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TOTALLY GEODESIC HYPERBOLIC 3-MANIFOLDS IN HYPERBOLIC LINK COMPLEMENTS OF TORI IN S^4

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We prove that certain hyperbolic link complements of 2-tori in S^4 do not contain closed embedded totally geodesic hyperbolic 3-manifolds.

1. Introduction

A classical problem in 4-dimensional topology is (see [14, Problem 3.20]): *Under what conditions does a closed, orientable 3-manifold M smoothly embed in S^4 ?* As an example of an obstruction, it is an old result of Hantzsche [8] that if a closed orientable 3-manifold M embeds in S^4 , then $\text{Tor}(H_1(M, \mathbb{Z})) \cong A \oplus A$ for some finite abelian group A . The focus of this paper is obstructing the embedding of closed hyperbolic 3-manifolds in S^4 via embeddings in hyperbolic link complements of 2-tori in S^4 .

A simple but elegant argument (see [9, Proposition 4.10]) shows that if X is a hyperbolic link complement of 2-tori in S^4 then $\chi(X) = \chi(S^4) = 2$, and so there are only finitely many hyperbolic link complements of 2-tori in S^4 . This finiteness statement holds more generally for hyperbolic link complements of 2-tori and Klein bottles in any fixed 4-manifold. By way of comparison, Thurston's hyperbolization theorem shows that many links in S^3 have hyperbolic complements, and although it is known that many hyperbolic link complements in S^3 do not contain a closed embedded totally geodesic surface (e.g., alternating links [17]), examples do exist (see [15; 17]). The main result of this paper (see [Theorem 1.1](#) below) provides more examples of hyperbolic link complements of 2-tori in S^4 that do not contain a closed embedded totally geodesic hyperbolic 3-manifold (our previous paper [5] provided one such example). We note that [5] shows that the hyperbolic link complements of 2-tori in S^4 in [Theorem 1.1](#) do contain infinitely many immersed closed totally geodesic hyperbolic 3-manifolds. To state [Theorem 1.1](#) we need to recall some additional notation.

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Ratcliffe and Tschantz [19] provided a census of 1171 so-called integral congruence two hyperbolic 4-manifolds that are all obtained from face-pairings of the ideal 24-cell in \mathbb{H}^4 . These are all commensurable cusped, arithmetic, hyperbolic 4-manifolds of Euler characteristic 1 (i.e., minimal volume). Ivanšić [10] provided an example of a cusped, orientable, hyperbolic 4-manifold of Euler characteristic 2 that is the complement of five 2-tori in S^4 (with the standard smooth structure [11]) and is constructed as the orientable double cover of the nonorientable manifold 1011 in the census of integral congruence two hyperbolic 4-manifolds mentioned above. In [5] we proved that this link complement of 2-tori in S^4 does not contain any closed embedded totally geodesic hyperbolic 3-manifolds (it does contain embedded orientable noncompact finite volume totally geodesic hyperbolic 3-manifolds). In [12], four additional examples of link complements of 2-tori in manifolds homeomorphic to S^4 were found. These arise as the orientable double covers of the nonorientable manifolds in the census of [19] with numbers 23, 71, 1091 and 1092. The main result of this note is to extend the result of [5] to these four other examples.

We fix the following notation. For $n \in \{23, 71, 1091, 1092\}$, we denote by $p_n : W_n \rightarrow N_n$ the orientation double coverings of the nonorientable integral congruence two hyperbolic 4-manifolds N_n . By construction, $\chi(N_n) = 1$ and $\chi(W_n) = 2$, with W_n a link complement of 2-tori in S^4 .

Theorem 1.1. *For $n \in \{23, 71, 1091, 1092\}$ the manifolds W_n do not contain a closed embedded totally geodesic hyperbolic 3-manifold.*

As with the case of the manifold 1011 of [19], the W_n of Theorem 1.1 all contain embedded noncompact finite volume totally geodesic hyperbolic 3-manifolds.

The strategy of the proof of Theorem 1.1 is similar to that of [5] but additional complications arise with these four examples (see Section 2B for a fuller discussion). Moreover, a different argument is needed to handle these cases, and this requires a detailed analysis of possible closed totally geodesic surfaces that can embed in certain arithmetic hyperbolic 3-manifolds that cover the Picard orbifold $\mathbb{H}^3/\mathrm{PSL}(2, \mathbb{Z}[i])$.

We finish the introduction by posing a question prompted by our work:

Question 1.2. *Does there exist a hyperbolic link complement of 2-tori in a closed (smooth) simply connected 4-manifold that contains a closed embedded totally geodesic hyperbolic 3-manifold?*

Examples of hyperbolic link complements of 2-tori in closed (smooth) simply connected 4-manifolds are given in [12; 20]. Indeed, [20, Theorem 1.2] shows that such link complements exist only in S^4 , $\#_r(S^2 \times S^2)$, or $\#_r(\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2)$, with $r > 0$. Furthermore, using the examples of [10], examples of link complements of 2-tori in $\#_r(S^2 \times S^2)$ for r even were exhibited in [20] (these cover the link

complement of [10]). It is unknown whether there exists a finite volume hyperbolic link complement of 2-tori in $\#_r(\mathbb{C}\mathbb{P}^2\#\mathbb{C}\mathbb{P}^2)$, for some $r > 0$.

2. Recap from [5]

The hyperboloid model of \mathbb{H}^4 is defined using the quadratic form

$$J = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2$$

with

$$\mathbb{H}^4 = \{x \in \mathbb{R}^5 : J(x) = -1, x_5 > 0\}$$

equipped with the Riemannian metric induced from the Lorentzian inner product associated to J . The full group of isometries of \mathbb{H}^4 is then identified with $O^+(4, 1)$, the subgroup of

$$O(4, 1) = \{A \in GL(5, \mathbb{R}) : A^t J A = J\},$$

preserving the upper sheet of the hyperboloid $J(x) = -1$, and where we abuse notation and use J to denote the symmetric matrix associated to the quadratic form. The full group of orientation-preserving isometries is given by

$$SO^+(4, 1) = \{A \in O^+(4, 1) : \det(A) = 1\}.$$

The groups $O^+(3, 1)$ and $SO^+(3, 1)$ are defined in a similar manner.

