k;c,c',B_m}$, then similar cancellation to (4-15) occurs for them. Moreover we have

$$\bigcup_{m=1,2,3} (E_{k;c,c',A_m} \cup E_{k;c,c',B_m}) = E_{k;c,c'} \tag{4-16}$$

because no X_k has six or more i-vertices. Although $A_1, \dots, B_3$ are not disjoint, we can arrange them to be disjoint by considering their difference sets and intersections (on which the same argument is valid). Thus we have (4-13). □

Proof of Proposition 4.4 (2) for $k = 7, 8, 9$. It is enough to consider the case $c = c_1$; the cases $c = c_2, c_3$ can be proved similarly.

The similar argument in the proof of (1) also implies (4-15) with A_1 and h_1 replaced respectively by A_m (or B_m) and h_m, $m = 2, 3$. We thus complete the proof, because X_k ($k = 7, 8, 9$) has three or less i-vertices and we have

$$E_{k;c_1,c_1'} = \bigcup_{m=2,3} (E_{k;c_1,c_1',A_m} \cup E_{k;c_1,c_1',B_m}). \tag{4-17}$$ □

Proof of Proposition 4.4 (2) for $k = 5, 6$. It is enough to consider the case $c = c_1$.

Let $E \subset E_{k;c_1,c_1'}$ be the subspace of $E_{k;c_1,c_1'}$ consisting of (θ, y) with each of A_2, B_2, A_3 and B_3 containing at least one y_j corresponding to an i-vertex j of X_k. Then the integrations in (4-13) with $E_{k;c_1,c_1'}$ replaced by $E_{k;c_1,c_1'} \setminus E$ are defined even if we set $h_m = 0$ for at least one $m \in \{2, 3\}$, and the cancellation similar to (4-15) occurs, similarly as the above proof of (2) for $k = 7, 8, 9$. Thus it suffices to show (4-13) with $E_{k;c_1,c_1'}$ replaced by E. Since X_k ($k = 5, 6$) has four i-vertices, each of A_2, B_2, A_3 and B_3 contains exactly one point on E. We divide E into two subspaces:

Type I: The subspace E_I of E consisting of (θ, y) with $y_5 \in \mathbb{R}^3$ outside neighborhoods of c_2 and c_3. Then two i-vertices (4 and 5 in the case of Figure 12) corresponding to the points in $A_m \cup B_m$ are not joined by any edge, for at least one $m = 2, 3$.

Even if we set $h_m = 0$ and these two points may collide, all the maps φ_α and hence ω_{X_k} are still defined on E_I, and the cancellation similar to (4-15) occurs.

Type II: the subspace E_{II} of E consisting of (θ, y) with $y_5 \in \mathbb{R}^3$ in a neighborhood of c_m, $m \in \{2, 3\}$ (see Figure 13; setting $h_2 = 0$ or $h_3 = 0$ are problematic on this subspace).

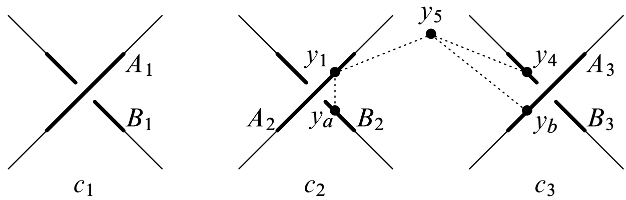

Figure 12. Proposition 4.4(2) for $k = 5, 6$, Type I subspace ($m = 3$, $\{a, b\} = \{2, 3\}$); one of the arcs A_1 and B_1 moves in the FH-move on c_1.

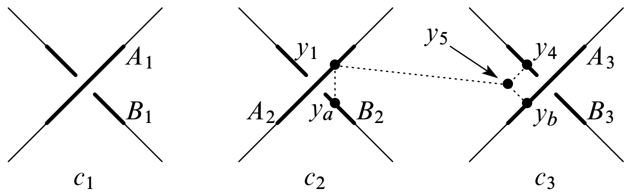

Figure 13. Proposition 4.4(2) for $k = 5, 6$, Type II subspace ($m = 3$, $\{a, b\} = \{2, 3\}$).

On E_{II} at least one edge α of X_k joins the vertex 5 and j with the corresponding point y_j not on $A_m \cup B_m$ ($j = 1$ in the case of Figure 13). Then the image of φ_α is not included in supp(vol) and hence $\varphi_\alpha^* \operatorname{vol} = 0$, because supp(vol) is assumed to be in neighborhoods of $(0, 0, \pm 1) \in S^2$ and our $\tilde{f}$ is almost planar. The integrand ω_{X_k} is therefore zero on E_{II}. $\square$

Proof of Proposition 4.4 (2) for $k = 4$. Consider the case $c = c_1$ (the same arguments are valid for $c = c_2, c_3$). Let $E \subset E_{4;c_1,c_1'}$ be the subspace consisting of (θ, y) where each of A_2, B_2, A_3 and B_3 contains at least one point. It is then enough to show (4-13) with $E_{4;c_1,c_1'}$ replaced by E, as in the above proofs.

As X_4 has four i-vertices, each of A_2, B_2, A_3 and B_3 contains exactly one point on E. In particular $y_1 \in A_2$, and the map φ_α for the loop α at the vertex 1 has the image outside supp(vol) by our assumption on $\tilde{f}$ and vol, and hence ω_{X_4} vanishes on E. $\square$

Proof of Proposition 4.4 (2) for $k = 3$. Again consider the case $c = c_1$. Let $E \subset E_{3;c_1,c_1'}$ be the subspace consisting of (θ, y) satisfying both (i) and (ii):

(i) y_1 is on the arc C that moves in the FH-moves on c_1.

(ii) Each of A_2, B_2, A_3 and B_3 contains exactly one of $y_2, \ldots, y_5$.

Then it suffices to show (4-13) with $E_{3;c_1,c_1'}$ replaced by E. This is because:

- If E' denotes the subspace of $E_{3;c_1,c_1'}$ consisting of (θ, y) that does not satisfy (ii), then the integrations in (4-13) with $E_{3;c_1,c_1'}$ replaced by E' are defined

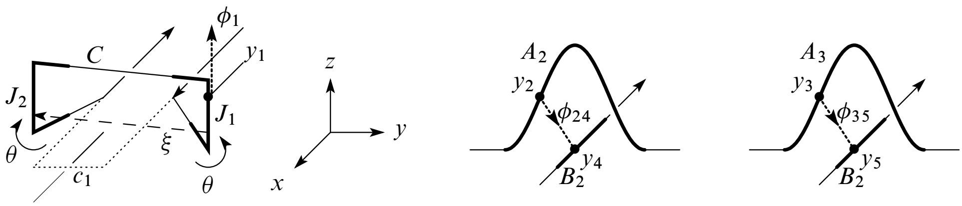

Figure 14. The configuration that can nontrivially contribute to $I(X_3)$.

even if we set $h_m = 0$ for at least one $m \in \{2, 3\}$ and the cancellation similar to (4-15) occurs, by the same reason as in the above proofs.

- If E'' denotes the subspace of $E_{3;c_1,c_1'}$ consisting of (θ, y) that satisfies (ii) but does not satisfy (i). then the map φ_α (α is the loop of X_3 at the i-vertex labeled by 1) has its image outside on supp(vol) since $\tilde{f}$ is supposed to be almost planar, and hence ω_{X_3} vanishes on E''.

Figure 14 shows the configurations in E that may nontrivially contribute to the integration of $I(X_3)$.

Let J_s $(s = 1, 2)$ be the unit intervals identified with those on C drawn with thick curves in Figure 14. We write $p_* FH_{\tilde{f}}(\theta)$ as f_θ for short. Define $\phi_1 \colon I_{c_1} \times J_s \to S^2$ $(s = 1, 2)$, $\phi_{24} \colon A_2 \times B_2 \to S^2$ and $\phi_{35} \colon A_3 \times B_3 \to S^2$ by

$$\phi_1(\theta, t) := \frac{f_\theta'(t)}{|f_\theta'(t)|}, \quad \phi_{ij}(t, u) := \frac{f(u) - f(t)}{|f(u) - f(t)|}, \quad (i, j) = (2, 4), (3, 5). \tag{4-18}$$

Then

$$\int_E \overline{p \circ FH_{\tilde{f}}}^* \omega_{X_3} = \int_{I_{c_1} \times (J_1 \sqcup J_2)} \phi_1^* \operatorname{vol} \int_{A_2 \times B_2} \phi_{24}^* \operatorname{vol} \int_{A_3 \times B_3} \phi_{35}^* \operatorname{vol}. \tag{4-19}$$

Define the diffeomorphisms $\xi \colon J_1 \to J_2$ and $\eta \colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\xi(t) = 1 - t, \quad \eta(x, y, z) := (-x, y, -z). \tag{4-20}$$

Then, with respect to the coordinates of $\mathbb{R}^3$ shown in Figure 14, the following diagram commutes:

$$\begin{CD} I_{c_1} \times J_1 @>{\phi_1}>> S^2 \\ @V{\mathrm{id}\times\xi}VV \circlearrowright @VV{\eta}V \\ I_{c_1} \times J_2 @>{\phi_1}>> S^2 \end{CD} \tag{4-21}$$

and since ξ reverses the orientation and η preserves the orientation, we have

$$\int_{I_{c_1} \times J_2} \phi_1^* \operatorname{vol} = -\int_{J_{c_1} \times J_1} \phi_1^* \operatorname{vol} \tag{4-22}$$

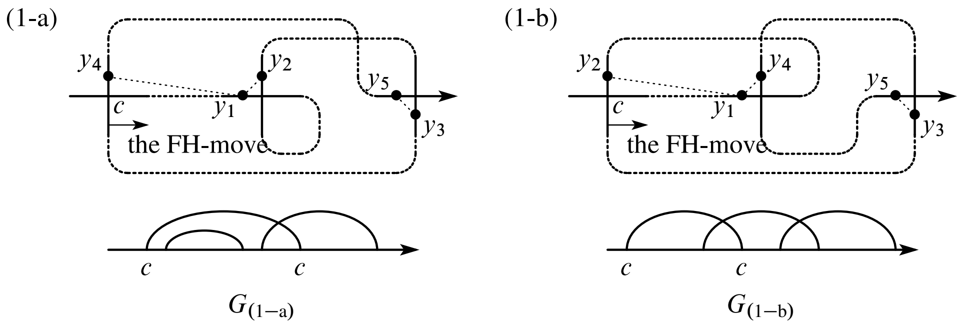

Figure 15. Configurations essentially contributing to $I(X_1)$; they can exist only if the three crossings under consideration respect the Gauss diagrams $G_{(1\text{-a})}$ or $G_{(1\text{-b})}$.

and hence

$$\int_{I_{c_1}\times(J_1\sqcup J_2)} \phi_1^* \operatorname{vol} = \sum_{s=1,2} \int_{I_{c_1}\times J_s} \phi_1^* \operatorname{vol} = 0. \tag{4-23}$$

Thus (4-19) is zero. □

Thus we only need to compute the alternating sums of the integrations of $I(X_1)$ and $I(X_2)$ in the limit $h_1, h_2, h_3 \to 0$.

4B2. *Computation of $I(X_1)$.* The following two subspaces of $E_{1;c_j,c'_j}$ $(j = 1, 2, 3)$ do not essentially contribute to the alternating sum of $I(X_1)$.

- The subspace where the arc near the left-most crossing moving in the FH-move contains no point; because the integrals on the subspace are the same for $\epsilon_j = +1$ and $\epsilon_j = -1$ and they cancel in the alternating sum.
- The subspace where no edge joins points on A_m and B_m $(m = 2, 3)$; because all the maps φ_α and hence the integrand ω_{X_1} can be defined even if $h_m = 0$ and thus the cancellation similar to (4-15) occurs.

Thus only the subspaces of types (1-a) and (1-b) consisting of (θ, y) as shown in Figure 15 can essentially contribute to the integrations of $I(X_1)$.

In both cases, the arc near the left-most crossing containing y_2 (case (1-a)) or y_4 (case (1-b)) moves to right in the FH-move, and when the arc comes over or under the middle crossing, the map φ_{12} or φ_{14} has its image in supp(vol) and the integrand is not zero at that moment.

If three crossings c_1, c_2, c_3 under consideration respect one of the Gauss diagrams in the Type I cycle (Figure 7), then in the FH-cycle we meet the situation (1-a) in Figure 15 once, because the Gauss diagram $G_{(1\text{-a})}$ appears once in the Type I cycle. If c_1, c_2, c_3 respect one of the Gauss diagrams in the Type II cycle (Figure 8), then in the FH-cycle we meet the situation (1-b) in Figure 15 twice, because the Gauss

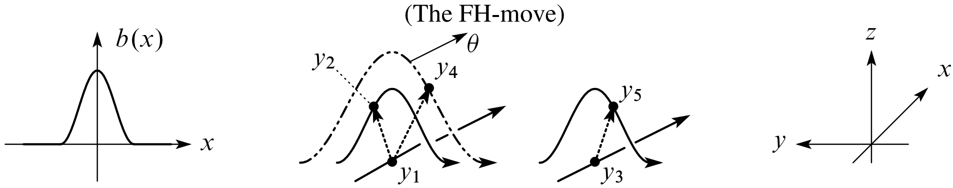

Figure 16. Proof of Proposition 4.5; the case (1-a).

diagram $G_{(\text{1-b})}$ appears twice in the Type II cycle. Otherwise we do not meet the situations (1-a) nor (1-b) and the integration vanishes.

Proposition 4.5. *We have*

$$\epsilon_1\epsilon_2\epsilon_3 \sum_{c\in\{c_1,c_2,c_3\}} \int_{E_{1;c,c'}} \overline{p\circ FH_{\tilde f_{\epsilon_1,\epsilon_2,\epsilon_3}}}^{\,*}\omega_{X_1} \tag{4-24}$$
$$= \begin{cases} \frac{1}{8} & \textit{if } c_1, c_2, c_3 \textit{ respect one of the Gauss diagrams in Type I cycle,} \\ -\frac{1}{4} & \textit{if } c_1, c_2, c_3 \textit{ respect one of the Gauss diagrams in Type II cycle,} \\ 0 & \textit{otherwise}; \end{cases}$$

see Figures 7 and 8 for Type I and II cycles, respectively.

Proof. Consider the first case; we may assume that c_1, c_2, c_3 respect the Gauss diagram $G_{(\text{1-a})}$. Then only $E_{1;c_1,c_1'}$ can contain the configurations of type (1-a) and nontrivially contribute to the alternating sum of the integrations of $I(X_1)$.

Let $b\colon \mathbb{R}^1 \to \mathbb{R}^1$ be a smooth even function whose graph looks as in Figure 16. For $(\theta, x_1, \dots, x_5) \in \mathbb{R}^6$, consider $y_1, \dots, y_5 \in \mathbb{R}^3$ given by

$$\begin{aligned} y_1 &= (x_1, 0, 0), \\ y_2 &= (0, -\epsilon_2 x_2, b(\epsilon_2 x_2)), \\ y_3 &= (x_3, 0, 0), \\ y_4 &= (\theta, -\epsilon_1 x_4, 2b(\epsilon_1 x_4/2)), \\ y_5 &= (0, -\epsilon_3 x_5, b(\epsilon_3 x_5)) \end{aligned} \tag{4-25}$$

and define $\varphi\colon \mathbb{R}^6 \to (S^2)^{\times 3}$ by

$$\varphi(\theta, x_1, \dots, x_5) := \left(\frac{y_2 - y_1}{|y_2 - y_1|}, \frac{y_5 - y_3}{|y_5 - y_3|}, \frac{y_4 - y_1}{|y_4 - y_1|} \right). \tag{4-26}$$

Then changing the variables suitably, the left hand side of (4-24) is equal to

$$\epsilon_1\epsilon_2\epsilon_3 \int_{\mathbb{R}^6} \varphi^*(\mathrm{vol}^{\times 3}), \tag{4-27}$$

where $\mathrm{vol}^{\times 3} = \mathrm{pr}_1^* \mathrm{vol} \wedge \mathrm{pr}_2^* \mathrm{vol} \wedge \mathrm{pr}_3^* \mathrm{vol} \in \Omega^6_{DR}((S^2)^{\times 3})$.

Define $\Phi\colon \mathbb{R}^6 \to (\mathbb{R}^2)^{\times 3}$ and $\psi_s\colon \mathbb{R}^2 \to S^2$ $(s = 1, 2)$ by respectively

$$\Phi(\theta, x_1, \dots, x_5) := ((x_1, \epsilon_2 x_2), (x_1 - \theta, \epsilon_1 x_4), (x_3, \epsilon_3 x_5)), \tag{4-28}$$

$$\psi_1(x, x') := \frac{y' - y}{|y' - y|}, \quad \psi_2(x, x') := \frac{y'' - y}{|y'' - y|}, \tag{4-29}$$

where $y := (x, 0, 0)$, $y' := (0, -x', b(x'))$, $y'' = (0, -x', 2b(x'/2))$. Then Φ is a linear diffeomorphism whose determinant is $\epsilon_1\epsilon_2\epsilon_3$, and the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{R}^6 \arrow[rr, "\varphi"] \arrow[dr, "\Phi"'] \arrow[rr, phantom, "\circlearrowleft"', yshift=-6ex] & & (S^2)^{\times 3} \\
& (\mathbb{R}^2)^{\times 3} \arrow[ur, "\psi_1^{\times 2} \times \psi_2"'] &
\end{tikzcd} \tag{4-30}$$

Thus (4-27) is equal to

$$(\epsilon_1\epsilon_2\epsilon_3)^2 \biggl(\int_{\mathbb{R}^2} \psi_1^* \operatorname{vol} \biggr)^2 \int_{\mathbb{R}^2} \psi_2^* \operatorname{vol} = \bigl(\tfrac{1}{2}\bigr)^3 = \tfrac{1}{8}, \tag{4-31}$$

here $\frac{1}{2}$ appears by exactly the same reason as in Example 3.2.

The second case that c_1, c_2, c_3 respect the Gauss diagram $G_{(1\text{-b})}$ can be similarly computed, replacing

- (4-25) and (4-26) respectively with

$$\begin{aligned}
y_1 &= (x_1, 0, 0),\\
y_2 &= (\theta, -\epsilon_2 x_2, b(\epsilon_2 x_2/2)),\\
y_3 &= (x_3, 0, 0),\\
y_4 &= (0, -\epsilon_1 x_4, b(\epsilon_1 x_4)),\\
y_5 &= (0, -\epsilon_3 x_5, b(\epsilon_3 x_5)),
\end{aligned} \tag{4-32}$$

$$\varphi(\theta, x_1, \dots, x_5) := \left(\frac{y_4 - y_1}{|y_4 - y_1|}, \frac{y_5 - y_3}{|y_5 - y_3|}, \frac{y_2 - y_1}{|y_2 - y_1|} \right) \tag{4-33}$$

  (namely y_2 and y_4 are swapped), and

- (4-28) with

$$\Phi(\theta, x_1, \dots, x_5) := ((x_1, \epsilon_2 x_2), (x_1 - \theta, \epsilon_1 x_4), (x_3, \epsilon_3 x_5)). \tag{4-34}$$

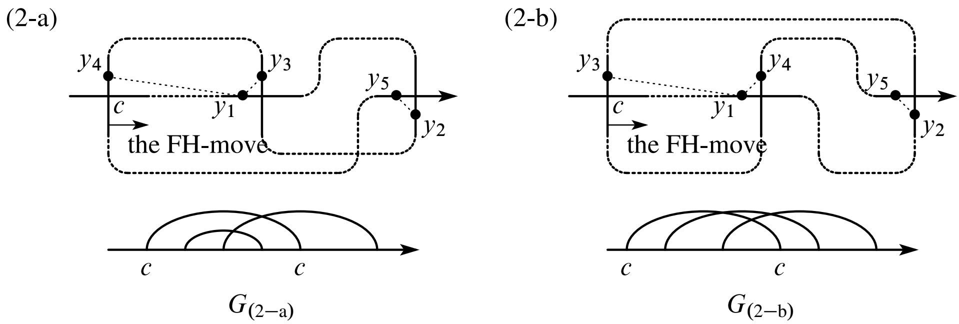

Figure 17. Configurations essentially contributing to $I(X_2)$; they can exist only if the three crossings under consideration respect the Gauss diagrams $G_{(2\text{-a})}$ or $G_{(2\text{-b})}$.

Then the determinant of Φ is $-\epsilon_1\epsilon_2\epsilon_3$, and because we meet the situation (1-b) twice in the FH-cycle, the left-hand side of (4-24) in this case is equal to

$$-2(\epsilon_1\epsilon_2\epsilon_3)^2\left(\int_{\mathbb{R}^2}\psi_1^*\,\mathrm{vol}\right)^2\int_{\mathbb{R}^2}\psi_2^*\,\mathrm{vol}=-\tfrac{1}{4}. \tag{4-35}$$ □

4B3. *Computation of $I(X_2)$.* The computation of $I(X_2)$ goes similarly to that of $I(X_1)$. Only the subspaces of types (2-a) and (2-b) consisting of (θ, y) as shown in Figure 17 can essentially contribute to the alternating sum of the integrations of $I(X_2)$.

If three crossings c_1, c_2, c_3 under consideration respect one of the Gauss diagrams in Type II cycle (Figure 8), then in the FH-cycle we meet the situation (2-a) in Figure 15 twice, because the Gauss diagram $G_{(2\text{-a})}$ appears twice in Type II cycle. If c_1, c_2, c_3 respect one of the Gauss diagrams in Type III cycle (Figure 9), then in the FH-cycle we meet the situation (2-b) in Figure 15 six times, because the Gauss diagram $G_{(2\text{-b})}$ appears six times in Type III cycle.

Proposition 4.6. *We have*

$$\epsilon_1\epsilon_2\epsilon_3\sum_{c\in\{c_1,c_2,c_3\}}\int_{E_{2;c,c'}}\overline{p\circ FH_{\tilde{f}_{\epsilon_1,\epsilon_2,\epsilon_3}}}^*\omega_{X_2} \tag{4-36}$$
$$=\begin{cases}-\frac{1}{4} & \text{if } c_1,c_2,c_3 \text{ respect one of the Gauss diagrams in Type II cycle},\\ \frac{3}{4} & \text{if } c_1,c_2,c_3 \text{ respect one of the Gauss diagrams in Type III cycle},\\ 0 & \text{otherwise};\end{cases}$$

see Figures 8 and 9 for Type II and III cycles, respectively.

Proof. Consider the first case that c_1, c_2, c_3 respect the Gauss diagram $G_{(2\text{-a})}$. Then only $E_{2;c_1,c_1'}$ can contain the configurations of type (2-a) and nontrivially contribute to the integral.

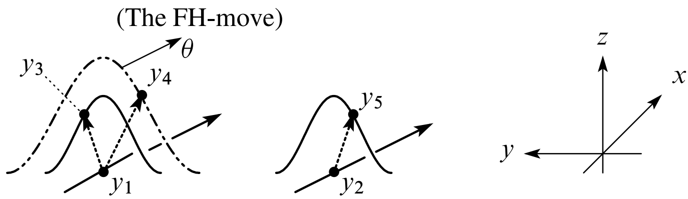

Figure 18. Proof of Proposition 4.6.

The proof of this case goes very similarly to the above ones; we just need to replace

- (4-25) and (4-26) respectively with

$$\begin{aligned} y_1 &= (x_1, 0, 0),\\ y_2 &= (x_2, 0, 0),\\ y_3 &= (0, -\epsilon_2 x_3, b(\epsilon_2 x_3)), \\ y_4 &= (\theta, -\epsilon_1 x_4, 2b(\epsilon_1 x_4/2)),\\ y_5 &= (0, \epsilon_3 x_5, b(\epsilon_3 x_5)), \end{aligned} \tag{4-37}$$

$$\varphi(\theta, x_1, \dots, x_5) := \left(\frac{y_3 - y_1}{|y_3 - y_1|}, \frac{y_5 - y_2}{|y_5 - y_2|}, \frac{y_4 - y_1}{|y_4 - y_1|} \right), \tag{4-38}$$

- (4-28) with

$$\Phi(\theta, x_1, \dots, x_5) := ((x_1, \epsilon_2 x_3), (x_1 - \theta, \epsilon_1 x_4), (x_2, \epsilon_3 x_5)). \tag{4-39}$$

Then Φ is a linear diffeomorphism with the determinant $-\epsilon_1\epsilon_2\epsilon_3$, and because we meet the situation (2-a) twice in the FH-cycle, the left hand side of (4-36) in this case is equal to

$$-2(\epsilon_1\epsilon_2\epsilon_3)^2 \left(\int_{\mathbb{R}^2} \psi_1^* \operatorname{vol} \right)^2 \int_{\mathbb{R}^2} \psi_2^* \operatorname{vol} = -\tfrac{1}{4}. \tag{4-40}$$

Consider the second case that c_1, c_2, c_3 respect the Gauss diagram $G_{(2\text{-b})}$. The proof of this case goes very similarly to that of the case (1-b) in Proposition 4.5; we just need to replace

- (4-25) and (4-26) respectively with

$$
\begin{aligned}
y_1 &= (x_1, 0, 0),\\
y_2 &= (x_2, 0, 0),\\
y_3 &= (\theta, -\epsilon_1 x_3, 2b(\epsilon_1 x_3/2)),\\
y_4 &= (0, -\epsilon_2 x_4, b(\epsilon_2 x_4)),\\
y_5 &= (0, \epsilon_3 x_5, b(\epsilon_3 x_5)),
\end{aligned}
\tag{4-41}
$$

$$
\varphi(\theta, x_1, \dots, x_5) := \left(\frac{y_4 - y_1}{|y_4 - y_1|}, \frac{y_5 - y_2}{|y_5 - y_2|}, \frac{y_3 - y_1}{|y_3 - y_1|}\right), \tag{4-42}
$$

- (4-28) with

$$
\Phi(\theta, x_1, \dots, x_5) := ((x_1 - \theta, \epsilon_2 x_3), (x_1, \epsilon_1 x_4), (x_2, \epsilon_3 x_5)). \tag{4-43}
$$

Then Φ is a linear diffeomorphism with the determinant $\epsilon_1\epsilon_2\epsilon_3$, and because we meet the situation (2-b) six times in the FH-cycle, the left hand side of (4-36) in this case is equal to

$$
6(\epsilon_1\epsilon_2\epsilon_3)^2\left(\int_{\mathbb{R}^2} \psi_1^* \operatorname{vol}\right)^2 \int_{\mathbb{R}^2} \psi_2^* \operatorname{vol} = \tfrac{3}{4}. \tag{4-44}
$$
□

Proof of Theorem 4.1. Let c_1, c_2 and c_3 respect one of the Gauss diagrams in Type I cycle (Figure 7). Then by (4-14) and Propositions 4.5, 4.6 we have

$$
\begin{aligned}
D^3 v(\tilde{f}) &= \sum_{k=1,2} a_k \sum_{c\in\{c_1,c_2,c_3\}} \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \epsilon_1\epsilon_2\epsilon_3 \int_{E_{k;c,c'}} \overline{p\circ FH_{\tilde{f}_{\epsilon_1,\epsilon_2,\epsilon_3}}}^* \omega_{X_k}\\
&= (-2)\cdot \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \tfrac{1}{8} + 1\cdot 0\\
&= -2\cdot\tfrac{1}{8}\cdot 8 = -2.
\end{aligned}
\tag{4-45}
$$

Next suppose that c_1, c_2 and c_3 respect one of the Gauss diagrams in Type II cycle (Figure 7). Then by (4-14) and Propositions 4.5 and 4.6,

$$
\begin{aligned}
D^3 v(\tilde{f}) &= \sum_{k=1,2} a_k \sum_{c\in\{c_1,c_2,c_3\}} \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \epsilon_1\epsilon_2\epsilon_3 \int_{E_{k;c,c'}} \overline{p\circ FH_{\tilde{f}_{\epsilon_1,\epsilon_2,\epsilon_3}}}^* \omega_{X_k}\\
&= (-2)\cdot \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \left(-\tfrac{1}{4}\right) + 1\cdot \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \left(-\tfrac{1}{4}\right)\\
&= 2.
\end{aligned}
\tag{4-46}
$$

Lastly suppose that c_1, c_2 and c_3 respect one of the Gauss diagrams in Type III cycle (Figure 7). Then

$$\begin{aligned} D^3 v(\tilde{f}) &= \sum_{k=1,2} a_k \sum_{c\in\{c_1,c_2,c_3\}} \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \epsilon_1\epsilon_2\epsilon_3 \int_{E_{k;c,c'}} \overline{p\circ FH_{\tilde{f}_{\epsilon_1,\epsilon_2,\epsilon_3}}}^{\,*}\omega_{X_k} \\ &= (-2)\cdot 0 + 1\cdot \sum_{\epsilon_1,\epsilon_2,\epsilon_3\in\{+1,-1\}} \frac{3}{4} \\ &= 6. \end{aligned} \tag{4-47}$$

If c_1, c_2 and c_3 respect no Gauss diagram in three cycles, then $D^3 v(\tilde{f}) = 0$. □

4C. *An explicit description of v.* It is known (see [12, page 215] for example) that the space of the Vassiliev invariants for framed knots of order less than or equal to three are multiplicatively generated by the framing number lk, the Casson invariant v_2 and the order three invariant v_3 (characterized by the conditions in Theorem 1.2). Thus all the Vassiliev invariants of order less than or equal to three are linear combinations of

$$\mathrm{lk}, \quad v_2, \quad \mathrm{lk}^2, \quad v_3, \quad \mathrm{lk}\cdot v_2, \quad \mathrm{lk}^3. \tag{4-48}$$

Lemma 4.7. *Our invariant v is of the form $v = av_3 + b\,\mathrm{lk}\cdot v_2 + cv_2$ for some constants $a, b, c \in \mathbb{R}$.*

Proof. The value of v on the trivial long knot $f_0(x) = (x, 0, 0)$ together with a framing number $w \in \mathbb{Z}$ is a linear combination of w, w^2 and w^3 because $v_2(f_0) = v_3(f_0) = 0$. But by the definition $p_* H_{(f_0,w)}$ is a constant loop of $\mathcal{K}$ for any $w \in \mathbb{Z}$. Thus $v(f_0, w) = 0$ for any $w \in \mathbb{Z}$, and the coefficients of lk, lk^2 and lk^3 must be zero. □

Below we compute the constants a, b, c in Lemma 4.7. We denote by 3_1^+ and 3_1^- respectively the right-handed and the left-handed trefoil knots, by 4_1 the figure eight knot. By the formulas for v_2 and v_3 in [15, Theorems 1 and 2] we have

$$v_2(3_1^+) = v_2(3_1^-) = 1, \quad v_2(4_1) = -1, \quad v_3(4_1) = 0. \tag{4-49}$$

Proposition 4.8. *We have $a = 6$, $b = -1$.*

Proof. Consider the "standard" diagram of 3_1^+ in Figure 2 and write it as $f = f_{+,+,+}$. This can be seen as a framed long knot with framing number $+3$. The diagram $f_{-,-,-}$ is 3_1^- with framing number -3 and all the other $f_{\epsilon_1,\epsilon_2,\epsilon_3}$ are trivial. The Gauss diagram in Figure 2 appears in the Type III cycle in Figure 9 and $D^3 v(f) = 6$

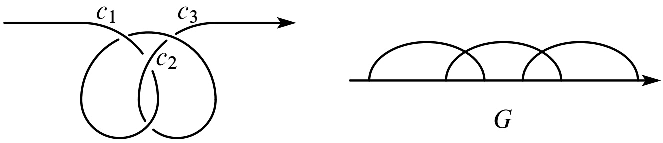

Figure 19. A diagram of $g = 4_1$ and the Gauss diagram that c_1, c_2, c_3 respect.

by Theorem 4.1. Thus we have

$$\begin{aligned} \text{(4-50)} \qquad 6 &= D^3 v(f) \\ &= (av_3(3_1^+) + b\cdot 3 + cv_2(3_1^+)) - (av_3(3_1^-) + b\cdot(-3) + cv_2(3_1^-)) \\ &= 2a + 6b, \end{aligned}$$

here the last equality holds by (4-49).

Next consider the diagram of 4_1 in Figure 19.

We write it as $g = g_{+,-,+}$, focusing on c_1, c_2, c_3. This can be seen as a framed long knot with framing number 0. Then $g_{-,-,-}$ is the 3_1^- with framing number -4 and all the other $g_{\epsilon_1,\epsilon_2,\epsilon_3}$ are trivial. The Gauss diagram G in Figure 19 appears in the Type II cycle in Figure 8 and $D^3 v(g) = 2$ by Theorem 4.1. Thus we have

$$\begin{aligned} \text{(4-51)} \qquad 2 &= D^3 v(g) \\ &= -(av_3(4_1) + b\cdot 0 + cv_2(4_1)) - (av_3(3_1^-) + b\cdot(-4) + cv_2(3_1^-)) \\ &= a + 4b, \end{aligned}$$

here the last equality holds again by (4-49). Therefore $a = 6$, $b = -1$ by (4-50) and (4-51). □

Corollary 4.9 [17, Theorem 3.1]. $\int_{G_f} I(X) = v_2(f)$ *for any* $f \in \mathcal{K}$.

Proof. It is not hard to see that $p_* FH_{(f,w+1)} = p_* FH_{(f,w)} - G_f$ for any $f \in \mathcal{K}$ and $w \in \mathbb{Z}$. Thus we have

$$\begin{aligned} \text{(4-52)} \qquad v(f, w+1) &= \int_{p_* FH_{(f,w+1)}} I(X) \\ &= \int_{p_* FH_{(f,w)}} I(X) - \int_{G_f} I(X) \\ &= v(f, w) - \int_{G_f} I(X). \end{aligned}$$

Since $v = 6v_3 - \mathrm{lk}\cdot v_2 + cv_2$,

$$\text{(4-53)} \quad 6v_3(f) - (w+1)v_2(f) + cv_2(f) = 6v_3(f) - wv_2(f) + cv_2 - \int_{G_f} I(X),$$

implying $\int_{G_f} I(X) = v_2(f)$. □

Proposition 4.10. *We have* $c = 0$.

Proof. Let $\tilde{f}$ be the knot 3_1^+ with the blackboard framing from the planar projection in Figure 2. Its framing number is $+3$, and as explained in [11], the FH-cycle $p_*FH_{\tilde{f}}$ is homologous to 3 times the Gramain cycle G_f (see Remark 3.4), where $f = p(\tilde{f}) \in \mathcal{K}$. This is because, as we can see in the figure in [11, page 4], the FH-move on each crossing of $\tilde{f}$ is the rotation around the long axis by degree π, and in the FH-cycle we perform the FH-moves six times. Thus

$$6v_3(3_1^+) - 3v_2(3_1^+) + cv_2(3_1^+) = v(\tilde{f}) = \int_{p_*FH_{\tilde{f}}} I(X) = 3\int_{G_f} I(X). \tag{4-54}$$

Corollary 4.9 allows us to rewrite (4-54) as

$$6 \cdot 1 - 3 \cdot 1 + c \cdot 1 = 3 \cdot 1, \tag{4-55}$$

and we have $c = 0$. □

This completes the proof of the formula $I(X) = 6v_3 - \mathrm{lk} \cdot v_2$ in Theorem 1.2.

Remark 4.11. In fact the proof of [17, Theorem 3.1] seems to contain an error. In [17, page 414] the second named author of the present paper claimed that "the zero-cycle e is given by $(\iota, 1)$", but e is indeed given by $(\iota, 2)$. Thus [17, Lemma 3.4] has to be corrected as "$D^2V(f) = \frac{1}{2}$" and consequently the evaluation of $I(X)$ over G_f should be $v_2(f)/2$, inconsistent with Corollary 4.9. Probably the proof of Corollary 4.9 is correct and this inconsistency comes from a missing factor of 2 in [16, Lemma 4.5], a special case of which ($n = 3$) is [17, Lemma 3.4].

Remark 4.12. An anonymous referee kindly suggested that the formula (1-1) in Theorem 1.2 can recover a result of Alvarez and Labastida [2]

$$v_3(T_{m,n}) = \frac{mn}{6} v_2(T_{m,n}) \tag{4-56}$$

for the (m, n)-torus (long) knot $T_{m,n}$.

The proof goes as follows. Let $\tilde{f}$ be a framed long knot whose underlying long knot is $f = p(\tilde{f}) = T_{m,n}$ and framing number $\mathrm{lk}(\tilde{f}) = w$. Then the formulas (1-1) and [17, Theorem 3.1] together with the fact that G_f generates $\pi_1(\mathcal{K}_f) \cong \mathbb{Z}$ if $f = T_{m,n}$ imply that $p_*FH_{\tilde{f}} = k(w)G_f$ for some $k(w) \in \mathbb{Z}$ and

$$6v_3(f) - w \cdot v_2(f) = \int_{p_*FH_{\tilde{f}}} I(X) = \int_{k(w)G_f} I(X) = k(w)v_2(f). \tag{4-57}$$

We can see that $k(mn) = 0$, proving the formula (4-56). To see this, we regard the space of framed long knots as that of framed embeddings $S^1 \hookrightarrow S^3$ that preserve the basepoints and have a prescribed framing at the basepoint, as explained in Section 2A. Then we have a homeomorphism

$$\widetilde{\mathrm{Emb}}(S^1, S^3) \approx \widetilde{\mathcal{K}} \times \mathrm{SO}(4), \quad \tilde{f} \mapsto (A^{-1} \cdot \tilde{f}, A(0)) \tag{4-58}$$

where $\widetilde{\mathrm{Emb}}(S^1, S^3)$ is the space of framed embeddings $S^1 \hookrightarrow S^3$ (without any basepoint conditions), $A\colon S^1 \to \mathrm{SO}(4)$ is the map given in Section 2A and $0 \in S^1 = [0, 1]/(0 \simeq 1)$ is the basepoint of S^1. This homeomorphism induces

$$\widetilde{\mathcal{K}} \approx \widetilde{\mathrm{Emb}}(S^1, S^3)/\,\mathrm{SO}(4) \tag{4-59}$$

and the Fox–Hatcher S^1-action on $\widetilde{\mathcal{K}}$ is interpreted as the reparametrization on the right hand side.

If $f = T_{m,n}$ is placed on the torus $\{(z, w) \in S^3 \mid |z| = |w| = 1/\sqrt{2}\}$ in the standard way, and if f is given the framing mn, then the reparametrization of $\tilde{f} = (f, mn) \in \widetilde{\mathrm{Emb}}(S^1, S^3)$ by $t \in S^1$ can be described as the multiplication of

$$r_{m,n}(t) = \begin{pmatrix} e^{2\pi\sqrt{-1}\,mt} & 0 \\ 0 & e^{2\pi\sqrt{-1}\,nt} \end{pmatrix} \in \mathrm{SO}(4). \tag{4-60}$$

In other words $FH_{(T_{m,n},mn)}$ is trivial on $\widetilde{\mathrm{Emb}}(S^1, S^3)/\,\mathrm{SO}(4)$ and thus on $\widetilde{\mathcal{K}}$. Therefore we have $k(mn) = 0$.

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ON THE THEORY OF GENERALIZED ULRICH MODULES

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Dedicated with gratitude to the memory of Professor Shiro Goto

We further develop the theory of generalized Ulrich modules introduced in 2014 by Goto et al. Our main goal is to address the problem as to when the operations of taking the Hom functor and horizontal linkage preserve the Ulrich property. One of the applications is a new characterization of quadratic hypersurface rings. Moreover, in the Gorenstein case, we deduce that applying linkage to sufficiently high syzygy modules of Ulrich ideals yields Ulrich modules. Finally, we explore connections to the theory of modules with minimal multiplicity, and as a byproduct we determine the Chern number of an Ulrich module as well as the Castelnuovo–Mumford regularity of its Rees module.

1. Introduction

This work is concerned with the theory of generalized Ulrich modules (over Cohen–Macaulay local rings) by Goto et al. [2014], which widely extended the classical study of maximally generated maximal Cohen–Macaulay modules — or *Ulrich modules*, as coined in [Herzog and Kühl 1987] — initiated in the 1980s by B. Ulrich [1984]. The term *generalized* refers to the fact that Ulrich modules are taken relatively to a zero-dimensional ideal which is not necessarily the maximal ideal, the latter situation corresponding to the classical theory; despite the apparent naivety of the idea, this passage adds considerable depth to the theory and enlarges its horizon of applications.

Motivated by the remarkable advances in [Goto et al. 2014], our purpose here is to present further progress which includes generalizations of several known results on Ulrich modules, from the above paper as well as [Kobayashi and Takahashi 2019; Ooishi 1991; Wiebe 2003], and connections to some other important classes such as that of modules with minimal multiplicity; for the latter task, we employ suitable numerical invariants such as the Castelnuovo–Mumford regularity of blowup modules.

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Keywords: Ulrich module, maximal Cohen–Macaulay module, horizontal linkage, module of minimal multiplicity, blowup module.

It is worth recalling that the original notion of an Ulrich module (together with the classical existence problem; see, however, Yhee's [2021] construction of local domains which do not admit Ulrich modules or (weakly) lim Ulrich sequences) has been extensively explored since its inception, in both commutative algebra and algebraic geometry. Echoing and complementing the second paragraph of the Introduction of [Yhee 2021], the applications include criteria for the Gorenstein property [Hanes and Huneke 2005; Ulrich 1984], the investigation of maximal Cohen–Macaulay modules over Gorenstein local rings and factoriality of certain rings [Herzog and Kühl 1987], the development of the theory of almost Gorenstein rings [Goto et al. 2015], strategies to tackle certain resistant conjectures in multiplicity theory — e.g., Ma's [2023] resolution of Lech's conjecture in the graded case by introducing and using the notion of (weakly) lim Ulrich sequences, which gives yet another way to generalize the classical Ulrich property — and methods for constructing resultants and Chow forms of projective algebraic varieties (see [Eisenbud and Schreyer 2003], where the concepts of Ulrich sheaf and Ulrich bundle were introduced).

In essence, the general approach suggested in [Goto et al. 2014] extended the definition of an Ulrich module M over a (commutative, Noetherian) Cohen–Macaulay local ring $(R, \mathscr{M})$ to a relative setting that takes into account an $\mathscr{M}$-primary ideal $\mathscr{I}$ containing a parameter ideal as a reduction, so that the case $\mathscr{I} = \mathscr{M}$ retrieves the standard theory. For instance, the condition of the freeness of $M/\mathscr{I}M$ over $R/\mathscr{I}$, which was hidden in the classical setting as $M/\mathscr{M}M$ is simply a vector space, is now required. Following this line of investigation, other works have appeared in the literature, including [Goto et al. 2016a; 2016b; 2019; Numata 2017].

We will briefly comment on our main results, section by section. Preliminary definitions and some known auxiliary results, which are used throughout the paper, are given in Section 2. The main goal of Section 3 is to investigate the Ulrich property under the Hom functor. In this regard, our main result is Theorem 3.2, which can be viewed as a generalization of [Goto et al. 2014, Theorem 5.1] and of [Kobayashi and Takahashi 2019, Proposition 4.1]. Moreover, Corollary 3.5 generalizes [Goto et al. 2014, Corollary 5.2], and Corollary 3.6 is a far-reaching extension of [Brennan et al. 1987, Lemma 2.2]. We also study a connection to the theory of semidualizing modules (see Corollary 3.8) and use it to derive a new characterization of when R is regular (see Corollary 3.9). In addition, in the last subsection, we provide some freeness criteria for $M/\mathscr{I}M$ over the Artinian local ring $R/\mathscr{I}$, which is one of the requirements for Ulrichness with respect to $\mathscr{I}$.

In Section 4 we are essentially interested in the behavior of the Ulrich property under the operation of horizontal linkage over Gorenstein local rings. The main result here is Theorem 4.1 (see also Corollary 4.4), from which we derive a curious characterization of quadratic hypersurface local rings (see Corollary 4.3). In

Corollary 4.7, we record the special case of sufficiently high syzygy modules of a nonparameter Ulrich ideal, in case R is Gorenstein.

In Section 5 we consider the class of modules with minimal multiplicity (in the sense of [Puthenpurakal 2003]) and then connect this concept to the Ulrich property, both taken with respect to $\mathscr{I}$. The basic relation is that Ulrich R-modules have minimal multiplicity (see Proposition 5.6), and as a consequence we use the Chern number — the first Hilbert coefficient — as an ingredient to obtain a characterization of Ulrichness (see Corollary 5.9) which generalizes [Ooishi 1991, Corollary 1.3(1)]. Under this perspective, modules with trivial Chern number are provided in Corollary 5.10, and considerations about the structure of the Hilbert–Samuel polynomial of an Ulrich module are given in Remarks 5.11. Our main technical result in this section is Theorem 5.14, which curiously does not contain Ulrich-like properties in its statement and, more precisely, characterizes modules with minimal multiplicity as follows:

Theorem 5.14. *Let $(R, \mathscr{M})$ be a Noetherian local ring with infinite residue field, M a Cohen–Macaulay R-module of dimension $t > 0$ and I an $\mathscr{M}$-primary ideal of R. Let $J = (z_1, \dots, z_t)$ be a minimal M-reduction of I. The following assertions are equivalent*:

(i) *M has minimal multiplicity with respect to I.*

(ii) $\operatorname{reg} \mathcal{R}(I, M) = \operatorname{reg} \mathcal{G}(I, M) = r_J(I, M) \leq 1$.

(iii) $r_J(I, M) \leq 1$.

Here, $\operatorname{reg}(-)$ denotes (Castelnuovo–Mumford) regularity, and $\mathcal{R}(I, M)$ and $\mathcal{G}(I, M)$ stand respectively for the Rees module and the associated graded module of I relative to M. Also, $r_J(I, M)$ is the reduction number of I with respect to J relative to M. We emphasize that Theorem 5.14 answers affirmatively the module-theoretic analogue of Sally's [1983] question about independence of reduction numbers for the class of modules with minimal multiplicity. Additionally, from this theorem we derive Corollary 5.15, which determines the regularity of the Rees and associated graded modules of $\mathscr{I}$ relative to an Ulrich module (this result partially generalizes [Ooishi 1991, Proposition 1.1]), and also Corollary 5.16, where we deal once again with high syzygy modules of Ulrich ideals.

Finally, Section 6 provides a detailed example to illustrate some of our main corollaries.

2. Conventions, preliminaries, and some auxiliary results

Throughout this paper, all rings are assumed to be commutative and Noetherian with 1, and by *finite* module we mean a finitely generated module.

In this section, we recall some of the basic notions and tools that will play an important role throughout the paper. Other auxiliary notions will be introduced as they become necessary.

2A. ***Ulrich ideals and modules.*** Let $(R, \mathscr{M})$ be a local ring, M a finite R-module, and $I \neq R$ an ideal of definition of M, i.e., $\mathscr{M}^n M \subset IM$ for some $n > 0$. Let us establish some notations. We denote by $\nu(M)$ and $e_I^0(M)$, respectively, the minimal number of generators of M and the multiplicity of M with respect to I. When $I = \mathscr{M}$, we simply write $e(M)$ in place of $e_{\mathscr{M}}^0(M)$.

Definition 2.1. Let $(R, \mathscr{M})$ be a local ring. A finite R-module M is *Cohen–Macaulay* if $\operatorname{depth}_R M = \dim M$, and *maximal Cohen–Macaulay* if $\operatorname{depth}_R M = \dim R$. Note the zero module is not maximal Cohen–Macaulay as its depth is set to be $+\infty$. Moreover, M is called *Ulrich* if M is a maximal Cohen–Macaulay R-module satisfying $\nu(M) = e(M)$.

For instance, if $(R, \mathscr{M})$ is a 1-dimensional Cohen–Macaulay local ring, then the power $\mathscr{M}^{e(R)-1}$ is an Ulrich module. Several other classes of examples can be found in [Brennan et al. 1987].

Ulrich modules are also dubbed *maximally generated maximal Cohen–Macaulay modules*. This is due to the fact that there is an inequality $\nu(M) \leq e(M)$ whenever the local ring R is Cohen–Macaulay and M is maximal Cohen–Macaulay; see [Brennan et al. 1987, Proposition 1.1].

Convention 2.2. Henceforth, in the entire paper, we adopt the following convention and notations. Whenever $(R, \mathscr{M})$ is a d-dimensional Cohen–Macaulay local ring, we will let $\mathscr{I}$ (to be distinguished from the notation I) stand for an $\mathscr{M}$-primary ideal that contains a parameter ideal

$$Q = (\boldsymbol{x}) = (x_1, \ldots, x_d)$$

as a reduction, i.e., $Q\mathscr{I}^r = \mathscr{I}^{r+1}$ for some integer $r \geq 0$. As is well known, any $\mathscr{M}$-primary ideal of R has this property provided that the residue class field $R/\mathscr{M}$ is infinite, or that R is analytically irreducible with $d = 1$.

Definition 2.3. Let R be a Cohen–Macaulay local ring. We say that the ideal $\mathscr{I}$ is *Gorenstein* if the quotient ring $R/\mathscr{I}$ is Gorenstein.

Next, we recall the general notions of Ulrich ideal and Ulrich module as introduced in [Goto et al. 2014], where in addition several explicit examples are given. As will be made clear, the latter Definition 2.7 below generalizes Definition 2.1.

Definition 2.4 [Goto et al. 2014]. Let R be a Cohen–Macaulay local ring. We say that the ideal $\mathscr{I}$ is *Ulrich* if $\mathscr{I}^2 = Q\mathscr{I}$ (the reduction number of $\mathscr{I}$ with respect to Q is at most 1) and $\mathscr{I}/\mathscr{I}^2$ is a free $R/\mathscr{I}$-module.

Examples 2.5. (i) [Kumashiro 2023, Proposition 3.10] Let $S = K[\![x, y, z]\!]$ be a formal power series ring over an infinite field K, and fix any regular sequence $\{f, g, h\} \subset (x, y, z)$. Then, $R = S/(f^2 - gh, g^2 - hf, h^2 - fg)$ is a 1-dimensional Cohen–Macaulay local ring and $\mathscr{I} = (f, g, h)R$ is an Ulrich ideal.

(ii) [Goto et al. 2014, Example 2.7(2)] One way to produce examples in arbitrary positive dimension is as follows. Given a field K and integers $d, s \geq 1$, consider the d-dimensional local hypersurface ring $R = K[\![z_1, \dots, z_{d+1}]\!]/(z_1^2 + \cdots + z_d^2 + z_{d+1}^{2s})$, where $z_1, \dots, z_{d+1}$ are formal indeterminates over k. Then, the ideal

$$\mathscr{I} = (z_1, \dots, z_d, z_{d+1}^s)R$$

is Ulrich and contains the parameter ideal $Q = (z_1, \dots, z_d)R$ as a reduction.

Remark 2.6. In a Gorenstein local ring, every Ulrich ideal is Gorenstein; see [Goto et al. 2014, Corollary 2.6].

Definition 2.7 [Goto et al. 2014]. Let R be a Cohen–Macaulay local ring and let M be a finite R-module. We say that M is *Ulrich with respect to* $\mathscr{I}$ if the following conditions hold:

(i) M is a maximal Cohen–Macaulay R-module.

(ii) $\mathscr{I}M = QM$.

(iii) $M/\mathscr{I}M$ is a free $R/\mathscr{I}$-module.

Remarks 2.8. (i) Let us recall the discussion in the paragraph after Definition 1.2 in [Goto et al. 2014]. Denote the length of R-modules by $\ell_R(-)$. If R is a Cohen–Macaulay local ring and M is a maximal Cohen–Macaulay R-module, then

$$e^0_{\mathscr{I}}(M) = e^0_Q(M) = \ell_R(M/QM) \geq \ell_R(M/\mathscr{I}M),$$

so that condition (ii) of Definition 2.7 is equivalent to saying that the equality $e^0_{\mathscr{I}}(M) = \ell_R(M/\mathscr{I}M)$ takes place. In particular, if $\mathscr{I} = \mathscr{M}$, condition (ii) is the same as $e(M) = \nu(M)$. Therefore, M is an Ulrich module with respect to $\mathscr{M}$ if and only if M is an Ulrich module in the sense of Definition 2.1.

(ii) Clearly, if $d = 1$ and $\mathscr{I}$ is an Ulrich ideal of R, then $\mathscr{I}$ is an Ulrich R-module with respect to $\mathscr{I}$.

(iii) Let us recall the following more general recipe to obtain Ulrich modules from Ulrich ideals (in the setting of Convention 2.2). If $\mathscr{I}$ is an Ulrich ideal of R which is not a parameter ideal, then for any $i \geq d$ the i-th syzygy module (see Section 2B below) of $R/\mathscr{I}$ is an Ulrich R-module with respect to $\mathscr{I}$, and conversely (we refer to [Goto et al. 2014, Theorem 4.1]). This is a very helpful property and will be explored in some of our results and examples.

2B. *Linkage.* The concepts recalled in this subsection can be described in the general context of semiperfect rings, but in this paper we focus on the special case of (finite modules over) a local ring R, since this is the setup where our results will be proved.

Given a finite R-module M, we write $M^* = \operatorname{Hom}_R(M, R)$. The (*Auslander*) *transpose* $\operatorname{Tr} M$ of M is defined as the cokernel of the dual $\partial_1^* = \operatorname{Hom}_R(\partial_1, R)$ of the first differential map ∂_1 in a minimal free resolution of M over R. Hence there is an exact sequence

$$0 \longrightarrow M^* \longrightarrow F_0^* \xrightarrow{\partial_1^*} F_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0$$

for suitable finite free R-modules F_0, F_1. The (*first*) *syzygy* module $\Omega^1 M = \Omega M$ of M is the image of ∂_1, hence a submodule of F_0. We recursively put $\Omega^k M = \Omega(\Omega^{k-1} M)$, the k-th syzygy module of M, for any $k \geq 2$.

Note that the modules $\operatorname{Tr} M$ and ΩM are uniquely determined up to isomorphism, since the same is true of a minimal free resolution of M. By [Auslander 1966, Proposition 6.3], we have an exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^1(\operatorname{Tr} M, R) \longrightarrow M \xrightarrow{e_M} M^{**} \longrightarrow \operatorname{Ext}_R^2(\operatorname{Tr} M, R) \longrightarrow 0, \tag{1}$$

where e_M is the evaluation map.

Martsinkovsky and Strooker [2004] generalized the classical theory of linkage for ideals to the context of modules by means of the operator $\lambda = \Omega \operatorname{Tr}$, i.e., a finite R-module M is sent to the composite $\Omega \operatorname{Tr} M$ defined from a minimal free presentation of M.

