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AN ISOPERIMETRIC INEQUALITY OF MINIMAL HYPERSURFACES IN SPHERES

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Let M^n be a closed immersed minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$. We establish a special isoperimetric inequality of M^n. As an application, if the scalar curvature of M^n is constant, then we get a uniform lower bound independent of M^n for the isoperimetric inequality. In addition, we obtain an inequality between Cheeger's isoperimetric constant and the volume of the nodal set of the height function.

1. Introduction

The isoperimetric inequalities have always been an important subject in differential geometry and they are bridges of analysis and geometry. There are some elegant works on isoperimetric inequalities; see [2; 7; 14; 24].

Let $x : M^n \looparrowright \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be a closed immersed minimal hypersurface in the unit sphere and denote by $\nu(x)$ a (local) unit normal vector field of M^n, ∇ and $\overline{\nabla}$ be the Levi–Civita connections on M^n and $\mathbb{S}^{n+1}$, respectively. Let A be the shape operator with respect to ν, i.e., $A(X) = -\overline{\nabla}_X \nu$. The squared length of the second fundamental form is $S = \|A\|^2$. For any unit vector $a \in \mathbb{S}^{n+1}$, the height functions are defined as

$$\varphi_a(x) = \langle x, a\rangle, \quad \psi_a(x) = \langle \nu, a\rangle.$$

These two functions are very basic and important. For instance, the well known Takahashi theorem [18] states that *M^n is minimal if and only if there exists a constant λ such that $\Delta\varphi_a = -\lambda\varphi_a$ for all $a \in \mathbb{S}^{n+1}$*. Analogously, Ge and Li [10] gave a Takahashi-type theorem, i.e., *an immersed hypersurface M^n in $\mathbb{S}^{n+1}$ is minimal and has constant scalar curvature (CSC) if and only if $\Delta\psi_a = \lambda\psi_a$ for some constant λ independent of $a \in \mathbb{S}^{n+1}$*. This condition is linked to the famous

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Chern conjecture (see [4; 15; 22; 20; 23]), which states that *a closed immersed minimal CSC hypersurface of* $\mathbb{S}^{n+1}$ *is isoparametric.*

Let $\{|\varphi_a| \geq t\} = \{x \in M^n : |\varphi_a| \geq t\}$ and $\{|\varphi_a| = t\} = \{x \in M^n : |\varphi_a| = t\}$. In particular, due to $\Delta\varphi_a = -n\varphi_a$ and $a \in \mathbb{S}^{n+1}$,

$$\{\varphi_a = 0\} = \{x \in M^n : \varphi_a = 0\}$$

is the nodal set of the eigenfunction φ_a. Here, the zero set of the eigenfunction of an elliptic operator, and its complement are called the nodal set, and nodal domain, respectively. Suppose $S_{\max} = \sup_{p \in M^n} S(p)$,

$$\theta_1 = \frac{\int_M S}{2nS_{\max}\operatorname{Vol}(M^n)}, \quad \theta_2 = \frac{n}{4n^2 - 3n + 1}\frac{\left(\int_M S\right)^2}{\operatorname{Vol}(M^n)\int_M S^2},$$

and

$$C_1 = \max\{\theta_1, \theta_2\}, \quad C_2 = \inf_{s \leq r \leq 1} \frac{2 + nr\ln((1-s^2)/(1-r^2))}{2 + n\ln((1-s^2)/(1-r^2))}.$$

We use Vol to represent the volume measure in this paper and the following special isoperimetric inequality is the main result.

Theorem 1.1. *Let M^n be a closed immersed, nontotally geodesic, minimal hypersurface in* $\mathbb{S}^{n+1}$:

(i) *For all* $0 \leq s < 1$ *and* $a \in \mathbb{S}^{n+1}$, *the following inequality holds*:

$$\operatorname{Vol}\{|\varphi_a| = s\} \geq C(n, s, S)\operatorname{Vol}\{|\varphi_a| \geq s\},$$

where

$$C(n,s,S) = \begin{cases} \frac{nC_1}{2C_2}, & s = 0; \\ \frac{nC_1}{C_2\sqrt{1-s^2}}, & 0 < s \leq \min\left\{\sqrt{C_1}, \frac{C_1}{C_2}\right\}; \\ \frac{ns}{\sqrt{1-s^2}}, & \min\left\{\sqrt{C_1}, \frac{C_1}{C_2}\right\} < s < 1. \end{cases}$$

(ii) $$\frac{(n+1)\operatorname{Vol}(\mathbb{S}^{n+1})}{n\operatorname{Vol}(\mathbb{S}^n)} \sup_{a \in \mathbb{S}^{n+1}} \operatorname{Vol}\{\varphi_a = 0\} \geq \operatorname{Vol}(M^n).$$

Obviously, if M^n is a closed immersed minimal CSC hypersurface (nontotally geodesic) in $\mathbb{S}^{n+1}$, then $C_1 = \theta_1 = 1/2n$ in Theorem 1.1 and one has