2A. Integral congruence two hyperbolic 4-manifolds. The manifolds $p_n : W_n \rightarrow N_n$ where $n \in \{23, 71, 1091, 1092\}$ of interest to us all arise as face-pairings of the regular ideal 24-cell in \mathbb{H}^4 (with all dihedral angles $\pi/2$), and are regular $(\mathbb{Z}/2\mathbb{Z})^4$ covers of the orbifold $\mathbb{H}^4/\Lambda(2)$, where $\Lambda(2)$ is the level two congruence subgroup of the group $O^+(J, \mathbb{Z}) = O^+(4, 1) \cap O(J, \mathbb{Z})$. These manifolds are referred to as *integral congruence two hyperbolic 4-manifolds* in [19, Table 1]. It will be useful to describe the $(\mathbb{Z}/2\mathbb{Z})^4$ action, and this is best described in the ball model as follows.

Locate the 24-cell in the ball model of hyperbolic space with vertices

$$(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1) \text{ and } (\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}).$$

The four reflections in the coordinate planes of \mathbb{R}^4 can be taken as generators of this $(\mathbb{Z}/2\mathbb{Z})^4$ group of isometries. Passing to the hyperboloid model, these reflections are elements of $\Lambda(2)$ and are listed as the first four matrices in [19, page 110]. Following [19] we denote this $(\mathbb{Z}/2\mathbb{Z})^4$ group of isometries by $K < \Lambda(2)$.

As noted in [19] (see also [18]) all of the face-pairings of *any* of the integral congruence two hyperbolic 4-manifolds are invariant under the group K . This implies that each of the coordinate hyperplane cross sections of the 24-cell extends in each of the integral congruence two hyperbolic 4-manifolds to a totally geodesic hypersurface which is the fixed point set of one of the reflections described above.

Following [18] we call these hypersurfaces *cross sections*. As described in [18], these cross sections can be identified with integral congruence two hyperbolic 3-manifolds which are also described in [19]. Moreover, it is possible to use [19] or [18] to identify these in any given example.

- Lemma 2.1.** (1) N_{23} has 4 nonorientable cross-sections all isometric to each other.
- (2) N_{71} has one orientable cross-section isometric to the complement of the link 8_2^4 and three nonorientable cross-sections.
- (3) N_{1091} has one orientable cross-section isometric to the complement of the link 8_2^4 and three nonorientable cross-sections, two of which are isometric to each other.
- (4) N_{1092} has two orientable cross-sections isometric to the complement of the link 8_2^4 and two nonorientable cross-sections which are isometric to each other.

Proof. The proof of this lemma is similar to the proof of [5, Lemma 7.1]. We give a fairly detailed discussion of case (1), and only mention salient points in the remaining cases.

- (1) In [19, Table 3] the manifold N_{23} is given by the code 1569A4 which represents the side pairing 1111555566669999AAAA4444 for the 24 sides of the ideal 24-cell Q^4 . In the notation of [19], the four cross sections have $k_1k_5k_9$ codes 352, 352, 156, 156, which correspond to the side pairings $r_i k_i$ for the 12 sides of the polytope Q^3 where r_i is the reflection in side i and $k_1 = k_2 = k_3 = k_4$, $k_5 = k_6 = k_7 = k_8$, $k_9 = k_{10} = k_{11} = k_{12}$. Since r_i is a reflection, the side pairing $r_i k_i$ is orientation preserving if and only if the corresponding k_i is orientation reversing. But this happens only if $k_i \in \{1, 2, 4, 7\}$ since then it corresponds to the diagonal matrices with $1 \leftrightarrow \text{diag}(-1, 1, 1, 1)$, $2 \leftrightarrow \text{diag}(1, -1, 1, 1)$, $4 \leftrightarrow \text{diag}(1, 1, -1, 1)$, $7 \leftrightarrow \text{diag}(-1, -1, -1, 1)$. Therefore, all four cross-sections of N_{23} are nonorientable.

From [19, Table 1], we see that the code 156 corresponds to the non-orientable integral congruence two 3-manifold M_4^3 of [19]. As in the proof of [5, Lemma 7.1], it can be checked that code 352 is equivalent to the code 156 via a symmetry of Q^3 (the polyhedron in [5, Figure 2]), and hence determine isometric manifolds.

- (2) In [19, Table 3] the manifold N_{71} is given by the code 13EB34. In the notation of [19], the four cross sections have $k_1k_5k_9$ codes 712, 152, 173, 136. It can be checked that the 3-manifold with code 712 is orientable and isometric to M_{10}^3 of [19], and that the remaining codes determine nonorientable manifolds. Thus, N_{71} has one orientable cross-section, and three nonorientable cross-sections.

- (3) In [19, Table 3] the manifold N_{1091} is given by the code 53FF35. In the notation of [19], the four cross sections have $k_1k_5k_9$ codes 712, 173, 173, 537. As noted in case (2), the code 712 determines an orientable 3-manifold, and it can be checked that the remaining determine nonorientable ones. Hence, N_{1091} has one orientable cross-section, and three nonorientable cross-sections.
- (4) In [19, Table 3] the manifold N_{1092} is given by the code 53FFCA. In the notation of [19], the four cross sections have $k_1k_5k_9$ codes 765, 174, 174, 537. Thus, the manifolds with code 174 is orientable and isometric to M_{10}^3 of [19], and the remaining two determine nonorientable manifolds. Furthermore, it can be checked that the codes 765 and 537 are equivalent via a symmetry of Q^3 and determine isometric manifolds. Thus, N_{1092} has two isometric orientable cross-sections, and two isometric nonorientable cross-sections. □

In what follows, let A (with orientable double cover A^+) denote the nonorientable manifold given by the code 537, and similarly let $B, C, D,$ and E (with orientable double covers $B^+, C^+, D^+,$ and E^+) denote the nonorientable manifolds given by the codes 152, 173, 136, and 156 respectively.

2B. Volume from tubular neighbourhoods. As in [5], to prove Theorem 1.1, we will make use of a result of Basmajian [2] which provides disjoint collars about closed embedded orientable totally geodesic hypersurfaces in hyperbolic manifolds. We state this only for hyperbolic 4-manifolds.

Following [2], let $r(x) = \log \coth(x/2)$, and let $V(r)$ denote the volume of a ball of radius r in \mathbb{H}^3 . It is noted in [2] that, $V(r) = \omega_3 \int_0^r \sinh^2(t) dt$, where ω_3 is the area of the unit sphere in \mathbb{R}^3 (i.e., $\omega_3 = 4\pi$).

