*Pacific Journal of Mathematics*

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GIULIO PERUGINELLI

Volume 324 No. 2 June 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 324, No. 2, 2023
https://doi.org/10.2140/pjm.2023.324.333

# POLYNOMIAL DEDEKIND DOMAINS WITH FINITE RESIDUE FIELDS OF PRIME CHARACTERISTIC

GIULIO PERUGINELLI

*To the everlasting memory of Robert Gilmer*

**We show that every Dedekind domain $R$ lying between the polynomial rings $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ with the property that its residue fields of prime characteristic are finite fields is equal to a generalized ring of integer-valued polynomials; that is, for each prime $p \in \mathbb{Z}$ there exists a finite subset $E_p$ of transcendental elements over $\mathbb{Q}$ in the absolute integral closure $\overline{\mathbb{Z}_p}$ of the ring of $p$-adic integers such that $R = \{f \in \mathbb{Q}[X] \mid f(E_p) \subseteq \overline{\mathbb{Z}_p},$ for each prime $p \in \mathbb{Z}\}$. Moreover, we prove that the class group of $R$ is isomorphic to a direct sum of a countable family of finitely generated abelian groups. Conversely, any group of this kind is the class group of a Dedekind domain $R$ between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.**

## 1. Introduction

Given a Dedekind domain $D$, the class group of $D$ measures how far $D$ is from being a UFD and it is therefore an important object in the study of factorization problems in the ring $D$. It is well-known that the class group of the ring of integers of a number field is a finite abelian group. In contrast with this result, Claborn [1966] proved the groundbreaking result that every abelian group occurs as the class group of a suitable Dedekind domain.

Eakin and Heinzer [1973] showed that every finitely generated abelian group is the class group of a Dedekind domain between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. More generally, they proved that if $V_1, \dots, V_n$ are distinct DVRs with same quotient field $K$ and, for each $i = 1, \dots, n$, $\{V_{i,j}\}_{j=1}^{g_i}$ is a finite collection of DVRs extending $V_i$ to $K(X)$, each of which is residually algebraic over $V_i$ (i.e., the extension of the residue fields is algebraic), then

$$R = \bigcap_{i,j} V_{i,j} \cap K[X]$$

is a Dedekind domain. They also give an explicit description of the class group of such a domain $R$, thanks to which they showed the quoted result by considering

*MSC2020:* 13B25, 13F05, 13F20, 20K99.
*Keywords:* Dedekind domain, class group, integer-valued polynomials.

suitable residually algebraic extensions of a finite set of DVRs of $\mathbb{Q}$ to $\mathbb{Q}(X)$.

Actually, if we suppose that each residue field extension of $V_{i,j}$ over $V_i$ is finite, a ring $R$ constructed as above can be represented as a ring of integer-valued polynomials in the following way. For each $i, j$, by [Peruginelli 2017, Theorem 2.5 and Proposition 2.2], there exists an element $\alpha_{i,j}$ in the algebraic closure $\overline{\widehat{K_i}}$ of the $V_i$-adic completion $\widehat{K_i}$ of $K$, $\alpha_{i,j}$ transcendental over $K$, such that

$$V_{i,j} = V_{i,\alpha_{i,j}} = \{\varphi \in K(X) \mid \varphi(\alpha_{i,j}) \in \overline{\widehat{V_i}}\},$$

where $\overline{\widehat{V_i}}$ is the absolute integral closure of $\widehat{V_i}$, the completion of $V_i$. Hence, the above ring $R$ can be represented as $R = \{f \in K[X] \mid f(\alpha_{i,j}) \in \overline{\widehat{V_i}}, \forall i, j\}$ (for more details, see [Peruginelli 2017, Remark 2.8]).

More recently, Glivický and Šaroch [2013] investigated a family of quasieuclidean subrings of $\mathbb{Q}[X]$ depending on a parameter $\alpha \in \widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. A ring of this family is always a Bézout domain (i.e., finitely generated ideals are principal) and might be a PID or not, according to the finiteness of some set of primes depending on $\alpha$ and the set of polynomials in $\mathbb{Z}[X]$. Glivická et al. [2023] observed that these rings can be realized as overrings of the classical ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$, which is a two-dimensional nonnoetherian Prüfer domain; such overrings have been completely characterized in [Chabert and Peruginelli 2016]. We will review this representation in Section 2.

In the same area, Chang [2022] generalized Eakin and Heinzer's result, proving that there exists an almost Dedekind domain $R$ (i.e., $R_M$ is a DVR for each maximal ideal $M$ of $R$) which is not noetherian, lies between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ and has class group isomorphic to a direct sum of a prescribed countable family of finitely generated abelian groups. As before, assuming the finiteness of the residue field extensions of the involved DVRs, Chang's construction falls in the class of integer-valued polynomial rings that we consider in this paper.

Here, we provide a complete description of the class of Dedekind domains $R$ lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ such that their residue fields of prime characteristic are finite fields. Throughout the paper, for short we denote the last property by saying that $R$ has finite residue fields of prime characteristic. We remark that the residue fields of such a domain $R$ cannot be all finite fields. In fact, since $R \subseteq \mathbb{Q}[X]_{(q)}$ for every irreducible $q \in \mathbb{Q}[X]$, the residue field of the center of the DVR $\mathbb{Q}[X]_{(q)}$ on $R$ is a finite extension of $\mathbb{Q}$, hence an infinite field. However, since $R$ is supposed to be Dedekind (in particular, a Prüfer domain) the residue fields of prime characteristic are algebraic extensions of the corresponding prime field (see, for example, [Peruginelli 2018, Theorem 3.14]). Infinite algebraic extensions of the prime fields of prime characteristic are also allowed, and that is the content of another work on this subject [Peruginelli 2023].

The paper is organized as follows. We first set the notation we will use throughout the paper and introduce the class of *generalized rings of integer-valued polynomials*, which are subrings of $\mathbb{Q}[X]$ formed by polynomials which are simultaneously integer-valued over different subsets of integral elements over $\mathbb{Z}_p$, the ring of $p$-adic integers, for $p$ running over the set of integer primes. In Section 2, we review Loper and Werner's construction [2012] of Prüfer domains and recall that it falls into the class of generalized rings of integer-valued polynomials, as already observed in [Peruginelli 2017, Remark 2.8]. We then characterize when a ring of their construction is a Dedekind domain in Theorem 2.15. In order to accomplish this objective, we introduce the definition of *polynomially factorizable* subsets $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}} = \prod_p \overline{\mathbb{Z}_p}$ (we refer to Section 1 for unexplained notation), which turns out to be the key assumption for such a ring to be of finite character (hence, a noetherian Prüfer domain, thus Dedekind). Furthermore, we show in Theorem 2.17 that every Dedekind domain $R$ with finite residue fields of prime characteristic lying between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ is equal to a generalized ring of integer-valued polynomials with class group equal to a direct sum of a countable family of finitely generated abelian groups (Recall that the Picard group of $\mathrm{Int}(\mathbb{Z})$ is a free abelian group of countably infinite rank [Gilmer et al. 1990]). Among other things, we will also characterize the PIDs among these class of domains, generalizing the aforementioned work of Glivický and Šaroch [2013] (see also [Glivická et al. 2023]). We will also give a criteria for when two such generalized rings of integer-valued polynomials are equal. Finally, in Section 3, by means of a suitable modification of Chang's construction, given a group $G$ which is the direct sum of a countable family of finitely generated abelian groups, we prove that there exists a Dedekind domain $R$ with finite residue fields of prime characteristic, $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$, with class group $G$, thus giving a positive answer to a question raised by Chang [2022]. By the previous results, such a domain is a generalized ring of integer-valued polynomials.

It has come to our attention that Theorem 7 of [Chang and Geroldinger 2024] shows the existence of a Dedekind domain with class group equal to a direct sum of a countable family of prescribed finitely generated abelian groups. However, that construction is based on a polynomial ring with an infinite set of indeterminates with the additional property that each ideal class contains infinitely many height-one prime ideals.

***Notation.*** The generalized rings of integer-valued polynomials considered in this paper fall into the class of integer-valued polynomials on algebras (see for example [Frisch 2013; 2014; Peruginelli and Werner 2017]), which encompasses also the classical definition of ring of integer-valued polynomials. We now recall the latter definition. Let $D$ be an integral domain with quotient field $K$ and $A$ a torsion-free $D$-algebra such that $A \cap K = D$. We may evaluate polynomials $f \in K[X]$ at any element $a \in A$ inside the extended algebra $A \otimes_D K$. The $D$-algebra $A$ clearly embeds into $A \otimes_D K$ and if $f(a) \in A$ we say that $f$ is integer-valued at $a$. In general, given a subset $S$ of $A$, we define the ring of integer-valued polynomials over $S$ as

$$\mathrm{Int}_K(S, A) = \{f \in K[X] \mid f(s) \in A, \forall s \in S\}.$$

Note that when $A = D$ we get the usual definition of ring of integer-valued polynomials on a subset $S$ of $D$, and in that case we omit the subscript $K$. If $S = D = A$, then we set $\mathrm{Int}(D, D) = \mathrm{Int}(D)$.