Definition 2.9 [Martsinkovsky and Strooker 2004]. Two finite R-modules M and N are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$. In the case where M and λM are horizontally linked, $M \cong \lambda^2 M$, we simply say that the module M is horizontally linked.

We also recall that a *stable* module is a finite module with no nonzero free direct summand. A finite R-module M is called a *syzygy module* if it is embedded in a finite free R-module, that is if $M \cong \Omega N$ for some finite R-module N. Here is a well-known characterization of horizontally linked modules.

Lemma 2.10 [Martsinkovsky and Strooker 2004, Theorem 2 and Corollary 6]. *A finite R-module M is horizontally linked if and only if it is stable and* $\operatorname{Ext}_R^1(\operatorname{Tr} M, R) = 0$, *if and only if M is a stable syzygy module.*

Lemma 2.11 [Martsinkovsky and Strooker 2004, Proposition 4]. *Suppose M is horizontally linked. Then, λM is also horizontally linked and, in particular, λM is stable.*

2C. ***Canonical modules.*** In the sequel we collect basic facts about canonical modules.

Lemma 2.12 [Bruns and Herzog 1993]. *Let R be a Cohen–Macaulay local ring with canonical module ω_R. Let M be a maximal Cohen–Macaulay R-module. Then the following statements hold*:

(i) $\operatorname{Hom}_R(M, \omega_R)$ *is a maximal Cohen–Macaulay R-module.*

(ii) $\operatorname{Ext}^i_R(M, \omega_R) = 0$ *for all $i > 0$.*

(iii) $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$.

(iv) *If $\boldsymbol{y}$ is an R-sequence, then $R/(\boldsymbol{y})$ has a canonical module $\omega_{R/(\boldsymbol{y})} \cong \omega_R/\boldsymbol{y}\omega_R$.*

(v) *Let $\varphi : R \to S$ be a local homomorphism of Cohen–Macaulay local rings such that S is a finite R-module. Then S has a canonical module $\omega_S \cong \operatorname{Ext}^t_R(S, \omega_R)$, where $t = \dim R - \dim S$.*

3. Hom functor and the Ulrich property

In this section we investigate, in essence, the behavior of the Ulrich property under the Hom functor.

3A. ***Key lemma, main result, and corollaries.*** We start with the following basic lemma, which will be a key ingredient in the proof of the main result of this section.

Lemma 3.1. *Let R be a Cohen–Macaulay local ring, M, N be maximal Cohen–Macaulay R-modules, and $\boldsymbol{y} = y_1, \dots, y_n$ be an R-sequence for some $n \geq 1$.*

(i) *If either $n = 1$ or $\operatorname{Ext}^i_R(M, N) = 0$ for all $i = 1, \dots, n-1$, there is an injection*

$$\operatorname{Hom}_R(M, N)/\boldsymbol{y}\operatorname{Hom}_R(M, N) \hookrightarrow \operatorname{Hom}_{R/(\boldsymbol{y})}(M/\boldsymbol{y}M, N/\boldsymbol{y}N).$$

(ii) *If $\operatorname{Ext}^i_R(M, N) = 0$ for all $i = 1, \dots, n$, there is an isomorphism*

$$\operatorname{Hom}_R(M, N)/\boldsymbol{y}\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_{R/(\boldsymbol{y})}(M/\boldsymbol{y}M, N/\boldsymbol{y}N).$$

Proof. We shall prove the assertion (i), which from the arguments below (essentially from (2)) will be easily seen to imply (ii). Set $R' = R/(y_1)$, $M' = M/y_1M$, and $N' = N/y_1N$. We will proceed by induction on n. Consider first the case $n = 1$, which is standard, but we supply the proof for convenience. Since M and N are maximal Cohen–Macaulay R-modules and $y_1 \in \mathscr{M}$ is R-regular, where $\mathscr{M}$ is the maximal ideal of R, it follows that y_1 is both M-regular and N-regular. In particular, we have the short exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow M' \longrightarrow 0,$$

which induces the exact sequence

$$(2)\quad 0 \to \operatorname{Hom}_R(M', N) \to \operatorname{Hom}_R(M, N) \xrightarrow{y_1} \operatorname{Hom}_R(M, N) \to \operatorname{Ext}^1_R(M', N) \\ \to \cdots \to \operatorname{Ext}^i_R(M, N) \to \operatorname{Ext}^{i+1}_R(M', N) \to \operatorname{Ext}^{i+1}_R(M, N) \to \cdots .$$

It follows an injection

$$(3)\qquad \operatorname{Hom}_R(M, N)/y_1 \operatorname{Hom}_R(M, N) \hookrightarrow \operatorname{Ext}^1_R(M', N).$$

Because y_1 is N-regular and $y_1 M' = 0$, there are isomorphisms

$$(4)\qquad \operatorname{Ext}^i_{R'}(M', N') \cong \operatorname{Ext}^{i+1}_R(M', N) \quad \text{for all } i \geq 0,$$

see [Bruns and Herzog 1993, Lemma 3.1.16]. In particular,

$$(5)\qquad \operatorname{Hom}_{R'}(M', N') \cong \operatorname{Ext}^1_R(M', N),$$

and the result follows by (3) and (5).

Now let $n \geq 2$. Clearly, R' is a Cohen–Macaulay ring and M', N' are maximal Cohen–Macaulay R'-modules. By assumption, $\operatorname{Ext}^i_R(M, N) = 0$ for all $i = 1, \ldots, n-1$. Thus, using (2) and (4), we obtain isomorphisms

$$(6)\qquad \operatorname{Ext}^i_{R'}(M', N') \cong \begin{cases} \operatorname{Hom}_R(M, N)/y_1 \operatorname{Hom}_R(M, N) & \text{if } i = 0, \\ 0 & \text{if } i = 1, \ldots, n-2. \end{cases}$$

Since $\boldsymbol{y}' = y_2, \ldots, y_n$ is an R'-sequence, the induction hypothesis yields an injection

$$\operatorname{Hom}_{R'}(M', N')/\boldsymbol{y}' \operatorname{Hom}_{R'}(M', N') \hookrightarrow \operatorname{Hom}_{R'/\boldsymbol{y}'R'}(M'/\boldsymbol{y}'M', N'/\boldsymbol{y}'N'),$$

where the latter module is clearly isomorphic to $\operatorname{Hom}_{R/(\boldsymbol{y})}(M/\boldsymbol{y}M, N/\boldsymbol{y}N)$. Now the conclusion follows by (6) with $i = 0$. □

The theorem below is our main result in this section.

Theorem 3.2. *Let R be a Cohen–Macaulay local ring of dimension d. Let M and N be maximal Cohen–Macaulay R-modules such that* $\operatorname{Hom}_R(M, N) \neq 0$ *and* $\operatorname{Ext}^i_R(M, N) = 0$ *for all $i = 1, \ldots, n$, where either $n = d-1$ or $n = d$. Let $\mathscr{I}$ and Q be as in Convention 2.2. Assume that M, resp. N, is an Ulrich R-module with respect to $\mathscr{I}$, and consider the following conditions*:

(i) $\operatorname{Hom}_R(M, N)$ *is an Ulrich R-module with respect to $\mathscr{I}$.*

(ii) $\operatorname{Hom}_R(M, N)/\mathscr{I} \operatorname{Hom}_R(M, N)$ *is a free $R/\mathscr{I}$-module.*

(iii) $\operatorname{Hom}_{R/Q}(R/\mathscr{I}, N/QN)$, *resp.* $\operatorname{Hom}_{R/Q}(M/QM, R/\mathscr{I})$, *is a free $R/\mathscr{I}$-module.*

Then the following statements hold:

(a) *If $n = d-1$ then (i) $\Longleftrightarrow$ (ii).*

(b) *If $n = d$ then (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).*

Proof. (a) Applying the functor $\mathrm{Hom}_R(-, N)$ to a free resolution

$$\cdots \longrightarrow F_{d+1} \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of the R-module M, and using the hypothesis that $\mathrm{Ext}^i_R(M, N) = 0$ for $i = 1, \dots, d-1$, we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(F_0, N) \to \cdots \to \mathrm{Hom}_R(F_{d-1}, N) \to \mathrm{Hom}_R(F_d, N).$$

Now set $X_0 := \mathrm{Hom}_R(M, N)$ and $X_i := \mathrm{Im}(\mathrm{Hom}_R(F_{i-1}, N) \to \mathrm{Hom}_R(F_i, N))$ for $i = 1, \dots, d$. Since N is maximal Cohen–Macaulay, then $\mathrm{depth}_R \mathrm{Hom}_R(F_i, N) = d$ for all $i = 0, \dots, d$. Thus, by the short exact sequence

$$0 \longrightarrow X_i \longrightarrow \mathrm{Hom}_R(F_i, N) \longrightarrow X_{i+1} \longrightarrow 0,$$

we get $\mathrm{depth}_R X_i \geq \min\{d, \mathrm{depth}_R X_{i+1} + 1\}$; see, e.g., [Bruns and Herzog 1993, Proposition 1.2.9]. Therefore,

$$\mathrm{depth}_R \mathrm{Hom}_R(M, N) \geq \min\{d, \mathrm{depth}_R X_d + d\} = d,$$

i.e., $\mathrm{Hom}_R(M, N)$ is a maximal Cohen–Macaulay R-module.

Now, as in Convention 2.2, let $\boldsymbol{x} = x_1, \dots, x_d$ be a generating set of the parameter ideal Q. Then $\boldsymbol{x}$ is an R-sequence (see [Bruns and Herzog 1993, Theorem 2.1.2(d)]), and so by Lemma 3.1(i) there is an injection

$$\mathrm{Hom}_R(M, N)/Q\,\mathrm{Hom}_R(M, N) \hookrightarrow \mathrm{Hom}_{R/Q}(M/QM, N/QN). \tag{7}$$

Because M (resp. N) is assumed to be Ulrich with respect to $\mathscr{I}$, the module M/QM (resp. N/QN) is annihilated by $\mathscr{I}$, and hence so is $\mathrm{Hom}_{R/Q}(M/QM, N/QN)$. In either case, it follows from (7) that the quotient $\mathrm{Hom}_R(M, N)/Q\,\mathrm{Hom}_R(M, N)$ is annihilated by $\mathscr{I}$. Thus,

$$\mathscr{I}\,\mathrm{Hom}_R(M, N) = Q\,\mathrm{Hom}_R(M, N).$$

Therefore, $\mathrm{Hom}_R(M, N)$ is Ulrich with respect to $\mathscr{I}$ if and only if the quotient module $\mathrm{Hom}_R(M, N)/\mathscr{I}\,\mathrm{Hom}_R(M, N)$ is $R/\mathscr{I}$-free, so (i) $\iff$ (ii).

(b) As seen above, there is an equality $\mathscr{I}\,\mathrm{Hom}_R(M, N) = Q\,\mathrm{Hom}_R(M, N)$. Notice that, furthermore, Lemma 3.1(ii) yields an isomorphism

$$\mathrm{Hom}_R(M, N)/Q\,\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{R/Q}(M/QM, N/QN). \tag{8}$$

Now suppose that M is Ulrich with respect to $\mathscr{I}$. From $M/QM = M/\mathscr{I}M \cong (R/\mathscr{I})^m$ for some integer $m > 0$, we deduce that

$$\mathrm{Hom}_{R/Q}(M/QM, N/QN) \cong (\mathrm{Hom}_{R/Q}(R/\mathscr{I}, N/QN))^m. \tag{9}$$

By (8) and (9), we get

$$\operatorname{Hom}_R(M, N)/\mathscr{I} \operatorname{Hom}_R(M, N) \cong (\operatorname{Hom}_{R/Q}(R/\mathscr{I}, N/QN))^m.$$

Therefore, the quotient $\operatorname{Hom}_R(M, N)/\mathscr{I} \operatorname{Hom}_R(M, N)$ is $R/\mathscr{I}$-free if and only if the module $\operatorname{Hom}_{R/Q}(R/\mathscr{I}, N/QN)$ is $R/\mathscr{I}$-free. The case where N is Ulrich with respect to $\mathscr{I}$ is completely similar. This shows (ii) $\Longleftrightarrow$ (iii) and concludes the proof of the theorem. □

Remark 3.3. It is worth observing that the condition $\operatorname{Hom}_R(M, N) = 0$ can hold even if M and N are both Ulrich. For instance, over the local ring $R = K[\![x, y]\!]/(xy)$, where x, y are formal variables over a field K, we have

$$\operatorname{Hom}_R(R/xR, R/yR) = 0.$$

We point out that Theorem 3.2 generalizes [Kobayashi and Takahashi 2019, Proposition 4.1] (see Corollary 3.7, to be given shortly) and, in addition, recovers the following result from [Goto et al. 2014]:

Corollary 3.4 [Goto et al. 2014, Theorem 5.1]. *Let R be a Cohen–Macaulay local ring with canonical module ω_R, and let M be an Ulrich R-module with respect to $\mathscr{I}$. Then the following assertions are equivalent*:

(i) $\operatorname{Hom}_R(M, \omega_R)$ *is an Ulrich R-module with respect to $\mathscr{I}$.*

(ii) $\mathscr{I}$ *is a Gorenstein ideal.*

Proof. By Lemma 2.12(ii), we have $\operatorname{Ext}^i_R(M, \omega_R) = 0$ for all $i > 0$. Since R/Q and $R/\mathscr{I}$ are zero-dimensional local rings and the ideal Q is generated by an R-sequence, there are isomorphisms

$$\omega_{R/\mathscr{I}} \cong \operatorname{Hom}_{R/Q}(R/\mathscr{I}, \omega_{R/Q}) \cong \operatorname{Hom}_{R/Q}(R/\mathscr{I}, \omega_R/Q\omega_R)$$

according to standard facts; see parts (iv) and (v) of Lemma 2.12. Now, applying Theorem 3.2(b) with $N = \omega_R$, we derive that $\operatorname{Hom}_R(M, \omega_R)$ is Ulrich with respect to $\mathscr{I}$ if and only if $\omega_{R/\mathscr{I}}$ is $R/\mathscr{I}$-free, or equivalently, $R/\mathscr{I}$ is a Gorenstein ring. □

Taking Remark 2.6 into account, the corollary below is readily seen to generalize [Goto et al. 2014, Corollary 5.2].

Corollary 3.5. *Let R be a Cohen–Macaulay local ring with canonical module ω_R, and let M be a maximal Cohen–Macaulay R-module. Assume that the ideal $\mathscr{I}$ is Gorenstein. Then the following assertions are equivalent*:

(i) *M is an Ulrich R-module with respect to $\mathscr{I}$.*

(ii) $\operatorname{Hom}_R(M, \omega_R)$ *is an Ulrich R-module with respect to $\mathscr{I}$.*

Proof. There is an isomorphism $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \omega_R), \omega_R)$ by Lemma 2.12(iii). The conclusion follows by Corollary 3.4. □

Our next result is a far-reaching extension of [Brennan et al. 1987, Lemma 2.2]; see also Corollary 3.9.

Corollary 3.6. *Let R be a Cohen–Macaulay local ring with canonical module ω_R. Assume that the ideal $\mathscr{I}$ is Gorenstein. Then the following assertions are equivalent*:

(i) *$\mathscr{I}$ is a parameter ideal.*

(ii) *R is an Ulrich R-module with respect to $\mathscr{I}$.*

(iii) *ω_R is an Ulrich R-module with respect to $\mathscr{I}$.*

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is immediate from Definition 2.7 and holds regardless of $\mathscr{I}$ being Gorenstein. Now, by virtue of the isomorphisms $\omega_R \cong \mathrm{Hom}_R(R, \omega_R)$ and $\mathrm{Hom}_R(\omega_R, \omega_R) \cong R$, our Corollary 3.5 yields (ii) $\Longleftrightarrow$ (iii). $\square$

As yet another byproduct of Theorem 3.2, we retrieve [Kobayashi and Takahashi 2019, Proposition 4.1], which in turn generalizes the local version of [Wiebe 2003, Proposition 3.5].

Corollary 3.7 [Kobayashi and Takahashi 2019, Proposition 4.1]. *Let R be a Cohen–Macaulay local ring of dimension d. Let M, N be maximal Cohen–Macaulay R-modules such that* $\mathrm{Hom}_R(M, N) \neq 0$ *and* $\mathrm{Ext}^i_R(M, N) = 0$ *for all $i = 1, \dots, d-1$. If either M or N is an Ulrich R-module*, *then so is* $\mathrm{Hom}_R(M, N)$.

Proof. As observed in Remarks 2.8(i), M is an Ulrich R-module if and only if M is an Ulrich R-module with respect to the maximal ideal $\mathscr{M}$ of R. Now, being a (finite-dimensional) vector space over the residue field $k = R/\mathscr{M}$, the module $\mathrm{Hom}_R(M, N)/\mathscr{M}\,\mathrm{Hom}_R(M, N)$ is k-free. Thus, $\mathrm{Hom}_R(M, N)$ is Ulrich by Theorem 3.2(a). $\square$

3B. ***Hom with values in a semidualizing module.*** Let us recall that a finite module $\mathscr{C}$ over a ring R is called *semidualizing* if the morphism $R \to \mathrm{Hom}_R(\mathscr{C}, \mathscr{C})$ given by homothety is an isomorphism and $\mathrm{Ext}^i_R(\mathscr{C}, \mathscr{C}) = 0$ for all $i > 0$. In this case, a finite R-module M is said to be *totally $\mathscr{C}$-reflexive* if the biduality map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathscr{C}), \mathscr{C})$ is an isomorphism and, in addition, $\mathrm{Ext}^i_R(M, \mathscr{C}) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, \mathscr{C}), \mathscr{C})$ for all $i > 0$. Note every totally $\mathscr{C}$-reflexive module is maximal Cohen–Macaulay by virtue of the relative Auslander–Bridger formula; see [Sather-Wagstaff 2010, Proposition 6.4.2]. A detailed account about the theory of semidualizing modules is given in [Sather-Wagstaff 2010].

As a matter of illustration, R is semidualizing as a module over itself, and, for any semidualizing R-module $\mathscr{C}$, both R and $\mathscr{C}$ are totally $\mathscr{C}$-reflexive. More interestingly, if R is a Cohen–Macaulay local ring possessing a canonical module ω_R, then ω_R is semidualizing and, in addition, every maximal Cohen–Macaulay R-module is totally ω_R-reflexive (to see this, use Lemma 2.12). It should also be pointed out, based on the existence of several examples in the literature, that not every semidualizing R-module must be isomorphic to R or ω_R; see for example [Araya and Iima 2018, Section 5; Sather-Wagstaff 2010, Section 2.3].

Corollary 3.8. *Let R be a Cohen–Macaulay local ring with a semidualizing module $\mathscr{C}$, and let M be a totally $\mathscr{C}$-reflexive R-module. Then, M is an Ulrich R-module if and only if $\operatorname{Hom}_R(M, \mathscr{C})$ is an Ulrich R-module.*

Proof. We have $M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \mathscr{C}), \mathscr{C})$, which in particular forces the module $\operatorname{Hom}_R(M, \mathscr{C})$ to be nontrivial, and in addition

$$\operatorname{Ext}^i_R(M, \mathscr{C}) = 0 = \operatorname{Ext}^i_R(\operatorname{Hom}_R(M, \mathscr{C}), \mathscr{C}) \quad \text{for all } i > 0.$$

Since $\mathscr{C}$ is semidualizing, $\operatorname{depth}_R \mathscr{C} = \operatorname{depth} R$ (see [Sather-Wagstaff 2010, Theorem 2.2.6(c)]) and hence $\mathscr{C}$ is maximal Cohen–Macaulay. The result is clear by Corollary 3.7. ☐

Note that Corollary 3.8 gives a different proof of the case $\mathscr{I} = \mathscr{M}$ of Corollary 3.5 by taking $\mathscr{C} = \omega_R$. Another byproduct of Corollary 3.8 is the following curious characterization of regular local rings.

Corollary 3.9. *Let R be a Cohen–Macaulay local ring with a semidualizing module $\mathscr{C}$. Then, R is regular if and only if $\mathscr{C}$ is an Ulrich R-module.*

Proof. According to [Sather-Wagstaff 2010, Proposition 2.1.12], saying that $\mathscr{C}$ is semidualizing is tantamount to R being a totally $\mathscr{C}$-reflexive R-module. Now, Corollary 3.8 yields that R is Ulrich over itself if and only if $\mathscr{C}$ is an Ulrich R-module. The former situation, as observed in [Brennan et al. 1987, Lemma 2.2], is equivalent to the regularity of R. ☐

We raise the following question and a related remark.

Question 3.10. Does Corollary 3.4 hold with $\mathscr{C}$ (a given semidualizing R-module) in place of ω_R?

Remark 3.11. An affirmative answer to Question 3.10 would imply the validity of Corollary 3.5 with $\mathscr{C}$ in place of ω_R as well, provided that R is a normal domain. Indeed, it suffices to note that in this case the maximal Cohen–Macaulay R-module M is necessarily reflexive in the usual sense, and thus by [Sather-Wagstaff 2010, Corollary 5.4.7], which also requires R to be normal, we have

$$M \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \mathscr{C}), \mathscr{C})$$

via the natural biduality map.

3C. ***Freeness criteria for $M/\mathscr{I}M$ via (co)homology vanishing.*** We close the section providing some criteria for the freeness of the $R/\mathscr{I}$-module $M/\mathscr{I}M$, which is of interest since this is one of the requirements for M to be Ulrich with respect to $\mathscr{I}$; see Definition 2.7.

As we have been investigating how Ulrichness behaves under the Hom ($=\mathrm{Ext}^0$) functor, it seems natural to wonder about the relevance of higher Ext modules in the theory, and in fact we shall see that the vanishing of finitely many "diagonal" Ext modules $\mathrm{Ext}^i_{R/\mathscr{I}}(M/\mathscr{I}M, M/\mathscr{I}M)$, under suitable hypotheses, can detect freeness over the Artinian local ring $R/\mathscr{I}$, which we will assume to be Gorenstein. Vanishing of homology modules, namely "diagonal" Tor modules $\mathrm{Tor}^{R/\mathscr{I}}_j(M/\mathscr{I}M, M/\mathscr{I}M)$, will also play a role. Essentially, our criteria will consist of adaptations of some results from [Huneke et al. 2004] and one from [Şega 2011].

In the proposition below, and as before, $(R, \mathscr{M})$ and $\mathscr{I}$ (also Q, which appears in the proof) are as in Convention 2.2, and $\ell_R(-)$ stands for length of R-modules.

Proposition 3.12. *Suppose $R/\mathscr{I}$ is Gorenstein* (*e.g.*, *R is Gorenstein and $\mathscr{I}$ is Ulrich*; *see Remark 2.6*) *and let M be a finite R-module. Assume any one of the following situations*:

(i) $\mathscr{M}^2M \subset \mathscr{I}M$ *and* $\mathrm{Ext}^i_{R/\mathscr{I}}(M/\mathscr{I}M, M/\mathscr{I}M) = 0$ *for all i satisfying* $1 \leq i \leq \max\{3, \nu(M), \ell_R(M/\mathscr{I}M) - \nu(M)\}$.

(ii) $\mathscr{M}^3 \subset \mathscr{I}$ *and* $\mathrm{Ext}^i_{R/\mathscr{I}}(M/\mathscr{I}M, M/\mathscr{I}M) = 0$ *for some $i > 0$.*

(iii) ($R/\mathscr{I}$ *need not be Gorenstein.*) *$R/\mathscr{M}$ is infinite, $\mathscr{I}$ is not a parameter ideal, $\mathscr{M}^3 \subset \mathscr{I}$, $e^0_{\mathscr{I}}(R) \leq 2\ell_R(\mathscr{M}/(\mathscr{M}^2 + \mathscr{I}))$, and* $\mathrm{Tor}^{R/\mathscr{I}}_j(M/\mathscr{I}M, M/\mathscr{I}M) = 0$ *for three consecutive values of $j \geq 2$.*

(iv) *$\mathscr{M}^4 \subset \mathscr{I}$, there exists $x \in \mathscr{M} \setminus \mathscr{I}$ such that the ideal $(\mathscr{I} : x)/\mathscr{I}$ is principal, and* $\mathrm{Tor}^{R/\mathscr{I}}_j(M/\mathscr{I}M, M/\mathscr{I}M) = 0$ *for all $j \gg 0$.*

Then, $M/\mathscr{I}M$ is $R/\mathscr{I}$-free.

Proof. For simplicity, set $\overline{R} = R/\mathscr{I}$, $\overline{\mathscr{M}} = \mathscr{M}/\mathscr{I}$, and $\overline{M} = M/\mathscr{I}M$. Let us assume (i). By assumption $\overline{\mathscr{M}}^2\overline{M} = 0$, hence

$$\nu(\overline{\mathscr{M}}\,\overline{M}) = \ell_R(\overline{\mathscr{M}}\,\overline{M}) = \ell_R(\mathscr{M}M/\mathscr{I}M).$$

On the other hand, by the short exact sequence

$$0 \longrightarrow \mathscr{M}M/\mathscr{I}M \longrightarrow M/\mathscr{I}M \longrightarrow M/\mathscr{M}M \longrightarrow 0,$$

we have $\ell_R(\mathscr{M}M/\mathscr{I}M) = \ell_R(M/\mathscr{I}M) - \ell_R(M/\mathscr{M}M)$. Therefore we obtain $\nu(\overline{\mathscr{M}}\,\overline{M}) = \ell_R(M/\mathscr{I}M) - \nu(M)$. In addition it is clear that $\nu(\overline{M}) = \nu(M)$. Now we can apply [Huneke et al. 2004, Proposition 4.4(1)], which ensures that the $R/\mathscr{I}$-module $M/\mathscr{I}M$ is either free or injective. Since $R/\mathscr{I}$ is Gorenstein, $M/\mathscr{I}M$ is necessarily free, as needed.

Assume that (ii) holds. Notice that $\overline{\mathscr{M}}^3 = 0$ by hypothesis. Now, since $R/\mathscr{I}$ is Gorenstein, the freeness of $M/\mathscr{I}M$ follows readily by [Huneke et al. 2004, Theorem 4.1(2)].

Now suppose (iii). Let $\ell\ell(\bar{R})$ denote the *Loewy length* of $\bar{R}$, which is the smallest integer n such that $\overline{\mathscr{M}}^n = 0$, i.e., $\mathscr{M}^n \subset \mathscr{I}$. Thus, by assumption, $\ell\ell(\bar{R}) \leq 3$. If $\ell\ell(\bar{R}) = 1$ ($\mathscr{I} = \mathscr{M}$), there is nothing to prove. If $\ell\ell(\bar{R}) = 2$, then $M/\mathscr{I}M$ is free by [Huneke et al. 2004, Remark 2.1]. So we can assume $\ell\ell(\bar{R}) = 3$. Using Remarks 2.8(i) and the hypothesis that $\mathscr{I}$ is not a parameter ideal (so that the inclusion $Q \subset \mathscr{I}$ is strict), we get $e^0_{\mathscr{I}}(R) = \ell_R(R/Q) \geq \ell_R(R/\mathscr{I}) + 1$. Therefore,

$$2\nu(\overline{\mathscr{M}}) = 2\ell_R(\mathscr{M}/(\mathscr{M}^2 + \mathscr{I})) \geq e^0_{\mathscr{I}}(R) \geq \ell_R(\bar{R}) + 1 = \ell_R(\bar{R}) - \ell\ell(\bar{R}) + 4.$$

We are now in a position to apply [Huneke et al. 2004, Theorem 3.1(2)] to conclude that $M/\mathscr{I}M$ is free.

Finally, suppose (iv). So $R/\mathscr{I}$ is Gorenstein and $\overline{\mathscr{M}}^4 = 0$, and in addition note that $(\mathscr{I} : x)/\mathscr{I}$ is the annihilator of $x\bar{R}$. Then $M/\mathscr{I}M$ is free by [Şega 2011, Theorem 3.3]. □

Remark 3.13. From the proof in the situation (iii) it is clear that, for general $\mathscr{I}$ (possibly a parameter ideal), the hypothesis on the multiplicity must be replaced with $e^0_{\mathscr{I}}(R) \leq 2\ell_R(\mathscr{M}/(\mathscr{M}^2 + \mathscr{I})) - 1$.

4. Horizontal linkage and the Ulrich property

We begin this section by pointing out the warming-up fact that, if the local ring R is Gorenstein, then it follows from [Martsinkovsky and Strooker 2004, Theorem 1] that every stable Ulrich R-module with respect to $\mathscr{I}$, where $\mathscr{I}$ is as in Convention 2.2, is horizontally linked (note that maximal Cohen–Macaulay modules are precisely the totally reflexive modules, since R is Gorenstein). See Section 2B for terminology.

In essence, our goal herein is to develop a further study of linkage of Ulrich modules with respect to $\mathscr{I}$, also assumed to be Ulrich but not a parameter ideal, the main result being the theorem below, which in particular shows that the operation of horizontal linkage over a Gorenstein local ring preserves the Ulrich property with respect to $\mathscr{I}$ for horizontally linked modules.

Theorem 4.1. *Let $(R, \mathscr{M})$ be a Cohen–Macaulay local ring of dimension d, and suppose the ideal $\mathscr{I}$ is Ulrich but not a parameter ideal. Consider the following assertions*:

(i) *R is Gorenstein.*

(ii) *M is Ulrich with respect to $\mathscr{I}$ if and only if λM is Ulrich with respect to $\mathscr{I}$, whenever M is a horizontally linked R-module.*

(iii) *λM is maximal Cohen–Macaulay, whenever M is a horizontally linked R-module which is Ulrich with respect to $\mathscr{I}$.*

(iv) $\operatorname{Ext}_R^{d+2}(R/\mathscr{I}, R) = 0$.

Then the following statements hold:

(a) *(i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).*

(b) *If $d \geq 2$, then (iii) $\Longrightarrow$ (iv).*

(c) *If $d \geq 2$ and $\mathscr{I} = \mathscr{M}$, then all the four conditions above are equivalent.*

Proof. (a) (i) $\Longrightarrow$ (ii). Let M be a horizontally linked R-module. By Lemma 2.10, M is a stable R-module. Assume that M is an Ulrich R-module with respect to $\mathscr{I}$. By [Goto et al. 2014, Corollary 5.3], the Auslander transpose $\operatorname{Tr} M$ is Ulrich with respect to $\mathscr{I}$. Moreover, since M is stable, we obtain by [Anderson and Fuller 1992, Theorem 32.13] that $\operatorname{Tr} M$ is stable as well. Applying [Goto et al. 2014, Corollary 5.3] we conclude that the syzygy module $\Omega \operatorname{Tr} M = \lambda M$ is Ulrich with respect to $\mathscr{I}$. Now, to see the converse, it suffices to apply Lemma 2.11 to the module λM and to use that $M \cong \lambda^2 M$. Notice that (ii) $\Longrightarrow$ (iii) is obvious. This concludes the proof of (a).

(b) (iii) $\Longrightarrow$ (iv). Let $\bar{R} = R/\mathscr{I}$, and assume on the contrary that $\operatorname{Ext}_R^{d+2}(\bar{R}, R) \neq 0$. First notice that $\Omega^{d+1}\bar{R}$ is stable, otherwise R would be a direct summand of $\Omega^{d+1}\bar{R}$ and then, by [Avramov 1998, Corollary 1.2.5],

$$d + 1 \leq \max\{0, \operatorname{depth} R - \operatorname{depth}_R \bar{R}\} = d - \operatorname{depth}_R \bar{R},$$

which is absurd. Now, by Lemma 2.10, $\Omega^{d+1}\bar{R}$ is a horizontally linked R-module. By [Goto et al. 2014, Theorem 3.2], $\Omega^{d+1}\bar{R}$ is an Ulrich R-module with respect to $\mathscr{I}$. It follows from the assumption of (iii) that $\lambda\Omega^{d+1}\bar{R}$ is a maximal Cohen–Macaulay R-module, which in turn fits into a short exact sequence

$$0 \longrightarrow \lambda\Omega^{d+1}\bar{R} \longrightarrow F \longrightarrow \operatorname{Tr}\Omega^{d+1}\bar{R} \longrightarrow 0$$

for some free R-module F. By [Bruns and Herzog 1993, Proposition 1.2.9],

$$\operatorname{depth}_R \operatorname{Tr}\Omega^{d+1}\bar{R} \geq \min\{\operatorname{depth}_R F, \operatorname{depth}_R \lambda\Omega^{d+1}\bar{R} - 1\} = d - 1 > 0. \tag{10}$$

Using (1), there is an exact sequence

$$0 \to \operatorname{Ext}_R^1(\operatorname{Tr}\operatorname{Tr}\Omega^{d+1}\bar{R}, R) \to \operatorname{Tr}\Omega^{d+1}\bar{R} \to (\operatorname{Tr}\Omega^{d+1}\bar{R})^{**} \to \operatorname{Ext}_R^2(\operatorname{Tr}\operatorname{Tr}\Omega^{d+1}\bar{R}, R) \to 0,$$

and since $\Omega^{d+1}\bar{R}$ is stable, we have $\operatorname{Tr}\operatorname{Tr}\Omega^{d+1}\bar{R} \cong \Omega^{d+1}\bar{R}$ by [Anderson and Fuller 1992, Corollary 32.14(4)]. Thus, we obtain the exact sequence

$$0 \to \operatorname{Ext}_R^{d+2}(\bar{R}, R) \to \operatorname{Tr}\Omega^{d+1}\bar{R} \to (\operatorname{Tr}\Omega^{d+1}\bar{R})^{**} \to \operatorname{Ext}_R^{d+3}(\bar{R}, R) \to 0. \tag{11}$$

As $\mathscr{I}$ is $\mathscr{M}$-primary, the nonzero module $\operatorname{Ext}_R^{d+2}(\bar{R}, R)$ must have finite length, which in particular implies $\operatorname{depth}_R \operatorname{Ext}_R^{d+2}(\bar{R}, R) = 0$. On the other hand, by virtue of (10) and (11), we get $\operatorname{depth}_R \operatorname{Ext}_R^{d+2}(\bar{R}, R) > 0$, a contradiction.

(c) (iv) $\Longrightarrow$ (i). If $\operatorname{Ext}_R^{d+2}(R/\mathscr{M}, R) = 0$ then, by [Matsumura 1986, Theorem 18.1], the local ring R is Gorenstein. □

In order to provide the first application of our theorem, we invoke the following classical concept:

Definition 4.2. A d-dimensional Cohen–Macaulay local ring R is said to have *minimal multiplicity* if its multiplicity and embedding dimension are related by $e(R) = \operatorname{edim} R - d + 1$. As is well known, there is in general an inequality $e(R) \geq \operatorname{edim} R - d + 1$, which originates the terminology.

Now recall that a local ring R is a *hypersurface* ring if $R \cong S/(f)$, where $(S, \mathscr{N})$ is a regular local ring and $f \in \mathscr{N}$. Such a ring is said to be a *quadratic* hypersurface ring if $f \in \mathscr{N}^2 \setminus \mathscr{N}^3$. Clearly, a hypersurface ring $R \cong S/(f)$ with $f \in \mathscr{N}^2$ is quadratic if and only if R has minimal multiplicity (equal to 2).

Our Theorem 4.1 yields a characterization of quadratic hypersurface rings in terms of linkage of Ulrich modules in the classical sense, in the case $\mathscr{I} = \mathscr{M}$. It is worth recalling an interesting connection, which we shall use in the proof of Corollary 4.5, between quadratic hypersurface rings and the Ulrich property. To wit, every nonfree maximal Cohen–Macaulay module over such a ring is a direct sum of an Ulrich module and a free module (see [Herzog and Kühl 1987, Corollary 1.4]); in particular, any such ring admits an Ulrich module.

Corollary 4.3. *Let R be a nonregular Cohen–Macaulay local ring of minimal multiplicity with dimension $d \geq 2$ and infinite residue field k. The following assertions are equivalent*:

(i) *R is a* (*quadratic*) *hypersurface ring.*

(ii) *M is Ulrich if and only if λM is Ulrich*, *whenever M is a horizontally linked R-module.*

(iii) *λM is maximal Cohen–Macaulay*, *whenever M is a horizontally linked Ulrich R-module.*

(iv) $\operatorname{Ext}_R^{d+2}(k, R) = 0$.

Proof. As before let $\mathscr{M}$ be the maximal ideal of R. Since $R/\mathscr{M}$ is infinite, it is well known that R has minimal multiplicity if and only if

$$\mathscr{M}^2 = (\boldsymbol{x})\mathscr{M}$$

with $\boldsymbol{x}$ an R-sequence (see [Bruns and Herzog 1993, Exercise 4.6.14]), which in turn means that $\mathscr{M}$ is an Ulrich ideal in the sense of Definition 2.4. Since R is nonregular,

$\mathscr{M}$ is not a parameter ideal. Therefore, as every hypersurface ring is Gorenstein, the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) follow readily by Theorem 4.1 with $\mathscr{I} = \mathscr{M}$. Now, as recalled in the proof of the theorem, condition (iv) forces R to be Gorenstein. But it is well known that a Gorenstein local ring having minimal multiplicity is just a quadratic hypersurface ring, as needed. □

Connections between a more general notion of minimal multiplicity and the Ulrich property with respect to $\mathscr{I}$ will be given in Section 5.

Before establishing another consequence of Theorem 4.1 over Gorenstein local rings, we invoke an auxiliary invariant which will be used in the proof, namely, the *Gorenstein dimension* of a finite R-module M, which is denoted by $\text{G-dim}_R\, M$ (for the definition, see, e.g., [Christensen 2000, Definition 1.2.3]). Recall that if R is Gorenstein then $\text{G-dim}_R\, M < \infty$ for every finite R-module M. If R is local and M is a finite R-module with $\text{G-dim}_R\, M < \infty$ then the so-called Auslander–Bridger formula states that $\text{G-dim}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M$. In particular, if R is Gorenstein, then $\text{G-dim}_R\, M = 0$ if and only if M is maximal Cohen–Macaulay. For details, see [Auslander and Bridger 1969; Christensen 2000].

Corollary 4.4. *Let R be a Gorenstein local ring, and suppose the ideal $\mathscr{I}$ is Ulrich but not a parameter ideal. Let M be a stable maximal Cohen–Macaulay R-module. Then, M is an Ulrich R-module with respect to $\mathscr{I}$ if and only if λM is an Ulrich R-module with respect to $\mathscr{I}$.*

Proof. Since R is Gorenstein and M is maximal Cohen–Macaulay, then as observed above we have $\text{G-dim}_R\, M = 0$. By [Martsinkovsky and Strooker 2004, Theorem 1], M is horizontally linked. Now the result follows from Theorem 4.1(a). □

Corollary 4.5. *Let R be a quadratic hypersurface local ring with infinite residue field, and let M be a stable maximal Cohen–Macaulay R-module. Then, λM is an Ulrich R-module.*

Proof. Over such a ring, any maximal Cohen–Macaulay module M is either free or satisfies

$$M \cong U \oplus F,$$

for some Ulrich module U and free module F, according to [Herzog and Kühl 1987, Corollary 1.4]. Thus, if in addition M is stable (in particular, nonfree), then it must be Ulrich. Also note the maximal ideal $\mathscr{M}$ of R is Ulrich but not a parameter ideal. Now we can apply Corollary 4.4 with $\mathscr{I} = \mathscr{M}$ to get the result. □

Before giving more consequences of Corollary 4.4, we recall a useful lemma.

Lemma 4.6 [Herzog and Kühl 1987, Lemma 1.2]. *Let R be a Gorenstein local ring. If M is a maximal Cohen–Macaulay R-module, then ΩM is a stable R-module.*

Corollary 4.7. *Let R be a Gorenstein local ring of dimension d, and suppose the ideal $\mathscr{I}$ is Ulrich but not a parameter ideal. Then, $\lambda(\Omega^k\mathscr{I})$ is an Ulrich R-module with respect to $\mathscr{I}$ for all $k \geq d$.*

Proof. First, as recalled in Remarks 2.8(iii), the R-module $\Omega^k(R/\mathscr{I})$ is Ulrich with respect to $\mathscr{I}$ (in particular, maximal Cohen–Macaulay) for all $k \geq d$. It follows by Lemma 4.6 that the R-module $\Omega^{k+1}(R/\mathscr{I}) = \Omega^k\mathscr{I}$ is stable for all $k \geq d$, and thus Corollary 4.4 concludes the proof. □

Corollary 4.8. *Let R be a 1-dimensional Gorenstein local ring. If $\mathscr{I}$ is an Ulrich ideal of R which is not a parameter ideal, then $\lambda\mathscr{I}$ is an Ulrich R-module with respect to $\mathscr{I}$.*

Proof. By Remarks 2.8(ii), $\mathscr{I}$ is an Ulrich R-module with respect to $\mathscr{I}$. Note $\mathscr{I}$ is stable as it is a nonprincipal ideal, hence a nonfree R-module. Now, apply Corollary 4.4. □

5. Minimal multiplicity and Ulrich properties

We start the section presenting a few preparatory definitions (Rees and associated graded modules, and relative reduction numbers) as well as some auxiliary facts.

Let I be a proper ideal of a ring R. Recall that the Rees algebra of I is the graded ring $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n$ (as usual, we put $I^0 = R$), which can be realized as the standard graded subalgebra $R[Iu] \subset R[u]$, where u is an indeterminate over R. The associated graded ring of I is given by $\mathcal{G}(I) = \bigoplus_{n\geq 0} I^n/I^{n+1} = \mathcal{R}(I) \otimes_R R/I$, which is standard graded over R/I.

Definition 5.1. If M is a finite R-module, the *Rees module* and the *associated graded module* of I relative to M are, respectively, given by

$$\mathcal{R}(I, M) = \bigoplus_{n\geq 0} I^n M, \quad \mathcal{G}(I, M) = \bigoplus_{n\geq 0} \frac{I^n M}{I^{n+1} M} = \mathcal{R}(I, M) \otimes_R R/I,$$

which are finite graded modules over $\mathcal{R}(I)$ and $\mathcal{G}(I)$, respectively.

Now consider a local ring $(R, \mathscr{M})$ with residue field k. For a proper ideal I of R, recall that the fiber cone of I is the special fiber ring of $\mathcal{R}(I)$, i.e., the standard graded k-algebra $\mathcal{F}(I) = \bigoplus_{n\geq 0} I^n/\mathscr{M}I^n = \mathcal{R}(I) \otimes_R k$. We can also consider the finite graded $\mathcal{F}(I)$-module $\mathcal{F}(I, M) = \bigoplus_{n\geq 0} I^n M/\mathscr{M}I^n M = \mathcal{R}(I, M) \otimes_R k$, whose Krull dimension (called *analytic spread* of I relative to M) is denoted by

$$s_M(I) = \dim \mathcal{F}(I, M).$$

Definition 5.2. Let I be a proper ideal of a ring R and let M be a nonzero finite R-module. An ideal $J \subset I$ is called an *M-reduction of I* if $JI^n M = I^{n+1} M$ for some integer $n \geq 0$. Such an M-reduction J is said to be *minimal* if it is minimal with respect to inclusion. If J is an M-reduction of I, we define the *reduction number of I with respect to J relative to M* as

$$r_J(I, M) = \min\{m \in \mathbb{N} \mid JI^m M = I^{m+1} M\}.$$

The lemma below detects a useful connection between minimal M-reductions and the so-called (*maximal*) *M-superficial sequences* of a given $\mathscr{M}$-primary ideal in a local ring $(R, \mathscr{M})$. For the definition and details about the latter concept, we refer to [Rossi and Valla 2010, Sections 1.2 and 1.3]; also see [Conti 2006].

Lemma 5.3 [Conti 2006, corollario 3.14]. *Let $(R, \mathscr{M})$ be a local ring with infinite residue field and let I be an $\mathscr{M}$-primary ideal. Let M be a finite R-module of positive dimension. Then, every minimal M-reduction of I can be generated by a maximal M-superficial sequence of I. Conversely, an ideal generated by a maximal M-superficial sequence of I is necessarily a minimal M-reduction of I.*

Next we invoke a central notion in this section, and a helpful lemma. As in Section 2A, if I is an ideal of definition of a finite R-module M then $e_I^0(M)$ denotes the multiplicity of M with respect to I. Moreover, we let $e_I^1(M)$ stand for the first Hilbert coefficient — the so-called *Chern number* — of M with respect to I.

Definition 5.4 [Puthenpurakal 2003, Definition 15]. Let $(R, \mathscr{M})$ be a local ring, M a Cohen–Macaulay R-module of dimension t and I a proper ideal of R such that $\mathscr{M}^n M \subset IM$ for some $n > 0$. Then M has *minimal multiplicity with respect to I* if

$$e_I^0(M) = (1-t)\ell_R(M/IM) + \ell_R(IM/I^2 M).$$

Notice that by taking $M = R$ and $I = \mathscr{M}$ we recover Definition 4.2.

Lemma 5.5 [Puthenpurakal 2003, Theorem 16]. *Let $(R, \mathscr{M})$ be a local ring, M a Cohen–Macaulay R-module of dimension t and I a proper ideal of R such that $\mathscr{M}^n M \subset IM$ for some $n > 0$. The following conditions are equivalent*:

(i) *M has minimal multiplicity with respect to I.*

(ii) $(z_1, \ldots, z_t)IM = I^2 M$, *for every maximal M-superficial sequence $z_1, \ldots, z_t$.*

(iii) $(z_1, \ldots, z_t)IM = I^2 M$, *for some maximal M-superficial sequence $z_1, \ldots, z_t$.*

(iv) $e_I^1(M) = e_I^0(M) - \ell_R(M/IM)$.

Here we observe that item (iii) above is not present in [Puthenpurakal 2003], but a simple inspection of the proof easily shows that this assertion is also equivalent to the ones given in Theorem 16 of that paper.

Our first result in this part is the following. As in the previous sections, we let $Q = (x_1, \ldots, x_d) \subset \mathscr{I}$ be as in Convention 2.2.

Proposition 5.6. *Suppose R is a Cohen–Macaulay local ring with infinite residue field. Then, every Ulrich R-module with respect to $\mathscr{I}$ has minimal multiplicity with respect to $\mathscr{I}$.*

Proof. Let M be an Ulrich module with respect to $\mathscr{I}$. In particular, M is maximal Cohen–Macaulay. Let $\operatorname{grade}(\mathscr{I}, M)$ denote the maximal length of an M-sequence contained in $\mathscr{I}$. By [Kadu 2011, Lemma 1.3 and Lemma 1.6], we have

$$\operatorname{grade}(\mathscr{I}, M) \leq s_M(\mathscr{I}) \leq \dim M.$$

As $\mathscr{I}$ is $\mathscr{M}$-primary, $\operatorname{grade}(\mathscr{I}, M) = \operatorname{depth} M = d$, where as before $d = \dim R$. Hence $s_M(\mathscr{I}) = d = \nu(Q)$, where $\nu(-)$ stands for minimal number of generators. As is well known (see, e.g., [Conti 2006, corollario 3.22]), this implies that Q is a minimal M-reduction of $\mathscr{I}$, and therefore Lemma 5.3 gives that $x_1, \dots, x_d$ is in fact a maximal M-superficial sequence of $\mathscr{I}$. On the other hand, because M is Ulrich, we have $QM = \mathscr{I}M$ and so

$$Q\mathscr{I}M = \mathscr{I}^2 M.$$

We conclude, by Lemma 5.5, that M has minimal multiplicity with respect to $\mathscr{I}$. $\square$

Remark 5.7. The converse of Proposition 5.6 fails even in the classical case $\mathscr{I} = \mathscr{M}$; see [Puthenpurakal 2005, Example 4.12].

Combining Proposition 5.6 and [Puthenpurakal 2003, Theorem 16], we immediately obtain the following property.

Corollary 5.8. *Suppose R is a Cohen–Macaulay local ring with infinite residue field. If M is an Ulrich R-module with respect to $\mathscr{I}$, then the associated graded $\mathcal{G}(\mathscr{I})$-module $\mathcal{G}(\mathscr{I}, M)$ is Cohen–Macaulay.*

The next consequence deals with the Chern number and gives a generalization of [Ooishi 1991, Corollary 1.3(1)].

Corollary 5.9. *Let $(R, \mathscr{M})$ be a Cohen–Macaulay local ring with infinite residue field and positive dimension, and let M be a maximal Cohen–Macaulay R-module. Then $e^1_{\mathscr{I}}(M) \geq 0$, and the following assertions are equivalent:*

(i) *M is an Ulrich R-module with respect to $\mathscr{I}$.*

(ii) *$M/\mathscr{I}M$ is a free $R/\mathscr{I}$-module and $e^1_{\mathscr{I}}(M) = 0$.*

Proof. Applying [Puthenpurakal 2003, Proposition 12] and Remarks 2.8(i), we get

$$e^1_{\mathscr{I}}(M) \geq e^0_{\mathscr{I}}(M) - \ell_R(M/\mathscr{I}M) \geq 0.$$

If M is Ulrich with respect to $\mathscr{I}$ then, by definition, the $R/\mathscr{I}$-module $M/\mathscr{I}M$ is free and in addition $e^0_{\mathscr{I}}(M) = \ell_R(M/\mathscr{I}M)$ (use again Remarks 2.8(i)). On the other hand, Proposition 5.6 ensures that M has minimal multiplicity with respect to $\mathscr{I}$, and therefore Lemma 5.5 gives $e^1_{\mathscr{I}}(M) = e^0_{\mathscr{I}}(M) - \ell_R(M/\mathscr{I}M) = 0$.

Conversely, suppose (ii). Since M is already assumed to be maximal Cohen–Macaulay, it remains to show that $\mathscr{I}M = QM$, which as we know is equivalent to the equality $e^0_{\mathscr{I}}(M) = \ell_R(M/\mathscr{I}M)$. But this follows from $0 \le e^0_{\mathscr{I}}(M) - \ell_R(M/\mathscr{I}M) \le e^1_{\mathscr{I}}(M) = 0$. This concludes the proof. □

Corollary 5.10. *Let $(R, \mathscr{M})$ be a Cohen–Macaulay local ring with infinite residue field and dimension $d \ge 1$. If $\mathscr{I}$ is an Ulrich ideal of R which is not a parameter ideal, then $e^1_{\mathscr{I}}(\Omega^k \mathscr{I}) = 0$ for all $k \ge d-1$. If in addition R is Gorenstein, then*

$$e^1_{\mathscr{I}}(\lambda(\Omega^k \mathscr{I})) = 0 \quad \textit{for all } k \ge d.$$

Proof. Recall that the R-module $\Omega^{k+1}(R/\mathscr{I}) = \Omega^k \mathscr{I}$ is Ulrich with respect to $\mathscr{I}$ (in particular, maximal Cohen–Macaulay) for all $k \ge d-1$; see Remarks 2.8(iii). Then the vanishing of $e^1_{\mathscr{I}}(\Omega^k \mathscr{I})$ follows by Corollary 5.9. Now if R is Gorenstein then, by Corollary 4.7, the module $\lambda(\Omega^k \mathscr{I})$ is Ulrich with respect to $\mathscr{I}$ for all $k \ge d$, and we again apply Corollary 5.9. □

Remarks 5.11. (i) Let M be a d-dimensional Cohen–Macaulay R-module (assume the setting of Convention 2.2, with $d > 0$ and $R/\mathscr{M}$ infinite). Recall that, for $k \gg 0$, the Hilbert–Samuel function $H^M_{\mathscr{I}}(k) = \ell_R(M/\mathscr{I}^k M)$ coincides with a degree d polynomial $P^M_{\mathscr{I}}(k)$, the Hilbert–Samuel polynomial of M with respect to $\mathscr{I}$, which can be expressed as

$$P^M_{\mathscr{I}}(k) = \sum_{i=0}^{d} (-1)^i e^i_{\mathscr{I}}(M) \binom{k+d-i-1}{d-i}.$$

Now if M is Ulrich with respect to $\mathscr{I}$, then in particular $M/\mathscr{I}M \cong (R/\mathscr{I})^{\nu(M)}$ and therefore, by Corollary 5.9, we get $e^0_{\mathscr{I}}(M) = \ell_R(M/\mathscr{I}M) = \nu(M)\ell_R(R/\mathscr{I})$ and $e^1_{\mathscr{I}}(M) = 0$. Thus, if for instance $d = 1$ then $P^M_{\mathscr{I}}(k) = \nu(M)\ell_R(R/\mathscr{I})k$. If $d = 2$, we have

$$P^M_{\mathscr{I}}(k) = \nu(M)\ell_R(R/\mathscr{I})\binom{k+1}{2} + e^2_{\mathscr{I}}(M),$$

which raises the problem of finding $e^2_{\mathscr{I}}(M)$. Of course, in case we know an integer k_0 satisfying $P^M_{\mathscr{I}}(k) = H^M_{\mathscr{I}}(k)$ for all $k \ge k_0$, then $e^2_{\mathscr{I}}(M)$ can be computed from the expression above by evaluating $k = k_0$.

(ii) If $d \ge 1$ and $\mathscr{I}$ is an Ulrich ideal of R then, as we know, the j-th syzygy module of $\mathscr{I}$ is Ulrich with respect to $\mathscr{I}$ for all $j \ge d-1$. Now assume $d = 1$. Applying the preceding part to the module $\Omega^j \mathscr{I}$ for any $j \ge 0$, and noticing that $\nu(\Omega^j \mathscr{I})$ is precisely the j-th Betti number $\beta_j(\mathscr{I})$ of $\mathscr{I}$, we obtain the simple formula

$$P^{\Omega^j \mathscr{I}}_{\mathscr{I}}(k) = \beta_j(\mathscr{I})\ell_R(R/\mathscr{I})k.$$

In addition, considering linkage and assuming that R is Gorenstein, our Corollary 4.7 yields that $\lambda(\Omega^j\mathscr{I})$ is also Ulrich with respect to $\mathscr{I}$ for any $j \geq 1$, and observe that $\nu(\lambda(\Omega^j\mathscr{I})) = \beta_j(\mathscr{I})$ as well. It follows that $P_{\mathscr{I}}^{\Omega^j\mathscr{I}}(k) = P_{\mathscr{I}}^{\lambda(\Omega^j\mathscr{I})}(k)$.

Our next result, Theorem 5.14 below, provides a characterization of modules of minimal multiplicity in terms of reduction number and Castelnuovo–Mumford regularity (of blowup modules). For completeness, we recall the definition of the latter, which is of great importance in commutative algebra and algebraic geometry, for instance in the study of degrees of syzygies over polynomial rings; we refer to [Brodmann and Sharp 1998, Chapter 15].