In [2, pages 213–214], the volume of a tubular neighbourhood of a closed embedded orientable totally geodesic hyperbolic 3-manifold of 3-dimensional hyperbolic volume A in a hyperbolic 4-manifold is given in terms of the 4-dimensional tubular neighbourhood function $c_4(A) = (\frac{1}{2})(V \circ r)^{-1}(A)$. Moreover, as noted in [2, Remark 2.1], when the totally geodesic submanifold separates, an improved estimate can be obtained using the tubular neighbourhood function $d_4(A) = (\frac{1}{2})(V \circ r)^{-1}(A/2)$ and we record this as follows.

Lemma 2.2. *Let X be an orientable finite volume hyperbolic 4-manifold containing a closed embedded separating orientable totally geodesic hyperbolic 3-manifold of 3-dimensional hyperbolic volume A . Then X contains a tubular neighbourhood of M of volume*

$$\mathcal{V}'(A) = 2A \int_0^{d_4(A)} \cosh^3(t) dt.$$

Moreover, [2] also proves that disjoint embedded closed orientable totally geodesic hyperbolic 3-manifolds in an orientable finite volume hyperbolic 4-manifold

have disjoint collars, thereby contributing additional volume. For our purposes we summarize what we need in the following.

Corollary 2.3. *Let X be an orientable finite volume hyperbolic 4-manifold of Euler characteristic χ containing K disjoint copies of a closed embedded orientable totally geodesic hyperbolic 3-manifold of 3-dimensional hyperbolic volume A . Assume that all of these disjoint copies separate X . Then*

$$\text{Vol}(X) = \left(\frac{4}{3}\pi^2\right)\chi \geq K\mathcal{V}'(A).$$

Given this set up we recall the basic strategy of [5]. To that end, let N (resp. W) denote one of the manifolds N_{23} , N_{71} , N_{1091} , or N_{1092} (resp. W_{23} , W_{71} , W_{1091} , or W_{1092}) and $M \hookrightarrow W$ a closed embedded totally geodesic hyperbolic 3-manifold. Since $W \subset S^4$, M is orientable and the embedding separates S^4 .

As in [5, Lemma 7.2] since W is the orientable double cover of N , it is a characteristic cover of N and hence a regular cover of $\mathbb{H}^4/\Lambda(2)$ (using Section 2A). If it can be shown that M is disjoint from the preimages of all of the cross-sections in N , then since W is a regular cover of $\mathbb{H}^4/\Lambda(2)$, using the isometries of W induced from the reflections in the coordinate hyperplanes we get 16 disjoint copies of M , all embedded and separating in W (since it is a submanifold of S^4).

Now the minimal volume of a closed hyperbolic 3-manifold is that of the Weeks manifold and is approximately 0.9427... [7]. Using this estimate for $\text{Vol}(M)$, and applying Corollary 2.3 we see that $\text{Vol}(W) \geq 16\mathcal{V}'(0.94)$, which is approximately 28.9. On the other hand, since $\chi(W) = 2$, $\text{Vol}(W) = \frac{8}{3}\pi^2$ which is approximately 26.3, a contradiction.

As proved in [5, Lemma 3.2], any M (as above) is disjoint from the lift of any orientable cross-section in N . To prove Theorem 1.1 we need to show that M is disjoint from the preimage of a nonorientable cross-section in N . This follows from our next lemma, since if M (as above) was not disjoint this would give rise to a closed embedded totally geodesic (possibly nonorientable) surface in the preimage of the cross-section.

Lemma 2.4. *Let Y be any of the nonorientable cross-sections listed in Lemma 2.1 and Y^+ the orientable double cover. Then Y^+ (and hence Y) does not contain a closed embedded totally geodesic surface.*

The strategy to prove Lemma 2.4 is this: we first identify $\Gamma = \pi_1(Y^+)$ as a congruence subgroup of the Picard group $\text{PSL}(2, \mathbb{Z}[i])$, and identify matrices (up to sign) that correspond to a generating set for Γ . We next use the classification of circles left invariant by nonelementary Fuchsian subgroups of $\text{PSL}(2, \mathbb{Z}[i])$ given in [16] to limit the possibilities for what circles can be associated to a closed embedded totally geodesic surface in Y^+ . Finally we use a criterion given by [13, Corollary 3.3] to prove that any candidate totally geodesic surface cannot be embedded.

The proof of [Lemma 2.4](#) occupies the remainder of this paper.

3. The Picard group and the fundamental groups of the cross-sections

The Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$ has a presentation from [\[21\]](#):

$$\mathrm{PSL}(2, \mathbb{Z}[i]) = \langle \alpha, l, t, u \mid \alpha^2 = l^2 = (\alpha l)^2 = (tl)^2 = (ul)^2 = (\alpha t)^3 = (u\alpha l)^3 = [t, u] = 1 \rangle,$$

where these generators can be represented by the matrices (up to sign) shown below:

$$\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad l = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

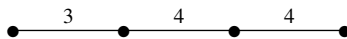
3A. Locating subgroups. It will be helpful to prove the following result which helps “locate” the fundamental groups of the manifolds Y^+ as subgroups of $\mathrm{PSL}(2, \mathbb{Z}[i])$.

