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CATENOID LIMITS OF SINGLY PERIODIC MINIMAL SURFACES WITH SCHERK-TYPE ENDS

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We construct families of embedded, singly periodic minimal surfaces of any genus g in the quotient with any even number $2n > 2$ of almost parallel Scherk ends. A surface in such a family looks like n parallel planes connected by $n - 1 + g$ small catenoid necks. In the limit, the family converges to an n-sheeted vertical plane with $n - 1 + g$ singular points, termed nodes, in the quotient. For the nodes to open up into catenoid necks, their locations must satisfy a set of balance equations whose solutions are given by the roots of Stieltjes polynomials.

Introduction

The goal of this paper is to construct families of singly periodic minimal surfaces (SPMSs) of any genus in the quotient with any even number $2n > 2$ of Scherk ends (asymptotic to vertical planes). Each family is parameterized by a small positive real number $\tau > 0$. In the limit $\tau \to 0$, the Scherk ends tend to be parallel, and the surface converges to an n-sheeted vertical plane with singular points termed nodes. As τ increases, the nodes open up into catenoid necks, and the surface looks like parallel planes connected by these catenoid necks.

There are many previously known examples of such SPMSs. Scherk [1835] discovered examples with genus zero and four Scherk ends. Karcher [1988] generalized Scherk's surface with any even number $2n > 2$ of Scherk ends. In this paper, examples of genus zero will be called "Karcher–Scherk saddle towers" or simply "saddle towers", and saddle towers with four Scherk ends will be called "Scherk saddle towers". Karcher also added handles between adjacent pairs of ends, producing SPMSs of genus n with $2n$ Scherk ends. Traizet glued Scherk saddle towers into SPMSs of genus $(n^2-3n+2)/2$ with $2n > 2$ Scherk ends because he was desingularizing simple arrangements of $n > 1$ vertical planes. Martín and Ramos Batista [2006] replaced the ends of Costa's surface by Scherk ends, thereby constructing an embedded SPMS of genus one with six Scherk ends and, for the first

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time, without any horizontal symmetry plane. Hauswirth, Morabito, and Rodríguez [Hauswirth et al. 2009] generalized this result, using an end-to-end gluing method to replace the ends of Costa–Hoffman–Meeks surfaces by Scherk ends, thereby constructing SPMSs of higher genus with six Scherk ends. Da Silva and Ramos Batista [2010] constructed an SPMS of genus two with eight Scherk ends based on Costa's surface. Also, Yucra Hancco, Lobos, and Ramos Batista [Yucra Hancco et al. 2014] constructed SPMSs with genus $2n$ and $2n$ Scherk ends.

The examples of da Silva and Ramos Batista as well as all examples of Traizet admit catenoid limits that can be constructed using techniques in the present paper.

One motivation of this work is an ongoing project to address various technical details in the gluing constructions.

Roughly speaking, given any "graph" G that embeds in the plane and minimizes the length functional, one could desingularize $G \times \mathbb{R}$ into an SPMS by placing a saddle tower at each vertex. Previously, this was only proved for simple graphs under the assumption of a horizontal reflection plane [Traizet 1996; 2001]. Recently, we managed to allow the graph to have parallel edges, to remove the horizontal reflection plane by Dehn twist [Chen and Traizet 2021], and to prove embeddedness by analyzing the bendings of Scherk ends [Chen 2021].

However, we still require that the vertices of G are neither "degenerate" nor "special". Here, a vertex of degree $2k$ is said to be *degenerate* (resp. *special*) if k (resp. $k-1$) of its adjacent edges extend in the same direction while the other k (resp. $k-1$) edges extend in the opposite direction. This limitation is due to the fact that a saddle tower with $2k$ Scherk ends cannot have $k-1$ ends extending in the same direction while the other $k-1$ ends extend in the opposite direction. Therefore, it is not possible to place a saddle tower at a degenerate or special vertex.

Nevertheless, we do know SPMSs that desingularize $G \times \mathbb{R}$ where G is a graph with a degenerate vertex. To include these in the gluing construction, we need to place catenoid limits of saddle towers, as those constructed in this paper, at degenerate vertices. From this point of view, the present paper can be seen as preparatory: the insight gained here will help us to glue saddle towers with catenoid limits of saddle towers in a future project.

This paper reproduces the main result of the thesis of Li [2012]. Technically, the construction implemented in [Li 2012] was in the spirit of [Traizet 2002b], which defines the Gauss map and the Riemann surface at the same time, and the period of the surface was assumed horizontal. Here, for the convenience of future applications, we present a construction in the spirit of [Traizet 2008; Chen and Traizet 2021; Chen 2021], which defines all three Weierstrass integrands by prescribing their periods, and the period of the surface is assumed vertical. In particular, we will reveal that a balance condition in [Li 2012] is actually a disguise of the balance of Scherk ends: the unit vectors in the directions of the ends sum up to zero.

1. Main result

1.1. ***Configuration.*** We consider $L+1$ vertical planes, $L \geq 1$, labeled by integers $l \in [1, L+1]$. Up to horizontal rotations, we assume that these planes are all parallel to the xz-plane, which we identify as the complex plane $\mathbb{C}$, with the x-axis (resp. z-axis) corresponding to the real (resp. imaginary) axis. We use the term "layer" for the space between two adjacent parallel planes. So there are L layers.

We want $n_l \geq 1$ catenoid necks on layer l, i.e., between the planes l and $l+1$, $1 \leq l \leq L$. For convenience, we adopt the convention that $n_l = 0$ if $l < 1$ or $l > L$, and write $N = \sum n_l$ for the total number of necks. Each neck is labeled by a pair (l, k), where $1 \leq l \leq L$ and $1 \leq k \leq n_l$.

To each neck is associated a complex number $q_{l,k} \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, $1 \leq l \leq L$, $1 \leq k \leq n_l$. Then the positions of the necks are prescribed at $\ln q_{l,k} + 2m\pi \mathrm{i}$, $m \in \mathbb{Z}$. Recall that the z-axis is identified as the imaginary axis of the complex plane $\mathbb{C}$, so the necks are periodic with period vector $(0, 0, 2\pi)$. Note that, if we multiply $q_{l,k}$'s by the same complex factor c, then the necks are all translated by $\ln c \pmod{2\pi \mathrm{i}}$. So we may quotient out translations by fixing $q_{1,1} = 1$.

Also, each plane has two ends asymptotic to vertical planes. We label the end of plane l that expands in the $-x$ (resp. x) direction by 0_l (resp. ∞_l). To be compatible with the language of graph theory that were used for gluing saddle towers [Chen and Traizet 2021], we use

$$\mathsf{H} = \{\eta_l : 1 \leq l \leq L+1,\ \eta \in \{0, \infty\}\}$$

to denote the set of ends. When 0_l is used as subscript for parameter x, we write $x_{l,0}$ instead of x_{0_l} to ease the notation; the same applies to ∞_l.

To each end is associated a real number $\dot\theta_h$, $h \in \mathsf{H}$. They prescribe infinitesimal changes of the directions of the ends. More precisely, for small τ, we want the unit vector in the direction of the end h to have a y-component of order $\tau\dot\theta_h + \mathcal{O}(\tau^2)$.

Remark 1. Multiplying $\dot\theta$ by a common real constant leads to a reparameterization of the family. Adding a common real constant to $\dot\theta_{l,0}$ and subtracting the same constant from $\dot\theta_{l,\infty}$ leads to horizontal rotations of the surface.

In the following, a *configuration* refers to the pair $(q, \dot\theta)$, where

$$q = (q_{l,k})_{\substack{1\leqslant l\leqslant L\\ 1\leqslant k\leqslant n_l}} \quad \text{and} \quad \dot\theta = (\dot\theta_h)_{h\in\mathsf{H}}.$$

1.2. ***Force.*** Given a configuration $(q, \dot\theta)$, let c_l be the real numbers that solve

$$-n_l c_l + n_{l-1} c_{l-1} + \dot\theta_{l,0} + \dot\theta_{l,\infty} = 0, \quad 1 \leq l \leq L+1. \tag{1}$$

Recall the convention $n_l = 0$ if $l < 1$ or $l > L$, so we also adopt the convention $c_l = 0$ if $l < 1$ or $l > L$. A summation over l yields

$$\Theta_1 = \sum_{h \in \mathsf{H}} \dot\theta_h = 0. \tag{2}$$

If (2) is satisfied, the real numbers c_l are determined by (1) as functions of $\dot\theta$.

For $1 \le l \le L+1$, let ψ_l be the meromorphic 1-form on the Riemann sphere $\hat{\mathbb{C}}$ with simple poles at $q_{l,k}$ with residue $-c_l$ for each $1 \le k \le n_l$, at $q_{l-1,k}$ with residue c_{l-1} for each $1 \le k \le n_{l-1}$, at 0 with residue $\dot\theta_{l,0}$, and at ∞ with residue $\dot\theta_{l,\infty}$. More explicitly,

$$\psi_l = \left(\sum_{k=1}^{n_l} \frac{-c_l}{z - q_{l,k}} + \sum_{k=1}^{n_{l-1}} \frac{c_{l-1}}{z - q_{l-1,k}} + \frac{\dot\theta_{l,0}}{z} \right) dz.$$

We then see that (1) arises from the residue theorem.

Remark 2. In the definition of configuration, we may replace $\dot\theta$ by the parameters $(c_l, \dot\theta_{l+1,0} - \dot\theta_{l,0})_{1 \le l \le L}$. Then $\dot\theta_{l,0}$'s are defined up to an additive constant (corresponding to a rotation), $\dot\theta_{l,\infty}$'s are determined by (1), and (2) is automatically satisfied. To quotient out reparameterizations of the family, we may assume that $c_l = 1$ for some $1 \le l \le L$.

We define the force $F_{l,k}$ by

$$F_{l,k} = \operatorname{Res}\left(\frac{\psi_l^2 + \psi_{l+1}^2}{2} \frac{z}{dz}, q_{l,k} \right). \tag{3}$$

Or, more explicitly,

$$\begin{aligned} F_{l,k} = \sum_{1 \le k \ne j \le n_l} \frac{2c_l^2 q_{l,k}}{q_{l,k} - q_{l,j}} &- \sum_{1 \le j \le n_{l+1}} \frac{c_l c_{l+1} q_{l,k}}{q_{l,k} - q_{l+1,j}} \\ &- \sum_{1 \le j \le n_{l-1}} \frac{c_l c_{l-1} q_{l,k}}{q_{l,k} - q_{l-1,j}} + c_l^2 + c_l(\dot\theta_{l+1,0} - \dot\theta_{l,0}). \end{aligned} \tag{4}$$

In [Li 2012], the force had two different formulas depending on the parity of l. One verifies that both are equivalent to (4).

Remark 3 (electrostatic interpretation). The force equation (4) can be expressed as

$$\begin{aligned} F_{l,k} = \sum_{1 \le k \ne j \le n_l} \frac{c_l^2 (q_{l,k} + q_{l,j})}{q_{l,k} - q_{l,j}} &- \sum_{1 \le j \le n_{l+1}} \frac{c_l c_{l+1} (q_{l,k} + q_{l+1,j})}{2(q_{l,k} - q_{l+1,j})} \\ - \sum_{1 \le j \le n_{l-1}} \frac{c_l c_{l-1} (q_{l,k} + q_{l-1,j})}{2(q_{l,k} - q_{l-1,j})} &+ \frac{c_l}{2} (\dot\theta_{l,\infty} - \dot\theta_{l,0} - \dot\theta_{l+1,\infty} + \dot\theta_{l+1,0}). \end{aligned}$$

Note that

$$\frac{a+b}{a-b} = \coth\frac{\ln a - \ln b}{2} = \frac{2}{\ln a - \ln b} + \sum_{m=1}^{\infty}\left(\frac{2}{\ln a - \ln b - 2m\pi\mathrm{i}} + \frac{2}{\ln a - \ln b + 2m\pi\mathrm{i}}\right).$$

Disregarding absolute convergence, we write this formally as

$$\frac{a+b}{a-b} = \sum_{m\in\mathbb{Z}} \frac{2}{\ln a - \ln b - 2m\pi\mathrm{i}}.$$

Then the force is given, formally, by

$$\begin{aligned} F_{l,k} = \sum_{0\neq m\in\mathbb{Z}} \frac{2c_l^2}{2m\pi\mathrm{i}} &+ \sum_{\substack{m\in\mathbb{Z}\\ 1\le k\neq j\le n_l}} \frac{2c_l^2}{\ln q_{l,k} - \ln q_{l,j} - 2m\pi\mathrm{i}} \\ &- \sum_{\substack{m\in\mathbb{Z}\\ 1\le j\le n_{l+1}}} \frac{c_l c_{l+1}}{\ln q_{l,k} - \ln q_{l+1.j} - 2m\pi\mathrm{i}} - \sum_{\substack{m\in\mathbb{Z}\\ 1\le j\le n_{l-1}}} \frac{c_l c_{l-1}}{\ln q_{l,k} - \ln q_{l-1.j} - 2m\pi\mathrm{i}} \\ &+ \frac{c_l}{2}(\dot\theta_{l,\infty} - \dot\theta_{l,0} - \dot\theta_{l+1,\infty} + \dot\theta_{l+1,0}). \end{aligned}$$

Recall that $\ln q_{l,k} + 2m\pi\mathrm{i}$ are the real positions of the necks. So this formal expression has an electrostatic interpretation similar to those in [Traizet 2002b; 2008]. Here, each neck interacts not only with all other necks in the same or adjacent layers, but also with background constant fields given by $\dot\theta$.