Let $S = \bigoplus_{n\geq 0} S_n$ be a finitely generated standard graded algebra over a ring S_0. As usual, we write $S_+ = \bigoplus_{n\geq 1} S_n$. For a graded S-module $A = \bigoplus_{n\in\mathbb{Z}} A_n$ satisfying $A_n = 0$ for all $n \gg 0$, we set

$$\operatorname{end} A = \begin{cases} \max\{n \mid A_n \neq 0\} & \text{if } A \neq 0, \\ -\infty & \text{if } A = 0. \end{cases}$$

Now fix a finite graded S-module $N \neq 0$. Given $j \geq 0$, let

$$H^j_{S_+}(N) = \varinjlim_k \operatorname{Ext}^j_S(S/S^k_+, N)$$

be the j-th local cohomology module of N. Recall $H^j_{S_+}(N)$ is a graded module such that $H^j_{S_+}(N)_n = 0$ for all $n \gg 0$; see [Brodmann and Sharp 1998, Proposition 15.1.5(ii)]. Thus, $\operatorname{end} H^j_{S_+}(N) < \infty$.

Definition 5.12. The *Castelnuovo–Mumford regularity* of the graded S-module N is given by

$$\operatorname{reg} N = \max\{\operatorname{end} H^j_{S_+}(N) + j \mid j \geq 0\}.$$

The following lemma will be very useful to the proof of Theorem 5.14, since it interprets the regularity of Rees modules as a relative reduction number in a suitable setting. It was originally stated in more generality (involving d-sequences) but here the special case of regular sequences suffices for our purposes.

Lemma 5.13 [Giral and Planas-Vilanova 2008, Theorem 5.3]. *Let R be a ring, I an ideal of R and M a finite R-module. Let $z_1, \dots, z_s$ be an M-sequence such that the ideal $J = (z_1, \dots, z_s)$ is an M-reduction of I. Let $r_J(I, M) = r$. Suppose either $s = 1$, or else $s \geq 2$ and*

$$(z_1, \dots, z_i)M \cap I^{r+1}M = (z_1, \dots, z_i)I^r M \quad \textit{for all } i = 1, \dots, s-1.$$

Then, $\operatorname{reg} \mathcal{R}(I, M) = r_J(I, M)$.

We are now ready for the main technical result of this section, which in particular will lead us to a byproduct on Ulrich modules. Note this theorem also gives a generalization of [Ooishi 1991, Proposition 1.2], where the situation $\mathscr{I} = \mathscr{M}$ was treated; more precisely, the condition $g_\Delta(M) = 0$ in that paper is equivalent to Puthenpurakal's notion of minimal multiplicity when $\mathscr{I} = \mathscr{M}$.

Theorem 5.14. *Let $(R, \mathscr{M})$ be a local ring with infinite residue field, M a Cohen–Macaulay R-module of dimension $t > 0$ and I an $\mathscr{M}$-primary ideal of R. Let $J = (z_1, \dots, z_t)$ be a minimal M-reduction of I. The following assertions are equivalent*:

(i) *M has minimal multiplicity with respect to I.*

(ii) $\operatorname{reg} \mathcal{R}(I, M) = \operatorname{reg} \mathcal{G}(I, M) = r_J(I, M) \leq 1$.

(iii) $r_J(I, M) \leq 1$.

Proof. First, notice that $z_1, \dots, z_t$ is a (maximal) M-superficial sequence of I by Lemma 5.3. As a consequence, since M is Cohen–Macaulay and I is $\mathscr{M}$-primary, $z_1, \dots, z_t$ must be in fact an M-sequence according to [Rossi and Valla 2010, Lemma 1.2]. Now, the core of the proof is the implication (i) $\Longrightarrow$ (ii), so assume first that (i) holds. In general, we have $\operatorname{reg} \mathcal{R}(I, M) = \operatorname{reg} \mathcal{G}(I, M)$, see [Zamani 2014, Corollary 3], and so it remains to prove that $\operatorname{reg} \mathcal{R}(I, M) = r_J(I, M)$, which we shall accomplish by means of Lemma 5.13.

Moreover, since $z_1, \dots, z_t$ is maximal M-superficial, Lemma 5.5 yields $JIM = I^2M$, i.e., $r_J(I, M) \leq 1$. Now, to simplify notation, set $\boldsymbol{z}_i = z_1, \dots, z_i$ for $i = 1, \dots, t-1$ (note we can assume $t > 1$ by Lemma 5.13). Since clearly $(\boldsymbol{z}_i)M \cap IM = (\boldsymbol{z}_i)M$ for all $i = 1, \dots, t-1$, the case $r_J(I, M) = 0$ is trivial by virtue of Lemma 5.13. Now suppose $r_J(I, M) = 1$. Again in view of Lemma 5.13, all we need to prove is that

$$(\boldsymbol{z}_i)M \cap I^2M = (\boldsymbol{z}_i)IM \quad \text{for all } i = 1, \dots, t-1.$$

First, it is clear that $(\boldsymbol{z}_i)IM \subset (\boldsymbol{z}_i)M \cap I^2M$. To show the other inclusion, take an arbitrary $f \in (\boldsymbol{z}_i)M \cap I^2M$. Because $JIM = I^2M$, we have

$$f = z_1m_1 + \cdots + z_im_i = z_1a_1m'_1 + \cdots + z_ta_tm'_t$$

with $m_j, m'_k \in M$ and $a_k \in I$. Hence

$$\overline{z_ta_tm'_t} = \bar{0} \in M/(\boldsymbol{z}_{t-1})M,$$

and since the sequence is regular on M, we have $\overline{a_tm'_t} = \bar{0} \in M/(\boldsymbol{z}_{t-1})M$, that is, $a_tm'_t = z_1w_{t,1} + \cdots + z_{t-1}w_{t,t-1}$ with $w_{t,j} \in M$. Therefore, f can be expressed as

$$z_1m_1 + \cdots + z_im_i = z_1(a_1m'_1 + z_tw_{t,1}) + \cdots + z_{t-1}(a_{t-1}m'_{t-1} + z_tw_{t,t-1}), \tag{12}$$

whose right-hand side shows $f \in (\boldsymbol{z}_{t-1})IM$, thus settling the case $i = t-1$. Next, for $i < t-1$, we reduce (12) modulo $(\boldsymbol{z}_{t-2})M$ and apply an analogous argument to the term $z_{t-1}(a_{t-1}m'_{t-1}+z_t w_{t,t-1})$ in order to obtain

$$a_{t-1}m'_{t-1}+z_t w_{t,t-1}=z_1 w_{t-1,1}+\cdots+z_{t-2}w_{t-1,t-2} \tag{13}$$

with $w_{t-1,j}\in M$. Thus, by (12) and (13),

$$f=z_1(a_1m'_1+z_t w_{t,1}+z_{t-1}w_{t-1,1})+\cdots+z_{t-2}(a_{t-2}m'_{t-2}+z_t w_{t,t-2}+z_{t-1}w_{t-1,t-2}).$$

Continuing with the argument, we get an equality

$$f=z_1(a_1m'_1+z_t w_{t,1}+\cdots+z_{i+1}w_{i+1,1})+\cdots+z_i(a_i m'_i+z_t w_{t,i}+\cdots+z_{i+1}w_{i+1,i}).$$

Since $a_1,\ldots,a_i,z_{i+1},\ldots,z_t\in I$, it follows that $f\in(\mathbf{z}_i)IM$, as needed.

The implication (ii) $\Longrightarrow$ (iii) is obvious. Finally, suppose (iii) holds. Then $JIM=I^2M$, and we have seen that $z_1,\ldots,z_t$ is a maximal M-superficial sequence. By Lemma 5.5, we conclude that M has minimal multiplicity with respect to I. □

As a consequence of Theorem 5.14, we determine the regularity of blowup modules of $\mathscr{I}$ relative to an Ulrich module. Also, taking $\mathscr{I}=\mathscr{M}$ the result retrieves part of [Ooishi 1991, Proposition 1.1].

Corollary 5.15. *Let $(R,\mathscr{M})$ be a Cohen–Macaulay local ring with infinite residue field and positive dimension, and let Q be as in Convention 2.2. If M is an Ulrich R-module with respect to $\mathscr{I}$, then*

$$\operatorname{reg}\mathcal{R}(\mathscr{I},M)=\operatorname{reg}\mathcal{G}(\mathscr{I},M)=r_Q(\mathscr{I},M)=0.$$

The converse holds in case M is maximal Cohen–Macaulay and $M/\mathscr{I}M$ is $R/\mathscr{I}$-free.

Proof. First, notice that Q is an M-reduction of $\mathscr{I}$, so the number $r_Q(\mathscr{I},M)$ makes sense. Now, because M is Ulrich with respect to $\mathscr{I}$, we have $QM=\mathscr{I}M$, which means $r_Q(\mathscr{I},M)=0$. On the other hand, Proposition 5.6 and its proof ensure that M has minimal multiplicity with respect to $\mathscr{I}$ and that Q is in fact a minimal M-reduction of $\mathscr{I}$, and so we can apply Theorem 5.14 to obtain $\operatorname{reg}\mathcal{R}(\mathscr{I},M)=\operatorname{reg}\mathcal{G}(\mathscr{I},M)=r_Q(\mathscr{I},M)$. The converse is clear. □

Corollary 5.16. *Let $(R,\mathscr{M})$ be a Cohen–Macaulay local ring with infinite residue field and positive dimension. Suppose $\mathscr{I}$ is an Ulrich ideal of R but not a parameter ideal. Then, $\operatorname{reg}\mathcal{R}(\mathscr{I},\Omega^k\mathscr{I})=0$ for all $k\geq d-1$. If in addition R is Gorenstein, then*

$$\operatorname{reg}\mathcal{R}(\mathscr{I},\lambda(\Omega^k\mathscr{I}))=0\quad \textit{for all } k\geq d.$$

Proof. As we know, the R-module $\Omega^{k+1}(R/\mathscr{I})=\Omega^k\mathscr{I}$ is Ulrich with respect to $\mathscr{I}$ for all $k\geq d-1$. Thus the first part follows from Corollary 5.15. If R is Gorenstein then by Corollary 4.7 the R-module $\lambda(\Omega^k\mathscr{I})$ is Ulrich with respect to $\mathscr{I}$ for all $k\geq d$. Now we again apply Corollary 5.15. □

Corollary 5.17. *Let $(R, \mathscr{M})$ be a 1-dimensional Cohen–Macaulay local ring with infinite residue field. If $\mathscr{I}$ is an Ulrich ideal, then*

$$\operatorname{reg} \mathcal{R}(\mathscr{I})_+ = 0.$$

Proof. Using Remarks 2.8(ii) and Corollary 5.15, we obtain $\operatorname{reg} \mathcal{R}(\mathscr{I}, \mathscr{I}) = 0$. On the other hand, we clearly have $\mathcal{R}(\mathscr{I}, \mathscr{I}) = \bigoplus_{i \geq 0} \mathscr{I}^{i+1} = \mathcal{R}(\mathscr{I})_+$. □

Example 5.18. Consider the local ring $R = K[[x, y]]/(x^2 + y^4)$, where K is an infinite field. The ideal $\mathscr{I} = (x, y^2)R$ is Ulrich (this is the case $d = 1$ and $s = 2$ of Examples 2.5(ii)). Then, Corollary 5.17 gives $\operatorname{reg} \mathcal{R}(\mathscr{I})_+ = 0$. To write this graded ideal explicitly, we can use (degree 1) variables T, U over R in order to determine a presentation of the Rees algebra

$$\mathcal{R}(\mathscr{I}) = R[T, U]/\mathscr{K}, \quad \mathscr{K} = (xT + y^2U, y^2T - xU, T^2 + U^2), \quad \mathcal{R}(\mathscr{I})_0 = R,$$

so that $\mathcal{R}(\mathscr{I})_+ = (T, U)R[T, U]/\mathscr{K}$.

Now let us use the same example to illustrate the determination of the Hilbert–Samuel polynomial $P_{\mathscr{I}}^{\mathscr{I}}(k)$. Notice that $\ell_R(R/\mathscr{I}) = \dim_K(K[[y]]/(y^2)) = 2$ and $\nu(\mathscr{I}) = 2$. By Remarks 5.11(i), we have $P_{\mathscr{I}}^{\mathscr{I}}(k) = \nu(\mathscr{I})\ell_R(R/\mathscr{I})k = 4k$, i.e.,

$$\ell_R(\mathscr{I}/\mathscr{I}^{k+1}) = 4k \quad \text{for all } k \gg 0.$$

6. A detailed example

In this last section, we fix formal indeterminates x, y, z over an infinite field K as well as the 2-dimensional local hypersurface ring $R = K[[x, y, z]]/(x^2 + y^2 + z^4)$. The ideal

$$\mathscr{I} = (x, y, z^2)R$$

is Ulrich — this is the case $d = s = 2$ of Examples 2.5(ii) — and not a parameter ideal. Our goal here is to find (explicit) Ulrich R-modules with respect to $\mathscr{I}$ and study their multiplicities, Chern numbers, and the regularity of the associated blowup modules.

First, $\mathscr{I}$ has an infinite (in fact, periodic) minimal R-free resolution

$$\cdots \longrightarrow R^4 \xrightarrow{\Phi} R^4 \xrightarrow{\Phi} R^4 \xrightarrow{\Phi} R^4 \xrightarrow{\Psi} R^3 \longrightarrow \mathscr{I} \longrightarrow 0, \tag{14}$$

where

$$\Phi = \begin{pmatrix} -z^2 & 0 & -y & x \\ 0 & -z^2 & x & y \\ -y & x & z^2 & 0 \\ x & y & 0 & z^2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} -z^2 & 0 & -y & x \\ 0 & -z^2 & x & y \\ x & y & 0 & z^2 \end{pmatrix}.$$

In what follows, as a matter of standard notation, whenever φ is a $p \times q$ matrix with entries in R, we let $\operatorname{Im}\varphi$ denote the R-submodule of R^p generated by the column vectors of φ. Below we observe a few facts.

- We claim that the R-submodules $\operatorname{Im}\Phi \subset R^4$ and $\operatorname{Im}\Psi \subset R^3$ are Ulrich with respect to $\mathscr{I}$. To see this, using Remarks 2.8(iii) we get that $\Omega^k\mathscr{I}$ is Ulrich with respect to $\mathscr{I}$ whenever $k \geq 1$. But in the present case, by (14), these modules are

$$\Omega\mathscr{I} = \operatorname{Im}\Psi, \quad \Omega^k\mathscr{I} = \operatorname{Im}\Phi, \quad \text{for all } k \geq 2,$$

thus showing the claim. Also notice (by the symmetry of Φ) that $\lambda(\Omega^k\mathscr{I}) = \lambda(\operatorname{Im}\Phi) = \operatorname{Im}\Phi^* = \operatorname{Im}\Phi$ for all $k \geq 2$. In particular, $\operatorname{Im}\Phi$ is horizontally linked.

- Let us compute multiplicities and Chern numbers. First, since $\operatorname{Im}\Psi$ is Ulrich with respect to $\mathscr{I}$, we must have $\operatorname{Im}\Psi/\mathscr{I}\operatorname{Im}\Psi \cong (R/\mathscr{I})^{\nu(\operatorname{Im}\Psi)}$. Note $\ell_R(R/\mathscr{I}) = \dim_K(K[\![z]\!]/(z^2)) = 2$. Thus, by Remarks 2.8(i),

$$e^0_{\mathscr{I}}(\operatorname{Im}\Psi) = \ell_R(\operatorname{Im}\Psi/\mathscr{I}\operatorname{Im}\Psi) = \nu(\operatorname{Im}\Psi)\ell_R(R/\mathscr{I}) = 4 \cdot 2 = 8.$$

Since $\nu(\operatorname{Im}\Phi) = 4$ as well, we have $e^0_{\mathscr{I}}(\operatorname{Im}\Phi) = 8$. As to the Chern numbers, Corollary 5.10 gives $e^1_{\mathscr{I}}(\Omega^k\mathscr{I}) = 0$ for all $k \geq 1$. Hence,

$$e^1_{\mathscr{I}}(\operatorname{Im}\Psi) = e^1_{\mathscr{I}}(\operatorname{Im}\Phi) = 0.$$

- For the Castelnuovo–Mumford regularity of blowup modules, Corollary 5.16 yields $\operatorname{reg}\mathcal{R}(\mathscr{I}, \Omega^k\mathscr{I}) = 0$ for all $k \geq 1$, and therefore

$$\operatorname{reg}\mathcal{R}(\mathscr{I}, \operatorname{Im}\Psi) = \operatorname{reg}\mathcal{R}(\mathscr{I}, \operatorname{Im}\Phi) = 0.$$

Finally, the associated graded $\mathcal{G}(\mathscr{I})$-modules $\mathcal{G}(\mathscr{I}, \operatorname{Im}\Psi)$ and $\mathcal{G}(\mathscr{I}, \operatorname{Im}\Phi)$ have regularity zero as well (see Corollary 5.15), and notice they are Cohen–Macaulay by Corollary 5.8.

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GROUPS WITH 2-GENERATED SYLOW SUBGROUPS AND THEIR CHARACTER TABLES

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Let G be a finite group with a Sylow p-subgroup P. We show that the character table of G determines whether P has maximal nilpotency class and whether P is a minimal nonabelian group. The latter result is obtained from a precise classification of the corresponding groups G in terms of their composition factors. For p-constrained groups G we prove further that the character table determines whether P can be generated by two elements.

1. Introduction

Recently, Navarro and Sambale [2023] have investigated finite groups G with a Sylow p-subgroup P such that $|P : P'| = p^2$ or $|P : \mathrm{Z}(P)| = p^2$ where $P' = [P, P]$ denotes the commutator subgroup and $\mathrm{Z}(P)$ is the center of P. It was proved that both properties can be read off from the character table $X(G)$ of G. This was another contribution to Richard Brauer's Problem 12 [1963], which asks what properties of a Sylow p-subgroup P are determined by $X(G)$. We refer the reader to the introduction of [Navarro and Sambale 2023] and [Sambale 2020] for a collection of the known results on this problem. We just mention that one important property is that $X(G)$ knows whether P is abelian. While there is an elementary proof of the case $p = 2$ by Camina and Herzog [1980], the full solution has required the classification of finite simple groups (see [Kimmerle and Sandling 1995; Navarro et al. 2015; Malle and Navarro 2021]).

After dealing with P' and $\mathrm{Z}(P)$, it is natural to turn our attention to the Frattini subgroup $\Phi(P)$ of P. Recall that $|P : \Phi(P)| \le p$ holds if and only if P is cyclic. It is easy to show that this property can be read off from $X(G)$ (see [Navarro 2018, Corollary 3.12]). In the first part of the present paper we consider groups G with $|P : \Phi(P)| = p^2$, i.e., P is generated by two elements, but not by one. For $p = 2$ this property is detectable by $X(G)$ as was shown in [Navarro et al. 2021]. We obtain the corresponding result for odd primes p provided that G is p-constrained in Corollary 5. In the general case we offer a partial solution depending on the socle of G (see Proposition 6 and the subsequent remark).

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Our next objective are groups with Sylow p-subgroups P of maximal nilpotency class. For $p=2$, this property is equivalent to $|P:P'|=4$. This case was previously handled in an elementary fashion by Navarro, Sambale, and Tiep [Navarro et al. 2018]. The general result is our first main theorem.

Theorem A. *The character table of a finite group G determines whether G has Sylow p-subgroups of maximal nilpotency class.*

It is known that $X(G)$ determines the isomorphism types of abelian Sylow subgroups. Of course we cannot expect this for maximal class Sylow subgroups as $X(D_8) = X(Q_8)$. Perhaps surprisingly, $X(G)$ does not even determine $X(P)$. Counterexamples for $p = 3$ arise as semidirect products of nonequivalent faithful actions of $\mathrm{SL}(2, 3)$ on $C_9 \times C_9$ (the groups are `SmallGroup`$(2^3 3^5, a)$ with $a \in \{2289, 2290\}$ in GAP [2020]). Here P indeed has maximal class. This is related to [Navarro et al. 2022, Question E].

We obtain Theorem A as a consequence of the following structure description, which might be of independent interest:

Theorem B. *Let G be a finite group with a Sylow p-subgroup P of maximal class. Suppose that $\mathrm{O}_{p'}(G) = 1$ and $\mathrm{O}^{p'}(G) = G$. Then one of the following holds*:

(i) *There exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.*

(ii) *G is quasisimple and $|\mathrm{Z}(G)| \le p$.*

The proof uses recent work by Grazian and Parker [2022] on fusion systems and is given in Section 3.

In the final part of the paper we study groups with minimal nonabelian Sylow p-subgroups P, i.e., P is nonabelian, but every proper subgroup of P is abelian. It is easy to see that this happens if and only if $|P : \mathrm{Z}(P)| = |P : \Phi(P)| = p^2$ (see Lemma 9 below). Refining [Navarro and Sambale 2023, Theorem 7.5], we obtain in Section 4 the following description:

Theorem C. *Let G be a finite group with a minimal nonabelian Sylow p-subgroup P and $\mathrm{O}_{p'}(G) = 1$. Then one of the following holds*:

(i) *$p = 2$, $P \in \{D_8, Q_8\}$ and $\mathrm{O}^{2'}(G) \in \{\mathrm{SL}(2, q), \mathrm{PSL}(2, q'), A_7\}$ where $q \equiv \pm 3 \pmod 8$ and $q' \equiv \pm 7 \pmod{16}$.*

(ii) *$|P| = p^3$ and $\exp(P) = p > 2$.*

(iii) *$G = P \rtimes Q$ where $Q \le \mathrm{GL}(2, p)$.*

(iv) *$p > 2$, $\mathrm{O}^{p'}(G) = S \rtimes C_{p^a}$ where S is a simple group of Lie type with cyclic Sylow p-subgroups. The image of C_{p^a} in $\mathrm{Out}(S)$ has order p.*

(v) *$p = 2$ and $G = \mathrm{PSL}(2, q^f) \rtimes C_{2^a d}$ where q is a prime, $q^f \equiv \pm 3 \pmod 8$ and $d \mid f$. Moreover, C_{2^a} acts as a diagonal automorphism of order 2 on $\mathrm{PSL}(2, q^f)$ and C_d induces a field automorphism of order d.*

(vi) $p = 3$ *and* $\mathrm{O}^{3'}(G) = \mathrm{PSL}^{\epsilon}(3, q^f) \rtimes C_{3^a}$ *where* $\epsilon = \pm 1$, q *is prime*, $(q^f - \epsilon)_3 = 3$ *and* $G/\mathrm{O}^{3'}(G) \leq C_f \times C_2$.

Here, PSL^{ϵ} stands for PSL if $\epsilon = 1$ and PSU otherwise. Again the proof is based on the classification of the corresponding fusion systems. To show that Case (iv) in Theorem C occurs for all odd primes p, we will exhibit appropriate examples after the proof.

Corollary D. *The character table of a finite group G determines whether G has minimal nonabelian Sylow p-subgroups.*

2. 2-generated Sylow subgroups

In the following G will always denote a finite group. The exponent of G is denoted by $\exp(G)$. The core of a subgroup $H \leq G$ is defined by $\mathrm{core}_G(H) := \bigcap_{g \in G} gHg^{-1} \trianglelefteq G$. For $x, y \in G$ let $[x, y] := xyx^{-1}y^{-1}$. The Fitting subgroup and the generalized Fitting subgroup of G are denoted by $\mathrm{F}(G)$ and $\mathrm{F}^*(G) = \mathrm{F}(G)\mathrm{E}(G)$ respectively. We write $\mathrm{Irr}(G)$ to denote the set of ordinary complex irreducible characters of G. For $g \in G$ and $\chi \in \mathrm{Irr}(G)$ let

$$\mathbb{Q}(g) := \mathbb{Q}(\chi(g) : \chi \in \mathrm{Irr}(G)),$$
$$\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) : g \in G).$$

It is well-known that $\mathbb{Q}(\chi)$ lies in the cyclotomic field $\mathbb{Q}_n$ where $n = |G|$. Let f_χ be the smallest positive integer such that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}_{f_\chi}$ (f_χ is called the *Feit number* in [Navarro 2018]). Let $\mathrm{Irr}_{p'}(G) := \{\chi \in \mathrm{Irr}(G) : p \nmid \chi(1)\}$ as usual. The p-part and the p'-part of an integer n are denoted by n_p and $n_{p'}$ respectively.

Our first lemma is applied multiple times throughout the paper.

Lemma 1. *Let A be an abelian normal subgroup of G such that $G = \langle x \rangle A$ for some $x \in G$. Then the map $A \to G'$, $a \mapsto [x, a]$ is an epimorphism with kernel $\mathrm{C}_A(x)$. In particular, $|G'| = |A/\mathrm{C}_A(x)|$.*

Proof. See [Isaacs 2008, Lemma 4.6]. □

To get from P' to $\Phi(P)$ we need the following variant:

Lemma 2. *Let P be a p-group with a proper normal subgroup Q and $x \in P$ such that $P = \langle x \rangle Q$ and $\langle x \rangle \cap Q \leq P'$. Then $|P : \Phi(P)| = p^2$ if and only if $|\mathrm{C}_{Q/\Phi(Q)}(x)| = p$.*

Proof. Since $\langle x \rangle \cap Q \leq P' \leq \Phi(P)$ and $Q < P$, we have

$$P/\Phi(P) = Q\Phi(P)/\Phi(P) \times \langle x \rangle \Phi(P)/\Phi(P) \cong Q/(Q \cap \Phi(P)) \times C_p.$$

Moreover,

$$\Phi(P) \cap Q = P'\Phi(Q)\langle x^p \rangle \cap Q = P'\Phi(Q)(\langle x^p \rangle \cap Q) = P'\Phi(Q).$$

Now $|P:\Phi(P)|=p^2$ if and only if

$$|Q/\Phi(Q):(P/\Phi(Q))'|=|Q:P'\Phi(Q)|=p.$$

By Lemma 1 applied to $Q/\Phi(Q)\trianglelefteq P/\Phi(Q)$, this is equivalent to

$$|\mathrm{C}_{Q/\Phi(Q)}(x)|=p. \qquad \square$$

The next result is a variation of [Navarro and Sambale 2023, Theorem 6.1].

Lemma 3. *Let G be a finite group with a Sylow p-subgroup P and $\mathrm{O}_{p'}(G)=1$. Then*

$$K:=\bigcap_{\substack{\chi\in\mathrm{Irr}_{p'}(G)\\ p^2\nmid f_\chi}}\mathrm{Ker}(\chi)=\mathrm{core}_G(\Phi(P)).$$

Proof. Let $n:=|G|$. If $n_p=1$, then the claim holds since $\bigcap_{\chi\in\mathrm{Irr}(G)}\mathrm{Ker}(\chi)=1=P$. Thus, let $n_p\neq 1$. Then $\mathcal{G}:=\mathrm{Gal}(\mathbb{Q}_n|\mathbb{Q}_{pn_{p'}})$ is a p-group. Let $N:=\mathrm{core}_G(\Phi(P))$ and $\chi\in\mathrm{Irr}_{p'}(G)$ with $p^2\nmid f_\chi$. Since $\mathbb{Q}(\chi_P)\subseteq\mathbb{Q}(\chi)\subseteq\mathbb{Q}_{pn_{p'}}$, $\mathcal{G}$ permutes the irreducible constituents of χ_P. Since the sizes of the $\mathcal{G}$-orbits are p-powers and $p\nmid\chi(1)$, there must be a linear constituent $\lambda\in\mathrm{Irr}(P|\chi)$ fixed by $\mathcal{G}$, i.e., $\mathbb{Q}(\lambda)\subseteq\mathbb{Q}_p$. It follows that $N\subseteq\Phi(P)\subseteq\mathrm{Ker}(\lambda)$. By Clifford theory, χ_N is a sum of conjugates of λ_N. Hence, $N\subseteq\mathrm{Ker}(\chi)$. This shows that $N\le K$.

Now let $\lambda\in\mathrm{Irr}(P/\Phi(P))$. This time, $\mathcal{G}$ acts on the irreducible constituents of λ^G. Since $p\nmid|G:P|=\lambda^G(1)$, there must be a constituent $\chi\in\mathrm{Irr}_{p'}(G|\lambda)$ fixed by $\mathcal{G}$, i.e., $p^2\nmid f_\chi$. This implies $\chi_{P\cap K}=\chi(1)1_{P\cap K}$. On the other hand, $\lambda_{P\cap K}$ is a constituent of $\chi_{P\cap K}$. Therefore, $P\cap K\subseteq\mathrm{Ker}(\lambda)$. Since $\lambda\in\mathrm{Irr}(P/\Phi(P))$ was arbitrary, we obtain $P\cap K\le\Phi(P)$. Now Tate's theorem (see [Huppert 1967, Satz IV.4.7]) yields that K is p-nilpotent. By hypothesis, $\mathrm{O}_{p'}(K)\le\mathrm{O}_{p'}(G)=1$ and K is a p-group. Finally, $K\le\mathrm{O}_p(G)\cap K\le P\cap K\le\Phi(P)$ and $K\le N$. $\square$

We mention that the characters χ with $p^2\nmid f_\chi$ are precisely the *almost p-rational* characters introduced in [Hung et al. 2022]. Lemma 3 allows to read off $K:=\mathrm{core}_G(\Phi(P))$ from the character table. Since $|P/K:\Phi(P/K)|=|P:\Phi(P)|$, it is therefore no loss to assume that $K=1$. The next theorem comes close to [Navarro and Sambale 2023, Theorem 3.1].

Theorem 4. *Let G be a finite group with a nonabelian Sylow p-subgroup P such that $|P:\Phi(P)|=p^2$ and $\mathrm{O}_{p'}(G)=1=\mathrm{core}_G(\Phi(P))$. Then $\mathrm{F}^*(G)$ is the unique minimal normal subgroup of G and $P\mathrm{F}^*(G)/\mathrm{F}^*(G)$ is cyclic. If $\mathrm{F}^*(G)$ is nonabelian, then P permutes the simple components of $\mathrm{F}^*(G)$ transitively. In particular, their number is a p-power in this case.*

Proof. Let N be a minimal normal subgroup of G. Then

$$\begin{aligned}|PN/N:\Phi(PN/N)| &= |P/P\cap N:\Phi(P/P\cap N)|\\ &= |P/P\cap N:\Phi(P)(P\cap N)/P\cap N|\\ &= |P:\Phi(P)(P\cap N)|\\ &\le |P:\Phi(P)|\\ &= p^2,\end{aligned}$$

where the second equality follows from [Isaacs 2008, Lemma 4.5], for instance. Suppose first that $P\cap N\le\Phi(P)$. Then by Tate's theorem (see [Huppert 1967, Satz IV.4.7]), N is a p-group and $N\le\Phi(P)$. This contradicts $\operatorname{core}_G(\Phi(P))=1$. Consequently, $|PN/N:\Phi(PN/N)|\le p$ and PN/N is cyclic. Let $M\ne N$ be another minimal normal subgroup of G. Then G/N and similarly G/M have cyclic Sylow p-subgroups. Since G is isomorphic to a subgroup of $G/M\times G/N$, G has abelian Sylow p-subgroups, which we have excluded explicitly. This shows that N is the unique minimal normal subgroup.

Assume now that N is nonabelian. Then $\mathrm{F}(G)\cap N=1$ implies $\mathrm{F}(G)=1=\mathrm{Z}(G)$ and $\mathrm{F}^*(G)=\mathrm{E}(G)=N$. Write $N=T_1\times\cdots\times T_n$ with nonabelian simple groups $T_1\cong\cdots\cong T_n$. If $P\le N$, using that P is 2-generated and nonabelian, we conclude that $n=1$ and P certainly acts transitively on $\{T_1,\dots,T_n\}$. Hence, we may assume that $P\nsubseteq N$ and $n\ge 2$. Let $Q_i:=P\cap T_i$ for $i=1,\dots,n$. Let $x\in P$ such that $PN/N=\langle xN\rangle$. Since $P\cap N\nsubseteq\Phi(P)$, there exists some $1\le i\le n$ with $Q_i\nsubseteq\Phi(P)$. Without loss of generality, let $i=1$. Choose $y\in Q_1\setminus\Phi(P)$. For all $j\in\mathbb{Z}$ we note that $xy^j\notin N\supseteq\Phi(P)$. Since $|P:\Phi(P)|=p^2$, it follows that $P=\langle x,y\rangle$. Without loss of generality, let $T_1,\dots,T_k$ be the orbit of T_1 under P. Suppose by way of contradiction that $k<n$. Then $Q_1\cdots Q_k\trianglelefteq P$ and $Q_{k+1}\times\cdots\times Q_n\le P/Q_1\cdots Q_k=\langle xQ_1\cdots Q_k\rangle$ is cyclic. This is only possible if $n=k+1$ and Q_n is cyclic. Moreover, $Q_n=\langle x^{p^a}z\rangle$ for some $a\ge 1$ and $z\in Q_1\cdots Q_k$. Since a nonabelian simple group cannot have a cyclic Sylow 2-subgroup, $p>2$. It follows from [Gross 1982, theorem A] that x induces an inner automorphism on T_n. This is impossible since x^{p^a} induces an inner automorphism of order $|T_n|_p$. This contradiction shows that P permutes the T_i transitively.

Finally, assume that N is elementary abelian. Since $\mathrm{O}_{p'}(G)=1$, we have $F:=\mathrm{F}(G)=\mathrm{O}_p(G)$. Suppose that $N<F$. Then $\Phi(F)\le\Phi(P)$ yields $\Phi(F)\le\operatorname{core}_G(\Phi(P))=1$, i.e., F is elementary abelian. Now the existence of an element of order p in $P\setminus N$ implies the existence of a (cyclic) complement of N in P. By a theorem of Gaschütz (see [Huppert 1967, Hauptsatz I.17.4]), N has a complement K in G. Since F centralizes N, we obtain $1\ne K\cap F\trianglelefteq NK=G$. This contradicts the fact that N is the unique minimal normal subgroup of G. Hence, $F=N$. Suppose that $\mathrm{E}(G)\ne 1$ and choose a central product $M\trianglelefteq G$ of quasisimple components. Then

$N \le \mathrm{Z}(M)$, because $1 \ne N \cap M \trianglelefteq G$. Since M/N has cyclic Sylow p-subgroups, the order of the Schur multiplier of M/N is not divisible by p. This contradicts $N \le \mathrm{Z}(M)$. We have therefore shown that $N = \mathrm{F}^*(G)$. □

In order to decide whether $|P : \Phi(P)| = p^2$, we may assume that the hypotheses of Theorem 4 are fulfilled. The situation now splits into two cases. When $\mathrm{F}^*(G)$ is abelian, the group G is p-constrained (recall that in general a group G is called p-constrained if $\mathrm{C}_{\overline{G}}(\mathrm{O}_p(\overline{G})) \le \mathrm{O}_p(\overline{G})$ where $\overline{G} := G/\mathrm{O}_{p'}(G)$). In this case we solve the problem completely. To do so, we will use a result of Higman (see [Navarro 2018, Corollary 7.18]) that allows to locate the p-elements in $X(G)$.

Corollary 5. *The character table of a p-constrained group G determines whether a Sylow p-subgroup P is generated by two elements.*

Proof. Let P be a Sylow p-subgroup of G. Since the character table $X(G)$ determines $X(G/\mathrm{O}_{p'}(G))$, we may assume that $\mathrm{O}_{p'}(G) = 1$. Since G is p-constrained, $\mathrm{O}_p(G) > 1$. By Lemma 3, we may assume that $\mathrm{core}_G(\Phi(P)) = 1$. Moreover, the orders and embeddings of the normal subgroups of G can be read off from $X(G)$. Hence by Theorem 4, we may assume that $N = \mathrm{O}_p(G) = \mathrm{F}(G)$ is the only minimal normal subgroup of G. If $P = N$, then $|P : \Phi(P)| = |P|$ and we are done. Hence, let $N < P$. By [Navarro 2018, Corollary 3.12], $X(G/N)$ detects whether P/N is cyclic. By Theorem 4, we can assume that this is the case. Choose $x \in P$ with $P/N = \langle xN \rangle$ (note that x can be spotted in $X(G)$ using [Navarro 2018, Corollary 3.12]). Since $P = N\langle x \rangle = \mathrm{O}_p(G)\langle x \rangle$ is the only Sylow p-subgroup of G containing x, $\mathrm{C}_P(x) = \mathrm{C}_N(x)\langle x \rangle$ is a Sylow p-subgroup of $\mathrm{C}_G(x)$. In particular, $|\mathrm{C}_N(x)| = |\mathrm{C}_G(x)|_p / |P/N|$ is determined by $X(G)$. By Lemma 1, we have

$$P' = [x, N] = \{[x, y] : y \in N\} \tag{2-1}$$

and $|P'| = |N/\mathrm{C}_N(x)|$ can be computed from $X(G)$. Let $|P/N| = p^a$ and $|N/P'| = p^n$. If $x^{p^a} \in P'$, then $P/P' \cong C_{p^a} \times C_p^n$ and otherwise $P/P' \cong C_{p^{a+1}} \times C_p^{n-1}$. Since $\mathbb{Q}(x)$ can be read off from $X(G)$, it suffices to show that

$$p|\mathbb{Q}(x) : \mathbb{Q}|_p = \exp(P/P').$$

Taking only $X(G/N)$ into account, we obtain $\mathbb{Q}(xN) = \mathbb{Q}_{p^a}$ or equivalently $|\mathbb{Q}(xN) : \mathbb{Q}|_p = p^{a-1}$ by [Navarro 2018, Theorem 3.11]. Thus $|\mathbb{Q}(x) : \mathbb{Q}|_p \ge p^{a-1}$. If $x^{p^a} = 1$, then $p|\mathbb{Q}(x) : \mathbb{Q}|_p = p^a = \exp(P/P')$ as desired. Now let $|\langle x \rangle| = p^{a+1}$. If $x^{p^a} \in P'$, then there exists $y \in N$ with $x^{p^a} = [x, y] = xyx^{-1}y^{-1}$ by (2-1). It follows that $yxy^{-1} = x^{1-p^a}$ and $|\mathrm{N}_G(\langle x \rangle) : \mathrm{C}_G(x)|_p = p$. Again by [Navarro 2018, Theorem 3.11], we have $p|\mathbb{Q}(x) : \mathbb{Q}|_p = p^a = \exp(P/P')$. Assume conversely that $|\mathbb{Q}(x) : \mathbb{Q}|_p = p^{a-1}$. Then there exists $y \in G$ with $yxy^{-1} = x^{1+kp^a}$ for some $0 < k < p$. We observe that $y \in \mathrm{N}_G(\langle x \rangle N) = \mathrm{N}_G(P)$. Replacing y by its p-part, we get $y \in P$. Now $x^{-kp^a} = [x, y] \in P'$ and $\exp(P/P') = p^a$ as desired. □

If G is p-solvable in the situation of Corollary 5 (recall that every p-solvable group is p-constrained), then $\mathrm{O}_p(G)$ has a complement K in $\mathrm{O}_{pp'}(G)$ by the Schur–Zassenhaus theorem. Using the Frattini argument, it is easy to show that $\mathrm{N}_G(K)$ is a complement of N in G. In this situation, G is a primitive permutation group on N of affine type.

On the other hand, every nonabelian simple group S gives rise to a nonsplit extension $G = N.S$ where $N = \Phi(G)$ is elementary abelian without complement (see [Doerk and Hawkes 1992, Theorem B.11.8]). Garrison [1976] has exhibited examples to show that $X(G)$ does not determine whether G splits over N. For instance,

$$\texttt{PerfectGroup}(7500, 1) \cong C_5^3 \rtimes A_5 \quad \text{and} \quad \texttt{PerfectGroup}(7500, 2) \cong C_5^3.A_5$$

in GAP [2020] have the same character table and the Sylow 5-subgroup is 2-generated in both cases.

Now assume that $N = \mathrm{F}^*(G)$ in the situation of Theorem 4 is nonabelian. If $N \cap P$ is abelian, then N has a complement in PN by [Huppert 1967, Satz IV.3.8]. In this case PN is a twisted wreath product. The nonsplit extension $M_{10} = A_6.C_2$ with $P = SD_{16}$, a semidihedral group, shows that this is not always the case. Even when N is not simple, $P \cap N$ is not always abelian (as in [Navarro and Sambale 2023, Theorem 3.1]). One example is

$$G = \mathrm{PSL}(2,7)^2 \rtimes \langle x \rangle \cong \mathrm{PSL}(2,7)^2 \rtimes C_4 \le \mathrm{PGL}(2,7) \wr C_2,$$

where x^2 acts as a diagonal automorphism on both factors $\mathrm{PSL}(2,7)$ simultaneously. Here $P = D_8^2 \rtimes C_4$ is 2-generated. Nevertheless, we provide the following reduction theorem:

Proposition 6. *Let G be a finite group with Sylow p-subgroup P such that $\mathrm{O}_{p'}(G) = 1$ and $N = \mathrm{F}^*(G)$ is the unique minimal normal subgroup of G. Suppose that N is nonabelian and PN/N is cyclic. Let S be a simple component of N. Assume that $|G : \mathrm{N}_G(S)|$ is a p-power. Then the following hold*:

(i) *$G = \mathrm{N}_G(S)P$.*

(ii) *$\widetilde{P} := \mathrm{N}_P(S)\mathrm{C}_G(S)/\mathrm{C}_G(S)$ is a Sylow p-subgroup of the almost simple group $\mathrm{N}_G(S)/\mathrm{C}_G(S)$ with socle $\widetilde{S} := S\mathrm{C}_G(S)/\mathrm{C}_G(S) \cong S$. Moreover, $\widetilde{P}\widetilde{S}/\widetilde{S}$ is cyclic.*

(iii) *$|P : \Phi(P)| \le p^2$ if and only if $|\widetilde{P} : \Phi(\widetilde{P})| \le p^2$.*

(iv) *S and $|\widetilde{P}|$ are determined by $X(G)$.*

Proof. (i) Since $|G : \mathrm{N}_G(S)|$ is a p-power, $|\mathrm{N}_G(S)P| = |\mathrm{N}_G(S) : \mathrm{N}_P(S)||P| = |G|$ and $G = \mathrm{N}_G(S)P$.

(ii) By (i), $\mathrm{N}_P(S)$ is a Sylow p-subgroup of $\mathrm{N}_G(S)$. Hence, $\widetilde{P}$ is a Sylow p-subgroup of $\mathrm{N}_G(S)/\mathrm{C}_G(S)$. Let $Q := N \cap P \trianglelefteq P$. Then $P/Q \cong PN/N$ is cyclic by hypothesis. Let $x \in P$ such that $P = \langle x \rangle Q$. Then $\widetilde{P}\widetilde{S}/\widetilde{S} \cong \mathrm{N}_P(S)S\mathrm{C}_G(S)/S\mathrm{C}_G(S) \le \langle x \rangle S\mathrm{C}_G(S)/S\mathrm{C}_G(S)$ is cyclic.

(iii) If $P \le N \le \mathrm{N}_G(S)$, then $S \trianglelefteq G$ and $N = S$. Here, $P \cong \widetilde{P}$, so we are done. Now assume $PN/N \neq 1$. As in (ii), let $Q := N \cap P \trianglelefteq P$. Since $\mathrm{O}^p(PN) = N$, there exists $x \in P$ such that $P = \langle x \rangle Q$ and $\langle x \rangle \cap Q \le P'$ (see [Brandis 1978, Satz 3.3]). Lemma 2 yields $|P : \Phi(P)| = p^2$ if and only if $|\mathrm{C}_{Q/\Phi(Q)}(x)| = p$.

By (i), we may write $N = T_1 \times \cdots \times T_{p^a}$ such that $T_i = x^{i-1}Sx^{1-i}$ for $i = 1, \dots, p^a$. Let $Q_i := T_i \cap P \le Q$. Then $\widetilde{Q} := Q_1\mathrm{C}_G(S)/\mathrm{C}_G(S) \cong Q_1$ is a normal subgroup of $\widetilde{P}$. Since $\mathrm{N}_P(S) = \langle x^{p^a} \rangle Q$, we have $\widetilde{P} = \langle \tilde{x} \rangle \widetilde{Q}$ where $\tilde{x} := x^{p^a}\mathrm{C}_G(S)$. It is easy to see that the map

$$\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a}) \to \mathrm{C}_{Q/\Phi(Q)}(x), \quad y\Phi(Q_1) \mapsto \prod_{i=0}^{p^a-1} x^i y x^{-i} \Phi(Q)$$

is an isomorphism. In particular, $|\mathrm{C}_{Q/\Phi(Q)}(x)| = |\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})|$. Assume for the moment that $x^{p^a} \in Q$. Then

$$\widetilde{P} = \widetilde{Q} \le \widetilde{S} \quad \text{and} \quad |\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})| = |Q_1/\Phi(Q_1)| = |\widetilde{P}/\Phi(\widetilde{P})|.$$

In this case, $|P : \Phi(P)| = p^2$ if and only if $\widetilde{P}$ is cyclic, i.e., $|\widetilde{P} : \Phi(\widetilde{P})| = p$. Now let $x^{p^a} \notin Q$. By way of contradiction, suppose that $x^{p^a} \in Q_1\mathrm{C}_G(S)$. Then there exists $y \in Q_1$ such that $x^{p^a}y \in \mathrm{C}_G(S)$. Now also

$$z := x^{p^a} \prod_{i=0}^{p^a-1} x^i y x^{-i} \in \mathrm{C}_G(S).$$

Since z is centralized by x, it follows that $z \in x^i\mathrm{C}_G(S)x^{-i} = \mathrm{C}_G(T_i)$ for $i = 1, \dots, p^a$. Hence, $z \in \mathrm{C}_G(N) = 1$ and $x^{p^a} \in Q$, a contradiction. Thus, $\widetilde{Q} < \widetilde{P}$ and

$$\widetilde{Q} \cap \langle \tilde{x} \rangle = (Q \cap \langle x^{p^a} \rangle)\mathrm{C}_G(S)/\mathrm{C}_G(S) \le P'\mathrm{C}_G(S)/\mathrm{C}_G(S) = \widetilde{P}'.$$

Lemma 2 shows that $|\widetilde{P} : \Phi(\widetilde{P})| = p^2$ if and only if

$$|\mathrm{C}_{Q_1/\Phi(Q_1)}(x^{p^a})| = |\mathrm{C}_{\widetilde{Q}/\Phi(\widetilde{Q})}(\tilde{x})| = p.$$

Now the claim follows.

(iv) The isomorphism types of N and S are determined by $X(G)$ according to [Navarro and Sambale 2023, Theorem 4.1]. We obtain $|\mathrm{N}_P(S)|$ from $|N| = |S|^{|P:\mathrm{N}_P(S)|}$. Arguing as in (iii), shows that $\mathrm{C}_P(S) = \mathrm{C}_Q(S) = Q_2 \cdots Q_{p^a}$. Hence, $|\mathrm{C}_P(S)| = |S|_p^{p^a-1}$ is computable from $X(G)$. The claim follows from $\widetilde{P} \cong \mathrm{N}_P(S)/\mathrm{C}_P(S)$. ☐

To decide whether $|P : \Phi(P)| = p^2$ holds, it suffices to obtain the structure of $\widetilde{P}$ with the notation from Proposition 6. If $p \ge 5$ and S is neither a linear nor a unitary group, then $\mathrm{Out}(S)$ has a cyclic Sylow p-subgroup by [Conway et al. 1985, Table 5]. In this case the isomorphism type of $\widetilde{P}$ is uniquely determined by $X(G)$ and the problem is solved. On the other hand, the proof of [Navarro and Sambale 2023, Lemma 5.1] shows that for linear and unitary groups S the condition $|P:\Phi(P)| = p^2$ is not determined by $|\widetilde{P}|$ alone. It remains a challenge to settle these cases (and $p = 3$ with $S = D_4(q)$, $E_6(q)$ and ${}^2E_6(q)$).

3. p-groups of maximal class

We start by introducing some terminology of (saturated, nonexotic) fusion systems. Let P be a Sylow p-subgroup of G as before. The *fusion system* $\mathcal{F} = \mathcal{F}_P(G)$ of G on P is a category whose objects are the subgroups of P and the morphisms of $\mathcal{F}$ have the form $f : S \to T$, $x \mapsto gxg^{-1}$ where $S, T \leq P$ and $g \in G$. Then $\mathrm{Aut}_{\mathcal{F}}(S) \cong \mathrm{N}_G(S)/\mathrm{C}_G(S)$ and $\mathrm{Out}_{\mathcal{F}}(S) \cong \mathrm{N}_G(S)/S\mathrm{C}_G(S)$. Elements $x, y \in P$ (or subsets $S, T \subseteq P$) are called $\mathcal{F}$*-conjugate* if there exists a morphism f such that $f(x) = y$ (or $f(S) = T$). A subgroup $S \leq P$ is called

- *fully normalized*, if $|\mathrm{N}_P(T)| \leq |\mathrm{N}_P(S)|$ for all $\mathcal{F}$-conjugates T of S,
- *centric*, if $\mathrm{C}_P(T) = \mathrm{Z}(T)$ for all $\mathcal{F}$-conjugates T of S,
- *radical*, if $\mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(S)) = \mathrm{Inn}(S)$ (equivalently, $\mathrm{O}_p(\mathrm{Out}_{\mathcal{F}}(S)) = 1$),
- *essential*, if S is fully normalized, centric and $\mathrm{Out}_{\mathcal{F}}(S)$ contains a strongly p-embedded subgroup (see [Aschbacher et al. 2011, Definition A.6]). For our purpose, it is enough to know that S is radical in this case.

By Alperin's fusion theorem, every morphism in $\mathcal{F}$ is a composition of restrictions of morphisms $f \in \mathrm{Aut}_{\mathcal{F}}(S)$ where $S = P$ or S is essential (see [Aschbacher et al. 2011, Theorem I.3.5]). Note that $\mathrm{Aut}_{\mathcal{F}}(P)$ permutes the essential subgroups by conjugation. Hence, if $Q \leq P$ does not lie in any essential subgroup, then Q is fully normalized. In this case, $\mathrm{N}_P(Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(Q)$ (see [Aschbacher et al. 2011, Lemma I.1.2]). Consequently, $\mathrm{C}_P(Q) = \mathrm{N}_P(Q) \cap \mathrm{C}_G(P)$ is a Sylow p-subgroup of $\mathrm{C}_G(P)$.

We call $\mathcal{F}$ *controlled* if $\mathrm{N}_G(P)$ controls the fusion in P with respect to G, i.e., every morphism $S \to T$ has the form $x \mapsto gxg^{-1}$ for some $g \in \mathrm{N}_G(P)$. Abstractly, this means that there are no essential subgroups and $\mathcal{F} = \mathcal{F}_P(P \rtimes A)$ for some Schur–Zassenhaus complement A of $\mathrm{Inn}(P)$ in $\mathrm{Aut}_{\mathcal{F}}(P)$. More generally, $\mathcal{F}$ is called *constrained* if there exists $Q \trianglelefteq P$ such that $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$ and $\mathrm{N}_G(Q)$ controls the fusion in P. By the model theorem (see [Aschbacher et al. 2011, Theorem I.4.9]), a constrained fusion system is realized by a unique group G such that $\mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$ (note that G is p-constrained). The largest subgroup $Q \trianglelefteq P$ such that $\mathrm{N}_G(Q)$ controls the fusion in P is denoted by $\mathrm{O}_p(\mathcal{F})$. Note that $\mathrm{O}_p(G) \leq \mathrm{O}_p(\mathcal{F})$.

It is well-known that a p'-automorphism of $Q \le P$ acts nontrivially on $Q/\Phi(Q)$. If Q is radical, it follows that $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q/\Phi(Q)$. Now assume that there exists a series of characteristic subgroups $\Phi(Q) = Q_0 < \cdots < Q_n = Q$ of Q. Then $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $Q_n/Q_{n-1} \times \cdots \times Q_1/Q_0$ by [Gorenstein 1980, 5.3.2]. This argument will often be applied in the following to exclude same candidates of essential subgroups.

We say that a p-group P of order p^n has *maximal class* if the nilpotency class is $n-1$. This may include the case $|P| = p^2$. The 2-groups of maximal class are the dihedral groups (including C_2^2), the semidihedral groups, the (generalized) quaternion groups and C_4 (see [Huppert 1967, Satz III.11.9]). Now assume that $n \ge 4$ and $p > 2$ to avoid some degenerate cases. Let $\mathrm{K}_2(P) = P'$ and $\mathrm{K}_{i+1}(P) = [P, \mathrm{K}_i(P)]$ for $i \ge 2$. Let $\mathbb{Z}_0(P) := 1$ and $\mathbb{Z}_{i+1}(P/\mathbb{Z}_i(P)) := \mathrm{Z}(P/\mathbb{Z}_i(P))$ for $i \ge 0$. Then $\mathrm{K}_i(P) = \mathbb{Z}_{n-i}(P)$ is the only normal subgroup of P of index p^i by [Huppert 1967, Hilfssatz III.14.2]. It is easy to see that the characteristic subgroups $P_1 := \mathrm{C}_P(\mathrm{K}_2(P)/\mathrm{K}_4(P))$ and $P_2 := \mathrm{C}_P(\mathbb{Z}_2(P))$ are maximal in P.

