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We propose a reduction scheme for polydifferential operators phrased in terms of L_∞-morphisms. The desired reduction L_∞-morphism has been obtained by applying an explicit version of the homotopy transfer theorem. Finally, we prove that the reduced star product induced by this reduction L_∞-morphism and the reduced star product obtained via the formal Koszul complex are equivalent.

1. Introduction

This paper aims to propose a reduction scheme for equivariant polydifferential operators that is phrased in terms of L_∞-morphisms, generalizing the results from [Esposito et al. 2022b], obtained for polyvector fields. Our main motivation comes from formal deformation quantization: deformation quantization has been introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [Bayen et al. 1978a; 1978b] and it relies on the idea that the quantization of a phase space described by a Poisson manifold M is described by a formal deformation, so-called *star product*, of the commutative algebra of smooth complex-valued functions $\mathscr{C}^\infty(M)$ in a formal parameter $\hbar$. The existence and classification of star products on Poisson manifolds has been provided by Kontsevich's formality theorem [2003], whereas the invariant setting of Lie group actions has been treated by Dolgushev

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[2005a; 2005b]. More explicitly, the formality theorem provides an L_∞-quasi-isomorphism between the differential graded Lie algebra (DGLA) of polyvector fields $T_{\mathrm{poly}}(M)$ and polydifferential operators $D_{\mathrm{poly}}(M)$ as well as between their invariant versions. As such, it maps Maurer–Cartan elements in the DGLA of polyvector fields, i.e., (formal) Poisson structures, to Maurer–Cartan elements in the DGLA of polydifferential operators, which correspond to star products.

One open question and our main motivation is to investigate the compatibility of deformation quantization and phase space reduction in the Poisson setting, and in this present paper we propose a way to describe the reduction on the quantum side by an L_∞-morphism. Given a Lie group G acting on a manifold M, we aim to reduce *equivariant star products* $(\star, H)$, that is, pairs consisting of an invariant star product $\star$ and a quantum momentum map $H = \sum_{r=0}^\infty \hbar^r J_r : \mathfrak{g} \longrightarrow \mathscr{C}^\infty(M)[\![\hbar]\!]$, where $\mathfrak{g}$ is the Lie algebra of G. In this case, J_0 is a classical momentum map for the Poisson structure induced by $\star$. Interpreting it as smooth map $J_0 : M \longrightarrow \mathfrak{g}^*$ and assuming that $0 \in \mathfrak{g}^*$ is a value and regular value, it follows that $C = J^{-1}(\{0\})$ is a closed embedded submanifold of M and by the Poisson version of the Marsden–Weinstein reduction [1974] we know that under suitable assumptions the reduced manifold $M_{\mathrm{red}} = C/\,\mathrm{G}$ is again a Poisson manifold if the action on C is proper and free. In this setting, there is a well-known BRST-like reduction procedure [Bordemann et al. 2000; Gutt and Waldmann 2010] of equivariant star products on M to star products on M_{red}.

In order to describe this reduction by an L_∞-morphism, we have to fix at first the DGLA controlling Hamiltonian actions in the quantum setting, i.e., a DGLA whose Maurer–Cartan elements correspond to equivariant star products. We denote it by

$$(D_{\mathfrak{g}}(M)[\![\hbar]\!], \hbar\lambda, \partial^{\mathfrak{g}} - [J_0, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}),$$

where $\lambda = \sum_i e^i \otimes (e_i)_M$ is given by the fundamental vector fields of the G-action in terms of a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ with dual basis $e^1, \dots, e^n$ of $\mathfrak{g}^*$. It is called the DGLA of *equivariant polydifferential operators*.

The construction of the desired L_∞-morphism to $(D_{\mathrm{poly}}(M_{\mathrm{red}}), \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}})$ is then based on the following steps:

- Assuming for simplicity $M = C \times \mathfrak{g}^*$, which always holds locally in suitable situations, we can perform a Taylor expansion around C and end up with a DGLA $D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$. Using a ‘partial homotopy’, we find a deformation retract to a DGLA structure on the space $\big(\prod_{i=0}^\infty (\mathrm{S}^i \mathfrak{g} \otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}$, that is, we get rid of differentiations in the $\mathfrak{g}^*$-direction.

- For the polyvector fields in [Esposito et al. 2022b] we used the canonical linear Poisson structure π_{KKS} on the dual of the action Lie algebroid $C \times \mathfrak{g}$ for the reduction. The analogue structure in our quantum setting is the product on the quantized universal enveloping algebra $U_\hbar(C \times \mathfrak{g})$ of the action Lie algebroid. We use this product to perturb the deformation retract from the last point. This is more complicated than the polyvector field case since we have to use now the homological perturbation lemma to perturb the involved chain maps, and the deformed maps are no longer compatible with the Lie brackets.

• We use the homotopy transfer theorem to construct the L_∞-projection from the Taylor expansion to $\big(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}$ with transferred L_∞-structure. Notice that in the polyvector field case it was not necessary to transfer the DGLA structure.

• We check in Proposition 3.10 that the transferred L_∞-structure is just a DGLA structure, and in Proposition 3.11 that the transferred Lie bracket is compatible with the projection to $D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]]$. Thus we get the reduction L_∞-morphism from the Taylor expansion to the polydifferential operators on M_{red}. Twisting it by the product on the universal enveloping algebra ensures that we start in the right curved DGLA structure.

Finally, the morphism can be globalized to general smooth manifolds M with sufficiently nice Lie group actions and we get the following result (Theorem 3.15):

Theorem. *There exists an L_∞-morphism*

$$D_{\mathrm{red}}:(D_{\mathfrak{g}}(M)[[\hbar]],\hbar\lambda,\partial^{\mathfrak{g}}-[J_0,\cdot\,]_{\mathfrak{g}},[\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]],0,\partial,[\,\cdot\,,\cdot\,]_{\mathrm{G}}), \tag{1-1}$$

*called the **reduction L_∞-morphism**.*

Finally, we compare the reduction of equivariant star products via D_{red} to a slightly modified version of the BRST reduction from [Bordemann et al. 2000; Gutt and Waldmann 2010]; see Theorem 4.4:

Theorem. *Let $(\star, H)$ be an equivariant star product on M. Then the reduced star product induced by D_{red} from (1-1) and the reduced star product via the formal Koszul complex are equivalent.*

Together with [Esposito et al. 2022b, Theorem 5.1] we have now the diagram

$$\begin{tikzcd}
(T^\bullet_{\mathfrak{g}}(M)[[\hbar]],\hbar\lambda,[-J_0,\cdot\,]_{\mathfrak{g}},[\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \arrow[d, "\mathrm{T}_{\mathrm{red}}"] & (D^\bullet_{\mathfrak{g}}(M)[[\hbar]],\hbar\lambda,\partial^{\mathfrak{g}}-[J_0,\cdot\,]_{\mathfrak{g}},[\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \arrow[d, "D_{\mathrm{red}}"] \\
(T^\bullet_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]],0,0,[\,\cdot\,,\cdot\,]_S) \arrow[r, "F_{\mathrm{red}}"] & (D^\bullet_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]],0,\partial,[\,\cdot\,,\cdot\,]_{\mathrm{G}})
\end{tikzcd}$$

where F_{red} is the standard Dolgushev formality with respect to a torsion-free covariant derivative on M_{red}. Also, in [Esposito et al. 2022a] we show that the Dolgushev formality is compatible with λ under suitable flatness assumptions. In these flat cases it induces an L_∞-morphism

$$F^{\mathfrak{g}} : (T^\bullet_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, [-J_0, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \longrightarrow (D^\bullet_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, \partial^{\mathfrak{g}} - [J_0, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}),$$

which gives the fourth arrow in the above diagram, and we plan to investigate its commutativity (up to homotopy) in future work.

The results of this paper are partially based on [Kraft 2021] and the paper is organized as follows. In Section 2 we recall the basic notions of (curved) L_∞-algebras, L_∞-morphisms and twists and fix the notation. Then we introduce in Section 2B the curved DGLA of equivariant polydifferential operators and show that they indeed control Hamiltonian actions. In Section 3 we construct the global reduction L_∞-morphism to the polydifferential operators on the reduced manifold. Finally, we compare in Section 4 the reduction via this reduction morphism D_{red} with a slightly modified BRST reduction of equivariant star products as explained in Appendix A, where we also recall the homological perturbation lemma. In Appendix B we give explicit formulas for the transferred L_∞-structure and the L_∞-projection induced by the homotopy transfer theorem.

2. Preliminaries

2A. *L_∞-algebras, Maurer–Cartan elements and twisting.* In this section we recall the notions of (curved) L_∞-algebras, L_∞-morphisms and their twists by Maurer–Cartan elements to fix the notation. Proofs and further details can be found in [Dolgushev 2005a; 2005b; Esposito and de Kleijn 2021].

We denote by $V^\bullet$ a graded vector space over a field $\mathbb{K}$ of characteristic 0 and define the *shifted* vector space $V[k]^\bullet$ by

$$V[k]^\ell = V^{\ell+k}.$$

A degree $+1$ coderivation Q on the coaugmented counital conilpotent cocommutative coalgebra $S^c(\mathfrak{L})$ cofreely cogenerated by the graded vector space $\mathfrak{L}[1]^\bullet$ over $\mathbb{K}$ is called an L_∞-*structure* on the graded vector space $\mathfrak{L}$ if $Q^2 = 0$. The (universal) coalgebra $S^c(\mathfrak{L})$ can be realized as the symmetrized deconcatenation coproduct on the space $\bigoplus_{n\geq 0} \mathrm{S}^n\mathfrak{L}[1]$ where $\mathrm{S}^n\mathfrak{L}[1]$ is the space of coinvariants for the usual (graded) action of S_n (the symmetric group in n letters) on $\otimes^n(\mathfrak{L}[1])$; see, for example, [Esposito and de Kleijn 2021]. Any degree $+1$ coderivation Q on $S^c(\mathfrak{L})$ is uniquely determined by the components

$$Q_n : \mathrm{S}^n(\mathfrak{L}[1]) \longrightarrow \mathfrak{L}[2] \tag{2-1}$$

through the formula

$$Q(\gamma_1 \vee \cdots \vee \gamma_n) = \sum_{k=0}^{n} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \epsilon(\sigma) Q_k(\gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(k)}) \vee \gamma_{\sigma(k+1)} \vee \cdots \vee \gamma_{\sigma(n)}. \tag{2-2}$$

Here $\mathrm{Sh}(k, n-k)$ denotes the set of $(k, n-k)$ shuffles in S_n, $\epsilon(\sigma) = \epsilon(\sigma, \gamma_1, \dots, \gamma_n)$ is a sign given by the rule $\gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(n)} = \epsilon(\sigma)\gamma_1 \vee \cdots \vee \gamma_n$ and we use the conventions that $\mathrm{Sh}(n, 0) = \mathrm{Sh}(0, n) = \{\mathrm{id}\}$ and that the empty product equals the unit. Note in particular that we also consider a term Q_0 and thus we are actually considering curved L_∞-algebras. Sometimes we also write $Q_k = Q_k^1$ and, following [Canonaco 1999], we denote by Q_n^i the component of $Q_n^i : \mathrm{S}^n \mathfrak{L}[1] \to \mathrm{S}^i \mathfrak{L}[2]$ of Q. It is given by

$$Q_n^i(x_1 \vee \cdots \vee x_n) = \sum_{\sigma \in \mathrm{Sh}(n+1-i, i-1)} \epsilon(\sigma) Q^1_{n+1-i}(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(n+1-i)}) \vee x_{\sigma(n+2-i)} \vee \cdots \vee x_{\sigma(n)}, \tag{2-3}$$

where Q^1_{n+1-i} are the usual structure maps.

Example 2.1 (curved DGLA). A basic example of an L_∞-algebra is that of a (curved) differential graded Lie algebra $(\mathfrak{g}, R, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ obtained by setting $Q_0(1) = -R$, $Q_1 = -\mathrm{d}$, $Q_2(\gamma \vee \mu) = -(-1)^{|\gamma|}[\gamma, \mu]$ and $Q_i = 0$ for all $i \geq 3$. Note that we denoted by $|\cdot|$ the degree in $\mathfrak{g}[1]$.

Let us consider two L_∞-algebras $(\mathfrak{L}, Q)$ and $(\widetilde{\mathfrak{L}}, \widetilde{Q})$. A degree-0 counital coalgebra morphism

$$F : S^c(\mathfrak{L}) \longrightarrow S^c(\widetilde{\mathfrak{L}})$$

such that $FQ = \widetilde{Q}F$ is said to be an *L_∞-morphism.* A coalgebra morphism F from $S^c(\mathfrak{L})$ to $S^c(\widetilde{\mathfrak{L}})$ such that $F(1) = 1$ is uniquely determined by its components (also called *Taylor coefficients*)

$$F_n : \mathrm{S}^n(\mathfrak{L}[1]) \longrightarrow \widetilde{\mathfrak{L}}[1],$$

where $n \geq 1$. Namely, we set $F(1) = 1$ and use the formula

$$F(\gamma_1 \vee \cdots \vee \gamma_n) = \sum_{p \geq 1} \sum_{\substack{k_1, \dots, k_p \geq 1 \\ k_1 + \cdots + k_p = n}} \sum_{\sigma \in \mathrm{Sh}(k_1, \dots, k_p)} \frac{\epsilon(\sigma)}{p!} F_{k_1}(\gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(k_1)}) \vee \cdots \vee F_{k_p}(\gamma_{\sigma(n-k_p+1)} \vee \cdots \vee \gamma_{\sigma(n)}),$$

where $\mathrm{Sh}(k_1, \dots, k_p)$ denotes the set of $(k_1, \dots, k_p)$-shuffles in S_n (again we set $\mathrm{Sh}(n) = \{\mathrm{id}\}$). We also write $F_k = F_k^1$ and similarly to (2-3) we get coefficients $F_n^j : \mathrm{S}^n \mathfrak{L}[1] \to \mathrm{S}^j \widetilde{\mathfrak{L}}[1]$ of F by taking the corresponding terms in [Dolgushev 2006, Equation (2.15)]. Note that F_n^j only depends on $F_k^1 = F_k$ for $k \leq n - j + 1$. Given an L_∞-morphism F of (noncurved) L_∞-algebras $(\mathfrak{L}, Q)$ and $(\widetilde{\mathfrak{L}}, \widetilde{Q})$, we obtain the map of complexes

$$F_1 : (\mathfrak{L}, Q_1) \longrightarrow (\widetilde{\mathfrak{L}}, \widetilde{Q}_1).$$