Remark 4 (another electrostatic interpretation). In fact, (4)$/q_{l,k}$ has a similar electrostatic interpretation. But this time, the necks are seen as placed at $q_{l,k}$. Each neck interacts with all other necks in the same and adjacent layers, as well as a virtual neck at 0 with "charge" $c_l + \dot\theta_{l+1,0} - \dot\theta_{l,0}$. This is no surprise, as electrostatic laws are known to be preserved under conformal mappings (such as $\ln z$).

1.3. ***Main result.*** In the following, we write $F = (F_{l,k})_{1\le l\le L, 1\le k\le n_l}$.

Definition 5. The configuration is *balanced* if $F = 0$ and $\Theta_1 = 0$.

Summing up all forces yields a necessary condition for the configuration to be balanced, namely

$$\begin{aligned} \Theta_2 = \sum_{\substack{1\le l\le L\\ 1\le k\le n_l}} F_{l,k} &= \sum_{1\le l\le L+1} -\frac{1}{2}\left(\mathrm{Res}\left(\frac{z\psi_l^2}{dz}, 0\right) + \mathrm{Res}\left(\frac{z\psi_l^2}{dz}, \infty\right)\right) \\ &= \sum_{1\le l\le L+1} \frac{\dot\theta_{l,\infty}^2 - \dot\theta_{l,0}^2}{2} = 0. \end{aligned}$$

Lemma 6. *The Jacobian matrix* $\partial(\Theta_1, \Theta_2)/\partial\dot\theta$ *has real rank* 2 *as long as* $c_l \neq 0$ *for some* $1 \le l \le L$.

The assumption of the lemma simply says that the surface does not remain a degenerate plane to the first order.

Proof. The proposition says that the matrix has an invertible minor of size 2×2. Explicitly, we have

$$\frac{\partial(\Theta_1, \Theta_2)}{\partial(\dot\theta_{l,0}, \dot\theta_{l,\infty})} = \begin{pmatrix} 1 & 1 \\ -\dot\theta_{l,0} & \dot\theta_{l,\infty} \end{pmatrix}.$$

This minor is invertible if and only if $\dot\theta_{l,0} + \dot\theta_{l,\infty}$ does not equal 0. This must be the case for at least one $1 \le l \le L$ because, otherwise, we have $c_l = 0$ for all $1 \le l \le L$. □

Definition 7. The configuration is *rigid* if the complex rank of $\partial F/\partial q$ is $N - 1$.

Remark 8. In fact, the complex rank of $\partial F/\partial q$ is at most $N-1$. We have seen that a complex scaling of q corresponds to a translation of $\ln q_{l,k} + 2m\pi\mathrm{i}$, $m \in \mathbb{Z}$, which does not change the force. It then makes sense to normalize q by fixing $q_{1,1} = 1$.

Theorem 9. *Let $(q, \dot\theta)$ be a balanced and rigid configuration such that $c_l \neq 0$ for $1 \le l \le L$. Then for $\tau > 0$ sufficiently small, there exists a smooth family M_τ of complete singly periodic minimal surfaces of genus $g = N - L$, period $(0, 0, 2\pi)$, and $2(L + 1)$ Scherk ends such that, as $\tau \to 0$:*

- *M_τ converges to an $(L+1)$-sheeted xz-plane with singular points at*

  $$\ln q_{l,k} + 2m\pi\mathrm{i}, \quad m \in \mathbb{Z}.$$

  Here, the xz-plane is identified as the complex plane $\mathbb{C}$, with the x-axis (resp. z-axis) identified as the real (resp. imaginary) axis.
- *After suitable scaling and translation, each singular point opens up into a neck that converges to a catenoid.*
- *The unit vector in the direction of each Scherk end h has the y-component $\tau\dot\theta_h + \mathcal{O}(\tau^2)$.*

Also, M_τ is embedded if

$$\dot\theta_{1,0} > \cdots > \dot\theta_{L+1,0} \quad \textit{and} \quad \dot\theta_{1,\infty} > \cdots > \dot\theta_{L+1,\infty}. \tag{5}$$

Remark 10. The family M_τ also depends smoothly on $\dot\theta$ belonging to the local smooth manifold defined by $\Theta_1 = 0$ and $\Theta_2 = 0$. Up to reparameterizations of the family and horizontal rotations, we obtain families parameterized by $2L - 1$ parameters. Since we have $2(L + 1)$ Scherk ends, this parameter count is compatible with the fact that Karcher–Scherk saddle towers with $2k$ ends form a family parameterized by $2k - 3$ parameters.

Remark 11. If the embeddedness condition (5) is satisfied and $\Theta_1 = 0$, the sequence $\dot\theta_{l,0} + \dot\theta_{l,\infty}$ is strictly monotonically decreasing, and changes sign once and only once. Then the sequence $n_l c_l$ is strictly concave (that is, $n_{l-1}c_{l-1} + n_{l+1}c_{l+1} < 2n_l c_l$ for $1 \le l \le L$). Hence c_l, $1 \le l \le L$, are strictly positive, and the condition of Lemma 6 is satisfied.

Remark 12. We could allow some c_l to be negative, with the price of losing embeddedness. Even worse, with negative c_l, the vertical planes in the limit will not be geometrically ordered as they are labeled. For instance, if $L = 2$, $c_1 > 0$, but $c_2 < 0$, then the catenoid necks, as well as the first and third "planes", will all lie on the same side of the second "plane".

Remark 13. We did not allow any c_l to be 0 in Theorem 9. Otherwise, the surface might still have nodes. In that case, the claimed family might not be smooth, and the claimed genus would be incorrect.

2. Examples

2.1. ***Surfaces of genus zero.*** When the genus satisfies $g = N - L = 0$, we have $n_l = 1$ for all $1 \le l \le L$, i.e., there is only one neck on every layer. It then makes sense to drop the subscript k. For instance, the position and the force for the neck on layer l are simply denoted by q_l and F_l, respectively. We assume $L > 1$ in this part.

In this case, if $\Theta_1 = 0$, (1) can be explicitly solved by

$$c_l = \sum_{i=1}^{l} (\dot\theta_{i,0} + \dot\theta_{i,\infty}), \quad 1 \le l \le L,$$

and the force can be written in the form

$$F_l = -\widetilde{Q}_l + \widetilde{Q}_{l-1} + c_l(\dot\theta_{l,\infty} + \dot\theta_{l+1,0}), \quad 1 \le l \le L,$$

where we changed to the parameters

$$\widetilde{Q}_l = \frac{c_{l+1}c_l}{1 - q_{l+1}/q_l}, \quad 1 \le l < L,$$

with the convention that $\widetilde{Q}_0 = \widetilde{Q}_L = 0$. Then the forces are linear in $\widetilde{Q}$ and, if $\Theta_2 = 0$, the balance condition $F = 0$ is uniquely solved by

$$\widetilde{Q}_l = \sum_{i=1}^{l} c_i(\dot\theta_{i+1,0} + \dot\theta_{i,\infty}) = -\sum_{i=l+1}^{L} c_i(\dot\theta_{i+1,0} + \dot\theta_{i,\infty}), \quad 1 \le l < L. \tag{6}$$

Therefore, if we fix $q_1 = 1$, all other q_l, $1 < l \le L$, are uniquely determined.

Recall from Remark 11 that, under the embeddedness condition (5), the numbers c_l, $1 \le l \le L$, are positive. Furthermore, the summands in (6) change sign at most once, so the sequence $\widetilde{Q}$ is unimodal, i.e., there exists $1 \le l' < L$ such that

$$0 = \widetilde{Q}_0 \le \widetilde{Q}_1 \le \cdots \le \widetilde{Q}_{l'} \ge \cdots \ge \widetilde{Q}_{L-1} \ge \widetilde{Q}_L = 0.$$

Hence $\widetilde{Q}_l$, $1 \le l \le L$, are nonnegative. Lastly,

$$\widetilde{Q}_l < \sum_{i=1}^{l} c_i(\dot\theta_{i,0} + \dot\theta_{i,\infty}) = \sum_{i=1}^{l}(c_i^2 - c_{i-1}c_i) \le c_l^2 \le c_{l+1}c_l \qquad \text{if } l < l',$$

$$\widetilde{Q}_l < -\sum_{i=l+1}^{L} c_i(\dot\theta_{i+1,0} + \dot\theta_{i+1,\infty}) = \sum_{i=l+1}^{L}(c_i^2 - c_{i+1}c_i) \le c_{l+1}^2 \le c_{l+1}c_l \quad \text{if } l \ge l'.$$

So q consists of real numbers and $q_{l+1}/q_l < 0$ for all $1 \le l < L$.

We have proved the following:

Proposition 14. *If the genus satisfies $g = N - L = 0$, and $\dot\theta$ satisfies the balancing condition $\Theta_1 = \Theta_2 = 0$ as well as the embeddedness condition (5), then up to complex scalings, there exist unique values for the parameters q, depending analytically on $\dot\theta$, such that the configuration $(q, \dot\theta)$ is balanced. All such configurations are rigid. If we fix $q_1 = 1$, then q consist of real numbers, and we have $q_l > 0$ (resp. < 0) if l is odd (resp. even).*

2.2. ***Surfaces with four ends.*** When $L = 1$, $\Theta_1 = \Theta_2 = 0$ implies that

$$\dot\theta_{1,0} + \dot\theta_{2,\infty} = \dot\theta_{2,0} + \dot\theta_{1,\infty} = 0.$$

Up to reparameterizations of the family, we may assume that $c_1 = 1$. It makes sense to drop the subscript l, and write F_k for $F_{1,k}$, q_k for $q_{1,k}$, and n for n_1. The goal of this part is to prove the following classification result.

Proposition 15. *Up to a complex scaling, a configuration with $L = 1$ and n nodes must be given by $q_k = \exp(2\pi \mathrm{i} k/n)$, and such a configuration is rigid.*

Such a configuration is an n-covering of the configuration for Scherk saddle towers. As a consequence, the arising minimal surfaces are n-coverings of Scherk saddle towers. This is compatible with the result of [Meeks and Wolf 2007] that the Scherk saddle towers are the only connected SPMSs with four Scherk ends.

Proof. To find the positions q_k such that

$$F_k = \sum_{1 \le k \ne j \le n} \frac{2q_k}{q_k - q_j} - (n-1) = 0, \quad 1 \le k \le n, \tag{7}$$

we use the polynomial method. Consider the polynomial

$$P(z) = \prod_{k=1}^{n}(z - q_k).$$

Then we have

$$\begin{aligned}
P' &= P\sum_{k=1}^{n}\frac{1}{z-q_k},\\
P'' &= P\sum_{k=1}^{n}\sum_{1\le k\ne j\le n}\frac{1}{z-q_j}\frac{1}{z-q_k} = 2P\sum_{k=1}^{n}\frac{1}{z-q_k}\sum_{1\le k\ne j\le n}\frac{1}{q_k-q_j}\\
&= P\sum_{k=1}^{n}\frac{n-1}{q_k(z-q_k)} = (n-1)P\sum_{k=1}^{n}\frac{1}{z}\left(\frac{1}{q_k}+\frac{1}{z-q_k}\right) \quad \text{(by (7))}\\
&= \frac{n-1}{z}\left(P' - \frac{P'(0)}{P(0)}P\right).
\end{aligned}$$

For the last equation to have a polynomial solution, we must have $P'(0) = 0$. Otherwise, the left-hand side would be a polynomial of degree $n-2$, but the right-hand side would be a polynomial of degree $n-1$.

Consequently, $F_k = 0$ if and only if

$$zP''(z) - (n-1)P'(z) = 0,$$

which, up to a complex scaling, is uniquely solved by

$$P(z) = z^n - 1.$$

So a balanced 4-end configuration must be given by the roots of unity $q_k = \exp(2\pi \mathrm{i} k/n)$, $0 \le k \le n-1$.

We now verify that the configuration is rigid. For this purpose, we compute

$$\frac{\partial F_k}{\partial q_j} = \begin{cases} 2\frac{q_k}{(q_k-q_j)^2}, & j \ne k,\\ 2\sum_{1\le k\ne i\le n}\frac{-q_i}{(q_k-q_i)^2}, & j = k.\end{cases}$$

Note that $\sum_{j=1}^{n} q_j \partial F_k/\partial q_j = 0$ while

$$q_j\frac{\partial F_k}{\partial q_j} = 2\frac{q_j q_k}{(q_k-q_j)^2} = 2\frac{e^{2\pi \mathrm{i}\frac{j+k}{n}}}{(e^{2\pi \mathrm{i}\frac{j}{n}} - e^{2\pi \mathrm{i}\frac{k}{n}})^2} \in \mathbb{R}_{<0}$$

when $j \ne k$, so the matrix

$$\frac{\partial F}{\partial q}\operatorname{diag}(q_1,\dots,q_n)$$

has real entries, has a kernel of complex dimension 1 (spanned by the all-one vector), and any of its principal submatrices are diagonally dominant. We then conclude that the matrix, as well as the Jacobian $\partial F/\partial q$, has a complex rank $n-1$. This finishes the proof of rigidity. □

Remark 16. The perturbation argument as in the proof of [Traizet 2002b, Proposition 1] also applies here, word by word, to prove the rigidity.

2.3. ***Gluing two saddle towers of different periods.*** We want to construct a smooth family of configurations depending on a positive real number λ such that, for small λ, the configuration looks like two columns of nodes far away from each other, one with period $2\pi/n_1$, and the other with period $2\pi/n_2$. If balanced and rigid, these configurations would give rise to minimal surfaces that look like two Scherk saddle towers with different periods that are glued along a pair of ends. The construction is in the same spirit as [Traizet 2002b, §2.5; 2008, §4.3.4].

