Pacific Journal of Mathematics

Volume 326 No. 2 October 2023

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2023 is US $605/year for the electronic version, and $820/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 326, No. 2, 2023
https://doi.org/10.2140/pjm.2023.326.187

SMOOTH LOCAL SOLUTIONS TO SCHRÖDINGER FLOWS WITH DAMPING TERM FOR MAPS INTO SYMPLECTIC MANIFOLDS

BO CHEN AND YOUDE WANG

We show the existence of short-time very regular solutions to the initial Neumann boundary value problem of Schrödinger flows with damping term (or Landau–Lifshitz–Gilbert flows) for maps from a 3-dimensional compact Riemannian manifold with smooth boundary into a compact symplectic manifold.

1. Introduction

Let (M, g) be a compact Riemannian manifold with smooth boundary and (N, J, ω) be a symplectic manifold, where ω is the symplectic form and $J : TN \to TN$ with $J^2 = -\,\mathrm{id}$ is an ω-tamed almost complex structure. For a smooth map $u \in C^2(M, N)$, the tension field is defined by

$$\tau(u) = \mathrm{tr}_g(\nabla du),$$

where ∇ denotes the induced connection on the pullback bundle u^*TN.

Recently, in [Chen and Wang 2023b; 2023a] we have addressed the local existence of strong or even smooth solutions to the initial Neumann boundary value problems to the Schrödinger flows from a smooth bounded domain Ω^m ($m = 2, 3$) into a standard sphere $\mathbb{S}^2$. A natural problem is whether or not one can extend the local existence of smooth solutions to the initial Neumann boundary value problem to the following Schrödinger flow from a compact Riemannian manifold with boundary (M, g) into a general symplectic manifold (N, J, ω):

$$\begin{cases} \partial_t u = J(u)\tau(u), & (x,t) \in M \times \mathbb{R}^+, \\ \partial u/\partial \nu = 0, & (x,t) \in \partial M \times \mathbb{R}^+, \\ u(x,0) = u_0 : M \to N. \end{cases}$$

Wang is the corresponding author.

MSC2020: 35K51, 35Q60, 58J35.

Keywords: Schrödinger flow with damping term, initial Neumann boundary value problem.

In this paper, we are concerned with a geometric flow for maps between (M, g) and (N, J, ω), which is a close relative of the Schrödinger flow. If u is a time-dependent map from (M, g) into N satisfying

$$\partial_t u + \gamma \nabla_v u = \alpha\tau(u) - \beta J(u)\tau(u),$$

we call this geometric flow a Schrödinger flow with damping term $\alpha\tau(u)$ (or a Landau–Lifshitz–Gilbert (LLG) geometric flow) for maps from (M, g) into (N, J, ω), where $\alpha > 0$, β and γ are fixed real numbers, $v : M \times \mathbb{R}^+ \to TM$ is a vector field satisfying $\mathrm{div}(v) = 0$ inside M for any $t \in \mathbb{R}^+$, and $\nabla_v u$ is defined by

$$\nabla_v u = du(v).$$

We are interested in the well-posedness to the initial Neumann boundary value problem of the above geometric flow

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha\tau(u) - \beta J(u)\tau(u), & (x, t) \in M \times \mathbb{R}^+, \\ \quad \partial u/\partial \nu = 0, & (x, t) \in \partial M \times \mathbb{R}^+, \\ \quad u(x, 0) = u_0 : M \to N. \end{cases} \tag{1-1}$$

In fact, the study of system (1-1) above can be regarded as the first step to approach the previous initial Neumann boundary value problem on the Schrödinger map flow. This is also the main motivation of this paper.

On the other hand, system (1-1) is of strong physical background. Now, let us recall some background materials and related equations of this flow.

1A. ***Background: Landau–Lifshitz–Gilbert equation and the Schrödinger map flow.*** Let Ω be a bounded domain in $\mathbb{R}^3$. In physics, for a map u from Ω into a standard sphere $\mathbb{S}^2$, the Landau–Lifshitz (LL) equation

$$\partial_t u = -u \times \Delta u \tag{1-2}$$

is a fundamental evolution equation for the ferromagnetic spin chain and was proposed on the phenomenological ground in studying the dispersive theory of magnetization of ferromagnets. It was first derived by Landau and Lifshitz [1935], and then proposed by Gilbert [1955] with dissipation as the form

$$\partial_t u = -\alpha u \times (u \times \Delta u) - \beta u \times \Delta u, \tag{1-3}$$

where β is a real number and $\alpha \geq 0$ is called the Gilbert damping coefficient. Hence, equation (1-3) above is also called the Landau–Lifshitz–Gilbert (LLG) equation if $\alpha > 0$. Here "$\times$" denotes the cross product in $\mathbb{R}^3$ and Δ is the Laplace operator in $\mathbb{R}^3$.

Let $i : \mathbb{S}^2 \to \mathbb{R}^3$ be the canonical inclusion map, which induces an embedding $i_* : T\mathbb{S}^2 \to \mathbb{S}^2 \times \mathbb{R}^3$, namely $i_*(p, v) = (p, di_p(v))$ for any $p \in \mathbb{S}^2$ and $v \in T_p\mathbb{S}^2$.

Let $\iota : \mathbb{R}^3 \setminus \{0\} \to \mathbb{S}^2$ be the projection defined by $\iota(y) = y/|y|$. Then a direct calculation shows

$$d\iota|_y(w) = \pi_y(w) = w - \langle w, y\rangle y$$

for $y \in \mathbb{S}^2$ and $w \in \mathbb{R}^3$, where π is the orthogonal projection from $\mathbb{R}^3$ to $T_y\mathbb{S}^2$. Moreover, it satisfies

$$i_{*y} \circ \pi_y = \pi_y, \quad \pi_y \circ i_{*y} = \mathrm{id}.$$

Then $u\times$ has the intrinsic form

$$u\times = i_{*u} \circ J(u) \circ \pi_u.$$

Here J is the complex structure on $\mathbb{S}^2$, i.e., $J(u) : T_u\mathbb{S}^2 \to T_u\mathbb{S}^2$ rotates vectors $\frac{\pi}{2}$ radians counterclockwise in the tangent space of $\mathbb{S}^2$. Therefore, (1-3) can be written as

$$\partial_t u = \alpha\pi_u \Delta u - \beta i_*(u) \circ J(u) \circ \pi_u \Delta u.$$

Since $\tau(u) = \pi_u \Delta u \in T_u\mathbb{S}^2$ (i.e., the tension field) and $\pi_u \partial_t u = \partial_t u$, we get the intrinsic version of (1-3) as

$$\partial_t u = \alpha\tau(u) - \beta J(u)\tau(u). \tag{1-4}$$

In the case $\alpha = 0$, it is just the Schrödinger flow into $\mathbb{S}^2$, which is introduced independently in [Ding and Wang 2001] and [Terng and Uhlenbeck 2006] as a geometric Hamiltonian flow of maps between manifolds. The intrinsic equation (1-4) can be defined between general manifolds and gives a natural generalization of the LLG equation, which is a parabolic perturbation of the Schrödinger flow. Namely, suppose that (M, g) is a Riemannian manifold and (N, J, ω) is a symplectic manifold, the LLG geometric flow for map $u : M \times \mathbb{R}^+ \to N \hookrightarrow \mathbb{R}^K$ is defined by

$$\partial_t u = \alpha\tau(u) - \beta J(u)\tau(u), \tag{1-5}$$

where

$$\tau(u) = \Delta u + A(u)(\nabla u, \nabla u)$$

is the tension field, $A(u)(\,\cdot\,,\cdot\,)$ is the second fundamental form of N in $\mathbb{R}^K$. Here we have embedded isometrically N into $\mathbb{R}^K$ by applying the well-known Nash embedding theorem. In the following, we always assume that $N \subset \mathbb{R}^K$ is just a submanifold in $\mathbb{R}^K$ for the sake of convenience and without loss of generality.

Let $v : M \times \mathbb{R}^+ \to TM$ be a vector field with $\operatorname{div}(v) \equiv 0$ inside M. The equation

$$\partial_t u + \gamma\nabla_v u = \alpha\tau(u) - \beta J(u)\tau(u) \tag{1-6}$$

appears in magnetoelastic theory, where $\gamma \in \mathbb{R}$ is a constant. One can refer to [Benešová et al. 2018; Kalousek et al. 2021] for more details.

In the special case of $\alpha = 0$, the equation

$$\partial_t u + \gamma \nabla_v u = -\beta J(u)\tau(u)$$

is called an incompressible Schrödinger flow, which was derived for the purely Eulerian simulation of incompressible fluids by Chern et al. [2016].

We should mention that (1-5) and (1-6) are gauge equivalent. Let $\phi_t : M \to M$ be a family of diffeomorphisms of M generated by γv, which preserves the volume element. Namely, ϕ_t is the solution to the ODE

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \gamma v(\phi_t(x), t), \\ \phi(\,\cdot\,, 0) = \phi_0, \end{cases} \tag{1-7}$$

where $\phi_0 : M \to M$ is a given diffeomorphism. If $\partial M \neq \varnothing$, we additionally assume $\gamma \langle v, \nu \rangle|_{\partial M} = 0$, where ν is the outer normal vector of ∂M. Let u solve (1-6), and set $\tilde{u}(x, t) = u(\phi_t(x), t)$. Then we have

$$\partial_t \tilde{u} = (\partial_t u + \gamma \nabla_v u) \circ \phi_t(x) = \phi_t^*(\alpha \tau(u) - \beta J(u)\tau(u)) = \alpha \tau(\tilde{u}) - \beta J(\tilde{u})\tau(\tilde{u}).$$

This is the standard LLG equation

$$\partial_t \tilde{u} = \alpha \tau(\tilde{u}) - \beta J(\tilde{u})\tau(\tilde{u})$$

with respect to the pullback metric $g_t = \phi_t^* g$.

It is worthy to point out that if the vector field v is the velocity field in magnetic fluid, which satisfies a Navier–Stokes equation involving a magnetic term, we can derive the so-called magnetic elasticity system (see [Benešová et al. 2018] for more details)

$$\begin{cases} \partial_t v + \nabla_v v + \nabla P = \mu \Delta v - \nabla \cdot (\nabla u \odot \nabla u - W'(F)F), \\ \operatorname{div}(v) = 0, \\ \partial_t F + (v \cdot \nabla)F - \nabla v F = \kappa \Delta F, \\ \partial_t u + \gamma \nabla_v u = \alpha \tau(u) - \beta u \times \Delta u, \end{cases} \tag{1-8}$$

accompanied by some suitable initial-boundary value conditions. Here μ, κ are two positive constants, $u : \Omega^m \times \mathbb{R}^+ \to \mathbb{S}^2$ is the magnetization field, $v : \Omega^m \times \mathbb{R}^+ \to \mathbb{R}^m$ is the velocity field of the fluid, P is the pressure function, and $F : \Omega^m \to \mathbb{R}^{m \times m}$ is the deformation gradient, where Ω^m is a domain in $\mathbb{R}^m$ with $m = 2, 3$. The term $\nabla u \odot \nabla u$ is an $m \times m$ matrix with (i, j)-th entry

$$(\nabla u \odot \nabla u)_{ij} = \langle \nabla_i u, \nabla_j u \rangle,$$

W is the elastic energy which satisfies $W(RS) = W(S)$ for all $R \in \mathbb{SO}(m)$ (and thus $W'(RS) = RW'(S)$) for all matrices $S \in \mathbb{R}^{m\times m}$, and

$$\tau(u) = \Delta u + |\nabla u|^2 u.$$

In the special case $\alpha = 0$ and $F \equiv 0$, equation (1-8) is the Navier–Stokes–Schrödinger flow, which can be used to describe the dispersive theory of magnetization of ferromagnets with quantum effects.

Next, we briefly recall a few results that are closely related to our work in the present paper. In 1985, the existence of global weak solutions to the LLG equation (i.e., (1-3) with $\alpha > 0$) was established by Visintin [1985]. P.L. Sulem, C. Sulem, and C. Bardos [Sulem et al. 1986] employed a difference method to prove that the LL equation (1-2) without a dissipation term defined on $\mathbb{R}^n$ admits a global weak solution and a smooth local solution. Later, Alouges and Soyeur [1992] showed the nonuniqueness of weak solutions to the LLG equation defined on a bounded domain $\Omega \subset \mathbb{R}^3$. Y.D. Wang [1998] adopted a more geometric approximation method (i.e., the complex structure approximation method) than the Ginzburg–Landau penalized method used for the LLG equation in [Alouges and Soyeur 1992; Bonithon 2007; Tilioua 2011] to obtain the global existence of weak solutions to the Schrödinger flow for maps from a closed Riemannian manifold or a bounded domain in $\mathbb{R}^n$ into $\mathbb{S}^2$. For recent developments of weak solutions to a class of generalized LL equations and related flows, we refer to [Jia and Wang 2019; 2020; Chen and Wang 2021] for various results.

The global well-posedness result for the LL equation on $\mathbb{R}^n$ with $n \geq 2$ was well studied by Ionescu, Kenig, and Bejanaru et al., we refer to [Bejenaru 2008; Bejenaru et al. 2007; 2011; Ionescu and Kenig 2007] for more details. For the Schrödinger flow from a closed manifold or $\mathbb{R}^n$ onto a compact Kähler manifold (i.e., (1-9) with $\alpha = 0$), the existence of local smooth solutions was obtained by Ding and Wang et al., one can refer to [Ding and Wang 1998; 2001; Sulem et al. 1986; Pang et al. 2000; 2001; 2002; Zhou et al. 1991].

In the case the domain manifold is a smooth bounded domain in $\mathbb{R}^3$, Carbou and Fabrie [2001] proved the local existence and uniqueness of regular solutions of the initial Neumann boundary value problem to the LLG equation. Recently, the local existence of very regular solutions to the LLG equation with $\alpha > 0$ was addressed by applying the delicate Galerkin approximation method and adding initial Neumann boundary compatibility conditions on the initial map [Carbou and Jizzini 2018]. Inspired by this method, which essentially stems from [Sulem et al. 1986], we obtained local-in-time very regular solutions to the LLG equation with spin-polarized transport in [Chen and Wang 2023c].

Very recently, the authors of this paper studied the most challenging LL equation (i.e., the Schrödinger flow into $\mathbb{S}^2$) on a smooth bounded domain in $\mathbb{R}^3$, and proved the existence and uniqueness of local-in-time strong solutions and local very regular solutions to its initial Neumann boundary value problem (see [Chen and Wang 2023b; 2023a]).

1B. ***Motivations and main results.*** Although we have proved the existence and uniqueness of local-in-time strong solutions and local very regular solutions to the initial Neumann boundary value problem of the Schrödinger flow from a smooth bounded domain in $\mathbb{R}^3$ into $\mathbb{S}^2$ (see [Chen and Wang 2023b; 2023a]), the existence of the initial Neumann boundary value problem of the Schrödinger flow from a smooth bounded domain M in $\mathbb{R}^3$ into a compact Kähler manifold N is still an open problem:

$$\begin{cases} \partial_t u = J(u)\tau(u), & (x,t) \in M \times \mathbb{R}^+, \\ \partial u/\partial\nu = 0, & (x,t) \in \partial M \times \mathbb{R}^+, \\ u(x,0) = u_0 : M \to N. \end{cases}$$

To this end, the first step is to extend Carbou's work [Carbou and Jizzini 2018] on the LLG equation for maps from a smooth bounded domain in $\mathbb{R}^3$ into $\mathbb{S}^2$ to the case from a compact Riemannian manifold with smooth boundary into a symplectic manifold. So, in this paper we consider the existence of regular solutions to the initial Neumann boundary value problem of (1-5) with $\alpha > 0$.