For an integral domain $D$, we define the Picard group of $D$, denoted by $\mathrm{Pic}(D)$, as the quotient of the abelian group of the invertible fractional ideals of $D$ by the subgroup generated by the nonzero principal fractional ideals, where the operation is the ideal multiplication (see [Cahen and Chabert 1997, §VIII.1]). If $D$ is a Dedekind domain, then $\mathrm{Pic}(D)$ is the usual ideal class group of $D$.

Let $\mathbb{P}$ be the set of all prime numbers. For a fixed $p \in \mathbb{P}$, we adopt the following notation:

- $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at $p\mathbb{Z}$.
- $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of $p$-adic integers and the field of $p$-adic numbers, respectively.
- $\overline{\mathbb{Q}_p}$ and $\overline{\mathbb{Z}_p}$ denote a fixed algebraic closure of $\mathbb{Q}_p$ and the absolute integral closure of $\mathbb{Z}_p$, respectively.
- For a finite extension $K$ of $\mathbb{Q}_p$, we denote by $O_K$ the ring of integers of $K$.
- $v_p$ denotes the unique extension of the $p$-adic valuation on $\mathbb{Q}_p$ to $\overline{\mathbb{Q}_p}$.
- If $\alpha \in \overline{\mathbb{Q}_p}$, we denote the ramification index $e(\mathbb{Q}_p(\alpha) \mid \mathbb{Q}_p)$ by $e_\alpha$.
- $\widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$, the profinite completion of $\mathbb{Z}$.
- $\widehat{\overline{\mathbb{Z}}} = \prod_{p \in \mathbb{P}} \overline{\mathbb{Z}_p}$.
- For $\alpha \in \overline{\mathbb{Q}_p}$, we set

$$V_{p,\alpha} = \{\varphi \in \mathbb{Q}(X) \mid \varphi(\alpha) \in \overline{\mathbb{Z}_p}\}.$$

Clearly, $V_{p,\alpha}$ is a valuation domain of $\mathbb{Q}(X)$ extending $\mathbb{Z}_{(p)}$ with maximal ideal equal to $M_{p,\alpha} = \{\varphi \in V_{p,\alpha} \mid v_p(\varphi(\alpha)) > 0\}$. Moreover, $V_{p,\alpha}$ is a DVR if $\alpha$ is transcendental over $\mathbb{Q}$ and it has rank 2 otherwise. In the former case, the ramification index $e(V_{p,\alpha} \mid \mathbb{Z}_{(p)})$ is equal to $e_\alpha$. In either case, let $O_\alpha$ and $M_\alpha$ be the valuation domain and maximal ideal of $\mathbb{Q}_p(\alpha)$, respectively. Then, the residue field of $V_{p,\alpha}$ is equal to $O_\alpha / M_\alpha$ and $pO_\alpha = M_\alpha^e$, for some integer $e$, which is equal to $e_\alpha$ (for all these results, see [Peruginelli 2017, Proposition 2.2 and Theorem 2.5]).

The following result, mentioned in the introduction, characterizes residually algebraic extensions of $\mathbb{Z}_{(p)}$ to $\mathbb{Q}(X)$ of a certain kind; the valuation overrings of the Dedekind domains we are dealing with belong to this class.

**Theorem 1.1** [Peruginelli 2017, Theorems 2.5 and 3.2]. *Let $W \subset \mathbb{Q}(X)$ be a valuation domain with maximal ideal $M$ extending $\mathbb{Z}_{(p)}$ for some $p \in \mathbb{P}$. If $pW = M^e$ for some $e \geq 1$ and $W/M \supseteq \mathbb{Z}/p\mathbb{Z}$ is a finite extension, then there exists $\alpha \in \overline{\mathbb{Q}_p}$ such that $W = V_{p,\alpha}$. Moreover, for $\alpha, \beta \in \overline{\mathbb{Q}_p}$, we have $V_{p,\alpha} = V_{p,\beta}$ if and only if $\alpha, \beta$ are conjugate over $\mathbb{Q}_p$.*

Clearly, if $W$ is as in the assumptions of Theorem 1.1 and $\mathbb{Z}[X] \subset W$, then $\alpha \in \overline{\mathbb{Z}_p}$.

Given $f \in \mathbb{Q}[X]$, the evaluation of $f(X)$ at an element $\alpha = (\alpha_p) \in \widehat{\overline{\mathbb{Z}}}$ is done componentwise:

$$f(\alpha) = (f(\alpha_p)) \in \prod_{p \in \mathbb{P}} \overline{\mathbb{Q}_p}.$$

We say that $f$ is *integer-valued* at $\alpha$ if $f(\alpha) \in \widehat{\overline{\mathbb{Z}}}$, which is equivalent to $f \in V_{p,\alpha_p}$ for all $p \in \mathbb{P}$.

**Definition 1.2.** Given a subset $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}}$, we define the *generalized ring of integer-valued polynomials on* $\underline{E}$ as:

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}}) = \{f \in \mathbb{Q}[X] \mid f(\alpha) \in \widehat{\overline{\mathbb{Z}}}, \forall \alpha \in \underline{E}\}.$$

If $\underline{E} = \widehat{\mathbb{Z}}$, then $\mathrm{Int}_{\mathbb{Q}}(\widehat{\mathbb{Z}}, \widehat{\overline{\mathbb{Z}}}) = \mathrm{Int}_{\mathbb{Q}}(\widehat{\mathbb{Z}}, \widehat{\mathbb{Z}}) = \mathrm{Int}(\mathbb{Z})$; in fact, the first equality follows easily from the fact that the polynomials have rational coefficients; for the last equality, see [Chabert and Peruginelli 2016, Remark 6.4] (essentially, $\mathbb{Z}$ is dense in $\widehat{\mathbb{Z}}$). We recall that the family of overrings of $\mathrm{Int}(\mathbb{Z})$ which are contained in $\mathbb{Q}[X]$ is formed exactly by the rings $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$, as $\underline{E}$ ranges through the subsets of $\widehat{\mathbb{Z}}$ of the form $\prod_{p \in \mathbb{P}} E_p$, where for each prime $p$, $E_p$ is a closed (possibly empty) subset of $\mathbb{Z}_p$ [Theorem 6.2]. In the study of a generalized ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$, without loss of generality we may suppose that the subset $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}}$ is of the form $\underline{E} = \prod_{p \in \mathbb{P}} E_p$ (see the arguments given in [Remark 6.3]). Note that we allow each component $E_p$ of $\underline{E}$ to be equal to the empty set.

## 2. Polynomial Dedekind domains

Loper and Werner [2012] exhibited a construction of Prüfer domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ in order to show the existence of a Prüfer domain strictly contained in $\mathrm{Int}(\mathbb{Z})$. As earlier in [Eakin and Heinzer 1973], their construction is obtained by intersecting a suitable family of valuation domains of $\mathbb{Q}(X)$ indexed by $\mathbb{P}$ with $\mathbb{Q}[X]$. A valuation domain of this family is equal to $V_{p,\alpha}$, for some $\alpha \in \overline{\mathbb{Z}_p}$, by Theorem 1.1 and the fact that $X$ is in every valuation domain of this family. By [Peruginelli 2017, Remark 2.8], a ring in Loper and Werner's construction can be represented as a generalized ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$, for a suitable subset $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}}$ which satisfies the following definition.

**Definition 2.1.** Let $\underline{E}=\prod_{p\in\mathbb{P}} E_p\subset\overline{\overline{\mathbb{Z}}}$. We say that $\underline{E}$ is locally bounded, if, for each prime $p$, $E_p$ is a subset of $\overline{\mathbb{Z}_p}$ of bounded degree, that is, $\{[\mathbb{Q}_p(\alpha):\mathbb{Q}_p]\mid\alpha\in E_p\}$ is bounded.

As we have already said above, some of the components $E_p$ of $\underline{E}$ may be equal to the empty set. Since $\mathbb{Q}_p$ has at most finitely many extensions of degree bounded by some fixed positive integer, if $E_p\subset\overline{\mathbb{Z}_p}$ has bounded degree then $E_p$ is contained in a finite extension of $\mathbb{Q}_p$.