Corollary 1.2. *Let M^n be a closed immersed, nontotally geodesic, minimal CSC hypersurface in* $\mathbb{S}^{n+1}$. *Then for all* $0 \leq s < 1$ *and* $a \in \mathbb{S}^{n+1}$, *the following inequality holds*:

$$\operatorname{Vol}\{|\varphi_a| = s\} \geq C(n, s)\operatorname{Vol}\{|\varphi_a| \geq s\},$$

where

$$C(n,s)=\begin{cases}\frac{1}{4C_2}, & s=0;\\ \frac{1}{2C_2\sqrt{1-s^2}}, & 0<s\le\min\left\{\sqrt{\frac{1}{2n}},\,\frac{1}{2nC_2}\right\};\\ \frac{ns}{\sqrt{1-s^2}}, & \min\left\{\sqrt{\frac{1}{2n}},\,\frac{1}{2nC_2}\right\}<s<1.\end{cases}$$

More precisely, Corollary 1.2 implies that the condition of constant scalar curvature has strong rigidity for minimal hypersurfaces, since the constant $C(n,s)$ depends only on n and s. Hence, the volume of M^n is strongly restricted by the volume of nodal set of the eigenfunctions φ_a ($a\in\mathbb{S}^{n+1}$) for minimal CSC hypersurfaces (nontotally geodesic), i.e.,

$$C_0(n)\,\mathrm{Vol}\,\{\varphi_a=0\}\ge\mathrm{Vol}(M^n),$$

where $C_0(n)=C(n,0)=4\inf_{0\le r\le 1}(2-nr\ln(1-r^2))/(2-n\ln(1-r^2))$. Besides, this rigid property provides some evidence for the Chern conjecture.

Remark 1.3. Under the conditions of Corollary 1.2, if M^n is an integral-Einstein (see Definition 3.1) minimal CSC hypersurface in $\mathbb{S}^{n+1}$ (or CSC hypersurface with $S>n$ and constant third mean curvature), then the constant $C(n,s)$ can be improved (see Corollary 3.2).

In 1984, Cheng, Li and Yau [6] proved that *if M^n is a closed immersed minimal hypersurface in $\mathbb{S}^{n+1}$ and M^n is nontotally geodesic*, *then*

$$\mathrm{Vol}(M^n)>\left(1+\frac{3}{\widetilde{B}_n}\right)\mathrm{Vol}(\mathbb{S}^n),$$

where $\widetilde{B}_n=2n+3+2\exp(2n\widetilde{C}_n)$ and $\widetilde{C}_n=\frac{1}{2}n^{n/2}e\Gamma(n/2,1)$. Thus, we have:

Corollary 1.4. *Let M^n be a closed immersed*, *nontotally geodesic*, *minimal CSC hypersurface in $\mathbb{S}^{n+1}$. Then there is a positive constant $\epsilon(n)>0$*, *depending only on n*, *such that*

$$\mathrm{Vol}\{\varphi_a=0\}\ge\epsilon(n)\,\mathrm{Vol}(\mathbb{S}^n)\quad\text{for all } a\in\mathbb{S}^{n+1},$$

where $\epsilon(n)>\frac{1}{4}(1+3/\widetilde{B}_n)\sup_{0\le r\le 1}\big((2-n\ln(1-r^2))/(2-nr\ln(1-r^2))\big)$.

Let $h(M)$ denote the Cheeger isoperimetric constant (see Definition 4.1), we have:

Theorem 1.5. *Let M^n be a closed immersed*, *nontotally geodesic*, *minimal hypersurface in $\mathbb{S}^{n+1}$. Then for all $a\in\mathbb{S}^{n+1}$ we have*

$$\mathrm{Vol}\,\{\varphi_a=0\}\ge\frac{2\sqrt{n+1}C_1}{C_0(n)}h(M)\,\mathrm{Vol}(M^n).$$

In particular, *we have the following assertions*:

(i) *If M^n is embedded, then* $h(M) > \frac{1}{10}(-\delta(n-1)+\sqrt{\delta^2(n-1)^2+5n})$, *where* $\delta = \sqrt{(S_{\max}-n)/n}$.

(ii) *If the image of M^n is invariant under the antipodal map* (*i.e.*, *M^n is radially symmetrical*), *then* $\mathrm{Vol}\{\varphi_a = 0\} \geq \frac{1}{2} h(M)\,\mathrm{Vol}(M^n)$.

2. Preliminary lemmas

In this section, we will prove Lemma 2.3 by Proposition 2.1 and Lemma 2.2. A direct calculation shows:

Proposition 2.1 [10; 13]. *For all $a \in \mathbb{S}^{n+1}$, we have*

$$\begin{aligned} &\nabla\varphi_a = a^{\mathrm{T}}, && \nabla\psi_a = -A(a^{\mathrm{T}}),\\ &\Delta\varphi_a = -n\varphi_a + nH\psi_a, && \Delta\psi_a = -n\langle\nabla H, a\rangle + nH\varphi_a - S\psi_a. \end{aligned}$$

where $a^{\mathrm{T}} \in \Gamma(TM)$ denotes the tangent component of a along M^n; A is the shape operator with respect to ν, i.e., $A(X) = -\overline{\nabla}_X\nu$; $S = \|A\|^2 = \mathrm{tr}(AA^t)$ and $H = \frac{1}{n}\,\mathrm{tr}\,A$ is the mean curvature.