In this case the L_∞-morphism F is called an *L_∞-quasi-isomorphism* if F_1 is a quasi-isomorphism of complexes. Given a DGLA $(\mathfrak{g}, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ and an element $\pi \in \mathfrak{g}[1]^0$ we can obtain a curved Lie algebra by defining a new differential $\mathrm{d} + [\pi, \cdot\,]$ and considering the curvature $R^\pi = \mathrm{d}\pi + \frac{1}{2}[\pi, \pi]$. In fact the same procedure can be applied to a curved Lie algebra $(\mathfrak{g}, R, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ to obtain the *twisted* curved Lie algebra $(\mathfrak{L}, R^\pi, \mathrm{d} + [\pi, \cdot\,], [\,\cdot\,,\,\cdot\,])$, where

$$R^\pi := R + \mathrm{d}\pi + \tfrac{1}{2}[\pi, \pi]. \tag{2-4}$$

The element π is called a *Maurer–Cartan element* if it satisfies the equation

$$R + \mathrm{d}\pi + \tfrac{1}{2}[\pi, \pi] = 0. \tag{2-5}$$

Finally, it is important to recall that given a DGLA morphism, or more generally an L_∞-morphism, $F : \mathfrak{g} \to \mathfrak{g}'$ between two DGLAs, one may associate to any (curved) Maurer–Cartan element $\pi \in \mathfrak{g}[1]^0$ a (curved) Maurer–Cartan element

$$\pi_F := \sum_{n \geq 1} \frac{1}{n!} F_n(\pi \vee \cdots \vee \pi) \in \mathfrak{g}'[1]^0. \tag{2-6}$$

In order to make sense of these infinite sums we consider DGLAs with complete descending filtrations

$$\cdots \supseteq \mathcal{F}^{-2}\mathfrak{g} \supseteq \mathcal{F}^{-1}\mathfrak{g} \supseteq \mathcal{F}^{0}\mathfrak{g} \supseteq \mathcal{F}^{1}\mathfrak{g} \supseteq \cdots, \qquad \mathfrak{g} \cong \varprojlim \mathfrak{g}/\mathcal{F}^n\mathfrak{g} \tag{2-7}$$

and

$$\mathrm{d}(\mathcal{F}^k\mathfrak{g}) \subseteq \mathcal{F}^k\mathfrak{g} \quad \text{and} \quad [\mathcal{F}^k\mathfrak{g}, \mathcal{F}^\ell\mathfrak{g}] \subseteq \mathcal{F}^{k+\ell}\mathfrak{g}. \tag{2-8}$$

In particular, $\mathcal{F}^1\mathfrak{g}$ is a projective limit of nilpotent DGLAs. In most cases the filtration is bounded below, i.e., bounded from the left with $\mathfrak{g} = \mathcal{F}^k\mathfrak{g}$ for some $k \in \mathbb{Z}$. If the filtration is unbounded, then we assume always that it is exhaustive, i.e., that

$$\mathfrak{g} = \bigcup_n \mathcal{F}^n\mathfrak{g}, \tag{2-9}$$

even if we do not mention it explicitly. Also, we assume that the DGLA morphisms are compatible with the filtrations. Considering only Maurer–Cartan elements in $\mathcal{F}^1\mathfrak{g}^1$ ensures the well-definedness of (2-6). Mainly, the filtration is induced by formal power series in a formal parameter $\hbar$. Starting with a DGLA $(\mathfrak{g}, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$, its $\hbar$-linear extension to formal power series $\mathfrak{G} = \mathfrak{g}[[\hbar]]$ of a DGLA $\mathfrak{g}$ has the complete descending filtration $\mathcal{F}^k\mathfrak{G} = \hbar^k\mathfrak{G}$.

One cannot only twist the DGLAs and L_∞-algebras, but also the L_∞-morphisms between them. Below we need the following result; see [Dolgushev 2006, Proposition 2; 2005b, Proposition 1].

Proposition 2.2. *Let $F : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ be an L_∞-morphism of DGLAs, $\pi \in \mathcal{F}^1\mathfrak{g}^1$ a Maurer–Cartan element and $S = F^1(\overline{\exp}(\pi)) \in \mathcal{F}^1\mathfrak{g}'^1$.*

(i) *The map*

$$F^\pi = \exp(-S\vee) F \exp(\pi\vee) : \bar{\mathrm{S}}(\mathfrak{g}[1]) \longrightarrow \bar{\mathrm{S}}(\mathfrak{g}'[1])$$

defines an L_∞-morphism between the DGLAs $(\mathfrak{g}, \mathrm{d}+[\pi, \cdot\,])$ and $(\mathfrak{g}', \mathrm{d}+[S, \cdot\,])$.

(ii) *The structure maps of F^π are given by*

$$F^\pi_n(x_1, \dots, x_n) = \sum_{k=0}^{\infty} \frac{1}{k!} F_{n+k}(\pi, \dots, \pi, x_1, \dots, x_n). \tag{2-10}$$

(iii) *Let F be an L_∞-quasi-isomorphism where F^1_1 is not only a quasi-isomorphism of filtered complexes $L \to L'$ but even induces a quasi-isomorphism*

$$F^1_1 : \mathcal{F}^k L \longrightarrow \mathcal{F}^k L'$$

for each k. Then F^π is an L_∞-quasi-isomorphism.

2B. ***Equivariant polydifferential operators.*** In the following we present some basic results concerning equivariant polydifferential operators, which are basically folklore knowledge and are based on [Tsygan 2010].

Let us consider the DGLA of *polydifferential operators* on a smooth manifold M

$$(D^\bullet_{\mathrm{poly}}(M), \partial = [\mu, \cdot\,]_{\mathrm{G}}, [\,\cdot\,, \cdot\,]_{\mathrm{G}}) \tag{2-11}$$

Here

$$D^\bullet_{\mathrm{poly}}(M) = \bigoplus_{n=-1}^{\infty} D^n_{\mathrm{poly}}(M),$$

where $D^n_{\mathrm{poly}}(M) = \mathrm{Hom}_{\mathrm{diff}}(\mathscr{C}^\infty(M)^{\otimes n+1}, \mathscr{C}^\infty(M))$ are the differentiable Hochschild cochains vanishing on constants. We use the sign convention from [Bursztyn et al. 2012] for the Gerstenhaber bracket $[\,\cdot\,, \cdot\,]$, not the original one from [Gerstenhaber 1963]. Explicitly

$$[D, E]_{\mathrm{G}} = (-1)^{|E||D|}(D \circ E - (-1)^{|D||E|} E \circ D) \tag{2-12}$$

with

$$D \circ E(a_0, \dots, a_{d+e}) = \sum_{i=0}^{|D|} (-1)^{i|E|} D(a_0, \dots, a_{i-1}, E(a_i, \dots, a_{i+e}), a_{i+e+1}, \dots, a_{d+e}) \tag{2-13}$$

for homogeneous $D, E \in D^{\bullet}_{\mathrm{poly}}(M)$ and $a_0, \dots, a_{d+e} \in \mathscr{C}^\infty(M)$. Also, μ denotes the commutative pointwise product on $\mathscr{C}^\infty(M)[[\hbar]]$ and ∂ is the usual Hochschild differential.

We are interested in the case of group actions where we always consider a (left) action $\Phi : \mathrm{G} \times M \to M$ of a connected Lie group G. Let M be now equipped with a G-invariant star product $\star$, that is, an associative product $\star = \mu + \sum_{r=1}^\infty \hbar^r C_r = \mu_0 + \hbar m_\star \in (D^1_{\mathrm{poly}}(M))^{\mathrm{G}}[[\hbar]]$. Recall that a linear map $H : \mathfrak{g} \to \mathscr{C}^\infty(M)[[\hbar]]$ is called a *quantum momentum map* if

$$\mathscr{L}_{\xi_M} = -\frac{1}{\hbar}[H(\xi), \cdot\,]_\star \quad \text{and} \quad \frac{1}{\hbar}[H(\xi), H(\eta)]_\star = H([\xi, \eta]),$$

where ξ_M denotes the fundamental vector field corresponding to the action Φ.

A pair $(\star, H)$ consisting of an invariant star product $\star = \mu + \hbar m_\star$ and a quantum momentum map H is also called *equivariant star product*. They are useful since they allow for a BRST like reduction scheme; see Appendix A. We introduce now the DGLA that contains the data of Hamiltonian actions, i.e., of equivariant star products. Here we follow [Tsygan 2010].

Definition 2.3 (equivariant polydifferential operators). The DGLA of *equivariant polydifferential operators* $(D^\bullet_{\mathfrak{g}}(M), \partial^{\mathfrak{g}}, [\,\cdot\,,\,\cdot\,]_{\mathfrak{g}})$ is defined by

$$D^k_{\mathfrak{g}}(M) = \bigoplus_{2i+j=k} (\mathrm{S}^i \mathfrak{g}^* \otimes D^j_{\mathrm{poly}}(M))^{\mathrm{G}} \tag{2-14}$$

with bracket

$$[\alpha \otimes D_1, \beta \otimes D_2]_{\mathfrak{g}} = \alpha \vee \beta \otimes [D_1, D_2]_{\mathrm{G}} \tag{2-15}$$

and differential

$$\partial^{\mathfrak{g}}(\alpha \otimes D_1) = \alpha \otimes \partial D_1 = \alpha \otimes [\mu, D_1]_{\mathrm{G}} \tag{2-16}$$

for $\alpha \otimes D_1, \beta \otimes D_2 \in D^\bullet_{\mathfrak{g}}(M)$. Here we denote by ∂ and $[\,\cdot\,,\,\cdot\,]_{\mathrm{G}}$ the usual Hochschild differential and Gerstenhaber bracket on the polydifferential operators, respectively, and by μ the pointwise multiplication of $\mathscr{C}^\infty(M)$.

Notice that invariance with respect to the group action means invariance under the transformations $\mathrm{Ad}^*_g \otimes \Phi^*_g$ for all $g \in G$, and that the equivariant polydifferential operators can be interpreted as equivariant polynomial maps $\mathfrak{g} \to D_{\mathrm{poly}}(M)$. We introduce the canonical linear map

$$\lambda : \mathfrak{g} \ni \xi \longmapsto \mathscr{L}_{\xi_M} \in D^0_{\mathrm{poly}}(M),$$

and see that $\lambda \in D^2_{\mathfrak{g}}(M)$ is central and moreover $\partial^{\mathfrak{g}}\lambda = 0$. This implies that we can see $D^\bullet_{\mathfrak{g}}(M)$ either as a flat DGLA with the above structures or as a curved DGLA with the above structures and curvature λ. In the case of formal power series we rescale the curvature again by $\hbar^2$ and obtain the following characterization of Maurer–Cartan elements:

Lemma 2.4. *A curved formal Maurer–Cartan element* $\Pi \in \hbar D^1_{\mathfrak{g}}(M)[[\hbar]]$, *that is, an element* Π *satisfying*

$$\hbar^2\lambda + \partial^{\mathfrak{g}}\Pi + \tfrac{1}{2}[\Pi, \Pi]_{\mathfrak{g}} = 0, \tag{2-17}$$

is equivalent to a pair $(m_\star, H)$, *where* $m_\star \in D^1_{\mathrm{poly}}(M))^{\mathrm{G}}[[\hbar]]$ *defines a* G*-invariant star product via* $\star = \mu + \hbar m_\star$ *with quantum momentum map* $H : \mathfrak{g} \to \mathscr{C}^\infty(M)[[\hbar]]$. *In other words*, $(\star, H)$ *is an equivariant star product.*

Proof. We have the decomposition

$$\Pi = \hbar m_\star - \hbar H \in \hbar(D^1_{\mathrm{poly}}(M))^{\mathrm{G}} \oplus (\mathfrak{g}^* \otimes D^{-1}_{\mathrm{poly}}(M))^{\mathrm{G}}[[\hbar]].$$

Then the curved Maurer–Cartan equation applied to an element $\xi \in \mathfrak{g}$ reads

$$\begin{aligned} -\hbar^2\mathscr{L}_{\xi_M} &= -\hbar^2\lambda(\xi) = \partial^{\mathfrak{g}}\Pi(\xi) + \tfrac{1}{2}[\Pi, \Pi]_{\mathfrak{g}}(\xi) \\ &= \hbar[\mu, m_\star]_{\mathrm{G}} + \tfrac{1}{2}\hbar^2[m_\star, m_\star]_{\mathrm{G}} - \hbar^2[m_\star, H(\xi)]_{\mathrm{G}}. \end{aligned}$$

This is equivalent to the fact that $\hbar m_\star$ is Maurer–Cartan in the flat setting and that $\mathscr{L}_{\xi_M} = -\frac{1}{\hbar}[H(\xi), -]_\star$, since $\hbar[m_\star, H(\xi)]_{\mathrm{G}}(f) = -[H(\xi), f]_\star$ for $f \in \mathscr{C}^\infty(M)$. Then the invariance of both elements implies that $\star = \mu + \hbar m_\star$ is a G-invariant star product with quantum momentum map H. □

Two equivariant star products $\hbar(m_\star - H)$ and $\hbar(m'_\star - H')$ are called *equivariantly equivalent* if they are gauge equivalent, i.e., if there exists an $\hbar T \in \hbar D^0_{\mathrm{poly}}(M)^{\mathrm{G}}[[\hbar]] \subset D^0_{\mathfrak{g}}(M)$ such that

$$\hbar(m'_\star - H') = \exp(\hbar[T, \cdot\,]_{\mathfrak{g}}) \triangleright \hbar(m_\star - H) = \exp(\hbar[T, \cdot\,]_{\mathfrak{g}})(\mu + \hbar(m_\star - H)) - \mu.$$

This means that $S = \exp(\hbar T)$ satisfies for all $f, g \in \mathscr{C}^\infty(M)[[\hbar]]$

$$S(f \star g) = Sf \star' Sg \quad \text{and} \quad SH = H'.$$

3. Reduction of the equivariant polydifferential operators

Now we aim to describe a reduction scheme for general equivariant polydifferential operators via an L_∞-morphism denoted by D_{red}, generalizing the results for the polyvector fields from [Esposito et al. 2022b].

Let M be a smooth manifold with action $\Phi : \mathrm{G} \times M \to M$ of a connected Lie group and let $(\star, H = J + \hbar J')$ be an equivariant star product, that is, a curved formal Maurer–Cartan element in the equivariant polydifferential operators; see Lemma 2.4. Here the component $J : M \to \mathfrak{g}^*$ of the quantum momentum map H in $\hbar$-order zero is a classical momentum map with respect to the Poisson structure induced by the skew-symmetrization of the $\hbar^1$-part of $\star$. We assume from now on that $0 \in \mathfrak{g}^*$ is a value and a regular value of J and set $C = J^{-1}(\{0\})$. In addition, we require the action to be proper around C and free on C. Then $M_{\mathrm{red}} = C/\mathrm{G}$ is a smooth manifold and we denote by $\iota : C \to M$ the inclusion and by $\mathrm{pr} : C \to M_{\mathrm{red}}$ the projection on the quotient. Moreover, the properness around C implies that there exists an G-invariant open neighborhood $M_{\mathrm{nice}} \subseteq M$ of C and a G-equivariant diffeomorphism $\Psi : M_{\mathrm{nice}} \to U_{\mathrm{nice}} \subseteq C \times \mathfrak{g}^*$, where U_{nice} is an open neighborhood of $C \times \{0\}$ in $C \times \mathfrak{g}^*$. Here the Lie group G acts on $C \times \mathfrak{g}^*$ as $\Phi_g = \Phi^C_g \times \mathrm{Ad}^*_{g^{-1}}$, where Φ^C is the induced action on C, and the momentum map on U_{nice} is the projection to $\mathfrak{g}^*$ (see [Bordemann et al. 2000, Lemma 3; Gutt and Waldmann 2010]).