Proposition 3.1. *Let Y be any of the nonorientable cross-sections listed in [Lemma 2.1](#) and Y^+ the orientable double cover. Then $\pi_1(Y^+)$ admits a faithful representation with image group Γ of index 48 in $\mathrm{PSL}(2, \mathbb{Z}[i])$ and $[\Gamma(1+i), \Gamma(1+i)] \triangleleft \Gamma \triangleleft \Gamma(1+i)$.*

Proof. To establish that $\pi_1(Y^+)$ admits a faithful representation with image group Γ with $\Gamma \triangleleft \Gamma(1+i)$, recall from [\[19, Section 3\]](#) that these integral congruence two hyperbolic 3-manifolds are constructed as follows. As in the case of dimension 4 described above these manifolds arise as regular covers (all with covering group $(\mathbb{Z}/2\mathbb{Z})^3$) of a certain congruence quotient of \mathbb{H}^3 , namely the subgroup $\Lambda(2) < \mathrm{O}^+(3, 1; \mathbb{Z})$. As shown in [\[19, page 105\]](#):

$$[\mathrm{O}^+(3, 1; \mathbb{Z}) : \Lambda(2)] = 12 \quad \text{with } \mathrm{O}^+(3, 1; \mathbb{Z})/\Lambda(2) \cong S_3 \times \mathbb{Z}/2\mathbb{Z}.$$

In addition, [\[19, Section 3\]](#) identifies the group $\mathrm{O}^+(3, 1; \mathbb{Z})$ with the Coxeter group T , generated by reflections in the faces of the noncompact tetrahedron with Coxeter diagram:



Using a presentation of this Coxeter group, one can find a presentation for the subgroup $\mathrm{SO}^+(3, 1; \mathbb{Z})$ (consisting of orientation preserving isometries) of index 2 in $\mathrm{O}^+(3, 1; \mathbb{Z})$, and it follows from this that the abelianization of $\mathrm{SO}^+(3, 1; \mathbb{Z})$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Indeed, it is known (see, e.g., [\[4\]](#)) that the subgroup $\mathrm{SO}^+(3, 1; \mathbb{Z})$ can be identified with $\mathrm{PGL}(2, \mathbb{Z}[i])$, which in turn contains $\mathrm{PSL}(2, \mathbb{Z}[i])$ of index 2.

Since $\Lambda(2)$ contains elements of determinant -1 , the group

$$\Lambda^+(2) = \Lambda(2) \cap \mathrm{SO}^+(3, 1; \mathbb{Z})$$

has index 2 in $\Lambda(2)$, and so $\Lambda^+(2)$ is isomorphic to a normal subgroup of index 12 in $\mathrm{PGL}(2, \mathbb{Z}[i])$ with quotient group $S_3 \times \mathbb{Z}/2\mathbb{Z}$.

As noted above, the abelianization of $SO^+(3, 1; \mathbb{Z}) \cong PGL(2, \mathbb{Z}[i])$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We claim that this implies that $\Lambda^+(2)$ is contained in the commutator subgroup of $SO^+(3, 1; \mathbb{Z})$. To see this, first recall that the abelianization of S_3 is $\mathbb{Z}/2\mathbb{Z}$ so we get the epimorphism $a : S_3 \times \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since the surjective composition

$$SO^+(3, 1; \mathbb{Z}) \xrightarrow{p} S_3 \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{a} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

lands in an abelian group, it factors through the abelianization of $SO^+(3, 1; \mathbb{Z})$, with the second map an isomorphism. It then follows that $\Lambda^+(2)$ must be contained in the commutator subgroup of $SO^+(3, 1; \mathbb{Z})$.

Furthermore, via the above identifications, $\Lambda^+(2)$ must be isomorphic to a normal subgroup of index 6 in $PSL(2, \mathbb{Z}[i])$. By [6, Theorem 2] there is a unique normal subgroup of index 6 in $PSL(2, \mathbb{Z}[i])$ and it is the principal congruence subgroup $\Gamma(1+i)$ (i.e., those elements in $PSL(2, \mathbb{Z}[i])$ congruent to the identity modulo the ideal $(1+i)$).

From [19, page 105], the group $O^+(3, 1; \mathbb{Z})$ is generated by the 4 reflections

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 2 \end{pmatrix},$$

and the subgroup $SO^+(3, 1; \mathbb{Z})$ can be identified with the group $PGL(2, \mathbb{Z}[i])$ via the isomorphism defined by

$$ab \mapsto \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad ac \mapsto \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}, \quad ad \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From [19, page 106] the group $\Lambda(2)$ is generated by the reflections $r_1 = abcba$, $r_2 = bcb$, $r_3 = c$, $r_4 = abdcdba$, $r_5 = bdcdb$, $r_6 = dcd$, and the subgroup $\Lambda^+(2)$ can thus be identified with the group $\Gamma(1+i)$ via the induced isomorphism defined by

$$r_1 r_2 \mapsto \begin{pmatrix} -1 & 1+i \\ -1+i & 1 \end{pmatrix}, \quad r_1 r_3 \mapsto \begin{pmatrix} -i & -1+i \\ 0 & i \end{pmatrix}, \quad r_1 r_4 \mapsto \begin{pmatrix} -i & 2i \\ 0 & i \end{pmatrix}, \\ r_1 r_5 \mapsto \begin{pmatrix} 2-i & -1+i \\ 1-i & i \end{pmatrix}, \quad r_1 r_6 \mapsto \begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix}.$$

As noted above, $\pi_1(Y) \triangleleft \Lambda(2)$ with quotient group $(\mathbb{Z}/2\mathbb{Z})^3$, so it follows that $\pi_1(Y^+) \triangleleft \Lambda^+(2) \cong \Gamma(1+i)$ with quotient group $(\mathbb{Z}/2\mathbb{Z})^3$.

Now $\Gamma(1+i)/[\Gamma(1+i), \Gamma(1+i)] \cong (\mathbb{Z}/2\mathbb{Z})^5$ (see the Magma [3] routine following this proof). Since the map $\Gamma(1+i) \rightarrow \Gamma(1+i)/\pi_1(Y^+) \cong (\mathbb{Z}/2\mathbb{Z})^3$ goes to an abelian group, the commutator subgroup $[\Gamma(1+i), \Gamma(1+i)]$ is sent to 1 and it follows that $[\Gamma(1+i), \Gamma(1+i)] \triangleleft \pi_1(Y^+)$.

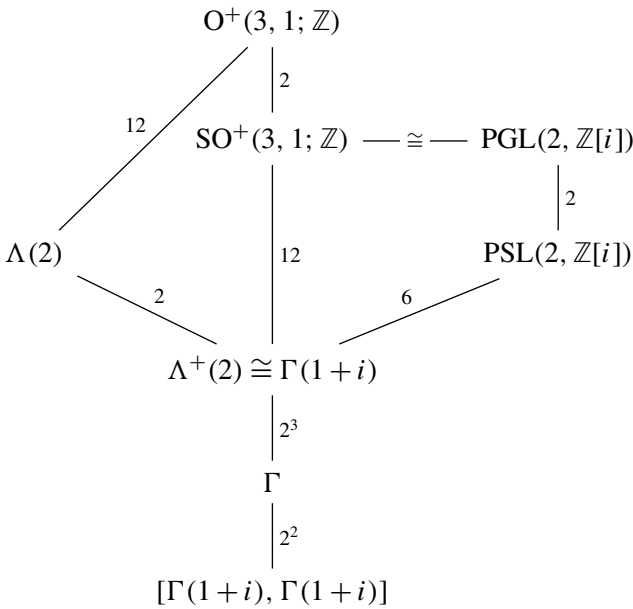


Figure 1. Lattice of subgroups.