Lemma 2.2 [10]. *Let M^n be a closed immersed minimal hypersurface in $\mathbb{S}^{n+1}$ with the squared length of the second fundamental form S*:

(i) *If $S \not\equiv 0$, then*

$$\frac{\int_M S}{2nS_{\max}} \leq \inf_{a\in\mathbb{S}^{n+1}} \int_M \varphi_a^2.$$

The equality holds if and only if $S \equiv n$ and M^n is the minimal Clifford torus $S^1(\sqrt{1/n}) \times S^{n-1}(\sqrt{(n-1)/n})$.

(ii) *If S has no restrictions, then*

$$\frac{n}{4n^2-3n+1}\left(\int_M S\right)^2 \leq \int_M S^2 \inf_{a\in\mathbb{S}^{n+1}} \int_M \varphi_a^2.$$

The equality holds if and only if M^n is an equator.

Lemma 2.3. *Let M^n be a closed immersed, nontotally geodesic, minimal hypersurface in $\mathbb{S}^{n+1}$. Then for all $0 \leq s \leq r \leq 1$ and $a \in \mathbb{S}^{n+1}$, the following inequality holds*:

$$\int_{\{|\varphi_a|\geq s\}} \varphi_a^2 \leq \frac{2+nr\ln((1-s^2)/(1-r^2))}{2+n\ln((1-s^2)/(1-r^2))} \int_{\{|\varphi_a|\geq s\}} |\varphi_a|.$$

Proof. By Proposition 2.1, we have

$$\nabla\varphi_a = a^{\mathrm{T}}, \quad \Delta\varphi_a = -n\varphi_a,$$

for all $a \in \mathbb{S}^{n+1}$. Hence, by the divergence theorem and

$$|a^{\mathrm{T}}|^2 + \varphi_a^2 + \psi_a^2 = 1, \tag{2-1}$$

for all $0 < t \le 1$ one has

$$\text{(2-2)}\quad \int_{\{|\varphi_a|\ge t\}} |\varphi_a| = \int_{\{|\varphi_a|=t\}} \frac{|a^{\mathrm{T}}|}{n} = \int_{\{|\varphi_a|=t\}} \frac{\sqrt{1-\varphi_a^2-\psi_a^2}}{n} \le \int_{\{|\varphi_a|=t\}} \frac{\sqrt{1-t^2}}{n},$$

where $\{|\varphi_a| \ge t\} = \{x \in M^n : |\varphi_a| \ge t\}$ and $\{|\varphi_a| = t\} = \{x \in M^n : |\varphi_a| = t\}$. Due to the coarea formula, (2-1) and (2-2), for all $0 \le s < r \le 1$ we obtain

$$\begin{aligned}\text{(2-3)}\quad \int_{\{s\le|\varphi_a|\le r\}} |\varphi_a| &= \int_s^r \int_{\{|\varphi_a|=t\}} \frac{|\varphi_a|}{|a^{\mathrm{T}}|} = \int_s^r \int_{\{|\varphi_a|=t\}} \frac{|\varphi_a|}{\sqrt{1-\varphi_a^2-\psi_a^2}}\\
&\ge \int_s^r \int_{\{|\varphi_a|=t\}} \frac{t}{\sqrt{1-t^2}} \ge \int_s^r \int_{\{|\varphi_a|\ge t\}} \frac{t}{\sqrt{1-t^2}} \frac{n}{\sqrt{1-t^2}} |\varphi_a|\\
&= \int_s^r \int_{\{|\varphi_a|\ge t\}} \frac{nt}{1-t^2} |\varphi_a| \ge \int_{\{|\varphi_a|\ge r\}} |\varphi_a| \int_s^r \frac{nt}{1-t^2}\\
&= \frac{n}{2} \ln\left(\frac{1-s^2}{1-r^2}\right) \int_{\{|\varphi_a|\ge r\}} |\varphi_a|.\end{aligned}$$

For all $0 \le s < r \le 1$, by $0 \le \varphi_a^2 \le |\varphi_a| \le 1$ we have

$$\begin{aligned}\text{(2-4)}\quad \int_{\{|\varphi_a|\ge s\}} \varphi_a^2 &= \int_{\{|\varphi_a|\ge r\}} \varphi_a^2 + \int_{\{s\le|\varphi_a|<r\}} \varphi_a^2\\
&\le \int_{\{|\varphi_a|\ge r\}} \varphi_a^2 + \int_{\{s\le|\varphi_a|<r\}} r|\varphi_a|\\
&= \int_{\{|\varphi_a|\ge r\}} \varphi_a^2 + r\int_{\{|\varphi_a|\ge s\}} |\varphi_a| - r\int_{\{|\varphi_a|\ge r\}} |\varphi_a|\\
&\le (1-r)\int_{\{|\varphi_a|\ge r\}} \varphi_a^2 + r\int_{\{|\varphi_a|\ge s\}} |\varphi_a|\\
&\le (1-r)\int_{\{|\varphi_a|\ge r\}} |\varphi_a| + r\int_{\{|\varphi_a|\ge s\}} |\varphi_a|.\end{aligned}$$