Lemma 7. *Let P be a p-group with a nonabelian subgroup $Q \le P$ of order p^3 and exponent p. If $\mathrm{C}_P(Q) = \mathrm{Z}(Q)$, then $\mathbb{Z}_2(P) \le Q$.*

Proof. Since $\mathrm{Z}(P) \le \mathrm{C}_P(Q)$, we have $Z := \mathrm{Z}(P) = \mathrm{Z}(Q) \cong C_p$. Let $xZ \in \mathrm{C}_{P/Z}(Q/Z)$. Then $x \in \mathrm{N}_P(Q)$. By [Winter 1972], $\mathrm{N}_P(Q)/Q \le \mathrm{Out}(Q) \cong \mathrm{GL}(2, p)$. As mentioned above, the kernel of the action of $\mathrm{Aut}(Q)$ on Q/Z is a p-group. Since $\mathrm{O}_p(\mathrm{GL}(2, p)) = 1$, we obtain $x \in Q$. Hence, $\mathbb{Z}_2(P)/Z = \mathrm{Z}(P/Z) \le \mathrm{C}_{P/Z}(Q/Z) = Q/Z$ and $\mathbb{Z}_2(P) \le Q$. □

Lemma 8. *Let G be a finite group with Sylow p-subgroup P of maximal class. Let $N \trianglelefteq G$ such that $p^2 \le |N|_p < |P|$. Then there exists $x \in P$ such that $|\mathrm{C}_G(x)|_p = p^2$.*

Proof. By hypothesis, $|P| \ge p|N|_p \ge p^3$. In particular, $\mathrm{Z}(P)$ is the unique normal subgroup of order p of P. Since $M := P \cap N \trianglelefteq P$, we have $\mathrm{Z}(P) \le N$. If $|P| = p^3$, every element $x \in P \setminus N$ cannot be conjugate to an element of $\mathrm{Z}(P) \le N$. Hence, $|\mathrm{C}_G(x)|_p = p^2$. Now assume that $|P| \ge p^4$. If $p = 2$, P is a dihedral, semidihedral or quaternion group and we choose $x \in P$ outside the cyclic maximal subgroup of P. For $p > 2$, let $x \in P \setminus (P_1 \cup P_2)$. By [Huppert 1967, Hilfssatz III.14.13], we have $|\mathrm{C}_P(x)| = p^2$. Since $|P| \ge p^4$, $\mathbb{Z}_2(P)$ is the unique normal subgroup of order p^2 in P. In particular, $\mathbb{Z}_2(P) \le M$ since $|M| \ge p^2$. If $p = 2$, we may assume that $x \notin M$. For $p > 2$, we have $P_1 \cup P_2 \cup M \subsetneq P$. Again we may choose $x \notin M$.

Let $\mathcal{F}$ be the fusion system of G on P. If x is not contained in any essential subgroup, then $\langle x \rangle$ is fully normalized as explained above. It follows that $|\mathrm{C}_G(x)|_p = |\mathrm{C}_P(x)| = p^2$ and we are done. Now let $Q < P$ be essential containing x. By [Grazian and Parker 2022, Theorem D], Q is a so-called pearl, i.e., Q is elementary abelian of order p^2 or nonabelian of order p^3 and exponent p (or $Q = Q_8$ if $p = 2$, see [Grazian and Parker 2022, Lemma 6.1]). As an essential subgroup, Q is centric and $\mathrm{C}_P(Q)=\mathrm{Z}(Q)$. Assume first that $|Q|=p^2$. Then

$$Z:=\mathrm{Z}(P)=M\cap Q=N\cap Q\trianglelefteq \mathrm{N}_G(Q).$$

Since Q is radical, $\mathrm{Out}_{\mathcal{F}}(Q)\cong \mathrm{N}_G(Q)/Q$ acts faithfully on $Z\times Q/Z\cong C_p^2$. But then $\mathrm{Out}_{\mathcal{F}}(Q)$ would be a p'-group in contradiction to $Q<\mathrm{N}_P(Q)$. Next let $|Q|=p^3$. Here, Lemma 7 shows that $\mathbb{Z}_2(P)=M\cap Q=N\cap Q\trianglelefteq \mathrm{N}_G(Q)$. Then $\mathrm{Out}_{\mathcal{F}}(Q)$ acts faithfully on $\mathbb{Z}_2(P)/Z\times Q/\mathbb{Z}_2(P)\cong C_p^2$ and we derive another contradiction. □

Theorem B. *Let G be a finite group with a Sylow p-subgroup P of maximal class. Suppose that $\mathrm{O}_{p'}(G)=1$ and $\mathrm{O}^{p'}(G)=G$. Then one of the following holds*:

(i) *There exists $x\in P$ such that $|\mathrm{C}_G(x)|_p=p^2$.*

(ii) *G is quasisimple and $|\mathrm{Z}(G)|\leq p$.*

Proof. We may assume that G is not simple and $|P|\geq p^3$. Let $N<G$ be a maximal normal subgroup. Then $1<|N|_p<|P|$ as $\mathrm{O}_{p'}(G)=1$ and $\mathrm{O}^{p'}(G)=G$. If $|N|_p\geq p^2$, then the claim follows from Lemma 8. Hence, let $|N|_p=p$. Then $P\cap N\trianglelefteq P$ has index $p^s\geq p^2$ and therefore $P\cap N=\mathrm{K}_s(P)\leq P'$. By Tate's theorem (see [Huppert 1967, Satz IV.4.7]), N has a normal p-complement. Since $\mathrm{O}_{p'}(G)=1$, this forces $|N|=p$. Since $|G:\mathrm{C}_G(N)|$ divides $p-1$, we further have $N\leq \mathrm{Z}(G)$. Since G/N is simple, G is quasisimple with $|\mathrm{Z}(G)|\leq p$. □

If Case (ii) in Theorem B applies with $|\mathrm{Z}(G)|=p$ and (i) fails, then Robinson's ordinary weight conjecture predicts the existence of an irreducible character χ in the principal p-block such that $p^2\chi(1)_p=|G|_p$ (see [Robinson 2008, Lemma 4.7]). Conversely, such a character can only appear when P has maximal class. Examples are $\mathrm{SL}(2,9)$ for $p=2$, $\mathrm{SL}(3,19)$ for $p=3$ and $\mathrm{SL}(p,q)$ for $p\geq 5$ where $q-1$ is divisible by p just once. Our proof of Theorem A does however not rely on any conjecture.

Theorem A. *The character table of a finite group G determines whether G has Sylow p-subgroups of maximal class.*

Proof. Let P be a Sylow p-subgroup of G. We may assume that $\mathrm{O}_{p'}(G)=1$ and $|P|\geq p^3$. Let $K:=\mathrm{O}^{p'}(G)$. The character table detects elements $x\in P$ such that $|\mathrm{C}_G(x)|_p=|\mathrm{C}_K(x)|_p=p^2$. In this case $|\mathrm{C}_P(x)|=p^2$ and P has maximal class by [Huppert 1967, Satz III.14.23]. Hence, by Theorem B we may assume that K is quasisimple with $|\mathrm{Z}(K)|\leq p$. Note that the character table of G determines the isomorphism type of the simple chief factor $K/\mathrm{Z}(K)$ (see [Navarro and Sambale 2023, Theorem 4.1]). In this way we confirm that the Sylow p-subgroup $P/\mathrm{Z}(K)$ of $K/\mathrm{Z}(K)$ has maximal class. If $\mathrm{Z}(K)=1$, then we are done. Otherwise, P

has maximal class if and only if $\mathrm{Z}(K) = \mathrm{Z}(P)$. This happens if and only if $|\mathrm{C}_G(x)|_p < |P|$ for all $x \in P \setminus \mathrm{Z}(K)$. □

4. Minimal nonabelian Sylow subgroups

The following elementary lemma underlines the importance of minimal nonabelian groups. For elements x, y, z of a group we use the commutator convention $[x, y, z] := [x, [y, z]]$.

Lemma 9. *For a p-group P the following assertions are equivalent*:

(1) *P is minimal nonabelian.*

(2) $|P : \Phi(P)| = |P : \mathrm{Z}(P)| = p^2$.

(3) $|P : \Phi(P)| = p^2$ *and* $|P'| = p$.

Proof. (1) ⇒ (2): Since P is nonabelian, there exist noncommuting elements $x, y \in P$. Since $\langle x, y\rangle$ is nonabelian, we have $P = \langle x, y\rangle$. By Burnside's basis theorem, $|P : \Phi(P)| = p^2$. Choose distinct maximal subgroups $S, T < P$. Since S and T are abelian and $P = ST$, it follows that $\Phi(P) = S \cap T \subseteq \mathrm{Z}(P)$. It is well-known that $P/\mathrm{Z}(P)$ cannot be a nontrivial cyclic group. In particular, $|P : \mathrm{Z}(P)| \geq p^2$ and $\Phi(P) = \mathrm{Z}(P)$.

(2) ⇒ (3): Let $\mathrm{Z}(P) < S < P$. Since $S/\mathrm{Z}(P)$ is cyclic and $\mathrm{Z}(P) \leq \mathrm{Z}(S)$, we obtain that S is abelian. Pick $x \in P \setminus S$. Then Lemma 1 yields that $|P'| = |S : \mathrm{Z}(P)| = p$.

(3) ⇒ (1): Obviously, P is nonabelian since $P' \neq 1$. For $g, x \in P$ we have $gxg^{-1} = [g, x]x \in P'x$. Thus, every conjugacy class lies in a coset of P'. The hypothesis $|P'| = p$ implies $|P : \mathrm{C}_P(x)| \leq p$ for every $x \in P$. Since $\Phi(P)$ is the intersection of the maximal subgroups of P, we deduce $\Phi(P) \leq \bigcap_{x \in P} \mathrm{C}_P(x) = \mathrm{Z}(P)$. Now for every maximal subgroup $S < P$, we see that $S/\mathrm{Z}(S)$ is cyclic and S must be abelian. We conclude that P is minimal nonabelian. □

The nonnilpotent, minimal nonabelian groups were classified by Miller and Moreno [1903]. The nilpotent ones are p-groups and have been determined by Rédei [1947]. For the convenience of the reader we give a proof.

Lemma 10 (Rédei). *Every minimal nonabelian p-group belongs to one of the following classes*:

(i) $\Gamma(a, b) := \langle x, y \mid x^{p^a} = y^{p^b} = 1,\ yxy^{-1} = x^{1+p^{a-1}}\rangle$ *a metacyclic group where* $a \geq 2$ *and* $b \geq 1$,

(ii) $\Delta(a, b) := \langle x, y \mid x^{p^a} = y^{p^b} = [x, y]^p = [x, x, y] = [y, x, y] = 1\rangle$ *where* $a \geq b \geq 1$,

(iii) Q_8.

Proof. Let P be minimal nonabelian. By Lemma 9, there exist $x, y \in P$ such that $P/P' = \langle xP'\rangle \times \langle yP'\rangle \cong C_{p^a} \times C_{p^b}$. Since $|P'| = p$, we have $P' = \langle z\rangle$ where $z := [x, y]$. Note that $P' \leq \Phi(P) = \mathrm{Z}(P)$ and $[x, z] = [y, z] = 1$. We distinguish three cases:

Case 1: $x^{p^a} = y^{p^b} = 1$. Here P fulfills the same relations as $\Delta(a, b)$, so it must be a quotient of the latter group. Moreover, every element of P can be written uniquely in the form $x^i y^j z^k$ with $1 \leq i \leq p^a$, $1 \leq j \leq p^b$ and $1 \leq k \leq p$. Consequently, $|P| = p^{a+b+1}$. For the same reason we have $|\Delta(a, b)| \leq p^{a+b+1}$. Therefore, $P \cong \Delta(a, b)$.

Case 2: Either $x^{p^a} = 1$ or $y^{p^b} = 1$. Without loss of generality, let $x^{p^a} \neq 1$ and $y^{p^b} = 1$. Then $P' \leq \langle x\rangle \trianglelefteq P$ and $yxy^{-1} = x^k$ for some $k \in \mathbb{Z}$. Since $\langle x^p, y\rangle < P$ is abelian, $x^p = yx^py^{-1} = x^{kp}$ and $p \equiv kp \pmod{p^{a+1}}$ as $|\langle x\rangle| = p^{a+1}$. Hence, we may assume that $k = 1 + p^a l$ for some $0 < l < p$. Let $0 < l' < p$ such that $ll' \equiv 1 \pmod p$. Then $y^{l'}xy^{-l'} = x^{(1+p^a l)^{l'}} = x^{1+p^a}$. Thus, after replacing y by $y^{l'}$, we obtain $yxy^{-1} = x^{1+p^a}$. Now P satisfies the relations of $\Gamma(a+1, b)$. It is clear that these groups have the same order, so $P \cong \Gamma(a+1, b)$.

Case 3: $x^{p^a} \neq 1 \neq y^{p^b}$. Without loss of generality, let $a \geq b$. Let $x^{p^a} = z^i$ and $y^{p^b} = z^j$ where $0 < i, j < p$. Then $(x^j)^{p^a} = z^{ij}$, $(y^i)^{p^b} = z^{ij}$ and $[x^j, y^i] = z^{ij}$ by [Huppert 1967, Hilfssatz III.1.3] (using $z \in \mathrm{Z}(P)$). Hence, replacing x by x^j and y by y^i, we may assume that $x^{p^a} = z = y^{p^b}$. Again by [Huppert 1967, Hilfssatz III.1.3],

$$(x^{p^{a-b}}y^{-1})^{p^b} = x^{p^a}y^{-p^b}[y^{-1}, x^{p^{a-b}}]^{\binom{p^b}{2}} = z^{p^{a-b}\binom{p^b}{2}} = 1$$

unless $p^b = p^a = 2$. In this exceptional case, $P \cong Q_8$. Otherwise, we replace y by $x^{p^{a-b}}y^{-1}$. Afterwards we still have $P/P' = \langle xP'\rangle \times \langle yP'\rangle$, but now $y^{p^b} = 1$. Thus, we are in Case (2). □

The metacyclic groups $\Gamma(a, b)$ can of course be constructed as semidirect products, while the groups $\Delta(a, b)$ can be constructed as subgroups of $\Gamma(a, b) \times C_{p^a}$. For $p = 2$, note that $\Gamma(2, 1) \cong D_8 \cong \Delta(1, 1)$. Apart from that, the groups in Lemma 10 are pairwise nonisomorphic (for different parameters a, b).

We digress slightly to present a counterexample to a related question. Since for p-groups P in general we have $\Phi(P) = P'\mho(P)$ where $\mho(P) = \langle x^p : x \in P\rangle$, one might wonder if $X(G)$ determines the property $|P : \mho(P)| = p^2$. For $p = 2$, it is well-known that $\mho(P) = \Phi(P)$, so the answer is yes in this case. For $p > 2$, $|P : \mho(P)| = p^2$ holds if and only if P is metacyclic (see [Huppert 1967, Satz III.11.4]). The following example shows that this is not even determined by $X(P)$.

Proposition 11. *For $a \geq 2$ and all primes p the groups $\Gamma(2, a)$ and $\Delta(a, 1)$ have the same character table.*

Proof. We denote the generators of $P := \Gamma(2, a)$ by x, y and those of $\widetilde{P} := \Delta(a, 1)$ by $\tilde{x}$, $\tilde{y}$ as in Lemma 10. Additionally, let $\tilde{z} := [\tilde{x}, \tilde{y}]$. We consider the maximal subgroups $Q := \langle x^p, y\rangle \le P$ and $\widetilde{Q} := \langle \tilde{x}, \tilde{z}\rangle \le \widetilde{P}$. Since $xyx^{-1} = x^{-p}y$ and $\tilde{y}\tilde{x}\tilde{y}^{-1} = \tilde{z}^{-1}\tilde{y}$, the map

$$Q \to \widetilde{Q}, \quad x^p \mapsto z,\ y \mapsto \tilde{x}$$

is an isomorphism compatible with the action of P and $\widetilde{P}$. The irreducible characters of P of degree p are induced from linear characters of Q, which are not P-invariant. Since these characters vanish outside Q, they correspond naturally to irreducible characters of $\widetilde{P}$. On the other hand, the linear characters of P are extensions of characters of Q with x^p in their kernel. For $\lambda \in \mathrm{Irr}(Q/P')$ the extensions $\hat{\lambda}$ are determined by $\hat{\lambda}(x) = \zeta$ where ζ is a p-th root of unity. Similarly, for $\lambda \in \mathrm{Irr}(\widetilde{Q}/\widetilde{P}')$ the extensions are determined by $\hat{\lambda}(\tilde{y}) = \zeta$. Therefore, the bijection $P \to \widetilde{P}$, $x^{i+jp}y^k \mapsto \tilde{x}^k\tilde{y}^i\tilde{z}^j$ where $0 \le i, j < p$ and $0 \le k < p^a$ induces the equality of the matrices $X(P)$ and $X(\widetilde{P})$. □

The second author has investigated fusion systems on minimal nonabelian 2-groups in order to classify blocks with such defect groups (see e.g., [Sambale 2016]). We now determine the fusion systems for odd primes too (partial results were obtained in [Yang and Gao 2011]). It turns out that they all come from finite groups unless $|P| = 7^3$. We make use of the Frobenius group $M_9 \cong \mathrm{PSU}(3, 2) \cong C_3^2 \rtimes Q_8$ with $\mathrm{Out}(M_9) \cong S_3$.

Theorem 12. *Let $\mathcal{F}$ be a saturated fusion system on a minimal nonabelian p-group P. Then one of the following holds*:

(i) $P \in \{D_8, Q_8\}$ *and* $\mathcal{F} = \mathcal{F}_P(G)$ *where* $G \in \{P, S_4, \mathrm{GL}(3, 2), \mathrm{SL}(2, 3)\}$.

(ii) $|P| = p^3$, $\exp(P) = p > 2$ *and the possibilities for $\mathcal{F}$ are given in* [Ruiz and Viruel 2004].

(iii) $P \cong \Gamma(a, b)$, $a \ge 2, b \ge 1$ *and* $\mathcal{F} = \mathcal{F}_P(C_{p^a} \rtimes C_{p^b d})$ *for some* $d \mid p-1$.

(iv) $P \cong \Delta(a, b)$, $a > b$ *and* $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ *where* $Q \le C_{p-1}^2$.

(v) $P \cong \Delta(a, a)$, $a \ge 2$ *and* $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ *for some p'-group* $Q \le \mathrm{GL}(2, p)$.

(vi) $p = 2$, $P \cong \Delta(a, 1)$, $a \ge 2$ *and* $\mathcal{F} = \mathcal{F}_P(A_4 \rtimes C_{2^a})$ *where C_{2^a} acts as a transposition in* $\mathrm{Aut}(A_4) = S_4$.

(vii) $p = 3$, $P \cong \Delta(a, 1)$, $a \ge 2$ *and* $\mathcal{F} = \mathcal{F}_P(G)$ *where* $G \in \{M_9 \rtimes C_{3^a}, M_9 \rtimes D_{2\cdot 3^a}\}$. *Here the image of C_{3^a} and $D_{2\cdot 3^a}$ in* $\mathrm{Out}(M_9)$ *is C_3 and S_3 respectively.*

Proof. The case $P \in \{D_8, Q_8\}$ is well-known and can be found in [Craven and Glesser 2012, Theorem 5.3], for instance. If $p = 2$ and $P = \Gamma(a, b)$ with $|P| \ge 16$, then $\mathcal{F}$ is trivial, i.e., $\mathcal{F} = \mathcal{F}_P(P)$ by [Craven and Glesser 2012, Theorem 3.7]. Then (iii) holds. Now suppose that $p > 2$ and $P = \Gamma(a, b)$. Then $\mathcal{F}$ is controlled, i.e., $\mathcal{F} = \mathcal{F}_P(P \rtimes Q)$ for some p'-group $Q \le \operatorname{Aut}(P)$ by [Stancu 2006] (see also [Craven and Glesser 2012, Theorem 3.10]). By [Sasaki 1997, Lemma 2.4], $\operatorname{Aut}(P) = A \rtimes \langle\sigma\rangle$ where A is a p-group, $|\langle\sigma\rangle| = p-1$, $\sigma(x) \in \langle x\rangle$ and $\sigma(y) = y$. Hence, Q is conjugate to a subgroup of $\langle\sigma\rangle$. After renaming the generators of P, we may assume that $Q \le \langle\sigma\rangle$. Now (iii) holds.

Next assume that $P \cong \Delta(a, b)$ for some $a \ge b \ge 1$. If $a = 1$ and $p > 2$, then $|P| = p^3$ and $\exp(P) = p$, so (ii) holds. Hence, let $a \ge 2$. Set $z := [x, y] \in P$. Since the p'-group $\operatorname{Out}_{\mathcal{F}}(P)$ acts faithfully on $P/\Phi(P) \cong C_p^2$, we have $\operatorname{Out}_{\mathcal{F}}(P) \le \mathrm{GL}(2, p)$. If $a > b$, then $\operatorname{Out}_{\mathcal{F}}(P)$ acts on $P/\Omega_{a-1}(P) \times \Omega_{a-1}(P)/\Phi(P)$ where $\Omega_{a-1}(P) = \langle g \in P : g^{p^{a-1}} = 1\rangle = \langle x^p, y, z\rangle$. In this case $\operatorname{Out}_{\mathcal{F}}(P) \le C_{p-1}^2$. If $\mathcal{F}$ is controlled, then we are in Case (iv) or (v). Hence, we may assume that $\mathcal{F}$ is not controlled. Then there exists an essential subgroup $Q \le P$. Since Q is centric and $\Phi(P) = \mathrm{Z}(P) \le \mathrm{C}_P(Q) \le Q$, Q is a maximal subgroup. Those are given by

$$\begin{aligned} \langle xy^i, y^p, z\rangle &\cong C_{p^a} \times C_{p^{b-1}} \times C_p, \quad i = 0, \dots, p-1, \\ \langle x^p, y, z\rangle &\cong C_{p^{a-1}} \times C_{p^b} \times C_p. \end{aligned}$$

By [Gorenstein 1980, Theorem 5.2.4], $A := \operatorname{Aut}_{\mathcal{F}}(Q)$ acts faithfully on $\Omega(Q) = \{g \in Q : g^p = 1\}$. Since $P/Q \le A$, this implies $\Omega(Q) \nsubseteq \mathrm{Z}(P)$ and $Q = \langle x^p, y, z\rangle$ with $b = 1$. Now Q is the only maximal subgroup of P isomorphic to $C_{p^{a-1}} \times C_p^2$. In particular, Q is characteristic in P. By Alperin's fusion theorem, $\mathcal{F}$ is constrained with $\mathrm{O}_p(\mathcal{F}) = Q$. By the model theorem, there exists a unique p-constrained group H with $P \in \mathrm{Syl}_p(H)$, $\mathrm{O}_{p'}(H) = 1$ and $\mathcal{F} = \mathcal{F}_P(H)$. We will construct H in the following.

By [Oliver 2014, Lemma 1.11], there exists an A-invariant decomposition $Q = Q_1 \times Q_2$ with $Q_1 \cong C_p^2$ and $Q_2 \cong C_{p^{a-1}}$. Moreover, $\mathrm{O}^{p'}(A) \cong \mathrm{SL}(2, p)$ acts faithfully on Q_1 and trivially on Q_2. Since $P/Q \le \mathrm{O}^{p'}(A)$, it follows that $Q_2 \le \mathrm{Z}(P) = \langle x^p, z\rangle$. Moreover, $xyx^{-1} = yz$ implies $z \in Q_1$. Let $\alpha \in A$ be a p'-automorphism acting trivially on Q_1. Then α commutes with the action of P/Q. Since Q is receptive (see [Aschbacher et al. 2011, Definition I.2.2]), α extends to an automorphism of P. Suppose that $\alpha \ne 1$. Since $Q_2 \le \mathrm{Z}(P) = \Phi(P)$, α must act nontrivially on P/Q_2. Note that P/Q_2 is nonabelian of order p^3 as $z \in Q_1$. An analysis of $\operatorname{Aut}(P/Q_2)$ reveals that α cannot act trivially on $Q/Q_2 \cong Q_1$. Hence, $\alpha = 1$ and A acts faithfully on Q_1. In particular, $A \le \mathrm{GL}(2, p)$. If $p = 2$, then

$$A \cong \mathrm{SL}(2, 2) = \mathrm{GL}(2, 2) \cong S_3.$$

It is easy to see that (vi) holds here. If $p = 3$, then $\mathrm{SL}(2, 3) \cong Q_8 \rtimes C_3$, $\mathrm{GL}(2, 3) \cong Q_8 \rtimes S_3$ and (vii) is satisfied. Thus, let $p \ge 5$. Then the Sylow normalizer in $\mathrm{SL}(2, p)$ acts nontrivially on a Sylow p-subgroup of $\mathrm{SL}(2, p)$. Hence, there exists $\alpha \in \mathrm{O}^{p'}(A)$

acting nontrivially P/Q. But then α acts nontrivially on $\langle x^p\rangle Q_1/Q_1 = Q/Q_1 \cong Q_2$. This contradicts [Oliver 2014, Lemma 1.11]. $\square$

The groups $A_4 \rtimes C_4$, $M_9 \rtimes C_9$ and $M_9 \rtimes D_{18}$ can be constructed in GAP [2020] as $\texttt{SmallGroup}(n,k)$ where $(n,k) \in \{(48,39), (648,534), (6^4, 2892)\}$ respectively.

Corollary 13. *Let $\mathcal{F}$ be a fusion system on a minimal nonabelian p-group P with $|P| \geq p^4$. Then $\mathcal{F}$ is constrained. If $p \geq 5$, then $\mathcal{F}$ is controlled.*

We now gather some information on simple groups in order to prove Theorem C. As customary, if q is a prime power, let

$$\mathrm{PSL}^\epsilon(n,q) := \begin{cases} \mathrm{PSL}(n,q) & \text{if } \epsilon = 1, \\ \mathrm{PSU}(n,q) & \text{if } \epsilon = -1. \end{cases}$$

The following is certainly known, but included for convenience.

Lemma 14. *Let q be a prime power. Let $S = \mathrm{PSL}^\epsilon(n,q)$ with a cyclic Sylow p-subgroup and $n \geq 3$. Then there exists a unique integer $2 \leq d \leq n$ such that p divides $q^d - \epsilon^d$.*

Proof. Since a nonabelian simple group cannot have cyclic Sylow 2-subgroups, we have $p > 2$. If $p \mid q$, then a Sylow p-subgroup of S is given by the set of unitriangular matrices. This subgroup is nonabelian since $n \geq 3$. Now let $p \nmid q$. If $q \equiv \epsilon \pmod p$, then S contains a subgroup of diagonal matrices isomorphic to C_p^2. Hence, let $q \not\equiv \epsilon \pmod p$. In the following we write $q^* := q$ if $\epsilon = 1$ and $q^* := q^2$ if $\epsilon = -1$. Let $x \in S$ be a generator of a Sylow p-subgroup of S. We identify x with a preimage in $\mathrm{GL}(n, q^*)$. We may assume that x has order p^k. Let e be the order of q^* modulo p^k. Then x has an eigenvalue $\zeta \in \mathbb{F}_{(q^*)^e}^\times$ of order p^k. Since $\mathrm{tr}(x) \in \mathbb{F}_{q^*}$, the elements $\zeta^{(q^*)^i}$ for $i = 0, \ldots, e-1$ are distinct eigenvalues of x. In particular, $e \leq n$. If $\epsilon = 1$, then $e \geq 2$ we can choose $d := e$ in the statement. If $2e \leq n$, we obtain $q^d \equiv 1 \equiv \epsilon^d \pmod p$ for $d := 2e$.

Now suppose that $\epsilon = -1$ and $2e > n$. Since x is a unitary matrix, we have $\bar{x}x^{\mathrm{t}} = 1$ where $\bar{x} = (x_{ij}^q)_{i,j}$ and x^{t} is the transpose of x. It follows that ζ^{-q} is an eigenvalue of x. Since $n < 2e$, there must be some i with $\zeta^{q^{2i}} = \zeta^{-q}$. This shows that $q^{2i-1} \equiv -1 \equiv \epsilon^{2i-1} \pmod{p^k}$. Since $q^{2(2i-1)} \equiv 1 \pmod{p^k}$, we have

$$e \mid 2i-1 \leq 2(e-1)-1 < 2e \quad \text{and} \quad e = 2i-1.$$

Hence, we can set $d := e$.

For the uniqueness of d, we note that

$$|S| = \frac{q^{n(n-1)/2}}{\gcd(n, q-\epsilon)} \prod_{i=2}^{n} (q^i - \epsilon^i),$$

is not divisible by p^{k+1}, since $p^k = |\langle x\rangle| = |S|_p$. $\square$

Lemma 15. *Let S be a finite simple group with Sylow 3-subgroup C_3^2 and outer automorphism of order 3. Then $S \cong \mathrm{PSL}^{\epsilon}(3, p^f)$ where $\epsilon = \pm 1$, p is a prime and $(p^f - \epsilon)_3 = 3$. Moreover*, $\mathrm{Out}(S) \cong C_3 \rtimes (C_f \times C_2)$.

Proof. The simple groups with Sylow 3-subgroup C_3^2 were classified in [Koshitani and Miyachi 2001, Proposition 1.2]. The alternating groups and sporadic groups do not have outer automorphisms of order 3. Now let S be a classical group of dimension d over $\mathbb{F}_{p^f}$. Then $p^f \not\equiv \pm 1 \pmod 9$. This implies $3 \nmid f$ and S does not have field automorphisms of order 3. According to [Conway et al. 1985, Table 5], there must be a diagonal automorphism of order 3. This forces $d = 3$ and $S = \mathrm{PSL}^{\epsilon}(3, p^f)$ such that $(p^f - \epsilon)_3 = 3$. If $\epsilon = 1$, then $\mathrm{Out}(S) = C_3 \rtimes (C_f \times C_2)$ as desired. If $\epsilon = -1$, then there is no graph automorphism and instead we have a field automorphism of order $2f$. However, since $p^f \equiv 2, 5 \pmod 9$, f must be odd and $C_{2f} \cong C_f \times C_2$. □

Theorem C. *Let G be a finite group with a minimal nonabelian Sylow p-subgroup P and $\mathrm{O}_{p'}(G) = 1$. Then one of the following holds*:

(i) $p = 2$, $P \in \{D_8, Q_8\}$ *and* $\mathrm{O}^{2'}(G) \in \{\mathrm{SL}(2, q), \mathrm{PSL}(2, q'), A_7\}$ *where* $q \equiv \pm 3 \pmod 8$ *and* $q' \equiv \pm 7 \pmod{16}$.

(ii) $|P| = p^3$ *and* $\exp(P) = p > 2$.

(iii) $G = P \rtimes Q$ *where* $Q \leq \mathrm{GL}(2, p)$.

(iv) $p > 2$, $\mathrm{O}^{p'}(G) = S \rtimes C_{p^a}$ *where S is a simple group of Lie type with cyclic Sylow p-subgroups. The image of C_{p^a} in* $\mathrm{Out}(S)$ *has order p.*

(v) $p = 2$ *and* $G = \mathrm{PSL}(2, q^f) \rtimes C_{2^a d}$ *where q is a prime*, $q^f \equiv \pm 3 \pmod 8$ *and $d \mid f$. Moreover, C_{2^a} acts as a diagonal automorphism of order 2 on* $\mathrm{PSL}(2, q^f)$ *and C_d induces a field automorphism of order d.*

(vi) $p = 3$ *and* $\mathrm{O}^{3'}(G) = \mathrm{PSL}^{\epsilon}(3, q^f) \rtimes C_{3^a}$ *where $\epsilon = \pm 1$, q is prime*, $(q^f - \epsilon)_3 = 3$ *and* $G/\mathrm{O}^{3'}(G) \leq C_f \times C_2$.

Proof. By Lemma 9, $|P : \mathrm{Z}(P)| = p^2$ and G is described in [Navarro and Sambale 2023, Theorem 7.5]. We go through the various cases in the notation used there:

In Case (A), using that P is 2-generated and $\mathrm{O}_p(G)$ is not cyclic, we deduce that $S = 1$. Here $P = \mathrm{F}^*(G) \trianglelefteq G$ and $\mathrm{C}_G(P) \leq P$. Since G/P acts faithfully on $P/\Phi(P) \cong C_p^2$, we have $G/P \leq \mathrm{GL}(2, p)$ and (iii) holds. Assume now that $P < G$. In Case (B), the quasisimple group C has a nonabelian Sylow p-subgroup of order p^3 which must coincide with P. If $P = D_8$, then (i) or (v) holds by the Gorenstein–Walter theorem (there are no field automorphisms of order 2) [Gorenstein 1980, p. 462]. If $P = Q_8$, the claim follows from the Brauer–Suzuki theorem [Gorenstein 1980, Theorem 12.1.1] and Walter's theorem [Gorenstein 1980, p. 485]. If $p > 2$, then we must have $\exp(P) = p$, since otherwise the focal subgroup theorem [Isaacs

2008, Theorem 5.21] and Theorem 12 lead to the contradiction $|P| = |P : P \cap G'| \geq p$. Thus, (ii) holds. Case (D) is impossible, since then P has a nonabelian maximal subgroup.

Now consider Case (C), i.e., $\mathrm{F}^*(G) = \mathrm{O}_p(G) \times S$ has abelian Sylow p-subgroups, S is a direct product of simple groups and $|G : \mathrm{F}^*(G)|_p = p$. Let $x \in P \setminus \mathrm{F}^*(G)$.

Case 1: $S = 1$. Since $S = 1$, $\mathrm{F}^*(G) = \mathrm{O}_p(G)$ and so

$$\mathrm{C}_G(\mathrm{F}^*(G)) = \mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G).$$

Therefore, $\mathrm{C}_P(\mathrm{O}_p(G)) = \mathrm{O}_p(G)$ and we have $\mathrm{C}_G(\mathrm{O}_p(G)) = \mathrm{O}_p(G) \times K$ where $K \leq \mathrm{O}_{p'}(G) = 1$. Hence, G is p-constrained and $\mathcal{F}_P(G)$ is given by (vi) or (vii) of Theorem 12. By the model theorem, the isomorphism type of G is uniquely determined by $\mathcal{F}_P(G)$. Since $\mathrm{PSL}(2, 3) \cong A_4$ and $\mathrm{PSU}(3, 2) \cong M_9$, we obtain (v) or (vi).

Case 2: $S \neq 1$ is not simple. By Lemma 10, the maximal subgroups of P are generated by at most three elements. Hence, S is a direct product of two or three simple groups, say $S = T_1 \times T_2$ or $T_1 \times T_2 \times T_3$. Since a Sylow 2-subgroup of a simple group cannot be generated by less than 2 elements, we deduce that $p > 2$ and the T_i have cyclic Sylow p-subgroups. If x does not normalize some T_i, then $p = 3$ and x permutes $T_1 \cong T_2 \cong T_3$. However, $C_{3^n} \wr C_3$ is not minimal nonabelian. Hence, x acts on each T_i. If x acts nontrivially on $\mathrm{O}_p(G)$, then $\mathrm{O}_p(G)\langle x \rangle$ is nonabelian and $P = \mathrm{O}_p(G)\langle x \rangle$. But then S would be simple. Similarly, if x acts nontrivially on $Q_1 := P \cap T_1$, then $P = Q_1\langle x \rangle$. Write $Q_2 := P \cap T_2 = \langle y \rangle$ such that $x^p \in yQ_1$. Then x centralizes y. By [Gross 1982, Theorem B], this implies that x induces an inner automorphism on T_2. However, x^p induces the inner automorphism by y. Hence, x cannot have order greater than $|T_2|_p$. Another contradiction.

Case 3: S is simple. Let $Q := P \cap S \trianglelefteq P$ be a Sylow p-subgroup of S. Arguing as in Case 2, we see that x acts nontrivially on Q and therefore $P = Q\langle x \rangle$. First let Q be cyclic. Then $p > 2$ and P is metacyclic. Since $\mathrm{Out}(S)$ needs to have an element of order p, S must be of Lie type. To obtain (iv), it remains to show that PS is normal in G. Assume the contrary. By the structure of $\mathrm{Out}(S)$ (see [Conway et al. 1985, Table 5]), P induces a field or graph automorphism of order p on S which acts nontrivially on the subgroup of outer diagonal automorphisms of S. In particular, the diagonal automorphism group must have order at least $p + 1$, in fact $2p + 1 \geq 7$ since $p > 2$. This excludes all families of simple groups except $S = \mathrm{PSL}^\epsilon(d, q^f)$ where $p \mid f$ and $d \geq 2p + 1$. Since Q is cyclic and $f > 1$, we have $q \neq p$. By Fermat's little theorem,

$$q^{(p-1)f} \equiv q^{2(p-1)f} \equiv 1 \pmod{p}.$$

This contradicts Lemma 14 (note that $p - 1$ is even). Hence, $PS \trianglelefteq G$ and (iv) holds.

Let Q be noncyclic. Recall that in general Q is homocyclic and $\mathrm{N}_S(Q)$ acts irreducibly on $\Omega(Q)$ (see [Flores and Foote 2009, Proposition 2.5]). This implies that P cannot be metacyclic, as otherwise the fusion in P is controlled by $\mathrm{N}_G(P)$ and $\mathrm{N}_G(Q) = \mathrm{N}_G(P)\mathrm{C}_G(Q)$ acts reducibly on Q according to Theorem 12. Hence, let $P \cong \Delta(a, b)$. Then P' is a direct factor of Q and we obtain $Q = \Omega(Q)$. If Q has rank 3, then $P \cong \Delta(2, 1)$. However, by Theorem 12, $\mathrm{N}_G(Q)/\mathrm{C}_G(Q) \leq \mathrm{GL}(2, p)$ does not act irreducibly on Q. Hence, we may assume that Q has rank 2. Now $P \cong \Delta(a, 1)$ with $a \geq 2$. If $\mathrm{N}_G(P)$ controls the fusion in P, then $\mathrm{N}_G(Q)$ would fix P'. Hence, we are in Case (vi) or (vii) of Theorem 12. Consider $p = 2$ first. By Walter's theorem (see [Gorenstein 1980, p. 485]), $S \cong \mathrm{PSL}(2, q^f)$ with $q^f \equiv \pm 3 \pmod 8$. It follows that f is odd and $G/PS \leq \mathrm{Out}(S) \leq C_{2f}$ by [Conway et al. 1985, Table 5]. Here C_2 induces a diagonal automorphism and C_f is caused by a field automorphism. So (v) holds. Finally, let $p = 3$. Here the claim follows easily from Lemma 15. □

Examples for Theorem C (iv) can be constructed as follows: Let $p > 2$ and $a \geq 2$. By Dirichlet's theorem, there exists a prime $q \equiv 1 + p^{a-1} \pmod{p^{a+1}}$. Then $q^p \equiv 1 + p^a \pmod{p^{a+1}}$ and $S := \mathrm{PSL}(2, q^p)$ has a cyclic Sylow p-subgroup Q of order p^a. Let $R \cong C_{p^b}$ and construct $G := S \rtimes R$ where R acts as the field automorphism $\mathbb{F}_{q^p} \to \mathbb{F}_{q^p}, \lambda \mapsto \lambda^q$ on S. By [Gross 1982], R acts nontrivially on Q and $P := Q \rtimes R \cong \Gamma(a, b)$. A different example is $G = \mathrm{Sz}(2^5) \rtimes C_5$ for $p = 5$.

Corollary 16. *Let G be a finite group with a minimal nonabelian Sylow p-subgroup and $\mathrm{O}_{p'}(G) = 1$. Then G has at most one nonabelian composition factor.*

Proof. We may assume that G is nonsolvable. If $|G|_p = p^3$, then $\mathrm{F}^*(G)$ is quasisimple and $G/\mathrm{F}^*(G) \leq \mathrm{Aut}(\mathrm{F}^*(G)) \leq \mathrm{Aut}(\mathrm{F}^*(G)/\mathrm{Z}(F^*(G))$ is solvable by Schreier's conjecture. Otherwise we have $\mathrm{F}^*(G) = S \times C_{p^b}$ for a simple group S and $b \geq 0$ by the proof of Theorem C. Since $\mathrm{Aut}(C_{p^b})$ is abelian, the claim follows again from Schreier's conjecture. □

Corollary D. *The character table of a finite group G determines whether G has minimal nonabelian Sylow p-subgroups.*

Proof. Let P be a Sylow p-subgroup of G. We may assume that $\mathrm{O}_{p'}(G) = 1$. By [Navarro and Sambale 2023, Theorem B], the character table determines whether $|P : \mathrm{Z}(P)| = p^2$. Suppose that this is the case. By Lemma 9, it remains to detect whether $|P : \Phi(P)| = p^2$. This is true for $|P| = p^3$, so let $|P| \geq p^4$. By Theorem 4 and Corollary 5, we may assume that $\mathrm{O}_p(G) = 1$. Now by Theorem C we expect that $\mathrm{O}^{p'}(G) = S \rtimes C_p$ for a simple group S with a cyclic Sylow p-subgroup Q. As usual, $X(G)$ determines the isomorphism type of S. If Q is indeed cyclic, then clearly P is 2-generated and we are done. □

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UNIVERSAL WEIL MODULE

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The classical construction of the Weil representation, with complex coefficients, has long been expected to work for more general coefficient rings. This paper exhibits the minimal ring $\mathcal{A}$ for which this is possible, the integral closure of $\mathbb{Z}\left[\frac{1}{p}\right]$ in a cyclotomic field, and carries out the construction of the Weil representation over $\mathcal{A}$-algebras. As a leitmotif all along the work, most of the problems can actually be solved over the base ring $\mathcal{A}$ and transferred to any $\mathcal{A}$-algebra by scalar extension. The most striking fact is that all these Weil representations arise as the scalar extension of a single one with coefficients in $\mathcal{A}$. In this sense, the Weil module obtained is universal. Building upon this construction, we speculate and make predictions about an integral theta correspondence.

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Introduction

This paper is intended as a stepping-stone in the direction of an "integral theta correspondence". Whatever this may be, it will require a theory of the Weil representation over rings and the purpose of this paper is to carry this out on rings with minimal hypotheses. When the coefficient ring is the field of complex numbers, this representation originated in problems related to θ-series and was first constructed in the seminal paper of André Weil [1964].

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There is another way, as opposed to the original approach of Weil, to build this representation. Because of its relations with quantum physics, it appears often in older literature as the so-called oscillator representation and involves the famous Stone–von Neumann theorem as a cornerstone in this alternative construction [Howe 1979]. It plays a pivotal role in the theta correspondence, where the interplay between this representation and the dual pairs introduced by Roger Howe [1979] led to a conjectural bijective correspondence between some subsets of irreducible representations for each member of the dual pair.

This correspondence, known in older literature as *Howe duality* or *the Howe correspondence*, took almost 40 years to be completely proven, and is now usually known as *the theta correspondence*. The main works which led to its proof include [Howe 1979; Rallis 1984; Kudla 1986; Mœglin et al. 1987; Waldspurger 1990; Mínguez 2008; Gan and Takeda 2016; Gan and Sun 2017] and we refer to [Trias 2020] for a more detailed exposition of these contributions to the classical theta correspondence. This celebrated bijection plays a central role in number theory as it encodes a lot of arithmetic information and allows one to build automorphic forms. It is the centre of a highly active research field in the topic.

In the 1980's, Marie-France Vignéras studied, in relation to Serre's conjectures, congruences between automorphic representations by means of the modular representation theory of their local factors. She considered smooth representations of connected reductive p-adic groups with coefficients in fields that are more general than the complex numbers, allowing in particular fields of positive characteristic. The theory splits into two different aspects, depending on whether the characteristic of the coefficient is different from p or not. In the first case, which we study here, we talk about ℓ-modular representations by implicitly meaning that $\ell \neq p$. (The second case is referred to as modulo p representation theory and requires completely different techniques.)

An important result about these ℓ-modular representations is the compatibility of the classical local Langlands correspondence for general linear groups with a certain ℓ-modular one as described in [Vignéras 2001]. In recent years, there has been a growing interest in studying representations in families i.e., over coefficient rings where p is invertible. For general linear groups, families of representations with coefficients in a Witt ring $W(\overline{\mathbb{F}}_\ell)$ are quite well understood [Helm 2016] and provide a local Langlands correspondence in families [Emerton and Helm 2014] compatible with (a modified version due to Breuil–Schneider of) the classical one and the one constructed by Vignéras.

In terms of the theta correspondence and the Weil representation, a generalisation to ℓ-modular representation theory has already been considered in the thesis of Alberto Mínguez [2006]. Taking an *ad-hoc* analogue of the Weil representation for type II dual pairs, he proves that a bijective correspondence holds when the characteristic is big enough compared to the size of the dual pair at play. In order to develop a modular theory of the theta correspondence, this analogue is not sufficient and one needs a proper construction of the Weil representation for coefficient fields, or even coefficient rings.

Sug Woo Shin [2012] achieves this for coefficient rings such that the associated affine scheme is locally noetherian, by the use of geometric methods such as a Stone–von Neumann theorem involving abelian schemes. Chinello and Turchetti [2015] built a Weil representation with coefficients in integral domains following the original approach of Weil. The other representation-theoretic strategy, using a nongeometric version of the Stone–von Neumann theorem, has been carried out in [Trias 2020]. The latter allows one to recover most of the classical objects and study them in detail, such as the metaplectic group, the metaplectic cocycle, and the lifts of dual pairs. Furthermore, this approach generalises in a nice way in families without the need of particular assumptions on the coefficient rings, improving the first two mentioned papers whose hypotheses (locally noetherian affine scheme, or, integral domains) turn out to be more restrictive.

The present paper brings a broader construction of the Weil representation with coefficients in any $\mathcal{A}$-algebra, where $\mathcal{A}$ is a minimal ring specified below. In addition, exhibiting a minimal Weil representation, called "universal" below, does not appear in any previous work; nor the focus on extending scalars. The rest of the introduction is split into two parts: in the first half we give more detail about the results we obtain along these lines, as well as considerations about the metaplectic group and the metaplectic cocycle; in the second half, we explain how we expect to use this to study an integral theta correspondence, with particular focus on the special case of $(\mathrm{GL}_1, \mathrm{GL}_1)$.

Let F be either a local nonarchimedean field, or a finite field, of residual characteristic p and residual cardinality q, but of characteristic not 2. The minimal condition mentioned above amounts to requiring two things: first that a nontrivial smooth additive character ψ of F exists, allowing Fourier transform techniques; second that p is invertible, that is a condition in terms of Haar measures. Write $\mathcal{K} = \mathbb{Q}[\zeta_p]$ when F has positive characteristic, and $\mathcal{K} = \mathbb{Q}[\zeta_{p^\infty}]$ when F has characteristic 0. The minimal ring $\mathcal{A}$ satisfying the previous two conditions is the integral closure of $\mathbb{Z}\left[\frac{1}{p}\right]$ in $\mathcal{K}$.

Fix from now on a nontrivial smooth character $\psi : F \to \mathcal{A}^\times$. Notations will be simplified in the introduction to be lighter than that in the main body of the paper. For any $\mathcal{A}$-algebra $\mathcal{B}$ with structure morphism φ, the character $\psi^{\mathcal{B}} = \varphi \circ \psi$ is a nontrivial character of F with values in $\mathcal{B}$. More generally, if χ is a character of any group with values in $\mathcal{A}$, we write

$$\chi^{\mathcal{B}} = \varphi \circ \chi.$$

0A. ***Theory of the Weil representation over an $\mathcal{A}$-algebra $\mathcal{B}$.*** The theory developed in [Mœglin et al. 1987, Chapter 2] for complex representations and in [Trias 2020] for ℓ-modular representations finds a natural generalisation for an $\mathcal{A}$-algebra $\mathcal{B}$. Note that there are no restrictive assumptions on the $\mathcal{A}$-algebra considered. In particular, it is not necessarily an integral domain.

Stone–von Neumann over $\mathcal{A}$-algebras. Let A be a self-dual subgroup in the symplectic space $(W, \langle \cdot , \cdot \rangle)$ and ψ_A a character of the group $A_H = A \times F$ extending ψ. Here A_H is considered as a subgroup of the Heisenberg group H which is the set $W \times F$ endowed with the law

$$(w, t) \cdot (w', t') = \big(w + w', t + t' + \tfrac{1}{2}\langle w, w' \rangle\big).$$

The theorem below gathers together in a succinct way the main results we obtain in Sections 2B and 2C. It is the core part of the classical Stone–von Neumann theorem when $\mathcal{B} = \mathbb{C}$ [Mœglin et al. 1987, Chapter 2, Theorem I.2] and its generalisation when $\mathcal{B}$ is a field of characteristic different from p [Trias 2020, Theorem 2.1]. Working over a general ring, the notion of "irreducible representation" is too restrictive. Instead, when G is a group, we say that a $\mathcal{B}[G]$-module V is *everywhere irreducible* if the representation $V \otimes_{\mathcal{B}} k(\mathcal{P})$ is irreducible for all $\mathcal{P} \in \mathrm{Spec}(\mathcal{B})$, where $k(\mathcal{P})$ is the field of fractions of $\mathcal{B}/\mathcal{P}$. This definition is very convenient to state a Stone–von Neumann theorem over general rings that includes the situation over coefficient fields.

Theorem A. *Set* $V_A^{\mathcal{B}} = \mathrm{ind}_{A_H}^H(\psi_A^{\mathcal{B}}) \in \mathrm{Rep}_{\mathcal{B}}(H)$:

(a) $V_A^{\mathcal{B}}$ *is everywhere irreducible, and is admissible.*

(b) *We have* $V_A^{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B} = V_A^{\mathcal{B}}$.

(c) *For A' any self-dual subgroup in W and $\psi_{A'}$ an extension of ψ to A'_H, one has*

$$\mathrm{Hom}_{\mathcal{B}[H]}(V_A^{\mathcal{B}}, V_{A'}^{\mathcal{B}}) \simeq \mathcal{B}.$$

A consequence of (a) and (c) is that the isomorphism class of the representation $V_A^{\mathcal{B}}$ does not depend on the choices of A and ψ_A. When $\mathcal{B}$ is a field, this representation is also irreducible.

The full Stone–von Neumann Theorem for fields $\mathcal{B}$ also asserts that any irreducible $V \in \mathrm{Rep}_{\mathcal{B}}(H)$, such that $V|_F$ is $\psi^{\mathcal{B}}$-isotypic, is in the isomorphism class defined by $V_A^{\mathcal{B}}$. We do not pursue such a precise result over rings. However, for most of the applications using Stone–von Neumann, and the Weil representation, one usually sticks to the explicit models given by the representations $V_A^{\mathcal{B}}$, where A is a self-dual lattice or a lagrangian, so our result is sufficient.

Weil representations over $\mathcal{A}$-algebras. Let A be a self-dual subgroup in W. According to Section 3, the action of $\mathrm{Sp}(W)$ on H induces a projective representation of $\sigma_{\mathcal{B}} : \mathrm{Sp}(W) \to \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ i.e., $\sigma_{\mathcal{B}}$ is a group morphism. Denote by $\mathrm{RED} : \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \to \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ the quotient morphism. To lift a projective representation, one uses the fibre product construction to obtain a representation of some central extension. Looking at the fibre product of $\sigma_{\mathcal{B}}$ and RED above $\mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$, Proposition 3.2 defines:

$$\begin{tikzcd} \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi,A}(W) \arrow[r, "\omega^{\mathcal{B}}_{\psi,A}"] \arrow[d, "p_{\mathcal{B}}"'] & \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \arrow[d, "\mathrm{RED}"] \\ \mathrm{Sp}(W) \arrow[r, "\sigma_{\mathcal{B}}"] & \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \end{tikzcd}$$

Definition. The Weil representation associated to ψ and A with coefficients in $\mathcal{B}$ is the representation $(\omega^{\mathcal{B}}_{\psi,A}, V_A^{\mathcal{B}})$ of the central extension $\widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi,A}(W)$ of $\mathrm{Sp}(W)$ by $\mathcal{B}^\times$.

Recalling the canonical identification $V_A^{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B} = V_A^{\mathcal{B}}$ from (b) of Theorem A above, our Theorem 3.4 ensures the compatibility.

Theorem B. *There exists a canonical morphism of central extensions*

$$\widetilde{\phi_{\mathcal{B}}} : \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi,A}(W) \to \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi,A}(W)$$

whose image is a central extension of $\mathrm{Sp}(W)$ *by* $\varphi(\mathcal{A})^\times$. *Moreover, there is a commuting diagram*:

$$\begin{tikzcd} \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi,A}(W) \arrow[r, "\omega^{\mathcal{A}}_{\psi,A}"] \arrow[d, "\widetilde{\phi_{\mathcal{B}}}"] & \mathrm{GL}_{\mathcal{A}}(V_A) \arrow[d] \\ \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi,A}(W) \arrow[r, "\omega^{\mathcal{B}}_{\psi,A}"] & \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \end{tikzcd}$$

Moreover there exist canonical identifications between these central extensions as A varies: for any other self-dual subgroup A', Corollary 3.6 defines a canonical morphism of central extension such that $\omega^{\mathcal{B}}_{\psi,A}$ and $\omega^{\mathcal{B}}_{\psi,A'}$ agree, where the term "agree" is made precise in the corollary mentioned. So the Weil representation $\omega^{\mathcal{B}}_{\psi}$ associated to ψ is well-defined in the sense that the isomorphism class of $\omega^{\mathcal{B}}_{\psi,A}$ does not depend on A.

The metaplectic group over $\mathcal{A}$-algebras. The isomorphism class of $\widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi,A}(W)$, as a central extension of $\mathrm{Sp}(W)$ by $\mathcal{B}^\times$, does not depend on the choice of A or ψ_A. In addition, the canonical isomorphism of central extensions induced by $V_A^{\mathcal{B}} \simeq V_{A'}^{\mathcal{B}}$ is compatible with the fibre product projections. Therefore one can speak of *the* metaplectic group over $\mathcal{B}$ associated to ψ as any element in the previous isomorphism class. Even if these groups may be isomorphic for different ψ, there does not necessarily exist an isomorphism compatible with the fibre product construction: in this sense these groups do depend on ψ.

We endow the module $V_A^{\mathcal{B}}$ with the discrete topology and the group $\mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ with the compact-open one. Then Corollary 4.2 compares the situation over $\mathcal{A}$ with that for the classical metaplectic group. Indeed if we endow $\mathbb{C}$ with a structure of $\mathcal{A}$-algebra, then:

Proposition. *The group* $\widetilde{\mathrm{Sp}}_{\psi,A}^{\mathcal{A}}(W)$ *is an open topological subgroup of* $\widetilde{\mathrm{Sp}}_{\psi,A}^{\mathbb{C}}(W)$.

Here the natural topology on $\widetilde{\mathrm{Sp}}_{\psi,A}^{\mathbb{C}}(W)$ is that as a subgroup of $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathbb{C}}(V_A^{\mathbb{C}})$. The classical metaplectic group is known to be locally profinite, and so is the metaplectic group over $\mathcal{A}$ because of the proposition. Define now the derived group

$$\widehat{\mathrm{Sp}}_{\psi,A}^{\mathcal{B}}(W) = [\widetilde{\mathrm{Sp}}_{\psi,A}^{\mathcal{B}}(W), \widetilde{\mathrm{Sp}}_{\psi,A}^{\mathcal{B}}(W)].$$

When $\mathcal{B} = \mathbb{C}$, this derived group is the reduced metaplectic group when F is local nonarchimedean, or the symplectic group when F is finite, except in the exceptional case $F = \mathbb{F}_3$ and $\dim_F(W) = 2$. According to Proposition 4.3, one has a canonical isomorphism of central extensions

$$\widehat{\mathrm{Sp}}_{\psi,A}^{\mathcal{A}}(W) \simeq \widehat{\mathrm{Sp}}_{\psi,A}^{\mathbb{C}}(W).$$

Proposition 4.4 sheds light on the structure of the metaplectic group.