Because the geometry of the domain manifold M does not affect our analysis and the main results, for simplicity, we assume that Ω is a smooth bounded domain in $\mathbb{R}^m$. Let u be a time-dependent map from Ω to N. We consider the initial Neumann boundary value problem of the general LLG flow (equation)

$$\begin{cases} \partial_t u + \gamma\nabla_v u = \alpha\tau(u) - \beta J(u)\tau(u), & (x,t) \in \Omega \times \mathbb{R}^+, \\ \partial u/\partial\nu = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to N \hookrightarrow \mathbb{R}^K, \end{cases} \tag{1-9}$$

where $\alpha > 0$, γ and β are fixed real numbers. Here $v : \Omega \times \mathbb{R}^+ \to \mathbb{R}^m$ is a vector field satisfying $\mathrm{div}(v) = 0$, and $\nabla_v u$ is defined by

$$\nabla_v u = du(v).$$

No doubt, the initial Neumann boundary value problem of the corresponding incompressible Schrödinger flow

$$\begin{cases} \partial_t u + \gamma\nabla_v u = -\beta J(u)\tau(u), & (x,t) \in \Omega \times \mathbb{R}^+, \\ \partial u/\partial\nu = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to N, \end{cases} \tag{1-10}$$

and related problems are more challenging and will be carried out in our forthcoming papers.

Our main results are the following two theorems:

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and N be a compact symplectic manifold. Let $u_0 \in H^2(\Omega, N)$ satisfy the compatibility condition*

$$\frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0.$$

Suppose $v \in L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))$, $\operatorname{div}(v) = 0$ for any $t \in \mathbb{R}^+$, and $\langle v, \nu\rangle|_{\partial\Omega\times\mathbb{R}^+} = 0$. Then there exists a constant $T_0 > 0$ depending only on γ, α, β, $\|u_0\|_{H^2(\Omega)}$, and $\|v\|_{L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))}$ such that (1-9) *admits a unique local solution u for any $T < T_0$ which satisfies*

$$u \in C^0\big([0, T], H^2(\Omega, N)\big) \cap L^2\big([0, T], H^3(\Omega, N)\big). \tag{1-11}$$

Furthermore, if $u_0 \in H^3(\Omega, N)$, $v \in C^0(\mathbb{R}^+, H^1(\Omega))$, and $\partial_t v \in L^2(\mathbb{R}^+, H^1(\Omega))$, then this solution u satisfies

$$\partial_t^i u \in C^0\big([0, T], H^{3-2i}(\Omega)\big) \cap L^2\big([0, T], H^{4-2i}(\Omega)\big) \tag{1-12}$$

for $T < T_0$ and $i = 0, 1$.

Moreover, we can obtain a very regular solution to (1-9) by adding higher order compatibility conditions on an initial map:

Theorem 1.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and N be a compact symplectic manifold. Let $k \geq 4$, $u_0 \in H^k(\Omega, N)$ satisfy the compatibility condition at $\left[\frac{k}{2}\right] - 1$ order, which is given in the Definition 5.1. Suppose that $\operatorname{div}(v) = 0$ for any $t \in \mathbb{R}^+$ and $\langle v, \nu\rangle|_{\partial\Omega\times\mathbb{R}^+} = 0$, and for any $i \leq \left[\frac{k}{2}\right] - 1$,*

$$\partial_t^i v \in C^0\big(\mathbb{R}^+, H^{k-2(i+1)}(\Omega, \mathbb{R}^3)\big) \cap L^2\big(\mathbb{R}^+, H^{2[k/2]-2i}(\Omega, \mathbb{R}^3)\big);$$

moreover, if k is odd, we additionally assume that $\partial_t^{[k/2]} v \in L^2(\mathbb{R}^+, L^2(\Omega))$. Then, for u and $T_0 > 0$ which are given in Theorem 1.1, we have that for any $T < T_0$ and $0 \leq i \leq \left[\frac{k}{2}\right] - 1$,

$$\partial_t^i u \in C^0\big([0, T], H^{k-2i}(\Omega, N)\big) \cap L^2\big([0, T], H^{k+1-2i}(\Omega, N)\big).$$

Remark 1.3. (1) Theorems 1.1 and 1.2 still hold true when Ω is a compact 3-dimensional Riemannian manifold with smooth boundary.

(2) By almost the same arguments as in the proofs of Theorem 1.1 and Theorem 1.2, we can also get a short-time very regular solution to the equation

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha(\tau(u) + \gamma J(u)\nabla_v u) + J(u)\tau(u), & (x, t) \in \Omega \times \mathbb{R}^+, \\ \quad \partial u/\partial\nu = 0, & (x, t) \in \partial\Omega \times \mathbb{R}^+, \\ \quad u(x, 0) = u_0 : \Omega \to N \hookrightarrow \mathbb{R}^K, \end{cases} \tag{1-13}$$

on $\Omega \times \mathbb{R}^+$, provided that u_0 satisfies some suitable compatibility conditions on the boundary. Here $\alpha > 0$ and $\gamma \in \mathbb{R}$.

To prove Theorem 1.1, we need to consider an extrinsic version (see (3-1)) of (1-9) and then use the solution of the auxiliary equation

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha(\Delta u + \mathcal{P}(u)(\nabla u, \nabla u)) - \beta \mathfrak{J}(u)\Delta u, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \quad \partial u / \partial \nu = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ \quad u(x,0) = u_0 : \Omega \to N \hookrightarrow \mathbb{R}^K, \end{cases} \tag{1-14}$$

which preserves the original geometric structures of (1-9), to approximate a solution of (1-9). Here $\mathcal{P}(\cdot,\cdot)$ and $\mathfrak{J}(u)$ are the extensions of $A(\cdot,\cdot)$ and J defined in Section 3A, respectively. We then prove the main result Theorem 1.1 by the following process $\mathscr{T}(1)$:

(1) We apply Galerkin approximation to (1-14), and then estimate some suitable energies directly to get a unique solution u to (1-14) satisfying

$$u \in C^0\big([0,T], H^2(\Omega, \mathbb{R}^K)\big) \cap L^2\big([0,T], H^3(\Omega, \mathbb{R}^K)\big).$$

Since $u_0 \in H^2(\Omega, N)$, the geometric structures of the above auxiliary equation (1-14) guarantee $u(x,t) \in N$ for a.e. $(x,t) \in \Omega \times [0, T_0)$. Therefore, u is also a solution to (1-9) satisfying (1-11).

(2) Since the space of test functions associated to (1-14) is small, we cannot get higher energy estimates directly to improve the regularity of u. We then consider the differential of Galerkin approximation to (1-14) with respect to time and then apply an energy method to show (1-12).

Next, with higher order compatibility conditions on initial data at hand we can prove Theorem 1.2 by following the ideas in [Carbou and Jizzini 2018; Chen and Wang 2023c]. More precisely, we consider the equation satisfied by $\partial_t^k u$ (i.e., (5-9)) with $k \geq 1$ and repeat the process $\mathscr{T}(1)$ in the proof of Theorem 1.1 with $\partial_t^k u$ in place of u. Namely, we prove the main result Theorem 1.2 by showing the so-called property $\mathscr{T}(k)$ which is defined in Section 5.

Our proof of Theorems 1.1 and 1.2 is similar to that of [Carbou and Jizzini 2018; Chen and Wang 2023c], but is more complicated. There are two technical issues we need to address in our presentation. The first one is that we obtain the extensions of $A(\cdot,\cdot)$ and J in a tubular neighborhood $U_{2\delta}(N)$ of N by using the canonical projections $\iota : U_{2\delta}(N) \to N$ and $\pi : N \times \mathbb{R}^K \to TN$, which satisfy the original geometric structures of $A(\cdot,\cdot)$ and J, respectively. Then by multiplying a truncation function involving the distance function $\mathrm{dist}(\cdot, N)$, we get the desired extensions (i.e., $\mathcal{P}(\cdot,\cdot)$ and $\mathfrak{J}(u)$) on $\mathbb{R}^K$ (see Section 3A). In particular, the extension $\mathfrak{J}$ of J is still antisymmetric, which plays an essential role in our proof. The second one is that the property $\mathrm{div}(v) = 0$ can be applied to eliminate some terms involving v in the process of the energy estimate. This makes the assumptions on regularity for v in Theorems 1.1 and 1.2 weaker than those for the electric current in [Carbou and Jizzini 2018], one can refer to [Carbou and Jizzini 2018] for more details.

The rest of our paper is organized as follows: In Section 2, we introduce basic notations on Sobolev spaces and some preliminary lemmas. In Section 3 and Section 4, we give the proof of Theorem 1.1. Finally, the proof of Theorem 1.2 is given in Section 5.

2. Preliminary

2A. ***Notations.*** In this subsection, we fix some notations on manifolds and Sobolev spaces which will be used in the following context:

Let (N, J, ω) be an n-dimensional symplectic manifold, where ω is the symplectic form and $J : TN \to TN$ with $J^2 = -\operatorname{id}$ is an ω-tamed almost complex structure, that is, for any $X, Y \in \Gamma(TN)$,

$$\omega(JX, JY) = \omega(X, Y).$$

Then ω and J induce a canonical Riemannian metric g on N as

$$g(X, Y) = \omega(X, JY),$$

which also satisfies

$$g(JX, JY) = g(X, Y).$$

By the Nash embedding theorem, we always embed isometrically (N, g) into $\mathbb{R}^K$ hence without loss of generality we assume $N \subset \mathbb{R}^K$ is an embedded submanifold of $\mathbb{R}^K$ with the induced metric. Let Ω be a smooth bounded domain in $\mathbb{R}^m$ with $m \geq 1$. Let $u = (u^1, \dots, u^K) : \Omega \to N \hookrightarrow \mathbb{R}^K$ be a map. We set

$$H^k(\Omega) = W^{k,2}(\Omega, \mathbb{R}^K)$$

and

$$H^k(\Omega, N) = \{u \in H^k(\Omega) : u(x) \in N \text{ for a.e. } x \in \Omega\}.$$

Moreover, let $(B, \|\cdot\|_B)$ be a Banach space and $f : [0, T] \to B$ be a map. For any $p > 0$ and $T > 0$, we define

$$\|f\|_{L^p([0,T],B)} := \left(\int_0^T \|f\|_B^p \, dt\right)^{1/p},$$

and set

$$L^p([0, T], B) := \{f : [0, T] \to B : \|f\|_{L^p([0,T],B)} < \infty\}.$$

In particular, we set

$$\begin{aligned} &L^p([0, T], H^k(\Omega, N)) \\ &\qquad = \{u \in L^p([0, T], H^k(\Omega)) : u(x, t) \in N \text{ for a.e. } (x, t) \in \Omega \times [0, T]\}, \end{aligned}$$

where $k \in \mathbb{N}$ and $p \geq 1$.

2B. ***Some basic lemmas.*** Next, we recall some crucial lemmas which will be used later. The following lemma of equivalent norms for Sobolev functions with Neumann boundary condition can be found in [Wehrheim 2004]:

Lemma 2.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^m$ and $k \in \mathbb{N}$. There exists a constant $C_{k,m}$ such that, for all $u \in H^{k+2}(\Omega)$ with $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$,*

$$\|u\|_{H^{k+2}(\Omega)} \le C_{k,m}(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^k(\Omega)}). \tag{2-1}$$

Here, for simplicity we define $H^0(\Omega) := L^2(\Omega)$.

In particular, the above lemma implies that we can define the H^{k+2}-norm of u as follows:

$$\|u\|_{H^{k+2}(\Omega)} := \|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^k(\Omega)}.$$

We also need to use the following ODE comparison theorem and the classical compactness results in [Boyer and Fabrie 2013; Simon 1987] to show the uniform estimates and the convergence of solutions to the approximate equation constructed in the coming sections:

Lemma 2.2. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function, which is locally Lipschitz in the second variable. Let $z : [0, T^*) \to \mathbb{R}$ be the maximal solution of the Cauchy problem*

$$\begin{cases} z' = f(t, z), \\ z(0) = z_0. \end{cases}$$

Let $y : \mathbb{R}^+ \to \mathbb{R}$ be a C^1 function such that

$$\begin{cases} y' \le f(t, y), \\ y(0) \le z_0. \end{cases}$$

Then, we have

$$y(t) \le z(t), \quad t \in [0, T^*).$$

Lemma 2.3 (Aubin–Lions–Simon compactness lemma, see [Simon 1987]). *Let $X \subset B \subset Y$ be Banach spaces with compact embedding $X \hookrightarrow B$. Let $1 \le p, q, r \le \infty$. For $T > 0$, we define*

$$E_{p,r} = \left\{ f : f \in L^p((0, T), X) \quad \text{and} \quad \frac{df}{dt} \in L^r((0, T), Y) \right\},$$

which is equipped with a norm $\|f\| := \|f\|_{L^p((0,T),X)} + \|df/dt\|_{L^r((0,T),Y)}$. Then the following properties hold true:

1. *If $p < \infty$ and $p < q$, the embedding $E_{p,r} \cap L^q((0, T), B)$ in $L^s((0, T), B)$ is compact for all $1 \le s < q$.*
2. *If $p = \infty$ and $r > 1$, the embedding of $E_{p,r}$ in $C^0([0, T], B)$ is compact.*

Lemma 2.4 [Boyer and Fabrie 2013, Theorem II.5.14]. *Let $k \in \mathbb{N}$, then the space*

$$E_{2,2} = \left\{ f : f \in L^2((0,T), H^{k+2}(\Omega)),\ \frac{\partial f}{\partial t} \in L^2((0,T), H^k(\Omega)) \right\}$$

is continuously embedded in $C^0([0,T], H^{k+1}(\Omega))$.

2C. ***Galerkin basis and Galerkin projection.*** Let Ω be a smooth bounded domain in $\mathbb{R}^m$, λ_i be the i-th eigenvalue of the operator $\Delta - I$ with Neumann boundary condition. We denote the corresponding eigenfunction of λ_i by f_i, that is,

$$(\Delta - I) f_i = -\lambda_i f_i \quad \text{with } \frac{\partial f_i}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

Without loss of generality, we assume that $\{f_i\}_{i=1}^{\infty}$ is a complete, standard orthonormal basis of $L^2(\Omega, \mathbb{R}^1)$. Let $H_n = \operatorname{span}\{f_1, \dots f_n\}$ be a finite subspace of L^2, $P_n : L^2 \to H_n$ be the Galerkin projection such that for any $f \in L^2$, $f^n = P_n f = \sum_1^n \langle f, f_i \rangle_{L^2} f_i$. Then the following result is proved in [Carbou and Jizzini 2018]:

Lemma 2.5. *There exists a constant C such that for all n, the projection P_n satisfies the following properties*:

(1) *For $f \in H^1(\Omega, \mathbb{R}^1)$, $\|P_n(f)\|_{H^1(\Omega)} \le \|f\|_{H^1(\Omega)}$,*

(2) *For $f \in H^2(\Omega, \mathbb{R}^1)$ with $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0$, $\|P_n(f)\|_{H^2(\Omega)} \le C\|f\|_{H^2(\Omega)}$,*

(3) *For $f \in H^3(\Omega, \mathbb{R}^1)$ with $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = 0$, $\|P_n(f)\|_{H^3(\Omega)} \le C\|f\|_{H^3(\Omega)}$.*

Here we set $H^k(\Omega, \mathbb{R}^1) = W^{k,2}(\Omega, \mathbb{R}^1)$ for $k \in \mathbb{N}$.