By Theorem 1.1, a Prüfer domain constructed in [Loper and Werner 2012] can be represented as an intersection of valuation domains (see also [Chabert and Peruginelli 2016]):

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\overline{\mathbb{Z}}})=\bigcap_{p\in\mathbb{P}}\bigcap_{\alpha_p\in E_p}V_{p,\alpha_p}\cap\bigcap_{q\in\mathcal{P}^{\mathrm{irr}}}\mathbb{Q}[X]_{(q)}. \tag{2.2}$$

Here $\underline{E}=\prod_{p\in\mathbb{P}} E_p\subset\overline{\overline{\mathbb{Z}}}$ is locally bounded and $\mathcal{P}^{\mathrm{irr}}$ denotes the set of irreducible polynomials in $\mathbb{Q}[X]$; note that the intersection on the right in this display equals $\mathbb{Q}[X]$. Similarly, for the ring $\mathrm{Int}_{\mathbb{Q}}(E_p,\overline{\mathbb{Z}_p})=\{f\in\mathbb{Q}[X]\mid f(E_p)\subseteq\overline{\mathbb{Z}_p}\}$ we have

$$\mathrm{Int}_{\mathbb{Q}}(E_p,\overline{\mathbb{Z}_p})=\bigcap_{\alpha_p\in E_p}V_{p,\alpha_p}\cap\bigcap_{q\in\mathcal{P}^{\mathrm{irr}}}\mathbb{Q}[X]_{(q)}. \tag{2.3}$$

In particular, $\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\overline{\mathbb{Z}}})=\bigcap_{p\in\mathbb{P}}(\mathbb{Z}\setminus p\mathbb{Z})^{-1}\,\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\overline{\mathbb{Z}}})=\bigcap_{p\in\mathbb{P}}\mathrm{Int}_{\mathbb{Q}}(E_p,\overline{\mathbb{Z}_p})$ by Lemma 2.5.

By means of the representation (2.2), the main result of [Loper and Werner 2012, Corollary 2.12] can now be restated as follows:

**Theorem 2.4.** *Let $\underline{E}\subset\overline{\overline{\mathbb{Z}}}$ be locally bounded. Then the ring $\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\overline{\mathbb{Z}}})$ is a Prüfer domain.*

We want to characterize when a ring of the form $\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\overline{\mathbb{Z}}})$, $\underline{E}\subseteq\overline{\overline{\mathbb{Z}}}$, is a Dedekind domain. In order to accomplish this objective, we need to describe the prime spectrum of this ring when $E$ is locally bounded. It is customary for rings of integer-valued polynomials to distinguish the prime ideals into two different kinds, and we do the same here in our setting: given a prime ideal $P$ of $R=\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\overline{\mathbb{Z}}})$, we say that $P$ is *nonunitary* if $P\cap\mathbb{Z}=(0)$ and that $P$ is *unitary* if $P\cap\mathbb{Z}=p\mathbb{Z}$ for some $p\in\mathbb{P}$.

It is a classical result that each nonunitary prime ideal of $R$ is equal to

$$\mathfrak{P}_q=q(X)\mathbb{Q}[X]\cap R$$

for some $q\in\mathcal{P}^{\mathrm{irr}}$ (see for example [Cahen and Chabert 1997, Corollary V.1.2]).

If $P\cap\mathbb{Z}=p\mathbb{Z}$, $p\in\mathbb{P}$, and $\alpha\in E_p$, the following is a unitary prime ideal of $R$:

$$\mathfrak{M}_{p,\alpha}=\{f\in R\mid v_p(f(\alpha))>0\}.$$

If $E_p$ is a closed subset of $\overline{\mathbb{Z}_p}$ for each prime $p$, and $\underline{E} = \prod_p E_p$ is locally bounded, we are going to show that each unitary prime ideal of $R$ is equal to $\mathfrak{M}_{p,\alpha}$, for some $p \in \mathbb{P}$ and $\alpha \in E_p$.

**Lemma 2.5.** *Let $\underline{E} \subseteq \widehat{\overline{\mathbb{Z}}}$ be any subset, $P$ be a finite subset of $\mathbb{P}$ and $S$ the multiplicative subset of $\mathbb{Z}$ generated by $\mathbb{P} \setminus P$. Then $S^{-1}\,\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}}) = \bigcap_{p \in P} \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$.*

*In particular, for each $p \in \mathbb{P}$, $(\mathbb{Z} \setminus p\mathbb{Z})^{-1}\,\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}}) = \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$.*

*Proof.* The proof follows by an argument similar to the one of [Chabert and Peruginelli 2018, Proposition 4.2]. Let $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ and $R_p = \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$, for each $p \in P$. The containment $S^{-1}R \subseteq \bigcap_{p \in P} R_p$ is clear, since $R \subseteq R_p$ and for every $d \in S$, $d$ is a unit in $R_p$, for each $p \in P$. Conversely, let $f \in \bigcap_{p \in P} R_p$. Let $d \in \mathbb{Z}$, $d \neq 0$, be such that $df \in \mathbb{Z}[X]$ and let $d = t \prod_{p \in P} p^{a_p}$, $a_p \geq 0$ and $t \in \mathbb{Z}$ not divisible by any $p \in P$. Then, letting $g = tf$, we have that $g$ is in $\mathbb{Z}_{(q)}[X] \subset R_q$ for each $q \notin P$ and $g$ is in $R_p$ for each $p \in P$ because $t$ is a unit in $\mathbb{Z}_{(p)}$, for all $p \in P$. Hence, $f = \frac{g}{t} \in S^{-1}R$, as desired. □

**Proposition 2.6.** *Let $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ be locally bounded and closed. If $M$ is a unitary prime ideal of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ such that $M \cap \mathbb{Z} = p\mathbb{Z}$ for some $p \in \mathbb{P}$, then $M$ is maximal and there exists $\alpha \in E_p$ such that $M = \mathfrak{M}_{p,\alpha}$.*

*Proof.* Let $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$. We use the fact that $R$ is a Prüfer domain by Theorem 2.4.

Let $M$ be a unitary prime ideal of $R$ and let $V = R_M$. Then, by Lemma 2.5, we have $R_p = \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) \subset V$, since $(\mathbb{Z} \setminus p\mathbb{Z})^{-1}V = V$. Let $M'$ be the center of $V$ on $R_p$. Since $M' \cap R = M$, it is sufficient to show that

$$M' = \mathfrak{M}_{p,\alpha} = \{f \in R_p \mid v_p(f(\alpha)) > 0\},$$

for some $\alpha \in E_p$ (with a slight abuse of notation, we denote the unitary prime ideals of $R$ and $R_p$ in the same way). Let $f \in R_p$. Let $K$ be a finite extension of $\mathbb{Q}_p$ such that $O_K$ contains $E_p$ and let $i_0, \ldots, i_{q-1} \in O_K$ be a set of representatives for $O_K/\pi O_K \cong \mathbb{F}_q$, where $\pi$ is a uniformizer of $O_K$ (i.e., a generator of the maximal ideal of $O_K$). For each $\alpha \in E_p$, there exists some $j \in \{0, \ldots, q-1\}$ such that $f(\alpha) - i_j \in \pi O_K$. In particular, $\prod_{j=0}^{q-1}(f(\alpha) - i_j) \in \pi O_K$ for each $\alpha \in E_p$. Observe that the polynomials $X^q - X$ and $\prod_{j=0}^{q-1}(X - i_j)$ coincide modulo $\pi$, so in particular $f(\alpha)^q - f(\alpha) \in \pi O_K$. If $e = e(O_K \mid \mathbb{Q}_p)$, we have $(f(\alpha)^q - f(\alpha))^e \in pO_K$. Equivalently, $(f^q - f)^e \in pR_p$, which is contained in $M'$. Since $M'$ is a prime ideal, it follows that $f^q - f \in M'$, so modulo $M'$, $f$ satisfies the equation $X^q - X = 0$. This shows that $R_p/M'$ is contained in the finite field $\mathbb{F}_q$, so it is a finite domain, hence a field. This proves that $M'$ is maximal. Note that, since $R/M \subseteq R_p/M'$ and the latter is a finite field, it follows also that $M$ is a maximal ideal of $R$.