Putting all of this together we obtain the lattice of subgroups shown in [Figure 1](#). We have the image group $\Gamma = \pi_1(Y^+)$ as claimed. That the index in $\text{PSL}(2, \mathbb{Z}[i])$ is 48 is clear from the lattice of subgroups above (or from volume consideration discussed in [\[19, page 108\]](#)). \square

Remark 3.2. The subgroup $[\Gamma(1+i), \Gamma(1+i)]$ can also be identified as the principal congruence subgroup $\Gamma(2+2i)$ (i.e., those elements in $\text{PSL}(2, \mathbb{Z}[i])$ congruent to the identity modulo the ideal $(2+2i) = (1+i)^3$). This follows from [\[1, Proposition 3.1\]](#) where the group $\Gamma(2+2i)$ is identified as a link group, and arises as the normal closure in $\text{PSL}(2, \mathbb{Z}[i])$ of the subgroup $\langle t^2u^2, t^4 \rangle$. Now $t^2u^2, t^4 \in \Gamma(1+i)$, and since $\Gamma(1+i)^{\text{ab}} \cong (\mathbb{Z}/2\mathbb{Z})^5$, it follows that t^2u^2 and t^4 are both mapped trivially under the abelianization map $\Gamma(1+i) \rightarrow \Gamma(1+i)^{\text{ab}}$. Hence $\langle t^2u^2, t^4 \rangle \subset [\Gamma(1+i), \Gamma(1+i)]$.

Now the subgroup $[\Gamma(1+i), \Gamma(1+i)]$ is a characteristic subgroup of $\Gamma(1+i)$ and hence is normal in $\text{PSL}(2, \mathbb{Z}[i])$, and so it follows that the normal closure in $\text{PSL}(2, \mathbb{Z}[i])$ of the subgroup $\langle t^2u^2, t^4 \rangle$ is contained in $[\Gamma(1+i), \Gamma(1+i)]$; i.e., $\Gamma(2+2i) \subset [\Gamma(1+i), \Gamma(1+i)]$. However, both these groups have index 192 in $\text{PSL}(2, \mathbb{Z}[i])$.

We now provide the short Magma [\[3\]](#) routine referred to in the proof of [Proposition 3.1](#). Referring below, the group g is the group $\text{PSL}(2, \mathbb{Z}[i])$, and the presentation used is that given above. The group N is $\Gamma(1+i)$.

```

> g<a, l, t, u> := Group<a, l, t, u | a2, l2, (a * l)2, (t * l)2, (u * l)2, (a * t)3,
(u * a * l)3, (t, u)>;
> h := sub<g | t * u, t2, u2>;
> N := NormalClosure<g, h>;
> printIndex<g, N>;
6
> printAbelianQuotientInvariants<N>;
[2, 2, 2, 2, 2]

```

3B. Generators for the fundamental groups of the cross-sections. Using the description of each nonorientable cross-section from [Lemma 2.1](#), the notation of [Section 2A](#) together with the isomorphism described in the proof of [Proposition 3.1](#), we find generators for the fundamental groups of A , B , C , D and E as side pairings of the 24-cell. We then use Magma to find generators for the orientation double covers A^+ , B^+ , C^+ , D^+ and E^+ and also to eliminate redundant generators. The generators are written below both as elements of $\mathrm{PSL}(2, \mathbb{Z}[i])$ and as words in the reflections r_i 's:

$$\begin{aligned}
\pi_1(A^+) &= \left\langle \begin{pmatrix} -i & -1+i \\ 1+i & -i \end{pmatrix}, \begin{pmatrix} 3i & 1-i \\ 1+i & -i \end{pmatrix}, \begin{pmatrix} 1-2i & -2-4i \\ -2i & -3-2i \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} -1-4i & -4+4i \\ -2-2i & -1+4i \end{pmatrix}, \begin{pmatrix} -2-5i & -1+9i \\ -3-3i & 2+7i \end{pmatrix} \right\rangle \\
&= \langle r_3 r_6 r_2 r_1, r_1 r_3 r_6 r_2, r_2 r_1 r_5 r_3 r_4 r_3, r_2 r_1 r_5 r_3 r_2 r_5 r_3 r_1, r_2 r_1 r_5 r_1 r_6 r_2 r_3 r_5 r_1 r_2 \rangle, \\
\pi_1(B^+) &= \left\langle \begin{pmatrix} 3 & -1-i \\ 2-2i & -1 \end{pmatrix}, \begin{pmatrix} -2+i & 1+i \\ 1+i & -i \end{pmatrix}, \begin{pmatrix} 1+2i & 2-2i \\ 4i & 5-2i \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} 1+2i & -1-i \\ 4+4i & -3-2i \end{pmatrix}, \begin{pmatrix} -2-i & 1+i \\ -1-i & i \end{pmatrix} \right\rangle \\
&= \langle r_1 r_2 r_6 r_2, r_3 r_4 r_3 r_2, (r_5 r_3 r_1)^2, r_5 r_3 r_2 r_6 r_2 r_1 r_3 r_5, r_5 r_1 r_4 r_2 r_1 r_5 \rangle, \\
\pi_1(C^+) &= \left\langle \begin{pmatrix} 1 & -1-i \\ 2-2i & -3 \end{pmatrix}, \begin{pmatrix} -1 & 1+i \\ -2 & 1+2i \end{pmatrix}, \begin{pmatrix} 3+2i & -2-2i \\ 2+2i & -1-2i \end{pmatrix}, \begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} -3 & 5-3i \\ -2 & 3-2i \end{pmatrix} \right\rangle \\
&= \langle r_2 r_6 r_2 r_1, r_2 r_5 r_3 r_1, (r_1 r_4 r_2)^2, r_1 r_4 r_6 r_1 r_4 r_1, r_1 r_4 r_5 r_3 r_1 r_2 r_4 r_1 \rangle, \\
\pi_1(D^+) &= \left\langle \begin{pmatrix} 1 & -1-i \\ 2-2i & -3 \end{pmatrix}, \begin{pmatrix} -2+3i & -2i \\ -1+i & -i \end{pmatrix}, \begin{pmatrix} 2+i & -2 \\ 1-i & i \end{pmatrix}, \begin{pmatrix} 2+i & -2i \\ 3+i & -3i \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} -3+2i & 4 \\ -5+i & 5+2i \end{pmatrix} \right\rangle
\end{aligned}$$