3. Local strong solution

3A. ***Approximation equation.*** We start with constructing the approximation equation of (1-9). Let N be a complete compact Riemannian manifold, and $N \subset \mathbb{R}^K$. Let $\pi : N \times \mathbb{R}^K \to TN$ be the canonical orthonormal projection induced by the inclusion map $i : N \hookrightarrow \mathbb{R}^K$. Then there exists a positive constant δ such that there exists a canonical well-defined projection

$$\iota : U_{2\delta}(N) \to N, \quad x \longmapsto \iota(x),$$

satisfying $\operatorname{dist}(x, N) = |x - \iota(x)|$, where

$$U_{2\delta}(N) := \{x \in \mathbb{R}^K \mid \operatorname{dist}(x, N) < 2\delta\}.$$

Moreover, we have the following theorem (refer to [Simon 1996] for a proof):

Theorem 3.1. *Let N be a compact n-dimensional C^∞-submanifold embedded in $\mathbb{R}^K$. Then there exists a positive number $\delta(N) > 0$ and a smooth projection map*

$$\iota : U_{2\delta}(N) \to N \subset \mathbb{R}^K$$

such that the following properties hold:

(1) *For any* $y \in U_{2\delta}(N)$, *we have* $y - \iota(y) \in T^{\perp}_{\iota(y)}N$ *with* $|y - \iota(y)| = \mathrm{dist}(y, N)$. *Moreover, if* $z \in N \setminus \{\iota(y)\}$, *we have* $|y - z| > |y - \iota(y)|$.

(2) *For any* $y \in N$ *and* $z \in T^{\perp}_{y}N$ *with* $|z| < 2\delta$, *we have*

$$\iota(y + z) = y.$$

(3) *For* $v \in \mathbb{R}^K$ *and* $y \in N$, *we have*

$$d\iota|_y(v) = \pi_y(v) \in T_yN.$$

(4) *For* $y \in N$ *and* $v_1, v_2 \in T_yN$, *we have*

$$\mathrm{Hess}\,\iota|_y(v_1, v_2) = \nabla\pi_y(v_1, v_2) = -A(y)(v_1, v_2).$$

We next restrict to the case where (N, J, ω) is a compact symplectic manifold. The almost complex structure is a map $J : TN \to TN$ such that $J^2 = -\,\mathrm{id}$. Then we can define an extension $\bar{J}$ of J on $U \times \mathbb{R}^K$ by

$$\begin{tikzcd} U \times \mathbb{R}^K \arrow[r, "\bar{J}"] \arrow[d, "(\iota,\pi\circ\iota)"] & U \times \mathbb{R}^K \\ TN \arrow[r, "J"] & TN \arrow[u, "i_*"'] \end{tikzcd}$$

where we define $U := U_{2\delta}(N)$. That is $\bar{J}(u) = (\iota(u), i_* \circ J(\iota(u)) \circ \pi_{\iota(u)}w)$ for any $(u, w) \in U \times \mathbb{R}^K$. If we restrict $\bar{J}$ to $\mathbb{R}^K$, the second component of $\bar{J}$ can be interpreted as a map

$$\hat{J} = i_* \circ J(\iota(u)) \circ \pi_{\iota(u)} : U \to \mathbb{R}^K \otimes \mathbb{R}^K, \quad \hat{J}(u) = (\hat{J}_{\alpha,\beta}(u))_{K\times K}.$$

To proceed, the following property on $\hat{J}$ will be used:

Lemma 3.2. *Let* $\hat{J} : U \to \mathbb{R}^K \otimes \mathbb{R}^K$ *be the smooth map defined as above. Then* $\hat{J}$ *is antisymmetric. Namely, for any* $u \in U$ *and* $X, Y \in \mathbb{R}^K$,

$$\langle \hat{J}(u)X, Y\rangle = -\langle X, \hat{J}(u)Y\rangle.$$

Proof. For any $u \in U$ and $X, Y \in \mathbb{R}^K$,

$$\begin{aligned} \langle \hat{J}(u)X, Y\rangle &= \langle i_* \circ J(\iota(u)) \circ \pi_{\iota(u)}(X), i_* \circ \pi_{\iota(u)}Y\rangle \\ &= \langle J(\iota(u)) \circ \pi_{\iota(u)}(X), \pi_{\iota(u)}(Y)\rangle_{T_{\iota(u)}N} \\ &= -\langle \pi_{\iota(u)}(X), J(\iota(u)) \circ \pi_{\iota(u)}(Y)\rangle_{T_{\iota(u)}N} = -\langle X, \hat{J}(u)Y\rangle. \end{aligned}$$

Hence, the proof is completed. $\square$

Therefore, (1-9) has the following extrinsic version:

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha(\Delta u + P(u)(\nabla u, \nabla u)) - \beta \hat{J}(u)\Delta u, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \quad \partial u/\partial \nu = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ \quad u(x,0) = u_0 : \Omega \to N \hookrightarrow \mathbb{R}^K. \end{cases} \tag{3-1}$$

Here we set $P(u) = -\operatorname{Hess} \iota(u)$, and have used the facts

$$\pi \circ \Delta u = \tau(u) = \Delta u + A(u)(\nabla u, \nabla u)$$

and

$$\operatorname{Hess} \iota|_u(\nabla u, \nabla u) = -A(u)(\nabla u, \nabla u)$$

for $u : \Omega \to N$ (see Theorem 3.1).

Let ζ be a cut-off function such that $\zeta = 1$ on $[0, \delta^2]$ and $\zeta = 0$ on $[2\delta^2, \infty)$. Then the definition domains of $\hat{J}$ and P can be naturally extended to $\mathbb{R}^K$ in the following way:

$$\mathfrak{J}(u) = \begin{cases} \zeta(\operatorname{dist}(u,N)^2) i_* \circ J(\iota(u)) \circ \pi_{\iota(u)}, & \operatorname{dist}(u,N) \le \sqrt{2}\delta, \\ 0, & \operatorname{dist}(u,N) > \sqrt{2}\delta, \end{cases}$$

and

$$\mathcal{P}(u) = \begin{cases} -\zeta(\operatorname{dist}(u,N)^2) \operatorname{Hess} \iota(u), & \operatorname{dist}(u,N) \le \sqrt{2}\delta, \\ 0, & \operatorname{dist}(u,N) > \sqrt{2}\delta, \end{cases}$$

where $\mathfrak{J}(u)$ is still a smooth antisymmetric matrix-valued function with compact support set. Then we consider the following approximation equation of (1-9):

$$\begin{cases} \partial_t u + \gamma \nabla_v u = \alpha(\Delta u + \mathcal{P}(u)(\nabla u, \nabla u)) - \beta \mathfrak{J}(u)\Delta u, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \quad \partial u/\partial \nu = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ \quad u(x,0) = u_0 : \Omega \to N \hookrightarrow \mathbb{R}^K. \end{cases} \tag{3-2}$$

3B. ***Galerkin approximation of* (3-2) *and a priori estimates.*** Next, we seek a solution u^n in H_n to the Galerkin approximation equation associated to (3-2), i.e.,

$$\begin{cases} \partial_t u^n - \alpha \Delta u^n = P_n(-\gamma \nabla_v u^n + \alpha \mathcal{P}(u^n)(\nabla u^n, \nabla u^n)) - \beta P_n(\mathfrak{J}(u^n)\Delta u^n), \\ \quad u^n(x,0) = u_0^n : \Omega \to \mathbb{R}^K. \end{cases} \tag{3-3}$$

Here $u^n(x,t) = \sum_{i=1}^n g_i^n(t) f_i(x)$, $g^n(t) = \{g_1^n(t), \dots, g_n^n(t)\}$ is a vector-valued function. One can refer to Section 2C for the notions of H_n and f_i. A direct calculation shows that g^n satisfies the ODE

$$\begin{cases} \dfrac{\partial g^n}{\partial t} = F(g^n(t)), \\ g^n(0) = (\langle u_0, f_1 \rangle, \dots, \langle u_0, f_n \rangle), \end{cases}$$

where $F(y)$ is a smooth function of y because of the smoothness of $\mathcal{P}$ and $\mathfrak{J}$. Then there exists a regular solution $g^n(t)$ on $[0, T^n)$, where T^n is the maximal time of existence. So, we get a regular solution u^n to (3-3) on $[0, T^n)$.

Next, by taking u^n as a test function of (3-3), we can see that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |u^n|^2\,dx + \alpha\int_\Omega |\nabla u^n|^2\,dx = -\gamma\int_\Omega \langle \nabla_v u^n, u^n\rangle\,dx + \alpha\int_\Omega \langle \mathcal{P}(u^n)(\nabla u^n, \nabla u^n), u^n\rangle\,dx - \beta\int_\Omega \langle \mathfrak{J}(u^n)\Delta u^n, u^n\rangle\,dx.$$

First of all, we use the fact that $\operatorname{div}(v) = 0$ with $\langle v, \nu\rangle|_{\partial\Omega} = 0$ for all t to eliminate the term

$$\int_\Omega \langle \nabla_v u^n, u^n\rangle\,dx = \frac{1}{2}\int_\Omega \operatorname{div}(v|u^n|^2)\,dx = 0.$$

On the other hand, since $\mathfrak{J}(u^n)$ is antisymmetric and $u^n \in H_n$, we have

$$\int_\Omega \langle \mathfrak{J}(u^n)\Delta u^n, u^n\rangle\,dx = -\int_\Omega \langle \nabla(\mathfrak{J}(u^n))\cdot\nabla u^n, u^n\rangle\,dx.$$

It follows that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |u^n|^2\,dx + \alpha\int_\Omega |\nabla u^n|^2\,dx \le C_{\alpha,\beta}\int_\Omega |\nabla u^n|^2\,dx \tag{3-4}$$

since $\mathcal{P}$ and $\mathfrak{J}$ are smooth maps with compact supports.

Next, taking $\Delta^2 u^n$ as another test function of (3-3), we can show

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\Delta u^n|^2\,dx + \alpha\int_\Omega |\nabla\Delta u^n|^2\,dx &= \gamma\int_\Omega \langle \nabla(v\cdot\nabla u^n), \nabla\Delta u^n\rangle\,dx + \beta\int_\Omega \langle \nabla(\mathfrak{J}(u^n))\Delta u^n, \nabla\Delta u^n\rangle\,dx \\ &\qquad - \alpha\int_\Omega \langle \nabla(\mathcal{P}(u^n)(\nabla u^n, \nabla u^n)), \nabla\Delta u^n\rangle\,dx \\ &= I + II + III.\end{aligned}\tag{3-5}$$

To proceed, we estimate the above three terms as follows:

$$\begin{aligned}|I| &\le |\gamma|\Big(\int_\Omega |\nabla v||\nabla u^n||\nabla\Delta u^n|\,dx + \int_\Omega |v||\nabla^2 u^n||\nabla\Delta u^n|\,dx\Big)\\ &\le C|\gamma|\|\nabla v\|_{L^3}\|\nabla u^n\|_{L^6}\|\nabla\Delta u^n\|_{L^2} + C|\gamma|\|v\|_{L^6}\|\nabla^2 u^n\|_{L^3}\|\nabla\Delta u^n\|_{L^2}\\ &\le C_\alpha|\gamma|^2\|v\|^2_{W^{1,3}}\|u^n\|^2_{H^2} + \tfrac{1}{16}\alpha\|\nabla\Delta u^n\|^2_{L^2}\\ &\qquad + C|\gamma|\|v\|_{H^1}(\|u^n\|_{H^2}\|\nabla\Delta u^n\|_{L^2} + \|u^n\|^{1/2}_{H^2}\|\nabla\Delta u^n\|^{3/2}_{L^2})\\ &\le C_\alpha(|\gamma|^2\|v\|^2_{H^1}\|u^n\|^2_{H^2} + |\gamma|^4\|v\|^4_{H^1}\|u^n\|^2_{H^2})\\ &\qquad + C_\alpha|\gamma|^2\|v\|^2_{W^{1,3}}\|u^n\|^2_{H^2} + \tfrac{1}{8}\alpha\|\nabla\Delta u^n\|^2_{L^2},\end{aligned}$$

where we have used the interpolation inequality

$$\|\nabla^2 u^n\|_{L^3} \le \|\nabla^2 u^n\|^{1/2}_{L^2}\|\nabla^2 u^n\|^{1/2}_{L^6},$$

and the Sobolev embedding inequality

$$\|f\|_{L^6} \le C\|f\|_{H^1}$$

for any $f \in H^1(\Omega)$.

The second term II can be estimated as follows:

$$\begin{aligned}|II| &\le C|\beta|\int_\Omega |\nabla u^n||\Delta u^n||\nabla\Delta u^n|\,dx\\ &\le C|\beta|\|\nabla u^n\|_{L^6}\|\Delta u^n\|_{L^3}\|\nabla\Delta u^n\|_{L^2}\\ &\le C|\beta|\|u^n\|_{H^2}\|\Delta u^n\|_{L^2}^{1/2}\|\Delta u^n\|_{L^6}^{1/2}\|\nabla\Delta u^n\|_{L^2}\\ &\le C|\beta|\|u^n\|_{H^2}^{3/2}(\|u^n\|_{H^2}^{1/2}+\|\nabla\Delta u^n\|_{L^2}^{1/2})\|\nabla\Delta u^n\|_{L^2}\\ &\le C_{\alpha,\beta}(\|u^n\|_{H^2}^4+\|u^n\|_{H^2}^6)+\tfrac18\alpha\|\nabla\Delta u^n\|_{L^2}^2.\end{aligned}$$

Similarly, for the last term III, we have

$$\begin{aligned}|III| &\le C_\alpha\int_\Omega(|\nabla u^n|^3+|\nabla u^n||\nabla^2 u^n|)|\nabla\Delta u^n|\,dx\\ &\le C_\alpha\|\nabla u^n\|_{L^6}^6+\tfrac{1}{16}\alpha\|\nabla\Delta u^n\|_{L^2}^2+C\|\nabla u^n\|_{L^6}\|\nabla^2 u^n\|_{L^3}\|\nabla\Delta u^n\|_{L^2}\\ &\le C_\alpha\|u^n\|_{H^2}^6+C_\alpha(\|u^n\|_{H^2}^4+\|u^n\|_{H^2}^6)+\tfrac18\alpha\|\nabla\Delta u^n\|_{L^2}^2.\end{aligned}$$

In view of the above estimates of terms I, II, and III, we have

$$\frac{\partial}{\partial t}\int_\Omega|\Delta u^n|^2\,dx+\alpha\int_\Omega|\nabla\Delta u^n|^2\,dx\le C_{\alpha,\gamma,\beta}(\|v\|_{W^{1,3}(\Omega)}^2+1)^2(\|u^n\|_{H^2(\Omega)}^2+1)^3. \tag{3-6}$$

Therefore, by combining (3-4) with (3-6), we conclude

$$\frac{\partial}{\partial t}\|u^n\|_{H^2(\Omega)}^2+\alpha\int_\Omega|\nabla\Delta u^n|^2\,dx\le C_{\alpha,\gamma,\beta}(\|v\|_{W^{1,3}(\Omega)}^2+1)^2(\|u^n\|_{H^2(\Omega)}^2+1)^3. \tag{3-7}$$

Proposition 3.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Suppose that u_0 is in $H^2(\Omega)$ and*

$$\frac{\partial u_0}{\partial\nu}\Big|_{\partial\Omega}=0,$$

$v\in L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))$, and $\operatorname{div}(v)=0$ with $\langle v,\nu\rangle|_{\partial\Omega\times\mathbb{R}^+}=0$. Then, there exists a positive constant T_0 depending only on α, γ, β, and $\|u_0\|_{H^2}$, such that the above approximate solutions u^n satisfy

$$\sup_{0\le t\le T}(\|u^n\|_{H^2(\Omega)}^2+\|\partial_t u^n\|_{L^2(\Omega)}^2)+\alpha\int_0^T(\|u^n\|_{H^3(\Omega)}^2+\|\partial_t u^n\|_{H^1(\Omega)}^2)\,dt\le C(T) \tag{3-8}$$

for $0<T<T_0$, where $C(T)$ is a constant depending on T.

