ON SLICE ALTERNATING 3-BRAID CLOSURES

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We construct ribbon surfaces of Euler characteristic one for several infinite families of alternating 3-braid closures. We also use a twisted Alexander polynomial obstruction to conclude the classification of smoothly slice knots which are closures of alternating 3-braids with up to 20 crossings.

1. Introduction

By an alternating braid we mean a braid such that along any strand, over- and undercrossings alternate. Let $\sigma_1$ and $\sigma_2$ be the standard generators of the braid group on three strands $B_3$. If the closure of an alternating 3-braid has nonzero determinant, then it is isotopic to the closure of a braid

$$\sigma_1^{a_1} \sigma_2^{-b_1} \sigma_1^{a_2} \sigma_2^{-b_2} \ldots \sigma_1^{a_n} \sigma_2^{-b_n},$$

with $n \geq 1$ for some $a_i, b_i \geq 1$ for all $i$. Every 3-braid of the form $(\star)$ can be equivalently described by its associated string $a = (2^{a_1-1}, b_1 + 2, \ldots, 2^{a_n-1}, b_n + 2)$, where $2^{a_i-1}$ represents the substring consisting of the number 2 repeated $a_i - 1$ times. Cyclic rotations and reversals of $a$ do not change the isotopy class of respective braid closures in $S^3$, so we consider associated strings up to those two operations. The linear dual of a string $b = (b_1, \ldots, b_k)$ with all $b_i \geq 2$ is defined as follows: if $b_j \geq 3$ for some $j$, write $b$ in the form $b = (2^{m_1}, 3 + n_1, 2^{m_2}, 3 + n_2, \ldots, 2^{m_l}, 2 + n_l)$ with $m_i, n_i \geq 0$ for all $i$. Then its linear dual is $c = (2 + m_1, 2^{[n_1]}, 3 + m_2, 2^{[n_2]}, 3 + m_3, \ldots, 3 + m_l, 2^{[n_l]})$. If $b$ is $(2^{[k]})$ or $(1)$, define its linear dual as $(k + 1)$ or the empty string, respectively.

Given a link $L \subset S^3$, by a ribbon surface we mean a surface $F$ bounded by $L$ that is properly smoothly embedded in $D^4$, has no closed components, and may be isotoped rel boundary so that the radial distance function $D^4 \to [0, 1]$ induces a handle decomposition on $F$ with only 0- and 1-handles. By a slice surface we mean a surface $S$ bounded by $L$ that is properly smoothly embedded in $D^4$ and has no closed components; neither $F$ nor $S$ are required to be connected or orientable. Following [5], we say that $L$ which bounds a ribbon (or slice) surface of Euler characteristic one is $\chi$-ribbon (or $\chi$-slice); these definitions coincide with the usual

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definitions of ribbon and slice in the case of knots. Clearly, if \( L \) is \( \chi \)-ribbon, then it is also \( \chi \)-slice.

Simone [20] has classified associated strings of all alternating 3-braid closures \( L \) with nonzero determinant such that \( \Sigma_2(S^3, L) \), the double branched cover of \( S^3 \) over \( L \), is unobstructed by Donaldson’s theorem from bounding a rational ball, into five families:

\[
\begin{align*}
S_{2a} &= \{(b_1 + 3, b_2, \ldots, b_k, 2, c_l, \ldots, c_1)\}, \\
S_{2b} &= \{(3+x, b_1, \ldots, b_{k-1}, b_k+1, 2^{[x]}, c_l+1, c_{l-1}, \ldots, c_1) \mid x \geq 0 \text{ and } k+l \geq 2\}, \\
S_{2c} &= \{(3 + x_1, 2^{[x_2]}, 3 + x_3, 2^{[x_4]}, \ldots, 3 + x_{2k+1}, 2^{[x_1]}, 3 + x_2, 2^{[x_3]}, \\
&\quad \ldots, 3 + x_{2k}, 2^{[x_{2k+1}]}) \mid k \geq 0 \text{ and } x_i \geq 0 \text{ for all } i\}, \\
S_{2d} &= \{(2, 2 + x, 2, 3, 2^{[x-1]}, 3, 4) \mid x \geq 1\} \cup \{(2, 2, 2, 4, 4)\}, \\
S_{2e} &= \{(2, b_1 + 1, b_2, \ldots, b_k, 2, c_l, \ldots, c_2, c_1 + 1, 2) \mid k+l \geq 3\} \cup \{(2, 2, 2, 3)\}.
\end{align*}
\]

Here strings \( (b_1, \ldots, b_k) \) and \( (c_1, \ldots, c_l) \) are linear duals of each other. Since \( \Sigma_2(S^3, L) \) of a \( \chi \)-slice link \( L \) bounds a rational ball [5, Proposition 2.6], every \( \chi \)-slice alternating 3-braid closure with nonzero determinant has its associated string in one of these families. Moreover, Simone has explicitly constructed rational balls for all such alternating 3-braid closures.

We show that alternating 3-braid closures whose associated strings lie in \( S_{2a} \cup S_{2b} \cup S_{2d} \cup S_{2e} \) are \( \chi \)-ribbon by exhibiting band moves, defined in Section 2, which make their link diagrams isotopic to the two- or three-component unlink. In Section 3, we consider the set \( S_{2c} \setminus (S_{2a} \cup S_{2b} \cup S_{2d} \cup S_{2e}) \) that includes strings associated to known non-\( \chi \)-slice alternating 3-braid closures, such as certain Turk’s head knots, and list more examples of potentially non-\( \chi \)-slice knots and links. In Section 4 we follow [11] and [1] in applying a twisted Alexander polynomial obstruction to show that among these examples, three knots are indeed not slice; this concludes the classification of smoothly slice knots which are closures of alternating 3-braids with up to 20 crossings.

2. Ribbon surfaces for \( S_{2a} \cup S_{2b} \cup S_{2d} \cup S_{2e} \)

One may exhibit a ribbon surface for a link \( L \) as follows. By a band move on \( L \) we mean choosing an embedding \( \varphi : D^1 \times D^1 \hookrightarrow S^3 \) of a band so that the image of \( \varphi \) is disjoint from \( L \) except for \( \varphi(\partial D^1 \times D^1) \) coincident with two segments of \( L \), removing those segments, joining corresponding ends along \( \varphi(D^1 \times \partial D^1) \) and smoothing the corners. This operation amounts to removing a 1-handle in the putative ribbon surface \( F \). If after \( n \) band moves, the resulting link is isotopic to the \((n+1)\)-component unlink, one has indeed obtained a ribbon surface \( F \) of Euler characteristic one bounded by \( L \), since each component of the unlink bounds a 0-handle.
of $F$. Each band may be represented on a link diagram by an arc with endpoints on $L$ that crosses the strands of $L$ transversally, has no self-crossings, and is annotated by the number of half-twists in the band relative to the blackboard framing.

Given a 3-braid $\beta = \sigma_1^{a_1} \sigma_2^{-b_1} \ldots \sigma_1^{a_n} \sigma_2^{-b_n}$, we draw it from left to right, as shown in Figure 1, and orient all strings in the closure $\hat{\beta}$ clockwise. Choose the chessboard colouring of the diagram for $\hat{\beta}$ where the unbounded region is white. Then there are $m = \left( \sum_{i=1}^{n} a_i \right) + 1$ black regions. We can index the black regions, excluding the one not adjacent to the unbounded region (marked by $^*$ in Figure 1), by $\{1, \ldots, m - 1\}$ such that the number of crossings along the boundary of the region indexed by $i$ is given by the $i$-th entry of the associated string $a = (2^{[a_1-1]}, b_1 + 2, \ldots, 2^{[a_n-1]}, b_n + 2)$, and the region indexed by $i$ shares one crossing with each of the regions indexed by $i - 1$ and $i + 1$ (mod $m - 1$).

**Proposition 2.1.** Let $a$ be the associated string of an alternating 3-braid closure $\hat{\beta}$. If $a \in S_{2a} \cup S_{2d} \cup S_{2e}$, then $\hat{\beta}$ bounds a ribbon surface with a single 1-handle. If $a \in S_{2b}$, then $\hat{\beta}$ bounds a ribbon surface with at most two 1-handles.