Thus, for all $0 \le s, r, u \le 1$ and $s < r$, by (2-3) and (2-4) we have

$$\begin{aligned}\int_{\{|\varphi_a|\ge s\}} \varphi_a^2 &\le r\int_{\{|\varphi_a|\ge s\}} |\varphi_a| + (1-r)\int_{\{|\varphi_a|\ge r\}} |\varphi_a|\\
&= r\int_{\{|\varphi_a|\ge s\}} |\varphi_a| + (1-r)\left[u\int_{\{|\varphi_a|\ge r\}} |\varphi_a| + (1-u)\int_{\{|\varphi_a|\ge r\}} |\varphi_a|\right]\\
&\le r\int_{\{|\varphi_a|\ge s\}} |\varphi_a| + (1-r)\left[\frac{2u\int_{\{s\le|\varphi_a|\le r\}} |\varphi_a|}{n\ln((1-s^2)/(1-r^2))} + (1-u)\int_{\{|\varphi_a|\ge r\}} |\varphi_a|\right].\end{aligned}$$

Choosing

$$\frac{2u_0}{n\ln((1-s^2)/(1-r^2))} = 1-u_0,$$

we have

$$u_0=\frac{n\ln((1-s^2)/(1-r^2))}{2+n\ln((1-s^2)/(1-r^2))}. \tag{2-5}$$

Hence, by Section 2 and (2-5) we have

$$\begin{aligned}\int_{\{|\varphi_a|\ge s\}}\varphi_a^2 &\le r\int_{\{|\varphi_a|\ge s\}}|\varphi_a|+(1-r)(1-u_0)\Big(\int_{\{s\le|\varphi_a|\le r\}}|\varphi_a|+\int_{\{|\varphi_a|\ge r\}}|\varphi_a|\Big)\\ &=[r+(1-r)(1-u_0)]\int_{\{|\varphi_a|\ge s\}}|\varphi_a|\\ &=\frac{2+nr\ln((1-s^2)/(1-r^2))}{2+n\ln((1-s^2)/(1-r^2))}\int_{\{|\varphi_a|\ge s\}}|\varphi_a|. \qquad \square\end{aligned}$$

In particular, setting $s=0$ in Lemma 2.3, we obtain

Corollary 2.4. *Let M^n be a closed immersed, nontotally geodesic, minimal hypersurface in $\mathbb{S}^{n+1}$. Then for all $a\in\mathbb{S}^{n+1}$, the following inequality holds*:

$$\int_M\varphi_a^2\le\frac{C_0(n)}{4}\int_M|\varphi_a|,$$

where $C_0(n)=4\inf_{0\le r\le 1}(2-nr\ln(1-r^2))/(2-n\ln(1-r^2))$.

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by Lemmas 2.2 and 2.3.

Proof of Theorem 1.1. Case (i). Since M^n is a closed minimal hypersurface (nontotally geodesic) in $\mathbb{S}^{n+1}$, by Lemma 2.2 we have

$$\inf_{a\in\mathbb{S}^{n+1}}\int_M\varphi_a^2\ge C_1\,\mathrm{Vol}(M^n), \tag{3-1}$$

where $C_1=\max\{\theta_1,\theta_2\}$ and

$$\theta_1=\frac{\int_M S}{2nS_{\max}\,\mathrm{Vol}(M^n)},\quad \theta_2=\frac{n}{4n^2-3n+1}\frac{\left(\int_M S\right)^2}{\mathrm{Vol}(M^n)\int_M S^2}.$$

On one hand, if $C_1 \geq s^2$, then (3-1) shows

$$\begin{aligned}\int_{\{|\varphi_a|\geq s\}} \varphi_a^2 &= \int_M \varphi_a^2 - \int_{\{|\varphi_a|<s\}} \varphi_a^2 \\ &\geq \int_M C_1 - \int_{\{|\varphi_a|<s\}} s^2 \\ &= \int_{\{|\varphi_a|\geq s\}} C_1 + \int_{\{|\varphi_a|<s\}} (C_1 - s^2) \\ &\geq \int_{\{|\varphi_a|\geq s\}} C_1.\end{aligned} \tag{3-2}$$

By Lemma 2.3, (2-2) and (3-2) , we obtain

$$\int_{\{|\varphi_a|\geq s\}} C_1 \leq \int_{\{|\varphi_a|\geq s\}} \varphi_a^2 \leq C_2 \int_{\{|\varphi_a|\geq s\}} |\varphi_a| \leq C_2 \int_{\{|\varphi_a|=s\}} \frac{\sqrt{1-s^2}}{n},$$

where $C_2 = \inf_{s\leq r\leq 1}(2+nr\ln((1-s^2)/(1-r^2)))/(2+n\ln((1-s^2)/(1-r^2)))$. Thus

$$\operatorname{Vol}\{|\varphi_a| = s\} \geq \frac{nC_1}{C_2\sqrt{1-s^2}} \operatorname{Vol}\{|\varphi_a| \geq s\} \quad (\sqrt{C_1} \geq s > 0). \tag{3-3}$$