Proof. Let $f(t)=\|u^n\|_{H^2}^2+1$. Since $v\in L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))$, inequality (3-7) implies $f(t)$ satisfies

$$\begin{cases} f'(t)\le C(f(t)+1)^3,\\ f(0)=\|u_0^n\|_{H^2}^2+1\le C\|u_0\|_{H^2}^2+1.\end{cases}$$

Here we have used the inequality

$$\|u_0^n\|_{H^2}^2\le C\|u_0\|_{H^2}^2,$$

since $\frac{\partial u_0}{\partial\nu}|_{\partial\Omega}=0$.

Then, by Lemma 2.1 and the classical comparison theorem of ODE, Lemma 2.2, we can show that there exists a positive constant T_0 depending only on α, γ, β, and $\|u_0\|_{H^2}$, such that for any $0 < T < T_0$,

$$\sup_{0\le t\le T} \|u^n\|^2_{H^2(\Omega)} + \alpha \int_0^T \|u^n\|^2_{H^3(\Omega)}\,dt \le C(T).$$

By (3-3), it is not difficult to show

$$\sup_{0\le t\le T} \|\partial_t u^n\|^2_{L^2(\Omega)} + \alpha \int_0^T \|\partial_t u^n\|^2_{H^1(\Omega)}\,dt \le C(T).$$

Therefore, the proof is completed. □

With the above uniform estimate (3-8) of u^n at hand, we can show that there exists a local strong solution to (3-2) by applying the compactness Lemma 2.3 and letting $n \to \infty$. Therefore, we conclude:

Theorem 3.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and $u_0 \in H^2(\Omega)$ with*

$$\frac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

Suppose that $v \in L^\infty(\mathbb{R}^+, W^{1,3}(\Omega))$ and $\operatorname{div}(v) = 0$ with $\langle v, \nu\rangle|_{\partial\Omega\times\mathbb{R}^+} = 0$. Then there exists a positive constant T_0 depending only on α, γ, β, and $\|u_0\|_{H^2}$, such that the initial Neumann boundary value problem (3-2) *admits a local strong solution $u \in C^0([0,T], H^2(\Omega)) \cap L^2([0,T], H^3(\Omega))$, which satisfies*

$$\sup_{0\le t\le T} (\|u\|^2_{H^2(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)}) + \alpha \int_0^T (\|u\|^2_{H^3(\Omega)} + \|\partial_t u\|^2_{H^1(\Omega)})\,dt \le C(T) \tag{3-9}$$

for $0 < T < T_0$, where $C(T)$ is a constant depending on T.

Since the proof of the above theorem is almost the same as that in [Chen and Wang 2023c], we omit it. To show that u is a strong solution to (1-9) or (3-1), we need to prove $u(x,t) \in N$ for almost all $(x,t) \in \Omega \times [0, T_0)$.

Proposition 3.5. *The solution u constructed in Theorem 3.4 satisfies $u(x,t) \in N$ for almost every $(x,t) \in \Omega \times [0, T_0)$, and hence u is a local strong solution to* (1-9).

Proof. Since $u \in L^\infty([0,T], H^2(\Omega))$ and $\partial u/\partial t \in L^2([0,T], L^2(\Omega))$ for $T < T_0$, Lemma 2.3 implies

$$u \in C^0([0,T], W^{1,4}(\Omega)).$$

It follows that

$$\sup_{x\in\Omega} |u(x,t) - u(x,0)| \le C\|u(\cdot,t) - u(\cdot,0)\|_{W^{1,4}} \to 0$$

as $t \to 0$. Then there exists a positive number $t_1 \le T$ such that for $t \le t_1$, we have

$$\sup_{x\in\Omega} |u(x,t) - u(x,0)| \le \delta,$$

namely $u(x,t) \in U_\delta(N)$ for $(x,t) \in \Omega \times [0,t_1]$. Therefore, by the definition of the cut-off function ζ, u satisfies

$$\frac{\partial u}{\partial t} + \gamma \nabla_v u = \alpha(\Delta u - \operatorname{Hess} \iota(\nabla u, \nabla u)) - \beta i_* \circ J(\iota(u)) \circ \pi_{\iota(u)} \Delta u.$$

Let $\rho(u) = u - \iota(u)$, then we have

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\rho(u)|^2\,dx &= \int_\Omega \Big\langle \rho(u), \frac{\partial u}{\partial t}\Big\rangle\,dx \\
&= \int_\Omega \langle \rho(u), \alpha(\Delta\rho(u) + d\iota(\Delta u))\rangle\,dx \\
&\qquad - \beta\int_\Omega \langle \rho(u), \hat{J}(u)\Delta u\rangle\,dx - \gamma\int_\Omega v\cdot\langle \rho(u), \nabla u\rangle\,dx \\
&= \alpha\int_\Omega \langle \rho(u), \Delta\rho(u)\rangle\,dx - \frac{\gamma}{2}\int_\Omega v\cdot\nabla|\rho(u)|^2\,dx \\
&= -\alpha\int_\Omega |\nabla\rho(u)|^2\,dx.
\end{aligned}$$

Here we have used the following facts:

(1) Since $\Delta\iota(u) = d\iota(\Delta u) + \operatorname{Hess}\iota(\nabla u, \nabla u)$, we have

$$\Delta u - \operatorname{Hess}\iota(\nabla u, \nabla u) = \Delta\rho(u) + d\iota(\Delta u).$$

(2) Since $\rho(u) \in T^\perp_{\iota(u)}N$ and $\hat{J}(u)\Delta u \in T_{\iota(u)}N$,

$$\langle \rho(u), \hat{J}(u)\Delta u\rangle = 0 \quad \text{and} \quad \langle \rho(u), d\iota(\Delta u)\rangle = 0.$$

(3) Since $\operatorname{div}(v) = 0$ and $\langle v, \nu\rangle|_{\partial\Omega} = 0$,

$$\int_\Omega v\cdot\nabla|\rho(u)|^2\,dx = 0.$$

Then the Gronwall inequality implies $\rho(u) = 0$ for almost all $(x,t) \in \Omega \times [0,t_1]$. Finally, we can prove this proposition by repeating the above argument. □

To end this section, we show the uniqueness of the solution u constructed above:

Proposition 3.6. *The solution to* (3-1) *in* $L^\infty\big([0,T], H^2(\Omega)\big) \cap L^2\big([0,T], H^3(\Omega)\big)$ *is unique.*

Proof. Assume u_1 and u_2 are two solutions in $L^\infty\big([0,T], H^2\big) \cap L^2\big([0,T], H^3(\Omega)\big)$, then $\bar{u} = u_1 - u_2$ satisfies

$$\text{(3-10)}\quad \begin{cases} \partial_t \bar{u} = -\gamma\nabla_v\bar{u} + \alpha\Delta\bar{u} + \alpha(P(u_1)(\nabla u_1, \nabla u_1) - P(u_2)(\nabla u_2, \nabla u_2)) \\ \qquad\qquad - \beta(\hat{J}(u_1) - \hat{J}(u_2))\Delta u_1 - \beta\hat{J}(u_2)\Delta\bar{u}, \\ \bar{u}(x,0) = 0 \\ \partial\bar{u}/\partial\nu = 0. \end{cases}$$

By taking $\bar{u}$ as a test function to (3-10), we can show

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\bar{u}|^2\,dx + \alpha\int_\Omega |\nabla\bar{u}|^2\,dx = -\gamma\int_\Omega \langle \nabla_v\bar{u}, \bar{u}\rangle\,dx + I + II + III.$$

Here

$$|\gamma|\cdot\Big|\int_\Omega\langle\nabla_v\bar u,\bar u\rangle\,dx\Big|=\Big|\frac{\gamma}{2}\Big|\cdot\Big|\int_\Omega\operatorname{div}(v|\bar u|^2)\,dx\Big|=0,$$

since $\langle v,\nu\rangle|_{\partial\Omega}=0$ and $\operatorname{div}(v)=0$.

$$\begin{aligned}
|I|&=\alpha\Big|\int_\Omega\langle P(u_1)(\nabla u_1,\nabla u_1)-P(u_2)(\nabla u_2,\nabla u_2),\bar u\rangle\,dx\Big|\\
&\le C\alpha\Big(\int_\Omega|\bar u|^2|\nabla u_1|^2\,dx+\int_\Omega|\nabla\bar u|(|\nabla u_1|+|\nabla u_2|)|\bar u|\,dx\Big)\\
&\le C\alpha(\|u_1\|^2_{H^3}+\|u_2\|^2_{H^3})\int_\Omega|\bar u|^2\,dx+\frac{\alpha}{8}\int_\Omega|\nabla\bar u|^2\,dx,\\
|II|&=|\beta|\Big|\int_\Omega\langle(\hat J(u_1)-\hat J(u_2))\Delta u_1,\bar u\rangle\,dx\Big|\\
&\le C|\beta|\Big|\int_\Omega\langle\operatorname{div}((\hat J(u_1)-\hat J(u_2))\nabla u_1),\bar u\rangle\,dx\Big|\\
&\qquad\qquad+C|\beta|\Big|\int_\Omega\langle\nabla((\hat J(u_1)-\hat J(u_2))\cdot\nabla u_1),\bar u\rangle\,dx\Big|\\
&\le C|\beta|\int_\Omega|\hat J(u_1)-\hat J(u_2)||\nabla u_1||\nabla\bar u|\,dx\\
&\qquad\qquad+C|\beta|\Big|\int_\Omega\langle\nabla((\hat J(u_1)-\hat J(u_2))\cdot\nabla u_1),\bar u\rangle\,dx\Big|\\
&\le C|\beta|\|\nabla u_1\|_{L^\infty}\int_\Omega|\nabla\bar u||\bar u|\,dx+C|\beta|(|\nabla u_1|^2_\infty+|\nabla u_2|^2_\infty)\int_\Omega|\bar u|^2\,dx\\
&\le C_\alpha|\beta|(\|u_1\|^2_{H^3}+\|u_2\|^2_{H^3})\int_\Omega|\bar u|^2\,dx+\frac{\alpha}{8}\int_\Omega|\nabla\bar u|^2\,dx,
\end{aligned}$$

$$\begin{aligned}
|III|&=|\beta|\Big|\int_\Omega\langle\hat J(u_2)\Delta\bar u,\bar u\rangle\,dx\Big|\\
&\le|\beta|\Big|\int_\Omega\langle\nabla(\hat J(u_2))\cdot\nabla\bar u,\bar u\rangle\,dx\Big|+|\beta|\Big|\int_\Omega|\langle\operatorname{div}(\hat J(u_2)\nabla\bar u),\bar u\rangle\,dx\Big|\\
&\le C|\beta||\int_\Omega|\nabla u_2||\nabla\bar u||\bar u|\,dx|\\
&\le C_\alpha|\beta|\|u_2\|^2_{H^3}\int_\Omega|\bar u|^2\,dx+\frac{\alpha}{8}\int_\Omega|\nabla\bar u|^2\,dx.
\end{aligned}$$

In view of the above estimates of terms I, II and III, we get

$$\frac{\partial}{\partial t}\int_\Omega|\bar u|^2\,dx+\alpha\int_\Omega|\nabla\bar u|^2\,dx\le C_{\alpha,\beta,\gamma}(\|u_1\|^2_{H^3(\Omega)}+\|u_2\|^2_{H^3(\Omega)}+1)\int_\Omega|\bar u|^2\,dx.$$

Then, since $\|u_1\|^2_{H^3(\Omega)}+\|u_2\|^2_{H^3(\Omega)}\in L^1[0,T]$, the Gronwall inequality implies $u_1\equiv u_2$. □

4. Local regular solution

In the previous section, we obtained a strong solution u to the equation

$$\text{(4-1)}\quad \begin{cases} \partial_t u+\gamma\nabla_v u=\alpha(\Delta u+P(u)(\nabla u,\nabla u))-\beta\hat{J}(u)\Delta u, & (x,t)\in\Omega\times[0,T_0),\\ \quad \partial u/\partial\nu=0, & (x,t)\in\partial\Omega\times[0,T_0),\\ \quad u(x,0)=u_0:\Omega\to N\hookrightarrow\mathbb{R}^K. \end{cases}$$

Here $u:\Omega\times[0,T_0)\to N$,

$$P(u)=-\operatorname{Hess}\iota(u):\mathbb{R}^K\otimes\mathbb{R}^K\to\mathbb{R}^K$$

is a bilinear functional, and

$$\hat{J}(u)=i_*\circ J\circ d\iota(u):\mathbb{R}^K\to\mathbb{R}^K$$

is an antisymmetric matrix, since $d\iota(u)=\pi_u$.

Suppose $u_0\in H^3(\Omega,N)$ and $\frac{\partial u_0}{\partial\nu}|_{\partial\Omega}=0$, we can improve the regularity of u by applying the differential of Galerkin approximation to (3-2) with respect to the time variable t.

Theorem 4.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and $u_0\in H^3(\Omega,N)$ with*

$$\frac{\partial u_0}{\partial\nu}\Big|_{\partial\Omega}=0.$$

Suppose that $v\in L^\infty(\mathbb{R}^+,W^{1,3}(\Omega))\cap C^0(\mathbb{R}^+,H^1(\Omega))$, $\partial_t v\in L^2(\mathbb{R}^+,H^1(\Omega))$, and $\operatorname{div}(v)=0$ with $\langle v,\nu\rangle|_{\partial\Omega\times\mathbb{R}^+}=0$. Then, the solution u given in Theorem 3.4 satisfies

$$\partial_t^i u\in C^0([0,T],H^{3-2i}(\Omega))\cap L^2([0,T],H^{4-2i}(\Omega))$$

for $T<T_0$ and $i=0,1$.