Proposition 17. *For a real number $\lambda > 0$ sufficiently small, there are balanced and rigid configurations $(q(\lambda), \dot\theta(\lambda))$ with $L = 2$ depending smoothly on λ such that, at $\lambda = 0$,*

$$\frac{q_{2,j}}{q_{1,k}} = 0, \quad 1 \le k \le n_1, 1 \le j \le n_2.$$

Up to a complex scaling and reparameterization, we may fix $q_{1,1} = 1$, and write $q_{2,1} = \lambda \exp(\mathrm{i}\phi)$. Then, at $\lambda = 0$, we have

$$\dot\theta_{1,0} + \dot\theta_{2,\infty} = \dot\theta_{2,0} + \dot\theta_{3,\infty} = \dot\theta_{3,0} + \dot\theta_{1,\infty} = 0 \tag{8}$$

and,

$$q_{1,k} = \exp\left(\frac{k-1}{n_1} 2\pi \mathrm{i}\right), \quad 1 \le k \le n_1,$$
$$\widetilde{q}_{2,k} := q_{2,k}/q_{2,1} = \exp\left(\frac{k-1}{n_2} 2\pi \mathrm{i}\right), \quad 1 \le k \le n_2,$$

where $\phi\,\mathrm{lcm}(n_1, n_2)$ is necessarily a multiple of π.

In other words, the construction only works if the configuration admits a reflection symmetry.

Remark 18. H. Chen was shown a video suggesting that, when two Scherk saddle towers are glued into a minimal surface, one can slide one saddle tower with respect to the other while the surface remains minimal. The proposition above suggests that this is not possible.

In fact, the family of configurations also depends on $\dot\theta$ belonging to the local manifold defined by $\Theta_1 = \Theta_2 = 0$ and (one equation from) (8). Up to rotations of the configuration and reparameterizations of the family of minimal surfaces, the family of configurations is parameterized, as expected, by two parameters.

Proof. Let us first study the situation at $\lambda = 0$. We compute, at $\lambda = 0$,

$$\begin{aligned}
\frac{F_{1,k}}{c_1^2} &= \sum_{1\le k\ne j\le n_1} \frac{2q_{1,k}}{q_{1,k}-q_{1,j}} - \sum_{1\le j\le n_2} \frac{c_2}{c_1}\frac{q_{1,k}}{q_{1,k}-q_{2,j}} + 1 + \frac{\dot\theta_{2,0}-\dot\theta_{1,0}}{c_1} \\
&= \sum_{1\le k\ne j\le n_1} \frac{2q_{1,k}}{q_{1,k}-q_{1,j}} - n_2\frac{c_2}{c_1} + 1 + \frac{\dot\theta_{2,0}-\dot\theta_{1,0}}{c_1}, \\
\frac{F_{2,k}}{c_2^2} &= \sum_{1\le k\ne j\le n_2} \frac{2q_{2,k}}{q_{2,k}-q_{2,j}} - \sum_{1\le j\le n_1} \frac{c_1}{c_2}\frac{q_{2,k}}{q_{2,k}-q_{1,j}} + 1 + \frac{\dot\theta_{3,0}-\dot\theta_{2,0}}{c_2} \\
&= \sum_{1\le k\ne j\le n_2} \frac{2q_{2,k}}{q_{2,k}-q_{2,j}} + 1 + \frac{\dot\theta_{3,0}-\dot\theta_{2,0}}{c_2}.
\end{aligned}$$

Write $G_l = \sum_k F_{l,k}$. Summing the above over k gives, at $\lambda = 0$,

$$\frac{1}{n_1}\frac{G_1}{c_1^2} = n_1 - n_2\frac{c_2}{c_1} + \frac{\dot\theta_{2,0}-\dot\theta_{1,0}}{c_1}, \qquad \frac{1}{n_2}\frac{G_2}{c_2^2} = n_2 + \frac{\dot\theta_{3,0}-\dot\theta_{2,0}}{c_2}.$$

So $G_1 = G_2 = 0$, at $\lambda = 0$, only if

$$\begin{aligned}
0 &= -(\dot\theta_{2,0}+\dot\theta_{3,\infty}) = \dot\theta_{3,0}-\dot\theta_{2,0}+n_2c_2 \\
&= -(\dot\theta_{1,0}+\dot\theta_{2,\infty}) = n_1c_1 - n_2c_2 + \dot\theta_{2,0}-\dot\theta_{1,0}.
\end{aligned}$$

This together with $\Theta_1 = 0$ proves (8).

Now assume that (8) is satisfied. Then we have, at $\lambda = 0$,

$$\begin{aligned}
\frac{F_{1,k}}{c_1^2} &= \sum_{1\le k\ne j\le n_1} \frac{2q_{1,k}}{q_{1,k}-q_{1,j}} - (n_1-1), \\
\frac{F_{2,k}}{c_2^2} &= \sum_{1\le k\ne j\le n_2} \frac{2q_{2,k}}{q_{2,k}-q_{2,j}} - (n_2-1).
\end{aligned}$$

These expressions are identical to the force (7) for single layer configurations. So we know for $l = 1, 2$ that, at $\lambda = 0$, the configuration is balanced only if

$$\widetilde{q}_{l,k} := \frac{q_{l,k}}{q_{l,1}} = \exp\left(\frac{k-1}{n_l}2\pi\mathrm{i}\right).$$

Up to complex scaling, we may fix $q_{1,1} = 1$ so $\widetilde{q}_{1,k} = q_{1,k}$. And up to reparameterization of the family (of configurations), we write $q_{2,1} = \lambda\exp(\mathrm{i}\phi)$.

Now assume these initial values for $\widetilde{q}_{l,k}$. Then we have, at $\lambda = 0$,

$$\frac{G_2}{c_1 c_2} = -\sum_{k=1}^{n_2}\sum_{j=1}^{n_1} \frac{q_{2,k}}{q_{2,k} - q_{1,j}} = \sum_{k=1}^{n_2}\sum_{j=1}^{n_1}\sum_{m=1}^{\infty} \left(\frac{q_{2,k}}{q_{1,j}}\right)^m$$
$$= \sum_{k=1}^{n_2}\sum_{j=1}^{n_1}\sum_{m=1}^{\infty} q_{2,1}^m \exp\left(2mi\pi\left(\frac{k-1}{n_2} - \frac{j-1}{n_1}\right)\right).$$

Seen as a power series of $q_{2,1}$, the coefficient for $q_{2,1}^m$ is

$$\sum_{k=1}^{n_2}\sum_{j=1}^{n_1} \exp\left(2mi\pi\left(\frac{k-1}{n_2} - \frac{j-1}{n_1}\right)\right).$$

It is nonzero only if m is a common multiple of n_1 and n_2, in which case the coefficient of $q_{2,1}^m$ equals $n_1 n_2$. In particular, let $\mu = \operatorname{lcm}(n_1, n_2)$; then, at $\lambda = 0$,

$$\operatorname{Im} \frac{G_2}{\lambda^\mu} = c_1 c_2 n_1 n_2 \sin(\mu\phi) \tag{9}$$

vanishes if and only if $\mu\phi$ is a multiple of π.

Now we use the implicit function theorem to find balanced configurations with $\lambda > 0$. From the proof for Proposition 15, we know that $(\partial F_{l,k}/\partial \widetilde{q}_{l,j})_{2\le j,k\le n_l}$, $l = 1, 2$, are invertible. Hence for λ sufficiently small, there exist unique values for $(\widetilde{q}_{l,k})_{l=1,2;2\le k\le n_l}$, depending smoothly on λ, $\dot\theta$, and ϕ, where $(F_{l,k})_{l=1,2;2\le k\le n_l} = 0$. By (9), there exists a unique value for ϕ, depending smoothly on λ and $\dot\theta$, such that $\operatorname{Im} G_2/\lambda^\mu = 0$. Note also that $\operatorname{Re} G_2$ is linear in $\dot\theta$. By Lemma 6, the solutions $(\lambda, \dot\theta)$ to $\operatorname{Re} G_2 = 0$ and $\Theta_1 = \Theta_2 = 0$ form a manifold of dimension 4 (including multiplication by common real factor on $\dot\theta$ and rotation of the configuration). Finally, we have $G_1 = 0$ by the residue theorem, and the balance is proved.

For the rigidity of the configurations with sufficiently small λ, we need to prove that the matrix

$$\begin{pmatrix} \left(\frac{\partial F_{1,k}}{\partial q_{1,j}}\right)_{2\le j,k\le n_1} & & \\ & \left(\frac{\partial F_{2,k}}{\partial \widetilde{q}_{2,j}}\right)_{2\le j,k\le n_2} & \\ & & \frac{\partial G_2}{\partial q_{2,1}} \end{pmatrix}$$

is invertible. We know that the first two blocks are invertible at $\lambda = 0$. By continuity, they remain invertible for λ sufficiently small. The last block is clearly nonzero for $\lambda \neq 0$ sufficiently small. □

2.4. ***Surfaces with six ends of type (n, 1).*** In this section, we investigate examples with $L = 2$ (hence six ends), $n_1 = n$, $n_2 = 1$. Up to a reparameterization of the family, we may assume that $c_1 = 1$. Up to a complex scaling, we may assume that $q_{2,1} = 1$.

We will prove that the $q_{1,k}$'s are given by the roots of hypergeometric polynomials. Let us first recall their definitions. A *hypergeometric function* is defined by

$$_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

with $a, b, c \in \mathbb{C}$, c is not a nonpositive integer,

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)},$$

and $(a)_0 = 1$. The hypergeometric function $w = {}_2F_1(a,b;c;z)$ solves the *hypergeometric differential equation*

$$z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0. \tag{10}$$

If $a = -n$ is a negative integer,

$$_2F_1(-n,b;c;z) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(b)_k}{(c)_k} z^k$$

is a polynomial of degree n, and is referred to as a *hypergeometric polynomial.*

Proposition 19. *Let $(q,\dot\theta)$ be a balanced configuration with $L=2$, $c_1=1$, $n_1=n$, $n_2=1$. Then, up to a complex scaling, we have $q_{2,1}=1$ and $(q_{1,k})_{1\le k\le n}$ are the roots of the hypergeometric polynomial ${}_2F_1(-n,b;c;z)$ with*

$$b := n - c_2 + \dot\theta_{2,0} - \dot\theta_{1,0}, \quad c := 1 + \dot\theta_{2,0} - \dot\theta_{1,0}.$$

As long as b and c are not nonpositive integers, and $c-b$ is not a nonpositive integer bigger than $-n$, the configuration is rigid.

Proof. The force equations are

$$F_{1,k} = \sum_{1\le k\ne j\le n} \frac{2q_{1,k}}{q_{1,k}-q_{1,j}} - \frac{q_{1,k}c_2}{q_{1,k}-1} + c, \quad 1\le k\le n,$$

$$F_{2,1} = -\sum_{j=1}^{n} \frac{c_2}{1-q_{1,j}} + c_2^2 + c_2(\dot\theta_{3,0} - \dot\theta_{2,0}),$$

where $c := 1 + \dot\theta_{2,0} - \dot\theta_{1,0}$. To solve $F_{1,k}=0$ for $k=1,2,\ldots,n$, we use again the polynomial method. Let

$$P(z) = \prod_{k=1}^{n} (z - q_{1,k}).$$

Then we have

$$\begin{aligned}
P' &= P \sum_{k=1}^{n} \frac{1}{z-q_{1,k}};\\
P'' &= 2P \sum_{k=1}^{n} \frac{1}{z-q_{1,k}} \sum_{1\le k\ne j\le n} \frac{1}{q_{1,k}-q_{1,j}}\\
&= P \sum_{k=1}^{n} \frac{1}{(z-q_{1,k})}\left(\frac{c_2}{q_{1,k}-1} - \frac{c}{q_{1,k}}\right) \quad (\text{by } F_{1,k}=0)\\
&= P \sum_{k=1}^{n}\left(\frac{c_2}{(z-1)(z-q_{1,k})} + \frac{c_2}{(z-1)(q_{1,k}-1)} - \frac{c}{z(z-q_{1,k})} - \frac{c}{zq_{1,k}}\right).
\end{aligned}$$

So the configuration is balanced if and only if

$$P'' + \left(\frac{-c_2}{z-1} + \frac{c}{z}\right)P' + \left(\frac{c_2}{z-1}\frac{P'(1)}{P(1)} - \frac{c}{z}\frac{P'(0)}{P(0)}\right)P = 0. \tag{11}$$

Define

$$b := n-1-c_2+c.$$

For (11) to have a polynomial solution of degree n, we must have

$$c_2\frac{P'(1)}{P(1)} = c\frac{P'(0)}{P(0)} = -nb,$$

so that the leading coefficients cancel. Then (11) becomes the hypergeometric differential equation

$$z(1-z)P'' + [c-(-n+b+1)z]P' + nbP = 0$$

to which the only polynomial solution (up to a multiplicative constant) is given by the hypergeometric polynomial $P(z) = {}_2F_1(-n, b; c; z)$ of degree n.

Furthermore, in order for $F_{2,1}=0$, we must have

$$\dot\theta_{3,0} - \dot\theta_{2,0} = \sum_{j=1}^{n} \frac{1}{1-q_{1,j}} - c_2 = \frac{P'(1)}{P(1)} - c_2 = -\frac{nb}{c_2} - c_2. \tag{12}$$

Note that b and c are real. If b is not a nonpositive integer, and $c-b$ is not a nonpositive integer bigger than $-n$, then all the n roots of $P(z) = {}_2F_1(-n, b; c; z)$ are simple. Indeed, under these assumptions, we have $P(0) = 1$ and $P(1) = (c-b)_n/(c)_n \ne 0$ by the Chu–Vandermonde identity. If z_0 is a root of $P(z)$, then $z_0 \ne 0, 1$. In view of the hypergeometric differential equation, if z_0 is not simple, we have $P(z_0) = P'(z_0) = 0$; hence $P(z) \equiv 0$ by the uniqueness theorem.