Theorem C. *One has the following properties*:

(a) *The group* $\widetilde{\mathrm{Sp}}_{\psi,A}^{\mathcal{B}}(W)$ *is the fibre product in the category of topological groups of the morphisms* $\sigma_{\mathcal{B}}$ *and* RED, *having the subspace topology in* $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$.

(b) *The representation* $\omega_{\psi,A}^{\mathcal{B}} : \widetilde{\mathrm{Sp}}_{\psi,A}^{\mathcal{B}}(W) \to \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ *is smooth.*

(c) *The map* $\widetilde{\phi_{\mathcal{B}}}$ *of Theorem B is open and continuous, and* $\widetilde{\mathrm{Sp}}_{\psi,A}^{\mathcal{B}}(W)$ *is locally profinite.*

(d) *Considering derived groups, the map* $\widetilde{\phi_{\mathcal{B}}}$ *restricts to*:

  (i) *A surjection* $\widehat{\mathrm{Sp}}_{\psi,A}^{\mathcal{A}}(W) \to \widehat{\mathrm{Sp}}_{\psi,A}^{\mathcal{B}}(W)$ *with kernel* $\{\pm 1\}$ *and image isomorphic to* $\mathrm{Sp}(W)$ *if* F *is local nonarchimedean and* $\mathrm{char}(\mathcal{B}) = 2$.

  (ii) *An isomorphism* $\widehat{\mathrm{Sp}}_{\psi,A}^{\mathcal{A}}(W) \simeq \widehat{\mathrm{Sp}}_{\psi,A}^{\mathcal{B}}(W)$ *otherwise.*

Again exclude the exceptional case, which is considered in Remark 4.12. In Section 4B, we prove:

Theorem D. *There exists a section* $\varsigma^{\mathcal{B}} : \mathrm{Sp}(W) \to \widehat{\mathrm{Sp}}^{\mathcal{B}}_{\psi,A}(W)$ *compatible with that defined over* $\mathcal{A}$ *and such that the associated* 2*-cocycle* $\hat{c}_{\mathcal{B}}$ *has image*:

- $\{1\}$ *if* F *is finite or* $\mathrm{char}(\mathcal{B}) = 2$.
- $\{\pm 1\}$ *if* F *is local nonarchimedean and* $\mathrm{char}(\mathcal{B}) \neq 2$.

Families of Weil representations. The consequence of these results is that one may speak of a universal Weil module $\omega_\psi^{\mathcal{A}}$ over $\mathcal{A}$ associated to ψ: that is (see Proposition 5.4) any Weil representation $\omega_\psi^{\mathcal{B}}$ with coefficients in $\mathcal{B}$ arises from the scalar extension of this universal Weil module. Thus, according to the compatibility in Theorem B, the Weil representation $\omega_\psi^{\mathcal{A}}$ is a family of Weil representations over the residue fields of $\mathrm{Spec}(\mathcal{A})$.

0B. ***Towards an integral theta correspondence.*** In the rest of the introduction, we give some new ideas and speculate in the direction of an integral theta correspondence. As an illustration, we study in detail the case of the type II dual pair $(F^\times, F^\times)$ but it is only this example which is part of the main body of the paper. Thus the text below is a kind of story about the bigger picture to motivate our study and can be omitted if the reader is only interested in the Weil representation itself.

Suppose again F is local nonarchimedean. For more general dual pairs (H_1, H_2), one usually considers the Weil representation with coefficients in a field, along with its biggest π_1-isotypic quotients for π_1 running through the irreducible representations of H_1. However, there is no natural definition of what a good biggest isotypic quotient over a ring should be. But there is another approach with a coarser invariant in terms of the Bernstein centre, giving a bigger representation. In order to lighten notations further, we omit the reference to ψ from now on.

Replacing biggest isotypic quotients: a heuristic approach. Suppose in this paragraph that $\mathcal{B}$ is an algebraically closed field. Let $\mathfrak{z}_{\mathcal{B}}(H_1)$ be the Bernstein centre of H_1. A character of the Bernstein of the centre is a $\mathcal{B}$-algebra morphism $\eta_1 : \mathfrak{z}_{\mathcal{B}}(H_1) \to \mathcal{B}$. The set of such characters correspond bijectively to the points in $\mathrm{Specmax}(\mathfrak{z}_{\mathcal{B}}(H_1))$. Denote by $\eta_{\pi_1} : \mathfrak{z}_{\mathcal{B}}(H_1) \to \mathcal{B}$ the character associated to π_1. The construction of the biggest π_1-isotypic quotient factors through the biggest η_{π_1}-isotypic quotient, in the sense that for any $V \in \mathrm{Rep}_{\mathcal{B}}(H_1)$, the quotient $V \to V_{\pi_1}$ factors through $V \to V \otimes_{\mathfrak{z}_{\mathcal{B}}(H_1)} \eta_{\pi_1}$. Denote by $V_{\eta_{\pi_1}}$ the latter representation. Regardless of the characteristic of $\mathcal{B}$, and similarly to $V_{\pi_1} \in \mathrm{Rep}_{\mathcal{B}}(H_1 \times H_2)$ when $V \in \mathrm{Rep}_{\mathcal{B}}(H_1 \times H_2)$, one has $V_{\eta_{\pi_1}} \in \mathrm{Rep}_{\mathcal{B}}(H_1 \times H_2)$.

When the characteristic ℓ of $\mathcal{B}$ is banal with respect to H_1, that is when ℓ does not divide the pro-order $|H_1|$ of H_1, the set of characters of $\mathfrak{z}_{\mathcal{B}}(H_1)$ is in bijection with the set of cuspidal supports in $\mathrm{Rep}_{\mathcal{B}}(H_1)$ and we expect the following to hold for all η_1 in a Zariski open subset of $\mathrm{Specmax}(\mathfrak{z}^{\mathcal{B}}(H_1))$:

$$V_{\eta_1} \simeq \bigoplus_{\pi_1 \in \eta_1} V_{\pi_1}$$

where $\pi_1 \in \eta_1$ means $\eta_{\pi_1} = \eta_1$, that is π_1 has cuspidal support corresponding to η_1.

Outside the banal setting, it seems risky to state any precise results. Already some key facts fail: the maximal ideals of $\mathfrak{z}_{\mathcal{B}}(H_1)$ are no longer in bijection with cuspidal supports. However the biggest π_1-isotypic quotient V_{π_1} always is a quotient of the bigger representation $V_{\eta_{\pi_1}}$, so this last construction encapsulates more information. In addition, we expect this object to behave in a nicer way for coefficient rings as it keeps track of congruences.

Illustration for the type II dual pair $(F^\times, F^\times)$. The category $\mathrm{Rep}_{\mathcal{B}}(F^\times)$ can be decomposed according to the level and we denote by $\mathrm{Rep}^0_{\mathcal{B}}(F^\times)$ the level 0 direct factor category. This category is Morita-equivalent to the category of $\mathfrak{z}^0$-modules, where $\mathfrak{z}^0$ is the commutative ring $\mathcal{B}[F^\times/1+\varpi_F\mathcal{O}_F]$. Up to choosing a uniformiser ϖ_F and a primitive $(q-1)$-th root of unity ζ in $F^\times$, this ring is isomorphic to $\mathcal{B}[X^{\pm 1}, Z]/(Z^{q-1}-1)$ by sending X to ϖ_F and Z to ζ. Instead of considering biggest isotypic quotients associated to irreducible representations in $\mathrm{Rep}_{\mathcal{B}}(F^\times)$, Section 6A1 considers more general isotypic families of representations using the explicit description of (the centre of) $\mathfrak{z}^0$.

Definition. Let $V \in \mathrm{Rep}_{\mathcal{B}}(F^\times)$. When $\mathcal{C}$ is a commutative $\mathcal{B}$-algebra and $\eta : \mathfrak{z}^0 \to \mathcal{C}$ is a $\mathcal{B}$-algebra morphism, the representation $V_\eta = V \otimes_{\mathfrak{z}^0} \eta \in \mathrm{Rep}_{\mathcal{C}}(F^\times)$ may be thought as the "biggest η-isotypic quotient of V."

Remark. Unlike the situation of the biggest isotypic quotient, V does not necessarily surject onto V_η if η is not surjective. So in general V_η is not a quotient of V, but the image of V in V_η generates V_η as a $\mathcal{C}$-module.

When $\mathcal{B}'$ is a $\mathcal{B}$-algebra, denote by $(\mathbf{1}^{\mathcal{B}'}_{F^\times}, \mathcal{B}') \in \mathrm{Rep}^0_{\mathcal{B}}(F^\times)$ the trivial $\mathfrak{z}^0$-module isomorphic to $\mathcal{B}'$. Denote by $(\chi_{\mathcal{B}}, \mathcal{B})$ the character with $\chi_{\mathcal{B}}(\varpi_F) = q \in \mathcal{B}^\times$ and $\chi_{\mathcal{B}}|_{\mathcal{O}_F^\times} = 1_{\mathcal{B}}$. Thus $\chi_{\mathcal{B}}$ is the inverse of the norm $|\cdot|_F$.

Let $I_{\mathbf{1}}$ be the ideal in $\mathfrak{z}^0$ corresponding to $(X-1, Z-1)$ in $\mathcal{B}[X^{\pm 1}, Z]/(Z^{q-1}-1)$. Denote the quotient map $\eta_{\mathbf{1}} : \mathfrak{z}^0 \to \mathfrak{z}^0/I_{\mathbf{1}}$. Consider the isotypic family $V_{\eta_{\mathbf{1}}}$ associated to $\eta_{\mathbf{1}}$ with respect to the action of the first copy of $F^\times$ on V. Take the same convention for I corresponding to $(X-q, Z-1)$ with η being the quotient map.

Theorem E. *One has in* $\mathrm{Rep}_{\mathcal{B}}(F^\times)$ *the following isomorphisms*:

(a) $V_{\eta_{\mathbf{1}}} \simeq \mathbf{1}^{\mathcal{B}/(q-1)\mathcal{B}}_{F^\times} \oplus \mathbf{1}^{\mathcal{B}}_{F^\times}$.

(b) $V_\eta \simeq \mathbf{1}^{\mathcal{B}/(q-1)\mathcal{B}}_{F^\times} \oplus \chi_{\mathcal{B}}$.

The subrepresentation $\mathbf{1}_{F^\times}^{\mathcal{B}/(q-1)\mathcal{B}}$ is in a certain sense the “defect” in the theta correspondence. This is a pure $(q-1)$-torsion submodule, whereas the other part is a free $\mathcal{B}$-module of rank 1. When $\mathcal{B}$ is a field, this defect vanishes if and only if the characteristic ℓ of $\mathcal{B}$ does not divide $q-1$, that is ℓ is banal with respect to $F^\times$.

Further example. Using this interpretation in terms of the characters of the Bernstein centre seems to be more suitable when $\mathcal{B}$ is a ring. Indeed recall the situation in [Trias 2020, Section 5.3] where F has odd residual characteristic and (H_1, H_2) is a type I dual pair that is split in the metaplectic group. Let ℓ be a prime that does not divide the pro-order of H_1 and endow $\mathcal{B} = W(\overline{\mathbb{F}}_\ell)$ with an $\mathcal{A}$-algebra structure. Let K be the fraction field of $\mathcal{B}$. For any absolutely irreducible cuspidal $\Pi_1 \in \mathrm{Rep}_K(H_1)$, one has the equality $V_{\Pi_1} = V_{\eta_{\Pi_1}}$ for $V \in \mathrm{Rep}_K(H_1)$.

The reduction modulo ℓ of Π_1 is obtained by choosing a sable lattice L_{Π_1} in Π_1. The reduction modulo ℓ of this lattice is an irreducible representation π_1 whose isomorphism class does not depend on the choice of L_{Π_1}. We refer to [Trias 2020, Section 5.3] for more details, but what is important here is that similarly to Π_1, we have $V_{\pi_1} = V_{\eta_{\pi_1}}$ for $V \in \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(H_1)$. Actually this comes along with some compatibilities to scalar extension. Indeed there exists a character $\eta_1 : \mathfrak{z}_{W(\overline{\mathbb{F}}_\ell)}(H_1) \to W(\overline{\mathbb{F}}_\ell)$ of the integral Bernstein centre such that $\eta_1 \otimes_{W(\overline{\mathbb{F}}_\ell)} \overline{\mathbb{F}}_\ell = \eta_{\pi_1}$ and $\eta_1 \otimes_{W(\overline{\mathbb{F}}_\ell)} K = \eta_{\Pi_1}$. This yields, for any $V \in \mathrm{Rep}_{W(\overline{\mathbb{F}}_\ell)}(H_1 \times H_2)$, the following canonical morphisms in $\mathrm{Rep}_{W(\overline{\mathbb{F}}_\ell)}(H_1 \times H_2)$:

$$\begin{tikzcd} V_{\eta_1} \arrow[r] \arrow[d, two heads] & V_{\eta_1} \otimes_{W(\overline{\mathbb{F}}_\ell)} K = (V \otimes_{W(\overline{\mathbb{F}}_\ell)} K)_{\eta_{\Pi_1}} \\ V_{\eta_1} \otimes_{W(\overline{\mathbb{F}}_\ell)} \overline{\mathbb{F}}_\ell = (V \otimes_{W(\overline{\mathbb{F}}_\ell)} \overline{\mathbb{F}}_\ell)_{\eta_{\pi_1}} & \end{tikzcd}$$

When $V = \omega$ is the Weil representation with coefficients in $W(\overline{\mathbb{F}}_\ell)$, the Weil representations with coefficients in the residue fields $\overline{\mathbb{F}}_\ell$ and K of $W(\overline{\mathbb{F}}_\ell)$ are $\bar{\omega} = \omega \otimes_{W(\overline{\mathbb{F}}_\ell)} \overline{\mathbb{F}}_\ell$ and $\Omega = \omega \otimes_{W(\overline{\mathbb{F}}_\ell)} K$, respectively. The biggest isotypic quotients are

$$\Omega_{\eta_{\Pi_1}} \simeq \Pi_1 \otimes_K \Theta(\Pi_1) \quad \text{and} \quad \bar{\omega}_{\eta_{\pi_1}} \simeq \pi_1 \otimes_{\overline{\mathbb{F}}_\ell} \Theta(\pi_1),$$

where $\Theta(\Pi_1) \in \mathrm{Rep}_K(H_2)$ and $\Theta(\pi_1) \in \mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(H_2)$. So ω_{η_1} is a good family object because its generic fibre is $\Pi_1 \otimes_K \Theta(\Pi_1)$ and its special fibre is $\pi_1 \otimes_{\overline{\mathbb{F}}_\ell} \Theta(\pi_1)$. In addition $\Theta(\Pi_1)$ is irreducible, when it is nonzero and ω_{η_1} is a $W(\overline{\mathbb{F}}_\ell)[H_1 \times H_2]$-lattice in $\Pi_1 \otimes_K \Theta(\Pi_1)$. Furthermore, when ℓ is banal with respect to H_2 and $\Theta(\Pi_1)$ is cuspidal, the representation $\Theta(\pi_1)$ is the reduction modulo ℓ of $\Theta(\Pi_1)$ and is therefore irreducible [Trias 2020, Theorem 5.17]. To relate $\Theta(\Pi_1)$ and $\Theta(\pi_1)$ in general, one needs to explicitly know which lattice in $\Pi_1 \otimes_K \Theta(\Pi_1)$ is ω_{η_1}.

First expectations. Of course in the most general situation, i.e., when the coefficient ring $\mathcal{B}$ is $\mathbb{Z}[\frac{1}{p}]$ (or $\mathcal{A}$), exhibiting blocks, as well as their centres, is a daydream. However, one can play with:

- "Simpler" rings $\mathcal{B}$ (fields, local rings, banal characteristic, etc.).
- Special classes of representations (cuspidals, level 0, etc.).
- Easier groups in the dual pair (small dimension, general linear, etc.).

As recalled, this has been achieved in [Trias 2020, Section 5.3] for type I dual pairs (H_1, H_2) over the local ring $W(\overline{\mathbb{F}}_\ell)$ when ℓ is banal with respect to H_1, looking at the block defined by a (super)cuspidal representation. In Section 6, we consider the (very simple) pair $(F^\times, F^\times)$, especially for level 0 representations. For bigger type II dual pairs $(\mathrm{GL}_n(F), \mathrm{GL}_m(F))$ and coefficients rings being made of Witt vectors, the work [Helm 2016] seems to be the cornerstone to tackle the problem. Based on calculations in small dimensions, we make the following two conjectures.

Torsion principle. When the pro-order of H_1, or that of H_2, is not invertible in $\mathcal{B}$, we expect the failure of the theta correspondence to appear as some $|H_1|_f |H_2|_f$-torsion submodule in the family object, where $|H_i|_f$ denotes the prime-to-p part of the pro-order of H_i. Thanks to Theorem E, this principle is made a bit more precise when $(H_1, H_2) = (F^\times, F^\times)$.

Bijection principle for characters of the Bernstein centre. Another problem is the following. When $\eta_1 : \mathfrak{z}_\mathcal{B}(H_1) \to \mathcal{B}$ is a character, are there any nice properties of $(\omega^\mathcal{B})_{\eta_1}$, where $\omega^\mathcal{B}$ is the Weil representation over $\mathcal{B}$? For instance, it seems that the action of $\mathfrak{z}_\mathcal{B}(H_2)$ can also be described in terms of a character of $\mathfrak{z}_\mathcal{B}(H_2)$. Indeed one expects that there exists a character $\eta_2 : \mathfrak{z}_\mathcal{B}(H_2) \to \mathcal{B}$ such that $((\omega^\mathcal{B})_{\eta_1})_{\eta_2} = (\omega^\mathcal{B})_{\eta_1}$. Denoting by $\eta_1 \otimes_\mathcal{B} \eta_2$ the natural character $\mathfrak{z}_\mathcal{B}(H_1) \otimes_\mathcal{B} \mathfrak{z}_\mathcal{B}(H_2) \to \mathcal{B}$, we expect even more: $(\omega^\mathcal{B})_{\eta_1} = (\omega^\mathcal{B})_{\eta_2} = (\omega^\mathcal{B})_{\eta_1 \otimes_\mathcal{B} \eta_2}$. Writing $\eta_2 = \theta(\eta_1)$, one could then speak of a theta correspondence in terms of characters of the respective Bernstein centres because θ would induce a bijection

$$\{\eta_1 : \mathfrak{z}_\mathcal{B}(H_1) \to \mathcal{B} \mid (\omega^\mathcal{B})_{\eta_1} \neq 0\} \overset{\theta}{\simeq} \{\eta_2 : \mathfrak{z}_\mathcal{B}(H_2) \to \mathcal{B} \mid (\omega^\mathcal{B})_{\eta_2} \neq 0\}.$$

1. Preliminaries

1A. ***Notations.*** All along the paper F will be a field of characteristic not 2, which is either finite or local nonarchimedean. The residual characteristic and cardinality of F are denoted as usual p and q. To turn F into a topological field one considers the usual locally profinite topology. One of the many equivalent formulations of the latter is "locally compact and totally disconnected".

$\mathcal{K}$ and $\mathcal{A}$. Let $\mathcal{K}$ be the field defined in the following two cases:

- $\mathcal{K}$ is the cyclotomic extension of $\mathbb{Q}$ containing the p-th roots of unity, when the characteristic of F is positive.
- $\mathcal{K}$ is the algebraic extension of $\mathbb{Q}$ containing all the p power roots of unity, when the characteristic of F is zero.

One can write $\mathcal{K} = \mathbb{Q}(\zeta_p)$ by fixing a generator ζ_p in the first case; in the second however, no generator exists, though the notation $\mathcal{K} = \mathbb{Q}(\zeta_{p^\infty})$ is commonly used. Based on classical results for cyclotomic extensions, the integral closure $\mathcal{O}_\mathcal{K}$ of $\mathbb{Z}$ in $\mathcal{K}$ is, in the first case $\mathbb{Z}[\zeta_p]$, and in the second $\mathbb{Z}[\zeta_{p^\infty}]$. From now on, let $\mathcal{A}$ be the subring of $\mathcal{K}$ obtained from the ring of integers $\mathcal{O}_\mathcal{K}$ by inverting p, that is

$$\mathcal{A} = \mathcal{O}_\mathcal{K}\left[\tfrac{1}{p}\right].$$

$\mathcal{A}$-algebras. By convention, the term $\mathcal{A}$-algebra will refer to commutative rings $\mathcal{B}$ endowed with an $\mathcal{A}$-algebra structure, that is, a ring morphism $\varphi : \mathcal{A} \to \mathcal{B}$. In order to avoid confusion, those $\mathcal{B}$ considered always are unitary rings and φ maps the neutral multiplicative element of $\mathcal{A}$ to that of $\mathcal{B}$. Denote $\mathrm{char}(\mathcal{B})$ the characteristic of $\mathcal{B}$, that is the natural number such that $\{k \in \mathbb{Z} \mid \varphi(k) = 0\} = \mathrm{char}(\mathcal{B})\mathbb{Z}$. The ring morphism φ induces a group morphism $\mathcal{A}^\times \to \mathcal{B}^\times$ between the group of units of $\mathcal{A}$ and that of $\mathcal{B}$. Denote $\mu^p(\mathcal{B}) = \{\zeta \in \mathcal{B}^\times \mid \exists k \in \mathbb{Z}, \zeta^{p^k} = 1\}$ for the group of elements in $\mathcal{B}^\times$ having order a power of p.

Character $\psi^\mathcal{B}$. Let $\mathcal{B}$ be an $\mathcal{A}$-algebra. Then φ restricts injectively to the group of roots in $\mathcal{A}^\times$ having order a power of p, that is the group morphism $\varphi : \mu^p(\mathcal{A}) \to \mu^p(\mathcal{B})$ is injective. Indeed, given two distinct roots of unity ζ and ζ' in $\mu^p(\mathcal{A})$, their difference $\zeta - \zeta'$ is in $\mathcal{A}^\times$ because $p \in \mathcal{A}^\times$, so they define two distinct elements in $\varphi(\mathcal{A}) = \mathcal{A}/\operatorname{Ker}(\varphi)$. Therefore one can build, out of any nontrivial smooth character $\psi : F \to \mathcal{A}^\times$, a character $\varphi \circ \psi : F \to \mathcal{B}^\times$ which is still nontrivial and smooth. In order to keep track of the ring considered, one uses a superscript to refer to the $\mathcal{A}$-algebra at stake. From now on, fix such a nontrivial smooth $\psi^\mathcal{A} : F \to \mathcal{A}^\times$ and set

$$\psi^\mathcal{B} = \varphi \circ \psi^\mathcal{A}.$$

Smooth representations. Let G be a locally profinite group. Let R be a commutative unitary ring. An $R[G]$-module V is said to be smooth if for all $v \in V$, the stabiliser G_v of v is open in G. One denotes $\mathrm{Rep}_R(G)$ the category of smooth $R[G]$-modules. For any closed subgroup H in G, the induction functor Ind_H^G associates to any $(\sigma, W) \in \mathrm{Rep}_R(H)$, the representation $\mathrm{Ind}_H^G(W) \in \mathrm{Rep}_R(G)$ of locally constant functions on G taking values in W and satisfying $f(hg) = \sigma(h) \cdot f(g)$ for all $g \in G$ and $h \in H$. The compact induction ind_H^G is the subfunctor of Ind_H^G made of functions compactly supported modulo H, that is the subspace of functions $f \in \mathrm{Ind}_H^G(W)$ such that the image of $\mathrm{supp}(f)$ in $H\backslash G$ is a compact set. A representation $V \in \mathrm{Rep}_R(G)$ is said to be admissible if for all compact open subgroups K in G, the set of K-invariants $V^K = \{v \in V \mid g \cdot v = v\}$ is finitely generated as an R-module.

Haar measures. Let G be a locally profinite group. In the following, we use the notations of [Vignéras 1996, I.1 and I.2]. The pro-order $|G|$ of G is the least common multiple, in the sense of supernatural integers, of the orders of its open compact subgroups. To be more explicit, $|G|$ is a function $\mathcal{P} \to \mathbb{N} \cup \infty$ on the set of prime numbers $\mathcal{P}$. This decomposes in an obvious way into two parts having disjoint supports, namely the finite part $|G|_f$ and the infinite one $|G|_\infty$. The only situation occurring in the present work is $|G| = |G|_f \times |G|_\infty$ with $|G|_\infty \in \{1, p^\infty\}$, according to G being either a finite group or an infinite p-adic group; in the latter case, $|G|_f$ is prime-to-p. Let R be a commutative unitary ring. As long as all the primes in $|G|_\infty$ are invertible in R, there exists a Haar measure on G with values in R, that is a nonzero left G-equivariant morphism $C_c^\infty(G, R) \to 1_G^R$ where $C_c^\infty(G, R)$ is the space of locally constant and compactly supported functions in G with values in R, and $(1_G^R, R)$ is the trivial representation. A normalised Haar measure on G is a Haar measure taking the value 1 on a compact open subgroup of G. In particular such a compact open subgroup must be of invertible pro-order in R. Reciprocally, any normalised Haar measure arises as a Haar measure having value 1 on a compact open subgroup of invertible pro-order in R.

The space W. Let $(W, \langle \cdot , \cdot \rangle)$ be a symplectic vector space of finite dimension over F. Its isometry group is composed of the F-linear invertible endomorphisms preserving the form $\langle \cdot , \cdot \rangle$ and is classically denoted $\mathrm{Sp}(W)$. A lagrangian in W is a maximal totally isotropic subspace. Denote the dimension of W by $n = 2m$, then X is a lagrangian if and only if it is a vector subspace which is totally isotropic (i.e., $\forall x, x' \in X$, $\langle x, x' \rangle = 0$) of dimension m. A lattice L in W is a free $\mathcal{O}_F$-module of rank n. The locally profinite topology on the field F induces a locally profinite topology on the finite dimensional vector space W. As a result, a lattice in W is a compact open set. Furthermore the subspace topology induced from that of $\mathrm{End}_F(W)$ on the symplectic group $\mathrm{Sp}(W)$ is the locally profinite one as well.

2. Metaplectic representations over $\mathcal{A}$-algebras

The Heisenberg group H is the set $W \times F$ endowed with the product topology and the composition law

$$(w, t) \cdot (w', t') = \left(w + w', t + t' + \tfrac{1}{2}\langle w, w' \rangle\right)$$

for (w, t) and (w', t') in $H = W \times F$.

Let $\mathcal{B}$ be an $\mathcal{A}$-algebra with structure morphism φ. Let $\psi^{\mathcal{A}} : F \to \mathcal{A}^\times$ be a nontrivial smooth character. As already mentioned in Section 1A, this defines, by composing ψ and φ, a character $\psi^{\mathcal{B}} : F \to \mathcal{B}^\times$ which is smooth and nontrivial.

2A. ***A lemma for representations over rings.*** Let G be a group and $\mathcal{R}$ a commutative ring. For every prime ideal $\mathcal{P}$ in $\mathrm{Spec}(\mathcal{R})$, one denotes $k(\mathcal{P})$ the fraction field of $\mathcal{R}(\mathcal{P}) = \mathcal{R}/\mathcal{P}$. Both $k(\mathcal{P})$ and $\mathcal{R}(\mathcal{P})$ are endowed with an obvious structure of $\mathcal{R}$-algebras. For any $\mathcal{R}[G]$-module V, the tensor product $V \otimes_{\mathcal{R}} k(\mathcal{P})$ is a $k(\mathcal{P})[G]$-module in the obvious way. Of course, the latter is smooth if the former is.

Definition 2.1. An $\mathcal{R}[G]$-module V is said to be irreducible at $\mathcal{P} \in \mathrm{Spec}(\mathcal{R})$ if the representation $V \otimes_{\mathcal{R}} k(\mathcal{P}) \in \mathrm{Rep}_{k(\mathcal{P})}(G)$ is irreducible. By extension, V is everywhere irreducible if it is irreducible at any point of $\mathrm{Spec}(\mathcal{R})$.

There exists a simple sufficient condition to be everywhere irreducible.

Lemma 2.2. *Let V be an $\mathcal{R}[G]$-module and consider the map $I \mapsto IV$ that maps an ideal I of $\mathcal{R}$ to the sub-$\mathcal{R}[G]$-module IV of V. If the previous map defines a bijection between ideals of $\mathcal{R}$ and sub-$\mathcal{R}[G]$-modules of V, then V is everywhere irreducible.*

Proof. Using the bijection, one has $\mathcal{P}V \subsetneq V$ for any prime (proper) ideal $\mathcal{P}$, so the module $V \otimes_{\mathcal{R}} \mathcal{R}(\mathcal{P}) = V/\mathcal{P}V$ is nonzero. It is even $\mathcal{R}(\mathcal{P})$-torsion free because, if $av \in \mathcal{P}V$ for $a \in R$ and $v \in V$, then $aI_v \subset \mathcal{P}$ where $I_v V = \mathcal{R}[G] \cdot v$. In particular $V \otimes_{\mathcal{R}} \mathcal{R}(\mathcal{P})$ embeds in $V \otimes_{\mathcal{R}} k(\mathcal{P})$ by a localisation argument, so the latter representation is nonzero.

In order to prove that $V \otimes_{\mathcal{R}} k(\mathcal{P})$ is irreducible, let W be a nonzero subrepresentation of $V \otimes_{\mathcal{R}} k(\mathcal{P})$ and define $W' = \{v \in V \mid v \otimes_{\mathcal{R}} 1 \in W\}$. As a first elementary claim, this W' is a nonzero sub-$\mathcal{R}[G]$-module of V. In addition the bijection hypothesis yields the existence of an ideal I of $\mathcal{R}$ such that $W' = IV$. Observe furthermore thanks to the bijection that $I \subset \mathcal{P}$ if and only if $IV \subset \mathcal{P}V$. As a consequence, the image of IV in $V \otimes_{\mathcal{R}} k(\mathcal{P})$ generates $V \otimes_{\mathcal{R}} k(\mathcal{P})$ as a $k(\mathcal{P})$-vector space if and only if I is not contained in $\mathcal{P}$. Of course the image of W' in $V \otimes_{\mathcal{R}} k(\mathcal{P})$ is nonzero because W is not, so I is not contained in $\mathcal{P}$ i.e., the image of W' generates $V \otimes_{\mathcal{R}} k(\mathcal{P})$. Therefore $W = V \otimes_{\mathcal{R}} k(\mathcal{P})$. □

2B. ***Models $V_A^{\mathcal{B}}$ associated to self-dual subgroups.*** When A is a closed subgroup of W, define

$$A^\perp = \{w \in W \mid \psi^{\mathcal{A}}(\langle w, A \rangle) = 1\}.$$

In this definition, whether one uses $\psi^{\mathcal{A}}$ or $\psi^{\mathcal{B}}$ matters not. Now, the closed subgroup A of W is said to be self-dual if $A^\perp = A$. Lagrangians and self-dual lattices provide examples of such subgroups, so there always exist self-dual subgroups in W.

Lemma 2.3. *Let A be a self-dual subgroup of W. Then there exists a character $\psi_A^{\mathcal{A}}$ which extends $\psi^{\mathcal{A}}$ to the subgroup $A_H = A \times F$ of the Heisenberg group H. Furthermore, $\psi_A^{\mathcal{B}} = \varphi \circ \psi_A^{\mathcal{A}}$ provides the same kind of extension, that is, $\psi_A^{\mathcal{B}}$ extends $\psi^{\mathcal{B}}$ to A_H.*

This lemma can be proved in the exact same elementary way as [Trias 2020, Lemma 2.2(a)]. For the sake of shortness, we simply refer to the latter. The heart of the current section is the following proposition, generalising [loc. cit., Lemma 2.2(b)] where the $\mathcal{A}$-algebra $\mathcal{B}$ is a field.

Proposition 2.4. *Let $\psi_A^{\mathcal{A}}$ be as above and set $V_A^{\mathcal{B}} = \mathrm{ind}_{A_H}^H(\psi_A^{\mathcal{B}}) \in \mathrm{Rep}_{\mathcal{B}}(H)$:*

(a) *The map $I \mapsto I V_A^{\mathcal{B}}$ defines a bijection from the set of ideals of $\mathcal{B}$ to the set of sub-$\mathcal{B}[H]$-modules of $V_A^{\mathcal{B}}$. In particular, $V_A^{\mathcal{B}}$ is everywhere irreducible.*

(b) *The $\mathcal{B}[H]$-module $V_A^{\mathcal{B}}$ is admissible and $V_A^{\mathcal{B}} = \mathrm{Ind}_{A_H}^H(\psi_A^{\mathcal{B}})$.*

(c) *$V_A^{\mathcal{B}}$ satisfies Schur's lemma, that is $\mathrm{End}_{\mathcal{B}[H]}(V_A^{\mathcal{B}}) = \mathcal{B}$.*

Proof. The core idea of the proof comes from [Trias 2020, Lemma 2.2(b) and Proposition 2.4(c)], which was originally generalising [Mœglin et al. 1987, Chapter 2, I.3 and I.6]. As some differences occur when dealing with $\mathcal{A}$-algebras instead of fields, we carefully examine and detail them below:

(a) First remark that, assuming the bijection property holds, the second part of the statement is a mere application of Lemma 2.2. Therefore we focus our attention to proving that such a bijection holds.

The $\mathcal{B}[H]$-module $V_A^{\mathcal{B}}$ is generated as a $\mathcal{B}$-module by a family $(\chi_{w,L})$ we now describe. As $\psi_A^{\mathcal{B}}$ is smooth, there exists for all $w \in W$ an open compact subgroup L_w of W such that $\psi_A^{\mathcal{B}}(a) = 1$ for all $a \in A_H \cap (w,0)(L_w,0)(w,0)^{-1}$. Fix such choices of small enough lattices $(L_w)_{w \in W}$. Then if L is a sublattice of L_w, there exists a unique function in $V_A^{\mathcal{B}}$ which is supported on $A_H(w,0)(L,0)$, right invariant under $(L,0)$ and taking the value 1 at $(w,0)$. One denotes it $\chi_{w,L}$. The $\mathcal{B}[H]$-module $V_A^{\mathcal{B}}$ being smooth, any f in this compactly induced module can be written as a finite sum of such $\chi_{w,L}$, that is the family $(\chi_{w,L})_{w \in W, L \subset L_w}$ is generating $V_A^{\mathcal{B}}$. Actually we can give a more precise decomposition in terms of these functions. We claim that f can be written as a finite sum $\sum f((w,0)) \cdot \chi_{w,L}$ where L only depends on f and the functions $\chi_{w,L}$ have disjoint supports. Indeed, assume that f is right invariant by $(L,0)$ and $f((w,0)) \neq 0$. In order for $\chi_{w,L}$ to be well-defined, the condition $\psi_A^{\mathcal{B}}(a) = 1$ for all $a \in A_H \cap (w,0)(L,0)(w,0)^{-1}$ needs to be satisfied. Note that $(w,0)(L,0)(w,0)^{-1} = \{(l, \langle w,l \rangle) \mid l \in L\}$ so the intersection with A_H is simply $\{(l, \langle w,l \rangle) \mid l \in A \cap L\}$. By right invariance, we obtain for all $l \in A \cap L$ the equality $f((w,0)) = f((w,0)(l,0)) = \psi_A^{\mathcal{B}}((l, \langle w,l \rangle)) f((w,0))$. This implies that $\psi_A^{\mathcal{B}}(\langle l, \langle w,l \rangle) = 1$ for all $l \in A \cap L$ because $1 - \zeta$ is a regular element in $\mathcal{B}$ when $\zeta \in \mu^p(\mathcal{B})$ and $\zeta \neq 1$. Therefore $f = \sum f(w,0) \cdot \chi_{w,L}$ where the sum runs over a finite number of double cosets $A_H(w,0)(L,0)$ in H. Because the subspace of functions in $V_A^{\mathcal{B}}$ taking values in I and the space $IV_A^{\mathcal{B}}$ both contain $(i\cdot\chi_{w,L})_{i\in I,w\in W,L\subset L_w}$ as a generating family, they must agree. Consequently the injectivity of the map $I\mapsto IV_A^{\mathcal{B}}$ follows.

The surjectivity of $I\mapsto IV_A^{\mathcal{B}}$ amounts to proving that any sub-$\mathcal{B}[H]$-module of $V_A^{\mathcal{B}}$ is of the form $IV_A^{\mathcal{B}}$. For any subset X of $V_A^{\mathcal{B}}$, define $I_X=\langle f(h)\mid h\in H,\ f\in X\rangle$ the ideal in $\mathcal{B}$ generated by the set of values of functions in X. There is an obvious inclusion of $\mathcal{B}[H]$-modules $\mathcal{B}[H]\cdot X\subset I_XV_A^{\mathcal{B}}$. We claim even more: this inclusion actually is an equality. It is enough to prove it when X is a singleton to deduce the result general case because $\mathcal{B}[H]\cdot X=\sum\mathcal{B}[H]\cdot f$ and $I_XV_A^{\mathcal{B}}=\sum I_fV_A^{\mathcal{B}}$ where the sums run over all $f\in X$. So from now on, suppose that X is made of a single function f in $V_A^{\mathcal{B}}$. We would like to prove that the reverse inclusion holds, that is

$$I_fV_A^{\mathcal{B}}\subset\mathcal{B}[H]\cdot f.$$

As p is invertible in $\mathcal{B}$, there exists a Haar measure of H which takes values in $\mathcal{B}$ and is normalised over a compact open subgroup of H. Let μ be such a measure. The claim will then follow from the — technical-to-state but rather clear — observation below.

Lemma 2.5. *Let f be a nonzero function of $V_A^{\mathcal{B}}$. For any $w\in W$, fix a sufficiently small lattice L_w in W such that $(L_w,0)$ leaves f right invariant and $\psi_A^{\mathcal{B}}(a)=1$ for all $a\in A_H\cap(w,0)(L_w,0)(w,0)^{-1}$. Then for any sublattice L of L_w, there exists an element $\phi_{w,L}\in\mathcal{B}[H]$ such that $\phi_{w,L}\cdot f=f((w,0))\chi_{w,L}$.*

Proof. First of all, the fact that such a choice of lattices $(L_w)_{w\in W}$ exists comes for the smoothness of $V_A^{\mathcal{B}}$ and $\psi_A^{\mathcal{B}}$. Let L be sublattice of L_w and define

$$\phi:a\in A\mapsto\frac{\psi_A^{\mathcal{B}}((-a,0))}{\mathrm{vol}(L^\perp\cap A)}\psi^{\mathcal{B}}(\langle -w,a\rangle)1_{L^\perp\cap A}(a)\in\mathcal{B}$$

where 1_X is the characteristic function of X, μ_A is a Haar measure of A normalised over a compact open subgroup and $\mathrm{vol}(L^\perp\cap A)$ is a power of p. Then an explicit computation will show that the function

$$\phi\cdot f:h\in H\mapsto\int_A\phi(a)f(h(a,0))\,d\mu_A(a)\in\mathcal{B}$$

belongs to $\mathcal{B}[H]\cdot f$ and is a scalar multiple of $\chi_{w,L}$.

We give short but prompt explanation of this last computational claim. Given that the function ϕ is compactly supported and locally constant, one can write — up to some volume factor which is a mere power of p — the function $\phi\cdot f$ as a finite sum

$$\sum\phi(a_i)f(h(a_i,0))=\left(\sum\phi(a_i)(a_i,0)\right)\cdot f(h).$$

So $\phi \cdot f$ belongs to $\mathcal{B}[H] \cdot f$. For all $w' \in W$, the computation mentioned above reads

$$\phi \cdot f((w', 0)) = f((w', 0)) \times \frac{1}{\mathrm{vol}(L^\perp \cap A)} \int_{L^\perp \cap A} \psi^{\mathcal{B}}(\langle w' - w, a \rangle)\, d\mu_A(a).$$

A classical argument rewrites the last term as $1_{A+w+L}(w')$. Therefore $\phi \cdot f$ has support $A_H(w, 0)(L, 0)$, is right invariant under $(L, 0)$ and takes the value $f((w, 0))$ at $(w, 0)$. By unicity, one must have $\phi \cdot f = f((w, 0)) \cdot \chi_{w,L}$. Now $\phi_{w,L}$ exists because $\phi \cdot f \in \mathcal{B}[H] \cdot f$. □

Applying the previous lemma, we conclude that the reverse inclusion $I_f V_A^{\mathcal{B}} \subset \mathcal{B}[H] \cdot f$ holds. So the map $I \mapsto I V_A^{\mathcal{B}}$ is injective and surjective, that is being bijective.

(b) Let L be an open compact subgroup of W. Let $w \in W$. Consider the set of functions left $\psi_A^{\mathcal{B}}$-equivariant, supported on the double coset $A_H(w, 0)(L, 0)$ and right invariant under $(L, 0)$. Actually this space of functions is isomorphic to either $\mathcal{B}$ or 0 as a consequence of the formula for invariants vectors in compactly induced representations [Vignéras 1996, I.5.6]. Denote by $\chi_{w,L}$ the appropriate generator, meaning the function that takes value either 1 or 0 at $(w, 0)$. Fix representatives in W for the double coset $A_H \backslash H/(L, 0) \simeq A\backslash W/L = W/(A + L)$. Remark that the admissibility of $V_A^{\mathcal{B}}$ follows from the fact that, given some L, there are only finitely many representatives w giving rise to nonzero functions $\chi_{w,L}$. We are now proving this claim about functions $\chi_{w,L}$.

Suppose $\chi_{w,L}$ is nonzero. For all $l \in L \cap A$, one has

$$\begin{aligned} 1 = \chi_{w,L}((w, 0)) &= \chi_{w,L}((w, 0)(l, 0)) \\ &= \chi_{w,L}((l, \langle w, l \rangle)(w, 0)) \\ &= \psi^{\mathcal{B}}(\langle w, l \rangle)\psi_A^{\mathcal{B}}((l, 0)). \end{aligned}$$

Thus for all $l \in L \cap A$, the relation $\psi^{\mathcal{B}}(\langle w, l \rangle) = \psi_A^{\mathcal{B}}((-l, 0))$ must hold. It means that any two representatives w and w', giving rise to nonzero $\chi_{w,L}$ and $\chi_{w',L}$, must satisfy the relation $\psi^{\mathcal{B}}(\langle w - w', l \rangle) = 1$ for all $l \in L \cap A$ i.e., $w - w' \in (L \cap A)^\perp$. However

$$(L \cap A)^\perp = L^\perp + A^\perp = L^\perp + A.$$

As L is compact open, its orthogonal $L^\perp$ is compact open too because this holds for lattices in W. So the image of $L^\perp$ in the quotient $W/(A + L)$ is a finite set, which means the set of representatives w giving rise to nonzero $\chi_{w,L}$, when L is fixed, is finite.

To conclude, for any sufficiently small open compact subgroup L of W, the condition for smallness being $L \times \mathrm{Ker}(\psi^{\mathcal{B}})$ is a subgroup of H, one has

$$(V_A^{\mathcal{B}})^{L \times \mathrm{Ker}(\psi^{\mathcal{B}})} = \bigoplus_{\chi_{w,L} \neq 0} \mathcal{B} \cdot \chi_{w,L}$$

where the right-hand side sum is finite. So the smooth $\mathcal{B}[H]$-module $V_A^{\mathcal{B}}$ is admissible, and according to [Vignéras 1996, I.5.6 1)], it is equivalent to saying that $\mathrm{ind}_{A_H}^H(\psi_A^{\mathcal{B}}) = \mathrm{Ind}_{A_H}^H(\psi_A^{\mathcal{B}})$.

(c) As proved in the previous point, there exists a sufficiently small open compact subgroup L of W such that $K = L \times \mathrm{Ker}(\psi^{\mathcal{B}})$ is a subgroup of H and

$$(V_A^{\mathcal{B}})^K = \bigoplus_{\chi_{w,L} \neq 0} \mathcal{B} \cdot \chi_{w,L}$$

where the right-hand side sum is finite. In addition, there exists a nonzero $\chi_{w,L}$ for $w \in W$ if the condition "$\psi_A^{\mathcal{B}}(a) = 1$ for all $a \in A_H \cap (w,0)(L,0)(w,0)^{-1}$" is satisfied. Therefore, up to strengthening the sufficiently small condition, one may suppose that $(V_A^{\mathcal{B}})^K \neq 0$. Because every $\mathcal{B} \cdot \chi_{w,L}$ is isomorphic to $\mathcal{B}$, and the sum runs over functions with mutually disjoint supports, the $\mathcal{B}$-module $(V_A^{\mathcal{B}})^K$ is a free module of finite rank.

Thanks to point (a), the $\mathcal{B}[H]$-module $V_A^{\mathcal{B}}$ is generated by a single element $\chi_{w,L}$. Indeed, the ideal $I_{\chi_{w,L}} = \langle \chi_{w,L}(h) \mid h \in H \rangle$ satisfies $\mathcal{B}[H] \cdot \chi_{w,L} = I_{\chi_{w,L}} V_A^{\mathcal{B}}$ and contains 1 since $\chi_{w,L}((w,0)) = 1$. Thus the restriction to $(V_A^{\mathcal{B}})^K$ induces an injective morphism of $\mathcal{B}$-algebras

$$\xi : \mathrm{End}_{\mathcal{B}[H]}(V_A^{\mathcal{B}}) \hookrightarrow \mathrm{End}_{\mathcal{H}_{\mathcal{B}}(H,K)}((V_A^{\mathcal{B}})^K),$$

where $(V_A^{\mathcal{B}})^K$ is a module on the relative Hecke algebra $\mathcal{H}_{\mathcal{B}}(H, K)$ [Vignéras 1996, I.4.5].

The module $(V_A^{\mathcal{B}})^K$ being free over $\mathcal{B}$, write its basis $\mathfrak{B} = (\chi_{w,L})_w$. In this basis, the function $\phi_{w,L}$ defined above in the proof of Lemma 2.5 becomes the elementary projector E_w onto $\chi_{w,L}$ i.e., for all $w' \in \mathfrak{B}$ one has

$$\phi_{w,L} \cdot \chi_{w',L} = \chi_{w',L}((w,0)) \times \chi_{w,L} = \begin{cases} 0 & \text{if } w' \neq w; \\ \chi_{w,L} & \text{otherwise.} \end{cases}$$

Let now $\Phi \in \mathrm{End}_{\mathcal{B}[H]}(V_A^{\mathcal{B}})$. Then the image $\xi(\Phi)$ of Φ in $\mathrm{End}_{\mathcal{H}_{\mathcal{B}}(H,K)}((V_A^{\mathcal{B}})^K)$ commutes with E_w for all $w \in \mathfrak{B}$ as it commutes with the action of $\phi_{w,L}$. Because of this commutation relation between $\xi(\Phi)$ and E_w, there exists a scalar $\lambda_w \in \mathcal{B}$ such that $\xi(\Phi)(\chi_{w,L}) = \lambda_w \times \chi_{w,L}$. As any $\chi_{w,L}$ generates $V_A^{\mathcal{B}}$ as a $\mathcal{B}[H]$-module, it does generate $(V_A^{\mathcal{B}})^K$ as a $\mathcal{H}_{\mathcal{B}}(H, K)$-module. This last fact implies that all the λ_w are equal. Therefore there exists $\lambda \in \mathcal{B}$ such that $\xi(\Phi) = \lambda\, \mathrm{Id}_{(V_A^{\mathcal{B}})^K}$. So $\Phi = \lambda\, \mathrm{id}_{V_A^{\mathcal{B}}}$ because ξ is injective. □

The following can be easily deduced from Proposition 2.4 that has just been proved and the finiteness property of the compact induction:

Corollary 2.6. *Let $\mathcal{B}'$ be a $\mathcal{B}$-algebra given by the ring morphism $\varphi' : \mathcal{B} \to \mathcal{B}'$. Then the morphism φ' induces a canonical isomorphism of smooth $\mathcal{B}'[H]$-modules*

$$V_A^{\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{B}' \simeq V_A^{\mathcal{B}'}.$$

It is given on simple tensor elements by the map $f \otimes_{\mathcal{B}} b' \mapsto b' \times (\varphi' \circ f)$.

This result will allow to reduce any problem over an $\mathcal{A}$-algebra to a problem over $\mathcal{A}$, because applying the corollary leads to the canonical identification

$$V_A^{\mathcal{B}} \simeq V_A^{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B}.$$

Furthermore, one can consider $\mathcal{A}$-algebras that are not integral domains. For instance, if $\mathcal{B} = \prod_i \mathcal{B}_i$ is a finite product of $\mathcal{A}$-algebras $(\mathcal{B}_i)$, then

$$V_A^{\mathcal{B}} \simeq \bigoplus_i V_A^{\mathcal{B}_i}.$$

2C. *Changing models from $V_{A_1}^{\mathcal{B}}$ to $V_{A_2}^{\mathcal{B}}$.* Let A_1 and A_2 be two self-dual subgroups of W. Let $\psi_{A_1}^{\mathcal{A}}$ be a character that extends $\psi^{\mathcal{B}}$ to $A_{1,H}$ as in Lemma 2.3. Similarly, fix an extension $\psi_{A_2}^{\mathcal{A}}$ of $\psi^{\mathcal{B}}$ with respect to $A_{2,H}$. Once again, set $\psi_{A_1}^{\mathcal{B}} = \varphi \circ \psi_{A_1}^{\mathcal{A}}$ and $\psi_{A_2}^{\mathcal{B}} = \varphi \circ \psi_{A_2}^{\mathcal{A}}$, which are both smooth and nontrivial characters. Suppose $\omega \in W$ satisfies the condition

$$\psi_{A_1}^{\mathcal{B}}((a,0))\psi_{A_2}^{\mathcal{B}}((a,0))^{-1} = \psi^{\mathcal{B}}(\langle a, \omega \rangle) \quad \text{for all } a \in A_1 \cap A_2.$$

Note that such an ω always exist as the left-hand side defines a character of $A_1 \cap A_2$.

Let μ be a Haar measure with values in $\mathcal{B}$ of the quotient $A_1 \cap A_2 \backslash A_2$. Define

$$I_\mu = \langle \mu(K) \mid K \text{ open compact subgroup} \rangle$$

the ideal of $\mathcal{B}$ generated by the various values taken by μ on the open compact subgroups of $A_1 \cap A_2 \backslash A_2$. By unicity of the Haar measure, the ideal I_μ is principal and generated by any $\mu(K)$ as long as the pro-order of K is invertible in $\mathcal{B}$. The measure is said to be invertible if $I_\mu = \mathcal{B}$. Of course, every normalised Haar measure, that is a measure taking the value 1 on a compact open subgroup, is invertible. For μ to be invertible, it is necessary and sufficient that there exists a compact open subgroup whose measure is a unit in $\mathcal{B}$ i.e., μ is a unit multiple of a normalised Haar measure.

Proposition 2.7. *The map $I_{A_1,A_2,\mu,\omega}$ associating to $f \in V_{A_1}^{\mathcal{B}}$ the function*

$$I_{A_1,A_2,\mu,\omega} f : h \mapsto \int_{A_{1,H} \cap A_{2,H} \backslash A_{2,H}} \psi_{A_2}^{\mathcal{B}}(a)^{-1} f((\omega,0)ah)\, d\mu(a)$$

is a morphism of smooth $\mathcal{B}[H]$-modules from $V^{\mathcal{B}}_{A_1}$ to $V^{\mathcal{B}}_{A_2}$. Its image is $I_\mu V^{\mathcal{B}}_{A_2}$ and, as a result, $I_{A_1,A_2,\mu,\omega}$ is an isomorphism if and only if μ is an invertible measure. In addition, any invertible measure μ induces an isomorphism of $\mathcal{B}$-modules

$$\mathrm{Hom}_{\mathcal{B}[H]}(V^{\mathcal{B}}_{A_1}, V^{\mathcal{B}}_{A_2}) = \{\lambda I_{A_1,A_2,\mu,\omega} \mid \lambda \in \mathcal{B}\} \simeq \mathcal{B}.$$

Proof. On the one hand, the function $I_{A_1,A_2,\mu,\omega} f$ is well defined. Indeed for any $h \in H$, the function $a \in A_{2,H} \mapsto \psi^{\mathcal{B}}_{A_2}(a)^{-1} f((\omega,0)ah) \in \mathcal{B}$ is $(A_{1_H} \cap A_{2,H})$-left invariant and locally constant, so one can consider it is a function on $A_{1,H} \cap A_{2,H} \backslash A_{2,H} = A_1 \cap A_2 \backslash A_2$. The function $a \in A_{2,H} \mapsto f((\omega,0)ah) \in \mathcal{B}$ is compactly supported modulo $A_{1,H} \cap A_{2,H}$ because, as in [Trias 2020, Section 2.3], the sum $A_1 + A_2$ is a closed subgroup of H. Finally, a change of variables implies that $I_{A_1,A_2,\mu,\omega} f$ is left $\psi^{\mathcal{B}}_{A_2}$-equivariant. The map $I_{A_1,A_2,\mu,\omega}$ is clearly $\mathcal{B}$-linear and H-equivariant so that $I_{A_1,A_2,\mu,\omega} \in \mathrm{Hom}_{\mathcal{B}[H]}(V^{\mathcal{B}}_{A_1}, V^{\mathcal{B}}_{A_2})$.

As a result of point (a) from Proposition 2.4, the image of $I_{A_1,A_2,\mu,\omega}$ must be of the form $I V^{\mathcal{B}}_{A_2}$ for some ideal I in $\mathcal{B}$. Actually, we proved a sharper results in the proof of point (a) showing that

$$I = \{I_{A_1,A_2,\mu,\omega} f(h) \mid f \in V^{\mathcal{B}}_{A_1}, h \in H\}.$$

If μ is chosen to be invertible, then for any other measure μ', there exists $\lambda \in \mathcal{B}$ generating $I_{\mu'}$ and such that the image of $I_{A_1,A_2,\mu',\omega}$ is $I_{\mu'} I V^{\mathcal{B}}_{A_2} = \lambda I V^{\mathcal{B}}_{A_2}$. It reduces to consider the morphism $I_{A_1,A_2,\mu,\omega}$ when μ is invertible. In this case, we show below that the morphism is surjective and as injective.