Proof. We divide the proof into two steps.

Step 1: *H^2-estimate of $\partial_t u$.*

To get H^2-estimates of the solution $\partial_t u$, we consider the equation of $w^n=\partial_t u^n$ as follows, where u^n is the Galerkin approximation of u:

$$\text{(4-2)}\quad \begin{aligned}\partial_t w^n=\alpha\Delta w^n+P_n(-\gamma\nabla_v w^n-\gamma\nabla_{\partial_t v}u^n)&\\ +\alpha P_n(\mathcal{P}(u^n)(\nabla w^n,\nabla u^n)+\partial_t\mathcal{P}(u^n)(\nabla u^n,\nabla u^n))&\\ -\beta P_n(\partial_t\mathfrak{J}(u^n)\Delta u^n-\mathfrak{J}(u^n)\Delta w^n).&\end{aligned}$$

Then we take Δw^n as a test function for (4-2) to give

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega |\nabla w^n|^2\,dx + 2\alpha\int_\Omega |\Delta w^n|^2\,dx \\
&\quad\le C_{\alpha,\beta}\Big(\int_\Omega |w^n|^2|\Delta u^n|^2\,dx + \int_\Omega |w^n|^2|\nabla u^n|^4\,dx\Big) \\
&\qquad\qquad + C_{\alpha,\gamma}\Big(\int_\Omega |\nabla w^n|^2|\nabla u^n|^2\,dx + |\partial_t v|^2|\nabla u^n|^2\,dx\Big) \\
&\qquad\qquad\qquad + |\gamma|\int_\Omega |\nabla w^n|^2|\nabla v|\,dx + \frac{\alpha}{2}\int_\Omega |\Delta w^n|^2\,dx \\
&\quad\le C_{\alpha,\beta,\gamma}(\|u^n\|^2_{H^3} + \|u^n\|^4_{H^2} + 1)\|w^n\|^2_{H^1} + C_{\alpha,\gamma}\|u^n\|^2_{H^2}\|\partial_t v\|^2_{H^1} \\
&\qquad\qquad\qquad + |\gamma|\int_\Omega |\nabla w^n|^2|\nabla v|\,dx + \frac{\alpha}{2}\int_\Omega |\Delta w^n|^2\,dx.
\end{aligned}$$

Here we have used the fact $\operatorname{div}(v) = 0$ and $\langle v, \nu\rangle|_{\partial\Omega\times\mathbb{R}^+} = 0$ to show

$$\int_\Omega \langle v\cdot\nabla w^n, \Delta w^n\rangle\,dx = -\int_\Omega \langle \nabla v\cdot\nabla w^n, \nabla w^n\rangle\,dx.$$

On the other hand, we have

$$\begin{aligned}
\int_\Omega |\nabla w^n|^2|\nabla v|\,dx &\le \|\nabla w^n\|^2_{L^3}\|\nabla v\|_{L^3} \le \|\nabla w^n\|_{L^2}\|\nabla w^n\|_{L^6}\|\nabla v\|_{L^3} \\
&\le C\|v\|_{W^{1,3}}\|\nabla w^n\|_{L^2}(\|w^n\|_{L^2} + \|\Delta w^n\|_{L^2}) \\
&\le C_\alpha\|v\|^2_{W^{1,3}}\|w^n\|^2_{H^1} + \frac{\alpha}{2}\int_\Omega |\Delta w^n|^2\,dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega |\nabla w^n|^2\,dx + \alpha\int_\Omega |\Delta w^n|^2\,dx \\
&\le C(\alpha,\beta,\gamma)(\|u^n\|^2_{H^3} + \|u^n\|^4_{H^2} + \|v\|^2_{W^{1,3}} + 1)\|w^n\|^2_{H^1} + C(\alpha,\gamma)\|u^n\|^2_{H^2}\|\partial_t v\|^2_{H^1}.
\end{aligned}$$

By assumption we know that $v \in L^2(\mathbb{R}^+, W^{1,3}(\Omega))$ and $\partial_t v \in L^2(\mathbb{R}^+, H^1(\Omega))$. Hence, by applying the Gronwall inequality we can derive from (3-8) that

$$\sup_{0<t\le T}\|w^n\|_{H^1(\Omega)} + \alpha\int_0^T \|w^n\|^2_{H^2(\Omega)}\,dt \le C(\alpha,\beta,\gamma,T,\|w^n|_{t=0}\|_{H^1(\Omega)}). \tag{4-3}$$

Now, it remains to give a bound of $\|w^n|_{t=0}\|_{H^1}$. Since

$$w^n(\cdot,0) = \alpha\Delta u_0^n + P_n(-\gamma\nabla_v u_0^n + \alpha\mathcal{P}(u_0^n)(\nabla u_0^n, \nabla u_0^n) - \beta\mathfrak{J}(u_0^n)\Delta u_0^n),$$

it is not difficult to show

$$\|w^n|_{t=0}\|_{H^1(\Omega)} \le C(\|u_0\|^2_{H^3(\Omega)}, \|v(\cdot,0)\|^2_{H^1}).$$

Here we have used the fact

$$\|u_0^n\|^2_{H^3(\Omega)} \le C\|u_0\|^2_{H^3(\Omega)}$$

by providing $\frac{\partial u_0}{\partial\nu}|_{\partial\Omega} = 0$.

Without loss of generality, the estimate (4-3) implies w^n weakly converges to $\partial_t u$, which satisfies

$$\partial_t u \in L^\infty\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big)$$

for any $0 < T < T_0$.

Step 2: *H^4-estimate of u.*

Equation (1-5) is equivalent to

$$\Delta u = -A(u)(\nabla u, \nabla u) + \frac{1}{\alpha^2+\beta^2}(\alpha\partial_t u + \beta J(u)\partial_t u) + \frac{\gamma}{\alpha^2+\beta^2}(\alpha\nabla_v u + \beta J(u)\nabla_v u).$$

Under the assumption that $v \in L^\infty(\mathbb{R}^+, H^1(\Omega))$, one can easily show

$$\Delta u \in L^\infty([0,T], L^3(\Omega)),$$

the classical L^p-theory of elliptic equations gives

$$u \in L^\infty([0,T], W^{2,3}(\Omega)).$$

Hence, by using the assumption $v \in L^\infty\big(\mathbb{R}^+, W^{1,3}(\Omega)\big) \cap L^2\big(\mathbb{R}^+, H^2(\Omega)\big)$, we can take almost the same argument as in [Chen and Wang 2023c] to show

$$\Delta u \in L^\infty\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big),$$

hence the classical L^2-theory of elliptic equations again gives

$$u \in L^\infty\big([0,T], H^3(\Omega)\big) \cap L^2\big([0,T], H^4(\Omega)\big).$$

Moreover, since $u \in L^2([0,T], H^4(\Omega))$ and $\partial_t u \in L^2([0,T], H^2(\Omega))$, Lemma 2.4 tells us that $u \in C^0([0,T], H^3(\Omega))$. It follows that

$$\partial_t u \in C^0([0,T], H^1(\Omega))$$

by using (4-1) and the fact $v \in C^0(\mathbb{R}^+, H^1(\Omega))$. □

The proof of Theorem 1.1. By combining Theorem 3.4, Propositions 3.5, 3.6, and Theorem 4.1, we can obtain the results in Theorem 1.1 and finish its proof. □

5. Local very regular solution

In this section, we prove Theorem 1.2.

5A. ***Compatibility conditions of the initial data.*** In order to make the LLG equation (4-1) (an extrinsic version of the LLG equation (1-9)) admit a regular or smooth solution, we need to pose some compatibility conditions of the initial data. We start with a brief description of the compatibility conditions of the initial data. For the sake of convenience, we assume v is a smooth vector field and u is a smooth solution to the initial Neumann boundary value problem of the LLG equation (4-1). Then, for any $k \in \mathbb{N},\ u_k = \partial_t^k u$ satisfies the linear equation

$$\partial_t u_k = \alpha \Delta u_k - \beta \hat{J}(u)\Delta u_k - \gamma \nabla_v u_k + L_k(u_k, u) + F_k(u) \tag{5-1}$$

with the initial data

$$V_k(u_0) := \partial_t^k u|_{t=0}.$$

In particular, $V_0 = u_0$ and

$$V_1(u_0) = -\gamma \nabla_{v(x,0)} u_0 + \alpha\tau(u_0) - \beta J(u_0)\tau(u_0).$$

Here we set

$$L_k(u_k, u) = 2\alpha P(u)(\nabla u_k, \nabla u) + \alpha d P(u)(u_k, \nabla u, \nabla u) - \beta d\hat{J}(u)(u_k)\Delta u,$$

and

$$F_k(u) = -\gamma \sum_{\substack{i+j=k \\ i\ge 1}} C_k^i \nabla_{v_i} u_j + \alpha \sum_{\substack{i_1+\cdots+i_s+m+l=k \\ 1\le i_j<k}} \nabla^s P(u)\# u_{i_1}\#\cdots\# u_{i_s}\#\nabla u_m \#\nabla u_l$$
$$+ \beta \sum_{\substack{i_1+\cdots+i_s+m=k \\ 1\le i_j<k}} \nabla^s \hat{J}(u)\# u_{i_1}\#\cdots\# u_{i_s}\#\Delta u_m,$$

where $v_i = \partial_t^i v$ and # denotes the linear contraction.

Then the compatibility conditions of the initial data is defined as follows:

Definition 5.1. Let $k \in \mathbb{N},\ u_0 \in H^{2k+2}(\Omega, N)$. Suppose that for any $0 \le i \le k,\ v$ satisfies

$$\partial_t^i v \in C^0(\mathbb{R}^+, H^{2k-2i}(\Omega)).$$

We say u_0 satisfies the compatibility condition at order k, if for any $j \in \{0, 1, \dots, k\}$

$$\frac{\partial V_j}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{5-2}$$

Intrinsically, if we set

$$\widetilde{V}_k(u_0) = \nabla_t^k u(x, 0) \in \Gamma(u_0^*(TN))$$

where $\nabla_t = \nabla^N_{\partial u/\partial t}$, then the compatibility conditions defined in Definition 5.1 has the following equivalent characterization:

Proposition 5.2. *Let $k \in \mathbb{N},\ u_0 \in H^{2k+2}(\Omega, N)$. Suppose that for any $0 \le i \le k,\ v$ satisfies*

$$\partial_t^i v \in C^0(\mathbb{R}^+, H^{2k-2i}(\Omega)).$$

Then u_0 satisfies the compatibility condition of order k if and only if for any $j \in \{0, 1, \dots, k\}$,

$$\nabla_\nu \widetilde{V}_j|_{\partial\Omega} = 0. \tag{5-3}$$

Proof. The necessity is proved by induction on k. Since $V_1 = \widetilde{V}_1$, if we assume $\frac{\partial V_1}{\partial \nu}|_{\partial\Omega} = 0$, then we have

$$\nabla_\nu \widetilde{V}_1|_{\partial\Omega} = \frac{\partial \widetilde{V}_1}{\partial \nu}\Big|_{\partial\Omega} + A(u_0)\bigg(\frac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega}, \widetilde{V}_1\bigg) = 0.$$

Then, we assume that the result is true for $1 \le l \le k-1$. For the case $l = k \le 2$, by definition of $\widetilde{V}_k$, a simple calculations gives

$$\widetilde{V}_k = V_k + \sum_\sigma B_{\sigma(k)}(u_0)(V_{a_1}, \dots, V_{a_s})$$

where the sum is over all multi-indices $a_1, \dots, a_s$ such that $1 \le a_i \le k-1$ and $a_1 + \cdots + a_s = k$ for all i,

$$(a_1, \dots, a_s) = \sigma(k)$$

is a partition of k, and B is a multilinear functional on $u_0^*(TN)$.

Hence, by using the assumption of induction, we have

$$\begin{aligned}\nabla_\nu \widetilde{V}_k|_{\partial\Omega} =& \frac{\partial \widetilde{V}_k}{\partial \nu}\Big|_{\partial\Omega} + A(u_0)\bigg(\frac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega}, \widetilde{V}_k\bigg)\\ =& \frac{\partial V_k}{\partial \nu}\Big|_{\partial\Omega} + \sum_\sigma \nabla B_{\sigma(k)}(u_0)\bigg(\frac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega}, V_{a_1}, \dots, V_{a_s}\bigg) = 0.\end{aligned}$$

For the converse the proof is almost the same as above, so we omit it. □

Remark 5.3. If $\gamma = 0$ in (1-9) and $\nabla^N J = 0$, we set

$$W_k = \nabla_t^{k-1}\tau(u)(x,0) \quad \text{and} \quad \widetilde{W}_k = \partial_t^{k-1}\tau(u)(x,0)$$

for any $k \ge 1$, and set $W_0 = \widetilde{W}_0 = u_0$. Then, by taking a similar argument to that in the proof of Proposition 5.2 or Proposition 3.2 in [Chen and Wang 2023b], we can show the k-order compatibility condition defined in Definition 5.1 is equivalent to one of the following:

(1) For $1 \le j \le k$,

$$\nabla_\nu W_j|_{\partial\Omega} = 0.$$

(2) For $1 \le j \le k$,

$$\frac{\partial \widetilde{W}_j}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

Next, we apply the method of induction to show the existence of very regular solutions to (4-1) by considering the initial Neumann boundary value problem of equation of $\partial_t^k u$ for $k \ge 1$ with corresponding initial data V_k. For this purpose, we intend to prove the main result Theorem 1.2 by showing the following process $\mathscr{T}(k)$ with $k \ge 2$:

(1) Assume that $u_0 \in H^{2k}(\Omega)$ satisfies the $(k-1)$-order compatibility conditions. Suppose moreover

$$\partial_t^i v \in C^0\big([0,T], H^{2k-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i}(\Omega)\big)$$

for any $0 < T < T_0$ and $i \in \{0, 1, \dots, k-1\}$. Then for any $0 \le i \le k-1$, we have

$$\partial_t^i u \in C^0\big([0,T], H^{2k-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+1}(\Omega)\big),$$

and

$$\partial_t^k u \in L^\infty\big([0,T], L^2(\Omega)\big) \cap L^2\big([0,T], H^1(\Omega)\big).$$

(2) Additionally, if $u_0 \in H^{2k+1}(\Omega)$,

$$\partial_t^i v \in C^0\big([0,T], H^{2k+1-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i}(\Omega)\big)$$

for $i \in \{0, 1, \dots, k-1\}$ and $\partial_t^k v \in L^2([0,T], L^2(\Omega))$, then for any $0 \le i \le k$ we have

$$\partial_t^i u \in C^0\big([0,T], H^{2k-2i+1}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big).$$