$$\begin{aligned}
 &= \langle r_2 r_6 r_2 r_1, r_2 r_3 r_2 r_3 r_4 r_3 r_2 r_1 r_5 r_2, r_2 r_3 r_2 r_4 r_2 r_1 r_5 r_2, r_2 r_5 r_1 r_2 r_3 r_2 r_3 r_4 r_3 r_2, \\
 &\quad r_2 r_5 r_1 r_2 r_6 r_2 r_5 r_1 \rangle, \\
 \pi_1(E^+) &= \left\langle \begin{pmatrix} 1 & -1-i \\ 2-2i & -3 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1+2i & 2-2i \\ 4i & 5-2i \end{pmatrix}, \begin{pmatrix} -1+2i & 2 \\ -4+2i & 3+2i \end{pmatrix}, \right. \\
 &\quad \left. \begin{pmatrix} 3+2i & -1-i \\ 4+4i & -1-2i \end{pmatrix} \right\rangle \\
 &= \langle r_2 r_6 r_2 r_1, r_4 r_3 r_2 r_1 r_3 r_5, (r_5 r_3 r_1)^2, r_5 r_3 r_1 r_4 r_3 r_2, r_5 r_3 r_1 r_2 r_6 r_2 r_3 r_5 \rangle.
 \end{aligned}$$

4. Fuchsian subgroups of the Picard group

To understand totally geodesic surfaces in Y^+ , we recall from [16] that every nonelementary Fuchsian subgroup of $\text{PSL}(2, \mathbb{Z}[i])$ fixes a circle or straight line \mathcal{C} with an equation of the form: $a|z|^2 + \bar{B}z + B\bar{z} + c = 0$, with $a, c \in \mathbb{Z}$ and $B \in \mathbb{Z}[i]$, and vice versa. We caution the reader that the normalization of the equation of the circle follows [13] (which we will use in the proof) rather than [16], the normalization of [16] uses $Bz + \bar{B}\bar{z}$.

Two such circles (or straight-lines) \mathcal{C} and \mathcal{C}' are said to be *equivalent* if there exists $\gamma \in \text{PSL}(2, \mathbb{Z}[i])$ such that $\gamma\mathcal{C} = \mathcal{C}'$. Define $D = |B|^2 - ac$ to be the *discriminant* of \mathcal{C} . This is preserved by the action of $\text{PSL}(2, \mathbb{Z}[i])$, and hence equivalent circles have the same discriminant (see [16]). If $\Delta < \text{PSL}(2, \mathbb{Z}[i])$ is a torsion-free subgroup of finite index, then a Δ -equivalence class of circles and straight-lines can be associated to a totally geodesic surface in \mathbb{H}^3/Δ and vice versa. Hence we can also refer to the discriminant of the associated totally geodesic surface. When $a \neq 0$, \mathcal{C} is a circle centered at $-B/a$, with radius $\sqrt{D}/|a|$. This is the case when the totally geodesic surface is closed [13, Lemma 3.1]. Note that if the surface associated to a circle \mathcal{C} is closed and embedded in \mathbb{H}^3/Δ , then for every element $\delta \in \Delta$ we must have $\delta\mathcal{C} = \mathcal{C}$ or $\delta\mathcal{C} \cap \mathcal{C} = \emptyset$.

It is shown in [16] that every circle (or straight-line) as above is $\text{PSL}(2, \mathbb{Z}[i])$ -equivalent to one of the following:

- $\mathcal{C}_D : |z|^2 - D = 0$.
- $\mathcal{C}_{D,1} : 2|z|^2 + z + \bar{z} - \frac{1}{2}(D - 1) = 0$ (when $D \equiv 1 \pmod{4}$).
- $\mathcal{C}_{D,2} : 2|z|^2 + iz - i\bar{z} - \frac{1}{2}(D - 1) = 0$ (when $D \equiv 1 \pmod{4}$).
- $\mathcal{C}_{D,3} : 2|z|^2 + (1+i)z + (1-i)\bar{z} - \frac{1}{2}(D - 2) = 0$ (when $D \equiv 2 \pmod{4}$).

The radius of the first circle listed above is \sqrt{D} and for the others it is $\sqrt{D}/2$.

Lemma 4.1. *Let Δ be a normal subgroup of finite index in $\text{PSL}(2, \mathbb{Z}[i])$, and assume that $S \hookrightarrow \mathbb{H}^3/\Delta$ is an embedded totally geodesic surface associated to the circle \mathcal{C} of discriminant D . Then there exists an embedded totally geodesic surface S' associated to one of the circles \mathcal{C}_D or $\mathcal{C}_{D,j}$ for one of $j = 1, 2, 3$.*

Proof. Observe that, since \mathcal{C} is associated to an embedded totally geodesic surface in \mathbb{H}^3/Δ , for any $\alpha \in \mathrm{PSL}(2, \mathbb{Z}[i])$, the circle $\alpha\mathcal{C}$ is associated to an embedded totally geodesic surface in $\mathbb{H}^3/\alpha\Delta\alpha^{-1}$. Since Δ is assumed to be normal in $\mathrm{PSL}(2, \mathbb{Z}[i])$, the surface associated to $\alpha\mathcal{C}$ is actually embedded in \mathbb{H}^3/Δ .