Our main observation, previously used by Lisca [14] and Lecuona [13], is that if $a$ contains two disjoint linearly dual substrings (possibly perturbed on the ends), then the link diagram of $\hat{\beta}$ contains sub-braids which, if connected to each other by a half-twist $(\sigma_2 \sigma_1 \sigma_2)^{-1}$, may be cancelled out via successive isotopies. More precisely, suppose that $(b_1, \ldots, b_k)$ and $(c_1, \ldots, c_l)$ are linear duals. Let $b' = (b_1 + x_l, b_2, \ldots, b_k + x_r)$ and $c' = (c_l + y_l, c_{l-1}, \ldots, c_1 + y_r)$ with $x_l, y_l \geq 0$ for $i \in \{l, r\}$ and suppose that $a = b' | t | c' | s$, where $t$ and $s$ are arbitrary strings, the length of $t$ is $t \geq 0$, and $|$ denotes string concatenation. Consider the sub-braid $B$ in the link diagram of $\hat{\beta}$ that exactly contains all crossings along the boundary of black regions $2, \ldots, k - 1$, all but $x_l + 1$ leftmost crossings along the boundary of region $1$, and all but $x_r + 1$ rightmost crossings along the boundary of region $k$. Consider also the sub-braid $C$ that exactly contains all crossings along the boundary of regions $k + t + 2, \ldots, k + t + l - 1$, all but $y_l + 1$ leftmost crossings along the boundary of region $k + t + 1$, and all but $y_r + 1$ rightmost crossings along the boundary of
region $k + t + l$. Then $B(\sigma_2 \sigma_1 \sigma_2)^{-1} C = (\sigma_2 \sigma_1 \sigma_2)^{-1}$. Hence, if after applying a band move to $\hat{\beta}$ away from $B$ and $C$, they are connected by a half-twist of the three strands, one may remove all crossings in $B$ and $C$ via isotopies illustrated in Figure 2. We call $B$ and $C$ dual sub-braids and enclose them in all following figures in blue and chartreuse rectangles, respectively.

**Proof of Proposition 2.1.** See Figures 4–7.

In searching for the band moves in Figures 4–7, we have used the algorithm of Owens and Swenton implemented in the KLO program [16]. The band moves we exhibit for these four families of alternating 3-braid closures are algorithmic in the sense of [16].

![Figure 2](image1.png)

**Figure 2.** Undoing flyped tongues [22] to cancel dual sub-braids.

![Figure 3](image2.png)

**Figure 3.** Cancellation of dual sub-braids for $(b_1, \ldots, b_k) = (2, 2, 3, 3)$ and $(c_1, \ldots, c_1) = (2, 3, 4)$ with $x_l = x_r = y_l = y_r = 0$. Fixing the ends on the braid shown, one may remove all crossings in $B$ and $C$ via moves illustrated in Figure 2.

![Figure 4](image3.png)

**Figure 4.** Band move for the $S_{2a}$ case.
Figure 5. Band moves for the $S_{2b}$ case. Start with the top left diagram if the two segments highlighted in purple do not lie on the same strand, otherwise start with the top right; this ensures that after step (2), the tangle $T$ does not lie on the otherwise unknotted split component. The nontrivial component of the link obtained after step (3) is the connected sum $T(2, x + 2) \# T(2, -(x + 2))$ of two torus links.

3. The case of $S_{2c} \setminus (S_{2a} \cup S_{2b} \cup S_{2d} \cup S_{2e})$

The remaining $S_{2c}$ family is of special interest because it contains strings associated to known examples of nonslice, nonzero determinant alternating 3-braid closures, specifically Turk’s head knots $K_7$ [19], $K_{11}$, $K_{17}$ and $K_{23}$ [1]; the associated string of $K_i$ for $i \in \{7, 11, 17, 23\}$ is $(3^{[i]})$. Thus, we should not expect to find a set of band moves for all links with strings in $S_{2c}$. We also note that knots of finite concordance order belonging to Family (3) in [15] have associated strings in $S_{2c}$.

We have that $S_{2c} \cap S_{2d} = S_{2c} \cap S_{2e} = \emptyset$: this can be seen by computing the $I(a) = \sum_{a \in a} 3 - a$ invariant [14] which is 0 for strings in $S_{2c}$, but 1 or 3 for strings...
in $S_{2d}$ or $S_{2e}$, respectively.\footnote{Observe that if $b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_l)$ are linearly dual to each other and $k + l \geq 2$, then $I(b | c) = 2$.} However, $S_{2c}$ has nonzero intersection with $S_{2d}$ and $S_{2b}$: if one defines a \textit{palindrome} to be a string $(a_1, \ldots, a_n)$ such that $a_i = a_{n-(i-1)}$ for all $1 \leq i \leq n$, then the following lemma holds.

\textbf{Lemma 3.1} [20, Lemma 3.6]. \textit{Let} $a = (b_1 + 3, b_2, \ldots, b_k, 2, c_l, \ldots, c_1) \in S_{2a}$ \textit{and} $b = (3 + x, b_1, \ldots, b_{k-1}, b_k + 1, 2^{|x|}, c_l + 1, c_{l-1}, \ldots, c_1) \in S_{2b}$. \textit{Then} $a \in S_{2c}$ \textit{if and only if} $(b_1 + 1, b_2, \ldots, b_k)$ \textit{is a palindrome and} $b \in S_{2c}$ \textit{if and only if} $(b_1 \ldots, b_k)$ \textit{is a palindrome.}
We seek to find an easier description of the complement $S_{2c}^\dagger := S_{2c} \setminus (S_{2a} \cup S_{2b} \cup S_{2d} \cup S_{2e})$. Let

$$c = (3 + x_1, 2^{[x_2]}, 3 + x_3, 2^{[x_4]}, \ldots, 3 + x_{2k+1}, 2^{[x_1]}, 3 + x_2, 2^{[x_3]}, \ldots, 3 + x_{2k}, 2^{[x_{2k+1}}) \in S_{2c},$$

where $k \geq 0$ and $x_i \geq 0$ for all $i$. One can more compactly describe $c$ by its $x$-string $x(c) = [x_1, \ldots, x_{2k+1}]$ (we use square brackets to denote $x$-strings and, as with associated strings, consider them up to cyclic rotations and reversals). For example, the $x$-string of $(3^{[1]})$ associated with $K_t$ is $[0^{[1]}]$. Also, when writing $c$ in the form $(\ast)$ with the first element being at least 3, call every maximal substring of the form $(2^{[x_1]})$ or $(3 + x)$ for $x \geq 0$ an entry; the total number of entries $e(c)$ in $c$ is congruent to 2 mod 4.

**Lemma 3.2.** Let $a = (b_1 + 3, b_2, \ldots, b_k, 2, c_1, \ldots, c_1) \in S_{2a} \cap S_{2c}$ and $b = (3 + y, b_1, \ldots, b_{k-1}, b_k + 1, 2^{[y]}, c_1 + 1, c_{l-1}, \ldots, c_1) \in S_{2b} \cap S_{2c}$. Then:

- $x(a) = [z_1]$ with $z_1 \geq 1$ or $x(a) = [z_1, \ldots, z_{\lceil \frac{n}{2} \rceil}, z_{\lceil \frac{n}{2} \rceil + 1}, z_{\lceil \frac{n}{2} \rceil}, \ldots, z_2, z_1 - 2]$ with $z_1 \geq 2$ and $n \geq 3$ odd.
- $x(b) = [n, 0, z_1]$ or $x(b) = [n, 0, z_2, \ldots, z_2, z_2 + 1, \ldots, z_3, z_2 + 1]$ with $n \geq 4$ even.

**Proof.** Consider $a$ and define $a_c = (2, c_1, \ldots, c_1)$. Notice that $a_c$ is the linear dual of the string

$$a_c^\dagger = (b_k + 1, b_{k-1}, \ldots, b_1),$$

which by **Lemma 3.1** must be a palindromic, and that $a = (b_1 + 3, b_2, \ldots, b_k \mid a_c)$. If $(b_1, \ldots, b_k)$ is the empty string, then $a = (2, 1) \notin S_{2c}$. Otherwise, write

$$a_c = (2^{[z_1]}, 3 + z_2, \ldots, 2^{[z_n]})$$

![Figure 7](image-url)
for \( n \geq 1 \) odd and \( z_1 \geq 1 \). If \( n = 1 \), then \( \alpha_c = (2^{[z_1]}) \) and \( \alpha = (3 + z_1, 2^{[z_1]}) \), so \( x(\alpha) = [z_1] \). If \( n > 1 \), then

\[
\alpha^*_c = (2 + z_1, 2^{[z_2]}, 3 + z_3, \ldots, 2^{[z_{n-1}]}, 2 + z_n).
\]

Thus,

\[
\alpha = (3 + (z_n + 2), 2^{[z_{n-1}]}, \ldots, 2^{[z_2]}, 1 + z_1, 2^{[z_1]}, 3 + z_2, \ldots, 2^{[z_1]}).
\]

If \( z_1 = 1 \), then

\[
\alpha = (3 + (z_n + 2), 2^{[z_{n-1}]}, \ldots, 3 + z_3, 2^{[z_1+z_2+1]}, 3 + z_2, \ldots, 2^{[z_1]})
\]
does not belong to \( S_{2c} \) because \( e(\alpha) \equiv 0 \) mod 4. If \( z_1 > 1 \), then

\[
\alpha = (3 + (z_n + 2), 2^{[z_{n-1}]}, \ldots, 2^{[z_2]}, 3 + (z_1 - 2), 2^{[z_1]}, 3 + z_2, \ldots, 2^{[z_1]}).
\]

Now, by considering \( \alpha^*_c \) we see that \( \alpha^*_c \) is a palindrome if and only if

\[
z_1 = z_n + 2, \quad z_2 = z_{n-1}, \quad \ldots, \quad z_{\left\lceil \frac{n}{2} \right\rceil} = z_{\left\lfloor \frac{n}{2} \right\rfloor} + 2,
\]

so we conclude that \( \alpha \in S_{2a} \cap S_{2c} \) if and only if \( x(\alpha) = [z_1] \) for \( z_1 \geq 1 \) or \( x(\alpha) = [z_1, z_2, \ldots, z_{\left\lceil \frac{n}{2} \right\rceil}, z_{\left\lfloor \frac{n}{2} \right\rfloor} + 1, z_{\left\lfloor \frac{n}{2} \right\rfloor}, \ldots, z_2, z_1 - 2] \) for \( z_1 \geq 2 \) and \( n \geq 3 \) odd.