The rigidity means that no perturbation of $q_{1,k}$ preserve the balance to the first order. To prove this fact, we use a perturbation argument similar to that in the proof of [Traizet 2002b, Proposition 1].

Let $(q_{1,k}(t))_{1\le k\le n}$ be a deformation of the configuration such that $q_{1,k}(0)=q_{1,k}$ and $(\dot F_{1,k}(0))_{1\le k\le n}=0$, where dot denotes derivative with respect to t. Define

$$P_t(z)=\sum_{j=0}^{n}a_j(t)z^j:=\prod_{k=1}^{n}(z-q_{1,k}(t)).$$

Then we have

$$z(1-z)P_t''+[c-(-n+b+1)z]P_t'+nbP_t=o(t),$$

meaning that the coefficients from the left side are all $o(t)$. So the coefficients of P_t must satisfy

$$(b+j)(n-j)a_j(t)+(j^2+j+cj)a_{j+1}(t)=o(t),\quad 0\le j\le n. \tag{13}$$

Note that $P_t(z)$ is monic by definition, meaning that $a_n(t)\equiv 1$. Since b and c are not nonpositive integers, we conclude that $a_j(t)=o(t)$ for all $0\le j\le n$. The simple roots depend analytically on the coefficients, so $q_{1,k}(t)=q_{1,k}+o(t)$. □

The simple roots of ${}_2F_1(-n,b;c;z)$ are either real or form conjugate pairs. As a consequence, if rigid, the configurations in the proposition above will give rise to minimal surfaces with horizontal symmetry planes.

Example 20. For each integer $n\ge 2$, Dominici, Johnston, and Jordaan [Dominici et al. 2013] enumerated the real parameters (b,c) for which ${}_2F_1(-n,b;c;z)$ has only real simple roots. The results are plotted in blue in Figure 1. The embeddedness conditions (5) are

$$\begin{aligned}
\dot\theta_{1,0}>\dot\theta_{2,0} &\implies c<1,\\
\dot\theta_{1,\infty}>\dot\theta_{2,\infty} &\implies b>-n,\\
\dot\theta_{2,0}>\dot\theta_{3,0} &\implies c_2^2>-nb,\\
\dot\theta_{2,\infty}>\dot\theta_{3,\infty} &\implies c_2^2>n(c_2+b),
\end{aligned}$$

where $c_2=n-1-b+c$. The region defined by these is plotted in red in Figure 1. Then noninteger parameters (b,c) in the intersection of red and blue regions give rise to balanced and rigid configurations with real $q_{1,k}$.

Figure 2 shows the configurations of three examples with $n=5$. □

Remark 21. As $c\to 0$, ${}_2F_1(-n,b;c;z)/\Gamma(c)$ converges to a polynomial with a root at 0. One may interpret that, as c increases across 0, a root moves from the interval $(-\infty,0)$ to the interval $(0,1)$ through 0.

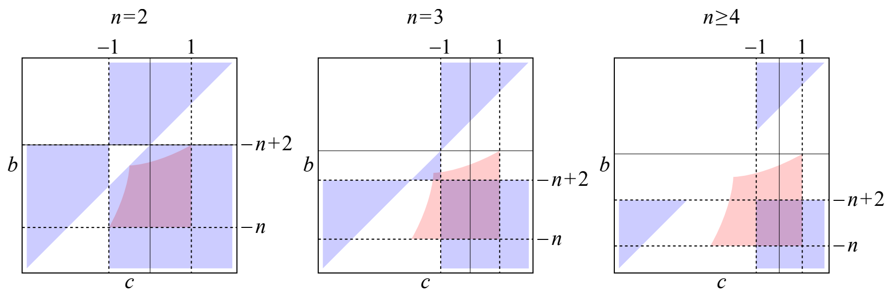

Figure 1. The set of b and c for which ${}_2F_1(-n, b; c; z)$ has only real simple roots (blue), and for which the embeddedness conditions are satisfied (red).

When $b = 1 - n$, ${}_2F_1(-n, b; c; z)$ becomes a polynomial of degree $n-1$. One may interpret that, as b increases across $1 - n$, a root moves from the interval $(-\infty, 0)$ to the interval $(1, \infty)$ through the infinity.

Example 22. Assume that $b + c = 1 - n$ (hence $c_2 = -2b$). Then by the identity

$$ {}_2F_1(-n, b; c; z) = \frac{(b)_n}{(c)_n}(-z)^n {}_2F_1\left(-n, 1 - c - n; 1 - b - n; \frac{1}{z}\right), $$

the simple roots must be symmetrically placed. That is, if z_0 is a root, so is $1/z_0$. This symmetry appears in the resulting minimal surfaces as a rotational symmetry. If the simple roots are real, the rotation reduces to a vertical reflectional. In view of Figure 1, we obtain the following concrete examples.

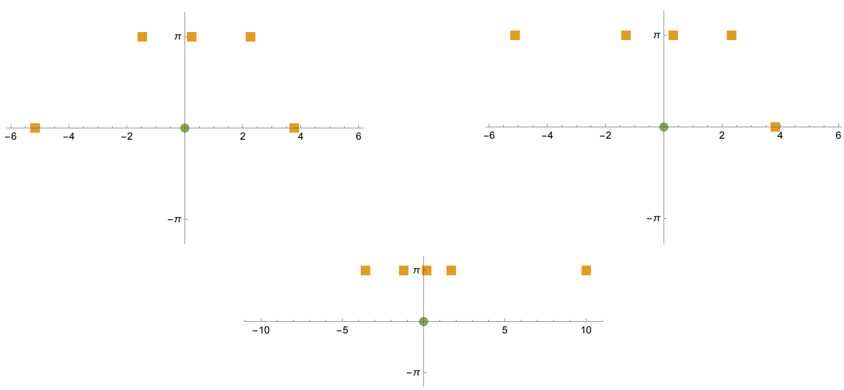

Figure 2. (5, 1) balanced configurations with $b = -3.4$, $c = -0.1$ (top left), $b = -3.4$, $c = 0.1$ (top right), and $b = -4.001$, $c = 0.5$ (bottom). The circles and squares represent the necks at levels one and two, respectively.

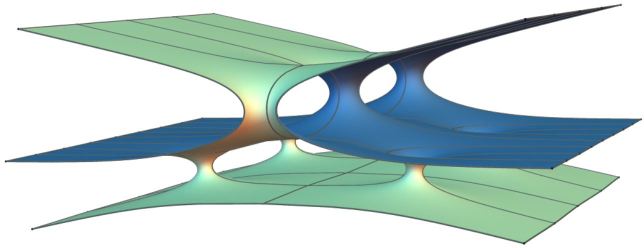

Figure 3. Genus one example with $n = 2$ and $0 < c < 1$.

- $n \geq 2$ and $0 < c < 1$. In this case ${}_2F_1(-n, b; c; z)$ has n simple negative roots. See Figure 3 for an example of this type with $n = 2$. Figure 4 shows the configurations of two examples with $n = 5$.
- $n \geq 3$ and $-1 < c < 0$, or $n = 3$ and $-\frac{5}{4} < c < -1$, or $n = 2$ and $-\frac{1}{2} < c < 0$. In these cases, ${}_2F_1(-n, b; c; z)$ has $n - 2$ simple negative roots, one root $0 < z_0 < 1$, and another root $1/z_0 > 1$. Figure 5 shows the configurations of two examples with $n = 5$. □

Remark 23. Examples with six Scherk ends are parameterized by three real parameters, here by b, c, and the family parameter τ. We see that the relation $b+c = 1-n$ imposes a rotational symmetry. It can be imagined that removing the relation would break this symmetry.

Remark 24. The polynomial method is often used to find balanced configurations of interacting points in the plane. In minimal surface theory, it has been employed in many implementations of Traizet's node-opening technique [Traizet 2002a; 2002b; Traizet and Weber 2005; Li 2012; Connor and Weber 2012; Connor 2017a; 2017b; Chen and Freese 2022].

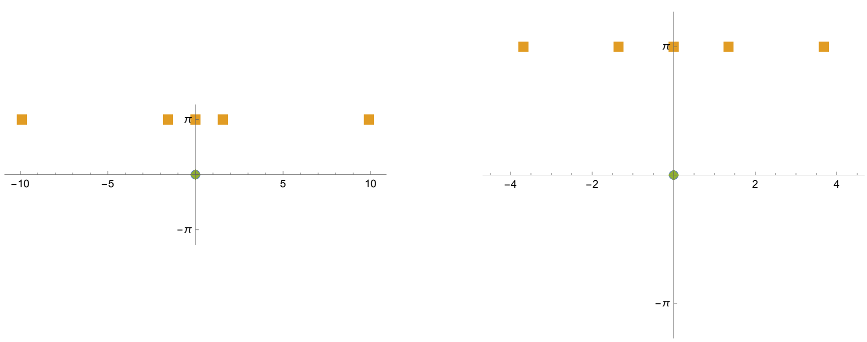

Figure 4. (5, 1) balanced configurations with $c = 0.001$ (left) and $c = 0.5$ (right). The circles and squares represent the necks at levels one and two, respectively.

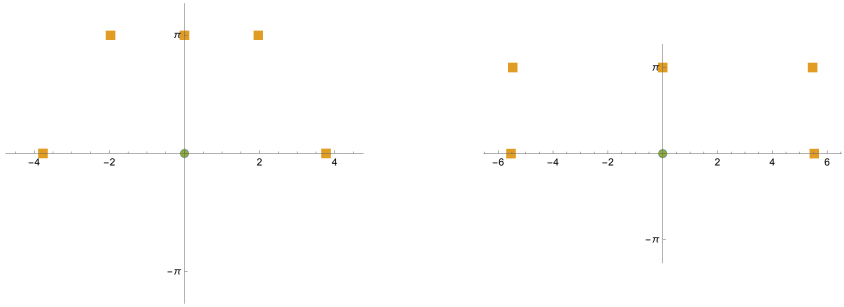

Figure 5. (5, 1) balanced configurations with $c = -0.5$ (left) and $c = -0.999$ (right). The circles and squares represent the necks at levels one and two, respectively.

2.5. ***Surfaces with eight ends of type (1, n, 1).*** Proposition 19 generalizes to the following lemma with similar proof:

Lemma 25. *We fix $q_{l\pm1,k}$'s and assume that $c_l = 1$. Then $q_{l,k}$'s in a balanced configuration are given by the roots of a Stieltjes polynomial $P(z)$ of degree n_l that solves the generalized Lamé equation (a.k.a. second-order Fuchsian equation) [Marden 1966]*

$$P'' + \left(\frac{c}{z} + \sum_{k=1}^{n_{l-1}} \frac{-c_{l-1}}{z - q_{l-1,k}} + \sum_{k=1}^{n_{l+1}} \frac{-c_{l+1}}{z - q_{l+1,k}}\right) P' + \left(\frac{\gamma_0}{z} + \sum_{k=1}^{n_{l-1}} \frac{\gamma_{l-1,k}}{z - q_{l-1,k}} + \sum_{k=1}^{n_{l+1}} \frac{\gamma_{l+1,k}}{z - q_{l+1,k}}\right) P = 0, \tag{14}$$

where $c = 1 + \dot\theta_{l+1,0} - \dot\theta_{l,0}$, subject to conditions

$$\gamma_0 + \sum_{k=1}^{n_{l-1}} \gamma_{l-1,k} + \sum_{k=1}^{n_{l+1}} \gamma_{l+1,k} = 0,$$

$$\sum_{k=1}^{n_{l-1}} \gamma_{l-1,k} q_{l-1,k} + \sum_{k=1}^{n_{l+1}} \gamma_{l+1,k} q_{l+1,k} =: -n_l b,$$

and

$$c - n_{l-1}c_{l-1} - n_{l+1}c_{l+1} = 1 - n_l + b.$$

Also, the matrix $(\partial F_{l,k}/\partial q_{l,j})_{1\le j,k\le n_l}$ is nonsingular as long as b is not a nonpositive integer bigger than n_l.

A root of $P(z)$ is simple if and only if it does not coincide with 0 or any $q_{l\pm1,k}$. If the roots $(q_{l,k})$ of $P(z)$ are all simple, then they solve the equations [Marden 1966]

$$\sum_{1\le k\ne j\le n_l} \frac{2}{q_{l,k} - q_{l,j}} + \sum_{1\le j\le n_{l+1}} \frac{-c_{l+1}}{q_{l,k} - q_{l+1,j}} + \sum_{1\le j\le n_{l-1}} \frac{-c_{l-1}}{q_{l,k} - q_{l-1,j}} + \frac{c}{q_{l,k}} = \frac{F_{l,k}}{q_{l,k}} = 0,$$

which is exactly our balance condition; see Remark 4. In addition, an equation system generalizing (13) has been obtained in [Heine 1878, §136], from which we may conclude the nonsingularity of the Jacobian. In fact, there are

$$\binom{n_{l-1}+n_l+n_{l+1}-1}{n_{l-1}+n_{l+1}-1}$$

choices of γ for which (14) has a polynomial solution of degree n_l [Heine 1878, §135].