From the classification of circles given above there is an $\alpha \in \mathrm{PSL}(2, \mathbb{Z}[i])$ such that $\alpha\mathcal{C}$ is one of \mathcal{C}_D or $\mathcal{C}_{D,j}$ for one of $j = 1, 2, 3$. The result follows. \square

Associated to the circle \mathcal{C} with equation $a|z|^2 + \bar{B}z + B\bar{z} + c = 0$ as above, is the Hermitian matrix $A = \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix}$ with an action of $\mathrm{PSL}(2, \mathbb{Z}[i])$ given by $\gamma^*A\gamma$, where $*$ denotes conjugate-transpose and the given action sends \mathcal{C} to $\gamma^{-1}\mathcal{C}$. Here \mathcal{C} is the set of all points $z \in \mathbb{C}$ such that $A\begin{pmatrix} z \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = 0$. Now [13, Corollary 3.3] provides a criterion for a totally geodesic surface S associated to a circle \mathcal{C} to be embedded in \mathbb{H}^3/Δ or not; namely if $\gamma \in \Delta$ does not leave \mathcal{C} invariant and satisfies

$$|\mathrm{tr}(\gamma^*A\gamma A^{-1})| < 2,$$

then $\gamma^{-1}\mathcal{C} \cap \mathcal{C} \neq \emptyset$ and S is not embedded. Furthermore, if S is closed, and γ does not leave \mathcal{C} invariant, then $\gamma^{-1}\mathcal{C} \cap \mathcal{C}$ is two points.

We will make use of the following lemma.

Lemma 4.2. *Let Δ be a subgroup of finite index in $\mathrm{PSL}(2, \mathbb{Z}[i])$ which contains the group $[\Gamma(1+i), \Gamma(1+i)]$, and let $M = \mathbb{H}^3/\Delta$. If $S \hookrightarrow M$ is a closed embedded totally geodesic surface (not necessarily orientable) associated to a circle \mathcal{C} , then there exists $\alpha \in \mathrm{PSL}(2, \mathbb{Z}[i])$ such that $\alpha\mathcal{C} = \mathcal{C}_{6,3}$.*

Proof. Recall from Remark 3.2 that $[\Gamma(1+i), \Gamma(1+i)] = \Gamma(2+2i)$, and so $\Gamma(2+2i) \subset \Delta$ by hypothesis. Therefore S gives rise to a closed embedded totally geodesic surface associated to the circle \mathcal{C} in the cover $\mathbb{H}^3/\Gamma(2+2i)$. Assuming that \mathcal{C} has discriminant D , there exists $\alpha \in \mathrm{PSL}(2, \mathbb{Z}[i])$ such that $\alpha\mathcal{C}$ is one of \mathcal{C}_D or $\mathcal{C}_{D,j}$ for $j = 1, 2, 3$

Now $\Gamma(2+2i) \triangleleft \mathrm{PSL}(2, \mathbb{Z}[i])$, and Lemma 4.1 shows that one of \mathcal{C}_D or $\mathcal{C}_{D,j}$ for $j = 1, 2, 3$ also gives rise to a closed embedded totally geodesic surface in $\mathbb{H}^3/\Gamma(2+2i)$.

Now the element $\begin{pmatrix} 1 & 2+2i \\ 0 & 1 \end{pmatrix}$ is in $\Gamma(2+2i)$, and using this element, a simple calculation shows that for the surface to be embedded, the radius of the associated circle must be $\leq \sqrt{2}$. From above, the radii of the circles \mathcal{C}_D or $\mathcal{C}_{D,j}$ for $j = 1, 2, 3$ is \sqrt{D} or $\sqrt{D}/2$. Hence, amongst the circles \mathcal{C}_D or $\mathcal{C}_{D,j}$, $j = 1, 2, 3$, the only possibilities are

$$\mathcal{C}_1, \quad \mathcal{C}_2, \quad \mathcal{C}_{1,1}, \quad \mathcal{C}_{1,2}, \quad \mathcal{C}_{2,3}, \quad \mathcal{C}_{5,1}, \quad \mathcal{C}_{5,2}, \quad \mathcal{C}_{6,3}$$

and the only one of these that can give rise to a *closed* surface is $\mathcal{C}_{6,3}$ (see [16, Lemma 8]).

The upshot of this discussion is that if $S \hookrightarrow M$ is a closed embedded totally geodesic surface with associated circle \mathcal{C} , then there exists $\alpha \in \text{PSL}(2, \mathbb{Z}[i])$ such that $\mathcal{C} = \alpha\mathcal{C}_{6,3}$. □

The proof of Lemma 2.4 will be completed in the sections below. To that end we make some additional comments and introduce some notation. From Proposition 3.1 and Remark 3.2, we need to consider certain groups Δ with $\Gamma(2+2i) \triangleleft \Delta \triangleleft \Gamma(1+i)$.

A complete system of (left or right) coset representatives for $\Gamma(1+i)$ in $\text{PSL}(2, \mathbb{Z}[i])$ is provided by the following 6 matrices:

$$\begin{aligned}
 T_0 &= \text{id}, & T_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & T_2 &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \\
 T_3 &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & T_4 &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, & T_5 &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.
 \end{aligned}$$

Using this and the normality of Δ in $\Gamma(1+i)$ (see the proof of Lemma 4.1), it follows that if $\mathcal{C} = \alpha\mathcal{C}_{6,3}$ corresponds to a closed embedded totally geodesic surface in \mathbb{H}^3/Δ then one of $T_i\mathcal{C}_{6,3}$ also corresponds to such a surface. Briefly, since \mathcal{C} corresponds to a closed embedded totally geodesic surface in \mathbb{H}^3/Δ , $\mathcal{C}_{6,3} = \alpha^{-1}\mathcal{C}$ corresponds to a closed embedded totally geodesic surface in $\mathbb{H}^3/\alpha^{-1}\Delta\alpha$. Writing $\alpha = \gamma T_i$ for some $\gamma \in \Gamma(1+i)$, and using $\Delta \triangleleft \Gamma(1+i)$ we deduce that $\mathcal{C}_{6,3}$ corresponds to a closed embedded totally geodesic surface in $\mathbb{H}^3/T_i^{-1}\Delta T_i$ from which it follows that $T_i\mathcal{C}_{6,3}$ corresponds to a closed embedded totally geodesic surface in \mathbb{H}^3/Δ .