Since $R_p$ is countable, $M'$ is countably generated, say $M' = \bigcup_{n\in\mathbb{N}} I_n$, where $I_n = (p, f_1, \ldots, f_n)$ for each $n \in \mathbb{N}$. By [Gilmer and Heinzer 1968, Proposition 1.4], for each $n \in \mathbb{N}$, there exists $\alpha_n \in E_p$ such that $I_n \subset \mathfrak{M}_{p,\alpha_n}$ (we may exclude the nonunitary prime ideals of $R_p$ because they do not contain $p$, hence neither $I_n$ for every $n$). Suppose first that $E_p$ is finite. Then there exists $\alpha \in E_p$ such that the set $J = \{n \in \mathbb{N} \mid I_n \subset \mathfrak{M}_{p,\alpha}\}$ is a cofinal subset of $\mathbb{N}$. Hence, for each $f \in M'$, there exists $n \in J$ such that $f \in I_n \subset \mathfrak{M}_{p,\alpha}$, so that $M' \subseteq \mathfrak{M}_{p,\alpha}$ and therefore equality holds since $M'$ is maximal. If $E_p$ is infinite, since it is a closed subset (because $\underline{E}$ is closed) contained in a finite extension of $\mathbb{Q}_p$, by compactness we may extract a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ from $E_p$ converging to some element $\alpha \in E_p$. Without loss of generality we suppose that $\alpha_n \to \alpha$. Now, for each $f \in M'$, $f \in I_n \subset \mathfrak{M}_{p,\alpha_n}$ for some $n$. Since $I_n \subseteq I_{n+1}$ for each $n \in \mathbb{N}$, $f \in \mathfrak{M}_{p,\alpha_m}$ for each $m \geq n$, that is, $v_p(f(\alpha_m)) > 0$. By continuity we get that $v_p(f(\alpha)) > 0$, that is, $f \in \mathfrak{M}_{p,\alpha}$. Therefore as before we conclude that $M' = \mathfrak{M}_{p,\alpha}$. □

Thus, if $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\overline{\mathbb{Z}}})$ is a Prüfer domain, given a maximal unitary ideal $\mathfrak{M}_{p,\alpha}$, $p \in \mathbb{P}$ and $\alpha \in E_p$, we have

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\overline{\mathbb{Z}}})_{\mathfrak{M}_{p,\alpha}} = V_{p,\alpha}. \tag{2.7}$$

Similarly, for $q \in \mathcal{P}^{\mathrm{irr}}$, we have

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\overline{\mathbb{Z}}})_{\mathfrak{P}_q} = \mathbb{Q}[X]_{(q)}. \tag{2.8}$$

We call the valuation domains $V_{p,\alpha}$ unitary, and the others $\mathbb{Q}[X]_{(q)}$ nonunitary. Similar equalities hold for the Prüfer domain $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$. Note that the residue field of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\overline{\mathbb{Z}}})$ at a unitary prime ideal is a finite field (by the property of the unitary valuation overrings we discussed about in Section 1), while the residue field of a nonunitary prime ideal of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\overline{\mathbb{Z}}})$ is a finite extension of the rationals, hence an infinite field.

We finish this section with the following remark.

**Remark 2.9.** By Theorem 1.1, given a ring $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$, without loss of generality we may assume that the elements of $E_p$ are pairwise nonconjugate over $\mathbb{Q}_p$. Under this further assumption and if $E_p$ is bounded (i.e., contained in a finite extension of $\mathbb{Q}_p$), Theorem 2.4, (2.7) and Proposition 2.6 imply that there is a one-to-one correspondence between the elements of $E_p$ and the unitary valuation overrings $V_{p,\alpha_p}$, $\alpha_p \in E_p$, of $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$.

**2A.** ***The local case.*** For a fixed $p \in \mathbb{P}$, we characterize in this section the subsets $E_p$ of $\overline{\mathbb{Z}_p}$ for which the corresponding ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a Dedekind domain. The following proposition is a generalization of [Chang 2022, Theorem 4.3 (2)].

**Proposition 2.10.** *Let $E_p$ be a subset of $\overline{\mathbb{Z}_p}$. Then $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a Dedekind domain with finite residue fields of prime characteristic if and only if $E_p$ is a finite subset of transcendental elements over $\mathbb{Q}$.*

*Suppose that $E_p = \{\alpha_1, \dots, \alpha_n\}$ and the $\alpha_i$'s are pairwise nonconjugate over $\mathbb{Q}_p$. Then, then the class group of $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is isomorphic to $\mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}^{n-1}$, where $e = \gcd\{e_{\alpha_i} \mid i = 1, \dots, n\}$. Thus $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a PID if and only if $E_p$ contains at most one element $\alpha_p \in \overline{\mathbb{Z}_p}$, such that $\alpha_p$ is transcendental over $\mathbb{Q}$ and unramified over $\mathbb{Q}_p$.*

*Proof.* Let $R_p = \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$. Note that, if $E_p$ is the empty set, then $R_p = \mathbb{Q}[X]$. We assume henceforth that $E_p \neq \varnothing$.

Suppose $R_p$ is a Dedekind domain with finite residue fields of prime characteristic. We show first that each maximal unitary ideal $M$ of $R_p$ is equal to $\mathfrak{M}_{p,\alpha_p}$, for some $\alpha_p \in E_p$. Let $V$ be a unitary valuation overring of $R_p$ which is centered on $M$. By Theorem 1.1, there exists $\alpha_0 \in \overline{\mathbb{Z}_p}$ such that $V = V_{p,\alpha_0}$. Then, $M = \mathfrak{M}_{p,\alpha_0}$. Since $M$ is finitely generated and $R_p$ is Prüfer, by [Gilmer and Heinzer 1968, Proposition 1.4] $M \subseteq \mathfrak{M}_{p,\alpha_p}$ for some $\alpha_p \in E_p$ (we may exclude the nonunitary prime ideals of $R_p$ because they do not contain $p$, hence neither $M$). Since $M$ is maximal, it follows that $M = \mathfrak{M}_{p,\alpha_p}$, which means that $\alpha_0$ and $\alpha_p$ are conjugate over $\mathbb{Q}_p$ by [Peruginelli 2017, Theorem 3.2]. Hence, without loss of generality, we may suppose that $\alpha_0 \in E_p$. Note that each $\alpha_p \in E_p$ is transcendental over $\mathbb{Q}$, otherwise the valuation overring $V_{p,\alpha_p}$ of $R_p$ would have rank 2. Since $R_p$ is Dedekind, $p$ is contained in only finitely many maximal ideals of this ring; necessarily, such ideals are unitary. By the previous argument, such ideals are equal to $\mathfrak{M}_{p,\alpha_p}$, for $\alpha_p \in E_p$. Since by Theorem 1.1 and (2.7), $\mathfrak{M}_{p,\alpha_p} = \mathfrak{M}_{p,\beta_p}$ if and only if $\alpha_p, \beta_p \in E_p$ are conjugate over $\mathbb{Q}_p$, it follows that $E_p$ is a finite subset of $\overline{\mathbb{Z}_p}$.

Conversely, suppose now that $E_p \subset \overline{\mathbb{Z}_p}$ is a finite subset of transcendental elements over $\mathbb{Q}$. The fact that $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a Dedekind domain follows from [Eakin and Heinzer 1973, Theorem], but we give a different self-contained argument based on the previous results. We know that $E_p$ has bounded degree, so $R_p$ is Prüfer, by Theorem 2.4. By (2.3), $R_p$ is equal to an intersection of DVRs which are essential over it. Moreover, each nonzero $f \in R_p$ belongs to finitely many maximal ideals, since $E_p$ is finite and $f$ has finitely many irreducible factors in $\mathbb{Q}[X]$. Hence, $R_p$ is a Krull domain, so, by [Gilmer 1992, Theorem 43.16], $R_p$ is a Dedekind domain. Finally, $R_p$ has finite residue fields of prime characteristic, because each of the unitary valuation overrings of $R_p$ (namely, $V_{p,\alpha_p}$, $\alpha_p \in E_p$) have finite residue field.

Assuming that the elements of $E_p$ are pairwise nonconjugate over $\mathbb{Q}_p$, the claim regarding the class group follows easily from [Eakin and Heinzer 1973, Theorem], taking into account the representation (2.3). If $E_p = \{\alpha_1, \dots, \alpha_n\}$, let

$\boldsymbol{e} = (e_{\alpha_1}, \dots, e_{\alpha_n}) \in \mathbb{Z}^n$ and $e = \gcd(e_{\alpha_1}, \dots, e_{\alpha_n})$. Then, the class group of $R_p$ is isomorphic to

$$\mathbb{Z}^n/\langle \boldsymbol{e} \rangle \cong \mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}^{n-1}.$$

The last claim follows at once from the description of the class group. □

**2B.** ***The global case.*** If, for each $p \in \mathbb{P}$, $E_p \subset \overline{\mathbb{Z}_p}$ is a finite subset of transcendental elements over $\mathbb{Q}$ and $\underline{E} = \prod_p E_p$, then, by [Chang 2022, Corollary 2.6], $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is an almost Dedekind domain. However, this ring might not be noetherian, that is, a Dedekind domain. See for example the construction of [Chang 2022, Theorem 3.1], in which the polynomial $X$ is divisible by infinitely many primes $p \in \mathbb{P}$. In general, an almost Dedekind domain $R$ is Dedekind if and only if it has finite character, that is, each nonzero $f \in R$ belongs to finitely many maximal ideals of $R$ [Gilmer 1992, Theorem 37.2], or, equivalently, $v(f) \neq 0$ only for finitely many valuation overrings $V$ of $R$ (which are only DVRs). We aim to characterize the subsets $\underline{E} = \prod_p E_p$ of $\widehat{\overline{\mathbb{Z}}}$ such that $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is Dedekind.