This observation allows us to easily construct balanced and rigid configurations of type $(1, n, 1)$. Up to reparametrizations and complex scalings, we may assume that $c_2 = 1$ and $q_{1,1} = 1$. Then $q_{3,1}$ must be real, and $(q_{2,k})_{1\le k\le n}$ are given by roots of a Heun polynomial. Such a configuration depends locally on four real parameters, namely $q_{3,1}$, c_1, c_3 and c (or b). When these are given, we have $n+1$ Heun polynomials, each of which gives balanced positions of $q_{2,k}$'s. For each of the Heun polynomials P, we have

$$\begin{aligned}
\dot\theta_{2,0} - \dot\theta_{1,0} &= \frac{P'(1)}{P(1)} - c_1,\\
\dot\theta_{3,0} - \dot\theta_{2,0} &= c - 1 = b + c_1 + c_3 - n,\\
\dot\theta_{4,0} - \dot\theta_{3,0} &= \frac{P'(q_{3,1})}{P(q_{3,1})} - c_3.
\end{aligned}$$

Together with the family parameter τ, the surface depends locally on five parameters, which is expected because there are eight ends.

Example 26 (symmetric examples). When $q_{3,1} = q_{1,1} = 1$, the Heun polynomial reduces to a hypergeometric polynomial ${}_2F_1(-n, b; c; z)$, where $c_1+c_3 = n-1-b+c$. Assume further that $b+c = 1-n$, so $c_1+c_3 = -2b$. This imposes a symmetry in the configuration. Because $(c_1+c_3)P'(1)/P(1) = -nb$, the embeddedness conditions simplify to

$$c_1 > \tfrac{1}{2}n, \quad c_3 > \tfrac{1}{2}n, \quad 1-\tfrac{1}{2}n < c < 1, \quad -n < b < -\tfrac{1}{2}n.$$

As explained in Example 22, the hypergeometric polynomial has real roots if b and c lie in the blue regions of Figure 1. More specifically:

- When $n \ge 2$ and $0 < c < 1$, ${}_2F_1(-n, b; c; z)$ has n simple negative roots. See Figure 6 for an example with $n = 5$.
- When $n \ge 4$ and $-1 < c < 0$, or $n = 3$ and $-\frac{1}{2} < c < 0$, ${}_2F_1(-n, b; c; z)$ has $n-2$ simple negative roots, one root $0 < z_0 < 1$, and another root $1/z_0 > 1$. □

Example 27 (offset handles). There are embedded examples in which the handles are not symmetrically placed. For instance, one balanced configuration of type

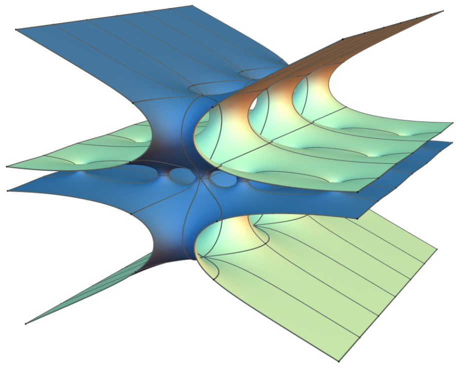

Figure 6. Genus four example with $n = 5$ and $0 < c < 1$.

(1, 2, 1) is given by

$$q_{3,1} = \tfrac{2}{3}, \quad q_{2,1} = -\tfrac{1}{3}, \quad q_{2,2} = -23,$$
$$c_1 = \tfrac{8}{5}, \quad c_3 = \tfrac{4189}{2890}, \quad b = -\tfrac{9857}{8670},$$

so $c = \frac{3956}{4335}$. □

2.6. ***Concatenating surfaces of type (1, n, 1).*** We describe a family of examples in the same spirit as [Traizet 2002a, Proposition 2.3]. Assume that we are in possession of R configurations of type $(1, n^{(r)}, 1)$, $n^{(r)} > 1$, $1 \le r \le R$. In the following, we use superscript (r) to denote the parameters of the r-th configuration. Up to reparameterizations and complex scalings, we may assume that $c_2^{(r)} = 1$ and $q_{1,1}^{(r)} = 1$. Then we may concatenate these configurations into one of type

$$(1, n_2, 1, n_4, 1, \dots, 1, n_{2R}, 1)$$

such that $q_{1,1} = 1$, $c_1 = 1$, and for $1 \le r \le R$, we have $n_{2r} = n^{(r)}$,

$$q_{2r,k} = q_{2r-1,1} q_{2,k}^{(r)}, \quad q_{2r+1,1} = q_{2r-1,1} q_{3,1}^{(r)}, \quad c_{2r} = \frac{c_{2r-1}}{c_1^{(r)}}, \quad c_{2r+1} = c_{2r-1} \frac{c_3^{(r)}}{c_1^{(r)}},$$

and

$$\dot\theta_{2r+1,0} - \dot\theta_{2r,0} = c_{2r}(c^{(r)} - 1).$$

The balance of even layers then follows from the balance of each subconfiguration. The balance of odd layers leads to

$$\dot\theta_{2r,0} - \dot\theta_{2r-1,0} = c_{2r}(\dot\theta_{2,0}^{(r)} - \dot\theta_{1,0}^{(r)} + c_1^{(r)}) + c_{2r-2}(\dot\theta_{4,0}^{(r-1)} - \dot\theta_{3,0}^{(r-1)} + c_3^{(r-1)}) - c_{2r-1}$$

for $1 \le r \le R+1$. As expected, such a configuration depends locally on $4R$ real parameters, namely $q_{3,1}^{(r)}$, $c_1^{(r)}$, $c_3^{(r)}$, and $c^{(r)}$, $1 \le r \le R$.

We may impose symmetry by assuming that $q^{(r)}_{3,1} = 1$, so $q_{2r+1,1} = 1$ for all $0 \le r \le R$, and that $b^{(r)} + c^{(r)} = 1 - n^{(r)}$, so $c^{(r)}_1 + c^{(r)}_3 = n^{(r)} - 1 - b^{(r)} + c^{(r)} = -2b^{(r)}$. Then $q_{2r,k} = q^{(r)}_{2,k}$, $1 \le k \le n^{(r)}$, are given by the roots of ${}_2F_1(-n^{(r)}, b^{(r)}; c^{(r)}; z)$, $1 \le r \le R$. Recall from Remark 11 that the embeddedness conditions simplifies to the concavity of the sequence $(n_l c_l)_{1\le l\le L}$. For even l, the concavity implies that $b^{(r)} > -n$; hence $c^{(r)} < 1$ for all $1 \le r \le R$. We may choose, for instance, $n_l c_l = \ln(1+l)$ or $n_l c_l = (\exp l - 1)/\exp(l-1)$ to obtain embedded minimal surfaces.

Remark 28. We can also append a configuration of type $(1, n^{(r)})$ to the sequence of $(1, n^{(r)}, 1)$-configurations to obtain a configuration of type

$$(1, n_2, 1, n_4, 1, \dots, 1, n_{2R-2}, 1, n_{2R}),$$

where the $q_{l,k}$, c_l, $\dot\theta_{l,0}$ terms are defined as above. Therefore, an embedded example of any genus with any even number (> 2) of ends can be constructed.

2.7. ***Numerical examples.*** The balance equations can be combined into one differential equation that is much easier to solve. A solution to this differential equation corresponds to several balance configurations that are equivalent by permuting the locations of the nodes.

Lemma 29. *Let L be a positive integer, $n_1, n_2, \dots, n_L \in \mathbb{N}$, and suppose $\{q_{l,k}\}$ is a configuration such that the $q_{l,k}$ are distinct. Let*

$$P_l(z) = \prod_{k=1}^{n_l}(z - q_{l,k}), \quad P(z) = \prod_{l=1}^{L} P_l(z), \quad P_0(z) = P_{L+1}(z) = 1,$$

and

$$\mathcal{F}P(z) = \sum_{l=1}^{L}\left(\frac{c_l^2 z P_l''(z)P(z)}{P_l(z)} - \frac{c_l c_{l+1} z P_l'(z)P_{l+1}'(z)P(z)}{P_l(z)P_{l+1}(z)} + (c_l^2 + c_l(\dot\theta_{l+1,0} - \dot\theta_{l,0}))\frac{P_l'(z)P(z)}{P_l(z)}\right).$$

Then the configuration $\{q_{l,k}\}$ is balanced if and only if $\mathcal{F}P(z) \equiv 0$.

Proof. We have seen that

$$\frac{P_l''(q_{l,k})}{P_l'(q_{l,k})} = \sum_{1\le k\ne j\le n_l} \frac{2}{q_{l,k} - q_{l,j}}, \quad \frac{P_{l\pm1}'(q_{l,k})}{P_{l\pm1}(q_{l,k})} = \sum_{j=1}^{n_{l\pm1}} \frac{1}{q_{l,k} - q_{l\pm1,j}}.$$

Define

$$F_l(z) = \frac{c_l^2 z P_l''(z)}{P_l'(z)} - \frac{c_l c_{l+1} z P_{l+1}'(z)}{P_{l+1}(z)} - \frac{c_l c_{l-1} z P_{l-1}'(z)}{P_{l-1}(z)} + c_l^2 + c_l(\dot\theta_{l+1,0} - \dot\theta_{l,0}).$$

Then $F_{l,k} = F_l(q_{l,k})$. Set

$$\begin{aligned} Q_l(z) &= \frac{P_l'(z)P(z)}{P_l(z)} F_l(z) \\ &= \frac{c_l^2 z P_l''(z)P(z)}{P_l(z)} - \frac{c_l c_{l+1} z P_l'(z) P_{l+1}'(z) P(z)}{P_l(z) P_{l+1}(z)} - \frac{c_l c_{l-1} z P_{l-1}'(z) P_l'(z) P(z)}{P_{l-1}(z) P_l(z)} \\ &\qquad + (c_l^2 + c_l(\dot\theta_{l+1,0} - \dot\theta_{l,0})) \frac{P_l'(z)P(z)}{P_l(z)}. \end{aligned}$$

Then $F_{l,k} = 0$ if and only if $Q_l(q_{l,k}) = 0$.

Now observe that $Q_l(z)$ and $Q(z) = \mathcal{F}P(z)$ are polynomials with degree strictly less than

$$\deg P = N = \sum_{l=1}^{L} n_l,$$

and $Q(q_{l,k}) = Q_l(q_{l,k})$ for $1 \le k \le n_l$ and $1 \le l \le L$. If $Q \equiv 0$ then $Q_l(q_{l,k}) = 0$ and so $\{q_{l,k}\}$ is a balanced configuration. If $\{q_{l,k}\}$ is a balanced configuration then $Q(q_{l,k}) = Q_l(q_{l,k}) = F_{l,k} = 0$. Hence, Q has at least N distinct roots. Since the degree of Q is strictly less than N, we must have $Q \equiv 0$. □

It is relatively easy to numerically solve $\mathcal{F}P(z) \equiv 0$ as long as we don't have too many levels and necks. So we use this lemma to find balanced configurations. Since all previous examples admit a horizontal reflection symmetry, we are most interested in examples without this symmetry, or with no nontrivial symmetry at all.

Figure 7 shows an example with $L = 3$,

$$\begin{gathered} n_1 = 1, \quad n_2 = 3, \quad n_3 = 2, \\ c_1 = 2, \quad c_2 = 1, \quad c_3 = \tfrac{13}{16}, \\ \theta_{1,0} = 0, \quad \theta_{2,0} = -\tfrac{1}{2}, \quad \theta_{3,0} = -\tfrac{27}{16}, \quad \theta_{4,0} = -\tfrac{29}{16}. \end{gathered}$$

This configuration corresponds to an embedded minimal surface with eight ends and genus three in the quotient. It has no horizontal reflectional symmetry, but does have a rotational symmetry.

Figure 8 shows two examples with $L = 3$,

$$\begin{gathered} n_1 = 1, \quad n_2 = 4, \quad n_3 = 3, \\ c_1 = \tfrac{7}{2}, \quad c_2 = 1, \quad c_3 = \tfrac{3}{4}, \\ \theta_{1,0} = 0, \quad \theta_{2,0} = -2, \quad \theta_{3,0} = -\tfrac{13}{5}, \quad \theta_{4,0} = -\tfrac{541}{180}. \end{gathered}$$

These configurations correspond to embedded minimal surfaces with eight ends and genus five in the quotient, with no nontrivial symmetry.

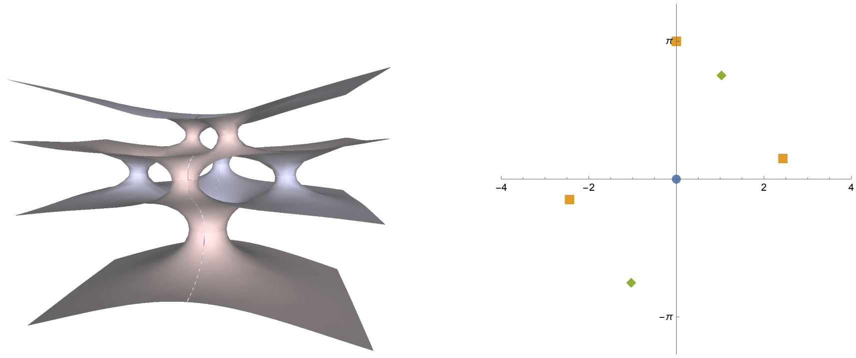

Figure 7. A $(1, 3, 2)$ balanced configuration with no horizontal reflectional symmetry. The circles, squares, and diamonds represent the necks at levels one, two, and three, respectively.