5B. ***H^5-regularity of u (i.e., the proof of property $\mathscr{T}(2)$).*** For any $T < T_0$, Theorem 4.1 implies that $\partial_t u \in C^0\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big)$ is a strong solution to

$$\text{(5-4)} \quad \begin{cases} \partial_t u_1 + \gamma \nabla_v u_1 \\ \qquad\qquad = \alpha \Delta u_1 - \beta \hat{J}(u) \Delta u_1 + L_1(u_1, u) + F_1(u), & (x,t) \in \Omega \times [0,T], \\ \partial u_1 / \partial \nu = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u_1(x,0) = V_1 \end{cases}$$

where

$$L_1(u_1, u) = 2\alpha P(u)(\nabla u_1, \nabla u) + \alpha dP(u)(u_1, \nabla u, \nabla u) - \beta d\hat{J}(u)(u_1)\Delta u,$$

and

$$F_1(u) = -\gamma \nabla_{\partial_t v} u.$$

To improve the regularity of $\partial_t u$, we solve the initial Neumann boundary value problem (5-4) with compatibility condition

$$\frac{\partial V_1}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

As before, we consider the Galerkin approximation equation of (5-4):

$$\text{(5-5)} \quad \begin{cases} \partial_t u_1^n + \gamma P_n(\nabla_v u_1^n) = \alpha \Delta u_1^n - \beta P_n(\hat{J}(u)\Delta u_1^n) + P_n(L_1(u_1^n, u) + F_1(u)), \\ \qquad\qquad u_1(x,0) = V_1^n. \end{cases}$$

Since the operators P and $\hat{J}$ satisfy

(1) $|P(u)| + |\hat{J}(u)| \le C$,

(2) $|\nabla(P(u))| + |\nabla(\hat{J}(u))| \le C|\nabla u|$,

(3) $\hat{J}$ is antisymmetric,

we can apply almost the same argument as that in [Chen and Wang 2023c] to give the estimate

$$\sup_{0\le t\le T}(\|u_1^n\|_{H^2}^2 + \|\partial_t u_1^n\|_{L^2}^2) + \alpha\int_0^T \|\Delta\nabla u_1^n\|_{L^2}^2 + \|\nabla\partial_t u_1^n\|_{L^2}^2\,dt \\ \le C(\|u_0\|_{H^3}, \|P_n(V_1)\|_{H^2}),$$

by providing $v \in C^0(\mathbb{R}^+, H^2(\Omega))$ and $\partial_t v \in L^\infty(\mathbb{R}^+, L^2(\Omega)) \cap L^2(\mathbb{R}^+, H^1(\Omega))$, then taking u_1^n and $\Delta^2 u_1^n$ as test functions to (5-5).

On the other hand, since

$$V_1 = -\gamma\nabla_{v(x,0)}u_0 + \alpha\tau(u_0) - \beta J(u_0)\tau(u_0)$$

and

$$\frac{\partial V_1}{\partial\nu}\Big|_{\partial\Omega} = 0,$$

Lemma 2.5 implies that

$$\|P_n(V_1)\|_{H^2(\Omega)} \le C\|V_1\|_{H^2} \le C(\|u_0\|_{H^4(\Omega)}, \|v(.,0)\|_{H^2(\Omega)}).$$

Without loss of generality, by using the compactness in Lemma 2.3, we can infer that u_1^n converges to a map $u_1 \in L^\infty([0,T], H^2(\Omega)) \cap L^2([0,T], H^3(\Omega))$ which solves (5-4). To show that $\partial_t u = u_1$ on $\Omega \times [0, T_0)$, we need to use the following result:

Proposition 5.4. *The solution to* (5-4) *in* $C^0([0,T], H^1) \cap L^2([0,T], H^2(\Omega))$ *is unique.*

Proof. Let v_1 and v_2 be two solutions of (5-4), which belong to the space

$$C^0([0,T], H^1(\Omega)) \cap L^2([0,T], H^2(\Omega)).$$

Then, $\omega = v_1 - v_2$ satisfies

$$\begin{cases} \partial_t\omega + \gamma\nabla_v\omega = \alpha\Delta\omega - \beta\hat{J}(u)\Delta\omega + L_1(\omega, u), & (x,t) \in \Omega\times[0,T], \\ \partial\omega/\partial\nu = 0, & (x,t)\in\partial\Omega\times[0,T], \\ \omega(x,0) = 0. \end{cases}$$

By choosing ω as a test function of this equation and taking a simple calculation we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\omega|^2\,dx + \alpha\int_\Omega |\nabla\omega|^2\,dx\\
&\qquad\le C_\alpha \int_\Omega (|\nabla u||\nabla\omega||\omega| + |\nabla u|^2|\omega|^2)\,dx - \beta\int_\Omega \langle d\hat{J}(u)(\omega)\Delta u,\,\omega\rangle\,dx\\
&\qquad\le C_\alpha \int_\Omega (|\nabla u||\nabla\omega||\omega| + |\nabla u|^2|\omega|^2)\,dx - \beta\int_\Omega \langle \nabla u,\,\nabla(d\hat{J}(u)(\omega)\omega)\rangle\,dx\\
&\qquad\le C_{\alpha,\beta} \int_\Omega (|\nabla u||\nabla\omega||\omega| + |\nabla u|^2|\omega|^2)\,dx\\
&\qquad\le C_{\alpha,\beta}\|u\|^2_{L^\infty([0,T],H^3(\Omega))}\int_\omega |\omega|^2\,dx + \frac{\alpha}{2}\int_\Omega |\nabla\omega|^2\,dx.
\end{aligned}
$$

Consequently, the Gronwall inequality implies $\omega \equiv 0$, completing the proof. □

It follows from Proposition 5.4 that

$$\partial_t u \in L^\infty\big([0,T],H^2(\Omega)\big)\cap L^2\big([0,T],H^3(\Omega)\big). \tag{5-6}$$

Additionally, if we provide $u_0 \in H^5(\Omega)$, $\partial_t v \in C^0([0,T],H^1(\Omega))$, and $\partial_t^2 v$ in $L^2(\mathbb{R}^+,L^2(\Omega))$, we can apply a similar argument to Step 1 of the proof of Theorem 4.1 to show

$$\partial_t^2 u \in L^\infty\big([0,T],H^1(\Omega)\big)\cap L^2\big([0,T],H^2(\Omega)\big) \tag{5-7}$$

by considering the equation of $\partial u_1^n/\partial t$.

To enhance the regularity of u, we need to use the following technical lemmas:

Lemma 5.5. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$, $n \ge 0$, and $m \ge 2$. If $f \in H^n(\Omega)$ (we set $H^0(\Omega) = L^2(\Omega)$) and $g \in H^m(\Omega)$, then $fg \in H^l(\Omega)$ with $l = \min\{n,m\}$. Moreover, there exists a constant $C(\|f\|_{H^n},\|g\|_{H^m})$ such that we have*

$$\|fg\|_{H^l(\Omega)} \le C(\|f\|_{H^n},\|g\|_{H^m}).$$

One can consult [Carbou and Jizzini 2018] for a proof. As a direct corollary, we have:

Corollary 5.6. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$ and N be a compact Riemannian submanifold of $\mathbb{R}^K$. If*

$$u \in L^\infty\big([0,T],H^k(\Omega,N)\big)\cap L^2\big([0,T],H^{k+1}(\Omega,N)\big)$$

with $k \ge 2$ and $L : N \to \mathbb{R}^K \otimes \mathbb{R}^K$ is a smooth map, then $L(u)$ belongs to $L^\infty\big([0,T],H^k(\Omega)\big)\cap L^2\big([0,T],H^{k+1}(\Omega)\big)$.

Proof. It is not difficult to show that the result holds true for $k = 2$. Hence, without loss of generality we can assume that $k \ge 3$.

Since $\nabla(L(u)) = \nabla L(u)\#\nabla u$, the fact that $u \in L^\infty([0, T], H^k(\Omega, N))$ with $k \geq 3$ implies

$$\nabla(L(u)) \in L^\infty([0, T], L^2(\Omega, N)).$$

On the other hand, a simple calculation gives

$$\begin{aligned}\nabla^l(L(u)) =& \sum_{\substack{i_1+\cdots+i_s=l\\ 1\leq s\leq l,\, i_j\geq 1}} \nabla^s L(u)\#\nabla^{i_1}u\#\cdots\#\nabla^{i_s}u\\ =& \nabla L(u)\#\nabla^l u + \nabla^2 L(u)\#\nabla^{l-1}u\#\nabla u + \sum_{\substack{i_1+\cdots+i_s=l\\ 2\leq s\leq l\\ 1\leq i_j\leq l-2}} \nabla^s L(u)\#\nabla^{i_1}u\#\cdots\#\nabla^{i_s}u\end{aligned}$$

for $2 \leq l \leq k+1$. Since $u \in L^\infty([0, T], H^k(\Omega, N))$ and $\sup_{y\in N} |\nabla^s L|(y) \leq C(s)$, Lemma 5.5 above implies

$$\nabla^l(L(u)) \in L^\infty([0, T], L^2(\Omega))$$

for $2 \leq l \leq k$.

To show $\nabla^{k+1}(L(u)) \in L^2([0, T], L^2(\Omega))$, we need only to deal with the following term of $\nabla^{k+1}(L(u))$:

$$I = \nabla^2 L(u)\#\nabla^{k-1}u\#\nabla^2 u,$$

since the other terms can be bounded directly by applying Lemma 5.5.

By using the facts $\nabla^{k-1}u \in L^2([0, T], H^2(\Omega))$ and $\nabla^2 u \in L^\infty([0, T], H^1(\Omega))$, we have

$$\begin{aligned}\int_0^T\int_\Omega |I|^2\,dxdt \leq& C\int_0^T \|\nabla^{k-1}u\|^2_{L^\infty(\Omega)}\,dt \sup_{t\in[0,T]}\int_\Omega |\nabla^2 u|^2\,dx\\ \leq& C\int_0^T \|\nabla^{k-1}u\|^2_{H^2}\,dt \sup_{t\in[0,T]}\int_\Omega |\nabla^2 u|^2\,dx < \infty.\end{aligned}$$

Therefore, we finish the proof. □

We are now in position to prove the main result (i.e., $\mathscr{T}(2)$) of this subsection:

Proposition 5.7. *Suppose that $u_0 \in H^4(\Omega, N)$ satisfies the 1-order compatibility condition defined in Definition 5.1, $v \in C^0\big(\mathbb{R}^+, H^2(\Omega)\big) \cap L^2\big(\mathbb{R}^+, H^3(\Omega)\big)$, and $\partial_t v \in L^\infty\big(\mathbb{R}^+, L^2(\Omega)\big) \cap L^2\big(\mathbb{R}^+, H^1(\Omega)\big)$. Then for any $0 < T < T_0$ we have*

$$\partial_t^i u \in C^0\big([0, T], H^{4-2i}(\Omega)\big) \cap L^2\big([0, T], H^{5-2i}(\Omega)\big)$$

for $i \in \{0, 1\}$, and

$$\partial_t^2 u \in L^\infty\big([0, T], L^2(\Omega)\big) \cap L^2\big([0, T], H^1(\Omega)\big).$$

Additionally, if $u_0 \in H^5(\Omega, N)$, $\partial_t^i v \in C^0\big(\mathbb{R}^+, H^{3-2i}(\Omega)\big) \cap L^2\big(\mathbb{R}^+, H^{4-2i}(\Omega)\big)$ *with* $i = 0, 1$, *and* $\partial_t^2 v \in L^2(\mathbb{R}^+, L^2(\Omega))$, *we obtain*

$$\partial_t^i u \in C^0\big([0, T], H^{5-2i}(\Omega)\big) \cap L^2\big([0, T], H^{6-2i}(\Omega)\big)$$

for $i \in \{0, 1, 2\}$.

Proof. Our proof is divided into two steps:

Step 1: H^5*-estimate of* u.

By using (4-1) and taking a simple computation we can show

$$\Delta u = -P(u)(\nabla u, \nabla u) + \frac{1}{\alpha^2+\beta^2}(\alpha\partial_t u + \beta\hat{J}(u)\partial_t u) + \frac{\gamma}{\alpha^2+\beta^2}(\alpha\nabla_v u + \beta\hat{J}(u)\nabla_v u). \tag{5-8}$$

In the case $u_0 \in H^4(\Omega, N)$, Lemma 5.5 and Corollary 5.6 tell us that

$$\Delta u \in L^\infty([0, T], H^2(\Omega)),$$

since $u \in L^\infty([0, T], H^3(\Omega))$, $v \in C^0(\mathbb{R}^+, H^2(\Omega))$ and by estimate (5-6). Hence, by the L^2-theory of elliptic equations we know that

$$u \in L^\infty([0, T], H^4(\Omega)).$$

Moreover, if we assume $v \in L^2([0, T], H^3(\Omega))$, we can apply Lemma 5.5 and Corollary 5.6 again to show

$$\Delta u \in L^2([0, T], H^3(\Omega)),$$

and hence we have $u \in L^2([0, T], H^5(\Omega))$. Consequently, Lemma 2.4 implies

$$\partial_t^i u \in C^0([0, T], H^{4-2i}(\Omega))$$

for $i = 0, 1$.

Step 2: H^6*-estimate of* u.

On the other hand, it follows from (5-8) that

$$\Delta\partial_t u = \frac{1}{\alpha^2+\beta^2}(\alpha\partial_t^2 u + \beta\hat{J}(u)\partial_t^2 u) + \frac{\beta}{\alpha^2+\beta^2} d\hat{J}(u)\#\partial_t u\#\partial_t u + \frac{\gamma}{\alpha^2+\beta^2}\partial_t(\alpha\nabla_v u + \beta\hat{J}(u)\nabla_v u) - \partial_t(P(u)(\nabla u, \nabla u)).$$

Then, by using estimate (5-7) and taking the same argument as above, we can show

$$\Delta\partial_t u \in L^\infty\big([0, T], H^1(\Omega)\big) \cap L^2\big([0, T], H^2(\Omega)\big);$$

hence the L^2-theory of the Laplace operator again implies

$$\partial_t u \in L^\infty\big([0, T], H^3(\Omega)\big) \cap L^2\big([0, T], H^4(\Omega)\big).$$

Finally, we can show

$$u \in L^\infty\big([0,T], H^5(\Omega)\big) \cap L^2\big([0,T], H^6(\Omega)\big),$$

by providing $v \in L^\infty\big([0,T], H^3(\Omega)\big) \cap L^2\big([0,T], H^4(\Omega)\big)$.

Now, by Lemma 2.4 we can also derive that

$$\partial_t^i u \in C^0([0,T], H^{5-2i}(\Omega))$$

for $i \in 0, 1$. Hence, it follows that $\partial_t^2 u \in C^0([0,T], H^1(\Omega))$ by using the equation of $\partial_t u$ and the fact $\partial_t^i v \in C^0(\mathbb{R}^+, H^{3-2i}(\Omega))$ with $i = 0, 1$. □

5C. ***Higher order regularity of u (i.e., the proof of $\mathscr{T}(k)$ with $k \geq 2$).*** In Section 5B, we have proved property $\mathscr{T}(k)$ in the case $k = 2$. Next, we assume that $\mathscr{T}(k)$ has been established for $k \geq 2$, then we intend to show $\mathscr{T}(k+1)$ is true. To this end, we assume that $u_0 \in H^{2(k+1)}(\Omega)$ satisfies the k-order compatibility conditions, and v satisfies

$$\partial_t^i v \in C^0\big([0,T], H^{2(k+1)-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2i}(\Omega)\big)$$

for any $0 < T < T_0$ and any $i \in \{0, 1, \ldots, k\}$. Moreover, property $\mathscr{T}(k)$ implies

$$\partial_t^i u \in C^0\big([0,T], H^{2k-2i+1}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big)$$

for any $0 \leq i \leq k$.