Similarly, if \( (b_1, \ldots, b_k) \) is empty, then \( b = (3 + y, 2^{[y]}, 2) = (3 + y, 2^{[y+1]}) \notin S_{2c} \). If \( k = 1 \), then the linear dual of \( (b_1) \) with \( b_1 \geq 2 \) is \( (2^{[b_1-1]}) \), so

\[
b = (3 + y, 2^{[0]}, b_1 + 1, 2^{[y]}, 3 + 0, 2^{[b_1-2]}) = (3 + y, 2^{[0]}, 3 + (b_1 - 2), 2^{[y]}, 3 + 0, 2^{[b_1-2]})
\]
is indeed in \( S_{2c} \) and \( x(b) = [y, 0, b_1 - 2] \). If \( k > 1 \), write

\[
(b_1, \ldots, b_k) = (2^{[z_1]}, 3 + z_2, \ldots, 2^{[z_{n-1}]}, 2 + z_n)
\]

for \( n \geq 2 \) even and \( z_1 \geq 1 \); its linear dual is

\[
(c_1, \ldots, c_l) = (2 + z_1, 2^{[z_2]}, 3 + z_3, \ldots, 2^{[z_{n-2}]}, 3 + z_{n-1}, 2^{[z_n]}).
\]

When \( n = 2 \), we recover the \( k = 1 \) case above, so suppose \( n > 2 \). Then we have

\[
b = (3 + y, 2^{[z_1]}, \ldots, 2^{[z_{n-1}]}, 3 + z_n, 2^{[y]}, 3 + 0, 2^{[z_{n-1}]}, 3 + z_{n-1}, 2^{[z_{n-2}]}, \ldots, 3 + z_3, 2^{[z_2 + 1]}).
\]

By comparing this with \( \alpha \), we see that \( z_1 \) (which corresponds to \( x_2 \)) must be zero, and

\[
(b_1, \ldots, b_k) = (3 + z_2, 2^{[z_3]}, \ldots, 2^{[z_{n-1}]}, 3 + (z_n - 1)).
\]

The string \( (b_1, \ldots, b_k) \) is thus a palindrome precisely when

\[
z_2 = z_n - 1, \quad z_3 = z_{n-1}, \quad \ldots, \quad z_{\frac{n}{2}} = z_{\frac{n}{2} + 2},
\]

i.e., \( x(b) = [y, 0, z_2, z_3, \ldots, z_{\frac{n}{2}}, z_{\frac{n}{2} + 1}, z_{\frac{n}{2}}, \ldots, z_3, z_2 + 1] \). \( \square \)
In particular, we can draw the easy conclusion that if \( x(c) \) contains neither two adjacent elements differing by 2 nor a 0, then \( c \in S_{2c}^\dagger \). We now show that infinitely many \( \chi \)-ribbon links have their associated strings in \( S_{2c}^\dagger \).

**Lemma 3.3.** Let \( \hat{\beta} \) be the closure of \( \beta = \sigma_1^{m+1}(\sigma_2^{-1}\sigma_1)^2\sigma_2^{-(m+1)}(\sigma_1\sigma_2^{-1})^2 \) with the associated string \( c = (3 + m, 3, 3, 2^m[l], 3, 3) \) and \( m \geq 3 \). Then \( c \in S_{2c}^\dagger \) and \( \hat{\beta} \) admits a ribbon surface with a single 1-handle.

**Proof.** We have \( x(c) = [m, 0, 0, 0, 0] \), so by Lemma 3.2, \( c \in S_{2c}^\dagger \). For the band move, see Figure 8. \( \square \)

Using KLO, we have found that 22 out of 33 closures of alternating 3-braids with up to 20 crossings whose associated strings belong to \( S_{2c}^\dagger \) are algorithmically ribbon, in each instance via at most two band moves. It is known that the Turk’s head knot \( K_7 \) with the associated string in \( S_{2c}^\dagger \) and 14 crossings is not slice [19]. The remaining 10 examples for which we were unable to find band moves exhibiting a \( \chi \)-ribbon surface are listed in Table 1. By a straightforward application of the Gordon–Litherland signature formula [10, Theorems 6 and 6′′], the signature of the closure of a braid \( \beta = \sigma_1^{a_1}\sigma_2^{b_1}\ldots\sigma_1^{a_n}\sigma_2^{b_n} \) with \( \sum_i a_i \) and \( \sum_i b_i \) both greater than one is

\[
\sigma(\hat{\beta}) = \sum_{i=1}^{n} b_i - a_i.
\]

Thus, for all links with associated strings in \( S_{2a} \cup S_{2b} \cup S_{2c} \) satisfying this condition (in particular, for those in Table 1), the signature vanishes, which means that for knots, so do the Ozsváth and Szabo’s \( \tau \) and Rasmussen’s \( s \) invariants [17; 18] without giving us any sliceness obstructions; Tristram–Levine signatures for knots in Table 1 are also zero. Moreover, by comparing their hyperbolic volumes, we have verified that none of the entries in Table 1 belong to the list of “escapee” \( \chi \)-ribbon links described in [16]: this further advances them as candidates for more careful study. In Section 4 we will show that the three knots \( K_1, K_2 \) and \( K_3 \) in Table 1 are not slice, which lets us conclude that every knot which is a closure of an alternating 3-braid with up to 20 crossings and whose double branched cover bounds a rational ball, except \( K_1, K_2, K_3 \) and \( K_7 \), is slice.

**Remark 3.4.** We note that not all alternating knots can be represented as closures of alternating braids. This implies that our list of smoothly nonslice knots which are closures of alternating 3-braids with up to 20 crossings does not include, for example, the nonslice alternating knot \( 5_2 \), which has braid index 3, but cannot be represented as a closure of any alternating braid [3]. A full classification of braid presentations of alternating links with braid index 3 has been given by Stoimenow in [21].
In this section we restrict our attention to the three knots in Table 1. Let
\[ \beta_1 = \sigma_1^2 \sigma_2^{-2} \sigma_1^2 \sigma_2^{-2} \sigma_1 \sigma_2^{-2} \sigma_1^2 \sigma_2^{-2} \sigma_1 \sigma_2^{-1}, \]
\[ \beta_2 = \sigma_1^3 \sigma_2^{-2} \sigma_1 \sigma_2^{-1} \sigma_1^2 \sigma_2^{-3} \sigma_1^2 \sigma_2^{-1} \sigma_1 \sigma_2^{-2}, \]
\[ \beta_3 = \sigma_1^2 \sigma_2^{-1} \sigma_1 \sigma_2^{-2} \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-2} \sigma_1 \sigma_2^{-1} \sigma_1^2 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}, \]
and let $K_i = \hat{\beta}_i$ for $i = 1, 2, 3$. We will show that the knots $K_i$ are not slice by adapting the approach of Aceto et al. [1], based in turn on work of Herald, Kirk and Livingston [11], and demonstrating that certain reduced twisted Alexander polynomials do not factor as norms; this is a generalisation of the Fox–Milnor condition on Alexander polynomials of $K_i$ which is passed by these knots. Fix distinct primes $p$ and $q$, and let $\zeta_q$ denote a primitive $q$-th root of unity. The general outline of the algorithm is the following.

(1) Construct the Seifert matrix $S_i$ for $K_i$ coming from the standard Seifert surface $F_i$ associated to $K_i$ viewed as a 3-braid closure.

(2) By considering the presentation matrix $P_i = tS_i - S_i^T \in \text{Mat}(\mathbb{Z}[t^\pm 1])$ of the Alexander module $\mathcal{A}(K_i)$, determine the structure of $H_1(\Sigma_p(K_i))$, the first homology of the $p$-fold cover of $S^3$ branched over $K_i$, as well as a basis of $H_1(\Sigma_p(K_i))$ given by lifts of curves in $S^3 \setminus \nu(F)$.

(3) Calculate the Blanchfield pairings $\text{Bl}_i : \mathcal{A}(K_i) \times \mathcal{A}(K_i) \to \mathbb{Q}(t)/\mathbb{Z}[t^\pm 1]$ and deduce the linking pairings $\lambda_i : H_1(\Sigma_p(K_i)) \times H_1(\Sigma_p(K_i)) \to \mathbb{Q}/\mathbb{Z}$.