Figure 9 shows two examples with $L = 3$,

$$n_1 = 1, \quad n_2 = 7, \quad n_3 = 3,$$
$$c_1 = \tfrac{17}{7}, \quad c_2 = 1, \quad c_3 = \tfrac{3}{2},$$
$$\theta_{1,0} = 0, \quad \theta_{2,0} = -\tfrac{1}{2}, \quad \theta_{3,0} = -\tfrac{3}{2}, \quad \theta_{4,0} = -\tfrac{2468}{441}.$$

These configurations correspond to embedded minimal surfaces with eight ends and genus eight in the quotient, with no nontrivial symmetry.

3. Construction

3.1. ***Opening nodes.*** To each vertical plane is associated a punctured complex plane $\mathbb{C}_l^\times \simeq \mathbb{C} \setminus \{0\}$, $1 \le l \le L+1$. They can be seen as Riemann spheres $\hat{\mathbb{C}}_l \simeq \mathbb{C} \cup \{\infty\}$ with two fixed punctures at $p_{l,0} = 0$ and $p_{l,\infty} = \infty$, corresponding to the two ends.

To each neck is associated a puncture $p^\circ_{l,k} \in \mathbb{C}_l^\times$ and a puncture $p'^\circ_{l,k} \in \mathbb{C}_{l+1}^\times$. Our initial surface at $\tau = 0$ is the noded Riemann surface Σ_0 obtained by identifying $p^\circ_{l,k}$ and $p'^\circ_{l,k}$ for $1 \le l \le L$ and $1 \le k \le n_l$.

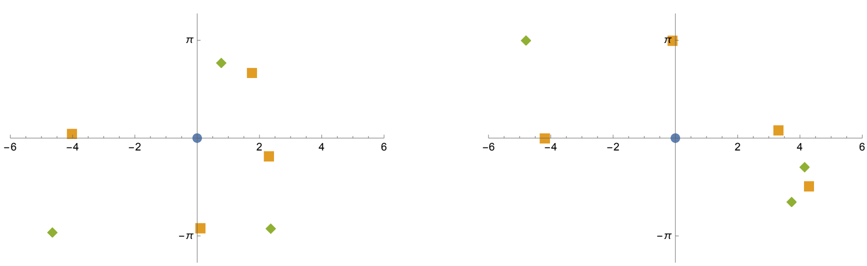

Figure 8. $(1, 4, 3)$ balanced configurations with no symmetries. The circles, squares, and diamonds represent the necks at levels one, two, and three, respectively.

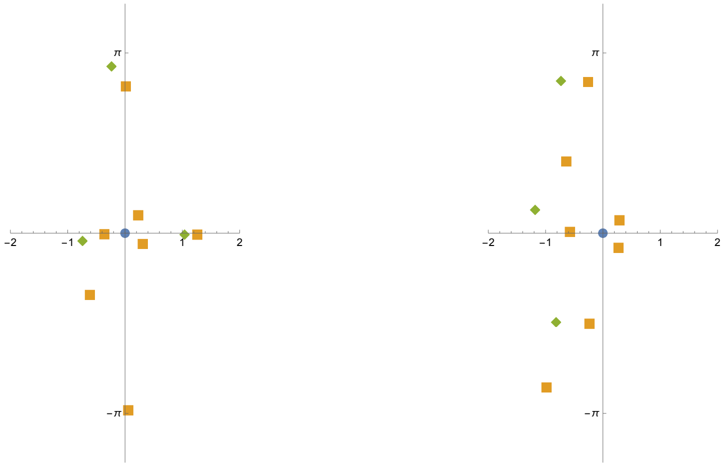

Figure 9. (1, 7, 3) balanced configurations with no symmetries. The circles, squares, and diamonds represent the necks at levels one, two, and three, respectively.

As τ increases, we open the nodes into necks as follows. Fix local coordinates $w_{l,0} = z$ in the neighborhood of $0 \in \hat{\mathbb{C}}_l$ and $w_{l,\infty} = 1/z$ in the neighborhood of $\infty \in \hat{\mathbb{C}}_l$. For each neck, we consider parameters $(p_{l,k}, p'_{l,k})$ in the neighborhoods of $(p^\circ_{l,k}, p'^\circ_{l,k})$ and local coordinates

$$w_{l,k} = \ln \frac{z}{p_{l,k}} \quad \text{and} \quad w'_{l,k} = \ln \frac{z}{p'_{l,k}}$$

in a neighborhood of $p_{l,k}$ and $p'_{l,k}$, respectively. In this paper, the branch cut of $\ln z$ is along the negative real axis, and we use the principal value of $\ln z$ with imaginary part in the interval $(-\pi, \pi]$.

As we only open finitely many necks, we may choose $\delta > 0$ independent of k and l such that the disks

$$|w_h| < 2\delta, \quad h \in \mathsf{H} \; (= [1, L+1] \times \{0, \infty\}),$$
$$|w_{l,k}| < 2\delta \quad \text{and} \quad |w'_{l,k}| < 2\delta, \qquad 1 \le l \le L, 1 \le k \le n_k$$

are all disjoint. For parameters $t = (t_{l,k})_{1 \le l \le L, 1 \le k \le n_l}$ in a neighborhood of 0 with $|t_{l,k}| < \delta^2$, we remove the disks

$$|w_{l,k}| < \frac{|t_{l,k}|}{\delta} \quad \text{and} \quad |w'_{l,k}| < \frac{|t_{l,k}|}{\delta}$$

and identify the annuli

$$\frac{|t_{l,k}|}{\delta} \le |w_{l,k}| \le \delta \quad \text{and} \quad \frac{|t_{l,k}|}{\delta} \le |w'_{l,k}| \le \delta$$

by

$$w_{l,k} w'_{l,k} = t_{l,k}.$$

If $t_{l,k} \neq 0$ for all $1 \leq l \leq L$ and $1 \leq k \leq n_l$, we obtain a Riemann surface denoted by Σ_t.

3.2. *Weierstrass data.* We construct a conformal minimal immersion using the Weierstrass parameterization in the form

$$z \mapsto \operatorname{Re} \int^z (\Phi_1, \Phi_2, \Phi_3),$$

where Φ_i are meromorphic 1-forms on Σ_t satisfying the conformality equation

$$Q := \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0. \tag{15}$$

3.2.1. *A-periods.* We consider the following fixed domains in all Σ_t:

$$U_{l,\delta} = \Big\{ z \in \hat{\mathbb{C}}_v : |w^\circ_{l,k}(z)| > \delta/2 \ \ \forall 1 \leq k \leq n_l \quad \text{if } 1 \leq l \leq L \\ \text{and} \quad |w'^\circ_{l,k}(z)| > \delta/2 \ \ \forall 1 \leq k \leq n_{l-1} \quad \text{if } 2 \leq l \leq L+1 \Big\}$$

and $U_\delta = \bigsqcup_{1 \leq l \leq L} U_{l,\delta}$.

Let $A_{l,k}$ denote a small counterclockwise circle in $U_{l,\delta}$ around $p_{l,k}$; it is then homologous in Σ_t to a clockwise circle in $U_{l+1,\delta}$ around $p'_{l,k}$. Moreover, let $A_{l,0}$ (resp. $A_{l,\infty}$) denote a small counterclockwise circle in $U_{l,\delta}$ around 0 (resp. ∞).

Recall that the vertical period vector is assumed to be $(0, 0, 2\pi)$, so we need to solve the A-period problems

$$\operatorname{Re} \int_{A_h} (\Phi_1, \Phi_2, \Phi_3) = (0, 0, 2\pi\sigma_h) \quad \text{and} \quad \operatorname{Re} \int_{A_{l,k}} (\Phi_1, \Phi_2, \Phi_3) = (0, 0, 0)$$

for $h \in \mathsf{H}$, $1 \leq l \leq L$, and $1 \leq k \leq n_l$. Here, the orientation $\sigma_h = \pm 1$ satisfies

$$\sigma_h = -\sigma_{\varsigma(h)},$$

where the “counterclockwise rotation” ς on H is defined by

$$\begin{cases} \varsigma(0_l) = 0_{l-1}, & 2 \leq l \leq L+1, \\ \varsigma(0_1) = \infty_1, & \\ \varsigma(\infty_l) = \infty_{l+1}, & 1 \leq l \leq L, \\ \varsigma(\infty_{L+1}) = 0_{L+1}. & \end{cases} \tag{16}$$

In particular, we have $\sigma_{l,0} = -\sigma_{l,\infty}$ for all $1 \leq l \leq L+1$.

Recall that the surface tends to an $(L+1)$-sheeted xz-plane in the limit $\tau \to 0$. So we define the meromorphic functions Φ_1, Φ_2 and Φ_3 as the unique regular 1-forms on Σ_t (see [Traizet 2013, §8]) with simple poles at p_h, $h \in \mathsf{H}$, and the A-periods

$$\int_{A_h} (\Phi_1, \widetilde{\Phi}_2, \Phi_3) = 2\pi\mathrm{i}(\alpha_h, \beta_h, \gamma_h - \mathrm{i}\sigma_h), \quad h \in \mathsf{H},$$
$$\int_{A_{l,k}} (\Phi_1, \widetilde{\Phi}_2, \Phi_3) = 2\pi\mathrm{i}(\alpha_{l,k}, \beta_{l,k}, \gamma_{l,k}), \quad 1 \le l \le L, 1 \le k \le n_l,$$

where $\Phi_2 = \tau\widetilde{\Phi}_2$ and, by the residue theorem, it is necessary that

$$\alpha_{l,0} + \alpha_{l,\infty} + \sum_{1\le k\le n_l} \alpha_{l,k} - \sum_{1\le k\le n_{l-1}} \alpha_{l-1,k} = 0, \tag{17}$$

$$\beta_{l,0} + \beta_{l,\infty} + \sum_{1\le k\le n_l} \beta_{l,k} - \sum_{1\le k\le n_{l-1}} \beta_{l-1,k} = 0, \tag{18}$$

$$\gamma_{l,0} + \gamma_{l,\infty} + \sum_{1\le k\le n_l} \gamma_{l,k} - \sum_{1\le k\le n_{l-1}} \gamma_{l-1,k} = 0, \tag{19}$$

for $1 \le l \le L+1$. Then the A-period problems are solved by definition.

3.2.2. *Balance of ends.* Summing up (18) over l gives

$$\sum_{h\in\mathsf{H}} \beta_h = 0, \tag{20}$$

which we use to replace (18) with $l = L+1$.

In this paper, the punctures $p_{l,0}$ and $p_{l,\infty}$ correspond to Scherk-type ends. Hence we fix

$$\alpha_h^2 + \tau^2\beta_h^2 \equiv 1 \quad \text{and} \quad \gamma_h \equiv 0 \tag{21}$$

for all $h \in \mathsf{H}$, so that (the stereographic projection of) the Gauss map

$$G = -\frac{\Phi_1 + \mathrm{i}\Phi_2}{\Phi_3}$$

extends holomorphically to the punctures p_h with unitary values. Then (19) is not independent: if it is solved for $1 \le l \le L$, it is automatically solved for $l = L+1$.

In particular, at $\tau = 0$, we have $\alpha_h^2 = 1$. In view of the orientation of the ends, we choose $\alpha_{l,0} = 1$ and $\alpha_{l,\infty} = -1$ so that $G(p_{l,\infty}) = G(p_{l,0}) = \mathrm{i}\sigma_{l,0}$.

Summing up (17) over l gives

$$\sum_{1\le l\le L+1} \left(\sqrt{1-\tau^2\beta_{l,\infty}^2} - \sqrt{1-\tau^2\beta_{l,0}^2}\right) = 0, \tag{22}$$

which we use to replace (17) with $l = L+1$.

Remark 30. The conditions (20) and (22) are disguises of the balance condition of Scherk ends, namely that the unit vectors in their directions should sum up to 0.

3.2.3. *B-periods.* For $1 \le l \le L+1$, we fix a point $O_l \in U_{l,\delta}$. For every $1 \le l \le L$ and $1 \le k \le n_l$ and $t_{l,k} \ne 0$, let $B_{l,k}$ be the concatenation of

(1) a path in $U_{l,\delta}$ from O_l to $w_{l,k} = \delta$,

(2) the path parameterized by $w_{l,k} = \delta^{1-2s}\, t_{l,k}^s$ for $s \in [0, 1]$, from $w_{l,k} = \delta$ to $w_{l,k} = t_h/\delta$, which is identified with $w'_{l,k} = \delta$, and

(3) a path in $U_{l+1,\delta}$ from $w'_{l,k} = \delta$ to O_{l+1}.

We need to solve the B-period problem, namely that

$$\operatorname{Re} \int_{B_{l,k}} (\Phi_1, \Phi_2, \Phi_3) = \operatorname{Re} \int_{B_{l,1}} (\Phi_1, \Phi_2, \Phi_3). \tag{23}$$

3.2.4. *Conformality.*

Lemma 31. *For t sufficiently close to* 0, *the conformality condition* (15) *is equivalent to*

$$\mathfrak{G}_{l,k} := \int_{A_{l,k}} \frac{w_{l,k} Q}{d w_{l,k}} = 0, \quad 1 \le l \le L,\ 1 \le k \le n_l, \tag{24}$$

$$\mathfrak{F}_{l,k} := \int_{A_{l,k}} \frac{Q}{d w_{l,k}} = 0, \quad 1 \le l \le L,\ 2 \le k \le n_l, \tag{25}$$

$$\mathfrak{F}'_{l,k} := \int_{A'_{l,k}} \frac{Q}{d w'_{l,k}} = 0, \quad 1 \le l \le L,\ 1+\delta_{l,L} \le k \le n_l, \tag{26}$$

where $A'_{l,k}$ in (26) *denotes a small counterclockwise circle in $U_{l+1,\delta}$ around $p'_{l,k}$* (*hence homologous to* $-A_{l,k}$), *and $\delta_{l,L} = 1$ if $l = L$ and* 0 *otherwise.*

Proof. By our choice of α_h and γ_h, the quadratic differential Q has at most simple poles at the $2L+2$ punctures p_h, $h \in \mathsf{H}$. The space of such quadratic differentials is of complex dimension $3(N-L)-3+(2L+2) = 3N-L-1$. We will prove that

$$Q \mapsto (\mathfrak{G}, \mathfrak{F}, \mathfrak{F}')$$

is an isomorphism. We prove the claim at $t = 0$; then the claim follows by continuity.