In particular, if $s = 0$, then

$$\lim_{s\to 0^+} \operatorname{Vol}\{|\varphi_a| = s\} = \lim_{s\to 0^+} \operatorname{Vol}\{\varphi_a = s\} + \lim_{s\to 0^+} \operatorname{Vol}\{\varphi_a = -s\} = 2\operatorname{Vol}\{\varphi_a = 0\},$$

and

$$\lim_{s\to 0^+} \operatorname{Vol}\{|\varphi_a| \geq s\} = \operatorname{Vol}\{|\varphi_a| \geq 0\} = \operatorname{Vol}(M^n).$$

By (3-3), one has

$$\operatorname{Vol}\{\varphi_a = 0\} \geq \frac{nC_1}{2C_2} \operatorname{Vol}\{|\varphi_a| \geq 0\} = \frac{nC_1}{2C_2} \operatorname{Vol}(M^n). \tag{3-4}$$

On the other hand, by (2-2), we have

$$\int_{\{|\varphi_a|\geq s\}} s \leq \int_{\{|\varphi_a|\geq s\}} |\varphi_a| \leq \int_{\{|\varphi_a|=s\}} \frac{\sqrt{1-s^2}}{n} \quad (1 > s > 0).$$

Hence

$$\operatorname{Vol}\{|\varphi_a| = s\} \geq \frac{ns}{\sqrt{1-s^2}} \operatorname{Vol}\{|\varphi_a| \geq s\} \quad (1 > s > 0). \tag{3-5}$$

Choose

$$\frac{ns}{\sqrt{1-s^2}} = \frac{nC_1}{C_2\sqrt{1-s^2}},$$

which implies that $s = C_1/C_2$. Then we have the following discussions:

(1) If $s=0$, (3-4) implies

$$\mathrm{Vol}\,\{\varphi_a=0\}\geq\frac{nC_1}{2C_2}\,\mathrm{Vol}\,\{|\varphi_a|\geq 0\}=\frac{nC_1}{2C_2}\,\mathrm{Vol}(M^n).$$

(2) If $0<s\leq\min\{\sqrt{C_1},C_1/C_2\}$, (3-3) implies

$$\mathrm{Vol}\,\{|\varphi_a|=s\}\geq\frac{nC_1}{C_2\sqrt{1-s^2}}\,\mathrm{Vol}\,\{|\varphi_a|\geq s\}.$$

(3) If $\min\{\sqrt{C_1},C_1/C_2\}<s<1$, (3-5) implies

$$\mathrm{Vol}\,\{|\varphi_a|=s\}\geq\frac{ns}{\sqrt{1-s^2}}\,\mathrm{Vol}\,\{|\varphi_a|\geq s\}.$$

Case (ii). By Proposition 2.1, we have

$$\nabla\varphi_a=a^{\mathrm{T}},\qquad\Delta\varphi_a=-n\varphi_a,$$

for all $a\in\mathbb{S}^{n+1}$. Hence, by the divergence theorem and $S\not\equiv 0$, one has

$$\int_M|\varphi_a|=\int_{\{\varphi_a>0\}}\varphi_a-\int_{\{\varphi_a\leq 0\}}\varphi_a=\int_{\{|\varphi_a|=0\}}\frac{2|a^{\mathrm{T}}|}{n}.$$

Since

$$\int_{a\in\mathbb{S}^{n+1}}|\varphi_a|=2\,\mathrm{Vol}(\mathbb{B}^{n+1})=\frac{2}{n+1}\,\mathrm{Vol}(\mathbb{S}^n),$$

we have

$$\frac{2}{n+1}\,\mathrm{Vol}(\mathbb{S}^n)\,\mathrm{Vol}(M^n)=\int_{a\in\mathbb{S}^{n+1}}\int_{x\in M}|\varphi_a|=\int_{a\in\mathbb{S}^{n+1}}\int_{\{|\varphi_a|=0\}}\frac{2|a^{\mathrm{T}}|}{n}.$$

By (2-1), one has

$$\mathrm{Vol}(M^n)\leq\frac{(n+1)\,\mathrm{Vol}(\mathbb{S}^{n+1})}{n\,\mathrm{Vol}(\mathbb{S}^n)}\sup_{a\in\mathbb{S}^{n+1}}\mathrm{Vol}\{\varphi_a=0\}.\qquad\square$$

Combining the intrinsic and extrinsic geometry, Ge and Li generalized Einstein manifolds to integral-Einstein (IE) submanifolds in [10].

Definition 3.1 [10]. Let M^n $(n\geq 3)$ be a compact submanifold in the Euclidean space $\mathbb{R}^N$. Then M^n is an IE submanifold if and only if for any unit vector $a\in\mathbb{S}^{N-1}$

$$\int_M\Big(\mathrm{Ric}-\frac{R}{n}\boldsymbol{g}\Big)(a^{\mathrm{T}},a^{\mathrm{T}})=0,$$

where $a^{\mathrm{T}}\in\Gamma(TM)$ denotes the tangent component of the constant vector a along M^n; Ric is the Ricci curvature tensor and R is the scalar curvature.