Using the action on the Hermitian forms described above, the action by the matrices T_i is given by $(T_i^{-1})^*AT_i^{-1}$, which, since the entries of the matrices T_i are integers, is simply $(T_i^{-1})^tAT_i^{-1}$. Hence, the circles $T_i\mathcal{C}_{6,3}$ for $i = 0, 1, \dots, 5$ are represented by the matrices

$$\begin{aligned}
 A &= A_0 = \begin{pmatrix} 2 & 1-i \\ 1+i & -2 \end{pmatrix}, & A_1 &= \begin{pmatrix} -2 & -1-i \\ -1+i & 2 \end{pmatrix}, & A_2 &= \begin{pmatrix} 2 & 3-i \\ 3+i & 2 \end{pmatrix}, \\
 A_3 &= \begin{pmatrix} 2 & -1-i \\ -1+i & -2 \end{pmatrix}, & A_4 &= \begin{pmatrix} -2 & -3-i \\ -3+i & -2 \end{pmatrix}, & A_5 &= \begin{pmatrix} -2 & 1-i \\ 1+i & 2 \end{pmatrix}.
 \end{aligned}$$

5. Proving no closed embedded totally geodesic surfaces

From Lemma 2.1 to prove Lemma 2.4 (i.e., that the link complements W_{23} , W_{71} , W_{1091} and W_{1092} do not contain a closed embedded totally geodesic hyperbolic 3-manifold), we are reduced to showing that the hyperbolic 3-manifolds A^+ , B^+ , C^+ , D^+ and E^+ do not contain a closed embedded totally geodesic surface (which could be nonorientable).

In what follows in each of the subsections below we list elements of the groups $\pi_1(A^+)$, $\pi_1(B^+)$, $\pi_1(C^+)$, $\pi_1(D^+)$ and $\pi_1(E^+)$ that provide self-intersections of the circles $T_i\mathcal{C}_{6,3}$. This is done using the matrix generators for each of $\pi_1(A^+)$, $\pi_1(B^+)$, $\pi_1(C^+)$, $\pi_1(D^+)$ and $\pi_1(E^+)$ listed in [Section 3B](#) and the criteria of [\[13\]](#) stated [Section 4](#):

$$|\text{tr}(\gamma^* A_i \gamma A_i^{-1})| < 2.$$

These calculations were performed in [Mathematica \[22\]](#) and the notebook is available from the authors upon request. For convenience, we shall simply denote the generators for each of the groups $\pi_1(A^+)$, $\pi_1(B^+)$, $\pi_1(C^+)$, $\pi_1(D^+)$ and $\pi_1(E^+)$ in [Section 3B](#) by g_1, g_2, \dots, g_5 in the order that they are listed. What is listed below are the Hermitian forms $A = A_0, A_1, \dots, A_5$ and those elements γ , written in terms of g_1, g_2, \dots, g_5 , for which $|\text{tr}(\gamma^* A_i \gamma A_i^{-1})| < 2$.

We will also make use of the element $l = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, which being an element of $\Gamma(1 + i)$ normalizes each of the groups $\pi_1(A^+)$, $\pi_1(B^+)$, $\pi_1(C^+)$, $\pi_1(D^+)$ and $\pi_1(E^+)$ by [Proposition 3.1](#). Additional explanation of elements that are not visibly in the groups $\pi_1(A^+)$, $\pi_1(B^+)$, $\pi_1(C^+)$, $\pi_1(D^+)$ and $\pi_1(E^+)$ is provided when needed.

Finally, we remark that we also need to ensure that the elements do not leave the circles in question invariant. This is clear if the elements are parabolic (since the surface is closed) and when the trace is a nonreal complex number that is not purely imaginary. In the cases where the element has trace that is pure imaginary we check to see whether the circle is left invariant.

The manifold A^+ :

circle/form	element	trace value
A	g_3	$2/3$
A_1	g_1	$2/3$
A_2	$g_1 g_4$	$2/3$
A_3	g_1	$2/3$
A_4	$l\beta l^{-1}$	$2/3$
A_5	g_3	$2/3$

Note that the element g_1 has trace $-2i$. However, a calculation shows that g_1 does not leave invariant any of the circles $T_i\mathcal{C}_{6,3}$ for $i = 0, 1, \dots, 5$. That $\beta \in \pi_1(A^+)$, can be checked by noting that

$$g_2^{-1} g_1 g_3 \beta = \begin{pmatrix} 5+8i & 10-2i \\ 18+6i & 13-16i \end{pmatrix} = \begin{pmatrix} 1+4(1+2i) & (2+2i)(2-3i) \\ (2+2i)(6-3i) & 1+4(3-4i) \end{pmatrix} \in \Gamma(2+2i).$$

By [Remark 3.2](#) and [Proposition 3.1](#), $\Gamma(2 + 2i) = [\Gamma(1 + i), \Gamma(1 + i)] < \pi_1(A^+)$.

None of the elements g_3 , g_1g_4 and β (and hence also $l\beta l^{-1} \in \pi_1(A^+)$) have purely imaginary trace.

The manifold B^+ :

circle/form	element	trace value
A	g_1	$2/3$
A_1	g_1	$2/3$
A_2	g_2	$2/3$
A_3	g_2	$2/3$
A_4	lg_1l^{-1}	$2/3$
A_5	g_1	$2/3$

From Section 3B, g_1 and g_2 are both parabolic, and as above $lg_1l^{-1} \in \pi_1(B^+)$.

The manifold C^+ : In this case the parabolic element g_4 works for all the forms A, A_1, \dots, A_5 with trace value $2/3$.

The manifold D^+ :

circle/form	element	trace value
A	g_1	$2/3$
A_1	g_1	$2/3$
A_2	lg_1l^{-1}	$2/3$
A_3	lg_1l^{-1}	$2/3$
A_4	lg_1l^{-1}	$2/3$
A_5	lg_1l^{-1}	$2/3$

From Section 3B, g_1 is parabolic, and as noted above $lg_1l^{-1} \in \pi_1(D^+)$.

The manifold E^+ : Since the parabolic element $g_1 \in \pi_1(E^+)$ is exactly the same as for $\pi_1(D^+)$, the same table holds for E^+ as that shown for D^+ .

Remark 5.1. From Remark 3.2 and Proposition 3.1 we know that $\mathbb{H}^3/\Gamma(2+2i)$ covers each of the manifolds A^+, B^+, C^+, D^+ , and E^+ , which we have shown do not contain a closed embedded totally geodesic surface. On the other hand, as pointed out in [13] the link complement $\mathbb{H}^3/\Gamma(2+2i)$ does contain a closed totally geodesic surface of genus 3 associated to $C_{6,3}$.

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