(4) Enumerate all $\mathbb{Z}[t^\pm 1]$-submodules $N$ of $H_1(\Sigma_p(K_i))$ with $|N|^2 = |H_1(\Sigma_p(K_i))|$ and thus find all metabolisers of $H_1(\Sigma_p(K_i))$, i.e., those $N$ on which $\lambda_i$ vanishes.

(5) Construct nontrivial characters $\chi : H_1(\Sigma_p(K_i)) \to \mathbb{Z}/q$ that vanish on the metabolisers.

(6) Using a Wirtinger presentation of $\pi_1(X_i)$, where $X_i$ is the knot complement of $K_i$, construct a certain homomorphism $\pi_1(X_i) \to \mathbb{Z} \times H_1(\Sigma_p(K_i))$ that induces a representation $\varphi_\chi : \pi_1(X_i) \to \text{GL}(p, \mathbb{Q}(\zeta_q)[t^\pm 1])$ for each character in (5).

<table>
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<th># of crossings</th>
<th>associated string</th>
<th>$x$-string</th>
<th># of components</th>
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<td>[0[0]]</td>
<td>3</td>
</tr>
<tr>
<td>18</td>
<td>(2, 4, 2, 4, 2, 4, 2, 3)</td>
<td>[1, 1, 1, 1, 0]</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>(2, 2, 4, 3, 2, 5, 2, 3, 4)</td>
<td>[2, 1, 0, 0, 1]</td>
<td>1</td>
</tr>
<tr>
<td>18</td>
<td>(2, 3, 4, 3, 2, 3, 2, 3, 3)</td>
<td>[1, 0, 0, 0, 1, 0, 0]</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>(2, 2, 2, 3, 3, 3, 6, 3, 3, 3)</td>
<td>[3, 0[6]]</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>(2, 4, 2, 4, 2, 4, 2, 4, 2, 4)</td>
<td>[1[5]]</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>(2, 4, 2, 3, 3, 4, 2, 4, 3, 3)</td>
<td>[1, 1, 1, 0, 0, 0, 0, 0]</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>(2, 4, 3, 2, 3, 4, 2, 3, 4, 3)</td>
<td>[1, 1, 0, 0, 1, 0, 0]</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>(2, 3, 2, 3, 2, 3, 4, 4, 4, 3)</td>
<td>[1, 0, 1, 0, 1, 0, 0]</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>(2, 2, 2, 4, 3, 2, 6, 2, 3, 4)</td>
<td>[3, 1, 0, 0, 1]</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1. Links in $S_{2c}^+$ with up to 20 crossings which are potentially non-$\chi$-slice. In the following we show that the three knots in this table are not slice.
Figure 9. Our choice of a Seifert surface $F_1$ for $K_1$. Lifts of Alexander dual curves $\hat{s}_{15}$ and $\hat{s}_{16}$ to generate $H_1(\Sigma_3(K_1))$.

(7) Use the Fox matrix for a Wirtinger presentation of $\pi_1(X_i)$ to obtain a matrix $\Phi_\chi$ for each $\chi$ in (5), whose determinant $\det \Phi_\chi$ is the reduced twisted Alexander polynomial $\bar{\Delta}_k^\chi(t)$.

(8) Verify that none of the $\bar{\Delta}_k^\chi(t)$ factor as norms, hence providing an obstruction to sliceness of all $K_i$.

For reference about various terms used in this outline, we direct the reader in the first instance to [11] and [1], as well as to the survey [9]. The computations were performed in SageMath notebooks available on the author’s website.2

4A. The Seifert matrix. Let $\beta$ be a 3-braid. A Seifert surface $F$ for $\hat{\beta}$ can be constructed by joining three discs $D_1$, $D_2$ and $D_3$ by half-twisted bands, where each band between $D_1$ and $D_2$ comes from a $\sigma_1$ term in $\beta$, and each band between $D_2$ and $D_3$ from a $\sigma_2$ term; identify the bands with $\sigma_i$’s. Let $g$ be the genus of $F$. We can choose the generators of $H_1(F)$ to be the loops running once through consecutive $\sigma_1$’s and $\sigma_2$’s, except for the loop between the first and last $\sigma_1$ and the first and last $\sigma_2$. We order these generators $s_1$, . . . , $s_{2g}$ by when the first $\sigma_i$ through which $s_j$ runs appears in $\beta$. With this setup, the Seifert matrix $S$ can be obtained using the algorithm of Collins [2]. Such $F$ with $s_1$, . . . , $s_{2g}$ for $K_1$ is shown in Figure 9. Also, for $\nu(F)$ an open tubular neighbourhood of $F$, denote by $\hat{s}_i$ a choice of a simple closed curve in $S^3 \setminus \nu(F)$ that is Alexander dual to $\{s_1, . . . , s_{2g}\}$, i.e., which satisfies $\text{lk}(s_i, \hat{s}_j) = \delta_{ij}$.

4B. Structure and bases of $H_1(\Sigma_3(K_i))$. We may perform column operations on the presentation matrices $P_i = tS_i - S_i^T$ of the Alexander modules $\mathcal{A}(K_i)$ to transform them into the forms

\[
\begin{pmatrix}
I & 0 \\
* & \begin{pmatrix} p_1(t) & 0 \\ 0 & p_1(t) \end{pmatrix}
\end{pmatrix},
\begin{pmatrix}
I & 0 \\
* & \begin{pmatrix} p_2(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & p_2(t) \end{pmatrix}
\end{pmatrix},
\begin{pmatrix}
I & 0 \\
* & \begin{pmatrix} p_3(t) & 0 \\ 0 & p_3(t) \end{pmatrix}
\end{pmatrix}
\]

2https://sites.google.com/view/vbrej
for \(i = 1, 2, 3\), respectively, where each \(p_i(t)\) is the square root of the untwisted Alexander polynomial \(\Delta_{K_i}(t)\), \(I\) is the identity matrix and \(*\) represents other entries. Specifically,

\[
\begin{align*}
p_1(t) &= 1 - 3t + 7t^2 - 10t^3 + 11t^4 - 10t^5 + 7t^6 - 3t^7 + t^8, \\
p_2(t) &= 1 - 3t + 6t^2 - 9t^3 + 11t^4 - 9t^5 + 6t^6 - 3t^7 + t^8, \\
p_3(t) &= 1 - 4t + 8t^2 - 11t^3 + 13t^4 - 11t^5 + 8t^6 - 4t^7 + t^8.
\end{align*}
\]

Recall that the Alexander module \(\mathcal{A}(K)\) of a knot \(K\) is the \(\mathbb{Z}[t^\pm 1]\)-module \(H_1(\tilde{X}_K)\), where \(\tilde{X}_K\) is the infinite cyclic cover of the knot complement \(X_K\) and \(t\) acts by deck transformations. Choose a preferred copy of \(S^3 \setminus \nu(F_i)\) in \(\tilde{X}_i^\infty\) for all \(i\). From [8, Theorems 1.3 and 1.4], summarised in the present context in [1, Theorem 3.6], it follows that

\[\mathcal{A}(K_i) \cong \mathbb{Z}[t^\pm 1]/\langle p_i(t) \rangle \oplus \mathbb{Z}[t^\pm 1]/\langle p_i(t) \rangle,\]

where \(\mathcal{A}(K_i)\) for \(i \in \{1, 3\}\) is generated by the lifts of \(\hat{s}_{15}\) and \(\hat{s}_{16}\) to the preferred copy of \(S^3 \setminus \nu(F_i)\) in \(\tilde{X}_i^\infty\), while \(\mathcal{A}(K_2)\) is generated by the lifts of \(\hat{s}_{14}\) and \(\hat{s}_{16}\); in each case, call these generators \(a\) and \(b\), respectively. Choose \(p = 3\). By, e.g., [6, Section 6.1], we have

\[
H_1(\Sigma_3(K_i)) \cong \mathcal{A}(K_i)/\langle t^2 + t + 1 \rangle \\
\cong \mathbb{Z}[t^\pm 1]/\langle p_i(t), t^2 + t + 1 \rangle \oplus \mathbb{Z}[t^\pm 1]/\langle p_i(t), t^2 + t + 1 \rangle \\
\cong \mathbb{Z}[t^\pm 1]/\langle 7t^, t^2 + t + 1 \rangle \oplus \mathbb{Z}[t^\pm 1]/\langle 7t, t^2 + t + 1 \rangle \\
\cong (\mathbb{Z}/7)[t^\pm 1]/\langle t^2 + t + 1 \rangle \oplus (\mathbb{Z}/7)[t^\pm 1]/\langle t^2 + t + 1 \rangle
\]

in each of the three cases, since all of \(p_i(t)\) are congruent to \(7t\) modulo \(t^2 + t + 1\). Hence, we fix \(q = 7\). The generators of \(\mathcal{A}(K_i)\) descend to \(H_1(\Sigma_3(K_i))\), so by abuse of notation we also denote them by \(a\) and \(b\). As a group, \(H_1(\Sigma_3(K_i)) \cong (\mathbb{Z}/7)^4\), and we may treat it as a \((\mathbb{Z}/7)\)-module generated by \(a, ta, b\) and \(tb\).