Corollary 3.2. *Let M^n be a closed immersed, nontotally geodesic, minimal hypersurface in $\mathbb{S}^{n+1}$. If it is IE and CSC (or CSC with $S > n$ and constant third mean curvature), then for all $0 \le s < 1$ and $a \in \mathbb{S}^{n+1}$, the following inequality holds:*

$$\mathrm{Vol}\,\{|\varphi_a| = s\} \ge C(n,s)\,\mathrm{Vol}\,\{|\varphi_a| \ge s\},$$

where

$$C(n,s) = \begin{cases} \frac{n}{2(n+2)C_2}, & s = 0; \\ \frac{n}{(n+2)C_2\sqrt{1-s^2}}, & 0 < s \le \min\Big\{\sqrt{\frac{1}{n+2}}, \frac{1}{(n+2)C_2}\Big\}; \\ \frac{ns}{\sqrt{1-s^2}}, & \min\Big\{\sqrt{\frac{1}{n+2}}, \frac{1}{(n+2)C_2}\Big\} < s < 1. \end{cases}$$

Proof. If M^n is minimal, IE and CSC, then [10] showed that

$$\int_M \varphi_a^2 = \frac{1}{n+2}\,\mathrm{Vol}(M^n), \quad a \in \mathbb{S}^{n+1}.$$

Thus, $C_1 = 1/(n+2)$ in Theorem 1.1. For a closed minimal CSC hypersurface in $\mathbb{S}^{n+1}$ with $S > n$ and constant third mean curvature, Ge and Li proved that it is an IE hypersurface in [10]. Thus, Corollary 3.2 is also true in this case. □

4. Proof of Theorem 1.5

In this section, we will discuss the Cheeger isoperimetric constant of minimal hypersurfaces in $\mathbb{S}^{n+1}$.

Definition 4.1 [5]. The Cheeger isoperimetric constant of a closed Riemannian manifold M^n is defined as

$$h(M) = \inf_H \frac{\mathrm{Vol}\,(H)}{\min\{\mathrm{Vol}(M_1), \mathrm{Vol}(M_2)\}},$$

where the infimum is taken over all the submanifolds H of codimension 1 of M^n; M_1 and M_2 are submanifolds of M^n with their boundaries in H and satisfy $M = M_1 \sqcup M_2 \sqcup H$ (a disjoint union).

Remark 4.2. Let M^n be a closed, immersed, minimal hypersurface in $\mathbb{S}^{n+1}$, which is nontotally geodesic. Since there is a vector $a \in \mathbb{S}^{n+1}$ such that $\mathrm{Vol}\{\varphi_a > 0\} = \mathrm{Vol}\{\varphi_a < 0\}$, we have

$$h(M) \le \sup_{a \in \mathbb{S}^{n+1}} \frac{2\,\mathrm{Vol}\{\varphi_a = 0\}}{\mathrm{Vol}(M^n)}.$$

Moreover, if the image of M^n is invariant under the antipodal map, then $\mathrm{Vol}\{\varphi_a > 0\} = \mathrm{Vol}\,\{\varphi_a < 0\}$ for all $a \in \mathbb{S}^{n+1}$ and

$$h(M) \le \inf_{a \in \mathbb{S}^{n+1}} \frac{2\,\mathrm{Vol}\{\varphi_a = 0\}}{\mathrm{Vol}(M^n)}.$$

In 1970, Cheeger [5] gave the famous inequality between the first positive eigenvalue $\lambda_1(M)$ of the Laplacian and the Cheeger isoperimetric constant $h(M)$ (see Definition 4.1):

$$h^2(M) \le 4\lambda_1(M).$$

Obviously, $\lambda_1(M) \le n$ for minimal hypersurfaces in $\mathbb{S}^{n+1}$ because of Proposition 2.1 and we have

$$h(M) \le 2\sqrt{\lambda_1(M)} \le 2\sqrt{n}.$$

The Yau conjecture [16] asserts that *if M^n is a closed embedded minimal hypersurface of $\mathbb{S}^{n+1}$, then $\lambda_1(M) = n$.* In particular, Choi and Wang [9] showed that $\lambda_1(M) \ge n/2$ and a careful argument (see [1, Theorem 5.1]) implied that the strict inequality holds, i.e., $\lambda_1(M) > n/2$. In addition, Tang and Yan [21; 19] proved the Yau conjecture in the isoparametric case. Choe and Soret [8] were able to verify the Yau conjecture for the Lawson surfaces and the Karcher-Pinkall-Sterling examples. For more details and references, please see the elegant survey by Brendle [1]. Besides, Buser [3] proved that:

Lemma 4.3 [3]. *If the Ricci curvature of a closed Riemannian manifold M^n is bounded below by $-(n-1)\delta^2$ ($\delta \ge 0$), then*

$$\lambda_1(M) \le 2\delta(n-1)h(M) + 10h^2(M). \tag{4-1}$$

Next, we will prove Theorem 1.5 by Lemmas 2.2, 4.3 and Corollary 2.4.