From now on we assume $M = M_{\mathrm{nice}}$. Then we can define an equivariant *prolongation map* by

$$\mathrm{prol} : \mathscr{C}^\infty(C) \ni \phi \longmapsto (\mathrm{pr}_1 \circ \Psi)^* \phi \in \mathscr{C}^\infty(M_{\mathrm{nice}})$$

and we directly get $\iota^* \,\mathrm{prol} = \mathrm{id}_{\mathscr{C}^\infty(C)}$.

Consider the Taylor expansion around C in the $\mathfrak{g}^*$-direction as in [Esposito et al. 2022b, Section 4.1], which is a map

$$D_{\mathfrak{g}^*} : D^k_{\mathrm{poly}}(C \times \mathfrak{g}^*) \longmapsto \prod_{i=0}^{\infty} (\mathrm{S}^i \mathfrak{g} \otimes T^{k+1}(\mathrm{S}\mathfrak{g}^*) \otimes D^k_{\mathrm{poly}}(C)),$$

where $T^\bullet(\mathrm{S}\mathfrak{g}^*)$ denotes the tensor algebra of $\mathrm{S}\mathfrak{g}^*$. Note that we are only interested in a subspace since we consider polydifferential operators vanishing on constants. Slightly abusing the notation, the Taylor expansion of the equivariant polydifferential operators takes then the following form:

$$D_{\mathrm{Tay}}(C \times \mathfrak{g}^*) = \Bigl(\mathrm{S}\mathfrak{g}^* \otimes \prod_{i=0}^{\infty} (\mathrm{S}^i \mathfrak{g} \otimes T(\mathrm{S}\mathfrak{g}^*) \otimes D_{\mathrm{poly}}(C)) \Bigr)^{\mathrm{G}} \tag{3-1}$$

and one easily checks that this yields an equivariant DGLA morphism

$$D_{\mathfrak{g}^*} : (D_{\mathfrak{g}}(M), \lambda, \partial^{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \longrightarrow (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*), \lambda, \partial, [\,\cdot\,,\cdot\,]). \tag{3-2}$$

Our goal consists in finding a reduction morphism from

$$D_{\mathrm{red}} : (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[\![\hbar]\!], \hbar\lambda, \partial+[-J,\cdot\,], [\,\cdot\,,\cdot\,]) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!], \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}}).$$

Following a similar strategy as in [Esposito et al. 2022b], we construct L_∞-morphisms

$$D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[\![\hbar]\!] \longrightarrow \left(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}[\![\hbar]\!] \longrightarrow D_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!] \tag{3-3}$$

with suitable L_∞-structures on the three spaces, where $\left(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}[\![\hbar]\!]$ is a candidate for a Cartan model.

3A. ***A 'partial' homotopy for the Hochschild differential.*** In order to find a suitable analogue of the Cartan model for the polydifferential operators, we need to understand the cohomology of

$$(D_{\mathfrak{g}}(M), \partial^{\mathfrak{g}}-[J,\cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}})$$

and in particular the role of the differential $[-J,\cdot\,]_{\mathfrak{g}}$. To this end we construct a 'partial' homotopy for $\partial^{\mathfrak{g}}-[J,\cdot\,]_{\mathfrak{g}}$. Here we use the results concerning the homotopy for the Hochschild differential from [De Wilde and Lecomte 1995]. In particular, we restrict ourselves to the subspace of normalized differential Hochschild cochains, i.e., polydifferential operators vanishing on constants. One can show that they are quasi-isomorphic to the differential ones. Recall the maps

$$\Phi : D^a_{\mathrm{poly}}(M) \longrightarrow D^{a-1}_{\mathrm{poly}}(M),$$

$$\Phi(A)(f_0,\dots,f_{a-1}) = \sum_{t=1}^{n}\sum_{i}\sum_{j=i}^{a-1}(-1)^i A\Big(f_0,\dots,f_{i-1},x^t,\dots,\frac{\partial}{\partial x^t}f_j,\dots,f_{a-1}\Big),$$

for $f_1,\dots,f_{a-1}\in\mathscr{C}^\infty(M)$, and

$$\Psi : D^a_{\mathrm{poly}}(M)\ni A\longmapsto (-1)^a[x^i,A]_{\mathrm{G}}\cup\frac{\partial}{\partial x^i} = (-1)^{a+1}\sum_{i=1}^{n}(A\circ x^i)\cup\frac{\partial}{\partial x^i}\in D^a_{\mathrm{poly}}(M),$$

for local coordinates $(x^1,\dots,x^n)$ of M. They satisfy, by [De Wilde and Lecomte 1995, Proposition 4.1], the condition

$$\Phi\circ\partial+\partial\circ\Phi = -(\deg_D\cdot\mathrm{id}+\Psi), \tag{3-4}$$

where $\deg_D$ is the order of the differential operator.

We assume from now on for simplicity $M=C\times\mathfrak{g}^*$ and $J=\mathrm{pr}_{\mathfrak{g}^*}$ and we want to find a suitable Cartan model for the polydifferential operators. Similarly to

[Esposito et al. 2022b, Definition 4.14] for the polyvector field case, we want to obtain a DGLA structure on

$$\left(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}. \tag{3-5}$$

Hence we adapt the maps Φ and Ψ in such a way that they only include coordinates $J_i=\alpha_i=e_i$ on $\mathfrak{g}^*$ with $i=1,\dots,n$:

$$\Phi(A)(f_0,\dots,f_{a-1})=\sum_{t=1}^{n}\sum_{i\le j<a}(-1)^i A\Big(f_0,\dots,f_{i-1},e_t,\dots,\frac{\partial}{\partial e_t}f_j,\dots,f_{a-1}\Big),$$

$$\Psi(A)=(-1)^{a+1}\sum_{i=1}^{n}(A\circ e_i)\cup\frac{\partial}{\partial e_i},$$

where $A\in D^a_{\mathrm{poly}}(C\times\mathfrak{g}^*)$ and $f_0,\dots,f_{a-i}\in\mathscr{C}^\infty(C\times\mathfrak{g}^*)$.

Proposition 3.1. *One has on* $D_{\mathrm{poly}}(C\times\mathfrak{g}^*)$

$$\Phi\circ\partial+\partial\circ\Phi=-(\deg_{\mathfrak{g}}\cdot\mathrm{id}+\Psi), \tag{3-6}$$

where $\deg_{\mathfrak{g}}$ *is the order of differentiations in the direction of* $\mathfrak{g}^*$*-coordinates.*

Proof. The proof follows the same lines as in [De Wilde and Lecomte 1995, Proposition 4.1]. It is proven by induction on the degree of a of $A\in D^a_{\mathrm{poly}}(C\times\mathfrak{g}^*)$. For $a=0$ and $A\in D^0_{\mathrm{poly}}(C\times\mathfrak{g}^*)$ as well as $f\in\mathscr{C}^\infty(C\times\mathfrak{g}^*)$ we get

$$\begin{aligned}((\Phi\circ\partial+\partial\circ\Phi)(A))(f)&=(\partial A)\Big(e_i,\frac{\partial}{\partial e_i}f\Big)\\&=e_iA\Big(\frac{\partial}{\partial e_i}f\Big)-A\Big(e_i\frac{\partial}{\partial e_i}f\Big)+A(e_i)\frac{\partial}{\partial e_i}f\\&=(-\deg_{\mathfrak{g}}(A)\,A-\Psi(A))(f).\end{aligned}$$

Note that Ψ has the following compatibility with the $\cup$-product:

$$\Psi(A\cup B)=(\Psi A)\cup B+A\cup(\Psi B)+(-1)^a(A\circ e_i)\cup\Big(\frac{\partial}{\partial e_i}\cup B+(-1)^bB\cup\frac{\partial}{\partial e_i}\Big).$$

Writing $\mathrm{i}(A)(\,\cdot\,)=(\,\cdot\,)\circ A$ one computes

$$\begin{aligned}(\Phi\circ\partial+\partial\circ\Phi)(A\cup B)&=((\Phi\circ\partial+\partial\circ\Phi)A)\cup B+A\cup((\Phi\circ\partial+\partial\circ\Phi)B)\\&\quad+((\mathrm{i}(e_i)\circ\partial+\partial\circ\mathrm{i}(e_i))A)\cup\mathrm{i}\Big(\frac{\partial}{\partial e_i}\Big)B\\&\quad+(-1)^a(\mathrm{i}(e_i)A)\cup\Big(\partial\circ\mathrm{i}\Big(\frac{\partial}{\partial e_i}\Big)-\mathrm{i}\Big(\frac{\partial}{\partial e_i}\Big)\circ\partial\Big)B.\end{aligned} \tag{3-7}$$

The operators $(\mathrm{i}(e_i)\circ\partial+\partial\circ\mathrm{i}(e_i))$ and $(\partial\circ\mathrm{i}(\partial/\partial e_i)-\mathrm{i}(\partial/\partial e_i)\circ\partial)$ are graded commutators of derivations of the $\cup$-product and are therefore graded derivations.

Thus they are determined by their action on $D^{-1}_{\mathrm{poly}}(C \times \mathfrak{g}^*)$ and $D^0_{\mathrm{poly}}(C \times \mathfrak{g}^*)$. The first one obviously vanishes. The second coincides on these generators with

$$A \longmapsto -\Big(\frac{\partial}{\partial e_i} \cup A + (-1)^a A \cup \frac{\partial}{\partial e_i}\Big)$$

and the proposition is shown. □

As in [Esposito et al. 2022b], we define a homotopy on the equivariant polydifferential operators

$$\begin{aligned}\hat{h} : (\mathrm{S}\mathfrak{g}^* \otimes D^d_{\mathrm{poly}}(C \times \mathfrak{g}^*))^{\mathrm{G}} \ni P \otimes D \longmapsto& \\ (-1)^{d+1}\, \mathrm{i_s}(e_i) P \otimes D \cup \frac{\partial}{\partial e_i} &\in (\mathrm{S}\mathfrak{g}^* \otimes D^{d+1}_{\mathrm{poly}}(C \times \mathfrak{g}^*))^{\mathrm{G}}.\end{aligned}$$

The fact that $\hat{h}$ maps invariant elements to invariant ones follows as in the case of polyvector fields. Finally, note that Φ and Ψ are equivariant, whence they can be extended to the equivariant polydifferential operators, where we can show:

Proposition 3.2. *One has on* $(\mathrm{S}\mathfrak{g}^* \otimes D_{\mathrm{poly}}(C \times \mathfrak{g}^*))^{\mathrm{G}}$

$$[\hat{h} - \Phi, \partial^{\mathfrak{g}} + [-J, \cdot\,]_{\mathfrak{g}}] = (\deg_{\mathrm{S}\mathfrak{g}^*} + \deg_{\mathfrak{g}})\,\mathrm{id}, \tag{3-8}$$

where $\deg_{\mathfrak{g}}$ *is again the order of differentiations in the direction of* $\mathfrak{g}^*$*-coordinates.*

Proof. From (3-6) we know $[\Phi, \partial^{\mathfrak{g}}] = -(\deg_{\mathfrak{g}} \cdot \mathrm{id} + \Psi)$. In addition, one has for homogeneous $P \otimes D$

$$\begin{aligned}\hat{h} \circ \partial^{\mathfrak{g}}(P \otimes D) &= (-1)^{d+2}\, \mathrm{i_s}(e_i) P \otimes (\partial D) \cup \frac{\partial}{\partial e_i} = -(-1)^{d+1}\, \mathrm{i_s}(e_i) P \otimes \partial\Big(D \cup \frac{\partial}{\partial e_i}\Big) \\ &= -\partial^{\mathfrak{g}} \circ \hat{h}(P \otimes D).\end{aligned}$$

Since we consider only differential operators vanishing on constants, one checks easily that also $[\Phi, [-J, \cdot\,]_{\mathfrak{g}}] = 0$. Finally,

$$\begin{aligned}[\hat{h}, [-J, \cdot\,]_{\mathfrak{g}}](P \otimes D) &= (-1)^d\, \mathrm{i_s}(e_i)(e^j \vee P) \otimes [-J_j, D] \cup \frac{\partial}{\partial e_i} \\ &\qquad + (-1)^{d+1} e^j \vee \mathrm{i_s}(e_i) P \otimes \Big[-J_j, D \cup \frac{\partial}{\partial e_i}\Big] \\ &= -\Psi(P \otimes D) + (-1)^d e^j \vee \mathrm{i_s}(e_i) P \otimes [-J_j, D] \cup \frac{\partial}{\partial e_i} \\ &\qquad + (-1)^{d+1} e^j \vee \mathrm{i_s}(e_i) P \otimes [-J_j, D] \cup \frac{\partial}{\partial e_i} + \deg_{\mathrm{S}\mathfrak{g}^*}(P) P \otimes D \\ &= (\deg_{\mathrm{S}\mathfrak{g}^*} \cdot \mathrm{id} - \Psi) P \otimes D.\end{aligned}$$

Thus the proposition is shown. □

The above constructions work also for the Taylor series expansion of the equivariant polydifferential operators, where we restrict ourselves again to polydifferential operators vanishing on constants. We slightly abuse the notation and denote them again by $D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$; see (3-1). Writing

$$h = \begin{cases} \frac{1}{\deg_{\mathrm{S}\mathfrak{g}^*} + \deg_{\mathfrak{g}}}(\hat{h} - \Phi) & \text{if } \deg_{\mathrm{S}\mathfrak{g}^*} + \deg_{\mathfrak{g}} \neq 0, \\ 0 & \text{else,} \end{cases} \tag{3-9}$$

we get the following result:

Proposition 3.3. *One has a deformation retract*

$$\Big(\big(\textstyle\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}[\![\hbar]\!], \partial\Big) \underset{p}{\overset{i}{\rightleftarrows}} (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], \partial + [-J, \cdot\,]) \circlearrowleft h \tag{3-10}$$

where p and i denote the obvious projection and inclusion. This means that one has $pi = \mathrm{id}$ and $\mathrm{id} - ip = [h, \partial + [-J, \cdot\,]]$. Also, the identities $hi = 0 = ph$ hold.

Remark 3.4. Note that one has $h^2 \neq 0$, i.e., the above retract is not a special deformation retract. However, by the results of [Huebschmann 2011b, Remark 2.1] we know that this could also be achieved.

The reduction works now in two steps. At first, we use the homological perturbation lemma from Proposition A.1 to deform the differential on $D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!]$, and in the second step we use the homotopy transfer theorem, see Theorem B.2, to extend the deformed projection to an L_∞-morphism. This will possibly give us higher brackets on $\big(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}[\![\hbar]\!]$ that we have to discuss.

3B. ***Application of the homological perturbation lemma.*** In our setting, the bundle $C \times \mathfrak{g} \to C$ can be equipped with the structure of a Lie algebroid since $\mathfrak{g}$ acts on C by the fundamental vector fields. The bracket of this *action Lie algebroid* is given by

$$[\xi, \eta]_{C \times \mathfrak{g}}(p) = [\xi(p), \eta(p)] - (\mathscr{L}_{\xi_C}\eta)(p) + (\mathscr{L}_{\eta_C}\xi)(p) \tag{3-11}$$

for $\xi, \eta \in \mathscr{C}^\infty(C, \mathfrak{g})$. The anchor is given by $\rho(p, \xi) = -\xi_C|_p$. In particular, one can check that π_{KKS} is the negative of the linear Poisson structure on its dual $C \times \mathfrak{g}^*$ in the convention of [Neumaier and Waldmann 2009].