### 4C. Blanchfield and linking forms.

From [8, Theorems 1.3, 1.4; 1, Theorem 3.6] and a calculation in the accompanying notebooks, we obtain that the Blanchfield pairings on \(\mathcal{A}(K_i)\) are given, with respect to the ordered basis \(\{a, b\}\) and after reducing both the numerators and denominators modulo \(t^3 - 1\), by

\[
\frac{1}{7} \begin{pmatrix} 2t^2 + 2t - 4 & -2t^2 + 4t - 2 \\ 4t^2 - 2t - 2 & -4t^2 - 4t + 8 \end{pmatrix}, \quad \frac{1}{7} \begin{pmatrix} -3t^2 - 3t + 6 & 3t^2 - 3t \\ -3t^2 + 3t & 3t^2 + 3t - 6 \end{pmatrix}, \quad \frac{1}{7} \begin{pmatrix} -4t^2 - 4t + 8 & 4t^2 - 2t - 2 \\ -2t^2 + 4t - 2 & 2t^2 + 2t - 4 \end{pmatrix}
\]

for \(i = 1, 2, 3\), respectively. Via [7, Chapter 2.6], applied similarly to [1, Proposition 3.7], we read off that the linking forms \(\lambda_i : H_1(\Sigma_3(K_i)) \times H_1(\Sigma_3(K_i)) \to \mathbb{Q}/\mathbb{Z}\)
with respect to the ordered basis \([a, ta, b, tb]\) are given by

\[
\frac{1}{7} \begin{pmatrix}
-4 & 2 & -2 & 4 \\
2 & -4 & -2 & -2 \\
-2 & -2 & 1 & -4 \\
4 & -2 & -4 & 1
\end{pmatrix}, \quad \frac{1}{7} \begin{pmatrix}
6 & -3 & 0 & -3 \\
-3 & 6 & 3 & 0 \\
0 & 3 & -6 & 3 \\
-3 & 0 & 3 & -6
\end{pmatrix}
\quad \text{and} \quad \frac{1}{7} \begin{pmatrix}
1 & -4 & -2 & -2 \\
-4 & 1 & 4 & -2 \\
-2 & 4 & -4 & 2 \\
-2 & -2 & 2 & -4
\end{pmatrix}.
\]

**4D. Metabolisers of** \(H_1(\Sigma_3(K_i))\). Write \(M = (\mathbb{Z}/7)[t^{\pm 1}]/\langle t^2 + t + 1 \rangle\) so that, as a \((\mathbb{Z}/7)[t^{\pm 1}]\)-module, \(H_1(\Sigma_3(K_i)) \cong M \oplus M\). Since the order \(|H_1(\Sigma_3(K_i))| = 7^4\), we seek to describe all its \((\mathbb{Z}/7)[t^{\pm 1}]\)-submodules of order \(7^2 = 49\). Since \(t^2 + t + 1\) has irreducible factors \((t - 2), (t + 3) \in (\mathbb{Z}/7)[t^{\pm 1}]\), the set \(\{0, 1, t - 2, t + 3\}\) contains precisely the \((\mathbb{Z}/7)[t^{\pm 1}]\)-submodules of \(M\); since the \((\mathbb{Z}/7)[t^{\pm 1}]\)-action on \(M\) factors through \((\mathbb{Z}/7)[t^{\pm 1}]\), these are also precisely the \((\mathbb{Z}/7)[t^{\pm 1}]\)-submodules of \(M\). Observe that \(|0)\rangle = 1, \langle 1)\rangle = 49\) and \(|t - 2)\rangle = |t + 3)\rangle = 7\). Now let \(N\) be a \((\mathbb{Z}/7)[t^{\pm 1}]\)-submodule of \(H_1(\Sigma_3(K_i))\), and consider the commutative diagram

\[
M \oplus \{0\} \longrightarrow M \oplus M \stackrel{\pi}{\longrightarrow} \{0\} \oplus M
\]

where \(\pi(x, y) = (0, y)\) for all \(x, y \in M\), and unlabelled arrows are inclusions; \(\ker \pi \mid N\) and \(\im \pi \mid N\) are submodules of \(M \oplus \{0\}\) and \(\{0\} \oplus M\), respectively. Since \(|N| = |\ker \pi \mid N| \cdot |\im \pi \mid N|\), we can deduce what \(N\) could be by order considerations.

- If \(|\ker \pi \mid N| = 49\), then \(|\im \pi \mid N| = 1\) and \(N = \ker \pi \mid N = \ spans(\mathbb{Z}/7)[t^{\pm 1}]\{1, 0\}\).
- If \(|\ker \pi \mid N| = 1\), then \(N \cong \im \pi \mid N = \ spans(\mathbb{Z}/7)[t^{\pm 1}]\{k, 1\}\) for some \(k \in (\mathbb{Z}/7)[t^{\pm 1}]\).

Now, let \(\{t - 2, t + 3\} = \{\langle \alpha \rangle, \langle \beta \rangle\}\); we have \(\Ann \alpha = \langle \beta \rangle\) and \(\Ann \beta = \langle \alpha \rangle\). There are two remaining cases to consider.

- Suppose \(\ker \pi \mid N \cong \im \pi \mid N \cong \langle \alpha \rangle\). Then \(N\) contains \(\{(\alpha, 0), (k, \alpha)\}\) for some \(k \in (\mathbb{Z}/7)[t^{\pm 1}]\). Since \(\beta(k, \alpha) = (\beta k, 0) \in \ker \pi \mid N\), we must have \(\beta k \in \langle \alpha \rangle\), so \(k \in \langle \alpha \rangle\), i.e., \(k = l \alpha\) for some \(l \in (\mathbb{Z}/7)[t^{\pm 1}]\). Then \(-l(\alpha, 0) + (k, \alpha) = (0, \alpha) \in N\), so \(N\) contains two linearly independent elements \((\alpha, 0)\) and \((0, \alpha)\) of order \(7\), and hence is generated by them for any choice of \(k\). This yields two submodules \(N = \ spans(\mathbb{Z}/7)[t^{\pm 1}]\{(t - 2, 0), (0, t - 2)\}\) and \(N = \ spans(\mathbb{Z}/7)[t^{\pm 1}]\{(t + 3, 0), (0, t + 3)\}\

- Suppose \(\ker \pi \mid N = \langle \alpha \rangle\) and \(\im \pi \mid N \cong \langle \beta \rangle\). We similarly observe that \(N\) contains \(\{(\alpha, 0), (k, \beta)\}\) for some \(k \in (\mathbb{Z}/7)[t^{\pm 1}]\). We have \(\alpha(k, \beta) = (\alpha k, 0) \in \ker \pi \mid N\), so we can take \(k\) modulo \(\alpha\), i.e., \(k \in \mathbb{Z}/7\). Then \(\{(\alpha, 0), (k, \beta)\}\) is a linearly independent set generating \(N\) for any choice of \(k \in \mathbb{Z}/7\). Thus, \(N = \ spans(\mathbb{Z}/7)[t^{\pm 1}]\{(t - 2, 0), (k, t + 3)\}\) or \(N = \ spans(\mathbb{Z}/7)[t^{\pm 1}]\{(t + 3, 0), (k, t - 2)\}\) for \(k \in \mathbb{Z}/7\).
We follow [1, Appendix A] and [11, Chapters 5–7] to construct representations $N$ of the knot group of $K$ are presented in Table 2.

By a direct computation carried out in the accompanying notebooks, the submodules $N_0^{\alpha}$ and $N_0^\beta$ are metabolisers for $K_i$ for all $i$; in addition, $K_1$ has metabolisers $N_6^{\alpha\beta}$ and $N_4^{\beta\alpha}$, $K_2$ has metabolisers $N_1^{\alpha\beta}$ and $N_1^{\beta\alpha}$, and $K_3$ has metabolisers $N_2^{\alpha\beta}$ and $N_3^{\beta\alpha}$.

### Table 2. Our choice of characters $\chi : H_1(\Sigma_3(K_i)) \to \mathbb{Z}/7$ vanishing on the metabolisers of $K_1$, $K_2$ and $K_3$; the characters $\chi_0^\alpha$ and $\chi_0^\beta$ are given for all $K_i$ by (1, 2, 1, 2) and (1, −3, 1, −3), respectively.