Proof of Theorem 1.5. Without loss of generality, assuming that $\mathrm{Vol}\,\{\varphi_a > 0\} \ge \mathrm{Vol}\,\{\varphi_a < 0\}$, one has

$$h(M) \le \frac{\mathrm{Vol}\,\{\varphi_a = 0\}}{\mathrm{Vol}\,\{\varphi_a < 0\}}. \tag{4-2}$$

For $\mathrm{Vol}\,\{\varphi_a > 0\} \le \mathrm{Vol}\,\{\varphi_a < 0\}$, the proof is similar and the following estimates of inequalities can be found in Ge and Li [11]. By Proposition 2.1, for any $a \in \mathbb{S}^{n+1}$, $\int_M \varphi_a = 0$. Thus

$$\int_{\{\varphi_a>0\}} \varphi_a = \int_{\{\varphi_a<0\}} -\varphi_a = \tfrac{1}{2}\int_M |\varphi_a|. \tag{4-3}$$

The divergence theorem shows that

$$\int_{\{\varphi_a<0\}} \Delta\varphi_a^2 = 0,$$

and by $\Delta\varphi_a^2 = -2n\varphi_a^2 + 2|a^{\mathrm{T}}|^2$, one has

$$n\int_{\{\varphi_a<0\}} \varphi_a^2 = \int_{\{\varphi_a<0\}} |a^{\mathrm{T}}|^2. \tag{4-4}$$

Then, due to (2-1) and (4-4), we have

$$(n+1)\int_{\{\varphi_a<0\}} \varphi_a^2 \le \int_{\{\varphi_a<0\}} 1. \tag{4-5}$$

By the Cauchy-Schwarz inequality and (4-5), one has

$$\sqrt{\frac{1}{n+1}}\int_{\{\varphi_a<0\}} 1 \ge \sqrt{\int_{\{\varphi_a<0\}} 1 \int_{\{\varphi_a<0\}} \varphi_a^2} \ge \int_{\{\varphi_a<0\}} -\varphi_a. \tag{4-6}$$

By Corollary 2.4, (4-2), (4-3) and (4-6), we have

$$\frac{\operatorname{Vol}\{\varphi_a=0\}}{h(M)} \ge \operatorname{Vol}\{\varphi_a<0\} \ge \frac{\sqrt{n+1}}{2}\int_M |\varphi_a| \ge \frac{2\sqrt{n+1}}{C_0(n)}\int_M \varphi_a^2.$$

Hence, by Lemma 2.2 we have

$$\operatorname{Vol}\{\varphi_a=0\} \ge \frac{2\sqrt{n+1}}{C_0(n)} h(M)\int_M \varphi_a^2 \ge \frac{2\sqrt{n+1}C_1}{C_0(n)} h(M)\operatorname{Vol}(M^n).$$

Case (i). Since M^n is a minimal hypersurface in $\mathbb{S}^{n+1}$, the Ricci curvature is given by

$$\operatorname{Ric}(X,Y) = (n-1)g(X,Y) - g(AX,AY), \quad X, Y \in \mathfrak{X}(M).$$

Let $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ denote the eigenvalues of the shape operator A. We obtain

$$\sum_{i=1}^n \lambda_i = 0, \quad \sum_{i=1}^n \lambda_i^2 = \|A\|^2 = S,$$

and

$$\begin{aligned} 0 = \sum_{i,j=1}^n \lambda_i\lambda_j \\ &= \lambda_1^2 + 2\sum_{j=2}^n \lambda_1\lambda_j + \sum_{i,j=2}^n \lambda_i\lambda_j \\ &\le -\lambda_1^2 + \sum_{i,j=2}^n \frac{\lambda_i^2+\lambda_j^2}{2} \\ &= (n-1)S - n\lambda_1^2. \end{aligned}$$

Thus

$$\mathrm{Ric}(X,X) \geq (n-1-\lambda_1^2)g(X,X) \geq -(n-1)\frac{S-n}{n}g(X,X).$$

By Lemma 4.3 and $\lambda_1(M) > n/2$ (see Choi–Wang [9] and Brendle [1]), one has

$$\frac{n}{2} < \lambda_1(M) \leq 2\delta(n-1)h(M) + 10h^2(M).$$

Note that $S_{\max} \geq n$ for all nontotally geodesic minimal hypersurfaces in $\mathbb{S}^{n+1}$ by Simons' inequality [17]

$$\int_M S(S-n) \geq 0.$$

Setting $\delta = \sqrt{(S_{\max}-n)/n}$, we have

$$h(M) > \frac{-\delta(n-1) + \sqrt{\delta^2(n-1)^2 + 5n}}{10}.$$

Case (ii). If the image of M^n is invariant under the antipodal map, the proof is complete by Remark 4.2. □

Remark 4.4. If M^n is a minimal isoparametric hypersurface with $g \geq 2$ distinct principal curvatures in $\mathbb{S}^{n+1}$, then $\lambda_1(M) = n$ (see Tang–Yan [19]), $S \equiv (g-1)n$ and $\delta = \sqrt{g-2}$ $(2 \leq g \leq 6)$. Thus, (4-1) implies that

$$h(M) \geq \frac{-\sqrt{g-2}(n-1) + \sqrt{(g-2)(n-1)^2 + 10n}}{10}.$$

In fact, Muto [12] carefully estimated the Cheeger isoperimetric constant of minimal isoparametric hypersurfaces and got better results.