For Lie algebroids there is a well-known construction of universal enveloping algebras [Moerdijk and Mrčun 2010; Neumaier and Waldmann 2009; Rinehart 1963]. It turns out that in our special case we get a simpler description of the universal enveloping algebra:

Proposition 3.5. *The universal enveloping algebra $U(C \times \mathfrak{g})$ of the action Lie algebroid $C \times \mathfrak{g}$ is isomorphic to $\mathscr{C}^\infty(C) \rtimes U(\mathfrak{g})$ with product*

$$(f, x) \cdot (g, y) = \sum (f \mathscr{L}(x_{(1)})(g), x_{(2)} y). \tag{3-12}$$

Here $y_{(1)} \otimes y_{(2)} = \Delta(y)$ denotes the coproduct on $U(\mathfrak{g})$ induced by extending $\Delta(\xi) = 1 \otimes \xi + \xi \otimes 1$ as an algebra morphism. Also, $\mathscr{L} : U(\mathfrak{g}) \to \mathrm{Diffop}(\mathscr{C}^\infty(C))$

is the extension of the anchor of the action algebroid, that is, of the negative fundamental vector fields, to the universal enveloping algebra. The same holds also in the formal setting of $U_\hbar(\mathfrak{g})$ with bracket rescaled by $\hbar$. Note that in this case one has to rescale $\mathscr{L}$ by powers of $\hbar$, that is, $\mathscr{L}_\xi = -\hbar\mathscr{L}_{\xi_C}$ for $\xi \in \mathfrak{g}$.

Proof. Note that the product is associative since

$$\begin{aligned}((f,x)\cdot(g,y))\cdot(h,z) &= \sum (f\mathscr{L}(x_{(1)})g, x_{(2)}y)\cdot(h,z)\\ &= \sum (f\mathscr{L}(x_{(1)})g\mathscr{L}(x_{(2)}y_{(1)})h, x_{(3)}y_{(2)}z)\\ &= \sum (f,x)\cdot(g\mathscr{L}(y_{(1)})h, y_{(2)}z) = (f,x)\cdot((g,y)\cdot(h,z)),\end{aligned}$$

where the penultimate identity follows with the coassociativity of Δ and the identity $\mathscr{L}(x)(fg) = \mathscr{L}(x_{(1)})(f)\mathscr{L}(x_{(2)})(g)$. The inclusions $\kappa_C : \mathscr{C}^\infty(C) \to \mathscr{C}^\infty(C) \rtimes U(\mathfrak{g})$ and $\kappa : \mathscr{C}^\infty(C)\otimes\mathfrak{g} \to \mathscr{C}^\infty(C)\rtimes U(\mathfrak{g})$ satisfy

$$[\kappa(s), \kappa_C(f)] = \kappa(\rho(s)f) \quad \text{and} \quad \kappa_C(f)\kappa(s) = \kappa(fs).$$

Thus the universal property gives the desired morphism $U(C\times\mathfrak{g}) \to \mathscr{C}^\infty(C)\rtimes U(\mathfrak{g})$. Recursively we can show that the right-hand side is generated by $u \in \mathscr{C}^\infty(C)$ and $\xi \in \mathscr{C}^\infty(C)\otimes\mathfrak{g}$ which gives the surjectivity of the morphism. Concerning injectivity, suppose $(f^{i_1}, e_{i_1})\cdots(f^{i_n}, e_{i_n}) = 0$ in $\mathscr{C}^\infty(C)\rtimes U(\mathfrak{g})$. We have to show that also $(f^{i_1}e_{i_1})\cdots(f^{i_1}e_{i_1}) = 0$ in $U(C\times\mathfrak{g})$. But this follows from a direct comparison of the terms in the corresponding associated graded algebras. □

It is worth mentioning that in [Huebschmann 1990] the above smashed product (used for Hopf algebras) is studied in a more general context.

Recall that by the Poincaré–Birkhoff–Witt theorem the map

$$\mathrm{S}(\mathfrak{g}) \ni x_1 \vee\cdots\vee x_n \longmapsto \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)}\cdots x_{\sigma(n)} \in U(\mathfrak{g})$$

is a coalgebra isomorphism with respect to the usual coalgebra structures induced by extending $\Delta(\xi) = \xi\otimes 1 + 1\otimes\xi$ for $\xi\in\mathfrak{g}$; see, for example, [Berezin 1967; Higgins 1969]. This statement holds also in the case of formal power series in $\hbar$ whence we can transfer the product on the universal enveloping algebra as in Proposition 3.5 to an associative product $\star_{\mathrm{G}} = \mu + \hbar m_{\mathrm{G}}$ on $\mathscr{C}^\infty(C)\otimes\mathrm{S}(\mathfrak{g})[[\hbar]]$.

Lemma 3.6. *The* ***Gutt product*** $\star_{\mathrm{G}}$ *on $\mathscr{C}^\infty(C)\otimes\mathrm{S}(\mathfrak{g})[[\hbar]]$ is* G*-invariant and $J = \mathrm{pr}_{\mathfrak{g}^*} : M = C\times\mathfrak{g}^* \to \mathfrak{g}^*$ is a momentum map, i.e.,*

$$-\mathscr{L}_{\xi_M} = \frac{1}{\hbar}\,\mathrm{ad}_{\star_{\mathrm{G}}}(J(\xi)). \tag{3-13}$$

Proof. The lemma follows directly from the explicit formula in Proposition 3.5. □

We deform the differential $\partial + [-J, \cdot\,]$ by $[\hbar m_{\mathrm{G}}, \cdot\,]$, that is, exactly by the higher orders of this product. The perturbed differential $\partial^{\mathfrak{g}} + [\hbar m_{\mathrm{G}} - J, \cdot\,] = [\star_{\mathrm{G}} - J, \cdot\,]$ squares indeed to zero since we have with the above lemma

$$[\star_{\mathrm{G}} - J, \cdot\,]^2 = \tfrac{1}{2}[[\star_{\mathrm{G}} - J, \star_{\mathrm{G}} - J], \cdot\,] = [-\hbar\lambda, \cdot\,] = 0,$$

where again $\lambda = e^i \otimes (e_i)_M$. By the homological perturbation lemma as formulated in Section A1 this yields a homotopy retract

$$\Big(\big(\textstyle\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}}[[\hbar]], \partial_\hbar\Big) \underset{p_\hbar}{\overset{i_\hbar}{\rightleftarrows}} (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[[\hbar]], [\star_{\mathrm{G}} - J, \cdot\,]) \circlearrowleft h_\hbar \tag{3-14}$$

with $B = [\hbar m_{\mathrm{G}}, \cdot\,]$ and

$$\begin{gathered} A = (\mathrm{id} + Bh)^{-1}B, \quad \partial_\hbar = \partial + pAi, \quad i_\hbar = i - hAi, \\ p_\hbar = p - pAh, \quad h_\hbar = h - hAh; \end{gathered} \tag{3-15}$$

compare with Proposition A.1. More explicitly, we have

$$i_\hbar = \sum_{k=0}^{\infty} (\widetilde{\Phi}\circ B)^k \circ i \quad \text{and} \quad h_\hbar = h \circ \sum_{k=0}^{\infty}(-Bh)^k, \tag{3-16}$$

where $\widetilde{\Phi}$ is the combination of Φ with the degree-counting coefficient from h from (3-9). We want to take a closer look at the induced differential:

Proposition 3.7. *One has*

$$p_\hbar = p \quad \textit{and} \quad \partial_\hbar = \partial + \delta \tag{3-17}$$

with

$$\delta(P\otimes D) = (-1)^d P_{(1)} \otimes D \cup \mathscr{L}_{P_{(2)}} - (-1)^d P \otimes D \cup \mathrm{id}$$

for homogeneous $P \otimes D \in \mathrm{S}\mathfrak{g} \otimes D^d_{\mathrm{poly}}(C)$.

Proof. The fact that $p_\hbar = p$ follows since Bh always adds differentials in the $\mathfrak{g}$-direction. For the deformed differential we compute for homogeneous $P \otimes D \in \mathrm{S}\mathfrak{g} \otimes D^d_{\mathrm{poly}}(C)$ and $f_i \in \mathscr{C}^\infty(C)$

$$\begin{aligned} (\delta(P\otimes D))(f_0, f_1, \dots, f_{d+1}) &= \bigg(p \circ \sum_{k=0}^{\infty} (B\circ\widetilde{\Phi})^k B \circ i(P\otimes D)\bigg)(f_0, f_1, \dots, f_{d+1}) \\ &= p(B(P\otimes D)(f_0, f_1, \dots, f_{d+1})) \\ &= (-1)^d p(\hbar m_{\mathrm{G}}(P\otimes D(f_0, \dots, f_d), f_{d+1}) \\ &= (-1)^d P_{(1)} \otimes D(f_0, \dots, f_d) \cdot \mathscr{L}_{P_{(2)}} f_{d+1} \end{aligned}$$

for all $P_{(2)} \neq 1$. Here we used the explicit form of the Gutt product as in Proposition 3.5 and the fact that $\mathrm{S}(\mathfrak{g})[[\hbar]]$ and $U_\hbar(\mathfrak{g})$ are isomorphic coalgebras. □

Since the classical homotopy equivalence data (3-10) is not a special deformation retract, the perturbed one is also not a special one. But it still has some nice properties.

Proposition 3.8. *One has*

$$p_\hbar \circ h_\hbar = 0 = h_\hbar \circ i_\hbar \quad \textit{and} \quad p_\hbar \circ i_\hbar = \mathrm{id}\,. \tag{3-18}$$

Proof. The properties follow from $p \circ h = 0 = h \circ i$, $p \circ i = \mathrm{id}$ and $\widetilde{\Phi}^2 = 0$. □

Thus the deformation retract (3-14) satisfies all properties of a special deformation retract except for $h_\hbar \circ h_\hbar = 0$, and we can still apply the homotopy transfer theorem.

3C. ***Application of the homotopy transfer theorem.*** We use the homotopy transfer theorem to extend $p_\hbar$ to an L_∞-morphism. We denote the L_∞-structure on the Taylor expansion by Q and the extension of $h_\hbar$ to the symmetric algebra as in (B-2) by H. Then applying the homotopy transfer theorem in the form of Theorem B.2 to the deformation retract (3-14) induces higher brackets $(Q_C)^1_k$ on $\left(\prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}[\![\hbar]\!]$:

Proposition 3.9. *The maps*

$$(Q_C)^1_1 = -\partial_\hbar, \quad (Q_C)^1_{k+1} = P^1_k \circ Q^k_{k+1} \circ i_\hbar^{\vee(k+1)}, \tag{3-19}$$

where

$$\begin{aligned} P^1_1 &= p_\hbar = p, \\ P^1_{k+1} &= \left(\sum_{\ell=2}^{k+1} Q^1_{C,\ell} \circ P^\ell_{k+1} - P^1_k \circ Q^k_{k+1}\right) \circ H_{k+1} \quad \textit{for } k \geq 1, \end{aligned} \tag{3-20}$$

induce a codifferential Q_C on the symmetric coalgebra of

$$\left(\prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}[\![\hbar]\!][1]$$

and an L_∞-quasi-isomorphism

$$P : (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[\![\hbar]\!], [\star_{\mathrm{G}} - J, \cdot\,], [\,\cdot\,,\cdot\,]) \longrightarrow \left(\left(\prod_{i=0}^\infty (\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}[\![\hbar]\!], Q_C\right).$$

Proof. The proposition follows directly from the homotopy transfer theorem as in Theorem B.2. Note that we do not need $h_\hbar \circ h_\hbar = 0$, only the other properties of a special deformation retract from Proposition 3.8. □

Let us take a closer look at the higher brackets Q_C induced by the homotopy transfer theorem. One can check that they vanish:

Proposition 3.10. *One has*

$$(Q_C)^1_{k+1} = 0 \quad \text{for all } k \geq 2. \tag{3-21}$$

Proof. In the higher brackets with $k \geq 2$ one has

$$H_k \circ Q^k_{k+1} \circ i_\hbar^{\vee(k+1)},$$

where in H_k one component consists of the application of $\widetilde{\Phi}$, that is, contains an insertion of a linear coordinate function e_t. We claim that it has to vanish. At first, it is clear that the image of i vanishes if one argument is e_t. Let us now show that $i_\hbar$ satisfies the same property, which directly gives the proposition since then also the bracket vanishes if one inserts a $\mathfrak{g}^*$-coordinate.

For homogeneous $D \in D^d_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ and $f_0, \dots, f_d \in \prod_i (\mathrm{S}^i \mathfrak{g} \otimes \mathscr{C}^\infty(C))$, we can compute

$$\begin{aligned}
&\Phi \circ B(D)(f_0, \dots, f_d) \\
&= \sum_{t=1}^n \sum_{j=1}^d \sum_{i=0}^j (-1)^i (B(D))\Big(f_0, \dots, f_{i-1}, e_t, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big) \\
&= \sum_{t=1}^n \sum_{j=1}^d \sum_{i=0}^j (-1)^i \Big(\hbar m_{\mathrm{G}}\Big(f_0, D(f_1, \dots, f_{i-1}, e_t, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d)\Big) \\
&\qquad\qquad - D\Big(\hbar m_{\mathrm{G}}(f_0, f_1), \dots, f_{i-1}, e_t, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big) + \cdots \\
&\qquad\qquad + (-1)^d \hbar m_{\mathrm{G}}\Big(D\Big(f_0, \dots, f_{i-1}, e_t, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_{d-1}\Big), f_d\Big)\Big).
\end{aligned}$$

If D vanishes if one of the arguments is a $\mathfrak{g}^*$-coordinate, then this simplifies to

$$\begin{aligned}
&\Phi \circ B(D)(f_0, \dots, f_d) \\
&\quad = \sum_{j=0}^d \Big(\hbar m_{\mathrm{G}}\Big(e_t, D\Big(f_0, \dots, f_{i-1}, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big)\Big) \\
&\qquad\qquad - D\Big(\hbar m_{\mathrm{G}}(e_t, f_0), \dots, f_{i-1}, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big)\Big) \\
&\qquad\qquad + \sum_{j=1}^d D\Big(\hbar m_{\mathrm{G}}(f_0, e_t), \dots, f_{i-1}, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big) + \cdots,
\end{aligned}$$

where e_t is always an argument of $\hbar m_{\mathrm{G}}$. In particular, we know $\hbar m_{\mathrm{G}}(e_i, e_j) = \frac{\hbar}{2}[e_i, e_j]$ and we see that the above sum vanishes if one of the functions f_i is a $\mathfrak{g}^*$-coordinate, that is, $\Phi \circ B(D)$ has the same vanishing property as D. The same holds for $\widetilde{\Phi} \circ B(D)$; hence by induction the image of $i_\hbar$ has the same property and the proposition is shown. □