Suppose μ is invertible. As in the proof of Proposition 2.4, choose a sufficiently small open compact subgroup L of W such that there exists a nonzero function $\chi_{\omega,L}$ supported on $A_{1,H}(\omega,0)(L,0)$, right invariant under $(L,0)$ and taking the value 1 at $(\omega,0)$. One may as well suppose that $\psi_{A_2}((l,0)) = 1$ for all $l \in L$, by choosing an even smaller L if needed. Then the formula for $\chi_{\omega,L}$ at $h = (0,0)$ reads

$$\begin{aligned} I_{A_1,A_2,\mu,\omega}\chi_{\omega,L}((0,0)) &= \int_{L\cap A_1\cap A_2\backslash L\cap A_2} \psi_{A_2}((l,0))^{-1}\chi_{\omega,L}((\omega,0)(l,0))\,d\mu(l) \\ &= \int_{L\cap A_1\cap A_2\backslash L\cap A_2} \chi_{\omega,L}((\omega,0))\,d\mu(l) \\ &= \mathrm{vol}(L\cap A_1\cap A_2\backslash L\cap A_2). \end{aligned}$$

The group $L \cap A_1 \cap A_2 \backslash L \cap A_2$ has pro-order a power of p, so its volume for the invertible measure μ is a unit i.e., $I_{A_1,A_2,\mu,\omega}\chi_{\omega,L}((0,0)) \in \mathcal{B}^\times$.

Therefore the previous unit $I_{A_1,A_2,\mu,\omega}\chi_{\omega,L}((0,0))$ belongs to I i.e., $I = \mathcal{B} = I_\mu$. It follows that the morphism $I_{A_1,A_2,\mu,\omega}$ is surjective. It is injective as well. Indeed, its kernel is of the form $I' V^{\mathcal{B}}_{A_1}$ for some ideal I' of $\mathcal{B}$, and for all $i' \in I'$, the function $i'\chi_{\omega,L}$ belongs to the kernel. However the function $I_{A_1,A_2,\mu,\omega}(i'\chi_{\omega,L}) =$

$i' I_{A_1,A_2,\mu,\omega}\chi_{\omega,L}$ takes the value i' at $(0,0)$ and is the zero function. So $i'=0$ and I' is the zero ideal of $\mathcal{B}$. $\square$

Consider the scalar extension functor

$$V \in \mathrm{Rep}_{\mathcal{A}}(H) \mapsto V \otimes_{\mathcal{A}} \mathcal{B} \in \mathrm{Rep}_{\mathcal{B}}(H)$$

and denote $\phi_{\mathcal{B}} : \mathrm{Hom}_{\mathcal{A}[H]}(V_{A_1}^{\mathcal{A}}, V_{A_2}^{\mathcal{A}}) \to \mathrm{Hom}_{\mathcal{B}[H]}(V_{A_1}^{\mathcal{B}}, V_{A_2}^{\mathcal{B}})$ the map that is induced by functoriality.

In particular for all $f \in \mathrm{Hom}_{\mathcal{A}[H]}(V_{A_1}^{\mathcal{A}}, V_{A_2}^{\mathcal{A}})$, the following diagram, where the vertical arrows are given by the canonical $V_A^{\mathcal{A}} \to V_A^{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B}$ of Corollary 2.6, is commutative:

$$\begin{tikzcd} V_{A_1}^{\mathcal{A}} \arrow[r, "f"] \arrow[d] & V_{A_2}^{\mathcal{A}} \arrow[d] \\ V_{A_1}^{\mathcal{B}} \arrow[r, "\phi_{\mathcal{B}}(f)"] & V_{A_2}^{\mathcal{B}} \end{tikzcd}$$

For $\omega \in W$, observe now that the two following conditions are equivalent:

- $\psi_{A_1}^{\mathcal{A}}((a,0))\psi_{A_2}^{\mathcal{A}}((a,0))^{-1} = \psi^{\mathcal{A}}(\langle a,\omega\rangle)$ for all $a \in A_1 \cap A_2$.
- $\psi_{A_1}^{\mathcal{B}}((a,0))\psi_{A_2}^{\mathcal{B}}((a,0))^{-1} = \psi^{\mathcal{B}}(\langle a,\omega\rangle)$ for all $a \in A_1 \cap A_2$.

Fix $\omega \in W$ satisfying one of the previous two. The corollary below is quite immediate.

Corollary 2.8. *Let $\mu^{\mathcal{A}}$ be an invertible Haar measure of $A_1 \cap A_2 \backslash A_2$ with values in $\mathcal{A}$. Set $\mu^{\mathcal{B}} = \varphi \circ \mu^{\mathcal{A}}$. This latter measure is an invertible $\mathcal{B}$-valued measure. Then for all $M \in \mathrm{Hom}_{\mathcal{B}[H]}(V_{A_1}^{\mathcal{B}}, V_{A_2}^{\mathcal{B}})$, there exists $\lambda \in \mathcal{B}$ such that*

$$M = \lambda \times I_{A_1,A_2,\mu^{\mathcal{B}},\omega} = \lambda \times \phi_{\mathcal{B}}(I_{A_1,A_2,\mu^{\mathcal{A}},\omega}).$$

3. Weil representations over $\mathcal{A}$-algebras

Let $\mathcal{B}$ be an $\mathcal{A}$-algebra. Let A a self-dual subgroup of W and $V_A^{\mathcal{B}} = \mathrm{ind}_{A_H}^{H}(\psi_A^{\mathcal{B}})$ the smooth $\mathcal{B}[H]$-module built in Section 2B, where $\psi_A^{\mathcal{B}}$ is an extension of $\psi^{\mathcal{B}}$ in the way of Lemma 2.3. The symplectic group $\mathrm{Sp}(W)$ is naturally acting on H through the first coordinate, that is

$$g \cdot (w,t) = (gw,t)$$

for $g \in \mathrm{Sp}(W)$ and $(w,t) \in H$. Of course, self-dual subgroups are preserved under this action, that is gA is self-dual for all $g \in \mathrm{Sp}(W)$.

In this section g always denotes an element of $\mathrm{Sp}(W)$. For $f \in V_A^{\mathcal{B}}$, the function

$$I_g f : h \in H \mapsto f(g^{-1}\cdot h) \in \mathcal{B}$$

belongs to $V_{gA}^{\mathcal{B}} = \mathrm{ind}_{(gA)_H}^{H}(\psi_{gA}^{\mathcal{B}})$ where $\psi_{gA}^{\mathcal{B}}(g \cdot a) = \psi_A^{\mathcal{B}}(a)$ for all $a \in A_H$. It is important to stress that $V_{gA}^{\mathcal{B}}$ depends on g, because even if $gA = A$, one may have that $\psi_{gA}^{\mathcal{B}} \neq \psi_A^{\mathcal{B}}$ as characters of A_H. Another caution is related to the map

$$I_g : f \in V_A^{\mathcal{B}} \mapsto I_g f \in V_{gA}^{\mathcal{B}}$$

that is not a morphism of $\mathcal{B}[H]$-modules. Indeed, for $h_0 \in H$, one has

$$I_g(h_0 \cdot f) = (g \cdot h_0) \cdot I_g f$$

whereas $h_0 \cdot (I_g f) = I_g((g^{-1} \cdot h_0) \cdot f)$.

Recall from Section 2C that there exists $\omega_g \in W$ such that the condition

$$\psi_{gA}^{\mathcal{B}}((a,0))\psi_A^{\mathcal{B}}((a,0))^{-1} = \psi^{\mathcal{B}}(\langle a, \omega_g \rangle)$$

holds for all $a \in gA \cap A$. Then for any Haar measure μ of $gA \cap A \backslash A$, one can compose the following morphisms of $\mathcal{B}$-modules

$$V_A^{\mathcal{B}} \xrightarrow{I_g} V_{gA}^{\mathcal{B}} \xrightarrow{I_{gA,A,\mu,\omega_g}} V_A^{\mathcal{B}}.$$

Therefore $I_{gA,A,\mu,\omega_g} \circ I_g \in \mathrm{End}_{\mathcal{B}}(V_A^{\mathcal{B}})$ is uniquely defined up to a scalar of $\mathcal{B}$, because the morphism I_{gA,A,μ,ω_g} is, thanks to Proposition 2.7.

Consider now the smooth $\mathcal{B}[H]$-module $(\rho_d, \mathrm{Ind}_F^H(\psi^{\mathcal{B}}))$ where F is identified with the centre of H. All the $\mathcal{B}[H]$-modules $V_A^{\mathcal{B}}$ naturally embed in the latter because the restriction of $\psi_A^{\mathcal{B}}$ to F is $\psi^{\mathcal{B}}$. Under this canonical identification for $V_A^{\mathcal{B}}$, one has

$$I_{gA,A,\mu,\omega_g} \circ I_g \circ \rho_d(h) = \rho_d(g \cdot h) \circ I_{gA,A,\mu,\omega_g} \circ I_g.$$

In other words $I_{gA,A,\mu,\omega_g} \circ I_g \in \mathrm{Hom}_{\mathcal{B}[H]}((\rho_d, V_A^{\mathcal{B}}), (\rho_d^g, V_A^{\mathcal{B}}))$ where $\rho_d^g : h \mapsto \rho_d(g \cdot h)$.

Again in Section 2C, invertible Haar measures are defined as unit multiples of normalised Haar measures. These exactly are the measures that can take unit values on compact open subgroups. As the linear map I_g is invertible, one easily deduces from Proposition 2.7 that the previous endomorphism is invertible:

Lemma 3.1. *If μ is invertible, then $I_{gA,A,\mu,\omega_g} \circ I_g \in \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$.*

As a result of the lemma, the image of the set $\{I_{gA,A,\mu,\omega_g} \circ I_g \mid \mu \text{ invertible}\}$ through the quotient map

$$\mathrm{RED} : \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \to \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})/\mathcal{B}^{\times} = \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$$

is well defined. As already mentioned the map $I_{gA,A,\mu,\omega_g} \circ I_g$ is unique up to a scalar, hence this image consists in a singleton; denote by M_g the single element it contains. Remark that M_g does not depend on the choice of ω_g because $\mathrm{Hom}_{\mathcal{B}[H]}(V_{gA}^{\mathcal{B}}, V_A^{\mathcal{B}}) \simeq \mathcal{B}$ once again by Proposition 2.7, and the set of invertible elements are those in $\mathcal{B}^\times$, which does correspond to the choice of an invertible Haar measure.

The proposition below allows to build Weil representations with coefficients in $\mathcal{B}$.

Proposition 3.2. *The map $\sigma_{\mathcal{B}} : g \in \mathrm{Sp}(W) \mapsto M_g \in \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ is a group morphism and defines a projective representation $V_A^{\mathcal{B}}$ of $\mathrm{Sp}(W)$. Using the fibre product construction, it lifts to a representation $\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}$ of a central extension of $\mathrm{Sp}(W)$ by $\mathcal{B}^\times$ in the following way:*

$$\begin{tikzcd}
\widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W) \arrow[r, "\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}"] \arrow[d, "p_{\mathcal{B}}"'] & \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \arrow[d, "\mathrm{RED}"] \\
\mathrm{Sp}(W) \arrow[r, "\sigma_{\mathcal{B}}"] & \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})
\end{tikzcd}$$

where $\widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W) = \mathrm{Sp}(W) \times_{\mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})} \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ is the fibre product defined by the group morphisms $\sigma_{\mathcal{B}}$ and RED, *together with the projection maps denoted $p_{\mathcal{B}}$ and $\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}$.*

Proof. The only point that needs explanation is the claim about $\sigma_{\mathcal{B}}$ being a group morphism. Let g and g' be two elements in $\mathrm{Sp}(W)$. By definition, there exists an invertible measure μ_g on $gA \cap A \backslash A$ and an element $\omega_g \in W$ such that

$$\mathrm{RED}(I_{gA,A,\mu_g,\omega_g} \circ I_g) = M_g.$$

Respectively, one can write the same type of relation for $M_{g'}$ with some $\mu_{g'}$ and $\omega_{g'}$.

An explicit computation of the composed map $I_g \circ I_{g'A,A,\mu_{g'},\omega_{g'}}$ gives the existence of an invertible measure μ on $gg'A \cap gA \backslash gA$ and an element $\omega \in W$ such that the commutation relation

$$I_g \circ I_{g'A,A,\mu_{g'},\omega_{g'}} = I_{gg'A,gA,\mu,\omega} \circ I_g$$

holds. In addition, the morphism

$$I_{gA,A,\mu_g,\omega} \circ I_{gg'A,gA,\mu,\omega} \in \mathrm{Hom}_{\mathcal{B}[H]}(V^{\mathcal{B}}_{gg'A}, \mathcal{V}_A^{\mathcal{B}})$$

is invertible because each one of the two is. Therefore Proposition 2.7 asserts the existence of an invertible measure $\mu_{gg'}$ on $A \cap gg'A \backslash gg'A$ and an element $\omega_{gg'} \in W$ such that

$$I_{gA,A,\mu_g,\omega} \circ I_{gg'A,gA,\mu,\omega} = I_{gA,A,\mu_{gg'},\omega_{gg'}}.$$

The claim hence follows by using the previous two relations and applying RED to

$$(I_{gA,A,\mu_g,\omega_g} \circ I_g) \circ (I_{g'A,A,\mu_{g'},\omega_{g'}} \circ I_{g'}). \qquad \square$$

Remark 3.3. Actually this fibre product makes sense in the category of topological groups in the following setting. Let $\mathcal{B}$ and $V_A^{\mathcal{B}}$ be endowed with the discrete topology. Then the compact-open topology on $\mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ is generated by the prebasis of open sets $S_{s,s'} = \{g \in \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \mid gs = s'\}$ for s and s' in $V_A^{\mathcal{B}}$. Similarly to [Trias 2020, Proposition 3.5], one can prove RED and $\sigma_{\mathcal{B}}$ are morphisms of topological groups. As a result of the continuity, the fibre product is a locally profinite group for the product topology and the representation $\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}$ is smooth. However, there is an interesting alternative way to prove it and that is developed in the next section. It illustrates the philosophy: any problem related to an $\mathcal{A}$-algebra $\mathcal{B}$ may be brought back to one directly involving $\mathcal{A}$.

Denote by $\phi_{\mathcal{B}} : \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}}) \to \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ the group morphism induced by the extension of scalars and the canonical identification $V_A^{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B} \simeq V_A^{\mathcal{B}}$ coming from Corollary 2.6.

Theorem 3.4. *The group morphism $\phi_{\mathcal{B}}$ induces a morphism of central extensions*

$$\widetilde{\phi_{\mathcal{B}}} : (g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \mapsto (g, \phi_{\mathcal{B}}(M)) \in \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W).$$

The image of $\widetilde{\phi_{\mathcal{B}}}$ is a central extension of $\mathrm{Sp}(W)$ by $\varphi(\mathcal{A})^\times$ where φ is the structure morphism $\varphi : \mathcal{A} \to \mathcal{B}$. Furthermore, the following diagram commutes:

$$\begin{tikzcd}
\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \arrow[r, "\omega_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}"] \arrow[d, "\widetilde{\phi_{\mathcal{B}}}"] & \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}}) \arrow[d, "\phi_{\mathcal{B}}"] \\
\widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W) \arrow[r, "\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}"] & \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})
\end{tikzcd}$$

Proof. By definition $(g, M) \in \mathrm{Sp}(W) \times \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}})$ belongs to $\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)$ if there exists an invertible Haar measure μ on $gA \cap A \backslash A$ with values in $\mathcal{A}$ and an element ω such that $M = I_{gA,A,\mu,\omega} \circ I_g$. Set $\mu^{\mathcal{B}} = \varphi \circ \mu$. Using the compatibility of Corollary 2.8, the equality $\phi_{\mathcal{B}}(I_{gA,A,\mu,\omega}) = I_{gA,A,\mu^{\mathcal{B}},\omega}$ holds and defines an isomorphism in $\mathrm{Hom}_{\mathcal{B}[H]}(V_{gA}^{\mathcal{B}}, V_A^{\mathcal{B}})$. Hence

$$\phi_{\mathcal{B}}(M) = I_{gA,A,\mu^{\mathcal{B}},\omega} \circ I_g$$

with $\mu^{\mathcal{B}}$ invertible, that is $(g, \phi_{\mathcal{B}}(M)) \in \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$.

The map $\widetilde{\phi_{\mathcal{B}}}$ thus defined clearly is a morphism of central extensions. In addition, an element (g, M) belongs to its kernel if and only if $g = \mathrm{Id}_W$ and $\phi_{\mathcal{B}}(M) = \mathrm{Id}_{V_A^{\mathcal{B}}}$. However

$$\{M \in \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}}) \mid (\mathrm{Id}_W, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)\} = \{\lambda\, \mathrm{Id}_{V_A^{\mathcal{A}}} \mid \lambda \in \mathcal{A}^\times\}.$$

Indeed M must be of the form $I_{gA,A,\mu,\omega} \circ I_g = I_{A,A,\mu,0} = \mu(\{0\}) \times \mathrm{Id}_{V_A^{\mathcal{A}}}$ where μ is an invertible measure of the singleton $\{0\}$, so there exists $\lambda \in \mathcal{B}^\times$ such that $M = \lambda\, \mathrm{Id}_{V_A^{\mathcal{A}}}$. Since $\phi_{\mathcal{B}}(\lambda\, \mathrm{Id}_{V_A^{\mathcal{A}}}) = \varphi(\lambda)\, \mathrm{Id}_{V_A^{\mathcal{B}}}$, the group $\{(\mathrm{Id}_W, \lambda\, \mathrm{Id}_{V_A^{\mathcal{A}}}) \mid \lambda \in \mathrm{Ker}(\varphi)\} \simeq \mathrm{Ker}(\varphi)$ is the kernel sought. The assertion on the image follows from the form of this kernel. □

Because of the previous compatibility, many problems over $\mathcal{B}$ reduce to those over the minimal ring $\mathcal{A}$. The corollary to the proposition below illustrates this philosophy.

Proposition 3.5. *Let A and A' be two self-dual subgroups of W. Let $\Phi_{A,A'}$ be an isomorphism in $\mathrm{Hom}_{\mathcal{A}[H]}(V_A^{\mathcal{A}}, V_{A'}^{\mathcal{A}})$. Then $\Phi_{A,A'}$ induces an isomorphism of central extensions*

$$(g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \mapsto (g, \Phi_{A,A'} M \Phi_{A,A'}^{-1}) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_{A'}^{\mathcal{A}}}(W)$$

compatible with the projections defining the fibre products. In particular, the equivalence class of the representation $\omega_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}$ does not on depend A in the sense that

$$\Phi_{A,A'} \circ \omega_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}((g, M)) \circ \Phi_{A,A'}^{-1} = \omega_{\psi^{\mathcal{A}}, V_{A'}^{\mathcal{A}}}((g, \Phi_{A,A'} M \Phi_{A,A'}^{-1}))$$

for all $(g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)$.

Proof. The existence of an isomorphism in $\mathrm{Hom}_{\mathcal{A}[H]}(V_A^{\mathcal{A}}, V_{A'}^{\mathcal{A}})$ is a consequence of Proposition 2.7. One can consider for example any $I_{A,A',\mu,\omega}$ as long as μ is invertible. The fact that $\Phi_{A,A'}$ induces an isomorphism of central extensions is quite clear when writing down the relations because $\Phi_{A,A'}$ is an isomorphism of $\mathcal{A}[H]$-modules. □

From Theorem 3.4 and the proposition above, one can deduce the exact same result for coefficients in any $\mathcal{A}$-algebra $\mathcal{B}$. Indeed, applying $\phi_{\mathcal{B}}$ to the last relation yields:

Corollary 3.6. *The equivalence class of $\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}$ does not depend on A, in the sense that for any other self-dual subgroup A' of W, there exists an isomorphism $\Phi'_{A,A'}$ in* $\mathrm{Hom}_{\mathcal{B}[H]}(V_A^{\mathcal{B}}, V_{A'}^{\mathcal{B}})$ *— one can take $\phi_{\mathcal{B}}(\Phi_{A,A'})$ for example — such that*

$$\Phi'_{A,A'} \circ \omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}((g, M)) \circ (\Phi'_{A,A'})^{-1} = \omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}((g, \Phi'_{A,A'} M (\Phi'_{A,A'})^{-1}))$$

for all $(g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_{A'}^{\mathcal{B}}}(W)$.

4. The metaplectic group over $\mathcal{A}$

The notations are those of Section 3. To quickly recall the context: let $\mathcal{B}$ be an $\mathcal{A}$-algebra, let A be a self-dual subgroup of W and $V_A^{\mathcal{B}} = \mathrm{ind}_{A_H}^H(\psi_A^{\mathcal{B}})$ be the smooth

$\mathcal{B}[H]$-module built in Section 2B, where $\psi_A^{\mathcal{B}}$ is an extension of $\psi^{\mathcal{B}}$ in the way of Lemma 2.3.

In Section 3, we constructed a projective representation $\sigma_{\mathcal{B}} : \mathrm{Sp}(W) \to \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ of the symplectic group and, in Proposition 3.2, we lifted it to a representation $(\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}, V_A^{\mathcal{B}})$ of a central extension of $\mathrm{Sp}(W)$ by $\mathcal{B}^\times$, namely

$$\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}} : \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W) \to \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}).$$

Recall that the group on the left-hand side is the fibre product in the category of groups of the group morphisms $\sigma_{\mathcal{B}} : \mathrm{Sp}(W) \to \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ and $\mathrm{RED} : \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \to \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$, together with the projection maps $p_{\mathcal{B}}$ and $\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}$. As a result of this construction, it is a subgroup of $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$. In particular, these constructions make sense over $\mathcal{A}$ itself, and Theorem 3.4 completes the picture relating the constructions over $\mathcal{A}$ and over any $\mathcal{A}$-algebra $\mathcal{B}$, yielding a morphism of central extensions

$$\widetilde{\phi_{\mathcal{B}}} : \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \to \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$$

compatible with the respective projection maps.

4A. ***A bit of topology.*** This section will shed some light on Remark 3.3 by bringing topology into the construction of Proposition 3.2. Endow $\mathcal{B}$ and $V_A^{\mathcal{B}}$ with the discrete topology. Then the open-compact topology on $\mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ is generated by the prebasis $S_{s,s'} = \{M \in \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \mid Ms = s'\}$ for s and s' running through $V_A^{\mathcal{B}}$.

The group $\mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ inherits the quotient topology, which is the finest making the quotient map $\mathrm{RED} : \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}}) \to \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ continuous. Recall from Theorem 3.4 that the projective representation $\sigma_{\mathcal{B}} : \mathrm{Sp}(W) \to \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ was defined in terms of the action of $\mathrm{Sp}(W)$ on H.

The complex case. The best-known feature comes when $\mathcal{B}$ is the field of complex numbers. Endowing $\mathbb{C}$ with a structure of $\mathcal{A}$-algebra amounts to fixing an embedding $\varphi : \mathcal{A} \to \mathbb{C}$. Observe that all such embeddings have the same image in $\mathbb{C}$, because $\mathcal{K}/\mathbb{Q}$ is a Galois extension. In particular, the image of the map $\mathcal{A}^\times \to \mathbb{C}^\times$ induced by φ does not depend on the choice of φ.

So when $\mathcal{B} = \mathbb{C}$ and φ is fixed, the representation $V_A^{\mathbb{C}} \in \mathrm{Rep}_{\mathbb{C}}(H)$ is irreducible as an application of Stone–von Neumann's theorem [Mœglin et al. 1987, Chapter 2, Theorem I.2] and

$$\omega_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}} \text{ is the Weil representation of the metaplectic group } \widetilde{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W).$$

The complex theory asserts that the Weil representation is smooth and the metaplectic group is a natural topological subgroup of $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathbb{C}}(V_A^{\mathbb{C}})$. To be more precise, the metaplectic group is a locally profinite group. Regarding the smoothness condition, this is equivalent to saying that the map $\omega_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}$ is continuous.

These topological properties are consequences of the continuity of the map $\sigma_{\mathbb{C}}$, which really is the cornerstone of the theory; and the metaplectic group inherits a natural topology as the fibre product in the category of topological groups of the continuous group morphisms RED and $\sigma_{\mathbb{C}}$.

Over $\mathcal{A}$. By analogy, one calls $\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)$ the metaplectic group over $\mathcal{A}$. Referring to Theorem 3.4, it is a subgroup of the metaplectic group because the group morphism

$$\widetilde{\phi_{\mathbb{C}}} : (g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \to (g, \phi_{\mathbb{C}}(M)) \in \widetilde{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W)$$

is injective.

Lemma 4.1. *The map $\phi_{\mathbb{C}} : M \in \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}}) \to \phi_{\mathbb{C}}(M) \in \mathrm{GL}_{\mathbb{C}}(V_A^{\mathbb{C}})$, coming from the scalar extension to $\mathbb{C}$, is continuous and defines an homeomorphism onto its image.*

Proof. The image of $\phi_{\mathbb{C}}$ is endowed with the subspace topology from $\mathrm{GL}_{\mathbb{C}}(V_A^{\mathbb{C}})$. The map $\phi_{\mathbb{C}}$ is continuous and injective, so it defines a bijection to its image, say $G_{\mathcal{A}}$. Denote $\phi'_{\mathbb{C}} : G_{\mathcal{A}} \to \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}})$ the inverse map. Then for all s and s' in $V_A^{\mathcal{A}}$, one has

$$\begin{aligned}(\phi'_{\mathbb{C}})^{-1}(S_{s,s'}) &= \{\phi_{\mathbb{C}}(M) \mid M \in \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}}) \text{ and } \phi_{\mathbb{C}}(M)(s \otimes_{\mathbb{C}} 1) = s' \otimes_{\mathbb{C}} 1\} \\ &= G_{\mathcal{A}} \cap S_{s \otimes_{\mathbb{C}} 1, s' \otimes_{\mathbb{C}} 1}\end{aligned}$$

that is the trace of an open set. So $(\phi'_{\mathbb{C}})^{-1}(S_{s,s'})$ is open in $G_{\mathcal{A}}$ and $\phi'_{\mathbb{C}}$ is continuous. □

Of course, the embedding $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}}) \to \mathrm{Sp}(W) \times \mathrm{GL}_{\mathbb{C}}(V_A^{\mathbb{C}})$ induced by $\phi_{\mathbb{C}}$ is an homeomorphism onto its image as well. As a result of the lemma, the subspace topology on $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}})$, inherited from that of $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathbb{C}}(V_A^{\mathbb{C}})$ using the previous embedding, coincides with the usual product topology. Restricting this morphism to the metaplectic group over $\mathcal{A}$, which is a subgroup of $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}})$, exactly yields $\widetilde{\phi_{\mathbb{C}}}$. Because of the homeomorphism property, one can identify the metaplectic group over $\mathcal{A}$ and its image under $\widetilde{\phi_{\mathbb{C}}}$, resulting in:

Corollary 4.2. *The group $\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)$ is a topological subgroup of $\widetilde{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W)$, the inclusion being canonically given by $\widetilde{\phi_{\mathbb{C}}}$. In addition $\widetilde{\phi_{\mathbb{C}}}$ is an open embedding.*

Proof. The fact that it is a topological subgroup follows from Lemma 4.1 and the subsequent discussion. The map $\widetilde{\phi_{\mathbb{C}}}$ is open because its image is open in the metaplectic group. Indeed the first projection of the fibre product yields an exact sequence

$$1 \to \mathbb{C}^\times \to \widetilde{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W) \to \mathrm{Sp}(W) \to 1.$$

Because $\mathcal{K}/\mathbb{Q}$ is a Galois extension, the image $\varphi(\mathcal{A}^\times)$ of $\mathcal{A}^\times$ does not depend on φ and always contains $\{\pm 1\}$. As a result, the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \widetilde{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W) & \longrightarrow & \mathrm{Sp}(W) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \widetilde{\phi_{\mathbb{C}}}} & & \uparrow{\scriptstyle \mathrm{Id}_{\mathrm{Sp}(W)}} & & \\ 1 & \longrightarrow & \mathcal{A}^\times & \longrightarrow & \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) & \longrightarrow & \mathrm{Sp}(W) & \longrightarrow & 1 \end{array}$$

and the group $\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)$ contains the reduced metaplectic group $\widehat{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W)$, that is the derived group of the metaplectic group.

When F is local nonarchimedean, this is the unique subgroup of the metaplectic group fitting into the exact sequence

$$1 \to \{\pm 1\} \to \widehat{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W) \to \mathrm{Sp}(W) \to 1.$$

Furthermore this reduced metaplectic group is open in the metaplectic group, so the claim follows because the metaplectic group over $\mathcal{A}$ contains it. When F is finite, the topology can just be ignored as these groups are finite and have discrete topology. $\square$

As above, denote by $\widehat{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W)$ the derived group of $\widetilde{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W)$. When F is finite, this group is the derived group of $\mathrm{Sp}(W)$. Except in the exceptional case $F = \mathbb{F}_3$ and $\dim(W) = 2$, the symplectic group is perfect i.e., equal to its own derived subgroup. When F is local archimedean it is the so-called reduced metaplectic group, which is a nontrivial extension of $\mathrm{Sp}(W)$ by $\{\pm 1\}$. Actually there exists a unique such (open) subgroup in the metaplectic group. Regardless of what F may be, we use brackets to define the derived group

$$\widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) = [\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W), \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)].$$

Recall that $\widetilde{\phi_{\mathbb{C}}}$ canonically identifies $\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)$ with its image in $\widetilde{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W)$. It also induces, by restriction, a map between the respective derived groups.

Proposition 4.3. *One has the following properties*:

(a) *The map $\sigma_{\mathcal{A}}$ is continuous and $\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)$ is the fibre product in the category of topological groups of the continuous morphisms* RED *and $\sigma_{\mathcal{A}}$.*

(b) *The representation $\omega_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}} : \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \to \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}})$ is smooth as this group morphism is the second projection of the fibre product.*

(c) *The group $\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)$ is open in $\widetilde{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}}, V_A^{\mathbb{C}}}(W)$ and therefore the metaplectic group over $\mathcal{A}$ is locally profinite.*

(d) *The map* $\widetilde{\phi_{\mathbb{C}}}$ *restricts to an isomorphism* $\widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}}(W) \simeq \widehat{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}},V_A^{\mathbb{C}}}(W)$ *and when*:

(i) F *is finite, it is the symplectic group except when* $F = \mathbb{F}_3$ *and* $\dim(W) = 2$.

(ii) F *is local nonarchimedean, it is the reduced metaplectic group.*

Proof. (a) The map $\sigma_{\mathcal{A}}$ is continuous, because $\sigma_{\mathbb{C}}$ itself is, and one has

$$\sigma_{\mathcal{A}} = \overline{\phi_{\mathbb{C}}} \circ \sigma_{\mathbb{C}}$$

where $\overline{\phi_{\mathbb{C}}} : \mathrm{PGL}_{\mathcal{A}}(V_A^{\mathcal{A}}) \to \mathrm{PGL}_{\mathbb{C}}(V_A^{\mathbb{C}})$ is the continuous group morphism defined from $\phi_{\mathbb{C}}$ by passing to the quotient. The fibre product of $\sigma_{\mathcal{A}}$ and RED in the category of topological groups defines a topological subgroup of $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathcal{A}}(V_A^{\mathcal{A}})$. In particular, this fibre product is, as a group, the metaplectic group over $\mathcal{A}$.

(b) The projection maps are continuous by definition of the fibre product.

(c) As a direct consequence of $\widetilde{\phi_{\mathbb{C}}}$ being an open embedding, the group $\widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}}(W)$ is an open subgroup of the metaplectic group, which is locally profinite. Hence it is a closed subgroup, so the subspace topology is the locally profinite one.

(d) The isomorphism follows considering the first projection $p_{\mathcal{A}} : \widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}}(W) \to \mathrm{Sp}(W)$. This map is surjective, and so is $p_{\mathbb{C}}$. In addition one has the equality

$$p_{\mathbb{C}} \circ \widetilde{\phi_{\mathbb{C}}} = p_{\mathcal{A}}.$$

Passing to derived groups yields

$$D(p_{\mathbb{C}}) : \widehat{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}},V_A^{\mathbb{C}}}(W) \to [\mathrm{Sp}(W), \mathrm{Sp}(W)].$$

It is an isomorphism in case (i) and a surjective morphism of kernel $\{\pm 1\}$ for (ii). But through the identification given by $\widetilde{\phi_{\mathbb{C}}}$, one has the inclusion

$$\widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}}(W) \subset \widehat{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}},V_A^{\mathbb{C}}}(W)$$

and $D(p_{\mathbb{C}}) \circ \widetilde{\phi_{\mathbb{C}}}$ is surjective. In case (i), the previous inclusion is an equality and except in the exceptional case mentioned the symplectic group is perfect. In case (ii), this implies the following inequality for the index of the quotient

$$[\widehat{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}},V_A^{\mathbb{C}}}(W) : \widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}}(W)] \leq 2.$$

It must be 2 as the reduced metaplectic group cannot be split over $\mathrm{Sp}(W)$. □

Over $\mathcal{B}$. Call $\widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}},V_A^{\mathcal{B}}}(W)$ the metaplectic group over $\mathcal{B}$ and define its derived group

$$\widehat{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}},V_A^{\mathcal{B}}}(W) = [\widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}},V_A^{\mathcal{B}}}(W), \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}},V_A^{\mathcal{B}}}(W)].$$

As above, the morphism of central extensions of Theorem 3.4

$$\widetilde{\phi_{\mathcal{B}}} : (g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \to (g, \phi_{\mathcal{B}}(M)) \in \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$$

restricts to a morphism at the level of derived groups. As $\phi_{\mathcal{B}}$ is continuous, it defines a continuous map $\overline{\phi_{\mathcal{B}}} : \mathrm{PGL}_{\mathcal{A}}(V_A^{\mathcal{A}}) \to \mathrm{PGL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ at the level of quotients. Then one has the equality $\sigma_{\mathcal{B}} = \overline{\phi_{\mathcal{B}}} \circ \sigma_{\mathcal{A}}$ and one deduces from Proposition 4.3 that $\sigma_{\mathcal{B}}$ is continuous.

Proposition 4.4. *One has the following properties*:

(a) *The group* $\widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$ *is the fibre product in the category of topological group of the continuous morphisms* $\sigma_{\mathcal{B}}$ *and* RED, *its topology being the subspace topology in* $\mathrm{Sp}(W) \times \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$.

(b) *The representation* $\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}} : \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W) \to \mathrm{GL}_{\mathcal{B}}(V_A^{\mathcal{B}})$ *is smooth as this group morphism is the second projection of the fibre product*;

(c) *The map* $\widetilde{\phi_{\mathcal{B}}} : (g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \to (g, \phi_{\mathcal{B}}(M)) \in \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$ *is an open continuous map and therefore the metaplectic group over* $\mathcal{B}$ *is locally profinite.*

(d) *Considering derived groups, the map* $\widetilde{\phi_{\mathcal{B}}}$ *restricts to*:
   (i) *A surjection* $\widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \to \widehat{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$ *of kernel* $\{\pm 1\}$ *and image isomorphic to* $\mathrm{Sp}(W)$ *if* F *is local nonarchimedean and* $\mathrm{char}(\mathcal{B}) = 2$.
   (ii) *An isomorphism* $\widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \simeq \widehat{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$ *otherwise.*

Proof. (a) and (b) Obvious from the definition of fibre products and projections.

(c) This needs some explanation however. Once again when F is finite, the topology is discrete and the statement trivially holds. Suppose now that F is local nonarchimedean. As a first observation, remark that the equality $\phi_{\mathcal{B}} \circ \omega_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}} = \omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}} \circ \widetilde{\phi_{\mathcal{B}}}$ holds.

Let $v \in V_A^{\mathcal{A}}$ such that $v \otimes_{\mathcal{A}} 1 \in V_A^{\mathcal{B}}$ is nonzero. Because of the previous equality, the stabiliser of $v \otimes_{\mathcal{A}} 1$ will be contained in the image of $\widetilde{\phi_{\mathcal{B}}}$ as a result of the following two facts. First, one has

$$\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(g, \lambda M)(v \otimes_{\mathcal{A}} 1) = \lambda M(v \otimes_{\mathcal{A}} 1)$$

for all $(g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$ and $\lambda \in \mathcal{B}^\times$. Not much has been said so far. Second, the surjectivity of $p_{\mathcal{A}}$ and $p_{\mathcal{B}}$ onto $\mathrm{Sp}(W)$ implies that for all $(g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$, there exists $\lambda \in \mathcal{B}^\times$ such that $(g, \lambda M)$ is in the image of $\widetilde{\phi_{\mathcal{B}}}$.

Combining the previous two facts, the stabiliser of $v \otimes_{\mathcal{A}} 1$ must be included in the image of $\widetilde{\phi_{\mathcal{B}}}$. So the image of $\widetilde{\phi_{\mathcal{B}}}$ is open because the stabiliser of any element is open as a consequence of $\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}$ being smooth.

The image of $\widetilde{\phi_{\mathcal{B}}}$ is an open subgroup in the metaplectic group over $\mathcal{B}$. If this subgroup is a locally profinite group, then the metaplectic group will be too. Using

Theorem 3.4, one has an exact sequence

$$1 \to \operatorname{Ker}(\widetilde{\phi_{\mathcal{B}}}) \to \widetilde{\operatorname{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \xrightarrow{\widetilde{\phi_{\mathcal{B}}}} \operatorname{Im}(\widetilde{\phi_{\mathcal{B}}}) \to 1.$$

where $\operatorname{Ker}(\widetilde{\phi_{\mathcal{B}}}) = \{(\mathrm{Id}_W, \lambda \,\mathrm{Id}_{V_A^{\mathcal{A}}}) \mid \lambda \in \mathcal{A}^\times \text{ and } \varphi(\lambda) = 1\} \simeq \operatorname{Ker}(\mathcal{A}^\times \to \varphi(\mathcal{A})^\times)$ is a discrete subgroup, so a closed subgroup. Thanks to Proposition 4.3 the metaplectic group over $\mathcal{A}$ is locally profinite, so its quotient by the previous discrete subgroup is locally profinite and $\widetilde{\phi_{\mathcal{B}}}$ factors through it, inducing an homeomorphism of topological groups.

(d) First of all, there is an induced map between derived subgroups

$$D(\widetilde{\phi_{\mathcal{B}}}) : \widehat{\operatorname{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \to \widehat{\operatorname{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W).$$

But $p_{\mathcal{B}} \circ D(\widetilde{\phi_{\mathcal{B}}}) = D(p_{\mathcal{A}})$ is a surjective map $\widehat{\operatorname{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \to [\operatorname{Sp}(W), \operatorname{Sp}(W)]$, which is an isomorphism in case (i) and has kernel $\{\pm 1\}$ in case (ii) according to Proposition 4.3. Therefore

$$\widetilde{\operatorname{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W) / \operatorname{Im}(D(\widetilde{\phi_{\mathcal{B}}}))$$

is abelian. By minimality of the derived group, we must have $\operatorname{Im}(D(\widetilde{\phi_{\mathcal{B}}})) = \widehat{\operatorname{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$. Furthermore

$$\operatorname{Ker}(D(\widetilde{\phi_{\mathcal{B}}})) = \{(\mathrm{Id}_W, \lambda \,\mathrm{Id}_{V_A^{\mathcal{A}}}) \mid \lambda \in \mathcal{A}^\times \text{ and } \varphi(\lambda) = 1\} \cap \widehat{\operatorname{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W).$$

When F is finite, the group $\operatorname{Ker}(D(\widetilde{\phi_{\mathcal{B}}})) = \{(\mathrm{Id}_W, \mathrm{Id}_{V_A^{\mathcal{A}}})\}$ is trivial. When F is local nonarchimedean, it is included in $\{(\mathrm{Id}_W, \epsilon \,\mathrm{Id}_{V_A^{\mathcal{A}}}) \mid \epsilon \in \{\pm 1\}\} \simeq \{\pm 1\}$. But this kernel is nontrivial if and only if $\varphi(-1) = \varphi(1) = 1$ in $\mathcal{B}$, that is $\varphi(2) = 0$, and $\operatorname{char}(\mathcal{B}) = 2$. □

Definition 4.5. Let $\widehat{\phi_{\mathcal{B}}} : \widehat{\operatorname{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W) \to \widetilde{\operatorname{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}(W)$ be the restriction $\widetilde{\phi_{\mathcal{B}}}|_{\widehat{\operatorname{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)}$.

This map will be used later on. Proposition 4.4 has already given some key properties of this map: just to mention a few, it is an open map and its kernel is explicit.

4B. ***Reduced cocycle for $\mathcal{A}$-algebras.*** One deduces from Proposition 4.4 that the metaplectic group over $\mathcal{B}$ either:

- Contains the symplectic group as a subgroup, then $\operatorname{char}(\mathcal{B}) = 2$ or F is finite.
- Does not contain the symplectic group as a subgroup, in which case F is local nonarchimedean and $\operatorname{char}(\mathcal{B}) \neq 2$, and its derived group is canonically isomorphic to the so-called reduced metaplectic group.

In practice, it is important to describe the explicit group law of the metaplectic group for applications. In the first case for instance, it is useful to have a precise formula for the embedding of the symplectic group inside the (split) metaplectic group. In the second case, there are important subgroups that are known to be split, such as inverse images of compact open subgroups, parabolic subgroups, Levi subgroups and unipotent radicals. However, there is *a priori* no guarantee that these groups are split in the reduced metaplectic even though they may be split in the metaplectic group. In order to do computations, one needs to express the cocycle which controls the group law of the reduced metaplectic group. This cocycle usually involves the so-called Weil factor, which is ill-defined when the $\mathcal{A}$-algebra $\mathcal{B}$ does not contain a square root of q. This is the reason why we develop a nonnormalised version of it.

4B1. *Nonnormalised Weil factor over $\mathcal{B}$.* The definition of the nonnormalised Weil factor, achieved over fields in [Trias 2020, Section 1.1], generalises to $\mathcal{A}$-algebras as explained below. Let X be a vector space over F of finite dimension m. Let $\mu^{\mathcal{A}}$ be an invertible Haar measure of X with values in $\mathcal{A}$.

Proposition 4.6. *Let Q be a nondegenerate quadratic form on X. Then there exists a unique nonzero element $\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q)$ in $\mathcal{A}$ such that for all $f \in C_c^\infty(X, \mathcal{A})$, one has*

$$\int_X \int_X f(y-x)\psi^{\mathcal{A}}(Q(x))\, d\mu^{\mathcal{A}}(x) d\mu^{\mathcal{A}}(y) = \Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q) \int_X f(x)\, d\mu^{\mathcal{A}}(x).$$

For any sufficiently small open compact subgroup K in X, the condition for smallness being "$\psi^{\mathcal{A}}(Q(u)) = 1$ for all $u \in K$", this factor explicitly reads

$$\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q) = \sum_{\bar{x} \in K'/K} \psi^{\mathcal{A}}(Q(\bar{x}))$$

where $K' = \{y \in X \mid \forall u \in K, \psi^{\mathcal{A}}(Q(y-u) - Q(y)) = 1\}$ is a compact open subgroup too.

Proof. The existence of such an element $\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q)$ comes from the definition of the nonnormalised Weil factor over fields and from computation, as examined below.

Indeed, the ring $\mathcal{A}$ is naturally contained in its field of fractions $\mathcal{K}$, and the measure $\mu^{\mathcal{A}}$ can be thought of as having values in $\mathcal{K}$. So there exists [Trias 2020, Proposition 1.2] a nonzero element $\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q)$ in $\mathcal{K}$, which achieves the first equality of the statement. A direct computation when $f = 1_K$ and $\psi^{\mathcal{A}}(Q(K)) = 1$ gives

$$\int_X 1_K(y-x)\psi^{\mathcal{A}}(Q(x))\, d\mu^{\mathcal{A}}(x) = \psi^{\mathcal{A}}(Q(y))\mu^{\mathcal{A}}(K) \times 1_{K'}(y)$$

where one easily checks from the definition that K' is a compact open subgroup of X. In addition it contains K. Applying $\mu^{\mathcal{A}}$ to the previous equality leads to

$$\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q) \times \mu^{\mathcal{A}}(1_K) = \operatorname{vol}(K) \sum_{\bar{x} \in K'/K} \psi^{\mathcal{A}}(Q(\bar{x}))$$

where $\mu^{\mathcal{A}}(1_K) = \operatorname{vol}(K) \in \mathcal{A}^\times$ because μ is invertible, resulting in the last equality. $\square$

Let now μ be a Haar measure of X with values in $\mathcal{B}$. Denote λ_μ the unique element in $\mathcal{B}$ such that $\mu = \lambda_\mu \times \mu^{\mathcal{B}}$, where $\mu^{\mathcal{B}} = \varphi \circ \mu^{\mathcal{A}}$ is an invertible Haar measure. Applying φ to the equalities in the previous proposition yields:

Corollary 4.7. *Let Q be a nondegenerate quadratic form on X. Then there exists a unique element $\Omega_\mu(\psi^{\mathcal{B}} \circ Q)$ in $\mathcal{B}$ such that for all $f \in C_c^\infty(X, \mathcal{B})$, one has*

$$\int_X \int_X f(y-x)\psi^{\mathcal{B}}(Q(x))\, d\mu(x)\, d\mu(y) = \Omega_\mu(\psi^{\mathcal{B}} \circ Q) \int_X f(x)\, d\mu(x).$$

Furthermore,

$$\Omega_\mu(\psi^{\mathcal{B}} \circ Q) = \lambda_\mu \times \varphi(\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q)).$$

When Q is a quadratic form on X, one denotes $\operatorname{rad}(Q)$ its radical. Observe that Q is nondegenerate if and only if $\operatorname{rad}(Q) = 0$. The nondegenerate quadratic form Q_{nd} associated to Q is the nondegenerate quadratic form induced by Q on $X/\operatorname{rad}(Q)$.

Definition 4.8. Let Q be a quadratic form on X. Let μ be Haar measure of $X/\operatorname{rad}(Q)$ with values in $\mathcal{B}$. The nonnormalised Weil factor is defined by:

- $\Omega_\mu(\psi^{\mathcal{B}} \circ Q) := \mu(\{0\})$ if Q is the zero quadratic form.
- $\Omega_\mu(\psi^{\mathcal{B}} \circ Q) := \Omega_\mu(\psi^{\mathcal{B}} \circ Q_{\mathrm{nd}})$ otherwise.

Lemma 4.9. *One has*

$$\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q) \in \mathcal{A}^\times.$$

In particular for any invertible Haar measure μ with values in $\mathcal{B}$

$$\Omega_\mu(\psi^{\mathcal{B}} \circ Q) \in \mathcal{B}^\times.$$

Proof. Let $\mathcal{K} \to \mathbb{C}$ be an embedding of $\mathcal{K}$ into $\mathbb{C}$ and $\varphi_{\mathbb{C}}$ its restriction to $\mathcal{A}$. The factor $\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q)$ can be thought of as the factor $\Omega_{\mu^{\mathbb{C}}}(\psi^{\mathbb{C}} \circ Q) = \varphi_{\mathbb{C}}(\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q))$ where $\mu^{\mathbb{C}} = \varphi_{\mathbb{C}} \circ \mu^{\mathcal{A}}$ is an invertible Haar measure. Then point (f) of [Trias 2020, Proposition 1.5] gives

$$\Omega_{\mu^{\mathbb{C}}}(\psi^{\mathbb{C}} \circ Q) = \omega_{\psi^{\mathbb{C}}}(\psi^{\mathbb{C}} \circ Q) \times |\rho|_{\mu^{\mathbb{C}}}^{1/2}$$

where $\omega_{\psi^{\mathbb{C}}}(\psi^{\mathbb{C}} \circ Q)$ is an eigth root of unity and $|\rho|_{\mu^{\mathbb{C}}} = \mu^{\mathbb{C}}(K)(q^{1/2})^k$, with K a compact open subgroup of X, a square root $q^{1/2}$ of q in $\mathbb{C}$ and an integer $k \in \mathbb{Z}$. So

$$\Omega_{\mu^{\mathbb{C}}}(\psi^{\mathbb{C}} \circ Q)^8 = (\mu^{\mathbb{C}}(K))^8 q^{4k}.$$

Therefore $\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q)^8 = (\mu^{\mathcal{A}}(K))^8 q^{4k} \in \mathcal{A}^\times$ because $\varphi_{\mathbb{C}}$ is injective and $\mathbb{Q}$-linear, implying the result about the factor being invertible. Hence the second equality results from applying φ and Corollary 4.7, given the fact that $\lambda_\mu \in \mathcal{B}^\times$. $\square$

Define for a in $F^\times$ the quadratic form $Q_a : x \in F \mapsto ax^2 \in F$. Then the factor

$$\Omega_{a,b}^{\mathcal{A}} = \frac{\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q_a)}{\Omega_{\mu^{\mathcal{A}}}(\psi^{\mathcal{A}} \circ Q_b)} \in \mathcal{A}^\times$$

does not depend on the choice of the invertible Haar measure $\mu^{\mathcal{A}}$, as the notation suggests. One can define $\Omega_{a,b}^{\mathcal{B}}$ in the obvious way, either as a quotient of two nonnormalised Weil factors or as the image of the previous using the map φ.

4B2. *Section $\varsigma^{\mathcal{B}}$ giving the cocycle.* Let X be a lagrangian of W. In particular this provides an instance of a self-dual subgroup in W. A nice section $\varsigma^{\mathcal{A}} : \mathrm{Sp}(W) \to \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_X^{\mathcal{A}}}(W)$ of $p_{\mathcal{A}}$ is defined below. It is nice in the sense that it will give the explicit group law in the metaplectic group over $\mathcal{A}$.

First of all, observe that, using the notation of Section 3, any section ς of $p_{\mathcal{A}}$ is given by a family $(\mu_g)_{g \in \mathrm{Sp}(W)}$ of measures where μ_g is an invertible measure of $gX \cap X \backslash X$. Namely it reads $\varsigma : g \mapsto (g, I_{gX,X,\mu_g,0} \circ I_g)$. One defines the section $\varsigma^{\mathcal{A}}$ mentioned above in the following way. The stabiliser $P(X)$ of X in $\mathrm{Sp}(W)$ is a maximal parabolic subgroup. For $g \in \mathrm{Sp}(W)$, let μ_g be the invertible measure on $gX \cap X \backslash X$ defined by

$$\mu_g = \Omega^{\mathcal{A}}_{1,\det_X(p_1 p_2)} \times \phi_1 \cdot \mu^{\mathcal{A}}_{w_j}$$

where:

- $(w_j)_{j=0,\dots,m}$ is a system of representatives in $\mathrm{Sp}(W)$ for $P(X) \backslash \mathrm{Sp}(W) / P(X)$.
- The element $g = p_1 w_j p_2 \in P(X) w_j P(X)$ with p_1 and p_2 in $P(X)$.
- $\det_X(p) = \det_F(p|_X)$ where $p|_X \in \mathrm{GL}(X) \simeq \mathrm{GL}_m(F)$.
- $gX \cap X \backslash X \overset{\phi_1}{\simeq} w_j X \cap X \backslash X$ is induced by $x \in X \mapsto \overline{p_1^{-1} x} \in w_j X \cap X \backslash X$.
- $Q_j(x) = \frac{1}{2}\langle w_j x, x \rangle$ is the nondegenerate quadratic form on $w_j X \cap X \backslash X$.
- For any invertible μ, set $\mu^{\mathcal{A}}_{w_j} = \Omega_\mu(\psi^{\mathcal{A}} \circ Q_j)^{-1} \mu$ which does not depend on μ.

See [Trias 2020, Section 3.5] to get a more detailed explanation about the previous definitions. Exclude the exceptional case $F = \mathbb{F}_3$ and $\dim(W) = 2$ from now on.

Proposition 4.10. *With the previous choice of μ_g, the section*

$$\varsigma^{\mathcal{A}} : g \in \mathrm{Sp}(W) \mapsto (g, I_{gX,X,\mu_g,0} \circ I_g) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_X^{\mathcal{A}}}(W)$$

has values in $\widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_X^{\mathcal{A}}}(W)$, except in the exceptional case $F = \mathbb{F}_3$ and $\dim(W) = 2$. The 2-cocycle defined by this section

$$\hat{c}^{\mathcal{A}} : (g_1, g_2) \in \mathrm{Sp}(W) \times \mathrm{Sp}(W) \mapsto \varsigma^{\mathcal{A}}(g_1)\varsigma^{\mathcal{A}}(g_2)\varsigma^{\mathcal{A}}(g_1g_2)^{-1} \in \mathcal{A}^\times$$

is trivial when F is finite, and has image $\{\pm 1\}$ when F is local nonarchimedean.

Proof. Consider an embedding $\mathcal{K} \to \mathbb{C}$ and denote $\varphi_{\mathbb{C}}$ its restriction to $\mathcal{A}$. The map

$$\widetilde{\phi_{\mathbb{C}}} : (g, M) \in \widetilde{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}}(W) \to (g, \phi_{\mathbb{C}}(M)) \in \widetilde{\mathrm{Sp}}^{\mathbb{C}}_{\psi^{\mathbb{C}},V_A^{\mathbb{C}}}(W)$$

and the compatibility $\phi_{\mathbb{C}}(I_{gX,X,\mu^{\mathcal{A}},0}) = I_{gX,X,\mu^{\mathbb{C}},0}$ from Corollary 2.8 where $\mu^{\mathbb{C}} = \varphi_{\mathbb{C}} \circ \mu^{\mathcal{A}}$, leads to

$$\widetilde{\phi_{\mathbb{C}}} \circ \varsigma^{\mathcal{A}}(g) = (g, I_{gX,X,\varphi_{\mathbb{C}}\circ\mu_g,0} \circ I_g).$$

But the measure $\varphi_{\mathbb{C}} \circ \mu_g$ above is the one defined in [Trias 2020, Lemma 3.23], and according to [loc. cit., Theorem 3.27], the map $\widetilde{\phi_{\mathbb{C}}} \circ \varsigma^{\mathcal{A}}$ is a section of $p_{\mathbb{C}}$ whose associated cocycle is trivial when F is finite, and has values in the reduced metaplectic group when F is local nonarchimedean. The associated cocycle $\hat{c}^{\mathbb{C}}$ is trivial when F is finite and has image $\{\pm 1\}$ when F is local nonarchimedean. Using point (d) of Proposition 4.3, the image of $\varsigma^{\mathcal{A}}$ lies in $\widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_X^{\mathcal{A}}}(W)$, except in the exceptional case $F = \mathbb{F}_3$ and $\dim(W) = 2$. In any case, the map $\varsigma^{\mathcal{A}}$ is injective so this defines a section of $p_{\mathcal{A}}$. In particular, it is a group morphism when F is finite as a result of the cocycle $\hat{c}^{\mathbb{C}}$ being trivial. □

One easily deduces from the previous proposition and Proposition 4.4, the corollary:

Corollary 4.11. *The section $\varsigma^{\mathcal{B}} = \widetilde{\phi_{\mathcal{B}}} \circ \varsigma^{\mathcal{A}}$ has values in $\widehat{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}},V_X^{\mathcal{B}}}(W)$, except in the exceptional case $F = \mathbb{F}_3$ and $\dim(W) = 2$. The 2-cocycle defined by this section*

$$\hat{c}^{\mathcal{B}} : (g_1, g_2) \in \mathrm{Sp}(W) \times \mathrm{Sp}(W) \mapsto \varsigma^{\mathcal{B}}(g_1)\varsigma^{\mathcal{B}}(g_2)\varsigma^{\mathcal{B}}(g_1g_2)^{-1} \in \mathcal{B}^\times$$

5 *is trivial when F is finite or $\mathrm{char}(\mathcal{B}) = 2$, and has image $\{\pm 1\}$ otherwise.*

Remark 4.12. In the exceptional case, the section $\varsigma^{\mathcal{A}}$, resp. $\varsigma^{\mathcal{B}}$, can still be defined. However the derived group $[\mathrm{Sp}(W), \mathrm{Sp}(W)]$ is a strict subgroup of the symplectic group $\mathrm{Sp}(W)$. So the image of the previous sections, which are again group morphisms, is just a subgroup of the metaplectic group over $\mathcal{A}$, resp. over $\mathcal{B}$, that is isomorphic to $\mathrm{Sp}(W)$.