<table>
<thead>
<tr>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_6^{\alpha\beta} : (1, 2, 1, -2)$</td>
<td>$\chi_1^{\alpha\beta} : (1, 2, 1, -4)$</td>
<td>$\chi_2^{\alpha\beta} = \chi_0^\alpha : (1, 2, 1, 2)$</td>
</tr>
<tr>
<td>$\chi_4^{\beta\alpha} : (1, -3, 1, -2)$</td>
<td>$\chi_1^{\beta\alpha} : (1, -3, 1, 1)$</td>
<td>$\chi_3^{\beta\alpha} : (1, -3, 1, 1)$</td>
</tr>
</tbody>
</table>

To summarise, writing elements of $H_1(\Sigma_3(K_i)) \cong M \oplus M$ additively with the first copy of $M$ generated by $a$ and the second by $b$, the desired submodules are

$$N_0 = \text{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{a\},$$
$$N_{k_0,k_1} = \text{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{ka + b\} \text{ for } k \in (\mathbb{Z}/7)[t^{\pm 1}]$$
$$= \text{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(k_0 + k_1t)a + b\} \text{ for } k_0, k_1 \in \mathbb{Z}/7,$$
$$N_0^{\alpha} = \text{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t - 2)a, (t - 2)b\},$$
$$N_0^{\beta} = \text{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t + 3)a, (t + 3)b\},$$
$$N_{k_0}^{\alpha\beta} = \text{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t - 2)a, k_0a + (t + 3)b\} \text{ for } k_0 \in \mathbb{Z}/7,$$
$$N_{k_0}^{\beta\alpha} = \text{span}_{(\mathbb{Z}/7)[t^{\pm 1}]}\{(t + 3)a, k_0a + (t - 2)b\} \text{ for } k_0 \in \mathbb{Z}/7.$$

By a direct computation carried out in the accompanying notebooks, the submodules $N_0^{\alpha}$ and $N_0^{\beta}$ are metabolisers for $K_i$ for all $i$; in addition, $K_1$ has metabolisers $N_6^{\alpha\beta}$ and $N_4^{\beta\alpha}$, $K_2$ has metabolisers $N_1^{\alpha\beta}$ and $N_1^{\beta\alpha}$, and $K_3$ has metabolisers $N_2^{\alpha\beta}$ and $N_3^{\beta\alpha}$.

### 4E. Characters vanishing on the metabolisers. It is easy to define characters $\chi : H_1(\Sigma_3(K_i)) \to \mathbb{Z}/7$ that vanish on the metabolisers. Let subscripts and superscripts denote corresponding metabolisers and 4-tuples in parentheses represent the values a character takes on the ordered basis $\{a, ta, b, tb\}$. Then we can take $\chi_0^\alpha$ and $\chi_0^\beta$ as defined by (1, 2, 1, 2) and (1, −3, 1, −3), respectively. The rest of the characters are presented in Table 2.

### 4F. Representations of the knot groups into $\GL(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}])$. Let $K \in \{K_1, K_2, K_3\}$.

We follow [1, Appendix A] and [11, Chapters 5–7] to construct representations $\varphi_\chi : \pi_1(X_K) \to \GL(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}])$ of the knot group of $K$ that determine twisted Alexander polynomials for each character in Table 2. Fix a basepoint $x_0$ in $S^3 \setminus \nu(F)$ and let $\tilde{x}_0$ be its lift to the
preferred copy of $S^3 \setminus \nu(F)$ in $\widetilde{X}_K$, the triple cyclic cover of the knot complement $X_K$. Also fix a based meridian $\mu_0$ in $S^3 \setminus K$ and let $\varepsilon : \pi_1(X_K) \to \mathbb{Z}$ be the abelianisation homomorphism. Define a map $l : \ker \varepsilon \to H_1(\Sigma_3(K))$ that takes a simple closed curve $\gamma \subset S^3 \setminus K$ based at $x_0$ with $\text{lk}(K, \gamma) = 0$ to the homology class of the well-defined lift $\tilde{\gamma}$ in $X^3_K \subset \Sigma_3(K)$ based at $\tilde{x}_0$. In particular, $l$ has the property that for any $\gamma \in \ker \varepsilon$, we have

$$(\tilde{\gamma})$$

$$l(\mu_0 \gamma \mu_0^{-1}) = t \cdot l(\gamma).$$

Now consider the semidirect product $\mathbb{Z} \rtimes H_1(\Sigma_3(K))$, with $\mathbb{Z} = \langle t \rangle$, whose product structure is given by $(t^{m_1}, x_1) \cdot (t^{m_2}, x_2) = (t^{m_1 + m_2}, t^{-m_2} \cdot x_1 + x_2)$ with $t$ acting on elements of $H_1(\Sigma_3(K))$ by deck transformations. Fix a Wirtinger presentation of $\pi_1(X_K) \cong \langle g_1, \ldots, g_n | r_1, \ldots, r_n \rangle$ and define a homomorphism

$$\psi : \pi_1(X_K) \to \mathbb{Z} \rtimes H_1(\Sigma_3(K)), \quad g_i \mapsto (t, l(\mu_0^{-1} g_i)) =: (t, v_i)$$

on the generators of $\pi_1(X_K)$, since clearly $\mu_0^{-1} g_i \in \ker \varepsilon$. Observe that a relation $g_i g_j g_i^{-1} g_k^{-1} = 1$ imposes, via the group structure on $\mathbb{Z} \rtimes H_1(\Sigma_3(K))$, the condition

$$(\text{for } i \neq j) \quad (1 - t) v_i + tv_j - v_k = 0.$$  

Finally, for a character $\chi : H_1(\Sigma_3(K)) \to \mathbb{Z}/7$, we obtain a representation

$$\varphi_\chi : \pi_1(X_K) \to \text{GL}(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}])$$

by setting $\varphi_\chi = \tau_\chi \circ \psi$, where

$$\tau_\chi : \mathbb{Z} \rtimes H_1(\Sigma_3(K)) \to \text{GL}(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}]),$$

$$(t^m, v) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta_7^\chi(v) & 0 & 0 \\ 0 & \zeta_7^\chi(t^{-1}v) & 0 \\ 0 & 0 & \zeta_7^{\chi(t^2v)} \end{pmatrix}.$$  

We shall apply the equation $(\tilde{\gamma})$ to determine the form of the first few $v_k$ for $K$ in terms of the generators $\{a, b\}$ of $H_1(\Sigma_3(K))$ and then deduce the rest of $v_k$ using $(\text{for } i \neq j)$, giving us the desired $\varphi_\chi$. We illustrate the process in more detail for $K_1$, with $K_2$ and $K_3$ cases being analogous.

Recall that we orient $K_1$ clockwise. Index the arcs in the diagram of $K_1$ as shown in Figure 10, starting with 1 at the top left and increasing the index at every undercrossing. This yields the following Wirtinger presentation of $\pi_1(X_1)$, with generators being the meridians $g_i$ about each arc $i$ based at $x_0$:

$$\pi_1(X_1) = \langle g_1, \ldots, g_{18} \rangle$$

$$\begin{bmatrix}
    g_1 g_3 g_1^{-1} g_2^{-1}, & g_3 g_1 g_3^{-1} g_2^{-1}, & g_6 g_1 g_7^{-1} g_8^{-1}, & g_8 g_1 g_6 g_1 g_6^{-1} g_{10}^{-1}, \\
    g_8 g_1 g_6 g_1 g_6^{-1} g_{10}^{-1}, & g_1 g_3 g_1^{-1} g_2^{-1}, & g_3 g_1 g_3^{-1} g_2^{-1}, & g_6 g_1 g_7^{-1} g_8^{-1}, \\
    g_2 g_1 g_3 g_1^{-1} g_2^{-1}, & g_8 g_1 g_4 g_1 g_4^{-1} g_{14}^{-1}, & g_4 g_1 g_4 g_1 g_4^{-1} g_{14}^{-1}, & g_2 g_1 g_3 g_1^{-1} g_2^{-1}, \\
    g_1 g_3 g_1^{-1} g_2^{-1}, & g_8 g_1 g_4 g_1 g_4^{-1} g_{14}^{-1}, & g_4 g_1 g_4 g_1 g_4^{-1} g_{14}^{-1}, & g_2 g_1 g_3 g_1^{-1} g_2^{-1}, \\
    g_1 g_3 g_1^{-1} g_2^{-1}, & g_8 g_1 g_4 g_1 g_4^{-1} g_{14}^{-1}, & g_4 g_1 g_4 g_1 g_4^{-1} g_{14}^{-1}, & g_2 g_1 g_3 g_1^{-1} g_2^{-1} \end{bmatrix}.$$
while for $K$, (natural extension to the group ring $\mathbb{Z}\phi$) fix the Wirtinger presentation of $\pi_1(X_1)$. Calculating twisted Alexander polynomials.