Remark 4.5. Let M^n be a closed embedded minimal hypersurface in $\mathbb{S}^{n+1}$. If $S < c(n)$ and $c(n)$ depends only on n, then there is a positive constant $\eta(n) > 0$, depending only on n, such that $h(M) > \eta(n)$.

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References

[1] S. Brendle, "Minimal surfaces in S^3: a survey of recent results", *Bull. Math. Sci.* **3**:1 (2013), 133–171. MR Zbl

[2] S. Brendle, "The isoperimetric inequality for a minimal submanifold in Euclidean space", *J. Amer. Math. Soc.* **34**:2 (2021), 595–603. MR Zbl

[3] P. Buser, “A note on the isoperimetric constant”, *Ann. Sci. École Norm. Sup.* (4) **15**:2 (1982), 213–230. MR Zbl

[4] S. Chang, “On minimal hypersurfaces with constant scalar curvatures in S^4”, *J. Differential Geom.* **37**:3 (1993), 523–534. MR Zbl

[5] J. Cheeger, “A lower bound for the smallest eigenvalue of the Laplacian”, pp. 195–199 in *Problems in analysis* (Sympos. in honor of Salomon Bochner, Princeton Univ., N.J., 1969), edited by R. C. Gunning, Princeton Univ. Press, 1970. MR Zbl

[6] S. Y. Cheng, P. Li, and S.-T. Yau, “Heat equations on minimal submanifolds and their applications”, *Amer. J. Math.* **106**:5 (1984), 1033–1065. MR Zbl

[7] J. Choe and R. Gulliver, “Isoperimetric inequalities on minimal submanifolds of space forms”, *Manuscripta Math.* **77**:2-3 (1992), 169–189. MR Zbl

[8] J. Choe and M. Soret, “First eigenvalue of symmetric minimal surfaces in $\mathbb{S}^3$”, *Indiana Univ. Math. J.* **58**:1 (2009), 269–281. MR Zbl

[9] H. I. Choi and A. N. Wang, “A first eigenvalue estimate for minimal hypersurfaces”, *J. Differential Geom.* **18**:3 (1983), 559–562. MR Zbl

[10] J. Ge and F. Li, “Integral-Einstein hypersurfaces in spheres”, preprint, 2021. arXiv 2101.03753

[11] J. Ge and F. Li, “Volume gap for minimal submanifolds in spheres”, preprint, 2022. Zbl arXiv 2210.04654

[12] H. Muto, “The first eigenvalue of the Laplacian of an isoparametric minimal hypersurface in a unit sphere”, *Math. Z.* **197**:4 (1988), 531–549. MR Zbl

[13] K. Nomizu and B. Smyth, “On the Gauss mapping for hypersurfaces of constant mean curvature in the sphere”, *Comment. Math. Helv.* **44** (1969), 484–490. MR

[14] R. Osserman, “The isoperimetric inequality”, *Bull. Amer. Math. Soc.* **84**:6 (1978), 1182–1238. MR Zbl

[15] M. Scherfner, S. Weiss, and S.-T. Yau, “A review of the Chern conjecture for isoparametric hypersurfaces in spheres”, pp. 175–187 in *Advances in geometric analysis*, edited by S. Janeczko et al., Adv. Lect. Math. (ALM) **21**, International Press, Somerville, MA, 2012. MR Zbl

[16] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology **1**, International Press, Cambridge, MA, 1994. MR Zbl

[17] J. Simons, “Minimal varieties in riemannian manifolds”, *Ann. of Math.* (2) **88** (1968), 62–105. MR Zbl

[18] T. Takahashi, “Minimal immersions of Riemannian manifolds”, *J. Math. Soc. Japan* **18** (1966), 380–385. MR Zbl

[19] Z. Tang and W. Yan, “Isoparametric foliation and Yau conjecture on the first eigenvalue”, *J. Differential Geom.* **94**:3 (2013), 521–540. MR Zbl

[20] Z. Tang and W. Yan, “On the Chern conjecture for isoparametric hypersurfaces”, *Sci. China Math.* **66**:1 (2023), 143–162. MR Zbl

[21] Z. Tang, Y. Xie, and W. Yan, “Isoparametric foliation and Yau conjecture on the first eigenvalue, II”, *J. Funct. Anal.* **266**:10 (2014), 6174–6199. MR Zbl

[22] Z. Tang, D. Wei, and W. Yan, “A sufficient condition for a hypersurface to be isoparametric”, *Tohoku Math. J.* (2) **72**:4 (2020), 493–505. MR Zbl

[23] H. Yang and Q.-M. Cheng, “Chern’s conjecture on minimal hypersurfaces”, *Math. Z.* **227**:3 (1998), 377–390. MR Zbl

[24] S. T. Yau, “Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold”, *Ann. Sci. École Norm. Sup.* (4) **8**:4 (1975), 487–507. MR Zbl

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