**Definition 2.11.** We say that $\underline{E}$ is *polynomially factorizable* if, for each $g \in \mathbb{Z}[X]$ and $\alpha = (\alpha_p) \in \underline{E}$, there exist $n, d \in \mathbb{Z}$, $n, d \geq 1$ such that $g(\alpha)^n/d$ is a unit of $\widehat{\overline{\mathbb{Z}}}$, that is, $v_p(g(\alpha_p)^n/d) = 0$, for all $p \in \mathbb{P}$.

Note that $g(\alpha)^n = (g(\alpha_p)^n) \in \widehat{\overline{\mathbb{Z}}}$. Loosely speaking, a subset $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}}$ is polynomially factorizable if, for every $g \in \mathbb{Z}[X]$ and $\alpha \in \underline{E}$, $g(\alpha) \in \widehat{\overline{\mathbb{Z}}}$ is divisible only by finitely many primes $p \in \mathbb{P}$ (up to some exponent $n \geq 1$), or, equivalently, all but finitely many components of $g(\alpha)$ are units. Note that, if the above condition of the definition holds, then $g(\alpha)^n$ and $d$ generate the same principal ideal of $\widehat{\overline{\mathbb{Z}}}$.

The next lemma gives a simple characterization of polynomially factorizable subsets $\underline{E}$ of $\widehat{\overline{\mathbb{Z}}}$ in terms of the finiteness of some sets of primes associated to every polynomial in $\mathbb{Z}[X]$. For every $g \in \mathbb{Z}[X]$ and subset $\underline{E} = \prod_p E_p \subseteq \widehat{\overline{\mathbb{Z}}}$, we set

$$\mathbb{P}_{g,\underline{E}} = \{p \in \mathbb{P} \mid \exists \alpha_p \in E_p \text{ such that } v_p(g(\alpha_p)) > 0\}.$$

The next result shows that $\underline{E}$ is polynomially factorizable if and only if $\mathbb{P}_{g,\underline{E}}$ is finite for every $g \in \mathbb{Z}[X]$.

**Lemma 2.12.** *Let $g \in \mathbb{Z}[X]$ and $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$, where each $E_p \subset \overline{\mathbb{Z}_p}$ is a closed set of transcendental elements over $\mathbb{Q}$. Then the following conditions are equivalent*:

i) *The set $\mathbb{P}_{g,\underline{E}}$ is finite.*

ii) *For each $\alpha \in \underline{E}$, there exist $n, d \in \mathbb{Z}$, $n, d \geq 1$ such that $g(\alpha)^n/d$ is a unit of $\widehat{\overline{\mathbb{Z}}}$.*

*Proof.* We use the following easy remark: for $\alpha = (\alpha_p) \in \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, the set $\{p \in \mathbb{P} \mid v_p(\alpha_p) > 0\}$ is finite if and only if there exists $d \in \mathbb{Z}$, $d \geq 1$, such that $\alpha\widehat{\mathbb{Z}} = d\widehat{\mathbb{Z}}$.

Suppose i) holds and let $\alpha = (\alpha_p) \in \underline{E}$. By assumption, there are only finitely many $p \in \mathbb{P}$ such that $v_p(g(\alpha_p)) > 0$, for some $\alpha_p \in E_p$, say, $p_1, \dots, p_k$. Let $\alpha \in \underline{E}$ be fixed; in particular, there exists $n \in \mathbb{N}$ such that $n v_p(g(\alpha_p)) = a_p \in \mathbb{Z}$ for each prime $p$ (where $a_p = 0$ for all $p \notin \{p_1, \dots, p_k\}$). Hence, if we let $d = \prod_{i=1}^{k} p_i^{a_{p_i}}$ we get $v_p(g(\alpha_p)^n) = v_p(d)$ for all $p \in \mathbb{P}$, thus ii) holds.

Assume now that ii) holds and suppose that $\mathbb{P}_{g,\underline{E}}$ is infinite. For each $p \in \mathbb{P}_{g,\underline{E}}$, let $\alpha_p \in E_p$ be such that $v_p(g(\alpha_p)) > 0$ and consider the element $\alpha = (\alpha_p) \in \underline{E}$, where $\alpha_p$ is any element of $E_p$ for $p \notin \mathbb{P}_{g,\underline{E}}$. If there is no $n \geq 1$ such that $n v_p(g(\alpha_p)) = a_p \in \mathbb{Z}$ for all $p \in \mathbb{P}$ we immediately get a contradiction. Suppose instead that such an $n$ exists. Since $a_p$ is nonzero for infinitely many $p \in \mathbb{P}$, there is no $d \in \mathbb{Z}$ such that $v_p(g(\alpha_p)^n/d) = 0$ for each $p \in \mathbb{P}$, which again is a contradiction. □

**Remark 2.13.** By Lemma 2.12, it follows easily that a subset $\underline{E} \subseteq \widehat{\overline{\mathbb{Z}}}$ is polynomially factorizable if and only if $\mathbb{P}_{g,\underline{E}}$ is finite for each irreducible $g \in \mathbb{Z}[X]$. In fact, if $g = \prod_i g_i$, where $g_i \in \mathbb{Z}[X]$ are irreducible, then $\mathbb{P}_{g,\underline{E}} = \bigcup_i \mathbb{P}_{g_i,\underline{E}}$.

It is well-known that, given a nonconstant $q \in \mathbb{Z}[X]$, there exist infinitely many $p \in \mathbb{P}$ for which there exists $n \in \mathbb{Z}$ such that $q(n)$ is divisible by $p$ (see for example the proof of [Cahen and Chabert 1997, Proposition V.2.8]). In particular, $\widehat{\mathbb{Z}}$ is not polynomially factorizable by Lemma 2.12.

The next lemma describes the Picard group of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ in terms of the Picard groups of the localizations $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$, $p \in \mathbb{P}$ (see Lemma 2.5).

**Lemma 2.14.** *Let $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ be a subset. Then*

$$\mathrm{Pic}(\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})) \cong \bigoplus_{p \in \mathbb{P}} \mathrm{Pic}(\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})).$$

*Proof.* Let $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ and $R_p = (\mathbb{Z} \setminus p\mathbb{Z})^{-1} R$, for $p \in \mathbb{P}$; by Lemma 2.5, $R_p = \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$. Since the proof follows by the same arguments of [Gilmer et al. 1990, Theorem 1], we just sketch it and refer to the cited paper for the details. By a classical argument (see for example [McQuillan 1985, Lemma 1]), every finitely generated ideal $J$ of $R$ (in particular, every invertible ideal of $R$) is isomorphic to a finitely generated unitary ideal $I$, that is, $I \cap \mathbb{Z} = d\mathbb{Z} \neq (0)$. For such an ideal, $(I \cap \mathbb{Z})_{(p)} = \mathbb{Z}_{(p)}$ for all $p \in \mathbb{P}$ not dividing $d$, so $IR_p = R_p$. This argument shows that we have a well-defined map from $\mathrm{Pic}(R)$ to $\bigoplus_{p \in \mathbb{P}} \mathrm{Pic}(R_p)$.

If $I$ is a unitary ideal of $R$, say $I \cap \mathbb{Z} = d\mathbb{Z}$, such that $IR_p$ is principal, it is generated by $d$. Hence, $I$ and $dR$ have the same localizations at each prime $p \in \mathbb{P}$, so they are equal. This shows that the previous map is injective.

For the surjectivity, it is sufficient to show that, if $J_p$ is an invertible unitary ideal of $R_p$, for some $p \in \mathbb{P}$, then there exists an invertible ideal $J$ of $R$ such that

$JR_p = J_p$ and $JR_q = R_q$ for each $q \in \mathbb{P} \setminus \{p\}$. The ideal $J = J_p \cap R$ has the required properties. □

Now we may characterize when a generalized ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is Dedekind and describe its class group.

**Theorem 2.15.** *Let* $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ *be a subset. Then* $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ *is a Dedekind domain with finite residue fields of prime characteristic if and only if* $E_p$ *is a finite set of transcendental elements over* $\mathbb{Q}$ *for each* $p \in \mathbb{P}$ *and* $\underline{E}$ *is polynomially factorizable.*

*In this case, the class group of* $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}})$ *is equal to a direct sum of a countable family of finitely generated abelian groups.*

*Proof.* Let $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ and suppose the conditions for $\underline{E}$ in the statement are satisfied. Then $\underline{E}$ is locally bounded and closed so, by Theorem 2.4, $R$ is Prüfer. For $R$ to be Dedekind, it is sufficient to show that it is a Krull domain [Gilmer 1992, Theorem 43.16]. By assumption, each of the unitary valuation overrings of $R$ in the representation (2.2) is a DVR with finite residue field, so $R$ has finite residue fields of prime characteristic by Proposition 2.6. We have to show that $R$ has finite character, that is, for each nonzero $f = \frac{g}{n} \in R$, $g \in \mathbb{Z}[X]$ and $n \in \mathbb{Z} \setminus \{0\}$, $f$ is contained in only finitely many maximal ideals of $R$. As in the proof of Proposition 2.10, $f$ is contained in only finitely many nonunitary prime ideals of $R$. We now check the maximal unitary ideals of $R$, described in the Proposition 2.6, which contain $f$. Since the denominator $n$ of $f$ is divisible by only finitely many $p \in \mathbb{P}$, $f$ is contained in only finitely many maximal unitary ideals if and only if the same condition holds for $g$. Since $E_p$ is finite for each $p \in \mathbb{P}$, this is equivalent to the finiteness of the set $\mathbb{P}_{g,\underline{E}}$. Since $\underline{E}$ is polynomially factorizable, by Lemma 2.12, $\mathbb{P}_{g,\underline{E}}$ is finite.