5. Families of Weil representations

Consider the map $\widehat{\phi_{\mathcal{B}}} : \widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}}(W) \to \widetilde{\mathrm{Sp}}^{\mathcal{B}}_{\psi^{\mathcal{B}},V_A^{\mathcal{B}}}(W)$ of Definition 4.5. The exceptional case $F = \mathbb{F}_3$ and $\dim(W) = 2$ needs separate treatment, which will be done as a quick remark, so we exclude it from now on.

Let H be a closed subgroup of $\mathrm{Sp}(W)$ and set

$$\widetilde{H}^{\mathcal{A}} = p_{\mathcal{A}}^{-1}(H) \quad \text{and} \quad \widetilde{H}^{\mathcal{B}} = p_{\mathcal{B}}^{-1}(H).$$

Denote by $\widehat{H}^{\mathcal{A}}$ the intersection of $\widetilde{H}^{\mathcal{A}}$ and $\widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}}(W)$. Recall that $\varphi : \mathcal{A} \to \mathcal{B}$ is the structure morphism of the $\mathcal{A}$-algebra $\mathcal{B}$ and consider the categories

$$\mathrm{Rep}'_{\mathcal{B}}(\widehat{H}^{\mathcal{A}}) = \{(\pi, V) \in \mathrm{Rep}_{\mathcal{B}}(\widehat{H}^{\mathcal{A}}) \mid \pi((\mathrm{Id}_W, \epsilon\,\mathrm{Id}_{V_A^{\mathcal{A}}})) = \varphi(\epsilon)\,\mathrm{Id}_V \text{ for } \epsilon \in \{\pm 1\}\}$$

and

$$\mathrm{Rep}'_{\mathcal{B}}(\widetilde{H}^{\mathcal{B}}) = \{(\pi, V) \in \mathrm{Rep}_{\mathcal{B}}(\widetilde{H}^{\mathcal{B}}) \mid \pi((\mathrm{Id}_W, \lambda\,\mathrm{Id}_{V_A^{\mathcal{B}}})) = \lambda\,\mathrm{Id}_V \text{ for } \lambda \in \mathcal{B}^\times\}.$$

Proposition 5.1. *The functor*

$$(\pi, V) \in \mathrm{Rep}'_{\mathcal{B}}(\widetilde{H}^{\mathcal{B}}) \mapsto (\pi \circ \widehat{\phi_{\mathcal{B}}}, V) \in \mathrm{Rep}'_{\mathcal{B}}(\widehat{H}^{\mathcal{A}})$$

defines an equivalence of categories.

Proof. This map is a functor and its inverse is given by the extension of scalars to $\mathcal{B}^\times$, that is for any $(\pi', V') \in \mathrm{Rep}'_{\mathcal{B}}(\widehat{H}^{\mathcal{A}})$, the representation

$$\pi'' : (\hat{h}, \lambda) \in \widehat{H}^{\mathcal{A}} \times \mathcal{B}^\times \mapsto \lambda\pi'(\hat{h}) \in \mathrm{GL}_{\mathcal{B}}(V')$$

factorises as a representation of $\widetilde{H}^{\mathcal{B}}$. Indeed, the surjective group morphism

$$(\hat{h}, \lambda) \in \widehat{H}^{\mathcal{A}} \times \mathcal{B}^\times \to \widehat{\phi_{\mathcal{B}}}(\hat{h}) \times (\mathrm{Id}_W, \lambda\,\mathrm{Id}_{V_A^{\mathcal{B}}}) \in \widetilde{H}^{\mathcal{B}}$$

is an isomorphism when F is finite and has kernel $\{((\mathrm{Id}_W, \epsilon\,\mathrm{Id}_{V_A^{\mathcal{A}}}), \varphi(\epsilon)) \mid \epsilon \in \{\pm 1\}\}$ when F is local nonarchimedean. But $\mathrm{Ker}(\pi'')$ contains the kernel of the surjective map above, that is it factorises as claimed. □

Remark 5.2. The reason for proving such a result is to consider the "same" group for any $\mathcal{A}$ -algebra $\mathcal{B}$, which is particularly convenient when looking at scalar extension for representations. For instance, the representation $\omega_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}} \otimes_{\mathcal{A}} \mathcal{B} \in \mathrm{Rep}'_{\mathcal{B}}(\widehat{H}^{\mathcal{A}})$, which is the scalar extension of $\omega_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}} \in \mathrm{Rep}'_{\mathcal{A}}(\widehat{H}^{\mathcal{A}})$, should be the "same" — the proposition below making this "same" precise — representation as $\omega_{\psi^{\mathcal{B}},V_A^{\mathcal{B}}} \in \mathrm{Rep}'_{\mathcal{B}}(\widetilde{H}^{\mathcal{B}})$.

Remark 5.3. In the exceptional case however, because the symplectic group $\mathrm{Sp}(W)$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$, the derived group $\widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}},V_A^{\mathcal{A}}}(W)$ is a strict subgroup of the symplectic group. One needs to replace $\widehat{\phi_{\mathcal{B}}}$ by any morphism that embeds $\mathrm{Sp}(W)$

in the metaplectic group over $\mathcal{A}$, composed with $\widetilde{\phi_{\mathcal{B}}}$. One can take for example the embeddings $\varsigma^{\mathcal{A}}$ and $\varsigma^{\mathcal{B}}$ according to Remark 4.12.

From the previous proposition and Theorem 3.4, the following compatibility holds:

Proposition 5.4. *The representations $\omega_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}} \otimes_{\mathcal{A}} \mathcal{B}$ and $\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}}$ are isomorphic, in the sense that the canonical identification $V_A^{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B} \simeq V_A^{\mathcal{B}}$ of Corollary 2.6 induces an isomorphism in $\mathrm{Rep}'_{\mathcal{B}}(\widehat{H}^{\mathcal{A}})$, namely*

$$(\omega_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}} \otimes_{\mathcal{A}} \mathcal{B},\, V_A^{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{B}) \simeq (\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}} \circ \widehat{\phi_{\mathcal{B}}},\, V_A^{\mathcal{B}}).$$

Of course when R is a field endowed with an $\mathcal{A}$-algebra structure, the representation $(\omega_{\psi^R, V_A^R}, V_A^R)$ is the modular Weil representation on W associated to ψ^R and V_A^R, in the way they are defined in [Mœglin et al. 1987, Chapter 2, II] for $R = \mathbb{C}$ and in [Trias 2020, Section 3] for more general fields. Recall that in this situation V_A^R is the metaplectic representation associated to ψ^R.

Dual pairs. When (H_1, H_2) is a dual pair in $\mathrm{Sp}(W)$, one may fix a model for the Weil representation and "embed" the lift of the dual pairs in the derived subgroup of the metaplectic group over $\mathcal{A}$ through the natural multiplication map. One can also use the lifts in the metaplectic group over $\mathcal{A}$ instead of the derived subgroup. This means looking at the representation

$$\omega_{\psi^{\mathcal{B}}, V_A^{\mathcal{B}}} \circ \widehat{\phi_{\mathcal{B}}}|_{\widehat{H_1}^{\mathcal{A}} \times \widehat{H_2}^{\mathcal{A}}} \in \mathrm{Rep}_{\mathcal{B}}(\widehat{H_1}^{\mathcal{A}} \times \widehat{H_2}^{\mathcal{A}})$$

where the restriction $\widehat{H_1}^{\mathcal{A}} \times \widehat{H_2}^{\mathcal{A}} \to \widehat{\mathrm{Sp}}^{\mathcal{A}}_{\psi^{\mathcal{A}}, V_A^{\mathcal{A}}}(W)$ is achieved using the natural multiplication map. Of course, when these lifts of dual pairs are split, one can always compose with their splittings to get representations of H_1 or H_2 themselves. It may happen that splittings do not exist in the derived subgroup even if they do exist in the metaplectic group itself [Mœglin et al. 1987, Chapter 2, Remark II.9] and [Trias 2020, Section 4]. So one may switch hats for tildes depending on the dual pair one wants to consider.

6. Features of the pair $(\mathrm{GL}_1(F), \mathrm{GL}_1(F))$

Suppose F is a local nonarchimedean field. Let W be a symplectic space over F of dimension 2 and $W = X + Y$ be a complete polarisation. For $a \in F^\times$, define m_a to be the unique endomorphism in $\mathrm{Sp}(W)$ such that in the previous basis:

$$m_a = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

The pair $(H_1, H_2) = (F^\times, F^\times)$ is defined by $(a_1, a_2) \mapsto m_{a_1} m_{a_2^{-1}}$. Up to some smooth characters of H_1 and H_2, the Weil representation ω_{H_1,H_2} is the "geometric" representation $(\rho, C_c^\infty(F, \mathcal{B}))$ where H_1 and H_2 act respectively on the left and on the right on the locally profinite space F. For $f \in C_c^\infty(F, \mathcal{B})$ and $a_1, a_2 \in F^\times$, it reads

$$\rho(a_1, a_2) \cdot f : x \in F \mapsto f(a_1^{-1} x a_2) \in \mathcal{B}.$$

6A. ***Level* 0 *part.*** The category $\mathrm{Rep}_{\mathcal{B}}(F^\times)$ is decomposed as a product of categories $\prod_{k \in \mathbb{N}} \mathrm{Rep}_{\mathcal{B}}^k(F^\times)$ where the index k is also known as the level. In this picture, the level 0 subcategory has the most direct description as it corresponds to representations with trivial action of the biggest pro-p-subgroup K of $F^\times$ which is, after choosing a uniformiser ϖ_F of F, the group $K = 1 + \varpi_F \mathcal{O}_F$. In addition the isomorphism $(k, u) \in \mathbb{Z} \times \mathcal{O}_F^\times \mapsto \varpi_F^k u \in F^\times$ induces an isomorphism $F^\times / K \simeq \mathbb{Z} \times (\mathbb{Z}/(q-1)\mathbb{Z})$. Suppose from now on a choice of uniformiser ϖ_F is made as well as a choice of a primitive $(q-1)$-root of unity ζ_{q-1} in F. Hence in the free part $\mathbb{Z}$ is generated by ϖ_F and the torsion part $\mathbb{Z}/(q-1)\mathbb{Z}$ is generated by ζ_{q-1}. So the group algebra $\mathcal{B}[F^\times/K]$ is isomorphic to the $\mathcal{B}$-algebra $\mathcal{B}[X^{\pm 1}, Z]/(Z^{q-1} - 1)$, where ϖ_F corresponds to X and ζ_{q-1} to Z.

The level* 0 *category. As we are only interested in the level 0 part, we shall only consider, for any $V \in \mathrm{Rep}_{\mathcal{B}}(F^\times)$, the direct factor representation V^K made of K-invariant vectors. As for the representation $(\rho_l, C_c^\infty(F, \mathcal{B}))$ given by the left $F^\times$-action, this level 0 part is the subspace of bi-K-invariant functions

$$C_c^\infty(F, \mathcal{B})^K = \{ f \in C_c^\infty(F, \mathcal{B}) \mid \forall x \in F \text{ and } k \in K,\ f(xk) = f(kx) = f(x) \}.$$

In addition, the centre $\mathfrak{z}^0$ of the level 0 category $\mathrm{Rep}_{\mathcal{B}}^0(F^\times)$ is, because the group $F^\times$ is abelian, equal to the endomorphism ring of a minimal progenerator of $\mathrm{Rep}_{\mathcal{B}}^0(F^\times)$. Let $(\mathbf{1}_K, \mathcal{B})$ be the free module $\mathcal{B}$ of rank 1 with trivial K-action. Then $\mathrm{ind}_K^{F^\times}(\mathbf{1}_K)$ is known to be a progenerator of $\mathrm{Rep}_{\mathcal{B}}^0(F^\times)$. As a space of functions this also is $C_c^\infty(F^\times/K, \mathcal{B})$, which is a free module of rank 1 over $\mathcal{B}[F^\times/K]$ generated by the characteristic function 1_K. Therefore

$$\mathrm{End}_{F^\times}(\mathrm{ind}_K^{F^\times}(\mathbf{1}_K)) = \mathrm{End}_{\mathcal{B}[F^\times/K]}(\mathrm{ind}_K^{F^\times}(\mathbf{1}_K)) \simeq \mathcal{B}[F^\times/K]$$

thanks to $\mathrm{ind}_K^{F^\times}(\mathbf{1}_K)$ being free of rank 1. So one can consider that the centre $\mathfrak{z}^0$ is $\mathcal{B}[F^\times/K] \simeq \mathcal{B}[X^{\pm 1}, Z]/(Z^{q-1} - 1)$. Eventually, the level 0 category is equivalent to the category of modules over the latter commutative ring.

6A1. *Specialisation using the centre.*

Morphism of the centre. Let $\mathcal{C}$ be a commutative $\mathcal{B}$-algebra. Let $\eta \in \mathrm{Hom}_{\mathcal{B}\text{-alg}}(\mathfrak{z}^0, \mathcal{C})$ be a morphism of $\mathcal{B}$-algebras. Of course η naturally endows $\mathcal{C}$ with a $\mathfrak{z}^0$-algebra structure. In addition, any representation in $\mathrm{Rep}_{\mathcal{B}}^0(F^\times)$ is canonically endowed with a $\mathfrak{z}^0$-module structure. By definition, this $\mathfrak{z}^0$-module structure commutes with the $F^\times$-action.

Definition 6.1. For any $V \in \mathrm{Rep}^0_{\mathcal{B}}(F^\times)$, one defines the representation

$$V_\eta = V \otimes_{\mathfrak{z}^0} \eta \in \mathrm{Rep}_{\mathcal{C}}(F^\times).$$

Examples. Recall $\mathfrak{z}^0 = \mathcal{B}[X^{\pm 1}, Z]/(Z^{q-1}-1)$. The following are easy claims:

- When $\mathcal{B}$ is a field and $\chi : F^\times/K \to \mathcal{B}^\times$ is a character, the $\mathcal{B}$-algebra morphism

  $$\eta_\chi : P \in \mathcal{B}[X^{\pm 1}, Z]/(Z^{q-1}-1) \mapsto P(\chi(\varpi_F), \chi(\zeta)) \in \mathcal{B}$$

  provides the biggest χ-isotypic quotient $V_{\eta_\chi} = V_\chi$. Furthermore

  $$\mathrm{Ker}(\eta_\chi) = (X - \chi(\varpi_F), Z - \chi(\zeta)).$$

- When $\varphi : \mathcal{B} \to \mathcal{B}'$ is a morphism of $\mathcal{B}$-algebras, the $\mathcal{B}$-algebra morphism

  $$\eta_\varphi : P \in \mathcal{B}[X^{\pm 1}, Z]/(Z^{q-1}-1) \to \varphi(P) \in \mathcal{B}'[X^{\pm 1}, Z]/(Z^{q-1}-1)$$

  provides the extension of scalars $V_{\eta_\varphi} = V \otimes_{\mathcal{B}} \mathcal{B}'$. Furthermore

  $$\mathrm{Ker}(\eta_\varphi) = \mathrm{Ker}(\varphi) \cdot \mathfrak{z}^0.$$

- Let χ be a character with values in $\mathcal{B}^\times$, let $\mathfrak{m}$ a maximal ideal in $\mathcal{B}$, and denote by $\varphi_{\mathfrak{m}}$ the quotient morphism $\mathcal{B} \to \mathcal{B}/\mathfrak{m}$ and $\chi_{\mathfrak{m}} = \varphi_{\mathfrak{m}} \circ \chi$, then

  $$(V_{\eta_\chi})_{\eta_{\varphi_{\mathfrak{m}}}} = (V_{\eta_{\varphi_{\mathfrak{m}}}})_{\eta_{\chi_{\mathfrak{m}}}} \quad \text{i.e., } V_{\eta_\chi} \otimes_{\mathcal{B}} (\mathcal{B}/\mathfrak{m}) = (V \otimes_{\mathcal{B}} (\mathcal{B}/\mathfrak{m}))_{\chi_{\mathfrak{m}}}.$$

  Therefore the representation V_{η_χ} may be viewed as a family of representations specialising at maximal ideals to biggest isotypic quotients, whereas it is less clear how direct methods would give a good definition of an isotypic quotient over a ring.

Remark 6.2. Unlike the construction of the biggest isotypic quotient for irreducible representations with coefficients in a field, the natural map $V \mapsto V_\eta$ is not surjective in general. Of course if η is surjective, the previous map is a quotient map.

6A2. *Isotypic families of the Weil representation.*

Level* 0 *Weil representation. Instead of considering representations with coefficients over different rings, this approach benefits from a greater flexibility when dealing with the level 0 Weil representation $C_c^\infty(F, \mathcal{B})^K$. For example in the second situation with $\varphi \in \mathrm{Hom}_{\mathcal{B}-\mathrm{alg}}(\mathcal{B}, \mathcal{B}')$, and thanks to the description as spaces of functions, one has

$$(C_c^\infty(F, \mathcal{B})^K)_{\eta_\varphi} = C_c^\infty(F, \mathcal{B}')^K.$$

Family for the trivial representation. Set $V = C_c^\infty(F, \mathcal{B})^K$ and $V_0 = C_c^\infty(F^\times, \mathcal{B})^K$. Recall there is an exact sequence of representations, that is given by the function restriction to the closed set $\{0\}$ in F, namely

$$0 \to V_0 \to V \to \mathbf{1}_{F^\times}^{\mathcal{B}} \to 0$$

where $\mathbf{1}_{F^\times}^{\mathcal{B}}$ is a free $\mathcal{B}$-module of rank 1 endowed with the trivial $F^\times$-action. Consider now the ideal $I_\mathbf{1} = (X-1, Z-1)$ in $\mathfrak{z}^0$ and the morphism $\eta_\mathbf{1} : \mathfrak{z}^0 \to \mathfrak{z}^0/I_\mathbf{1}$. As $(\mathbf{1}_{F^\times}^{\mathcal{B}})_{\eta_\mathbf{1}} = \mathbf{1}_{F^\times}^{\mathcal{B}}$, and V_0 is free of rank 1 over $\mathfrak{z}^0$, it induces an exact sequence

$$\mathbf{1}_{F^\times}^{\mathcal{B}} \to V_{\eta_\mathbf{1}} \to \mathbf{1}_{F^\times}^{\mathcal{B}} \to 0.$$

The kernel of the map $\mathcal{B} \to V_{\eta_\mathbf{1}}$ is $(q-1)\mathcal{B}$ because

$$((X-1)V + (Z-1)V) \cap V_0 = (X-1)V_0 + (Z-1)V_0 + (q-1)V_0.$$

So the following sequence is exact:

$$0 \to \mathbf{1}_{F^\times}^{\mathcal{B}/(q-1)\mathcal{B}} \to V_{\eta_\mathbf{1}} \to \mathbf{1}_{F^\times}^{\mathcal{B}} \to 0.$$

Denoting by β the image of $1_{\mathcal{O}_F}$ in V_{η_1}, the above sequence splits as $b \in \mathcal{B} \mapsto b \cdot \alpha \in V_{\eta_1}$ is a section of $V_{\eta_1} \to \mathbf{1}_{F^\times}^{\mathcal{B}}$. So one has $V_{\eta_\mathbf{1}} \simeq \mathbf{1}_{F^\times}^{\mathcal{B}/(q-1)\mathcal{B}} \oplus \mathbf{1}_{F^\times}^{\mathcal{B}}$.

***The family for* $(X-q, Z-1)$.** It does not coincide with the family for the trivial representation, except at the nonbanal prime ideals. These are the prime ideals $\mathcal{P}$ in $\mathcal{B}$ such that $\mathcal{P} \cap \mathbb{Z}$ is generated by a prime ℓ dividing $q-1$. Denoting η the character $\mathfrak{z}^0 \to \mathfrak{z}^0/(X-q, Z-1)$ and $(\chi_\mathcal{B}, \mathcal{B})$ the character such that $\chi_\mathcal{B}(\zeta) = 1$ and $\chi_\mathcal{B}(\varpi_F) = q$, one similarly has

$$0 \to \chi_\mathcal{B} \to V_\eta \to \mathbf{1}_{F^\times}^{\mathcal{B}/(q-1)\mathcal{B}} \to 0.$$

Indeed on the one hand $(X-1)\mathbf{1}_{F^\times}^{\mathcal{B}} + (Z-1)\mathbf{1}_{F^\times}^{\mathcal{B}} = (1-q)\mathbf{1}_{F^\times}^{\mathcal{B}}$ so $(\mathbf{1}_{F^\times}^{\mathcal{B}})_\eta = \mathbf{1}_{F^\times}^{\mathcal{B}/(q-1)\mathcal{B}}$. On the other hand V_0 is $\mathfrak{z}^0$-free so $(V_0)_\eta = \chi_\mathcal{B}$. Denote by α and β the images of $1_{1+\varpi_F\mathcal{O}_F}$ and $1_{\mathcal{O}_F}$ in V_η. The following computation helps identifying the $(q-1)$-torsion

$$(X-1)\beta = (q-1)\alpha = (X-q)\beta + (1-q)\beta = (q-1)\beta.$$

Then $\lambda \in \mathcal{B} \mapsto \lambda(\beta - \alpha) \in V_\eta$ factorises as a section of $V_\eta \to \mathbf{1}_{F^\times}^{\mathcal{B}/(q-1)\mathcal{B}}$. As a consequence, one has $V_\eta \simeq \mathbf{1}_{F^\times}^{\mathcal{B}/(q-1)\mathcal{B}} \oplus \chi_\mathcal{B}$.

Remark 6.3. We interpret $\mathbf{1}_{F^\times}^{\mathcal{B}/(q-1)\mathcal{B}}$ as the greatest common quotient of $\mathbf{1}_{F^\times}^{\mathcal{B}}$ and $\chi_\mathcal{B}$.

More general families. One can look at any ideal in $\mathfrak{z}^0$ to get more new families of representations. For example, instead of only looking at characters with values 1 at ζ, one can look at irreducible factors Q of $Z^{q-1} - 1$ that are different from $Z - 1$, and consider the ideal (P, Q) for an irreducible polynomial P in $\mathcal{B}[X^{\pm 1}]$.

Remark 6.4. Even when $\mathcal{B}$ is an integral domain, the previous classes of ideals (P, Q) are not necessarily prime ideals in $\mathfrak{z}^0$. The irreducibility has therefore to be considered over the field extension $\mathrm{Frac}(\mathcal{B})[Z]/(Q)$ of $\mathrm{Frac}(\mathcal{B})$ i.e., P is irreducible as a polynomial over this bigger field. Furthermore, letting P be a nonunitary polynomial allows to consider characters with coefficients in $\mathrm{Frac}(\mathcal{B})$ e.g., $\overline{\mathbb{Z}}_\ell[X^{\pm 1}]/(\ell X - 1) = \overline{\mathbb{Q}}_\ell$ when $\mathcal{B} = \overline{\mathbb{Z}}_\ell$.

6B. ***Positive level part.*** Let $k \in \mathbb{N}^*$. As a first observation, the level k parts of the representations $C_c^\infty(F, \mathcal{B})$ and $C_c^\infty(F^\times, \mathcal{B})$ are equal. Therefore the problem reduces to understand the level k part of the regular representation. The same techniques as in the previous paragraph apply once the centre $\mathfrak{z}^k$ of the category has been made explicit. The study will not be developed in the present work for the sake of shortness. But in order to flag some differences, here are some remarks below:

- If $\mathcal{B}$ does not have enough p-power roots of unity, the situation is more complicated as no characters of level k may exist, that is there does not exist a group morphism $\chi : 1 + \varpi_F \mathcal{O}_F \to \mathcal{B}^\times$ such that $1 + \varpi_F^{k+1} \mathcal{O}_F \subset \mathrm{Ker}(\chi) \subsetneq 1 + \varpi_F^k \mathcal{O}_F$.
- Provided $\mathcal{B}$ has enough p-power roots of unity, the set of characters
$$\mathrm{Char}_{\mathcal{B}}^k = \{\chi : 1 + \varpi_F \mathcal{O}_F \to \mathcal{B}^\times \mid \chi \in \mathrm{Rep}_{\mathcal{B}}^k(1 + \varpi_F \mathcal{O}_F)\}$$
is not empty and decomposes the category $\mathrm{Rep}_{\mathcal{B}}^k(F^\times)$ as product of categories $\prod_{\chi \in \mathrm{Char}_{\mathcal{B}}^k} \mathrm{Rep}_{\mathcal{B}}^\chi(F^\times)$, where each category factor is equivalent to $\mathrm{Rep}_{\mathcal{B}}^0(F^\times)$.

In the first situation, the situation may be quite complicated to write down, though this first situation only occurs when F has positive characteristic. Indeed, $\mathcal{A}$ is isomorphic to $\mathbb{Z}\big[\frac{1}{p}, \zeta_p\big]$ in this case, whereas it is $\mathbb{Z}\big[\frac{1}{p}, \zeta_{p^\infty}\big]$ for characteristic zero F. In the event of $\mathcal{B}$ having enough p-power roots of unity, one can reduce the situation to the level 0 part of $C_c^\infty(F^\times, \mathcal{B})$ as it is isomorphic to the χ-part of $C_c^\infty(F^\times, \mathcal{B})$ for $\chi \in \mathrm{Char}_{\mathcal{B}}^k$. This latter has been studied in the previous section.

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LOEWNER CHAINS APPLIED TO g-STARLIKE MAPPINGS OF COMPLEX ORDER OF COMPLEX BANACH SPACES

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This paper is devoted to studying geometric and analytic properties of g-starlike mappings of complex order λ. By using Loewner chains, we obtain the growth theorems for g-starlike mappings of complex order λ on the unit ball in reflexive complex Banach spaces, which generalize some results of Graham, Hamada and Kohr. As applications, several different kinds of distortion theorems for g-starlike mappings of complex order λ are obtained. Finally, we prove that the Roper–Suffridge extension operators preserve the property of g-starlike mappings of complex order λ in complex Banach spaces, which generalizes many classical results.

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1. Introduction

Let $f = z + \sum_{n=2}^{\infty} a_n z^n$ be a normalized univalent function on the unit disk $\mathbb{D}$ in $\mathbb{C}$. The growth theorem shows that the modulus of a normalized univalent function $|f|$ has a finite upper and positive lower bound depending only on the modulus of the variable $|z|$, and the image of f contains a disk centered at origin with radius $\frac{1}{4}$. The distortion theorem gives explicit upper and lower bounds on $|f'(z)|$ in terms of $|z|$. The term distortion arises from the geometric interpretation of $|f'(z)|$ as the

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infinitesimal magnification factor of arc length and the interpretation of the square of $|f'(z)|$ as the infinitesimal magnification factor of area.

However, in the case of several complex variables, H. Cartan pointed out that the growth theorem and distortion theorem do not hold for normalized biholomorphic mappings. In addition, he suggested that one should investigate the important geometrically defined subfamilies of convex and starlike mappings. As a matter of fact, there was little work in the geometric directions suggested by Cartan, until the 1970s, when a number of results dealing with the convex and starlike biholomorphic mappings appeared. As a direct generalization of the growth theorem for univalent function on the unit disk $\mathbb{D}$, the growth theorem for normalized biholomorphic starlike mappings on the unit ball $\mathbb{B}_n$ was obtained by Barnard, Fitzgerald and Gong [Barnard et al. 1991] using the analytical characterization of starlikeness, and by Kubicka and Poreda [1988] using the method of Loewner chains.

Theorem A [Barnard et al. 1991; Kubicka and Poreda 1988]. *Let f be a starlike mapping on the unit ball $\mathbb{B}_n$. Then, for any point $z \in \mathbb{B}_n$, we have*

$$\frac{\|z\|}{(1+\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^2}.$$

Furthermore, the above estimates are sharp.

If the convexity restriction is attached to the family of normalized locally biholomorphic mapping f, the following growth theorem for convex mappings is due to Suffridge [1977], Thomas [1991], Liu [1989] or Liu and Ren [1998].

Theorem B. *Let f be a convex mapping on the unit ball $\mathbb{B}_n$. Then, for any point $z \in \mathbb{B}_n$, we have*

$$\frac{\|z\|}{1+\|z\|} \le \|f(z)\| \le \frac{\|z\|}{1-\|z\|}.$$

Moreover, the above estimates are sharp.

In several complex variables, Barnard, Fitzgerald and Gong [Barnard et al. 1994] were the first to show that the version of the distortion theorem for the determinant of the Jacobian of normalized biholomorphic convex mappings holds on the unit ball $\mathbb{B}_2$ in $\mathbb{C}^2$, but there does not exist a direct generalization of the distortion theorem in the case of the family of starlike mappings. The monograph of Graham and Kohr [2003, Chapter 7] and Gong [1998, Chapter 3, Chapter 4] contain a nice development of the growth theorem and distortion theorem for starlike mappings and convex mappings. And for a more classical results concerning starlike mappings and convex mappings in n-dimensional Euclidean space or complex Banach space; see [Gurganus 1975; Kikuchi 1973; Pfaltzgraff 1974; Poreda 1989; Roper and Suffridge 1995; Suffridge 1970; 1973; 1977].

Hamada and Honda [2008] introduced a subfamily of starlike mappings on the unit ball in complex Banach spaces, which is called g-starlike mappings. They also obtained a sharp growth theorem for this mappings by using the method of parametric representation. Recently, the distortion theorem for g-starlike mappings on the unit ball $\mathbb{B}_n$ was obtained by Graham, Hamada and Kohr [Graham et al. 2020a] using the Schwarz lemma at the boundary. As a generalization of spirallike mappings, Bălăeţi and Nechita [2010] defined almost starlike mappings of complex order λ on the unit ball $\mathbb{B}_n$ and gave an equivalent characterization in terms of Loewner chains. It is interesting to note that the family of g-starlike mappings gives a unified representation of some well-known subfamilies of starlike mappings, and the family of almost starlike mappings of complex order λ gives a unified expression of some well-known subfamilies of spirallike mappings of type β. There is a lot of results concerning g-starlike mappings and almost starlike mappings of complex order λ; see [Chirilă 2014; 2015; Graham et al. 2002a; 2020b; Hamada and Kohr 2004; Hamada et al. 2006; 2021; Li and Zhang 2019; Zhang et al. 2018].

In view of the above results, the motivation for this paper can be summarized in terms of the following question:

Question. Can we unify g-starlike mappings and complex order λ on the unit ball of complex Banach spaces and characterize their geometric and analytic properties?

We manage to answer the above questions affirmatively in the case of the unit ball of some complex Banach space. In Section 2, the definition of g-starlike mappings of complex order λ is given by combining the definition of g-starlike mappings with the definition of almost starlike mappings of complex order λ on the unit ball in complex Banach spaces. As mentioned in Remark 2.4, it gives a unified expression of a variety of biholomorphic mappings, which includes g-starlike mappings, almost starlike mappings of complex order λ as the special case. In Section 3, by using Loewner chains idea, we establish a growth theorem of g-starlike mappings of complex order λ in reflexive complex Banach spaces, which is a generalization of [Hamada and Honda 2008, Theorem 3.1]. Because the family of g-starlike mappings of complex order λ contains most of the biholomorphic mappings that have geometry meaning in higher dimensions, this result essentially corresponds to giving a unified form of the growth theorems for some subfamilies of starlike mappings and spirallike mappings. As applications, in Section 4, we obtain distortion theorems for g-starlike mappings of complex order λ on the unit polydisk $\mathbb{D}^n$ and the unit ball $\mathbb{B}_n$ respectively, which is a generalization of [Graham et al. 2020a, Theorem 5.6, Theorem 5.11; Liu et al. 2015, Theorem 4.2; 2011, Theorem 3.1, Theorem 3.2]. In Section 5, we will prove that the Roper–Suffridge type extension operator and the Muir type extension operator preserve g-starlike mappings of complex order λ on domain Ω_r respectively, where g is a univalent convex function on $\mathbb{D}$. In particular, if $\lambda = 0$, then the results obtained in this paper are generalizations of results in [Graham et al. 2020b; Muir 2005].

2. Preliminaries

2A. ***Notations and definitions.*** Let $\mathbb{D}_r = \{\zeta \in \mathbb{C} : |\zeta| < r\}$ be the disk of radius r in the complex plane $\mathbb{C}$, and let $\mathbb{D}_1 = \mathbb{D}$. Let $\mathbb{C}^n$ denote the space of n complex variables $u = (u_1, \dots, u_n)'$ equipped with inner product $\langle u, v\rangle = \sum_{k=1}^{n} u_k \overline{v_k}$, and the Euclidean norm $\|u\| = \sqrt{\sum_{k=1}^{n} |u_k|^2}$, the symbol $'$ means the transpose of vectors and matrices. The open ball centered at zero and radius r in $\mathbb{C}^n$ is denoted by $\mathbb{B}_n(0, r) = \{u \in \mathbb{C}^n : \|u\| < r\}$, the closed ball is denoted by $\overline{\mathbb{B}_n(0, r)}$, the unit ball is denoted by $\mathbb{B}_n$. Let $\mathbb{D}^n(0, r) = \{u = (u_1, \dots, u_n)' \in \mathbb{C}^n : |u_k| < r, k = 1, \dots, n\}$ be the polydisk of radius r. The unit polydisk is denoted by $\mathbb{D}^n$. The boundary of $\mathbb{B}_n$ is denoted by $\partial\mathbb{B}_n = \left\{u \in \mathbb{C}^n : \sum_{k=1}^{n} |u_k|^2 = 1\right\}$, the distinguished boundary of the polydisk $\mathbb{D}^n$ is denoted by $(\partial\mathbb{D})^n = \{u \in \mathbb{C}^n : |u_k| = 1, k = 1, \dots, n\}$. Let X be a complex Banach space with respect to the norm $\|\cdot\|_X$. Let $\mathcal{B}_r = \{x \in X : \|x\|_X < r\}$ be the open ball centered at zero and of radius r, and let $\mathcal{B}$ be the open unit ball in X. Let $\overline{\mathcal{B}_r}$ be the closed ball centered at zero and of radius r. Let $\Omega \subseteq X$ be a domain which contains the origin, we denote by $H(\Omega)$ the set of holomorphic mappings from Ω to X. If $f \in H(\Omega)$, and $f(0) = 0$, $Df(0) = I$, then we say that f is normalized, where $Df(0)$ is the Fréchet derivative of f at 0, I is the identity operator on X. A holomorphic mapping $f \in H(\Omega)$ is said to be biholomorphic if the inverse f^{-1} exists and it is holomorphic on the open set $f(\Omega)$. A mapping $f \in H(\Omega)$ is said to be locally biholomorphic if each $x \in \Omega$ has a neighborhood V such that $f\,|_V$ is biholomorphic. If $X = \mathbb{C}^n$, then $Df(z) = J_f(z)$ is the Jacobian matrix of f.

Let $T : X \to \mathbb{C}$ be a continuous linear functional. Then

$$\|T\| = \sup\{|Tx| : x \in \partial\mathcal{B}\}.$$

For each $x \in X\backslash\{0\}$, we define $T(x) = \{T_x \in X^* : \|T_x\| = 1, \|T_x(x)\| = \|x\|\}$. According to the Hahn–Banach theorem, $T(x)$ is nonempty. For any fixed $x \in X$, $\zeta \in \mathbb{C}\backslash\{0\}$, we have $T_{\zeta x} = (|\zeta|/\zeta)T_x$. In particular, $T_{rx} = T_x$ when $r > 0$.

The following elementary definitions are used:

- If for any $x \in \Omega$, $t \in [0, 1]$, $(1-t)x \in \Omega$ holds, then Ω is said to be starlike (with respect to the origin).

- A domain $\Omega \subseteq X$ is said to be convex if given $x_1, x_2 \in \Omega$, $tx_1 + (1-t)x_2 \in \Omega$, for all $t \in [0, 1]$.

- A domain $\Omega \subseteq X$ is said to be ε-starlike if there exists a positive number $\varepsilon \in [0, 1]$, such that for any $z, w \in \Omega$, one has $(1-t)z + \varepsilon t w \in \Omega$ for all $t \in [0, 1]$.

In particular, if $\varepsilon = 0$ or $\varepsilon = 1$, then the ε-starlike domain reduces to starlike domain with respect to the origin or convex domain, respectively.

• Let $f \in H(\Omega)$ be biholomorphic mapping with $0 \in f(\Omega)$. If $f(\Omega)$ is starlike (with respect to the origin), then f is said to be starlike. If $f(\Omega)$ is convex, then f is said to be convex. If $f(\Omega)$ is ε-starlike, then f is said to be ε-starlike, where $\varepsilon \in [0, 1]$.

• Let $g : \mathbb{D} \to \mathbb{C}$ be a holomorphic univalent function, $g(0) = 1$ and $\Re g(\zeta) > 0$. Furthermore, let g be symmetric along the real axis, i.e., $g(\bar{\zeta}) = \overline{g(\zeta)}$, and satisfy the condition

$$\begin{cases} \min_{|\zeta|=r} \Re g(\zeta) = \min\{g(r), g(-r)\}; \\ \max_{|\zeta|=r} \Re g(\zeta) = \max\{g(r), g(-r)\}. \end{cases}$$

Let $G(\mathbb{D})$ denote the family of holomorphic functions g defined as above.

2B. ***Loewner chains.*** We next recall the notions of subordination and Loewner chains on the unit ball $\mathcal{B}$ in X. Some results may be found in [Graham et al. 2013; 2020b].

A mapping $v \in H(\mathcal{B})$ is called a Schwarz mapping if $v(0) = 0$ and $\|v(x)\|_X < 1$, $x \in \mathcal{B}$.

If $f, g \in H(\mathcal{B})$, and there exists a Schwarz mapping v such that $f = g \circ v$, then we say that f is subordinate to g, denoted by $f \prec g$.

If g is biholomorphic on $\mathcal{B}$, then $f \prec g$ is equivalent to requiring that $f(0) = g(0)$ and $f(\mathcal{B}) \subseteq g(\mathcal{B})$.

Definition 2.1. Let $\mathcal{B}$ be the unit ball of a complex Banach space X. A mapping $f : \mathcal{B} \times [0, \infty) \to X$ is called a univalent subordination chain if $f(\cdot, t)$ is univalent on $\mathcal{B}$, $f(0, t) = 0$ for $t \geq 0$, and $f(\cdot, s) \prec f(\cdot, t)$ when $0 \leq s \leq t < \infty$. A univalent subordination chain $f : \mathcal{B} \times [0, \infty) \to X$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on $\mathcal{B}$ and $Df(0, t) = e^t I$, for all $t \geq 0$.

The subordination condition of Loewner chain is equivalent to the existence of a unique biholomorphic Schwarz mapping $v = v(\cdot, s, t)$, called the transition mapping associated with $f(x, t)$, such that $f(x, s) = f(v(x, s, t), t)$ for $x \in \mathcal{B}$ and $0 \leq s \leq t$.

Let $g \in G(\mathbb{D})$ be defined as above. The family $\mathcal{M}_g(\mathcal{B})$ of holomorphic mappings $h : \mathcal{B} \to X$ that is analogous to the analytic functions on the unit disk in the complex plane, with positive real part, is defined as follows.

$$\mathcal{M}_g(\mathcal{B}) = \left\{ h \in H(\mathcal{B}) : h(0) = 0, Dh(0) = I, \frac{1}{\|x\|_X} T_x\{h(x)\} \in g(\mathbb{D}), T_x \in T(x), x \in \mathcal{B}\setminus\{0\} \right\}.$$

If $g(\zeta) = (1 + \zeta)/(1 - \zeta)$, $\zeta \in \mathbb{D}$, then $\mathcal{M}_g(\mathcal{B})$ reduces to the Carathéodory family $\mathcal{M}(\mathcal{B})$ on the unit ball $\mathcal{B}$ in a complex Banach space.

We know that both $\mathcal{M}(\mathcal{B})$ and $\mathcal{M}_g(\mathcal{B})$ consist of so-called holomorphically accretive mappings, which were intensively studied in Euclidean space $\mathbb{C}^n$ or complex Banach spaces during the last decades. Some related results may be found in [Duren et al. 2010; Elin et al. 2019; Graham et al. 2002a; 2013; Hamada and Kohr 2004; Pfaltzgraff 1974; Reich and Shoikhet 1996; 2005; Suffridge 1973].

Definition 2.2 [Bracci et al. 2009; Duren et al. 2010; Graham et al. 2002a]. A Herglotz vector field associated with the family $\mathcal{M}(\mathcal{B})$ on $\mathcal{B}$ is a mapping $h = h(x,t) : \mathcal{B} \times [0,\infty) \to X$ satisfying the following conditions:

(i) $h(\cdot\,, t) \in \mathcal{M}(\mathcal{B})$, for a.e. $t \geq 0$.

(ii) $h(x, \cdot)$ is strongly measurable on $[0,\infty)$, for all $x \in \mathcal{B}$.

Hamada and Kohr [2004] proved that if X is a reflexive complex Banach space, and $h(x,t) : \mathcal{B} \times [0,\infty) \to X$ is a Herglotz vector field, then for each $s \geq 0$ and $x \in \mathcal{B}$, the initial value problem

$$\begin{cases} \frac{\partial v}{\partial t} = -h(v,t), & a.e.\ s \leq t, \\ v(x,s,s) = x, & t = s \end{cases}$$

has a unique solution $v = v(x,s,t)$ such that $v(\cdot\,, s, t)$ is a univalent Schwarz mapping, $v(x,s,\cdot)$ is Lipschitz continuous on $[s,\infty)$ uniformly with respect to $x \in \overline{\mathcal{B}_r}$, $r \in (0,1)$, $Dv(0,s,t) = e^{s-t}I$ for $0 \leq s \leq t$. Furthermore, the following limit

$$\lim_{t\to\infty} e^t v(x,s,t) = f(x,s)$$

exists uniformly on each closed ball $\overline{\mathcal{B}_r}$ for $r \in (0,1)$, $s \in [0,\infty)$. And $f(x,t)$ is a univalent subordination chain.

2C. *g-starlike mappings of complex order* λ.

Definition 2.3. Let $g \in G(\mathbb{D})$, $\lambda \in \mathbb{C}$ with $\Re\lambda \leq 0$. And let $f : \mathcal{B} \to X$ be a normalized locally biholomorphic mapping. If

$$\{(1-\lambda)(Df(x))^{-1} f(x) + \lambda x\} \in \mathcal{M}_g(\mathcal{B}),$$

then f is called a g-starlike mapping of complex order λ.

We denote by $\mathcal{S}^*_{g,\lambda}(\mathcal{B})$ the family of g-starlike mapping of complex order λ on $\mathcal{B}$.

Obviously, for the case of $X = \mathbb{C}, \mathcal{B} = \mathbb{D}$, the above definition shows that $f \in \mathcal{S}^*_{g,\lambda}(\mathbb{D})$ if and only if $(1-\lambda) f(z)/(zf'(z)) + \lambda \prec g$.

Remark 2.4. (i) Let $\lambda = 0$. Then $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ is a g-starlike mapping on the unit ball $\mathcal{B}$, some results of g-starlike mappings may be found in [Chirilă 2014; 2015; Graham et al. 2002a; Hamada et al. 2021].

(ii) Let $\alpha \in [0, 1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\lambda = (\alpha - i \tan\beta)/(\alpha - 1)$. Then $\mathcal{S}^*_{g,\lambda}(\mathcal{B}) = \widehat{\mathcal{S}}_g^{\alpha,\beta}(\mathcal{B})$, the definition on the unit ball $\mathbb{B}_n$ in Euclidean space can be found in [Tu and Xiong 2019].

(iii) Let $g(\zeta) = (1+\zeta)/(1-\zeta)$, $\zeta \in \mathbb{D}$. Then $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ means that

$$\frac{1}{\|x\|} T_x\{(1-\lambda)(Df(x))^{-1} f(x) + \lambda x\}$$

maps the unit ball $\mathcal{B}\backslash\{0\}$ into the right half plane, i.e.,

$$\Re T_x\{(1-\lambda)(Df(x))^{-1} f(x)\} \geq -\|x\| \Re\lambda,\ x \in \mathcal{B}\backslash\{0\}.$$

This is the definition of almost starlike mappings of complex order λ; see [Bălăeţi and Nechita 2010; Zhang et al. 2018].

(iv) Let $g(\zeta) = (1 + A\zeta)/(1 + B\zeta)$, $-1 \leq B < A \leq 1$, $\zeta \in \mathbb{D}$. Then $\mathcal{S}^*_{g,\lambda}(\mathcal{B}) = \mathcal{S}^*_{\mathcal{B}}[A, B, \lambda]$ is the Janowski-starlike mappings of complex order λ on the unit ball $\mathcal{B}$; see [Li and Zhang 2019].

Let $\alpha \in (0, 1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $A = 1$, $B = 2\alpha - 1$, $\lambda = i \tan\beta$, then $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ means that $\frac{1}{\|x\|} T_x\{(1-\lambda)(Df(x))^{-1} f(x) + \lambda x\}$ maps the unit ball $\mathcal{B}\backslash\{0\}$ into the domain $\Sigma_1 = \left\{\zeta \in \mathbb{C} : \left|\zeta - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}\right\}$, i.e.,

$$\left| e^{-i\beta} \frac{1}{\|x\|} T_x\{(Df(x))^{-1} f(x)\} - \left(\frac{\cos\beta}{2\alpha} - i \sin\beta\right) \right| < \frac{\cos\beta}{2\alpha}, \quad x \in \mathcal{B}\backslash\{0\}.$$

This is the definition of spirallike mappings of type β and order α; see [Feng et al. 2007].

(v) Let $\rho \in [0, 1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\lambda = i \tan\beta$. If

$$g(\zeta) = 1 + 4(1-\rho)/\pi^2 (\log(1+\sqrt{\zeta})/(1-\sqrt{\zeta}))^2, \quad \zeta \in \mathbb{D},$$

then $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ means that $\frac{1}{\|x\|} T_x\{(1-\lambda)(Df(x))^{-1} f(x) + \lambda x\}$ maps the unit ball $\mathcal{B}\backslash\{0\}$ into the domain $\Sigma_2 = \{\zeta \in \mathbb{C} : |\zeta - 1| < (1-2\rho) + \Re\{\zeta\}\}$, i.e.,

$$\left| \frac{1}{\|x\|} T_x\{(Df(x))^{-1} f(x) - 1\} \right| < (1-2\rho)\cos\beta + \Re\left\{ e^{-i\beta} \frac{1}{\|x\|} T_x\{(Df(x))^{-1} f(x)\} \right\}, \quad x \in \mathcal{B}\backslash\{0\},$$

where the branch of the logarithm function is chosen such that $\log 1 = 0$, which reduces to the definition of parabolic spirallike mappings of type β and order ρ; see [Zhang and Yan 2016].

Next, we give two examples in higher dimensions.

Example 2.5. Assume $\lambda \in \mathbb{C}$, $\Re\lambda \leq 0$ and $g \in G(\mathbb{D})$ is a convex function. Suppose that $f : \mathbb{B}_n \to \mathbb{C}^n$ is holomorphic with $f(z) = (f_1(z_1), f_2(z_2), \dots, f_n(z_n))'$, where $f_j(z_j)$, $j = 1, 2, \dots, n$, are normalized biholomorphic functions on $\mathbb{D}$. If

$$(1-\lambda)\frac{f_j(z_j)}{z_j f_j'(z_j)} + \lambda \prec g(z_j), \quad z_j \in \mathbb{D}, \ j = 1, 2, \dots, n,$$

then $f \in S^*_{g,\lambda}(\mathbb{B}_n)$.

Proof. Since

$$\frac{1}{\sum_{j=1}^n |z_j|^2}\langle (1-\lambda)(Df(z))^{-1} f(z) + \lambda z, z\rangle$$

$$= \frac{1}{\sum_{j=1}^n |z_j|^n}\sum_{j=1}^n |z_j|^2 \left((1-\lambda)\frac{f_j(z_j)}{z_j f_j'(z_j)} + \lambda\right) \in g(\mathbb{D}),$$

we have $f \in S^*_{g,\lambda}(\mathbb{B}_n)$. □

Example 2.6. Let $a, \lambda \in \mathbb{C}$, $\Re\lambda \leq 0$, $g \in G(\mathbb{D})$. Assume that $f : \mathbb{B}_n \to \mathbb{C}^n$ is a holomorphic mapping with $f(z) = (z_1 + a z_2^2, z_2, \dots, z_n)'$. If

$$|a| \leq \frac{3\sqrt{3}}{2}\frac{1}{|1-\lambda|}\operatorname{dist}(1, \partial g(\mathbb{D})),$$

then $f \in S^*_{g,\lambda}(\mathbb{B}_n)$.

Proof. By some elementary calculations, we get

$$(Df(z))^{-1} = \begin{pmatrix} 1 & -2az_2 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then

$$\frac{1}{\|z\|^2}\langle (1-\lambda)(Df(z))^{-1} f(z) + \lambda z, z\rangle = 1 - \frac{a(1-\lambda)\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}.$$

Since $|a| \leq \frac{3\sqrt{3}}{2}\frac{1}{|1-\lambda|}\operatorname{dist}(1, \partial g(\mathbb{D}))$, it yields that

$$\left|\frac{a(1-\lambda)\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}\right| \leq \left|\frac{a(1-\lambda)\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2}\right| < \frac{2}{3\sqrt{3}}|a||1-\lambda| \leq \operatorname{dist}(1, \partial g(\mathbb{D})).$$

This implies

$$1 - \frac{a(1-\lambda)\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2} \in g(\mathbb{D}).$$

Thus $f \in S^*_{g,\lambda}(\mathbb{B}_n)$. □

3. Growth theorems for g-starlike mappings of complex order λ

In the next subsection, we utilize the method of Loewner chains to deal with the growth theorem of g-starlike mappings of complex order λ on the unit ball in a reflexive complex Banach space X. The family of g-starlike mappings of complex order λ unifies the family of almost starlike mappings of complex order λ and the family of g-starlike mappings, and the result in the forthcoming subsection will lead to a number of well known statements.

3A. ***Several lemmas.*** We begin this subsection with the following equivalent characterization for almost starlike mappings of complex order λ in terms of Loewner chains on the unit ball $\mathcal{B}$.

Lemma 3.1 [Zhang et al. 2018]. *Let f be a normalized locally biholomorphic mapping on $\mathcal{B}$, and let $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$. Then f is an almost starlike mapping of complex order λ on $\mathcal{B}$ if and only if*

$$F(x,t) = e^{(1-\lambda)t} f(e^{\lambda t}x), \quad \forall x \in \mathcal{B}, t \in [0,+\infty)$$

is a Loewner chain.

The following lemma is due to Kato.

Lemma 3.2 [Kato 1967]. *Let $x : [0,+\infty) \to X$ be differentiable at the point $s \in (0,+\infty)$, and let $\|x(t)\|$ be also differentiable at the point s with respect to t. Then*

$$\Re\left\{T_{x(s)}\left[\frac{dx}{dt}(s)\right]\right\} = \frac{d\|x(s)\|}{dt}, \quad s \in [0,+\infty).$$

In fact, the following lemma shows that Loewner chain is generated by its transition mapping. It is due to Graham et al. [2013].

Lemma 3.3. *Suppose that X is a reflexive complex Banach space. Let $f(x,t) : \mathcal{B} \times [0,\infty) \to X$ be a Loewner chain. And let $v(x,s,t)$ be the transition mapping associated with $f(x,t)$. If for each $r \in (0,1)$, there exists $M = M(r) > 0$ such that*

$$\|e^{-t} f(x,t)\|_X \le M(r), \quad x \in \mathcal{B}_r, t \in [0,\infty),$$

then

$$f(x,s) = \lim_{t\to\infty} e^t v(x,s,t)$$

uniformly on $\overline{\mathcal{B}_r}$ for $r \in (0,1)$.

In fact, the following lemma plays an important role in the proof of growth theorem.

Lemma 3.4. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \leq 0$, $g \in G(\mathbb{D})$, and let $f : \mathcal{B} \to X$ be a g-starlike mapping of complex order λ on $\mathcal{B}$. Then*

$$\begin{aligned}(3.1)\quad -\|x\|\Re\lambda + \|x\| \min\{g(\|x\|), g(-\|x\|)\} &\leq \Re\{(1-\lambda)T_x[(Df(x))^{-1} f(x)]\} \\ &\leq -\|x\|\Re\lambda + \|x\| \max\{g(\|x\|), g(-\|x\|)\}.\end{aligned}$$

Moreover, if $g \in G(\mathbb{D})$ *also satisfies* $\max_{|\zeta|=r}|g(\zeta)| = \max\{g(r), g(-r)\}$, $r \in (0, 1)$, *then*

$$\begin{aligned}-\|x\|\Re\lambda + \|x\| \min\{g(\|x\|), g(-\|x\|)\} &\leq |(1-\lambda)T_x[(Df(x))^{-1} f(x)]| \\ &\leq \|x\||\lambda| + \|x\| \max\{g(\|x\|), g(-\|x\|)\}.\end{aligned}$$

Proof. Fixing $x \in \mathcal{B}\backslash\{0\}$, let $x_0 = \frac{x}{\|x\|}$. Then the holomorphic function

$$q(\zeta) = \begin{cases}(1-\lambda)\frac{1}{\zeta}T_{x_0}[(Df(\zeta x_0))^{-1} f(\zeta x_0)] + \lambda, & \zeta \in \mathbb{D}\backslash\{0\}, \\ 1, & \zeta = 0,\end{cases}$$

is well defined on the unit disk $\mathbb{D}$. Since

$$q(\zeta) = (1-\lambda)\frac{1}{|\zeta|}T_{\zeta x_0}[(Df(\zeta x_0))^{-1} f(\zeta x_0)] + \lambda, \zeta \neq 0,$$

from Definition 2.3, it yields that $q(0) = g(0) = 1$, $q(\mathbb{D}) \subseteq g(\mathbb{D})$, i.e., $q \prec g$.