Considering $(Q_C)_2^1$, we can simplify (3-19) to

$$(Q_C)_2^1 = \sum_{k=1}^{\infty} p \circ Q_2^1 \circ ((\widetilde{\Phi} \circ B)^k \circ i \vee i + i \vee (\widetilde{\Phi} \circ B)^k \circ i) + p \circ Q_2^1 \circ (i \vee i),$$

where the last term is the usual Gerstenhaber bracket. This is clear since $\widetilde{\Phi}$ adds a differential in the $\mathfrak{g}^*$-direction and the bracket can only eliminate it on one argument. Recall that we also have the canonical projection $\mathrm{pr}\colon \big(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\big)^{\mathrm{G}} \to D_{\mathrm{poly}}(M_{\mathrm{red}})$ which projects first to symmetric degree zero and then restricts to $\mathscr{C}^{\infty}(C)^{\mathrm{G}} \cong \mathscr{C}^{\infty}(M_{\mathrm{red}})$. It is a DGLA morphism with respect to classical structures, namely, Hochschild differentials and Gerstenhaber brackets. We extend it $\hbar$-linearly and can show that it is also a DLGA morphism with respect to the deformed DGLA structure Q_C:

Proposition 3.11. *The projection induces a DGLA morphism*

$$\mathrm{pr}\colon \left(\left(\prod_{i=0}^{\infty}(\mathrm{S}^i\mathfrak{g} \otimes D_{\mathrm{poly}}(C))\right)^{\mathrm{G}}[\![\hbar]\!], Q_C\right) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!], \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}}). \tag{3-22}$$

Proof. By the explicit form of the differential $(Q_C)_1^1 = -\partial_\hbar = -(\partial + \delta)$ from Proposition 3.7 we know that $\mathrm{pr} \circ \partial_\hbar = \mathrm{pr} \circ \partial = \partial \circ \mathrm{pr}$. Thus it only remains to show that $\mathrm{pr} \circ (Q_C)_2^1 = Q_2^1 \circ \mathrm{pr}^{\vee 2}$, which is equivalent to showing

$$\mathrm{pr} \circ \sum_{k=1}^{\infty} p \circ Q_2^1 \circ ((\widetilde{\Phi} \circ B)^k \circ i \vee i + i \vee (\widetilde{\Phi} \circ B)^k \circ i) = 0. \tag{$*$}$$

In the proof of Proposition 3.10 we computed $\Phi \circ B(D)$ of some $D \in D^d_{\mathrm{Tay}}(C \times \mathfrak{g}^*)$ and we saw that the image of i vanishes if one inserts a $\mathfrak{g}^*$-coordinate and that $\Phi \circ B$ preserves this property. Therefore, we got for such a D that vanishes if one of the arguments is e_t

$$\begin{aligned}
\Phi \circ B(D)&(f_0, \dots, f_d) \\
&= \sum_{j=0}^{d} \Big(\hbar m_{\mathrm{G}}\Big(e_t, D\Big(f_0, \dots, f_{i-1}, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big)\Big) \\
&\qquad - D\Big(\hbar m_{\mathrm{G}}(e_t, f_0), \dots, f_{i-1}, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big)\Big) \\
&\qquad + \sum_{j=1}^{d} D\Big(\hbar m_{\mathrm{G}}(f_0, e_t), \dots, f_{i-1}, \dots, \frac{\partial}{\partial e_t} f_j, \dots, f_d\Big) - \cdots \\
&\qquad\qquad - D\Big(f_0, \dots, , f_{d-1}, \hbar m_{\mathrm{G}}\Big(e_t, \frac{\partial}{\partial e_t} f_d\Big)\Big),
\end{aligned} \tag{$**$}$$

where $f_0, \dots, f_d \in \prod_i (\mathrm{S}^i\mathfrak{g} \otimes \mathscr{C}^\infty(C))$. Let us consider now $(*)$ applied to homogeneous $P \otimes D \vee Q \otimes D'$, where $P, Q \in \mathrm{S}\mathfrak{g}$ and $D, D' \in D_{\mathrm{poly}}(C)[[\hbar]]$. At first we note that this is zero if both $P \neq 1 \neq Q$ since the Gerstenhaber bracket can cancel at most one term. Similarly, it is zero if both $P = 1 = Q$. Thus we consider without loss of generality $D, Q \otimes D'$ with $Q \neq 1$ and $D \in (D^d_{\mathrm{poly}}(C))^{\mathrm{G}}[[\hbar]]$, where the only possible contributions are

$$\mathrm{pr} \circ p \circ Q_2^1\big(((\widetilde{\Phi} \circ B)^k D) \vee (Q \otimes D')\big) = (-1)^{d+(dd')}\, \mathrm{pr} \circ p\big(((\widetilde{\Phi} \circ B)^k D) \circ (Q \otimes D')\big)$$

for all $k \geq 1$. Note that, up to a sign, this is $((\widetilde{\Phi} \circ B)^k D) \circ (Q \otimes D')$ applied to invariant functions $\mathscr{C}^\infty(C)^{\mathrm{G}}[[\hbar]]$ and then projected to $\mathrm{S}^0\mathfrak{g}$. But on invariant functions the vertical vector fields and the differentials in the $\mathfrak{g}^*$-direction vanish, and we have only one slot where they can give a nontrivial contribution, namely $Q \otimes D'$. We fix the symmetric degree $Q \in \mathrm{S}^i\mathfrak{g}$ and get

$$\begin{aligned}
\mathrm{pr} \circ p \circ Q_2^1\big(((\widetilde{\Phi} \circ B)^k D) \vee (Q \otimes D')\big)& \\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\big((\Phi(B(\widetilde{\Phi} B)^{k-1} D)_i) \circ (Q \otimes D')\big) \\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\big((\Phi B(\widetilde{\Phi} B)^{k-1} D) \circ (Q \otimes D')\big).
\end{aligned}$$

Here $(B(\widetilde{\Phi} B)^{k-1} D)_i$ denotes the component of $B(\widetilde{\Phi} B)^{k-1} D$ with i differentiations in the $\mathfrak{g}^*$-direction. The $1/i$ comes from the degree of the homotopy (3-9) since we have no $\mathrm{S}\mathfrak{g}^*$-degree and since the only term that can be nontrivial is the one with i differentiations in the $\mathfrak{g}^*$-direction applied to Q. We compute with $(**)$

$$\begin{aligned}
\mathrm{pr} \circ p \circ & Q_2^1\big(((\widetilde{\Phi} \circ B)^k D) \vee (Q \otimes D')\big) \\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\big((\Phi B(\widetilde{\Phi} B)^{k-1} D) \circ (Q \otimes D')\big) \\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\Big(\Big(-\hbar \mathscr{L}_{(e_t)_C} \circ \mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D \circ \frac{\partial}{\partial e_t}\Big) \circ (Q \otimes D') \\
&\qquad\qquad - \Big(\mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D \circ \Big(\hbar m_{\mathrm{G}}\Big(e_t, \frac{\partial}{\partial e_t} \cdot\Big)\Big)\Big) \circ (Q \otimes D')\Big) \\
&= \frac{(-1)^{d+(dd')}}{i}\, \mathrm{pr} \circ p\Big((-\hbar \mathscr{L}_{(e_t)_C} \circ \mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D) \circ \Big(\frac{\partial}{\partial e_t} Q \otimes D'\Big) \\
&\qquad\qquad - (\mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D) \circ \Big(\Big(\hbar m_{\mathrm{G}}\Big(e_t, \frac{\partial}{\partial e_t} \cdot\Big)\Big) \circ (Q \otimes D')\Big)\Big).
\end{aligned}$$

But we know $\hbar m_{\mathrm{G}}(e_t, \cdot\,) = -\hbar \mathscr{L}_{(e_t)_C} + \hbar m_{\mathfrak{g}}(e_t, \cdot\,)$, where $\hbar m_{\mathfrak{g}}$ denotes the higher components of the Gutt product on $\mathfrak{g}^*$. Moreover, we have by the invariance

$$-[\mathscr{L}_{(e_t)_C}, \mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D]_{\mathrm{G}} = \Big[-f_{tk}^j e_j \frac{\partial}{\partial e_k}, \mathrm{pr}|_{\mathrm{S}^0\mathfrak{g}} (\widetilde{\Phi} \circ B)^{k-1} D\Big]_{\mathrm{G}}$$

and thus

$$\begin{aligned}\hbar \operatorname{pr}\circ p\Big(\Big(-[\mathscr{L}_{(e_t)_C}, \operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi}\circ B)^{k-1}D]_{\mathrm{G}}\Big)\circ\Big(\frac{\partial}{\partial e_t}Q\otimes D'\Big)\Big)\\ &= \hbar \operatorname{pr}\circ p\Big(\Big(\operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi}\circ B)^{k-1}D\circ\Big(f_{tk}^j e_j\frac{\partial}{\partial e_k}\Big)\Big)\circ\Big(\frac{\partial}{\partial e_t}Q\otimes D'\Big)\Big)\\ &= \hbar \operatorname{pr}\circ p\Big((\operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi}\circ B)^{k-1}D)\circ\Big(f_{tk}^j e_j\frac{\partial}{\partial e_k}\frac{\partial}{\partial e_t}Q\otimes D'\Big)\Big)=0.\end{aligned}$$

The only remaining terms are

$$\begin{aligned}\operatorname{pr}\circ p\circ Q_2^1\big(((\widetilde{\Phi}\circ B)^k D)\vee(Q\otimes D')\big)\\ &= (-1)^{d+(dd')}\operatorname{pr}\circ p\big((\operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi}\circ B)^k D)\circ(Q\otimes D')\big)\\ &= -\frac{(-1)^{d+(dd')}}{i}\operatorname{pr}\circ p\Big((\operatorname{pr}|_{\mathrm{S}^0\mathfrak{g}}(\widetilde{\Phi}\circ B)^{k-1}D)\circ\Big(\hbar m_{\mathfrak{g}}\big(e_t\frac{\partial}{\partial e_t}Q\big)\otimes D'\Big)\Big).\end{aligned}$$

We know that $\hbar m_{\mathfrak{g}}(e_t, (\partial/\partial e_t)Q)$ is either zero or in $\mathrm{S}^{>0}\mathfrak{g}$ and the statement follows by induction. □

In particular, we can compose this projection pr with the L_∞-projection from Proposition 3.9 that we constructed with the homotopy transfer theorem. Summarizing, we have shown:

Theorem 3.12. *There exists an L_∞-morphism*

$$D_{\mathrm{red}}=\operatorname{pr}\circ P:(D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[[\hbar]],[\star_{\mathrm{G}}-J,\cdot\,],[\,\cdot\,,\cdot\,])\longrightarrow(D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]],\partial,[\,\cdot\,,\cdot\,]_{\mathrm{G}}).$$

Finally, as in the polyvector field case in [Esposito et al. 2022b], we can twist the above morphism to obtain an L_∞-morphism from the curved equivariant polydifferential operators into the Cartan model and therefore also into the polydifferential operators on M_{red}, see Proposition 2.2 for the basics of the twisting procedure.

Proposition 3.13. *Twisting the reduction L_∞-morphism D_{red} from Theorem 3.12 with $-\hbar m_{\mathrm{G}}$ yields an L_∞-morphism*

$$D_{\mathrm{red}}^{-\hbar m_{\mathrm{G}}}:(D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[[\hbar]],\hbar\lambda,\partial+[-J,\cdot\,],[\,\cdot\,,\cdot\,])\longrightarrow(D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]],\partial,[\,\cdot\,,\cdot\,]_{\mathrm{G}}),$$

where $\lambda=\sum_i e^i\otimes(e_i)_M$ denotes the curvature.

Proof. At first we check that the curvature is indeed given by

$$e^i\otimes[-e_i,-\hbar m_{\mathrm{G}}]_{\mathrm{G}}=e^i\otimes-[e_i,\cdot\,]_{\star_{\mathrm{G}}}=e^i\otimes(\hbar\mathscr{L}_{(e_i)_C}-\hbar\operatorname{ad}(e_i))=\hbar\lambda; \tag{3-23}$$

see Lemma 3.6. The only thing left to show is that the DGLA structure on M_{red} is not changed, which is equivalent to

$$\sum_{k=1}^{\infty}\frac{(-\hbar)^k}{k!}(D_{\mathrm{red}})_k^1(m_{\mathrm{G}}\vee\cdots\vee m_{\mathrm{G}})=0. \tag{3-24}$$

But using the explicit form of P from Proposition 3.9 we see inductively that P vanishes if every argument has a differential in the $\mathfrak{g}^*$-direction and the statement is shown. $\square$

Remark 3.14. In the polyvector field case from [Esposito et al. 2022b, Proposition 4.29] we saw that the structure maps of the twisted morphism coincide with the structure maps of the original one. In our case it is not clear, that is, one might indeed have $D_{\mathrm{red}}^{-\hbar m_{\mathrm{G}}} \neq D_{\mathrm{red}}$.

This reduction morphism can be used to obtain a reduction morphism of the equivariant polydifferential operators $D_{\mathfrak{g}}^{\bullet}(M)$ of more general manifolds $M \neq C \times \mathfrak{g}^*$. More explicitly, assuming that the action is proper around C and free on C, we can restrict at first to $M_{\mathrm{nice}} \cong U_{\mathrm{nice}} \subset C \times \mathfrak{g}^*$, that is, we have

$$\begin{aligned} \cdot\,|_{U_{\mathrm{nice}}} : (D_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, \partial^{\mathfrak{g}} - [J, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \\ \longrightarrow (D_{\mathfrak{g}}(U_{\mathrm{nice}})[[\hbar]], \hbar\lambda|_{U_{\mathrm{nice}}}, \partial^{\mathfrak{g}} - [J|_{U_{\mathrm{nice}}}, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}). \end{aligned}$$

But on U_{nice} we can perform the Taylor expansion that is a morphism of curved DGLAs

$$\begin{aligned} D_{\mathfrak{g}^*} : (D_{\mathfrak{g}}(U_{\mathrm{nice}})[[\hbar]], \hbar\lambda|_{U_{\mathrm{nice}}}, \partial^{\mathfrak{g}} - [J|_{U_{\mathrm{nice}}}, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \\ \longrightarrow \big(D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], \hbar\lambda, \partial - [J, \cdot\,], [\,\cdot\,,\cdot\,]\big). \end{aligned}$$

Finally, we can compose it with $D_{\mathrm{red}}^{-\hbar m_{\mathrm{G}}}$ and obtain the following statement:

Theorem 3.15. *The composition of the above morphisms is an L_∞-morphism*

$$D_{\mathrm{red}} : (D_{\mathfrak{g}}(M)[[\hbar]], \hbar\lambda, \partial^{\mathfrak{g}} - [J, \cdot\,]_{\mathfrak{g}}, [\,\cdot\,,\cdot\,]_{\mathfrak{g}}) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[[\hbar]], 0, \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}}),$$

*called the **reduction L_∞-morphism**.*

Remark 3.16 (choices). Note that the only noncanonical choice we made is an open neighborhood of C in M which is diffeomorphic to a star shaped open neighborhood of C in $C \times \mathfrak{g}*$. Recall that the choice of this neighborhood works as follows. Take an arbitrary G-equivariant tubular neighborhood embedding $\psi : \nu(C) \to U \subseteq M$, where $\nu(C)$ denotes the normal bundle. Then define

$$\phi : \nu(C) \ni [v_p] \longmapsto \big(p, J(\psi([v_p]))\big) \in C \times \mathfrak{g}^*, \tag{3-25}$$

which is a diffeomorphism in a neighborhood of C. After some suitable restriction we obtain the identification. Nevertheless, we had to choose a G-equivariant tubular neighborhood and any two choices differ by a G-equivariant local diffeomorphism around C

$$A : C \times \mathfrak{g}^* \longrightarrow C \times \mathfrak{g}^*,$$

which is the identity when restricted to C. One can show that in the Taylor expansion

$$D_{\mathfrak{g}^*}(A^* f) = \mathrm{e}^X D_{\mathfrak{g}^*}(f)$$

for a vector field $X \in \prod_{i\geq 1}(\mathrm{S}^i\mathfrak{g}\otimes\mathfrak{X}(C))^{\mathrm{G}} \subseteq D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)$. Since any vector field is closed, X does not derive in the $\mathfrak{g}^*$-direction and λ is central, we obtain an inner automorphism

$$\mathrm{e}^{[X,\cdot\,]} : (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[\![\hbar]\!], \hbar\lambda, \partial - [J,\cdot\,], [\,\cdot\,,\cdot\,]) \longrightarrow (D_{\mathrm{Tay}}(C\times\mathfrak{g}^*)[\![\hbar]\!], \hbar\lambda, \partial - [J,\cdot\,], [\,\cdot\,,\cdot\,])$$

of curved Lie algebras which acts trivially on the level of equivalence classes of Maurer–Cartan elements. We are certain that the two reduction L_∞-morphisms are homotopic in a suitable curved setting, which, to our knowledge, is not developed yet.