Conversely, if $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ is a Dedekind domain with finite residue fields of prime characteristic, then, for each prime $p$, the overring $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is a Dedekind domain with finite residue fields of prime characteristic [Gilmer 1992, Theorem 40.1]. By Proposition 2.10, $E_p$ is a finite subset of $\overline{\mathbb{Z}_p}$ formed by transcendental elements over $\mathbb{Q}$ (so, in particular, $\underline{E}$ is locally bounded). If there exists some $g \in \mathbb{Z}[X]$ such that the set $\mathbb{P}_{g,\underline{E}}$ is infinite, then $g(X)$ would be contained in infinitely many unitary prime ideals of $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$, a contradiction with [Gilmer 1992, Theorem 37.2]. Therefore, $\underline{E}$ is polynomially factorizable by Lemma 2.12.

The final claim follows from Lemma 2.14 and Proposition 2.10. □

The next corollary is a generalization of [Glivický and Šaroch 2013, Lemma 3.3]: it characterizes the elements $\alpha$ in $\widehat{\overline{\mathbb{Z}}}$ for which the ring $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \widehat{\overline{\mathbb{Z}}})$ is a PID.

**Corollary 2.16.** *Let* $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ *be a subset such that, for each* $p \in \mathbb{P}$, *the elements of* $E_p$ *are pairwise nonconjugate over* $\mathbb{Q}_p$. *Then* $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$ *is a PID with* *finite residue fields of prime characteristic if and only if, for each prime $p$, $E_p$ contains at most one element of $\overline{\mathbb{Z}_p}$, unramified over $\mathbb{Q}_p$ and transcendental over $\mathbb{Q}$, and $\underline{E}$ is polynomially factorizable.*

Note that if the conditions of Corollary 2.16 occur, namely, $E_p = \{\alpha_p\}$ for each $p \in \mathbb{P}$, then $\underline{E}$ is the singleton $\{\alpha\}$, where $\alpha = (\alpha_p) \in \widehat{\overline{\mathbb{Z}}}$. The condition that $\underline{E}$ is polynomially factorizable appears in other equivalent forms in [Glivický and Šaroch 2013, Lemma 3.3] and [Glivická et al. 2023, Proposition 1.1], in the case $\alpha \in \widehat{\mathbb{Z}}$.

*Proof.* The proof follows from Theorem 2.15, Lemma 2.14 and Proposition 2.10. □

An argument similar to the one in the proof of [Eakin and Heinzer 1973, Theorem] shows that a PID $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \widehat{\overline{\mathbb{Z}}})$ as in the statement of Corollary 2.16 is never a Euclidean domain.

We now show that each Dedekind domain with finite residue fields of prime characteristic between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ is indeed a generalized ring of integer-valued polynomials.

**Theorem 2.17.** *Let $R$ be a Dedekind domain with finite residue fields of prime characteristic such that $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$. Then $R$ is equal to $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}})$, for some subset $\underline{E} = \prod_p E_p \subset \widehat{\overline{\mathbb{Z}}}$ such that $E_p$ is a finite set of transcendental elements over $\mathbb{Q}$ for each prime $p$ and $\underline{E}$ is polynomially factorizable.*

*In particular, the class group of $R$ is isomorphic to a direct sum of a countable family of finitely generated abelian groups.*

*Proof.* Let $\mathbb{P}_R = \{p \in \mathbb{P} \mid \exists P \in \mathrm{Spec}(R) \text{ such that } P \cap \mathbb{Z} = p\mathbb{Z}\}$. Clearly, $\mathbb{P}_R$ is empty if and only if $R = \mathbb{Q}[X]$; in this case for $\underline{E}$ equal to the empty set we have the claim. Suppose $\mathbb{P}_R$ is not empty. For each $p \in \mathbb{P}_R$, we denote by $\mathbb{P}_{R,p}$ the set of unitary prime ideals of $R$ lying above $p$. By assumption, for each $P \in \mathbb{P}_{R,p}$, $p \in \mathbb{P}$, $R_P$ is a DVR of $\mathbb{Q}(X)$ with finite residue field extending $\mathbb{Z}_{(p)}$. By Theorem 1.1, there exists $\alpha_p \in \overline{\mathbb{Z}_p}$, transcendental over $\mathbb{Q}$, such that $R_P = V_{p,\alpha_p}$. Let $E_p$ be the subset of $\overline{\mathbb{Z}_p}$ formed by such $\alpha_p$'s, for each $P \in \mathbb{P}_{R,p}$. Since $R$ is Dedekind and by (2.2) and (2.3), we have the equalities

$$\begin{aligned} R = \bigcap_{p \in \mathbb{P}_R} \bigcap_{P \in \mathbb{P}_{R,p}} R_P \cap \mathbb{Q}[X] &= \bigcap_{p \in \mathbb{P}_R} \bigcap_{\alpha_p \in E_p} V_{p,\alpha_p} \cap \mathbb{Q}[X] \\ &= \bigcap_{p \in \mathbb{P}_R} \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}}), \end{aligned}$$

where $\underline{E} = \prod_{p \in \mathbb{P}_R} E_p \subset \widehat{\overline{\mathbb{Z}}}$. By Theorem 2.15, for each $p \in \mathbb{P}$, $E_p$ is a finite subset of $\overline{\mathbb{Z}_p}$ of transcendental elements over $\mathbb{Q}$, $\underline{E}$ is polynomially factorizable and the class group of $R$ is isomorphic to a direct sum of a countable family of finitely generated abelian groups. □

It was shown in [Glivický and Šaroch 2013, Proposition 3.4] that the cardinality of the set of $\alpha \in \widehat{\mathbb{Z}}$ such that $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \widehat{\mathbb{Z}})$ is a PID is $2^{\aleph_0}$. The next corollary describes all the PIDs with finite residue fields of prime characteristic between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

**Corollary 2.18.** *Let $R$ be a PID with finite residue fields of prime characteristic such that $\mathbb{Z}[X] \subset R \subset \mathbb{Q}[X]$. Then $R$ is equal to $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \overline{\widehat{\mathbb{Z}}})$, for some $\alpha = (\alpha_p) \in \overline{\widehat{\mathbb{Z}}}$ such that, for each $p \in \mathbb{P}$, $\alpha_p$ is transcendental over $\mathbb{Q}$, $\alpha_p$ is unramified over $\mathbb{Q}_p$ and $\{\alpha\}$ is polynomially factorizable.*

*Proof.* The proof follows from Theorem 2.17 and Corollary 2.16. □

**2C. *Equality of generalized rings of integer-valued polynomials.*** Given two locally bounded closed subsets $\underline{E}, \underline{F}$ of $\overline{\widehat{\mathbb{Z}}}$, we characterize when the associated generalized ring of integer-valued polynomials $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$, $\mathrm{Int}_{\mathbb{Q}}(\underline{F}, \overline{\widehat{\mathbb{Z}}})$ are the same.

The following is a general result about integral extensions of rings of integer-valued polynomials. For an integral domain $D$ with quotient field $K$, let $\overline{K}$ and $\overline{D}$ be the algebraic closure of $K$ and the absolute integral closure of $D$, respectively. We let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of $K$. For a subset $\Omega$ of $\overline{K}$ we set $G_K(\Omega) = \{\sigma(a) \mid \sigma \in G_K, a \in \Omega\} = \bigcup_{\sigma \in G_K} \sigma(\Omega)$. We say that $\Omega$ is $G_K$*-invariant* if $G_K(\Omega) = \Omega$. Note that in general we have

$$\mathrm{Int}_K(\Omega, \overline{D}) = \mathrm{Int}_K(G_K(\Omega), \overline{D}) \tag{2.19}$$

because if $f(\alpha) \in \overline{D}$ for some $f \in K[X]$ and $\alpha \in \Omega$, then, for every $\sigma \in G_K$, we have $f(\sigma(\alpha)) = \sigma(f(\alpha)) \in \overline{D}$ because $\sigma(\overline{D}) \subseteq \overline{D}$.