In particular, $u_k = \partial_t^k u \in C^0\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big)$ is a strong solution to the equation

$$\text{(5-9)} \quad \begin{cases} \partial_t w + \gamma \nabla_v w = \alpha \Delta w - \beta \hat{J}(u) \Delta w + L_k(w,u) + F_k(u), & (x,t) \in \Omega \times [0,T], \\ \partial w / \partial \nu = 0, & (x,t) \in \partial\Omega \times [0,T], \\ w(x,0) = V_k(u_0) : \Omega \to \mathbb{R}^K. \end{cases}$$

In the following context, we improve the regularity of u by proving the following three claims:

(1) If $u_0 \in H^{2(k+1)}(\Omega)$ satisfies the k-order compatibility conditions, then we get a regular solution to (5-9):

$$w \in C^0\big([0,T], H^2(\Omega)\big) \cap L^2\big([0,T], H^3(\Omega)\big).$$

(2) It follows from an argument on uniqueness that $w = u_k$. Hence we can show

$$u_i \in C^0\big([0,T], H^{2(k+1)-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)+1-2i}(\Omega)\big)$$

for any $0 \leq i \leq k+1$, by using (4-1).

(3) Additionally if $u_0 \in H^{2(k+1)+1}(\Omega)$, we can further prove

$$u_{k+1} \in C^0\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big)$$

by considering differentiation of the Galerkin approximation equation to (5-9) in the time direction. This implies

$$u_i \in C^0\big([0,T], H^{2(k+1)+1-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)+2-2i}(\Omega)\big)$$

for any $0 \le i \le k+1$.

5D. *Regular solution to* (5-9). To show the existence of local regular solutions to (5-9) by applying a similar argument to that in Section 3, first of all, we estimate the nonhomogeneous term F_k by using the estimates given in Lemmas 5.5 and 5.6.

Lemma 5.8. *Assume that, for $0 \le i \le k$, the field v satisfies*

$$\partial_t^i v \in C^0\big([0,T], H^{2(k+1)-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2i}(\Omega)\big)$$

and property $\mathscr{T}(k)$ holds true. Then, we have

$$F_i \in L^\infty\big([0,T], H^{2k-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big)$$

for $0 \le i \le k$.

Proof. For any $0 \le i \le k$, by setting $v_i = \partial_t^i v$, we have

$$\begin{aligned} F_i(u) = -\gamma \sum_{\substack{m+j=i\\ m\ge 1}} v_m \# \nabla u_j + \alpha \sum_{\substack{i_1+\cdots+i_s+m+l=i\\ 1\le i_j<i}} \nabla^s P(u) \# u_{i_1} \# \cdots \# u_{i_s} \# \nabla u_m \# \nabla u_l \\ + \beta \sum_{\substack{i_1+\cdots+i_s+m=i\\ 1\le i_j<i}} \nabla^s \hat{J}(u) \# u_{i_1} \# \cdots \# u_{i_s} \# \Delta u_m \\ = I + II + III. \end{aligned}$$

Next we estimate the above three terms step by step. For term I: since $1 \le m \le i$ and $0 \le j = i-m \le i-1$, then we have

$$v_m \in L^\infty\big([0,T], H^{2k-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big)$$

and

$$u_j \in L^\infty([0,T], H^{2k-2i+3}(\Omega)).$$

Hence, Lemma 5.5 claims

$$I \in L^\infty\big([0,T], H^{2k-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big).$$

For term II: since $1 \le i_j \le i-1$ and $0 \le m \le i-1$, we have

$$u_{i_j} \in L^\infty([0,T], H^{2k-2i+3}(\Omega))$$

and

$$\nabla u_m \in L^\infty\big([0,T], H^{2k-2i+2}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+3}(\Omega)\big).$$

It follows from Lemma 5.5 that

$$II \in L^\infty([0,T], H^{2k-2i+2}(\Omega)).$$

Similarly, by applying Corollary 5.6 with $\nabla^s \hat{J}$ in place of L, we can also show

$$III \in L^\infty\big([0,T], H^{2k-2i+1}(\Omega)\big) \cap L^2\big([0,T], H^{2k-2i+2}(\Omega)\big).$$

Therefore, the desired results are proved. □

Now we turn to considering the Galerkin approximation of (5-9):

$$\begin{cases} \partial_t w^n + \gamma P_n(\nabla_v w^n) = \alpha \Delta w^n - \beta P_n(\hat{J}(u)\Delta w^n) + P_n(L_k(w^n, u) + F_k(u)), \\ w^n(x,0) = P_n(V_k(u_0)) : \Omega \to \mathbb{R}^{n+k}. \end{cases} \tag{5-10}$$

It is not difficult to show that there exists a unique solution $w^n \in H^n$ to (5-10) on a maximal interval $[0, T_*^n)$, and we will show $T_0 \le T_*^n$.

Next, we choose w^n and $\Delta^2 w^n$ as test functions of (5-10) and take a simple calculation to show

$$\frac{\partial}{\partial t}\int_\Omega |w^n|^2\,dx + \alpha\int_\Omega |\nabla w^n|^2\,dx \le C_\alpha(1+|\beta|^2)(\|u\|_{H^3}^2+1)\|w^n\|_{H^1}^2 + \int_\Omega |F_k|^2\,dx,$$

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega |\Delta w^n|^2\,dx &+ \alpha\int_\Omega |\nabla\Delta w^n|^2\,dx \\ &\le C_\alpha(1+|\beta|^2+|\gamma|^2)(\|u\|_{H^3}^6 + \|v\|_{H^2}^2+1)\|w^n\|_{H^2}^2 + C_\alpha\int_\Omega |\nabla F_k|^2\,dx.\end{aligned}$$

It follows that

$$\frac{\partial}{\partial t}\|w^n\|_{H^2}^2 + \alpha\int_\Omega |\nabla\Delta w^n|^2\,dx \le C_{\alpha,\beta,\gamma}\,p(t)\|w^n\|_{H^2}^2 + C_\alpha q(t),$$

where

$$p(t) := \|u\|_{H^3}^6 + \|v\|_{H^2}^2 + 1 \le C(T)$$

and

$$q(t) := \|F_k\|_{H^1}^2 \in L^1([0,T])$$

for any $T < T_0$.

On the other hand, since $u_0 \in H^{2k+2}(\Omega)$ and $v_i \in C^0([0,T], H^{2k-2i}(\Omega))$ with $0 \le i \le k$, it is not difficult to show

$$\|V_k^n\|_{H^2(\Omega)}^2 \le C\|V_k\|_{H^2(\Omega)}^2 \le C(T, \|u_0\|_{H^{2(k+1)}(\Omega)}^2).$$

Here we have used Lemma 2.5 in the first inequality.

Thus, by the Gronwall inequality we can infer from the above

$$\sup_{0<t\le T}(\|w^n\|_{H^2}^2 + \|\partial_t w^n\|_{L^2}^2) + \alpha\int_0^T (\|w^n\|_{H^3}^2 + \|\partial_t w^n\|_{H^1}^2)\,dt \le C(T).$$

Hence without loss of generality, we assume that w^n converges to a regular solution $w \in L^\infty\big([0,T], H^2\big) \cap L^2\big([0,T], H^3(\Omega)\big)$ to (5-9). Moreover, $\partial_t w$ is in $L^\infty\big([0,T], L^2(\Omega)\big) \cap L^2\big([0,T], H^1(\Omega)\big)$. By Lemma 2.4 we know that

$$w \in C^0([0,T], H^2(\Omega)).$$

5E. ***Uniqueness of strong solutions to equation* (5-9).**

Proposition 5.9. *There exists a unique solution to equation* (5-9) *in the space* $L^\infty([0,T],H^1(\Omega))\cap L^2([0,T],H^2(\Omega))$.

Proof. Suppose w_1 and w_2 are two solutions to (5-9) belonging to the space $L^\infty([0,T],H^1)\cap L^2([0,T],H^2(\Omega))$. Then, the difference $\bar{w}=w_1-w_2$ satisfies

$$\begin{cases} \partial_t\bar{w}+\gamma\nabla_v\bar{w}=\alpha\Delta\bar{w}-\beta\hat{J}(u)\Delta\bar{w},u)+F_k(u), & (x,t)\in\Omega\times[0,T],\\ \quad\partial\bar{w}/\partial\nu=0, & (x,t)\in\partial\Omega\times[0,T],\\ \quad\bar{w}(x,0)=0. \end{cases}$$

Taking $\bar{w}$ as a test function to the above equation, we can show

$$\begin{aligned} \text{(5-11)}\quad \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega|\bar{w}|^2\,dx+\alpha\int_\Omega|\nabla\bar{w}|^2\,dx &\\ =-\gamma\int_\Omega\langle v\cdot\nabla\bar{w},\bar{w}\rangle\,dx-\beta\int_\Omega\langle\hat{J}(u)\Delta\bar{w},\bar{w}\rangle\,dx+\int_\Omega\langle L_k(\bar{w},u),\bar{w}\rangle\,dx &\\ =I+II+III. & \end{aligned}$$

We estimates the above three terms as follows:

$$I=-\frac{\gamma}{2}\int_\Omega v\cdot\nabla|\bar{w}|^2\,dx=-\frac{\gamma}{2}\int_\Omega\operatorname{div}(v|\bar{w}|^2)\,dx=0,$$

since $\operatorname{div}(v)=0$ and $\langle v,\nu\rangle|_{\partial\Omega}=0$.

$$\begin{aligned} |II|&=|\beta|\left|\int_\Omega\langle\hat{J}(u)\Delta\bar{w},\bar{w}\rangle\,dx\right|\\ &\le C|\beta|\int_\Omega|\nabla\bar{w}||\nabla u||\bar{w}|\,dx\le C_\alpha\beta^2\|u\|^2_{H^3}\int_\Omega|\bar{w}|^2\,dx+\frac{\alpha}{4}\int_\Omega|\nabla\bar{w}|^2\,dx.\\ |III|&\le C\alpha\int_\Omega(|\bar{w}||\nabla\bar{w}||\nabla u|+|\bar{w}|^2|\nabla u|)\,dx+C|\beta|\left|\int_\Omega\langle d\hat{J}(\bar{w})\Delta u,\bar{w}\rangle\,dx\right|\\ &\le C_\alpha(1+\beta^2)\|u\|^2_{H^3}\int_\Omega|\bar{w}|^2\,dx+\frac{\alpha}{4}\int_\Omega|\nabla\bar{w}|^2\,dx. \end{aligned}$$

Here we have used the fact

$$\left|\int_\Omega\langle\nabla\hat{J}(\bar{w})\Delta u,\bar{w}\rangle\,dx\right|\le\left|\int_\Omega\langle\nabla(d\hat{J}(\bar{w}))\cdot\nabla u,\bar{w}\rangle\,dx\right|+\left|\int_\Omega\langle(d\hat{J}(\bar{w}))\cdot\nabla u,\nabla\bar{w}\rangle\,dx\right|$$

since $\frac{\partial u}{\partial\nu}|_{\partial\Omega}=0$.

By combining the estimates of I–III with (5-11), we get

$$\frac{\partial}{\partial t}\int_\Omega|\bar{w}|^2\,dx+\alpha\int_\Omega|\nabla\bar{w}|^2\,dx\le C_{\alpha,\beta}\int_\Omega|\bar{w}|^2\,dx.$$

It follows from the Gronwall inequality that $\bar{w}\equiv 0$. Therefore, the proof is completed. □

As a direct conclusion of the above proposition, we have $u_k\equiv w$ and hence

$$u_k\in C^0([0,T],H^2(\Omega))\cap L^2([0,T],H^3(\Omega)).$$

5F. *The proof of item (1) of property $\mathscr{T}(k+1)$.* Now we are in position to prove item (1) of property $\mathscr{T}(k+1)$ as follows:

Proposition 5.10. *Assume that $u_0 \in H^{2(k+1)}(\Omega)$ satisfies the k-order compatibility condition, for $i \in \{0, 1, \dots, k\}$,*

$$v_i = \partial_t^i v \in C^0\big([0, T], H^{2(k+1)-2(i+1)}(\Omega)\big) \cap L^2\big([0, T], H^{2(k+1)-2i}(\Omega)\big),$$

and property $\mathscr{T}(k)$ holds true. Then, for any $i \in \{0, 1, \dots, k+1\}$,

$$u_i \in L^\infty\big([0, T], H^{2(k+1)-2i}(\Omega)\big) \cap L^2\big([0, T], H^{2(k+1)+1}(\Omega)\big).$$

It follows that, for any $i \in \{0, 1, \dots, k\}$,

$$u_i \in C^0\big([0, T], H^{2(k+1)-2i}(\Omega)\big) \cap L^2\big([0, T], H^{2(k+1)+1}(\Omega)\big).$$

Proof. Since

$$u_{k+1} = \alpha \Delta u_k - \beta \hat{J}(u) \Delta u_k - \gamma \nabla_v u_k + L_k(u_k, u) + F_k(u)$$

and $u_k \in L^\infty\big([0, T], H^2\big) \cap L^2\big([0, T], H^3(\Omega)\big)$, a direct calculation shows

$$u_{k+1} \in L^\infty\big([0, T], L^2(\Omega)\big) \cap L^2\big([0, T], H^1(\Omega)\big).$$

Next we prove this proposition by inducting on $k+1-l$. We have shown the result is true for $l = 0$ and $l = 1$. Now, we assume that for $l = i \geq 1$ the result has been proved. Then, we need to establish it for $l = i+1$, where $i \leq k-1$. Thus, we consider the following equation of u_{k-i}:

$$\begin{aligned}\Delta u_{k-i} &= \frac{1}{\sigma}(\alpha u_{k-i+1} + \beta \hat{J}(u) u_{k-i+1}) + \frac{\alpha\gamma}{\sigma} \sum_{q+m=k-i} v_q \# u_m \\ &\quad + \sum_{i_1+\cdots+i_q+s+m=k-i} \nabla^q P \# u_{i_1} \# \cdots \# u_{i_q} \# \nabla u_s \# \nabla u_m \\ &\quad + \frac{\beta\gamma}{\sigma} \sum_{i_1+\cdots+i_q+s+m=k-i} \nabla^q \hat{J} \# u_{i_1} \# \cdots \# u_{i_q} \# v_s \# \nabla u_m \\ &\quad + \frac{\beta}{\sigma} \sum_{\substack{i_1+\cdots+i_q+m=k-i \\ m<k-i}} \nabla^q \hat{J} \# u_{i_1} \# \cdots \# u_{i_q} \# u_{m+1} \\ &= J_1 + J_2 + J_3 + J_4 + J_5,\end{aligned} \tag{5-12}$$

where σ denotes $\alpha^2 + \beta^2$.

Next we estimate the above five terms step by step. First of all, by the assumptions of induction, we have the following:

(1) For $i+1 \leq l \leq k+1$, $u_{k+1-l} \in L^\infty\big([0, T], H^{2l-1}(\Omega)\big) \cap L^2\big([0, T], H^{2l}(\Omega)\big)$.