As a last remark of this section, we want to mention a very interesting observation, which is not directly connected to the rest of this paper. Nevertheless, we felt that it can be interesting from many other perspectives.

Remark 3.17 (Cartan model). One can show that the DGLA structure Q_C from Proposition 3.9 on $\prod_{i=0}^\infty(\mathrm{S}^i\mathfrak{g}\otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}[\![\hbar]\!]$ restricts to $(\mathrm{S}\mathfrak{g}\otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}[\hbar]$ and hence can be evaluated at $\hbar = 1$. We still have the DGLA map

$$\mathrm{pr} : (\mathrm{S}\mathfrak{g}\otimes D_{\mathrm{poly}}(C))^{\mathrm{G}} \longrightarrow D_{\mathrm{poly}}(M_{\mathrm{red}}).$$

We want to sketch the proof of the fact that this is a quasi-isomorphism, which motivates us to interpret $(\mathrm{S}\mathfrak{g}\otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}$ as a Cartan model for equivariant polydifferential operators, generalizing the Cartan model for equivariant polyvector fields from [Esposito et al. 2022b, Section 4.2].

Picking a G-invariant covariant derivative (not necessarily torsion-free) for which the fundamental vector fields are flat in the fiber direction one can, using the PBW-isomorphism for Lie algebroids (see [Laurent-Gengoux et al. 2021; Nistor et al. 1999]), prove that there is an equivariant cochain map $K : D_{\mathrm{poly}}(C) \to T_{\mathrm{poly}}(C)$ and an equivariant homotopy $h : D^\bullet_{\mathrm{poly}}(C) \to D^{\bullet-1}_{\mathrm{poly}}(C)$, such that

$$\begin{tikzcd} T_{\mathrm{poly}}(C) \arrow[r, shift left, "\mathsf{hkr}"] & (D_{\mathrm{poly}}(C), \partial) \arrow[l, shift left, "K"] \arrow[loop right, "h"] \end{tikzcd} \tag{3-26}$$

is a special deformation retract. Additionally, one can show that

$$K(D_1 \cup D_2) = K(D_1)\wedge K(D_2) \quad \text{and} \quad K(\mathscr{L}_P) = \begin{cases} -P_C & \text{for } P\in\mathfrak{g}\subseteq\mathrm{S}\mathfrak{g}, \\ 0 & \text{else,}\end{cases}$$

for $D_1, D_2 \in D_{\mathrm{poly}}(C)$ and $P \in \mathrm{S}\mathfrak{g}$. We extend now (3-26) to

$$
\begin{tikzcd}
((\mathrm{S}\mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}, 0) \arrow[r, shift left, "\mathsf{hkr}"] & ((\mathrm{S}\mathfrak{g} \otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}, \partial) \arrow[l, shift left, "K"] \arrow[loop right, "h"]
\end{tikzcd}
$$

to obtain a special deformation retract. Now we include δ as in Proposition 3.7 and see it as a perturbation of ∂. One can show that the perturbation is small in the sense of the homological perturbation lemma as in [Crainic 2004], and we obtain

$$
\begin{tikzcd}
((\mathrm{S}\mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}, \delta) \arrow[r, shift left, "\widehat{\mathsf{hkr}}"] & ((\mathrm{S}\mathfrak{g} \otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}, \partial + \delta) \arrow[l, shift left, "\widehat{K}"] \arrow[loop right, "\hat{h}"]
\end{tikzcd}
$$

where δ is the differential

$$
\delta(P \otimes X) = \mathrm{i}(e^i)P \otimes (e_i)_C \wedge X
$$

obtained in [Esposito et al. 2022b, Definition 4.14] on $(\mathrm{S}\mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}$. Finally, one can show that

$$
\begin{tikzcd}
((\mathrm{S}\mathfrak{g} \otimes T_{\mathrm{poly}}(C))^{\mathrm{G}}, \delta) \arrow[r, "\widehat{\mathsf{hkr}}"] \arrow[d] & ((\mathrm{S}\mathfrak{g} \otimes D_{\mathrm{poly}}(C))^{\mathrm{G}}, \partial + \delta) \arrow[d] \\
(T_{\mathrm{poly}}(M_{\mathrm{red}}), 0) \arrow[r, "\mathsf{hkr}"] & (D_{\mathrm{poly}}(M_{\mathrm{red}}), \partial)
\end{tikzcd}
$$

commutes and both the horizontal maps, as well as the left-vertical map, are quasi-isomorphisms, which implies the claim.

4. Comparison of the reduction procedures

At the level of Maurer–Cartan elements, we know that the L_∞-morphism D_{red} from Theorem 3.15 induces a map from equivariant star products $(\star, H)$ with quantum momentum map $H = J + O(\hbar)$ on M to star products $\star_{\mathrm{red}}$ on the reduced manifold M_{red}. We conclude with a comparison of this reduction procedure with the reduction of formal Poisson structures via the quantized Koszul complex as in [Bordemann et al. 2000; Gutt and Waldmann 2010]; see also our adapted version in Appendix A.

We assume for simplicity $M = C \times \mathfrak{g}^*$ and work in the Taylor expansion of the equivariant polydifferential operators. We identify $\mathscr{C}^\infty(C)$ with $\mathrm{prol}\,\mathscr{C}^\infty(C) \subset \mathscr{C}^\infty(C \times \mathfrak{g}^*)$. Let us start with an equivariant star product $(\star, H = J + \hbar H')$ on $C \times \mathfrak{g}^*$, which means that $\hbar\pi_\star - \hbar H' = \star - \star_{\mathrm{G}} - (H - J)$ is a Maurer–Cartan element in

$$
(D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[[\hbar]], [\star_{\mathrm{G}} - J, \cdot\,], [\,\cdot\,,\cdot\,]).
$$

Proposition 4.1. *Defining* $I_1^1 = i_\hbar$ *and* $I_k^1 = h_\hbar \circ Q_2^1 \circ I_{k+1}^2$ *gives an* L_∞*-morphism*

$$I : \left(\left(\prod_{i=0}^{\infty} (\mathrm{S}^i \mathfrak{g} \otimes D_{\mathrm{poly}}(C)) \right)^{\mathrm{G}} [\![\hbar]\!], Q_C \right) \longrightarrow (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], [\star_{\mathrm{G}} - J, \cdot\,], [\,\cdot\,,\cdot\,]).$$

Moreover, one I *is a quasi-inverse of the* L_∞*-projection* P *from Proposition 3.9 and one has* $P \circ I = \mathrm{id}$.

Proof. Note that we have in general $h_\hbar^2 \neq 0$, but the only part of the homotopy that appears in the above recursions is $\widetilde{\Phi}$, where we know $\widetilde{\Phi} \circ \widetilde{\Phi} = 0$. Therefore, the statement follows from Proposition B.3. □

We get with Corollary B.5:

Corollary 4.2. *The* L_∞*-morphism* I *is compatible with the filtration induced by* $\hbar$ *and*

$$\hbar \widetilde{\pi}_\star = (I \circ P)^1 (\overline{\exp}(\hbar \pi_\star - \hbar H')) \in (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], [\star_{\mathrm{G}} - J, \cdot\,], [\,\cdot\,,\cdot\,])$$

is a well-defined Maurer–Cartan element that is equivalent to $\hbar\pi_\star - \hbar H'$. *In particular,* $(\tilde{\star} = \star_{\mathrm{G}} + \hbar\widetilde{\pi}_\star, J)$ *is a strongly invariant star product, that is, an equivariant star product such that the quantum momentum map is just the classical momentum map, and it is equivariantly equivalent to* $(\star, H)$.

The reduction of $(\tilde{\star}, J)$ via the reduction L_∞-morphism D_{red} is now easy:

Lemma 4.3. *The reduction* L_∞*-morphism*

$$D_{\mathrm{red}} = \mathrm{pr} \circ P : (D_{\mathrm{Tay}}(C \times \mathfrak{g}^*)[\![\hbar]\!], [\star_{\mathrm{G}} - J, \cdot\,], [\,\cdot\,,\cdot\,]) \longrightarrow (D_{\mathrm{poly}}(M_{\mathrm{red}})[\![\hbar]\!], \partial, [\,\cdot\,,\cdot\,]_{\mathrm{G}})$$

from Theorem 3.12 maps $\hbar\widetilde{\pi}_\star$ *to a Maurer–Cartan element* $\hbar m_{\mathrm{red}} = \mathrm{pr} \circ P^1(\exp \hbar\widetilde{\pi}_\star)$ *in the polydifferential operators on* M_{red}. *The corresponding star product* $\tilde{\star}_{\mathrm{red}} = \mu + \hbar m_{\mathrm{red}}$ *is given by*

$$\mathrm{pr}^*(u_1 \tilde{\star}_{\mathrm{red}} u_2) = \iota^*(\mathrm{prol}(\mathrm{pr}^* u_1) \tilde{\star}\, \mathrm{prol}(\mathrm{pr}^* u_2)) \tag{4-1}$$

for all $u_1, u_2 \in \mathscr{C}^\infty(M_{\mathrm{red}})[\![\hbar]\!]$.

Proof. By definition of $\hbar\widetilde{\pi}_\star$ we know $h_\hbar \hbar\widetilde{\pi}_\star = \widetilde{\Phi}(\hbar\widetilde{\pi}_\star) = 0$, and thus

$$\hbar m_{\mathrm{red}} = \mathrm{pr} \circ P^1(\exp \hbar\widetilde{\pi}_\star) = \mathrm{pr} \circ p(\hbar\widetilde{\pi}_\star).$$

Equation (4-1) follows since $\hbar m_{\mathrm{G}}(\mathrm{prol}(\mathrm{pr}^* u_1), \mathrm{prol}(\mathrm{pr}^* u_2)) = 0$. □

Moreover, we know by Lemma A.5 that the BRST reduction of $\mu + \hbar m_{\mathrm{G}} + \hbar\widetilde{\pi}_\star$ coincides with (4-1), and we have shown:

Theorem 4.4. *Let* $(\star, H)$ *be an equivariant star product on* M. *Then the reduced star product induced by* D_{red} *from Theorem 3.12 and the reduced star product via the formal Koszul complex (A-14) are equivalent.*

Proof. We know that both reduction procedures map equivalent equivariant star products to equivalent reduced star products. Moreover, we saw above that both reduction procedures coincide on $(\tilde{\star} = \star_{\mathrm{G}} + \hbar\widetilde{\pi}_{\star}, J)$ which is equivariantly equivalent to $(\star, H)$. ☐

Appendix A: BRST reduction of equivariant star products

We recall a slightly modified version of the reduction of equivariant star products as introduced in [Bordemann et al. 2000; Gutt and Waldmann 2010]; see also [Esposito et al. 2020] for a discussion of this reduction scheme in the context of Hermitian star products. It relies on the quantized Koszul complex and the homological perturbation lemma.

A1: *Homological perturbation lemma.* At first we recall from [Crainic 2004, Theorem 2.4; Reichert 2017, Chapter 2.4] a version of the homological perturbation lemma that is adapted to our setting. Let

$$\begin{tikzcd} (C, \mathrm{d}_C) \arrow[r, shift left, "i"] & (D, \mathrm{d}_D) \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd}$$

be a homotopy retract (also called homotopy equivalence data), i.e., let (C, d_C) and (D, d_D) be two chain complexes together with two quasi-isomorphisms

$$i : C \longrightarrow D \quad \text{and} \quad p : D \longrightarrow C \tag{A-1}$$

and a chain homotopy

$$h : D \longrightarrow D \quad \text{with} \quad \mathrm{id}_D - ip = \mathrm{d}_D h + h \mathrm{d}_D \tag{A-2}$$

between id_D and ip. Then we say that a graded map $B : D_\bullet \longrightarrow D_{\bullet-1}$ with $(\mathrm{d}_D + B)^2 = 0$ is a *perturbation* of the homotopy retract. The perturbation is called *small* if $\mathrm{id}_D + Bh$ is invertible, and the homological perturbation lemma states that in this case the perturbed homotopy retract is a again a homotopy retract; see [Crainic 2004, Theorem 2.4] for a proof.

Proposition A.1 (homological perturbation lemma). *Let*

$$\begin{tikzcd} (C, \mathrm{d}_C) \arrow[r, shift left, "i"] & (D, \mathrm{d}_D) \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd}$$

be a homotopy retract and let B be small perturbation of d_D. *Then the perturbed data*

$$\begin{tikzcd} (C, \hat{\mathrm{d}}_C) \arrow[r, shift left, "I"] & (D, \hat{\mathrm{d}}_D) \arrow[l, shift left, "P"] \arrow[loop right, "H"] \end{tikzcd} \tag{A-3}$$

with

$$
\begin{aligned}
A &= (\mathrm{id}_D + Bh)^{-1}B, & \hat{\mathrm{d}}_D &= \mathrm{d}_D + B, & \hat{\mathrm{d}}_C &= \mathrm{d}_C + pAi, \\
I &= i - hAi, & P &= p - pAh, & H &= h - hAh,
\end{aligned}
\tag{A-4}
$$

is again a homotopy retract.

Remark A.2. In [Crainic 2004] it is shown that perturbations of special deformation retracts are again special deformation retracts, which is in general not true for deformation retracts; see Appendix B for the different notions.