**Lemma 2.20.** *Let $D$ be an integrally closed domain with quotient field $K$. Let $\Omega \subset \overline{D}$ be $G_K$-invariant. Let $F$ be an algebraic extension of $K$ containing $\Omega$. Then $\mathrm{Int}_F(\Omega, \overline{D})$ is the integral closure of $\mathrm{Int}_K(\Omega, \overline{D})$ in $F(X)$.*

*Proof.* By [Cahen and Chabert 1997, Proposition IV.4.1], $\mathrm{Int}_{\overline{K}}(\Omega, \overline{D})$ is integrally closed. In particular, $\mathrm{Int}_F(\Omega, \overline{D}) = \mathrm{Int}_{\overline{K}}(\Omega, \overline{D}) \cap F(X)$ is integrally closed, too. Hence, we just need to show that $\mathrm{Int}_K(\Omega, \overline{D}) \subseteq \mathrm{Int}_F(\Omega, \overline{D})$ is an integral ring extension.

Without loss of generality, we may enlarge $F$ and suppose that $F$ is normal over $K$ (e.g., we may take $F = \overline{K}$). Let $f \in \mathrm{Int}_F(\Omega, \overline{D}) \subset F[X]$. In particular, $f$ is integral over $K[X]$, that is, it satisfies a monic equation of the form

$$f^n + g_{n-1} f^{n-1} + \cdots + g_1 f + g_0 = 0,$$

where $g_i \in K[X]$, for $i = 0, \ldots, n-1$. We claim that $g_i \in \mathrm{Int}_K(\Omega, \overline{D})$, for $i = 0, \ldots, n-1$, which will prove the claim. In fact, let

$$\Phi(T) = T^n + g_{n-1} T^{n-1} + \cdots + g_0 \in K[X][T],$$

and suppose that $\Phi(T)$ is irreducible over $K(X)$. The roots of $\Phi(T)$ are the conjugates of $f$ under the action of the Galois group $\mathrm{Gal}(F(X)/K(X)) \cong \mathrm{Gal}(F/K)$, which acts on the coefficients of the polynomial $f$. If $\sigma \in \mathrm{Gal}(F/K)$, then $\sigma(f) \in \mathrm{Int}_F(\Omega, \overline{D})$. In fact, for each $\alpha \in \Omega$, since $\Omega$ is $\mathrm{Gal}(F/K)$-invariant, we have $\alpha = \sigma(\alpha')$ for some $\alpha' \in \Omega$, therefore $\sigma(f)(\alpha) = \sigma(f(\alpha'))$ which still is an element of $\overline{D}$ (which likewise is left invariant under the action of $\mathrm{Gal}(F/K)$). Now, since each coefficient $g_i$ of $\Phi(T)$ is an elementary symmetric function of the elements $\sigma(f), \sigma \in \mathrm{Gal}(F/K)$, we have $g_i(\alpha) \in \overline{D}$, for each $\alpha \in \Omega$; thus $g_i \in \mathrm{Int}_K(\Omega, \overline{D})$, as claimed. □

To ease notation, we denote the absolute Galois group of $\mathbb{Q}_p$ ($p$ prime) by $G_p$.

**Theorem 2.21.** *Suppose $\underline{E} = \prod_p E_p$ and $\underline{F} = \prod_p F_p$ are locally bounded closed subsets of $\overline{\widehat{\mathbb{Z}}}$. Then the rings $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$ and $\mathrm{Int}_{\mathbb{Q}}(\underline{F}, \overline{\widehat{\mathbb{Z}}})$ are equal if and only if $G_p(E_p) = G_p(F_p)$, for each $p \in \mathbb{P}$.*

*Proof.* Clearly, $\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}}) = \mathrm{Int}_{\mathbb{Q}}(\underline{F}, \overline{\widehat{\mathbb{Z}}})$ if and only if the two rings have the same localization at each $p \in \mathbb{P}$, that is, by Lemma 2.5, $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}}(F_p, \overline{\mathbb{Z}_p})$. Such a condition is equivalent to $\mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}_p}(F_p, \overline{\mathbb{Z}_p})$. In fact, one implication is obvious because $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p})$ is the contraction to $\mathbb{Q}[X]$ of $\mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p})$. Conversely, suppose that $\mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}}(F_p, \overline{\mathbb{Z}_p})$ and let $f \in \mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p})$, say $f(X) = \sum_i \alpha_i X^i$. We can choose $g \in \mathbb{Q}[X]$ sufficiently $v_p$-adically close to $f(X)$, that is, $g(X) = \sum_i a_i X^i$, where $v_p(\alpha_i - a_i) \geq n$ for each $i \geq 0$, where $n \in \mathbb{N}$ is arbitrary large. Then $h = f - g \in p^n \mathbb{Z}_p[X]$, so, if $\alpha_p \in E_p$, it follows that $g(\alpha_p) = f(\alpha_p) + h(\alpha_p) \in \overline{\mathbb{Z}_p}$. Hence, $g \in \mathrm{Int}_{\mathbb{Q}}(E_p, \overline{\mathbb{Z}_p}) = \mathrm{Int}_{\mathbb{Q}}(F_p, \overline{\mathbb{Z}_p})$. If now $\beta_p \in F_p$, we have $f(\beta_p) = g(\beta_p) + h(\beta_p) \in \overline{\mathbb{Z}_p}$, which proves that $f \in \mathrm{Int}_{\mathbb{Q}_p}(F_p, \overline{\mathbb{Z}_p})$. The other containment $\mathrm{Int}_{\mathbb{Q}_p}(F_p, \overline{\mathbb{Z}_p}) \subseteq \mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p})$ follows in the same way.

Let $p \in \mathbb{P}$ be a fixed prime and set $\widehat{R}_{p,E_p} = \mathrm{Int}_{\mathbb{Q}_p}(E_p, \overline{\mathbb{Z}_p})$ and $\widehat{R}_{p,F_p} = \mathrm{Int}_{\mathbb{Q}_p}(F_p, \overline{\mathbb{Z}_p})$. Since $E_p, F_p$ are subsets of $\overline{\mathbb{Z}_p}$ of bounded degree, there exists a finite Galois extension $K$ of $\mathbb{Q}_p$ containing both of them. By (2.19), $\widehat{R}_{p,E_p} = \mathrm{Int}_{\mathbb{Q}_p}(G_p(E_p), \overline{\mathbb{Z}_p})$ and $\widehat{R}_{p,F_p} = \mathrm{Int}_{\mathbb{Q}_p}(G_p(F_p), \overline{\mathbb{Z}_p})$. Clearly, $\widehat{R}_{p,E_p}$ and $\widehat{R}_{p,F_p}$ are equal if and only if they have the same integral closure in $K(X)$. By Lemma 2.20, this amounts to say that

$$\mathrm{Int}_K(G_p(E_p), \overline{\mathbb{Z}_p}) = \mathrm{Int}_K(G_p(F_p), \overline{\mathbb{Z}_p}). \tag{2.22}$$

Note that the rings of (2.22) are equal to $\mathrm{Int}_K(G_p(E_p), O_K)$, $\mathrm{Int}_K(G_p(F_p), O_K)$, respectively, where $O_K$ is the ring of integers of $K$. Moreover, $G_p(E_p)$ is a closed subset of $O_K$, being a finite union of closed sets $\sigma(E_p)$, $\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)$. Similarly, $G_p(F_p)$ is closed.

Finally, by [McQuillan 1991, Lemma 2], (2.22) holds if and only if $G_p(E_p) = G_p(F_p)$. □

Theorem 2.21 implies that the rings $\mathrm{Int}_{\mathbb{Q}}(\{\alpha\}, \widehat{\overline{\mathbb{Z}}})$, $\alpha \in \widehat{\overline{\mathbb{Z}}}$, are in one-to-one correspondence with the elements of $\widehat{\overline{\mathbb{Z}}}$.

## 3. Construction of a Dedekind domain with prescribed class group

We review Chang's construction [2022] mentioned in the introduction and modify it in order to show that, given a group $G$ which is the direct sum of a countable family of finitely generated abelian groups, there exists a Dedekind domain $R$ with finite residue fields of prime characteristic, $\mathbb{Z}[X] \subset R \subseteq \mathbb{Q}[X]$, such that the class group of $R$ is $G$. As in [Eakin and Heinzer 1973], we show first that the ring constructed by Chang can also be represented as a generalized ring of integer-valued polynomials. In [Chang 2022, Lemma 3.4] it is proved that for each $n \in \mathbb{N}$ and $p \in \mathbb{P}$, there exists a DVR of $\mathbb{Q}(X)$ which is a residually algebraic extension of $\mathbb{Z}_{(p)}$ with ramification index equal to $n$; by means of Theorem 1.1, we can give an explicit representation of such an extension in terms of a valuation domain $V_{p,\alpha}$ associated to some $\alpha \in \overline{\mathbb{Z}_p}$ which generates a totally ramified extension of $\mathbb{Q}_p$ of degree $n$.