Consider Q in the kernel. Recall from [Traizet 2008] that a regular quadratic differential on Σ_0 has at most double poles at the nodes $p_{l,k}$ and $p'_{l,k}$. Then (24) guarantees that Q has at most simple poles at the nodes. By (25) and (26), Q may only have simple poles at $p_{l,1} \in \mathbb{C}_l^\times$, $1 \le l \le L$, and $p'_{L,1} \in \mathbb{C}_{L+1}^\times$. So, on each Riemann sphere $\hat{\mathbb{C}}_l$, Q is a quadratic differential with at most simple poles at three punctures; the other two being $0, \infty$. But such a quadratic differential must be 0. □

3.3. ***Using the implicit function theorem.*** All parameters vary in a neighborhood of their central values, denoted by a superscript ∘. We will see that

$$\beta_h^\circ = \dot\theta_h, \quad \alpha_{l,k}^\circ = \gamma_{l,k}^\circ = 0, \quad \beta_{l,k}^\circ = -c_l, \quad p_{l,k}'^\circ = \overline{p_{l,k}}.$$

Let us first solve (20) and (22).

Proposition 32. *Suppose we are given a configuration* $(q, \dot{\theta})$ *such that* $\Theta_1 = \Theta_2 = 0$. *For* τ *sufficiently small and* β_h *close to* $\beta_h^\circ = \dot{\theta}_h$, *the solutions* (τ, β) *to* (20) *and* (22) *form a smooth manifold of dimension* $2L+1$.

Proof. At $\tau = 0$, (20) is solved by $\beta_h^\circ = \dot{\theta}_h$ if $\Theta_1 = 0$. Taking the derivative of (22) with respect to τ^2 gives

$$\sum_{1\le l\le L+1} \frac{\beta_{l,\infty}^2 - \beta_{l,0}^2}{2} = 0, \tag{27}$$

which is solved by $\beta_h^\circ = \dot{\theta}_h$ if $\Theta_2 = 0$. The proposition then follows from Lemma 6 and the implicit function theorem. □

From now on, we assume that the parameters $(\tau, (\beta_h)_{h\in \mathsf{H}})$ are solutions to (20) and (22) in a neighborhood of $(0, \dot{\theta})$.

3.3.1. *Solving conformality problems.*

Proposition 33. *For* τ *sufficiently small and* $\beta_{l,k}$, $p_{l,k}$, *and* $p'_{l,k}$ *in a neighborhood of their central values, there exist unique values of* $t_{l,k}$, $\alpha_{l,k}$, *and* $\gamma_{l,k}$, *depending real-analytically on* (τ^2, β, p, p'), *such that the balance equations* (17) *and* (19) *with* $1 \le l \le L$ *and the conformality equations* (24) *and* (25) *are solved. Also, at* $\tau = 0$, *we have* $t_{l,k} = 0$, $\alpha_{l,k} = \gamma_{l,k} = 0$,

$$\frac{\partial t_{l,k}}{\partial(\tau^2)} = \frac{1}{4}\beta_{l,k}^2,$$

and, for $2 \le k \le n_l$,

$$\frac{\partial}{\partial(\tau^2)}(\alpha_{l,k} - \mathrm{i}\sigma_{l,0}\gamma_{l,k}) = -\frac{1}{2}\operatorname{Res}\left(\frac{\widetilde{\Phi}_2^2}{dw_{l,k}}, p_{l,k}\right) = -\frac{1}{2}\operatorname{Res}\left(\frac{z\widetilde{\Phi}_2^2}{dz}, p_{l,k}\right). \tag{28}$$

Note that, according to this proposition, if $\beta_{l,k}^\circ \ne 0$, then $t_{l,k} > 0$ for sufficiently small τ.

Proof. At $\tau = 0$, for $2 \le k \le n_l$ we have

$$\mathfrak{G}_{l,k} = \int_{A_{l,k}} \frac{w_{l,k} Q}{dw_{l,k}} = 2\pi\mathrm{i}(\alpha_{l,k}^2 + \gamma_{l,k}^2) = 0,$$

which vanishes when

$$\alpha_{l,k} = \gamma_{l,k} = 0.$$

Recall that $\alpha_h = \pm 1$ at $\tau = 0$ and that $\gamma_h \equiv 0$. Then by the residue theorem, we have

$$\alpha_{l,1} = \gamma_{l,1} = 0.$$

As a consequence, we have at $\tau = 0$

$$\Phi_1^\circ = \frac{dz}{z}, \quad \Phi_2^\circ = 0, \quad \text{and} \quad \Phi_3^\circ = -\mathrm{i}\sigma_{l,0}\frac{dz}{z};$$

so $Q = 0$ as we expect.

We then compute the partial derivatives at $\tau = 0$:

$$\begin{aligned}
\frac{\partial}{\partial\alpha_{l,k}}\mathfrak{F}_{l,k} &= \int_{A_{l,k}} 2\frac{\Phi_1^\circ}{dw_{l,k}}\frac{\partial\Phi_1}{\partial\alpha_{l,k}}\bigg|_{\tau=0} = \int_{A_{l,k}} 2\frac{dz/z}{dz/p}\frac{dz}{z-p_{l,k}} = 4\pi\mathrm{i},\\
\frac{\partial}{\partial\gamma_{l,k}}\mathfrak{F}_{l,k} &= \int_{A_{l,k}} 2\frac{\Phi_3^\circ}{dw_{l,k}}\frac{\partial\Phi_3}{\partial\gamma_{l,k}}\bigg|_{\tau=0} = \int_{A_{l,k}} -2i\sigma_{l,0}\frac{dz/z}{dz/p}\frac{dz}{z-p_{l,k}} = 4\pi\sigma_{l,0},\\
\frac{\partial}{\partial t_{l,k}}\mathfrak{G}_{l,k} &= \int_{A_{l,k}} \frac{2w_{l,k}}{dw_{l,k}}\left(\Phi_1^\circ\frac{\partial\Phi_1^\circ}{\partial t_{l,k}} + \Phi_3^\circ\frac{\partial\Phi_3^\circ}{\partial t_{l,k}}\right)\\
&= \frac{-1}{\pi\mathrm{i}}\left(\int_{A_{l,k}}\frac{\Phi_1^\circ}{w_{l,k}}\int_{A'_{l,k}}\frac{\Phi_1^\circ}{w'_{l,k}} + \int_{A_{l,k}}\frac{\Phi_3^\circ}{w_{l,k}}\int_{A'_{l,k}}\frac{\Phi_3^\circ}{w'_{l,k}}\right)\\
&= -8\pi\mathrm{i},
\end{aligned}$$

where the second to last line is true by [Traizet 2008, Lemma 3]. All other partial derivatives vanish. Therefore, by the implicit function theorem, there exist unique values of $\alpha_{l,k}$, $\gamma_{l,k}$ (with $2 \le k \le n_l$), and $t_{l,k}$ (with $1 \le k \le n_l$) that solve the conformality equations (24) and (25). Recall that α_h are determined by (21). Then $\alpha_{l,1}$ and $\gamma_{l,1}$ are uniquely determined by the linear balance equations (17) and (19).

Moreover,

$$\frac{\partial}{\partial(\tau^2)}\mathfrak{F}_{l,k} = \int_{A_{l,k}}\frac{\widetilde{\Phi}_2^2}{dw_{l,k}}, \qquad \frac{\partial}{\partial(\tau^2)}\mathfrak{G}_{l,k} = 2\pi\mathrm{i}\beta_{l,k}^2.$$

Hence the total derivatives satisfy

$$\frac{d}{d(\tau^2)}\mathfrak{F}_{l,k} = 4\pi\mathrm{i}\frac{\partial\alpha_{l,k}}{\partial(\tau^2)} + 4\pi\sigma_{l,0}\frac{\partial\gamma_{l,k}}{\partial(\tau^2)} + 2\pi\mathrm{i}\,\mathrm{Res}\left(\frac{\widetilde{\Phi}_2^2}{dw_{l,k}}, p_{l,k}\right) = 0$$

and

$$\frac{d}{d(\tau^2)}\int_{A_{l,k}}\mathfrak{G}_{l,k} = -8\pi\mathrm{i}\frac{\partial t_{l,k}}{\partial(\tau^2)} + 2\pi\mathrm{i}\beta_{l,k}^2 = 0.$$

This proves the claimed partial derivatives with respect to τ^2. □

Remark 34. We see from the computations that our local coordinates w and w' are chosen for convenience. Had we used other coordinates, the computations would be very different, but $\partial(\alpha_{l,k} - \mathrm{i}\sigma_{l,0}\gamma_{l,k})/\partial(\tau^2)$ would be invariant, and $\partial t_{l,k}/\partial(\tau^2)$ would be rescaled to keep the conformal type of Σ_t (to the first order). So the choice of local coordinates has no substantial impact on our construction.

3.3.2. *Solving B-period problems.* In the following, we make a change of variable $\tau = \exp(-1/\xi^2)$.

Proposition 35. *Let the parameters $t_{l,k}$, $\alpha_{l,k}$, and $\gamma_{l,k}$ be given by Proposition 33. For ξ sufficiently small and $p_{l,k}$ and $p'_{l,k}$ in a neighborhood of their central values, there exist unique values of $\beta_{l,k}$, depending smoothly on (ξ, p, p') and $(\beta_h)_{h\in\mathsf{H}}$, such that the balance equation (18) with $1 \le l \le L$ and the y-component of the B-period problem (23) are solved. In addition, at $\xi = 0$ and $\beta_h = \beta_h^\circ = \dot\theta_h$, we have $\beta_{l,k} = \beta_{l,1} = -c_l$ where c_l is given by (1).*

Proof. By Lemma 8.3 of [Chen and Traizet 2021],

$$\Big(\int_{B_{l,k}} \widetilde{\Phi}_2\Big) - \beta_{l,k} \ln t_{l,k}$$

extends holomorphically to $t = 0$ as bounded analytic functions of other parameters. We have seen that $t_{l,k} \sim \tau^2 \beta_{l,k}^2/4$. So

$$\mathfrak{H} := -\frac{\xi^2}{2} \operatorname{Re}\Big(\int_{B_{l,k}} \widetilde{\Phi}_2 - \int_{B_{l,1}} \widetilde{\Phi}_2\Big) = \beta_{l,k} - \beta_{l,1}$$

at $\xi = 0$. Therefore, $\mathfrak{H} = 0$ is solved at $\xi = 0$ by $\beta_{l,k} = \beta_{l,1}$ for all $2 \le k \le n_l$, and $\beta_{l,1} = -c_l$ follows as (1) is just a reformulation of (18). The proposition then follows by the implicit function theorem. □

Proposition 36. *Assume that the parameters $t_{l,k}$, $\alpha_{l,k}$, $\beta_{l,k}$ and $\gamma_{l,k}$ are given by Propositions 33 and 35. For ξ sufficiently small and $p_{l,k}$ in a neighborhood of their central values, there exist unique values of $p'_{l,k}$, depending smoothly on ξ, p, and $(\beta_h)_{h\in\mathsf{H}}$, such that the x- and z-components of the B-period problem (23) are solved. In addition, up to complex scalings on $\mathbb{C}^\times_{l+1}$, $1 \le l \le L$, we have $p'_{l,k} = \overline{p_{l,k}}$ at $\xi = 0$ for any $1 < k \le n_l$.*

Proof. At $\xi = 0$, recall that $\Phi_1 = dz/z$ and $\Phi_3 = -\mathrm{i}\sigma_{l,0}\,dz/z$. So

$$\operatorname{Re}\int_{B_{l,k}} \Phi_1 - \operatorname{Re}\int_{B_{l,1}} \Phi_1 = \operatorname{Re}\ln\frac{p_{l,k}}{p_{l,1}} - \operatorname{Re}\ln\frac{p'_{l,k}}{p'_{l,1}};$$

$$\operatorname{Re}\int_{B_{l,k}} \Phi_3 - \operatorname{Re}\int_{B_{l,1}} \Phi_3 = \sigma_{l,0}\Big(\operatorname{Im}\ln\frac{p_{l,k}}{p_{l,1}} + \operatorname{Im}\ln\frac{p'_{l,k}}{p'_{l,1}}\Big).$$

They vanish if and only if $\ln(p_{l,k}/p_{l,1}) = \overline{\ln(p'_{l,k}/p'_{l,1})}$. We normalize the complex scaling on $\mathbb{C}^\times_{l+1}$, $1 \le l \le L$, by fixing $p'_{l,1} = \overline{p_{l,1}}$. Then the B-period problem is solved at $\xi = 0$ with $p'_{l,k} = \overline{p_{l,k}}$. By the same argument as in [Traizet 2008], the integrals are smooth functions of ξ and other parameters, so the proposition follows by the implicit function theorem. □

3.3.3. *Balancing conditions.* Define

$$\mathfrak{R}_{l,k} = \operatorname{Res}\Big(\frac{z\widetilde{\Phi}_2^2}{dz}, p_{l,k}\Big) \quad \text{and} \quad \mathfrak{R}'_{l,k} = \operatorname{Res}\Big(\frac{z\widetilde{\Phi}_2^2}{dz}, p'_{l,k}\Big).$$