$\phi$ represents $v$, this lets us calculate the values of $\mu$, we have

\[ a = l(g_8g_{12}^{-1}) = l(g_8g_1^{-1}g_1g_{12}^{-1}) = l(g_8g_1^{-1}) + l(g_1g_{12}^{-1}) \]
\[ = l(g_1g_1^{-1}g_8g_1^{-1}) - l(g_1g_1^{-1}) \]
\[ = l(g_1g_1^{-1}g_8g_1^{-1}) - l(g_1g_1^{-1}g_1g_1^{-1}) = tv_8 - tv_{12}. \]

Applying (‡‡‡) to the relation $g_{12}g_7g_{12}^{-1}g_8^{-1} = 1$ and recalling we are working modulo $t^2 + t + 1$, we get

\[ (1 - t)v_{12} + tv_7 - v_8 = 0 \quad \Rightarrow \quad (1 - t)v_{12} - v_8 = -tb \cdot (-t) \]
\[ \Rightarrow \quad (tv_8 - tv_{12}) + t^2v_{12} = t^2b \]
\[ \Rightarrow \quad a + t^2v_{12} = t^2b \cdot t \]
\[ \Rightarrow \quad v_{12} = -ta + b. \]

Now we can use (‡‡‡) repeatedly to obtain the values of all $v_i$. With the same conventions and the choice $\mu_0 = g_1$, for $K_2$ we have

\[ l(\hat{s}_{14}) = l(g_1^{-1}g_6) = a \quad \text{and} \quad l(\hat{s}_{16}) = l(g_{14}g_7^{-1}) = b, \]

while for $K_3$,

\[ l(\hat{s}_{15}) = l(g_1^{-1}g_7) = a \quad \text{and} \quad l(\hat{s}_{16}) = l(g_8g_{13}^{-1}) = b; \]

this lets us calculate the values of $v_i$ in Table 3 analogously. With that, constructing representations $\varphi_\chi$ for the characters in Section 4E is mechanical.

4G. Calculating twisted Alexander polynomials. Again, let $K \in \{K_1, K_2, K_3\}$ and fix the Wirtinger presentation of $\pi_1(X_K)$ as in Section 4F. Given a representation $\varphi_\chi : \pi_1(X_K) \to \text{GL}(3, \mathbb{Q}(\zeta_7)[t^{\pm 1}])$, let $\Phi : \mathbb{Z}[\pi_1(X_K)] \to \text{Mat}_3(\mathbb{Q}(\zeta_7)[t^{\pm 1}])$ be its natural extension to the group ring $\mathbb{Z}[\pi_1(X_K)]$ taking values in the set of $3 \times 3$
matrices with $\mathbb{Q}(\zeta_7)[t^{\pm 1}]$ coefficients. Let

$$\Psi = \left( \begin{array}{c} \partial r_i \\ \partial g_j \end{array} \right)_{i,j=1,\ldots,18}$$

be the Fox matrix for the Wirtinger presentation of $\pi_1(X_K)$; the row of $\Psi$ corresponding to the relation $g_i g_j g_i^{-1} g_k^{-1}$ has $1 - g_k$ in the $i$-th column, $g_i$ in the $j$-th column, $-1$ in the $k$-th column and zeros elsewhere. Write $r(\Psi)$ for the reduced Fox matrix obtained by dropping the first row and column from $\Psi$ and let $\Phi_\chi$ be the $51 \times 51$ matrix obtained by applying $\Phi$ to $r(\Psi)$ entrywise. By [11, Section 9], the reduced twisted Alexander polynomial $\tilde{\Delta}_K^\chi(t)$ of $(K, \chi)$ (for nontrivial $\chi$) is given by

$$\tilde{\Delta}_K^\chi(t) = \frac{\det \Phi_\chi}{(t-1) \det(\varphi_\chi(g_1) - I)}.$$ 

Thus we obtain the 11 reduced twisted Alexander polynomials listed in the Appendix associated with our characters of interest.

4H. **Obstructing sliceness of $K_i$.** To show that $K_1$, $K_2$ and $K_3$ are not slice, we use the following generalisation of the Fox–Milnor condition, due to Kirk and Livingston [12].

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>$\pi_1(X_1)$</th>
<th>$\pi_1(X_2)$</th>
<th>$\pi_1(X_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_2$</td>
<td>$(6t + 5)a + (5t + 6)b$</td>
<td>$(5t + 6)a + (4t + 4)b$</td>
<td>$(5t + 6)a + (6t + 5)b$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$5ta + 5b$</td>
<td>$3a + (3t + 1)b$</td>
<td>$(4t + 3)a + (t + 1)b$</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$(2t + 5)a + 6b$</td>
<td>$(2t + 6)a + 2b$</td>
<td>$(6t + 3)a + b$</td>
</tr>
<tr>
<td>$v_5$</td>
<td>$(6t + 5)a + (5t + 3)b$</td>
<td>$(4t + 1)a + (6t + 5)b$</td>
<td>$(6t + 4)a + (4t + 6)b$</td>
</tr>
<tr>
<td>$v_6$</td>
<td>$5tb$</td>
<td>$a$</td>
<td>$(4t + 1)a + (t + 6)b$</td>
</tr>
<tr>
<td>$v_7$</td>
<td>$b$</td>
<td>$a + (6t + 1)b$</td>
<td>$a$</td>
</tr>
<tr>
<td>$v_8$</td>
<td>$(5t + 6)a + b$</td>
<td>$(6t + 6)a + (6t + 5)b$</td>
<td>$a + (5t + 6)b$</td>
</tr>
<tr>
<td>$v_9$</td>
<td>$(3t + 2)a + (4t + 1)b$</td>
<td>$5ta + (3t + 5)b$</td>
<td>$(3t + 6)a + (5t + 3)b$</td>
</tr>
<tr>
<td>$v_{10}$</td>
<td>$(t + 2)a + (5t + 1)b$</td>
<td>$(2t + 3)a + (3t + 3)b$</td>
<td>$(4t + 6)a + (3t + 3)b$</td>
</tr>
<tr>
<td>$v_{11}$</td>
<td>$6a + (4t + 1)b$</td>
<td>$(3t + 6)a + 5b$</td>
<td>$(3t + 6)a + 2tb$</td>
</tr>
<tr>
<td>$v_{12}$</td>
<td>$6ta + b$</td>
<td>$(6t + 2)a + (6t + 6)b$</td>
<td>$(6t + 2)a + 6b$</td>
</tr>
<tr>
<td>$v_{13}$</td>
<td>$6a + (6t + 6)b$</td>
<td>$a + b$</td>
<td>$a + 6tb$</td>
</tr>
<tr>
<td>$v_{14}$</td>
<td>$(3t + 4)a + (6t + 2)b$</td>
<td>$a + 5tb$</td>
<td>$(6t + 6)a + 6b$</td>
</tr>
<tr>
<td>$v_{15}$</td>
<td>$3a + (2t + 4)b$</td>
<td>$(5t + 3)a + 6b$</td>
<td>$(6t + 2)a + (3t + 4)b$</td>
</tr>
<tr>
<td>$v_{16}$</td>
<td>$5a + (2t + 3)b$</td>
<td>$(5t + 5)a + (3t + 5)b$</td>
<td>$(t + 1)a + (2t + 6)b$</td>
</tr>
<tr>
<td>$v_{17}$</td>
<td>$4a + (2t + 2)b$</td>
<td>$ta + (5t + 3)b$</td>
<td>$ta + (2t + 5)b$</td>
</tr>
<tr>
<td>$v_{18}$</td>
<td>$(6t + 1)b$</td>
<td>$(6t + 1)a$</td>
<td>$(6t + 1)a$</td>
</tr>
</tbody>
</table>

**Table 3.** Values of $v_k = l(\mu_0^{-1} g_k) \in H_1(\Sigma_3(K_i))$. 
Theorem 4.1 [12, Proposition 6.1]. Let $K \subset S^3$ be a slice knot and fix distinct primes $p$ and $q$. Then there exists a covering transformation invariant metaboliser $N$ in $H_1(\Sigma_p(K))$ such that the following condition holds: for every character $\chi : H_1(\Sigma_p(K)) \to \mathbb{Z}/q$ that vanishes on $N$, the associated reduced twisted Alexander polynomial $\tilde{\Delta}_K(\chi)(t) \in \mathbb{Q}(\zeta_q)[t^{\pm 1}]$ is a norm, i.e., $\tilde{\Delta}_K(\chi)(t)$ can be written as

$$\tilde{\Delta}_K(\chi)(t) = \lambda t^k f(t) \bar{f}(t)$$

for some $\lambda \in \mathbb{Q}(\zeta_q)$, $k \in \mathbb{Z}$ and $f(t) \in \mathbb{Q}(\zeta_q)[t^{\pm 1}]$ by the involution $t \mapsto t^{-1}$, $\zeta_q \mapsto \zeta_q^{-1}$.

Using the routine implemented in SnapPy [4] for determining whether an element of $\mathbb{Q}(\zeta_q)[t^{\pm 1}]$ is a norm, which relies on the SageMath algorithm for factoring polynomials over cyclotomic fields, we conclude via a calculation in the accompanying notebooks that none of the 11 polynomials in the Appendix are norms. This implies that $K_1$, $K_2$ and $K_3$ are not slice.