Let $I$ be a countable set and $G = \bigoplus_{i\in I} G_i$ be a direct sum of finitely generated abelian groups $G_i$. Suppose that for each $i \in I$ we have

$$G_i \cong \mathbb{Z}^{m_i} \oplus \mathbb{Z}/n_{i,1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_{i,k_i}\mathbb{Z}$$

for some uniquely determined nonnegative integers $m_i, n_{i,1}, \ldots, n_{i,k_i}$ satisfying $n_{i,j} \mid n_{i,j+1}$. We partition $\mathbb{P}$ into a family of finite subsets $\{\mathbb{P}_i\}_{i\in I}$ each of which contains arbitrary chosen $1+k_i$ primes, namely $\mathbb{P}_i = \{p_i, q_{i,1}, \ldots, q_{i,k_i}\}$ and correspondingly for each $i \in I$ we fix the following $1+k_i$ sets:

i) $E_{p_i}$ is a subset of $\mathbb{Z}_{p_i}$ of $m_i + 1$ elements $\{\alpha_{p_i,1}, \ldots, \alpha_{p_i,m_i+1}\}$ which are transcendental over $\mathbb{Q}$.

ii) For $j = 1, \ldots, k_i$, $E_{q_{i,j}} = \{\alpha_{q_{i,j}}\}$ a singleton of $\overline{\mathbb{Z}_{q_{i,j}}}$ such that $\alpha_{q_{i,j}}$ is transcendental over $\mathbb{Q}$ and $n_{i,j} = e_{\alpha_{q_{i,j}}}$, the ramification index of $\mathbb{Q}_p(\alpha_{q_{i,j}})$ over $\mathbb{Q}_p$.

We set $\underline{E}_i = E_{p_i} \times \prod_{j=1}^{k_i} E_{q_{i,j}}$ and also

$$R_i = \mathrm{Int}_{\mathbb{Q}}(E_{p_i}, \mathbb{Z}_{p_i}) \cap \bigcap_{j=1}^{k_i} \mathrm{Int}_{\mathbb{Q}}(E_{q_{i,j}}, \overline{\mathbb{Z}}_{q_{i,j}}) = \mathrm{Int}_{\mathbb{Q}}(\underline{E}_i, \widehat{\overline{\mathbb{Z}}}).$$

Since each of the unitary valuation overrings of $R_i$, namely $V_{p,\alpha_p}$, $p \in \mathbb{P}_i$ and $\alpha_p \in E_p$, is a DVR which is residually algebraic over $\mathbb{F}_p$ [Peruginelli 2017, Proposition 2.2], by [Eakin and Heinzer 1973, Theorem and Corollary] $R_i$ is a Dedekind domain with class group isomorphic to $G_i$.

We also set

$$R = \bigcap_{i\in I} R_i = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\overline{\mathbb{Z}}}),$$

where $\underline{E} = \prod_i \underline{E}_i$. By [Chang 2022, Corollary 2.6], $R$ is an almost Dedekind domain with class group isomorphic to $G$.

As we already mentioned at the beginning of Section 2B, the ring $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$ is not Dedekind in general. By Theorem 2.15, this happens precisely when $\underline{E}$ is polynomially factorizable. By a suitable modification of the above construction, we are going to show that there exists a polynomially factorizable subset $\underline{E}$ of $\overline{\widehat{\mathbb{Z}}}$ such that $R$ is Dedekind with class group isomorphic to $G$, thus giving a positive answer to [Chang 2022, Question 3.7].

**Theorem 3.1.** *Let $G$ be a direct sum of a countable family $\{G_i\}_{i\in I}$ of finitely generated abelian groups (which are not necessarily distinct). Then there exists a Dedekind domain $R$ between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ with class group isomorphic to $G$. Moreover, for each $i \in I$, there exists a multiplicative subset $S_i$ of $\mathbb{Z}$ such that $S_i^{-1}R$ is a Dedekind domain with class group $G_i$.*

*Proof.* We keep the notation used in the above construction. Let $\mathbb{P}_r = \bigcup_{i\in I}(\mathbb{P}_i \setminus \{p_i\})$. For each $q = q_{i,j} \in \mathbb{P}_r$, for some $i \in I$ and $j \in \{1, \dots, k_i\}$, we set $n_q = n_{i,j}$. We choose a uniformizer $\tilde{q}$ of $\mathbb{Z}_q$ which is transcendental over $\mathbb{Q}$. Let $\tilde{\alpha}_q \in \overline{\mathbb{Z}_q}$ be a root of the Eisenstein polynomial $X^{n_q} - \tilde{q}$. Clearly, $\tilde{\alpha}_q$ is still transcendental over $\mathbb{Q}$ and it is well-known that $\mathbb{Q}_q(\tilde{\alpha}_q)$ is a totally ramified extension of $\mathbb{Q}_q$ of degree $n_q$. We now let $\alpha_q = \tilde{\alpha}_q + \lfloor \log q \rfloor$: this is another generator of $\mathbb{Q}_q(\tilde{\alpha}_q)$ over $\mathbb{Q}_q$ which still is transcendental over $\mathbb{Q}$ and has $v_q$-adic valuation zero. We then set $E_q = \{\alpha_q\}$ in the above construction.

Similarly, for each $p = p_i \in \mathbb{P} \setminus \mathbb{P}_r$, for some $i \in I$, let $m_p = m_{p_i}$. We choose distinct elements $\alpha_{p,i} \in \lfloor \log p \rfloor + p\mathbb{Z}_p$, for $i = 1, \dots, m_p + 1$, which are transcendental over $\mathbb{Q}$ and set $E_p = \{\alpha_{p,1}, \dots, \alpha_{p,m_p+1}\}$.

We show now that with these choices the subset $\underline{E} = \prod_p E_p \subset \overline{\widehat{\mathbb{Z}}}$ is polynomially factorizable, and therefore the corresponding domain $R = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \overline{\widehat{\mathbb{Z}}})$ is a Dedekind domain by Theorem 2.15. By Lemma 2.12, we need to show that for each $g \in \mathbb{Z}[X]$, $\mathbb{P}_{g,\underline{E}}$ is finite. Let $g \in \mathbb{Z}[X]$ be a fixed polynomial. For $\alpha = (\alpha_p) \in \underline{E}$, we have:

- $\alpha_p = pa + \lfloor \log p \rfloor$, for some $a \in \mathbb{Z}_p$, if $p \in \mathbb{P} \setminus \mathbb{P}_r$.
- $\alpha_p = \tilde{\alpha}_p + \lfloor \log p \rfloor$, if $p \in \mathbb{P}_r$, where $\tilde{\alpha}_p$ is a root of an Eisenstein polynomial, so, in particular, $v_p(\tilde{\alpha}_p) > 0$.

For each $p \in \mathbb{P}$, let $\pi_p$ be a uniformizer of $\mathbb{Q}_p(\alpha_p)$ (which is just $p$ if $p \notin \mathbb{P}_r$). We then have

$$g(\alpha_p) \equiv g(\lfloor \log p \rfloor) \pmod{\pi_p}.$$

Now, for all $p$ sufficiently large, $g(\lfloor \log p \rfloor)$ is not divisible by $p$, since

$$\lim_{x\to\infty} \frac{g(\log x)}{x} = 0.$$

Hence, $\mathbb{P}_{g,\underline{E}}$ is finite.

The fact that $\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\overline{\mathbb{Z}}})$ has class group equal to $G$ follows either by [Chang 2022, Corollary 2.6] or by applying Lemma 2.14 and Proposition 2.10, by noting that $\mathrm{Pic}(\mathrm{Int}_{\mathbb{Q}}(E_p,\overline{\mathbb{Z}_p}))=\mathbb{Z}^{m_p}$ for each $p\in\mathbb{P}\setminus\mathbb{P}_r$ and $\mathrm{Pic}(\mathrm{Int}_{\mathbb{Q}}(E_q,\overline{\mathbb{Z}_q}))=\mathbb{Z}/n_q\mathbb{Z}$ for each $q\in\mathbb{P}_r$.

For the last claim, if $i\in I$, we let $S_i$ be the multiplicative subset of $\mathbb{Z}$ generated by $\mathbb{P}\setminus\mathbb{P}_i$. Then, by Lemma 2.5, $S_i^{-1}\,\mathrm{Int}_{\mathbb{Q}}(\underline{E},\overline{\overline{\mathbb{Z}}})=\mathrm{Int}_{\mathbb{Q}}(\underline{E}_i,\overline{\overline{\mathbb{Z}}})$ which has class group isomorphic to $G_i$ by Lemma 2.14 and Proposition 2.10. □

## Acknowledgments

The author wishes to thank Jean-Luc Chabert for reading a preliminary version of this paper. The author also wishes also to thank the anonymous referees for helpful remarks.

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Received July 9, 2022. Revised April 19, 2023.

GIULIO PERUGINELLI
DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"
UNIVERSITÀ DI PADOVA
PADOVA
ITALY
gperugin@math.unipd.it

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Volume 324 No. 2 June 2023


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