Let the central values $p^\circ_{l,k}$ equal $\operatorname{conj}^l q_{l,k}$, where q is from a balanced configuration. So the central values $p'^\circ_{l,k}$ equal $\operatorname{conj}^{l+1} q_{l,k}$ and

$$\widetilde{\Phi}_2^\circ = \begin{cases} \overline{\operatorname{conj}^* \psi_l} & \text{on } \mathbb{C}_l^\times \text{ for } l \text{ odd,} \\ \psi_l & \text{on } \mathbb{C}_l^\times \text{ for } l \text{ even.} \end{cases}$$

Then we have

$$\overline{\mathfrak{R}_{l,k}} + \mathfrak{R}'_{l,k} = 2\operatorname{conj}^{l+1} F_{l,k}$$

at the central values, where $F_{l,k}$ is the force given by (4). Also, by the residue theorem on $\mathbb{C}_l^\times$,

$$\sum_{k=1}^{n_{l-1}} \mathfrak{R}'_{l-1,k} + \sum_{k=1}^{n_l} \mathfrak{R}_{l,k} + \beta_{l,0}^2 - \beta_{l,\infty}^2 = 0. \tag{29}$$

Proposition 37. *Assume that the parameters $t_{l,k}$, $\alpha_{l,k}$, and $\gamma_{l,k}$ are given as analytic functions of τ^2 by Proposition 33. Then $\widetilde{\mathfrak{F}}'_{l,k} := \tau^{-2}\mathfrak{F}'_{l,k}$ extends analytically to $\tau = 0$ with the value*

$$\begin{cases} 4\pi \mathrm{i} \operatorname{conj}^{l+1} F_{l,k}, & 2 \le k \le n_l, \\ 4\pi \mathrm{i} \operatorname{conj}^{l+1}\big(F_{l,1} + \sum_{j=1}^{l-1} \sum_{k=1}^{n_j} F_{j,k}\big), & k = 1. \end{cases}$$

Proof. If $f(z)$ is an analytic function in z and $f(0) = 0$, then $f(z)/z$ extends analytically to $z = 0$ with the value $df/dz\,|_{z=0}$. We compute at $\tau = 0$ that

$$\frac{\partial}{\partial \alpha}\mathfrak{F}'_{l,k} = -4\pi \mathrm{i} \quad \text{and} \quad \frac{\partial}{\partial \gamma}\mathfrak{F}'_{l,k} = 4\pi\sigma_{l,0}.$$

Then

$$\frac{d}{d(\tau^2)}\mathfrak{F}'_{l,k} = -4\pi \mathrm{i}\frac{\partial \alpha_{l,k}}{\partial(\tau^2)} + 4\pi\sigma_{l,0}\frac{\partial \gamma_{l,k}}{\partial(\tau^2)} + 2\pi \mathrm{i}\mathfrak{R}'_{l,k}.$$

For $2 \le k \le n_l$, by (28), $\widetilde{\mathfrak{F}}'_{l,k} := \tau^{-2}\mathfrak{F}'_{l,k}$ extends to $\tau = 0$ with the value

$$\frac{d}{d(\tau^2)}\mathfrak{F}'_{l,k} = 2\pi \mathrm{i}(\overline{\mathfrak{R}_{l,k}} + \mathfrak{R}'_{l,k}) = 4\pi \mathrm{i} \operatorname{conj}^{l+1} F_{l,k}.$$

As for $k=1$ and $l<L$, we compute at $\tau=0$

$$\begin{aligned}
\sum_{k=1}^{n_l}\frac{d\mathfrak{F}'_{l,k}}{d(\tau^2)}+\sum_{k=1}^{n_{l-1}}\operatorname{conj}\Big(\frac{d\mathfrak{F}'_{l-1,k}}{d(\tau^2)}\Big)
&=-4\pi\mathrm{i}\frac{\partial}{\partial\tau^2}\Big(\sum_{k=1}^{n_l}\alpha_{l,k}-\sum_{k=1}^{n_{l-1}}\alpha_{l-1,k}\Big)\\
&\qquad+4\pi\sigma_{l,0}\frac{\partial}{\partial\tau^2}\Big(\sum_{k=1}^{n_l}\gamma_{l,k}-\sum_{k=1}^{n_{l-1}}\gamma_{l-1,k}\Big) \qquad \text{(because } \sigma_{l-1,0}=-\sigma_{l,0}\text{)}\\
&\qquad\qquad+2\pi\mathrm{i}\Big(\sum_{k=1}^{n_l}\mathfrak{R}'_{l,k}-\sum_{k=1}^{n_{l-1}}\overline{\mathfrak{R}'_{l-1,k}}\Big)\\
&=4\pi\mathrm{i}\frac{\partial}{\partial\tau^2}(\alpha_{l,0}+\alpha_{l,\infty})-4\pi\sigma_{l,0}\frac{\partial}{\partial\tau^2}(\gamma_{l,0}+\gamma_{l,\infty}) \qquad \text{(by (17) and (19))}\\
&\qquad+2\pi\mathrm{i}\Big(\sum_{k=1}^{n_l}\mathfrak{R}'_{l,k}+\sum_{k=1}^{n_l}\overline{\mathfrak{R}_{l,k}}+\beta_{l,0}^2-\beta_{l,\infty}^2\Big) \qquad \text{(by (29))}\\
&=2\pi\mathrm{i}\Big(\beta_{l,\infty}^2-\beta_{l,0}^2+\sum_{k=1}^{n_l}(\overline{\mathfrak{R}_{l,k}}+\mathfrak{R}'_{l,k})+\beta_{l,0}^2-\beta_{l,\infty}^2\Big) \qquad \text{(by (21))}\\
&=4\pi\mathrm{i}\sum_{k=1}^{n_l}\operatorname{conj}^{l+1}F_{l,k},
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{k=1}^{n_l}\frac{d\mathfrak{F}'_{l,k}}{d(\tau^2)}=\frac{d\mathfrak{F}'_{l,1}}{d(\tau^2)}&+4\pi\mathrm{i}\operatorname{conj}^{l+1}\sum_{k=2}^{n_l}F_{l,k}\\
&=(-\operatorname{conj})^l\sum_{m=1}^{l}(-\operatorname{conj})^m\Big(\sum_{k=1}^{n_m}\frac{d\mathfrak{F}'_{m,k}}{d(\tau^2)}+\sum_{k=1}^{n_{m-1}}\operatorname{conj}\Big(\frac{d\mathfrak{F}'_{m-1,k}}{d(\tau^2)}\Big)\Big)\\
&=(-\operatorname{conj})^l\sum_{m=1}^{l}(-\operatorname{conj})^m\Big(4\pi\mathrm{i}\sum_{k=1}^{n_m}\operatorname{conj}^{m+1}F_{m,k}\Big)\\
&=4\pi\mathrm{i}\operatorname{conj}^{l+1}\sum_{m=1}^{l}\sum_{k=1}^{n_m}F_{m,k},
\end{aligned}$$

so $\widetilde{\mathfrak{F}}'_{l,1}:=\tau^{-2}\mathfrak{F}'_{l,1}$ extends to $\tau=0$ with the value

$$\frac{d\mathfrak{F}'_{l,1}}{d(\tau^2)}=4\pi\mathrm{i}\operatorname{conj}^{l+1}\Big(F_{l,1}+\sum_{m=1}^{l-1}\sum_{k=1}^{n_m}F_{m,k}\Big). \qquad \square$$

Therefore, if $(q, \dot\theta)$ is balanced, $\widetilde{\mathfrak{F}}' = 0$ is solved at $\tau = 0$. Recall that we normalize the complex scaling on $\mathbb{C}_1^\times$ by fixing $p_{1,1}$. If $(q, \dot\theta)$ is rigid, because $\Theta_2 = \sum F_{l,k} = 0$ independent of p, the partial derivative of $(\widetilde{\mathfrak{F}}')_{(l,k)} \neq (L, 1)$ with respect to $(p_{l,k})_{(l,k)\neq(1,1)}$ is an isomorphism from $\mathbb{C}^{N-1}$ to $\mathbb{C}^{N-1}$. The following proposition then follows by the implicit function theorem.

Proposition 38. *Assume that the parameters $t_{l,k}$, $\alpha_{l,k}$, $\beta_{l,k}$, $\gamma_{l,k}$, $p'_{l,k}$ are given by Propositions 33, 35, and 36. Assume further that the central values $q_{l,k} = \operatorname{conj}^l p^\circ_{l,k}$ and $\dot\theta_h = \beta^\circ_h$ form a balanced and rigid configuration $(q, \dot\theta)$. Then for (τ, β) in a neighborhood of $(0, \dot\theta)$ that solves (20) and (22), there exists values for $p_{l,k}$, unique up to a complex scaling, depending smoothly on τ and $(\beta_h)_{h\in\mathsf{H}}$, such that $p_{l,k}(0, \dot\theta) = p^\circ_{l,k}$ and the conformality condition (26) is solved.*

3.4. ***Embeddedness.*** It remains to prove that:

Proposition 39. *The minimal immersion given by the Weierstrass parameterization is regular and embedded.*

Proof. The immersion is regular if $|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 > 0$. This is easily verified on U_δ. On the necks and the ends, the regularity follows if we prove that $\widetilde\Phi_2$ has no zeros outside U_δ. At $\tau = 0$, $\widetilde\Phi_2$ has $n_l + n_{l-1} + 2$ poles on $\hat{\mathbb{C}}_l$, hence $n_l + n_{l-1}$ zeros. By taking δ sufficiently small, we may assume that all these zeros lie in $U_{l,\delta}$. By continuity, $\widetilde\Phi_2$ has $n_l + n_{l-2}$ zeros in $U_{l,\delta}$ also for τ sufficiently small. But for $\tau \neq 0$, $\widetilde\Phi_2$ is meromorphic on a Riemann surface Σ_τ of genus $g = N - L$ and has $2L + 2$ simple poles, hence has $2(N - L) - 2 + 2L + 2 = 2N$ zeros. So $\widetilde\Phi_2$ has no further zeros in Σ_t, and, in particular, not outside U_δ.

We now prove that the immersion

$$z \mapsto \operatorname{Re} \int^z (\Phi_1, \widetilde\Phi_2, \Phi_3)$$

is an embedding, and the limit positions of the necks are as prescribed.

On $U_{l,\delta}$, the Gauss map $G = -(\Phi_1 + \mathrm{i}\Phi_2)/\Phi_3$ converges to $\mathrm{i}\sigma_{l,0}$, so the immersion is locally a graph over the xz-plane. Fix an orientation $\sigma_{1,0} = -1$; then up to translations, we have

$$\lim_{\tau\to 0} \Big(\operatorname{Re} \int^z \Phi_1 + \mathrm{i} \operatorname{Re} \int^z \Phi_3 \Big) = \operatorname{conj}^l(\ln z) + 2m\pi \mathrm{i},$$

where m depends on the integral path, and

$$\lim_{\tau\to 0} \operatorname{Re} \int^z \widetilde\Phi_2 = \operatorname{Re} \int^z (\operatorname{conj}^*)^l \psi_l =: \Psi_l(\operatorname{conj}^l z),$$

which is well defined for $z \in U_{l,\delta}$ because the residues of ψ_l are all real.

With a change of variable $z \mapsto \ln z$, we see that the immersion restricted to $U_{l,\delta}$ converges to a periodic graph over the xz-planes, defined within bounded

x-coordinate and away from the points $\ln q_{l,k} + 2m\pi\mathrm{i}$, and the period is $2\pi\mathrm{i}$. Here, again, we identified the xz-plane with the complex plane.

This graph must be included in a slab parallel to the xz-plane with bounded thickness. We have seen from the integration along B_k that the distance between adjacent slabs is of the order $\mathcal{O}(\ln\tau)$. So the slabs are disjoint for τ sufficiently small.

As for the necks and ends, note that there exists $Y > 0$ such that $\Psi_l^{-1}([-Y, Y])$ is bounded by $n_l + n_{l-1} + 2$ convex curves. After the change of variable $z \mapsto \ln z$, all but two of these curves remain convex; those around 0 and ∞ become periodic infinite curves. If Y is chosen sufficiently large, there exists $X > 0$ independent of l such that the curves $|z| = \exp(\pm X)$ are included in $\Psi_l^{-1}([-Y, Y])$ for every $1 \le l \le L+1$. After the change of variable $z \mapsto \ln z$, these curves become curves with $\operatorname{Re} z = \pm X$.

Hence for τ sufficiently small, we may find Y_l^+ and Y_l^-, with $Y_l^- < Y_l^+ < Y_{l+1}^-$, and $X > 0$, such that:

- The immersion with $Y_l^- < y < Y_l^+$ and $-X < x < X$ is a graph bounded by $n_l + n_{l-1}$ planar convex curves parallel to the xz-plane and two periodic planar infinite curves parallel to the yz-plane.
- The immersion with $Y_l^+ < y < Y_{l+1}^-$ and $-X < x < X$ consists of annuli, each bounded by two planar convex curves parallel to the xz-plane. These annuli are disjoint and, by a theorem of Schiffman [1956], all embedded.
- The immersion with $|x| > X$ are ends, i.e., graphs over vertical half-planes, extending in the direction $(-1, -\dot\theta_{l,0})$ and $(+1, -\dot\theta_{l,\infty})$, $1 \le l \le L+1$. If the inequality (5) is satisfied, these graphs are disjoint.

This finishes the proof of embeddedness. □

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