Appendix: Reduced twisted Alexander polynomials for $K_1$, $K_2$ and $K_3$

The following table contains reduced twisted Alexander polynomials for knots $K_1$, $K_2$ and $K_3$ associated to characters vanishing on the metabolisers of respective knots; for brevity, we write $\zeta = \zeta_7$ and $\theta = \zeta_7 + \zeta_7^2 + \zeta_7^4$.

<table>
<thead>
<tr>
<th>$(K_i, \chi)$</th>
<th>$\tilde{\Delta}_K(\chi)(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(K_1, \chi_0^a)$</td>
<td>$-t^{15} + (-2\theta - 1)t^{14} + (-8\theta - 3)t^{13} + 15t^{12} + (-3\theta + 48)t^{11}$ + $(-8\theta + 33)t^{10} + (-48\theta + 34)t^9 + 199t^8 + (48\theta + 82)t^7$ + $(8\theta + 41)t^6 + (3\theta + 51)t^5 + 15t^4 + (8\theta + 5)t^3 + (2\theta + 1)t^2 - t$</td>
</tr>
<tr>
<td>$(K_1, \chi_0^\beta)$</td>
<td>$-t^{15} + (-4\theta + 5)t^{14} + (24\theta - 15)t^{13} + (-93\theta - 14)t^{12} + (98\theta + 11)t^{11}$ + $(-2\theta + 71)t^{10} + (-11\theta - 154)t^9 + 360t^8 + (11\theta - 143)t^7 + (2\theta + 73)t^6$ + $(-98\theta - 87)t^5 + (93\theta + 79)t^4 + (-24\theta - 39)t^3 + (4\theta + 9)t^2 - t$</td>
</tr>
<tr>
<td>$(K_1, \chi_6^{a\beta})$</td>
<td>$-t^{15} + (2\zeta^5 - \zeta^4 + 4\zeta^3 - \zeta^2 - 2\zeta + 5)t^{14}$ + $(-3\zeta^5 + 7\zeta^4 - 24\zeta^3 - 3\zeta^2 + 2\zeta - 20)t^{13}$ + $(7\zeta^5 - 67\zeta^4 + 41\zeta^3 - 8\zeta^2 - 35\zeta + 71)t^{12}$ + $(-45\zeta^5 + 52\zeta^4 - 38\zeta^3 + 3\zeta^2 - \zeta + 19)t^{11}$ + $(68\zeta^5 + 51\zeta^4 + 114\zeta^3 + 24\zeta^2 + 95\zeta + 63)t^{10}$ + $(116\zeta^5 + 121\zeta^4 + 80\zeta^3 + 56\zeta^2 + 124\zeta + 65)t^9$ + $(149\zeta^5 - 3\zeta^4 - 3\zeta^3 + 149\zeta^2 + 19)t^8$ + $(-68\zeta^5 - 44\zeta^4 - 3\zeta^3 - 8\zeta^2 - 124\zeta - 59)t^7$ + $(-71\zeta^5 + 19\zeta^4 - 44\zeta^3 - 27\zeta^2 - 95\zeta - 32)t^6$ + $(4\zeta^5 - 37\zeta^4 + 53\zeta^3 + 44\zeta^2 + \zeta + 20)t^5$ + $(27\zeta^5 + 76\zeta^4 - 32\zeta^3 + 42\zeta^2 + 35\zeta + 42)t^4$ + $(-5\zeta^5 - 26\zeta^4 + 5\zeta^3 - 5\zeta^2 - 2\zeta - 22)t^3$ + $(\zeta^5 + 6\zeta^4 + \zeta^3 + 4\zeta^2 + 2\zeta + 7)t^2 - t$</td>
</tr>
</tbody>
</table>
\[(K_1, \chi_4^\beta) \quad t^{15} + (2\zeta^5 + \zeta^4 + 2\zeta^3 + \zeta^2 - \zeta + 2) t^{14} + (-5\zeta^5 - 2\zeta^4 - 3\zeta^3 - 6\zeta^2 - 2\zeta - 9) t^{13} + (10\zeta^5 + 4\zeta^4 + 9\zeta^2 + 20) t^{12} + (-35\zeta^5 - 36\zeta^4 - 30\zeta^3 - 35\zeta^2 - 4\zeta - 10) t^{11} + \ldots + (56\zeta^5 - 36\zeta^4 - 10\zeta^3 - 30\zeta^2 + 27\zeta + 38) t^{7} + (7\zeta^5 + 38\zeta^4 + 38\zeta^3 + 7\zeta^2 - 59) t^5 + (56\zeta^5 - 36\zeta^4 + 10\zeta^3 - 30\zeta^2 + 27\zeta + 38) t^3 + (2\zeta^5 + 3\zeta^4 + 2\zeta^3 + 3\zeta^2 + \zeta + 3) t^2 - t \]

\[(K_2, \chi_0^\alpha) \quad t^{15} + (-\theta - 2) t^{14} + (-2\theta - 1) t^{13} + (3\theta + 3) t^{12} + (-13\theta - 22) t^{11} + \ldots \]

\[(K_2, \chi_0^\beta) \quad t^{15} + (-4\theta - 7) t^{14} + (16\theta + 15) t^{13} + (-41\theta - 26) t^{12} + (55\theta + 5) t^{11} + \ldots \]

\[(K_2, \chi_1^\alpha) \quad t^{15} + (-3\zeta^5 + 3\zeta^4 - 2\zeta^3 + \zeta^2 - 2) t^{14} + (4\zeta^5 - 12\zeta^4 + 6\zeta^3 - 13\zeta^2 + \zeta + 13) t^{13} + \ldots \]

\[(K_2, \chi_1^\beta) \quad t^{15} + (-\zeta^5 - 2\zeta^4 + \zeta^3 + \zeta^2 + 3\zeta^2 - \zeta - 1) t^{13} + (\zeta^5 - 2\zeta^4 + 3\zeta^3 - 3\zeta^2 - 4) t^{12} + \ldots \]
\[(K_3, \chi_0^\alpha) \quad t^{15} + (-\theta - 3)t^{14} + (-3\theta - 1)t^{13} + (-2\theta - 22)t^{12} + (-73\theta - 8)t^{11} \]

\[= (K_3, \chi_2^{\alpha \beta}) \quad + (10\theta + 239)t^{10} + (362\theta + 223)t^9 - 675t^8 + (-362\theta - 139)t^7 \]

\[+ (-10\theta + 229)t^6 + (73\theta + 65)t^5 + (2\theta - 20)t^4 + (3\theta + 2)t^3 + (-\theta - 4)t^2 + t \]

\[(K_3, \chi^\beta_0) \quad t^{15} - 7t^{14} + (-2\theta + 17)t^{13} + (6\theta - 32)t^{12} + (-26\theta + 26)t^{11} \]

\[+ (24\theta + 8)t^{10} + (40\theta + 83)t^9 - 178t^8 + (-40\theta + 43)t^7 + (-24\theta - 16)t^6 \]

\[+ (260 + 52)t^5 + (-6\theta - 38)t^4 + (2\theta + 19)t^3 - 7t^2 + t \]

\[(K_3, \chi_3^{\beta \alpha}) \quad t^{15} + (-\xi^5 + 3\xi^4 + 2\xi^3 + 2\xi^2 + 4\xi - 3)t^{14} \]

\[+ (18\xi^5 + \xi^4 + 3\xi^3 + 3\xi^2 - 4\xi + 11)t^{13} \]

\[+ (-33\xi^5 - 17\xi^4 - 26\xi^3 - 21\xi^2 - 11\xi - 60)t^{12} \]

\[+ (-5\xi^5 - 52\xi^4 - 16\xi^3 - 3\xi^2 - 56\xi + 45)t^{11} \]

\[+ (-14\xi^5 + 48\xi^4 + 66\xi^3 - 18\xi^2 + 59\xi - 5)t^{10} \]

\[+ (106\xi^5 + 89\xi^4 - 10\xi^3 + 109\xi^2 + 18\xi + 101)t^9 \]

\[+ (-133\xi^5 - 123\xi^4 - 123\xi^3 - 133\xi^2 - 212)t^8 \]

\[+ (91\xi^5 - 28\xi^4 + 71\xi^3 + 88\xi^2 - 18\xi + 83)t^7 \]

\[+ (-77\xi^5 + 7\xi^4 - 11\xi^3 - 73\xi^2 - 59\xi - 64)t^6 \]

\[+ (53\xi^5 + 40\xi^4 + 4\xi^3 + 51\xi^2 + 56\xi + 101)t^5 \]

\[+ (-10\xi^5 - 15\xi^4 - 6\xi^3 - 22\xi^2 + 11\xi - 49)t^4 \]

\[+ (7\xi^5 + 7\xi^4 + 5\xi^3 + 22\xi^2 + 4\xi + 15)t^3 \]

\[+ (-2\xi^5 - 2\xi^4 - \xi^3 - 5\xi^2 - 4\xi - 7)t^2 + t \]

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