We are interested in even simpler complexes of the form

$$
\begin{tikzcd}
0 & D_0 \arrow[l] \arrow[d, shift right, "p"'] \arrow[r, shift right, "h_0"'] & D_1 \arrow[l, shift right, "\mathrm{d}_{D,1}"'] \arrow[r, shift right, "h_1"'] & \cdots \arrow[l, shift right, "\mathrm{d}_{D,2}"'] \\
0 & C_0 \arrow[l] \arrow[u, shift right, "i"'] & 0 \arrow[l]
\end{tikzcd}
\tag{A-5}
$$

In this case, the perturbed homotopy retract corresponding to a small perturbation B according to (A-4) is given by

$$
I = i, \quad P = p - p(\mathrm{id}_D + B_1 h_0)^{-1} B_1 h_0, \quad H = h - h(\mathrm{id}_D + Bh)^{-1}Bh
$$

and, using the geometric power series, this can be simplified to

$$
I = i, \quad P = p(\mathrm{id}_D + B_1 h_0)^{-1}, \quad H = h(\mathrm{id}_D + Bh)^{-1}. \tag{A-6}
$$

Here we denote by $B_1 : D_1 \longrightarrow D_0$ the degree one component of B, analogously for h. By Remark A.2 we know that deformation retracts are in general not preserved under perturbations. However, in this case we see that, starting with a deformation retract, the additional condition $h_0 i = 0$ suffices to guarantee

$$
PI = p(\mathrm{id}_D + B_1 h_0)^{-1} i = pi = \mathrm{id}_{C_0}.
$$

A2: *Quantized Koszul complex.* Let now $(M, \{\,\cdot\,,\cdot\,\})$ be a smooth Poisson manifold with a left action of the Lie group G. Moreover, let $J : M \to \mathfrak{g}^*$ be a classical (equivariant) momentum map. As usual, we assume that $0 \in \mathfrak{g}^*$ is a value and a regular value of J and set $C = J^{-1}(\{0\})$. In addition, we require the action to be proper on M (or at least around C) and free on C, which implies that $M_{\mathrm{red}} = C/\,\mathrm{G}$ is a smooth manifold. The reduction via the classical Koszul complex $\Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M)$ is one way to show that M_{red} is even a Poisson manifold, but we need the quantum version to show that we have an induced star product on M_{red}. The Koszul differential ∂ is given by

$$
\partial : \Lambda^q \mathfrak{g} \otimes \mathscr{C}^\infty(M) \longrightarrow \Lambda^{q-1}\mathfrak{g} \otimes \mathscr{C}^\infty(M), \quad a \mapsto \mathrm{i}(J)a = J_i \, \mathrm{i}_{\mathrm{a}}(e^i)a, \tag{A-7}
$$

where i denotes the left insertion and $J = J_i e^i$ the decomposition of J with respect to a basis $e^1, \dots, e^n$ of $\mathfrak{g}^*$. Then $\partial^2 = 0$ follows immediately with the commutativity of the pointwise product in $\mathscr{C}^\infty(M)$. The differential ∂ is also a derivation with respect to the associative and supercommutative product on the Koszul complex, consisting of the $\wedge$-product on $\Lambda^\bullet \mathfrak{g}$ tensored with the pointwise product on the functions. Also, it is invariant with respect to the induced $\mathfrak{g}$-representation

$$\mathfrak{g} \ni \xi \mapsto \rho(\xi) = \operatorname{ad}(\xi) \otimes \mathrm{id} - \mathrm{id} \otimes \mathscr{L}_{\xi_M} \in \operatorname{End}(\Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M)) \tag{A-8}$$

as we have

$$\begin{aligned} \partial \rho(e_a)(x \otimes f) &= f^k_{aj} e_k \wedge \mathrm{i}(e^j) \wedge \mathrm{i}(e^i) x \otimes J_{0,i} f + f^i_{aj}\, \mathrm{i}(e^j) x \otimes J_{0,i} f \\ &\qquad + \mathrm{i}(e^i) x \otimes J_{0,i} \{J_{0,a}, f\}_0 \\ &= \rho(e_a) \partial (x \otimes f) \end{aligned}$$

for all $x \in \Lambda^\bullet \mathfrak{g}$ and $f \in \mathscr{C}^\infty(M)$.

One can show that the Koszul complex is acyclic in positive degree with homology $\mathscr{C}^\infty(C)$ in order zero, and that one has a G-equivariant homotopy

$$h : \Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M) \longrightarrow \Lambda^{\bullet+1} \mathfrak{g} \otimes \mathscr{C}^\infty(M); \tag{A-9}$$

see [Bordemann et al. 2000, Lemma 6; Gutt and Waldmann 2010]. In other words, this means that

$$\mathrm{prol} : (\mathscr{C}^\infty(C), 0) \rightleftarrows (\Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M), \partial) : \iota^*, h$$

is a HE data of the special type of (A-5), that is, we have the diagram

$$\begin{tikzcd} 0 & \mathscr{C}^\infty(M) \arrow[l] \arrow[r, shift right, "h_0"'] \arrow[d, shift right, "\iota^*"'] & \Lambda^1 \mathfrak{g} \otimes \mathscr{C}^\infty(M) \arrow[l, shift right, "\partial_1"'] \arrow[r, shift right, "h_1"'] & \cdots \arrow[l, shift right, "\partial_2"'] \\ 0 & \mathscr{C}^\infty(C) \arrow[l] \arrow[u, shift right, "\mathrm{prol}"'] & 0 \arrow[l] \end{tikzcd}$$

For the reduction of equivariant star products, we need to deform it to the *quantized Koszul complex*. The *quantized Koszul differential*

$$\boldsymbol{\partial} : \Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M_{\mathrm{nice}})[[\hbar]] \longrightarrow \Lambda^{\bullet-1} \mathfrak{g} \otimes \mathscr{C}^\infty(M_{\mathrm{nice}})[[\hbar]]$$

is defined by

$$\begin{aligned} \boldsymbol{\partial}^{(\kappa)}(x \otimes f) &= \\ &\mathrm{i}(e^a) x \otimes H_a \star f - \frac{\hbar}{2} f^c_{ab} e_c \wedge \mathrm{i}(e^a)\, \mathrm{i}(e^b) x \otimes f + \hbar \kappa f^b_{ab}\, \mathrm{i}(e^a)\, (x \otimes f) \end{aligned} \tag{A-10}$$

for $\kappa \in \mathbb{C}[[\hbar]]$, $x \in \Lambda^\bullet \mathfrak{g}[[\hbar]]$ and $f \in \mathscr{C}^\infty(M_{\mathrm{nice}})[[\hbar]]$, where $\Delta = f^b_{ab} e^a$ is the modular one-form of $\mathfrak{g}$.

Remark A.3. Note that in the literature [Bordemann et al. 2000; Gutt and Waldmann 2010] a different convention is used:

$$\boldsymbol{\partial}'^{(\kappa)}(x\otimes f)=\mathrm{i}(e^a)x\otimes f\star H_a+\frac{\hbar}{2}f_{ab}^c e_c\wedge\mathrm{i}(e^a)\,\mathrm{i}(e^b)x\otimes f+\hbar\kappa\,\mathrm{i}(\Delta)\,(x\otimes f)$$

for $\kappa\in\mathbb{C}[[\hbar]]$. In particular, $\boldsymbol{\partial}'^{(\kappa)}$ is left $\star$-linear. However, in order to simplify the comparison of the BRST reduction with the reduction via D_{red} in Section 4, we want the quantized Koszul differential to be right $\star$-linear, which leads to our convention in (A-10).

The reduction of the star product in our convention works analogously to [Bordemann et al. 2000; Gutt and Waldmann 2010] since $\boldsymbol{\partial}^{(\kappa)}$ satisfies all the desired properties:

Lemma A.4. *Let* $(\star, H)$ *be an equivariant star product and* $\kappa\in\mathbb{C}[[\hbar]]$.

(i) *One has* $\boldsymbol{\partial}^{(0)}\circ\mathrm{i}(\Delta)+\mathrm{i}(\Delta)\circ\boldsymbol{\partial}^{(0)}=0$.

(ii) $\boldsymbol{\partial}^{(\kappa)}$ *is right* $\star$*-linear.*

(iii) $\boldsymbol{\partial}^{(\kappa)}=\partial+O(\hbar)$.

(iv) $\boldsymbol{\partial}^{(\kappa)}$ *is* G*-equivariant.*

(v) *One has* $\boldsymbol{\partial}^{(\kappa)}\circ\boldsymbol{\partial}^{(\kappa)}=0$.

Proof. The proof is analogous to [Gutt and Waldmann 2010, Lemma 3.4]. □

Assume that we have chosen a value $\kappa\in\mathbb{C}[[\hbar]]$ and write $\boldsymbol{\partial}=\boldsymbol{\partial}^{(\kappa)}$. Then by the homological perturbation lemma one gets a perturbed homotopy retract

$$\begin{tikzcd}
0 & \mathscr{C}^\infty(M_{\mathrm{nice}})[[\hbar]] \arrow[l] \arrow[r, shift right, "\boldsymbol{h}_0"'] \arrow[d, shift right, "\boldsymbol{\iota}^*"'] & \Lambda^1\mathfrak{g}\otimes\mathscr{C}^\infty(M_{\mathrm{nice}})[[\hbar]] \arrow[l, shift right, "\boldsymbol{\partial}_1"'] \arrow[r, shift right, "\boldsymbol{h}_1"'] & \cdots \arrow[l, shift right, "\boldsymbol{\partial}_2"'] \\
0 & \mathscr{C}^\infty(C)[[\hbar]] \arrow[l] \arrow[u, shift right, "\mathbf{prol}"'] & 0, \arrow[l] &
\end{tikzcd}$$

where

(A-11) $$\mathbf{prol}=\mathrm{prol},\qquad \boldsymbol{\iota}^*=\iota^*(\mathrm{id}+B_1h_0)^{-1},\qquad \boldsymbol{h}=h(\mathrm{id}+Bh)^{-1},$$

and where $\boldsymbol{\partial}-\partial=B$; see (A-6). One can show that the *deformed restriction map* $\boldsymbol{\iota}^*$ is given by

(A-12) $$\boldsymbol{\iota}^*=\iota^*\circ S=\sum_{r=0}\hbar^r\boldsymbol{\iota}^*{}_r:\mathscr{C}^\infty(M_{\mathrm{nice}})[[\hbar]]\longrightarrow\mathscr{C}^\infty(C)[[\hbar]]$$

with a G-equivariant formal series of differential operators $S=\mathrm{id}+\sum_{r=1}^\infty\hbar^r S_r$ on $\mathscr{C}^\infty(M_{\mathrm{nice}})$ and with S_r vanishing on constants. Also, it is uniquely determined by the properties

$$\boldsymbol{\iota}^*{}_0 = \iota^*, \quad \boldsymbol{\iota}^*\boldsymbol{\partial}_1 = 0 \quad \text{and} \quad \boldsymbol{\iota}^* \operatorname{prol} = \operatorname{id}_{\mathscr{C}^\infty(C)[\![\hbar]\!]}. \tag{A-13}$$

The reduced star product $\star_{\mathrm{red}}$ on $M_{\mathrm{red}} = C/\mathrm{G}$ is then given by

$$\mathrm{pr}^*(u_1 \star_{\mathrm{red}} u_2) = \boldsymbol{\iota}^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2)) \tag{A-14}$$

for all $u_1, u_2 \in \mathscr{C}^\infty(M_{\mathrm{red}})[\![\hbar]\!]$; compare with [Bordemann et al. 2000, Theorem 32]. In [Reichert 2017, Lemma 4.3.1] it has been shown that equivariantly equivalent star products reduce to equivalent star products on M_{red}.

For the comparison of the reduction procedures in Section 4 we need the following observation:

Lemma A.5. *Let $(\star = \mu + \hbar\pi_\star + \hbar m_{\mathrm{G}}, J)$ be an equivariant star product on $C \times \mathfrak{g}^*$, and choose $\kappa = -1$ for the quantized Koszul differential. If one has $\widetilde{\Phi}(\hbar\pi_\star) = 0 = \Phi(\hbar\pi_\star)$, then it follows for all $u_1, u_2 \in \mathscr{C}^\infty(M_{\mathrm{red}})[\![\hbar]\!]$*

$$\mathrm{pr}^*(u_1 \star_{\mathrm{red}} u_2) = \boldsymbol{\iota}^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2)) = \iota^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2)).$$

Proof. We have for a polynomial function $f = P \otimes \phi \in \mathrm{S}^j\mathfrak{g} \otimes \mathscr{C}^\infty(C) \subset \mathscr{C}^\infty(C \times \mathfrak{g}^*)$

$$\begin{aligned}
(\boldsymbol{\partial} - \partial)h_0(P \otimes \phi) &= \frac{1}{j}\big(\hbar(\pi_\star + m_{\mathrm{G}})(e_i, \mathrm{i}(e^i)P \otimes \phi) + \hbar\kappa f_{ib}^b\, \mathrm{i}(e^i)P \otimes \phi\big) \\
&= \frac{1}{j}(\Phi(\hbar\pi_\star + \hbar m_{\mathrm{G}})(P \otimes \phi) + \hbar\kappa f_{ib}^b\, \mathrm{i}(e^i)P \otimes \phi) \\
&= \frac{1}{j}\big(\hbar m_{\mathrm{G}}(e_i, \mathrm{i}(e^i)P \otimes \phi) + \hbar\kappa f_{ib}^b\, \mathrm{i}(e^i)P \otimes \phi\big) \\
&= \frac{1}{j}\big(\hbar m_{\mathfrak{g}}(e_i, \mathrm{i}(e^i)P) \otimes \phi - \mathrm{i}(e^i)P \otimes \hbar\mathscr{L}_{(e_i)_C}\phi + \hbar\kappa f_{ib}^b\, \mathrm{i}(e^i)P \otimes \phi\big),
\end{aligned}$$

where $\hbar m_{\mathfrak{g}}$ denotes the nontrivial part of the Gutt product on $\mathfrak{g}^*$. We know that $\operatorname{im}(\hbar m_{\mathfrak{g}}(e_i, \cdot\,)) \in \mathrm{S}^{>0}\mathfrak{g}[\![\hbar]\!]$, hence it follows

$$(*) \quad \iota^* \circ (\boldsymbol{\partial} - \partial)h_0(P \otimes \phi) = \frac{1}{j}\iota^*(-\mathrm{i}(e^i)P \otimes \hbar\mathscr{L}_{(e_i)_C}\phi + \hbar\kappa f_{ib}^b\, \mathrm{i}(e^i)P \otimes \phi).$$

On an invariant polynomial $P \otimes \phi \in (\mathrm{S}^j\mathfrak{g} \otimes \mathscr{C}^\infty(C))^{\mathrm{G}}$ we have

$$-\mathrm{i}(e^i)P \otimes \hbar\mathscr{L}_{(e_i)_C}\phi = -\hbar\, \mathrm{i}(e^i)\operatorname{ad}(e_i)P \otimes \phi = -\hbar f_{ij}^i\, \mathrm{i}(e^j)P \otimes \phi,$$

hence $(*)$ vanishes for $\kappa = -1$. Thus we have in this case

$$\mathrm{pr}^*(u_1 \star_{\mathrm{red}} u_2) = \boldsymbol{\iota}^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2)) = \iota^*(\operatorname{prol}(\mathrm{pr}^* u_1) \star \operatorname{prol}(\mathrm{pr}^* u_2))$$

and the statement is shown. □

Appendix B: Explicit formulas for the homotopy transfer theorem

In is well-known that L_∞-quasi-isomorphisms always admit L_∞-quasi-inverses. It is also well-known that given a homotopy retract one can transfer L_∞-structures; see, for instance, [Loday and Vallette 2012, Section 10.3]. Explicitly, a homotopy retract (also called homotopy equivalence data) consists of two cochain complexes (A, d_A) and (B, d_B) with chain maps i, p and homotopy h such that

$$
\begin{tikzcd} (A, \mathrm{d}_A) \arrow[r, shift left, "i"] & (B, \mathrm{d}_B) \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd} \tag{B-1}
$$

with $h \circ \mathrm{d}_B + \mathrm{d}_B \circ h = \mathrm{id} - i \circ p$, and such that i and p are quasi-isomorphisms. Then the homotopy transfer theorem states that if there exists a flat L_∞-structure on B, then one can transfer it to A in such a way that i extends to an L_∞-quasi-isomorphism. By the invertibility of L_∞-quasi-isomorphisms there also exists an L_∞-quasi-isomorphism into A denoted by P; see, for example, [Loday and Vallette 2012, Proposition 10.3.9].