By the subordination principle, it follows that $q(r\mathbb{D}) \subseteq g(r\mathbb{D})$, $r \in (0, 1)$. Hence

$$\min\{g(r), g(-r)\} \leq \Re q(\zeta) \leq \max\{g(r), g(-r)\}.$$

Let $\zeta = \|x\|$. Then

$$\begin{aligned}-\|x\|\Re\lambda+\|x\| \min\{g(\|x\|), g(-\|x\|)\} &\leq \Re\{(1-\lambda)T_x[(Df(x))^{-1} f(x)]\} \\ &\leq -\|x\|\Re\lambda+\|x\| \max\{g(\|x\|), g(-\|x\|)\}.\end{aligned}$$

If we impose the condition $\max_{|\zeta|=r}|g(\zeta)| = \max\{g(r), g(-r)\}$, $r \in (0, 1)$, then

$$\begin{aligned}-\|x\|\Re\lambda+\|x\| \min\{g(\|x\|), g(-\|x\|)\} &\leq |(1-\lambda)T_x[(Df(x))^{-1} f(x)]| \\ &\leq \|x\||\lambda|+\|x\| \max\{g(\|x\|), g(-\|x\|)\}. \quad \square\end{aligned}$$

Remark 3.5. If $\mathcal{B} = \mathbb{B}_n \subseteq \mathbb{C}^n$, then the inequality (3.1) is equivalent to the following form:

$$\begin{aligned}-\|z\|^2\Re\lambda+\|z\|^2 \min\{g(\|z\|), g(-\|z\|)\} &\leq \Re\{(1-\lambda)\bar{z}'[(Df(z))^{-1} f(z)]\} \\ &\leq -\|z\|^2\Re\lambda+\|z\|^2\max\{g(\|z\|), g(-\|z\|)\}.\end{aligned}$$

3B. *Growth theorems of the classes $\mathcal{S}^*_{g,\lambda}(\mathcal{B})$.* The method to approach the following theorem is analogous to that of [Zhang et al. 2018], although we are now considering normalized biholomorphic mappings on the unit ball $\mathcal{B}$ in an infinite dimensional complex Banach space.

Theorem 3.6. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$, $g \in G(\mathbb{D})$, and let $f : \mathcal{B} \to X$ be a g-starlike mapping of complex order λ on $\mathcal{B}$ in reflexive complex Banach space. Then*

$$\begin{aligned}\|x\| \exp\biggl(\int_0^{\|x\|}\biggl[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\}-\Re\lambda}-1\biggr]\frac{dy}{y}\biggr) &\\ \le \|f(x)\| &\\ \le \|x\| \exp\biggl(\int_0^{\|x\|}\biggl[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\}-\Re\lambda}-1\biggr]\frac{dy}{y}\biggr).&\end{aligned}$$

Proof. Since $\Re g(\zeta) > 0$, $\zeta \in \mathbb{D}$, we have $f \in \mathcal{S}^*_{g,\lambda}(\mathcal{B})$ is also an almost starlike mapping of complex order λ. By Lemma 3.1 we know that

$$F(x,t) = e^{(1-\lambda)t} f(e^{\lambda t}x)$$

is a Loewner chain, hence we have

$$F(x,s) \prec F(x,t), \quad \forall 0 \le s \le t,$$

i.e., there is a Schwarz mapping $v(x,s,t)$ such that $F(x,s) = F(v(x,s,t),t)$. By some calculation, we obtain

$$\begin{aligned}&\frac{\partial F}{\partial t}(x,t) = (1-\lambda)e^t e^{-\lambda t} f(e^{\lambda t}x) + \lambda e^t Df(e^{\lambda t}x)x,\\ &DF(x,t) = e^t Df(e^{\lambda t}x).\end{aligned}$$

Let $\frac{\partial F}{\partial t}(x,t) = DF(x,t)h(x,t)$. Then

$$h(x,t) = (1-\lambda)e^{-\lambda t}(Df(e^{\lambda t}x))^{-1} f(e^{\lambda t}x) + \lambda x.$$

For fixed $x \in \mathcal{B}\backslash\{0\}$, $s \ge 0$, let $v(t) = v(x,s,t)$. Then

$$\frac{\partial v}{\partial t}(t) = -(DF(v(t),t))^{-1}\frac{\partial F}{\partial t}(v(t),t) = -h(v(t),t).$$

Since for all $x \in \mathcal{B}$, we have

$$\|e^{-t}F(x,t)\|_X \le \begin{cases}\frac{\|x\|_X}{(1-\|x\|_X)^{2/(1+\Re\lambda)}}, & \Re\lambda \ne -1,\\ \|x\|_X \exp(\|x\|_X), & \Re\lambda = -1,\end{cases}$$

here we use the fact that f is also an almost starlike mapping of complex order λ and the upper bound of $\|f(x)\|_X$; see [Zhang et al. 2018, Theorem 3.1]. By

Lemma 3.3, we obtain

$$\lim_{t\to\infty} e^t v(x,s,t) = F(x,s).$$

Furthermore, by Lemmas 3.2 and 3.4 we see that

$$\begin{aligned}
(3.2)\quad \frac{d\|v(t)\|}{dt} &\\
&= \Re T_{v(t)}\left[\frac{dv(t)}{dt}\right]\\
&= -\Re T_{v(t)}[(1-\lambda)e^{-\lambda t}(Df(e^{\lambda t}v(t)))^{-1}f(e^{\lambda t}v(t)) + \lambda v(t)]\\
&= -\Re \frac{e^{\lambda t}}{|e^{\lambda t}|} T_{e^{\lambda t}v(t)}[(1-\lambda)e^{-\lambda t}(Df(e^{\lambda t}v(t)))^{-1}f(e^{\lambda t}v(t))] - \|v(t)\|\Re\lambda\\
&= -\frac{1}{|e^{\lambda t}|}\Re\{(1-\lambda)T_{e^{\lambda t}v(t)}[(Df(e^{\lambda t}v(t)))^{-1}f(e^{\lambda t}v(t))]\} - \|v(t)\|\Re\lambda\\
&\le -\|v(t)\|\min\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\}.
\end{aligned}$$

By Lemma 3.4 and equality (3.2), we have

$$\begin{aligned}
-\|e^{\lambda t}v(t)\|\max\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\} &\\
&\le -\Re\{(1-\lambda)T_{e^{\lambda t}v(t)}[(Df(e^{\lambda t}v(t)))^{-1}f(e^{\lambda t}v(t))]\} - \|e^{\lambda t}v(t)\|\Re\lambda\\
&= |e^{\lambda t}|\frac{d\|v(t)\|}{dt}\\
&\le -\|e^{\lambda t}v(t)\|\min\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\}.
\end{aligned}$$

Since

$$\frac{d\|e^{\lambda t}v(t)\|}{dt} = |e^{\lambda t}|\frac{d\|v(t)\|}{dt} + \|e^{\lambda t}v(t)\|\Re\lambda,$$

we have

$$\begin{aligned}
(3.3)\quad \|e^{\lambda t}v(t)\|\Re\lambda - \|e^{\lambda t}v(t)\|\max\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\} &\\
&\le \frac{d\|e^{\lambda t}v(t)\|}{dt}\\
(3.4)\quad &\le \|e^{\lambda t}v(t)\|\Re\lambda - \|e^{\lambda t}v(t)\|\min\{g(\|e^{\lambda t}v(t)\|), g(-\|e^{\lambda t}v(t)\|)\}\\
&< 0,
\end{aligned}$$

which implies that $\|e^{\lambda t}v(t)\|$ is decreasing on $[s,\infty)$.

Integrating on both sides of the inequality (3.4) with respect to $\tau \in [s,t]$, we infer that

$$\begin{aligned}(1-\Re\lambda)(t-s) &\le \int_s^t \frac{1-\Re\lambda}{\|e^{\lambda\tau}v(\tau)\|\Re\lambda - \|e^{\lambda\tau}v(\tau)\|\min\{g(\|e^{\lambda\tau}v(\tau)\|), g(-\|e^{\lambda\tau}v(\tau)\|)\}} \\ &\qquad\qquad \times \frac{d\|e^{\lambda\tau}v(\tau)\|}{d\tau}d\tau \\ &= \int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|} \frac{1-\Re\lambda}{y\Re\lambda - y\min\{g(y), g(-y)\}}dy \\ &= \int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|} \left[\frac{1-\Re\lambda}{\Re\lambda - \min\{g(y), g(-y)\}} + 1\right]\frac{dy}{y} - \int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|} \frac{1}{y}dy,\end{aligned}$$

hence

$$(1-\Re\lambda)(t-s) \le \int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|} \left[\frac{1-\Re\lambda}{\Re\lambda - \min\{g(y), g(-y)\}} + 1\right]\frac{dy}{y} + \log\frac{\|e^{\lambda s}x\|}{\|e^{\lambda t}v(t)\|},$$

i.e.,

$$e^{(t-s)} \le \frac{\|x\|}{\|v(t)\|}\exp\left(\int_{\|e^{\lambda s}x\|}^{\|e^{\lambda t}v(t)\|} \left[\frac{1-\Re\lambda}{\Re\lambda - \min\{g(y), g(-y)\}} + 1\right]\frac{dy}{y}\right). \tag{3.5}$$

By using Lemma 3.3 we have $\|e^t v(t)\| \to \|e^{(1-\lambda)s} f(e^{\lambda s}x)\|$ as $t \to +\infty$. Because $\lim_{t\to+\infty}\|e^{\lambda t}v(t)\| = 0$, then taking $t \to +\infty$ on the both sides of the inequality (3.5), and taking $s = 0$, we see that

$$\|f(x)\| \le \|x\|\exp\left(\int_0^{\|x\|}\left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right).$$

By using the same method for obtaining inequality (3.3), we get

$$\|f(x)\| \ge \|x\|\exp\left(\int_0^{\|x\|}\left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right). \qquad \square$$

Remark 3.7. In particular, if $g \in G(\mathbb{D})$ and $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$ are some special functions and special complex number, such as in Remark 2.4, we can get the growth theorems of starlike mappings, spirallike mappings of type β, etc. This is one of the reasons for the interest in this normalized biholomorphic mappings.

4. Distortion theorems for g-starlike mappings of complex order λ

4A. ***Distortion theorems along a unit direction.*** In this subsection, we obtain the distortion theorems for g-starlike mappings of complex order λ along a unit direction on the unit polydisk $\mathbb{D}^n$ and the unit ball $\mathcal{B}$, respectively.

Theorem 4.1. *Let* $\lambda \in \mathbb{C}$ *with* $\Re\lambda \le 0$, $g \in G(\mathbb{D})$ *with*

$$\max_{|\zeta|=r}|g(\zeta)| = \max\{g(r), g(-r)\},$$

$r \in (0,1)$, *and let* $f : \mathbb{D}^n \to \mathbb{C}^n$ *be a g-starlike mapping of complex order* λ. *Then, for all* $z \in \mathbb{D}^n \setminus \{0\}$, *there exists a unit vector* $\zeta(z)$ *such that*

$$\begin{aligned}\frac{|1-\lambda|}{|\lambda|+\max\{g(\|z\|), g(-\|z\|)\}} &\exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\}-\Re\lambda}-1\right]\frac{dy}{y}\right)\\ &\le \|Df(z)\zeta(z)\|\\ &\le \frac{|1-\lambda|}{\min\{g(\|z\|), g(-\|z\|)\}-\Re\lambda}\exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\}-\Re\lambda}-1\right]\frac{dy}{y}\right).\end{aligned}$$

Proof. The proof is divided into the following two steps:

Step 1. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{D}^n$ with $|\xi_1| = |\xi_2| = \cdots = |\xi_n| = \|\xi\|$. Then $T_\xi = \left(0, \ldots, 0, \frac{\|\xi\|}{\xi_j}, 0, \ldots, 0\right) \in T(\xi)$.

Taking $w(z) = (w_1(z), \ldots, w_n(z))' = (Df(z))^{-1} f(z)$, then there exists $1 \le j \le n$ such that

$$\begin{aligned}\|w(z)\| &= |w_j(z)|\\ &\le \max_{\xi \in (\partial\mathbb{D}(0,\|z\|))^n} |w_j(\xi)|\\ &= \max_{\xi \in (\partial\mathbb{D}(0,\|z\|))^n} \left|\frac{\|\xi\|}{\xi_j} w_j(\xi)\right|\\ &= \max_{\xi \in (\partial\mathbb{D}(0,\|z\|))^n} |T_\xi[w(\xi)]|\\ &= \max_{\xi \in (\partial\mathbb{D}(0,\|z\|))^n} |T_\xi[(Df(\xi))^{-1} f(\xi)]|.\end{aligned}$$

By using Lemma 3.4, we have

$$\begin{aligned}|1-\lambda||T_\xi[(Df(\xi))^{-1} f(\xi)]| &\le \|\xi\||\lambda| + \|\xi\| \max\{g(\|\xi\|), g(-\|\xi\|)\}\\ &\le \|z\||\lambda| + \|z\| \max\{g(\|z\|), g(-\|z\|)\}.\end{aligned}$$

Hence

$$\|(Df(z))^{-1} f(z)\| \le \frac{\|z\|}{|1-\lambda|}(|\lambda| + \max\{g(\|z\|), g(-\|z\|)\}). \tag{4.1}$$

Since $\|T_z\| \le 1$, by Lemma 3.4 we get

$$\begin{aligned}\|(Df(z))^{-1} f(z)\| &\ge |T_z[(Df(z))^{-1} f(z)]|\\ &\ge \frac{\|z\|}{|1-\lambda|}(-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}).\end{aligned} \tag{4.2}$$

Step 2. Let $\zeta(z) = (Df(z))^{-1} f(z)/\|(Df(z))^{-1} f(z)\|$, $z \in \mathbb{D}^n\backslash\{0\}$. Then

$$f(z) = Df(z)(Df(z))^{-1} f(z) = \|(Df(z))^{-1} f(z)\| Df(z)\zeta(z).$$

Hence, by Theorem 3.6, (4.1) and (4.2), we have

$$\begin{aligned}
&\frac{|1-\lambda|}{|\lambda| + \max\{g(\|z\|), g(-\|z\|)\}} \exp\left(\int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right) \\
&\le \|Df(z)\zeta(z)\| = \frac{\|f(z)\|}{\|(Df(z))^{-1} f(z)\|} \\
&\le \frac{|1-\lambda|}{\min\{g(\|z\|), g(-\|z\|)\} - \Re\lambda} \exp\left(\int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right),
\end{aligned}$$

which completes the proof. □

Theorem 4.2. *Let* $\lambda \in \mathbb{C}$ *with* $\Re\lambda \le 0$, $g \in G(\mathbb{D})$ *with*

$$\max_{|\zeta|=r} |g(\zeta)| = \max\{g(r), g(-r)\},$$

$r \in (0, 1)$, *and let* $f : \mathcal{B} \to X$ *be a* g*-starlike mapping of complex order* λ *in reflexive complex Banach spaces. Then, for all* $x \in \mathcal{B}\backslash\{0\}$, *there exists a unit vector* $\zeta(x)$ *such that*

$$\begin{aligned}
&\|Df(x)\zeta(x)\| \\
&\le \frac{|1-\lambda|}{\min\{g(\|x\|), g(-\|x\|)\} - \Re\lambda} \exp\left(\int_0^{\|x\|} \left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right).
\end{aligned}$$

Proof. Let $\zeta(x) = \frac{(Df(x))^{-1} f(x)}{\|(Df(x))^{-1} f(x)\|} \in \partial\mathcal{B}$. Then

$$f(x) = Df(x)(Df(x))^{-1} f(x) = \|(Df(x))^{-1} f(x)\| Df(x)\zeta(x).$$

By using Lemma 3.4, we get

$$\begin{aligned}
-\|x\|\Re\lambda + \|x\| \min\{g(\|x\|), g(-\|x\|)\} &\le \left|(1-\lambda)T_x[(Df(x))^{-1} f(x)]\right| \\
&\le |1-\lambda| \|(Df(x))^{-1} f(x)\|.
\end{aligned}$$

Hence, by Theorem 3.6, we have

$$\begin{aligned}
\|Df(x)\zeta(x)\| &= \frac{\|f(x)\|}{\|(Df(x))^{-1} f(x)\|} \\
&\le \frac{|1-\lambda|}{\min\{g(\|x\|), g(-\|x\|)\} - \Re\lambda} \\
&\qquad \times \exp\left(\int_0^{\|x\|} \left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right). \quad \square
\end{aligned}$$

Remark 4.3. If $\lambda = 0$ and g is some biholomorphic function in Definition 2.3, we can get the results in [Liu et al. 2011; 2012] from Theorems 4.1 and 4.2.

4B. *Distortion theorems on the unit ball $\mathbb{B}_n$.* In this subsection, the distortion theorems for g-starlike mappings of complex order λ at extreme points are established on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. Denote by $T_{z_0}^{(1,0)}(\partial\mathbb{B}_n) = \{w \in \mathbb{C}^n : \overline{z_0}'w = 0\}$ the complex tangent space at $z_0 \in \partial\mathbb{B}_n$. The following boundary Schwarz lemma is due to Liu et al. [2015] and Graham et al. [2020a], which plays an important role in the proof of the following theorem.

Lemma 4.4 [Graham et al. 2020a; Liu et al. 2015]. *Let $f : \mathbb{B}_n \to \mathbb{B}_n$ be a holomorphic mapping. If f is holomorphic at $z_0 \in \partial\mathbb{B}_n$, $f(z_0) = w_0 \in \partial\mathbb{B}_n$, then $Df(z_0)$ has the following properties*:

(i) *There is a $\mu \in \mathbb{R}$ such that $\overline{Df(z_0)}'w_0 = \mu z_0$ and*

$$\mu = \overline{w_0}'Df(z_0)z_0 \geq \frac{|1 - \bar{c}'w_0|^2}{1 - \|c\|^2} > 0,$$

where $c = f(0)$.

(ii) $\|Df(z_0)\beta\| \leq \sqrt{\mu}$, *for all $\beta \in T_{z_0}^{(1,0)}(\partial\mathbb{B}_n)$ with $\|\beta\| = 1$.*

(iii) $|\det Df(z_0)| \leq \mu^{(n+1)/2}$.

Theorem 4.5. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \leq 0$, $g \in G(\mathbb{D})$, and let $f : \mathbb{B}_n \to \mathbb{C}^n$ be a g-starlike mapping of complex order λ*:

(1) *If $z \in \mathbb{B}_n$ satisfies $\max_{\|\zeta\|=\|z\|}\|f(\zeta)\| = \|f(z)\|$, then*

$$|\det Df(z)| \leq \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}\right)^{(n+1)/2} \times \exp\left(n\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right).$$

(2) *If $z \in \mathbb{B}_n$ satisfies $\min_{\|\zeta\|=\|z\|}\|f(\zeta)\| = \|f(z)\|$, then*

$$|\det Df(z)| \geq \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}\right)^{(n+1)/2} \times \exp\left(n\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right).$$

Proof. Without loss of generality, let $\|z\| = r \in (0, 1)$, $M = \max_{\|\zeta\|=r}\|f(\zeta)\|$ and $m = \min_{\|\zeta\|=r}\|f(\zeta)\|$:

(1) Let $\eta(w) = f(rw)/M$, $w \in \mathbb{B}_n$, then $\eta : \mathbb{B}_n \to \mathbb{B}_n$, $\eta(0) = 0$ and η is biholomorphic in a neighborhood of $\overline{\mathbb{B}_n}$. Take $z_0 = z/r$ and $w_0 = \eta(z_0) = f(z)/M$, then $z_0 \in \partial\mathbb{B}_n$, $w_0 \in \partial\mathbb{B}_n$. By Lemma 4.4, there is a $\mu \in \mathbb{R}$ such that $\overline{D\eta(z_0)}' w_0 = \mu z_0$ and $1 \le \mu = \overline{w_0}' D\eta(z_0) z_0 = \overline{f(z)}' Df(z) z / M^2$. Because $\overline{w_0}' = \mu \overline{z_0}' (D\eta(z_0))^{-1}$, we know that $\overline{f(z)}'$ and $\bar{z}'(Df(z))^{-1}$ have the same direction.

Furthermore, since

$$\begin{aligned} \mu &= \frac{\overline{f(z)}' Df(z) z}{M^2} \\ &= \frac{\|f(z)\| \bar{z}'(Df(z))^{-1} Df(z) z}{\|f(z)\|^2 \|\bar{z}'(Df(z))^{-1}\|} \\ &= \frac{\|z\|^2}{\|f(z)\| \|\bar{z}'(Df(z))^{-1}\|} \\ &= \frac{\|z\|^2}{\bar{z}'(Df(z))^{-1} f(z)} \\ &= \frac{\Re\{(1-\lambda)\|z\|^2\}}{\Re\{(1-\lambda)\bar{z}'(Df(z))^{-1} f(z)\}} \\ &\le \frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}, \end{aligned}$$

by Lemma 4.4 we have

$$|\det D\eta(z_0)| \le \mu^{(n+1)/2} \le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}} \right)^{(n+1)/2}.$$

Because $D\eta(z_0) = \frac{r}{M} Df(rz_0) = \frac{r}{M} Df(z)$, by Theorem 3.6, we obtain

$$\begin{aligned} |\det Df(z)| &= \left(\frac{M}{r}\right)^n |\det D\eta(z_0)| \\ &\le \left(\frac{\|f(z)\|}{\|z\|}\right)^n \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}} \right)^{(n+1)/2} \\ &\le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}} \right)^{(n+1)/2} \\ &\qquad \times \exp\left(n \int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1 \right] \frac{dy}{y} \right). \end{aligned}$$

(2) Let $h(w) = f(rw)/m$, $w \in \mathbb{B}_n$, then $h(0) = 0$ and h is biholomorphic in a neighborhood of $\overline{\mathbb{B}_n}$ with $h(\mathbb{B}_n) \supset \mathbb{B}_n$. Take $z_0 = z/r$ and $w_0 = h(z_0) = f(z)/m$, then $z_0 \in \partial\mathbb{B}_n$ and $w_0 \in \partial\mathbb{B}_n$. Furthermore, $h^{-1} : \mathbb{B}_n \to \mathbb{B}_n$, $h^{-1}(0) = 0$ and h^{-1} is holomorphic in a neighborhood of $\overline{\mathbb{B}_n}$ with $h^{-1}(w_0) = z_0$. For the same reason as in the proof of (1) we conclude that $\overline{f(z)}'$ and $\bar{z}'(Df(z))^{-1}$ have the same direction.

By Lemmas 4.4 and 3.4, there exists a $\mu \in \mathbb{R}$ such that

$$\begin{aligned}
1 \le \mu &= \overline{z_0}' Dh^{-1}(w_0) w_0 \\
&= \overline{z_0}'(Dh(z_0))^{-1} w_0 \\
&= \frac{\bar{z}'\big(\frac{r}{m} Df(z)\big)^{-1}}{r} \frac{f(z)}{m} \\
&= \frac{\bar{z}'(Df(z))^{-1} f(z)}{\|z\|^2} \\
&= \frac{\Re\{(1-\lambda)\bar{z}'(Df(z))^{-1} f(z)\}}{\Re\{(1-\lambda)\|z\|^2\}} \\
&\le \frac{1}{\Re(1-\lambda)}(-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}).
\end{aligned}$$

By Lemma 4.4 we have

$$\begin{aligned}
|\det Dh^{-1}(w_0)| &= \frac{1}{|\det Dh(z_0)|} \\
&\le \mu^{(n+1)/2} \\
&\le \left(\frac{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)} \right)^{(n+1)/2}.
\end{aligned}$$

Since $Dh(z_0) = \frac{r}{m} Df(z)$, we obtain

$$\begin{aligned}
\frac{1}{|\det Df(z)|} &= \left(\frac{r}{m}\right)^n \frac{1}{|\det Dh(z_0)|} \\
&\le \left(\frac{\|z\|}{\|f(z)\|} \right)^n \left(\frac{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)} \right)^{(n+1)/2} \\
&\le \left(\frac{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)} \right)^{(n+1)/2} \\
&\qquad \times \exp\left(-n \int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1 \right] \frac{dy}{y} \right),
\end{aligned}$$

where we have used Theorem 3.6, i.e.,

$$\begin{aligned}
|\det Df(z)| \ge \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}} \right)^{(n+1)/2} \\
\times \exp\left(n \int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1 \right] \frac{dy}{y} \right). \quad \square
\end{aligned}$$

Remark 4.6. Note that if $\lambda = 0$, Theorem 4.5 reduces to [Graham et al. 2020a, Theorem 5.6].

Theorem 4.7. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$, $g \in G(\mathbb{D})$, and let $f : \mathbb{B}_n \to \mathbb{C}^n$ be a g-starlike mapping of complex order λ:*

(1) *If $z \in \mathbb{B}_n$ satisfies*

$$\max_{\|\zeta\|=\|z\|} \|f(\zeta)\| = \|f(z)\|,$$

then for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$ there holds

$$\|Df(z)\beta\| \le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2} \times \exp\left(\int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right)\|\beta\|.$$

(2) *If $z \in \mathbb{B}_n$ satisfies*

$$\min_{\|\zeta\|=\|z\|} \|f(\zeta)\| = \|f(z)\|,$$

then for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$ there holds

$$\|Df(z)\beta\| \ge \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2} \times \exp\left(\int_0^{\|z\|} \left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right)\|\beta\|.$$

Proof. (1) From the proof of Theorem 4.5, we know that if $z \in \mathbb{B}_n$ is the maximum module point of f in the ball $\overline{\mathbb{B}_n(0, \|z\|)}$, there exists a real number $\mu > 0$ such that

$$\mu \le \frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}.$$

Using Lemma 4.4, we obtain

$$\|D\eta(z_0)\beta\| \le \sqrt{\mu} \le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2}$$

for all $\beta \in T_{z_0}^{(1,0)}(\partial\mathbb{B}_n)$ with $\|\beta\| = 1$, i.e.,

$$\|D\eta(z_0)\beta\| \le \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2}\|\beta\|, \quad \forall\beta \in T_{z_0}^{(1,0)}(\partial\mathbb{B}_n).$$

Since

$$T_{z_0}^{(1,0)}(\partial\mathbb{B}_n) = T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$$

and

$$D\eta(z_0) = \frac{r}{M} Df(rz_0) = \frac{\|z\|}{\|f(z)\|} Df(z),$$

we get

$$\begin{aligned}\|Df(z)\beta\| &= \frac{\|f(z)\|}{\|z\|}\|D\eta(z_0)\beta\| \\ &\le \frac{\|f(z)\|}{\|z\|}\left(\frac{\Re(1-\lambda)}{-\Re\lambda+\min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2}\|\beta\|,\end{aligned}$$

for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$. By Theorem 3.6 we can obtain

$$\begin{aligned}\|Df(z)\beta\| \le \left(\frac{\Re(1-\lambda)}{-\Re\lambda+\min\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2} \\ \times \exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\min\{g(y), g(-y)\}-\Re\lambda}-1\right]\frac{dy}{y}\right)\|\beta\|\end{aligned}$$

for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$.

(2) From the proof of Theorem 4.5, we know that if $z \in \mathbb{B}_n$ is the minimum module point of f in the ball $\overline{\mathbb{B}_n(0, \|z\|)}$, there is a real number $\mu > 0$ such that

$$\mu \le \frac{1}{\Re(1-\lambda)}(-\Re\lambda+\max\{g(\|z\|), g(-\|z\|)\}).$$

By Lemma 4.4 we have

$$\|Dh^{-1}(w_0)\gamma\| \le \sqrt{\mu} \le \left(\frac{-\Re\lambda+\max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)}\right)^{1/2},$$

for all $\gamma \in T_{w_0}^{(1,0)}(\partial\mathbb{B}_n)$ with $\|\gamma\| = 1$, i.e.,

$$\|Dh^{-1}(w_0)\gamma\| \le \left(\frac{-\Re\lambda+\max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)}\right)^{1/2}\|\gamma\|, \tag{4.3}$$

for all $\gamma \in T_{w_0}^{(1,0)}(\partial\mathbb{B}_n)$.

Noting that

$$Dh^{-1}(w_0) = (Dh(z_0))^{-1} = \left(\frac{r}{m}Df(z)\right)^{-1} = \frac{\|f(z)\|}{\|z\|}(Df(z))^{-1}$$

and

$$Dh^{-1}(w_0)T_{w_0}^{(1,0)}(\partial\mathbb{B}_n) = T_{z_0}^{(1,0)}(\partial\mathbb{B}_n) = T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|)),$$

we know that

$$Df(z)T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|)) = T_{w_0}^{(1,0)}(\partial\mathbb{B}_n).$$

Replacing $Df(z)\beta$ with γ in the inequality (4.3), where $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$, we obtain

$$\frac{\|f(z)\|}{\|z\|}\|\beta\| \le \left(\frac{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}{\Re(1-\lambda)}\right)^{1/2} \|Df(z)\beta\|,$$

i.e.,

$$\|Df(z)\beta\| \ge \frac{\|f(z)\|}{\|z\|}\left(\frac{\Re(1-\lambda)}{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2} \|\beta\|,$$

for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$. By Theorem 3.6 we see that

$$\|Df(z)\beta\| \ge \left(\frac{\Re(1-\lambda)}{-\Re\lambda + \max\{g(\|z\|), g(-\|z\|)\}}\right)^{1/2} \times \exp\left(\int_0^{\|z\|}\left[\frac{1-\Re\lambda}{\max\{g(y), g(-y)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right)\|\beta\|$$

for all $\beta \in T_z^{(1,0)}(\partial\mathbb{B}_n(0, \|z\|))$. □

Remark 4.8. Note that if $\lambda = 0$, Theorem 4.7 reduces to [Graham et al. 2020a, Theorem 5.11].

5. Roper–Suffridge extension operators and the families $\mathcal{S}^*_{g,\lambda}(\mathcal{B})$

5A. ***Roper–Suffridge extension operators.*** The challenge of constructing examples of starlike mappings and of convex mappings in higher dimensions was well-known, until the introduction of the Roper–Suffridge operator [Roper and Suffridge 1995]. This operator is used to construct starlike mappings and convex mappings in higher dimensions via starlike functions and convex functions in the unit disk, respectively. In the same paper, Roper and Suffridge proved that if f is a normalized locally biholomorphic convex function on the unit disk $\mathbb{D}$, then

$$\Phi_n(f)(u) = (f(u_1), \sqrt{f'(u_1)}\tilde{u}),\ u = (u_1, \tilde{u}) \in \mathbb{B}_n,$$

is a normalized locally biholomorphic convex mapping on the Euclidean unit ball $\mathbb{B}_n$, where $\tilde{u} \in \mathbb{C}^{n-1}$, $\sqrt{f'(0)} = 1$. Graham and Kohr [2000] used the analytic definition of starlike mappings to prove that if f is a starlike function on $\mathbb{D}$, then $\Phi_n(f)$ is a starlike mapping on the unit ball $\mathbb{B}_n$. Furthermore, Graham and Hamada et al. [2002b] proved that if f is a normalized locally biholomorphic starlike function on $\mathbb{D}$, then $\Phi_{n,\alpha,\beta}(f)$ is a normalized locally biholomorphic starlike mapping on $\mathbb{B}_n$ for $\alpha \in [0, 1]$, $\beta \in \left[0, \frac{1}{2}\right]$, $\alpha + \beta \le 1$; if f is a normalized locally biholomorphic convex function on $\mathbb{D}$, then $\Phi_{n,\alpha,\beta}(f)$ is a normalized locally biholomorphic convex mapping on $\mathbb{B}_n$ if and only if $(\alpha, \beta) = \left(0, \frac{1}{2}\right)$, where

$$\Phi_{n,\alpha,\beta}(f)(u) = \left(f(u_1), \left(\frac{f(u_1)}{u_1}\right)^{\alpha}(f'(u_1))^{\beta}\tilde{u}\right), \quad u = (u_1, \tilde{u}) \in \mathbb{B}_n,$$

$\alpha \in [0, 1]$, $\beta \in \left[0, \frac{1}{2}\right]$, $\alpha+\beta \le 1$, and the branches of the power functions are chosen such that $(f(u_1)/u_1)^{\alpha} |_{u_1=0}= 1$, $(f'(u_1))^{\beta} |_{u_1=0}= 1$.

In the above, the Roper–Suffridge operator is only defined on the unit ball $\mathbb{B}_n$. Graham and Kohr [2000], raised the following question:

Question. Consider the egg domain $\Omega_{2,p} = \{(u_1, u_2) \in \mathbb{C}^2 : |u_1|^2 + |u_2|^p < 1\}$, where $p > 1$. Does the operator

$$\Phi_{n,1/p}(f)(u) = (f(u_1), (f'(u_1))^{1/p} u_2), u = (u_1, u_2) \in \Omega_{2,p},$$

extend convex functions on $\mathbb{D}$ to convex mappings on the egg domain $\Omega_{2,p}$?

Gong and Liu [2002] gave an affirmative answer to the above question. They used the contractive property of Carathéodory metric under holomorphic mappings to show that if f is a normalized locally biholomorphic ε starlike function on $\mathbb{D}$, then

$$\Phi_{n,1/p}(f)(u) = (f(u_1), (f'(u_1))^{1/p} \tilde{u}), \quad u = (u_1, \tilde{u}) \in \Omega_p,$$

is a normalized locally biholomorphic ε starlike mapping on Ω_p, where $\Omega_p = \left\{(u_1, \dots, u_n) \in \mathbb{C}^n : |u_1|^2 + \sum_{j=2}^n |u_j|^p < 1\right\}$.

Muir [2005] introduced an extension operator from a new perspective as follows:

$$\Phi_{n,P}(f)(u) = (f(u_1) + P(\tilde{u}) f'(u_1), \sqrt{f'(u_1)}\tilde{u}), \quad u = (u_1, \tilde{u}) \in \mathbb{B}_n,$$

where f is a normalized locally biholomorphic function on $\mathbb{D}$ and $P : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial mapping of degree 2, and $\sqrt{f'(0)} = 1$. Furthermore, he showed that if f is a normalized locally biholomorphic starlike function on $\mathbb{D}$, then $\Phi_{n,P}(f)$ is a normalized locally biholomorphic starlike mapping on $\mathbb{B}_n$ if and only if $\|P\| \le \frac{1}{4}$; if f is a normalized locally biholomorphic convex function on $\mathbb{D}$, then $\Phi_{n,P}(f)$ is a normalized locally biholomorphic convex mapping on $\mathbb{B}_n$ if and only if $\|P\| \le \frac{1}{2}$.

Recently, Graham and Hamada et al. [Graham et al. 2020b] consider the extension operator $\Phi_{\alpha,\beta}$ and Φ_{P_r} on some unit ball in the complex Banach space $Z = \mathbb{C} \times Y$, where

$$\Phi_{\alpha,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} (f'(z_1))^{\beta} w\right), \qquad z= (z_1, w) \in \Omega_r,$$

$$\Phi_{P_r}(f)(z) = (f(z_1) + P_r(w) f'(z_1), (f'(u_1))^{1/r} w), \quad z= (z_1, w) \in \Omega_r,$$

$\alpha \in [0, 1]$, $\beta \in [, , 1/r]$, $\alpha+\beta \le 1$, f is a normalized locally biholomorphic function on $\mathbb{D}$, the branches of the power functions are chosen such that $(f(z_1)/z_1)^{\alpha} |_{z_1=0}= 1$, $(f'(z_1))^{\beta} |_{z_1=0}= 1$, $P_r : Y \to \mathbb{C}$ is a homogeneous polynomial mapping of degree r, $2 \le r$ and

$$\Omega_r = \{z = (z_1, w) \in Z = \mathbb{C} \times Y : |z_1|^2 + \|w\|_Y^r < 1\},$$

where Y is a complex Banach space. The Minkowski function of Ω_r is a complete norm $\|\cdot\|_Z$ on Z, Ω_r is the unit ball of Z with respect to this norm. They proved that these two extension operators had the following properties: Let $g \in G(\mathbb{D})$ be a convex function, and the normalized locally biholomorphic function f can be embedded as the first element of a g-Loewner chain on $\mathbb{D}$. Then $\Phi_{\alpha,\beta}(f)$ can be embedded as the first element of a g-Loewner chain on Ω_r. If $\|P_r\| \le \frac{1}{4}\,\mathrm{dist}(1, \partial g(\mathbb{D}))$, then $\Phi_{P_r}(f)$ can be embedded as the first element of a g-Loewner chain on Ω_r. The extension operators for normalized locally biholomorphic functions on the unit disk $\mathbb{D}$ to higher dimensional spaces have been extensively studied in the literature, see, e.g., [Elin 2011; Elin and Levenshtein 2014; Feng and Liu 2008; Gong and Liu 2003; Graham et al. 2012; Liu et al. 2019; Liu and Xu 2006; Wang 2013; Wang and Liu 2010; 2018].

In the next subsection, we study the extension operators $\Phi_{\alpha,\beta}$ and Φ_{P_r} associated with the g-starlike mappings of complex order λ on Ω_r by using two different methods.

5B. ***Some lemmas.*** In order to prove the main theorems in this subsection, we need the following lemmas.

Lemma 5.1 [Graham et al. 2020b]. *Let Y be a complex Banach space and let $\Omega_r = \{z = (z_1, w) \in \mathbb{C} \times Y : |z_1|^2 + \|w\|_Y^r < 1\}$ be the unit ball of $Z = \mathbb{C} \times Y$, where $r \ge 1$. Let $z = (z_1, w) \ne 0$. Then*

$$T_z((z_1, 0)) = \frac{2|z_1|^2\|z\|_Z}{2|z_1|^2 + r(\|z\|_Z^2 - |z_1|^2)}$$

and

$$T_z((0, w)) = \frac{r(\|z\|_Z^2 - |z_1|^2)\|z\|_Z}{2|z_1|^2 + r(\|z\|_Z^2 - |z_1|^2)}$$

for any $T_z \in T(z)$.

Lemma 5.2 [Muir 2008]. *Let $f : \mathbb{D} \to \mathbb{C}$ be a normalized biholomorphic function, $k \ge 2$. Then*

$$\left|(1 - |\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - k\bar{\zeta}\right| \le k + 2, \quad \forall \zeta \in \mathbb{D}.$$

Lemma 5.3 [Pommerenke 1975]. *Let $g : \mathbb{D} \to \mathbb{C}$ be a convex function. Then for any $a \in \mathbb{D}$, $g(\mathbb{D})$ contains the disk of radius $\frac{1}{2}|g'(a)|(1 - |a|^2)$ centered at $g(a)$.*

Lemma 5.4 [Graham et al. 2020b]. *Let $g \in G(\mathbb{D})$. We say that a mapping $f = f(x, t) : \mathcal{B} \times [0, \infty) \to X$ is a g-Loewner chain if the following conditions hold:*

(i) *$f(x, t)$ is a Loewner chain such that $\{e^{-t} f(\cdot, t)\}_{t \ge 0}$ is uniformly bounded on each ball $\mathcal{B}_r$ $(0 < r < 1)$;*

(ii) $\frac{\partial f}{\partial t}(x,t)$ *exists for a.e.* $t \geq 0$ *and for all* $x \in \mathcal{B}$, *and there exists a Herglotz vector field* $h = h(x,t) : \mathcal{B} \times [0,\infty) \to X$ *with* $h(\cdot,t) \in \mathcal{M}_g(\mathcal{B})$ *for a.e.* $t \geq 0$ *such that*

$$\frac{\partial f}{\partial t}(x,t) = Df(x,t)h(x,t), \quad a.e.\ t \geq 0, x \in \mathcal{B}.$$

Remark 5.5. Let $g \in G(\mathbb{D})$. It is not difficult to deduce that $f : \mathcal{B} \to X$ is a g-starlike mapping of complex order λ if and only if $F(x,t) = e^{(1-\lambda)t} f(e^{\lambda t}x)$, $\forall x \in \mathcal{B}$, $t \in [0,\infty)$, is a g-Loewner chain.

5C. ***Examples of*** $\mathcal{S}^*_{g,\lambda}(\Omega_r)$**.** By using Roper–Suffridge extension operators, we can construct many examples of $\mathcal{S}^*_{g,\lambda}(\Omega_r)$ via holomorphic functions of $\mathcal{S}^*_{g,\lambda}(\mathbb{D})$.

Theorem 5.6. *Let* $g \in G(\mathbb{D})$ *be a convex function , and let* Y *be a complex Banach space. Denote* $\Omega_r = \{z = (z_1, w) \in Z : |z_1|^2 + \|w\|_Y^r < 1\}$ *by the unit ball of* $Z = \mathbb{C} \times Y$, *where* $r \geq 1$. *Suppose that* $\alpha \in [0,1]$, $\beta \in [0, 1/r]$, $\alpha + \beta \leq 1$. *If* f *is a* g*-starlike function of complex order* λ *on* $\mathbb{D}$, *then*

$$F(z) = \Phi_{\alpha,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\alpha} (f'(z_1))^{\beta} w \right) \in \mathcal{S}^*_{g,\lambda}(\Omega_r),$$

where $z = (z_1, w) \in \Omega_r$ *and the branches of the power functions are chosen such that* $(f(z_1)/z_1)^{\alpha} |_{z_1=0} = 1$ *and* $(f'(z_1))^{\beta} |_{z_1=0} = 1$.

Proof. Note that

$$f(z_1,t) = e^{(1-\lambda)t} f(e^{\lambda t} z_1), \forall z_1 \in \mathbb{D}, t \in [0,\infty)$$

is a g-Loewner chain, since f is a g-starlike function of complex order λ on the unit disk $\mathbb{D}$. According to a result of Graham et al. [2020b, Theorem 3.1], we know that

$$\begin{aligned} F(z,t) &= e^t \Phi_{\alpha,\beta}(e^{-t} f(\cdot,t))(z,t) \\ &= \left(f(z_1,t), e^{(1-\alpha-\beta)t} \left(\frac{f(z_1,t)}{z_1}\right)^{\alpha} (f'(z_1,t))^{\beta} w \right) \end{aligned}$$

is a g-Loewner chain. Furthermore,

$$\begin{aligned} F(z,t) &= \left(e^{(1-\lambda)t} f(e^{\lambda t} z_1), e^{(1-\lambda)t} \left(\frac{f(e^{\lambda t} z_1)}{e^{\lambda t} z_1}\right)^{\alpha} (f'(e^{\lambda t} z_1))^{\beta} (e^{\lambda t} w) \right) \\ &= e^{(1-\lambda)t} \Phi_{\alpha,\beta}(f)(e^{\lambda t} z). \end{aligned}$$

It yields that $F = \Phi_{\alpha,\beta}(f) \in \mathcal{S}^*_{g,\lambda}(\Omega_r)$. □

Theorem 5.7. *Let Y be a complex Banach space and let $k \geq 2$ be an integer. Denote*

$$\Omega_r = \{z = (z_1, w) \in \mathbb{C} \times Y : |z_1|^2 + \|w\|_Y^r < 1\},$$

$r \geq 1$, and let $P_k : Y \to \mathbb{C}$ be a homogeneous polynomial mapping of degree k, $r \leq k$. Assume that f is a g-starlike function of complex order λ on $\mathbb{D}$, where $g \in G(\mathbb{D})$ is a convex function. If

$$\|P_k\| \leq r/(2(k+r)|1-\lambda|)\,\mathrm{dist}(1, \partial g(\mathbb{D})),$$

then

$$F(z) = \Phi_{P_k}(f)(z_1, w) = (f(z_1) + P_k(w) f'(z_1), (f'(z_1))^{1/k} w) \in \mathcal{S}^*_{g,\lambda}(\Omega_r),$$

where $(z_1, w) \in \Omega_r$ and the branch of the power function is chosen such that $(f'(z_1))^{1/k} |_{z_1=0} = 1$.

Proof. For any holomorphic mapping $\eta(z) = (\eta_1(z), \eta_0(z)) : \Omega_r \to \mathbb{C} \times Y$, we have

$$\begin{aligned} DF(z)\eta(z) = \big(&\eta_1(z)(f'(z_1) + P_k(w) f''(z_1)) + \nabla P_k(w) f'(z_1)\eta_0(z),\\ &\tfrac{1}{k}(f'(z_1))^{(1/k)-1} f''(z_1)\eta_1(z) w + (f'(z_1))^{1/k}\eta_0(z)\big). \end{aligned}$$

Let $DF(z)\eta(z) = F(z)$. Then

$$\begin{aligned} \eta(z) &= (DF(z))^{-1}F(z)\\ &= \left(\frac{f(z_1)}{f'(z_1)} - (k-1)P_k(w), \left(1 - \frac{1}{k}\frac{f(z_1)f''(z_1)}{(f'(z_1))^2} + \left(1-\frac{1}{k}\right)P_k(w)\frac{f''(z_1)}{f'(z_1)}\right)w\right). \end{aligned}$$

Hence,

$$\begin{aligned} (1-\lambda)&(DF(z))^{-1}F(z) + \lambda z\\ &= \left((1-\lambda)\left(\frac{f(z_1)}{f'(z_1)} - (k-1)P_k(w)\right) + \lambda z_1,\right.\\ &\qquad \left.(1-\lambda)\left(1 - \frac{1}{k}\frac{f(z_1)f''(z_1)}{(f'(z_1))^2} + \left(1-\frac{1}{k}\right)P_k(w)\frac{f''(z_1)}{f'(z_1)}\right)w + \lambda w\right). \end{aligned} \tag{5.1}$$

Next, we will show that

$$\frac{1}{\|z\|_Z} T_z\{(1-\lambda)(DF(z))^{-1}F(z) + \lambda z\} \in g(\mathbb{D}),\ z \in \Omega_r \backslash \{0\}.$$

It is equivalent to prove

$$\frac{1}{\rho} T_z\{(1-\lambda)(DF(\rho z))^{-1}F(\rho z) + \lambda \rho z\} \in g(\mathbb{D}),\ z \in \partial\Omega_r,\ \rho \in (0, 1).$$

Indeed, if $z = (z_1, 0) \in \partial\Omega_r$, then

$$\frac{1}{\rho} T_z\{(1-\lambda)(DF(\rho z))^{-1}F(\rho z) + \lambda \rho z\} = (1-\lambda)\frac{f(\rho z_1)}{\rho z_1 f'(\rho z_1)} + \lambda \in g(\mathbb{D}).$$

If $z=(z_1,w)\in\partial\Omega_r$ with $w\neq 0$, then by using Lemma 5.1 and (5.1), we have

$$\begin{aligned}
&\frac{(2|z_1|^2+r(1-|z_1|^2))}{\rho}T_z\{(1-\lambda)(DF(\rho z))^{-1}F(\rho z)+\lambda\rho z\}\\
&=2|z_1|^2\left((1-\lambda)\frac{f(\rho z_1)}{\rho z_1 f'(\rho z_1)}+\lambda\right)-2(1-\lambda)(k-1)\rho^{k-1}P_k(w)\overline{z_1}\\
&\quad+r(1-|z_1|^2)(1-\lambda)\left(1-\frac{1}{k}\frac{f(\rho z_1)f''(\rho z_1)}{(f'(\rho z_1))^2}+\left(1-\frac{1}{k}\right)\rho^k P_k(w)\frac{f''(\rho z_1)}{f'(\rho z_1)}\right)\\
&\quad+r\lambda(1-|z_1|^2)\\
&=2|z_1|^2\left((1-\lambda)\frac{f(\rho z_1)}{\rho z_1 f'(\rho z_1)}+\lambda\right)+\frac{r}{k}(1-|z_1|^2)\left(1-(1-\lambda)\frac{f(\rho z_1)f''(\rho z_1)}{(f'(\rho z_1))^2}\right)\\
&\quad+\frac{r(k-1)}{k}(1-|z_1|^2)\\
&\quad\times\left[1+(1-\lambda)\rho^{k-2}\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)}-\frac{2k}{r}\overline{\rho z_1}\right)\right].
\end{aligned}$$

Let

$$\psi(\zeta)=(1-\lambda)\frac{f(\zeta)}{\zeta f'(\zeta)}+\lambda,\quad \forall\zeta\in\mathbb{D}.$$

Since f is a g-starlike function of complex order λ on $\mathbb{D}$, we have

$$\psi(\zeta)\in g(\mathbb{D}), \tag{5.2}$$

and $\psi(0)=g(0)=1$, i.e., $\psi\prec g$. Hence, there is a Schwarz mapping $v:\mathbb{D}\to\mathbb{D}$ such that $v(0)=0$ and $\psi(\zeta)=g(v(\zeta))$. Furthermore, it is easy to see that, for all $\zeta\in\mathbb{D}$,

$$|v'(\zeta)|\le\frac{1-|v(\zeta)|^2}{1-|\zeta|^2}\quad\text{and}\quad \psi(\zeta)+\zeta\psi'(\zeta)=1-(1-\lambda)\frac{f(\zeta)f''(\zeta)}{(f'(\zeta))^2},$$

Therefore, we have

$$\begin{aligned}
&\frac{(2|z_1|^2+r(1-|z_1|^2))}{\rho}T_z\{(1-\lambda)(DF(\rho z))^{-1}F(\rho z)+\lambda\rho z\}\\
&=2|z_1|^2\psi(\rho z_1)+\frac{r}{k}(1-|z_1|^2)(\psi(\rho z_1)+\rho z_1\psi'(\rho z_1))+\frac{r(k-1)}{k}(1-|z_1|^2)\\
&\quad\times\left[1+(1-\lambda)\rho^{k-2}\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)}-\frac{2k}{r}\overline{\rho z_1}\right)\right]\\
&=\frac{2(k-r)}{k}|z_1|^2\psi(\rho z_1)+\frac{r}{k}(1+|z_1|^2)\\
&\quad\times\left(\psi(\rho z_1)+\frac{\rho z_1\psi'(\rho z_1)(1-|z_1|^2)}{1+|z_1|^2}\right)+\frac{r(k-1)}{k}(1-|z_1|^2)\\
&\quad\times\left[1+(1-\lambda)\rho^{k-2}\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)}-\frac{2k}{r}\overline{\rho z_1}\right)\right].
\end{aligned}$$

Since $g(\mathbb{D})$ contains a disk with $g(a)$ as center and $\frac{1}{2}|g'(a)|(1-|a|^2)$ as radius, where $a = v(\rho z_1)$, and

$$\left|\frac{\rho z_1 \psi'(\rho z_1)(1-|z_1|^2)}{1+|z_1|^2}\right| = \frac{\rho|z_1|}{1+|z_1|^2}|g'(a)||v'(\rho z_1)|(1-|z_1|^2) < \tfrac{1}{2}|g'(a)|(1-|a|^2),$$

we have

$$\psi(\rho z_1) + \frac{\rho z_1 \psi'(\rho z_1)(1-|z_1|^2)}{1+|z_1|^2} \in g(\mathbb{D}). \tag{5.3}$$

On the other hand, since

$$|P_k(w)| \le \|P_k\| \|w\|_Y^k \quad \text{and} \quad \|P_k\| \le \frac{r}{2(k+r)|1-\lambda|} \operatorname{dist}(1, \partial g(\mathbb{D})),$$

by using Lemma 5.2, we have

$$\begin{aligned} \rho^{k-2}\left|(1-\lambda)\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)} - \frac{2k}{r}\overline{\rho z_1}\right)\right| & \\ &\le \rho^{k-1}|1-\lambda|\|P_k\|\left(\frac{2k}{r}+2\right) \\ &\le \operatorname{dist}(1, \partial g(\mathbb{D})). \end{aligned}$$

It yields that

$$1 + (1-\lambda)\rho^{k-2}\frac{P_k(w)}{1-|z_1|^2}\left(\rho^2(1-|z_1|^2)\frac{f''(\rho z_1)}{f'(\rho z_1)} - \frac{2k}{r}\overline{\rho z_1}\right) \in g(\mathbb{D}). \tag{5.4}$$

Putting the equation (5.2), (5.3) and (5.4) together, we get

$$\frac{1}{\rho}T_z\{(1-\lambda)(DF(\rho z))^{-1}F(\rho z) + \lambda \rho z\} \in g(\mathbb{D}), z \in \partial\Omega_r, \quad \forall \rho \in (0,1). \qquad \square$$

Remark 5.8. If $r = k$ and $\lambda = 0$, then Theorem 5.7 reduces to [Graham et al. 2020b, Theorem 4.1].

In particular, when $Y = \mathbb{C}^{n-1}$, we do have the following corollary, which is a generalization of [Muir 2005, Theorem 4.1].

Corollary 5.9. *Let $k \ge 2$ be an integer. And let $P_k : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial mapping of degree k. Assume that f is a g-starlike function of complex order λ on $\mathbb{D}$, where $g \in G(\mathbb{D})$ is a convex function. If $\|P_k\| \le 1/((k+2)|1-\lambda|)$* $\operatorname{dist}(1, \partial g(\mathbb{D}))$, *then*

$$F(z_1, w) = \Phi_{P_k}(f)(z_1, w) = (f(z_1) + P_k(w) f'(z_1), \quad (f'(z_1))^{1/k} w) \in \mathcal{S}^*_{g,\lambda}(\mathbb{B}_n),$$

where $z = (z_1, w) \in \mathbb{B}_n$ and the branch of the power function is chosen such that $(f'(z_1))^{1/k}|_{z_1=0} = 1$.

Remark 5.10. Since the functions g in Remark 2.4 are all convex functions, the extension operators $\Phi_{\alpha,\beta}$ and Φ_{P_k} preserve the geometric properties of the normalized locally biholomorphic mappings, which we have displayed in Remark 2.4, respectively.

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