In this section we state a version of this statement adapted to our applications. For simplicity, we assume that we have a deformation retract satisfying

$$
p \circ i = \mathrm{id}_A .
$$

By [Huebschmann 2011b, Remark 2.1] we can assume that we have even a special deformation retract, also called *contraction*, where

$$
h^2 = 0, \quad h \circ i = 0 \quad \text{and} \quad p \circ h = 0.
$$

Assume now that (B, Q_B) is an L_∞-algebra with $(Q_B)^1_1 = -\mathrm{d}_B$. In the following we give a more explicit description of the transferred L_∞-structure Q_A on A and of the L_∞-projection $P : (B, Q_B) \to (A, Q_A)$ inspired by the symmetric tensor trick [Berglund 2014; Huebschmann 2011a; 2011b; Manetti 2010]. The map h extends to a homotopy $H_n : \mathrm{S}^n(B[1]) \to \mathrm{S}^n(B[1])[-1]$ with respect to $Q^n_{B,n} : \mathrm{S}^n(B[1]) \to \mathrm{S}^n(B[1])[1]$; see, for instance, [Loday and Vallette 2012, p. 383] for the construction on the tensor algebra, which we adapt to our setting as follows. We define the operator

$$
K_n : \mathrm{S}^n(B[1]) \longrightarrow \mathrm{S}^n(B[1])
$$

by

$$
K_n(x_1 \vee \cdots \vee x_n) = \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \frac{\epsilon(\sigma)}{n-i} ipX_{\sigma(1)} \vee \cdots \vee ipX_{\sigma(i)} \vee X_{\sigma(i+1)} \vee X_{\sigma(n)}.
$$

Note that here we sum over the whole symmetric group and not the shuffles, since in this case the formulas are easier. We extend $-h$ to a coderivation to $\mathrm{S}(B[1])$, i.e.,

$$\widetilde{H}_n(x_1 \vee \cdots \vee x_n) := -\sum_{\sigma \in \mathrm{Sh}(1,n-1)} \epsilon(\sigma)\, hx_{\sigma(1)} \vee x_{\sigma(2)} \vee \cdots \vee x_{\sigma(n)}$$

and define

$$H_n = K_n \circ \widetilde{H}_n = \widetilde{H}_n \circ K_n. \tag{B-2}$$

Since i and p are chain maps, we have $K_n \circ Q^n_{B,n} = Q^n_{B,n} \circ K_n$, where $Q^n_{B,n}$ is the extension of the differential $Q^1_{B,1} = -\mathrm{d}_B$ to $\mathrm{S}^n(B[1])$ as a coderivation. Hence we have

$$Q^n_{B,n} H_n + H_n Q^n_{B,n} = (n \cdot \mathrm{id} - ip) \circ K_n,$$

where ip is extended as a coderivation to $\mathrm{S}(B[1])$. A combinatorial and not very enlightening computation shows that finally

$$Q^n_{B,n} H_n + H_n Q^n_{B,n} = \mathrm{id} - (ip)^{\vee n}. \tag{B-3}$$

Now assume that we have a codifferential Q_A and a morphism of coalgebras P with structure maps $P^1_\ell : \mathrm{S}^\ell(B[1]) \to A[1]$ such that P is an L_∞-morphism up to order k, that is,

$$\sum_{\ell=1}^{m} P^1_\ell \circ Q^\ell_{B,m} = \sum_{\ell=1}^{m} Q^1_{A,\ell} \circ P^\ell_m$$

for all $m \leq k$. Then we have the following statement, whose proof can be found in [Esposito et al. 2022b].

Lemma B.1. *Let $P : \mathrm{S}(B[1]) \to \mathrm{S}(A[1])$ be an L_∞-morphism up to order $k \geq 1$. Then*

$$L_{\infty,k+1} = \sum_{\ell=2}^{k+1} Q^1_{A,\ell} \circ P^\ell_{k+1} - \sum_{\ell=1}^{k} P^1_\ell \circ Q^\ell_{B,k+1} \tag{B-4}$$

satisfies

$$L_{\infty,k+1} \circ Q^{k+1}_{B,k+1} = -Q^1_{A,1} \circ L_{\infty,k+1}. \tag{B-5}$$

This allows us to prove one version of the homotopy transfer theorem.

Theorem B.2 (homotopy transfer theorem). *Let (B, Q_B) be a flat L_∞-algebra with differential $(Q_B)^1_1 = -\mathrm{d}_B$ and contraction*

$$\begin{tikzcd} (A, \mathrm{d}_A) \arrow[r, shift left, "i"] & (B, \mathrm{d}_B) \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd} \tag{B-6}$$

Then

$$(Q_A)^1_1 = -\mathrm{d}_A, \quad (Q_A)^1_{k+1} = \sum_{i=1}^{k} P^1_i \circ (Q_B)^i_{k+1} \circ i^{\vee(k+1)},$$
$$P^1_1 = p, \qquad P^1_{k+1} = L_{\infty,k+1} \circ H_{k+1} \qquad \textit{for } k \geq 1$$

turns (A, Q_A) *into an* L_∞*-algebra with* L_∞*-quasi-isomorphism* $P : (B, Q_B) \to (A, Q_A)$*. In addition, one has* $P^1_k \circ i^{\vee k} = 0$ *for* $k \neq 1$.

Proof. We observe $P^1_{k+1}(ix_1 \vee \cdots \vee ix_{k+1}) = 0$ for all $k \geq 1$ and $x_i \in A$, which directly follows from $h \circ i = 0$, and thus $H_{k+1} \circ i^{\vee(k+1)} = 0$. Suppose that Q_A is a codifferential up to order $k \geq 1$, i.e., $\sum_{\ell=1}^{m} (Q_A)^1_\ell (Q_A)^\ell_m = 0$ for all $m \leq k$, and that P is an L_∞-morphism up to order $k \geq 1$. We know that these conditions are satisfied for $k = 1$ and we show that they hold for $k+1$. Starting with Q_A we compute

$$\begin{aligned}
(Q_A Q_A)^1_{k+1} &= (Q_A Q_A)^1_{k+1} \circ P^{k+1}_{k+1} \circ i^{\vee(k+1)} \\
&= \sum_{\ell=1}^{k+1} (Q_A Q_A)^1_\ell P^\ell_{k+1} i^{\vee(k+1)} \\
&= (Q_A Q_A P)^1_{k+1} i^{\vee(k+1)} \\
&= \sum_{\ell=2}^{k+1} (Q_A)^1_\ell (Q_A P)^\ell_{k+1} i^{\vee(k+1)} + (Q_A)^1_1 (Q_A P)^1_{k+1} i^{\vee(k+1)} \\
&= \sum_{\ell=2}^{k+1} (Q_A)^1_\ell (P Q_B)^\ell_{k+1} i^{\vee(k+1)} + (Q_A)^1_1 (Q_A)^1_{k+1} \\
&= (Q_A P Q_B)^1_{k+1} i^{\vee(k+1)} - (Q_A)^1_1 (Q_A)^1_{k+1} + (Q_A)^1_1 (Q_A)^1_{k+1} \\
&= \sum_{\ell=1}^{k} (Q_A P)^1_\ell (Q_B)^\ell_{k+1} i^{\vee(k+1)} + (Q_A P)^1_{k+1} (Q_B)^{k+1}_{k+1} i^{\vee(k+1)} \\
&= \sum_{\ell=1}^{k} (P Q_B)^1_\ell (Q_B)^\ell_{k+1} i^{\vee(k+1)} + (Q_A P)^1_{k+1} i^{\vee(k+1)} (Q_A)^{k+1}_{k+1} \\
&= -(P Q_B)^1_{k+1} i^{\vee(k+1)} (Q_A)^{k+1}_{k+1} + (Q_A P)^1_{k+1} i^{\vee(k+1)} (Q_A)^{k+1}_{k+1} \\
&= -(Q_A)^1_{k+1} (Q_A)^{k+1}_{k+1} + (Q_A)^1_{k+1} (Q_A)^{k+1}_{k+1} = 0.
\end{aligned}$$

By the same computation as in Lemma B.1, where one in fact only needs that Q_A is a codifferential up to order $k+1$, it follows that

$$L_{\infty,k+1} \circ Q^{k+1}_{B,k+1} = -Q^1_{A,1} \circ L_{\infty,k+1}.$$

It remains to show that P is an L_∞-morphism up to order $k+1$. We have

$$\begin{aligned}
P^1_{k+1} \circ (Q_B)^{k+1}_{k+1} &= L_{\infty,k+1} \circ H_{k+1} \circ (Q_B)^{k+1}_{k+1} \\
&= L_{\infty,k+1} - L_{\infty,k+1} \circ (Q_B)^{k+1}_{k+1} \circ H_{k+1} - L_{\infty,k+1} \circ (i \circ p)^{\vee(k+1)} \\
&= L_{\infty,k+1} + (Q_A)^1_1 \circ P^1_{k+1}
\end{aligned}$$

since

$$\begin{aligned} L_{\infty,k+1}\circ(i\circ p)^{\vee(k+1)} &= \Big(\sum_{\ell=2}^{k+1} Q^1_{A,\ell}\circ P^\ell_{k+1} - \sum_{\ell=1}^{k} P^1_\ell\circ Q^\ell_{B,k+1}\Big)\circ(i\circ p)^{\vee(k+1)} \\ &= (Q_A)^1_{k+1}\circ p^{\vee(k+1)} - (Q_A)^1_{k+1}\circ p^{\vee(k+1)} = 0. \end{aligned}$$

Therefore

$$P^1_{k+1}\circ(Q_B)^{k+1}_{k+1} - (Q_A)^1_1\circ P^1_{k+1} = L_{\infty,k+1},$$

i.e., P is an L_∞-morphism up to order $k+1$. The statement follows inductively. □

A special case of the above theorem, for i being a DGLA morphism, was proven in [Esposito et al. 2022b, Proposition 3.2]. We also want to give an explicit formula for a L_∞-quasi-inverse of P, generalizing [Esposito et al. 2022b, Proposition 3.3].

Proposition B.3. *The coalgebra map $I : \mathrm{S}^\bullet(A[1]) \to \mathrm{S}^\bullet(B[1])$ recursively defined by the maps $I^1_1 = i$ and $I^1_{k+1} = h\circ L_{\infty,k+1}$ for $k \geq 1$ is an L_∞-quasi inverse of P. Since $h^2 = 0 = h\circ i$, one even has $I^1_{k+1} = h\circ\sum_{\ell=2}^{k+1} Q^1_{B,\ell}\circ I^\ell_{k+1}$ and $P\circ I = \mathrm{id}_A$.*

Proof. We proceed by induction. Assume that I is an L_∞-morphism up to order k; then we have

$$\begin{aligned} I^1_{k+1}Q^{k+1}_{A,k+1} - Q^1_{B,1}I^1_{k+1} &= -Q^1_{B,1}\circ h\circ L_{\infty,k+1} + h\circ L_{\infty,k+1}\circ Q^{k+1}_{A,k+1} \\ &= -Q^1_{B,1}\circ h\circ L_{\infty,k+1} - h\circ Q^1_{B,1}\circ L_{\infty,k+1} \\ &= (\mathrm{id} - i\circ p)L_{\infty,k+1}. \end{aligned}$$

We used that $Q^1_{B,1} = -\mathrm{d}_B$ and the homotopy equation of h. Moreover, we get with $p\circ h = 0$

$$\begin{aligned} p\circ L_{\infty,k+1} &= p\circ\Big(\sum_{\ell=2}^{k+1} Q^1_{B,\ell}\circ I^\ell_{k+1} - \sum_{\ell=1}^{k} I^1_\ell\circ Q^\ell_{A,k+1}\Big) \\ &= \sum_{\ell=2}^{k+1}(P\circ Q_B)^1_\ell\circ I^\ell_{k+1} - \sum_{\ell=2}^{k+1}\sum_{i=2}^{\ell} P^1_i\circ Q^i_{B,\ell}\circ I^\ell_{k+1} - Q^1_{A,k+1} \\ &= \sum_{\ell=2}^{k+1}(Q_A\circ P)^1_\ell\circ I^\ell_{k+1} - \sum_{i=2}^{k+1}\sum_{\ell=i}^{k+1} P^1_i\circ Q^i_{B,\ell}\circ I^\ell_{k+1} - Q^1_{A,k+1} \\ &= Q^1_{A,k+1} - \sum_{i=2}^{k+1}\sum_{\ell=i}^{k+1} P^1_i\circ I^i_\ell\circ Q^\ell_{A,k+1} - Q^1_{A,k+1} = 0, \end{aligned}$$

and therefore I is an L_∞-morphism. □

Remark B.4. In the homotopy transfer theorem the property $h^2 = 0$ is not needed, and that one can also adapt the above construction of I to this more general case.

Note that there exists a homotopy equivalence relation $\sim$ between L_∞-morphisms, see, for example, [Dolgushev 2007], such that equivalent L_∞-morphisms map

Maurer–Cartan elements to equivalent Maurer–Cartan elements; see, for instance, [Bursztyn et al. 2012, Lemma B.5] for the case of DGLAs and [Kraft 2021, Proposition 1.4.6] for the case of flat L_∞-algebras.

Corollary B.5. *In the above setting one has* $P \circ I = \mathrm{id}_A$ *and* $I \circ P \sim \mathrm{id}_B$. *In particular, assume that one has complete descending filtrations on* A, B *such that all the maps are compatible. Then every Maurer–Cartan element* $\pi \in \mathcal{F}^1 B$ *is equivalent to* $(I \circ P)^1(\overline{\exp}(\pi))$.

Proof. By [Kraft and Schnitzer 2021, Proposition 3.8] P admits a quasi-inverse I' such that $P \circ I' \sim \mathrm{id}_A$ and $I' \circ P \sim \mathrm{id}_B$, which implies

$$I \circ P = \mathrm{id}_B \circ I \circ P \sim I' \circ P \circ I \circ P = I' \circ P \sim \mathrm{id}_B .$$

The rest of the statement is then clear. □

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