(2) For $0 \leq l \leq i < k$, $u_{k+1-l} \in L^\infty\big([0, T], H^{2l}(\Omega)\big) \cap L^2\big([0, T], H^{2l+1}(\Omega)\big)$.

(3) For $0 \le s \le k$,

$$v_s \in L^\infty\big([0,T], H^{2k-2s}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2l}(\Omega)\big).$$

The estimate of term J_1: since

$$u_{k-i+1} \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big)$$

and

$$u \in L^\infty\big([0,T], H^{2k+1}(\Omega)\big) \cap L^2\big([0,T], H^{2k+2}(\Omega)\big),$$

by applying Corollary 5.6 with L replaced by $\hat{J}$ and Lemma 5.5, we have

$$J_1 \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

The estimate of term J_3: A simple computation shows that J_3 satisfies

$$\begin{aligned} J_3 &= \nabla P \# u_{k-i} \# \nabla u \# \nabla u + P(u) \# \nabla u_{k-i} \# \nabla u \\ &\quad + \sum_{\substack{i_1+\cdots+i_q+s+m=k-i \\ i_j, m, s \le k-i-1}} \nabla^q P \# u_{i_1} \# \cdots \# u_{i_q} \# \nabla u_s \# \nabla u_m \\ &= a+b+c. \end{aligned}$$

Since $u_{k-i} \in L^\infty\big([0,T], H^{2i+1}\big) \cap L^2\big([0,T], H^{2i+2}\big)$ with $i \le k-1$ and

$$\nabla u \in L^\infty\big([0,T], H^{2k}(\Omega)\big) \cap L^2\big([0,T], H^{2k+1}(\Omega)\big),$$

Lemma 5.5 implies

$$a+b \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

On the other hand, by using the fact $i_j, m, s \le k-i-1$, we have

$$u_{i_j} \in L^\infty([0,T], H^{2(i+1)+1}(\Omega)) \quad \text{and} \quad \nabla u_m \in L^\infty([0,T], H^{2(i+1)}(\Omega)).$$

It follows that $c \in L^\infty([0,T], H^{2(i+1)})$. Consequently, we obtain

$$J_3 \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

Taking almost the same argument as in estimating J_3, we obtain

$$J_2 + J_4 \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

Then we show the last term:

$$\begin{aligned} J_5 &= \frac{\beta}{\sigma} \nabla \hat{J}(u) \# u_{k-i} \# u_1 + \frac{\beta}{\sigma} \sum_{\substack{i_1+\cdots+i_q+m=k-i \\ i_j, m<k-i}} \nabla^q \hat{J}(u) \# u_{i_1} \# \cdots \# u_{i_q} \# u_{m+1} \\ &= d+e. \end{aligned}$$

Since $u_{k-i} \in L^\infty([0,T], H^{2i+1}(\Omega))$ with $i \le k-1$ and

$$u_1 \in L^\infty([0,T], H^{2k-1}(\Omega)) \subset L^\infty([0,T], H^{2i+1}(\Omega)),$$

we have

$$d \in L^\infty([0,T], H^{2i+1}(\Omega)).$$

Since $m, i_j \le k-i-1$, by Lemma 5.5 and Corollary 5.6, it is not difficult to show

$$e \in L^\infty([0,T], H^{2i+1}(\Omega)).$$

Combining the above estimates of J_1–J_5 with formula (5-12), we conclude that

$$\Delta u_{k-i} \in L^\infty\big([0,T], H^{2i}(\Omega)\big) \cap L^2\big([0,T], H^{2i+1}(\Omega)\big).$$

Then, by the L^2-theory of Laplace operator we have

$$u_{k-i} \in L^\infty\big([0,T], H^{2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2(i+1)+1}(\Omega)\big)$$

for $1 \le i \le k-1$.

It remains to show the result in the case of $l = k+1$. Since

$$\Delta u = -P(u)(\nabla u, \nabla u) + \frac{1}{\sigma}(\alpha \partial_t u + \beta \hat{J}(u)\partial_t u) + \frac{\gamma}{\sigma}(\alpha \nabla_v u + \beta \hat{J}(u)\nabla_v u) \tag{5-13}$$

and

- $u \in L^\infty([0,T], H^{2k+1}(\Omega))$,
- $\partial_t u \in L^\infty\big([0,T], H^{2k}(\Omega)\big) \cap L^2\big([0,T], H^{2k+1}(\Omega)\big)$,
- $v \in L^\infty\big([0,T], H^{2k}(\Omega)\big) \cap L^2\big([0,T], H^{2k+2}(\Omega)\big)$,

we can apply Lemma 5.5 to show

$$\Delta u \in L^\infty([0,T], H^{2k}(\Omega)),$$

which gives $u \in L^\infty([0,T], H^{2k+2}(\Omega))$.

And again it follows that

$$\Delta u \in L^2([0,T], H^{2k+1}(\Omega)),$$

then the L^2-theory of the Laplace operator yields

$$u \in L^2([0,T], H^{2(k+1)+1}(\Omega)).$$

Therefore, the proof is completed. □

5G. ***The proof of item (2) in property $\mathscr{T}(k+1)$.*** In the last part, we assume that $u_0 \in H^{2(k+1)+1}(\Omega)$ satisfies the k-order compatibility conditions. Furthermore, suppose that there hold true the following properties $C(k)$:

- for any $i \in \{0, 1, \dots, k\}$,

  $$v_i \in C^0\big([0,T], H^{2(k+1)+1-2(i+1)}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2i}(\Omega)\big)$$

  and $\partial_t^{k+1} v \in L^2([0,T], L^2(\Omega))$;

- for any $i \in \{0, 1, \ldots, k+1\}$, we have

$$u_i \in C^0\big([0,T], H^{2(k+1)-2i}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)+1-2i}(\Omega)\big).$$

Next, we turn to proving item (2) of property $\mathscr{T}(k+1)$.

First of all, taking almost the same argument as in Lemma 5.8, we can show:

Proposition 5.11. *For any* $i \in \{0, 1, \ldots, k\}$,

$$\partial_t F_i \in L^2([0,T], H^{2k-2i}(\Omega)).$$

Next, we can also prove the following proposition, which is analogous to the main theorem in Section 4:

Proposition 5.12. *Assume that* $u_0 \in H^{2(k+1)+1}(\Omega)$ *satisfies the k-order compatibility conditions. If the properties* $C(k)$ *hold true*, *then we have*

$$u_{k+1} \in C^0\big([0,T], H^1(\Omega)\big) \cap L^2\big([0,T], H^2(\Omega)\big).$$

Proof. It follows from the Galerkin approximation equation (5-10) that $w_t^n := \partial_t w^n$ satisfies

$$\partial_t w_t^n - \alpha \Delta w_t^n = P_n \partial_t(-\gamma \nabla_v w^n - \beta \hat{J}(u)\Delta w^n + L_k(w^n, u) + F_k(u)).$$

Then, taking $-\Delta w_t^n$ as a test function to this equation, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla w_t^n|^2\,dx + \alpha \int_\Omega |\Delta w_t^n|\,dx
&= \gamma \int_\Omega \langle \partial_t(v \cdot \nabla w^n), \Delta w_t^n\rangle\,dx + \beta \int_\Omega \langle \partial_t(\hat{J}(u)\Delta w^n), \Delta w_t^n\rangle\,dx \\
&\qquad - \int_\Omega \langle \partial_t L_k(w^n, u), \Delta w_t^n\rangle\,dx - \int_\Omega \langle \partial_t F_k(u), \Delta w_t^n\rangle\,dx \\
&= M_1 + M_2 + M_3 + M_4.
\end{aligned}$$

By direct calculations, we show the estimates of M_1–M_4 as follows:

$$\begin{aligned}
|M_1| &\le C|\gamma| \int_\Omega (|\partial_t v||\nabla w^n| + |v||\nabla \partial_t w^n|)|\Delta w_t^n|\,dx \\
&\le C_\alpha |\gamma|^2 \|\partial_t v\|_{H^1}^2 \|w^n\|_{H^2}^2 + C_\alpha \|v\|_{H^2}^2 |\gamma|^2 \int_\Omega |\nabla w_t^n|^2\,dx + \frac{\alpha}{8}\int_\Omega |\Delta w_t^n|^2\,dx, \\
|M_2| &= |\beta| \left| \int_\Omega \langle \partial_t(\hat{J}(u))\Delta w^n, \Delta w_t^n\rangle\,dx \right| \\
&\le C_\alpha |\beta|^2 \|\partial_t u\|_{H^2}^2 \|w^n\|_{H^2}^2 + \frac{\alpha}{8}\int_\Omega |\Delta w_t^n|^2\,dx,
\end{aligned}$$

$$
\begin{aligned}
|M_3| &= \Big|\int_\Omega \langle \partial_t L_k(w^n, u), \Delta w_t^n\rangle\, dx\Big| \\
&\le C\alpha \int_\Omega (|\nabla w_t^n|^2|\nabla u|^2 + |u_t|^2|\nabla w^n|^2|\nabla u|^2 + |\nabla u_t|^2|\nabla w^n|^2)\, dx \\
&\quad + C\alpha \int_\Omega (|w_t^n|^2|\nabla u|^4 + |u_t|^2|w^n|^2|\nabla u|^4 + |\nabla u_t|^2|\nabla u|^2|w^n|^2)\, dx \\
&\qquad + C_\alpha|\beta|^2 \int_\Omega (|w_t^n|^2|\Delta u|^2 + |u_t|^2|w^n|^2|\Delta u|^2 + |\Delta u_t|^2|w^n|^2)\, dx \\
&\qquad\quad + \frac{\alpha}{8}\int_\Omega |\Delta w_t^n|^2\, dx \\
&\le C_\alpha(1+\beta^2) f(t)\Big(\int_\Omega |w_t^n|^2\, dx\Big) + C\alpha\|u\|_{H^3}^2 \int_\Omega |\nabla w_t^n|^2\, dx + \frac{\alpha}{8}\int_\Omega |\Delta w_t^n|^2\, dx,
\end{aligned}
$$

where

$$
f(t) := \|u_t\|_{H^2}^2 \|w^n\|_{H^2}^2 (\|u\|_{H^2}^2 + 1)^2 \le C(T).
$$

The last term satisfies the estimate

$$
|M_4| \le C(\alpha)\|\partial_t F_k\|_{L^2}^2 + \frac{\alpha}{8}\int_\Omega |\Delta w_t^n|^2\, dx.
$$

Hence, we conclude that

$$
\frac{\partial}{\partial t}\int_\Omega |\nabla w_t^n|^2\, dx + \alpha\int_\Omega |\Delta w_t^n|\, dx \le C_{\gamma,\alpha,\beta,T}\int_\Omega |\nabla w_t^n|^2\, dx + C_\alpha\|\partial_t F_k\|_{L^2}^2.
$$

It follows

$$
\sup_{0\le t\le T}\|\partial_t w^n\|_{H^1}^2 + \alpha\int_0^T\int_\Omega |\Delta w_t^n|^2\, dxdt \le C(T, \|V_k^n\|_{H^3}^2),
$$

since $\|\partial_t F_k\|_{L^2}^2 \in L^1([0, T])$.

Now, it remains to show there exists a uniform bound of $\|V_k^n\|_{H^3}^2$. By using the fact $v_i \in C^0([0, T], H^{2k-2i+1})$ with $0 \le i \le k$, we can show

$$
\|V_k^n\|_{H^3(\Omega)}^2 \le C\|V_k\|_{H^3(\Omega)}^2 \le C(\|u_0\|_{H^{2(k+1)+1}(\Omega)}^2).
$$

Hence, without loss of generality we can assume that w_t^n converges weakly to

$$
u_{k+1} \in L^\infty\big([0, T], H^1(\Omega)\big) \cap L^2\big([0, T], H^2(\Omega)\big).
$$

It follows that

$$
\partial_t u_{k+1} \in L^2([0, T], L^2(\Omega))
$$

by applying the equation of u_{k+1} and the fact $\partial_t F_k \in L^2([0, T], L^2(\Omega))$. Then, Lemma 2.4 gives

$$
u_{k+1} \in C^0([0, T], H^1(\Omega)). \qquad \square
$$

Consequently, taking the estimates in Propositions 5.10–5.12 into consideration, and adopting almost the same argument as in the proof of Proposition 5.10, we can see that it is not difficult to show

$$u_i \in L^\infty\big([0,T], H^{2(k+1)-2i+1}(\Omega)\big) \cap L^2\big([0,T], H^{2(k+1)-2i+2}(\Omega)\big)$$

for any $0 \le i \le k+1$. Hence, Lemma 2.4 implies that for any $i \in \{0,\dots,k\}$,

$$u_i \in C^0([0,T], H^{2(k+1)-2i+1}(\Omega)).$$

Therefore, the second term (2) in property $\mathscr{T}(k+1)$ is proved.

Acknowledgements

The authors are grateful to the referees for the insightful comments which intensely improved the presentation of this work. Chen is supported partially by NSFC (Grant 12141103), Wang is supported partially by NSFC (Grant 11971400) and National Key Research and Development Projects of China (Grant 2020YFA0712500).

References

[Alouges and Soyeur 1992] F. Alouges and A. Soyeur, "On global weak solutions for Landau–Lifshitz equations: existence and nonuniqueness", *Nonlinear Anal.* **18**:11 (1992), 1071–1084. MR Zbl

[Bejenaru 2008] I. Bejenaru, "Global results for Schrödinger maps in dimensions $n \ge 3$", *Comm. Partial Differential Equations* **33**:1-3 (2008), 451–477. MR Zbl

[Bejenaru et al. 2007] I. Bejenaru, A. D. Ionescu, and C. E. Kenig, "Global existence and uniqueness of Schrödinger maps in dimensions $d \ge 4$", *Adv. Math.* **215**:1 (2007), 263–291. MR Zbl

[Bejenaru et al. 2011] I. Bejenaru, A. D. Ionescu, C. E. Kenig, and D. Tataru, "Global Schrödinger maps in dimensions $d \ge 2$: small data in the critical Sobolev spaces", *Ann. of Math.* (2) **173**:3 (2011), 1443–1506. MR Zbl

[Benešová et al. 2018] B. Benešová, J. Forster, C. Liu, and A. Schlömerkemper, "Existence of weak solutions to an evolutionary model for magnetoelasticity", *SIAM J. Math. Anal.* **50**:1 (2018), 1200–1236. MR Zbl

[Bonithon 2007] G. Bonithon, "Landau–Lifschitz–Gilbert equation with applied electric current", *Discrete Contin. Dyn. Syst.* (2007), 138–144. MR Zbl

[Boyer and Fabrie 2013] F. Boyer and P. Fabrie, *Mathematical tools for the study of the incompressible Navier–Stokes equations and related models*, Applied Mathematical Sciences **183**, Springer, 2013. MR Zbl

[Carbou and Fabrie 2001] G. Carbou and P. Fabrie, "Regular solutions for Landau–Lifschitz equation in a bounded domain", *Differential Integral Equations* **14**:2 (2001), 213–229. MR Zbl

[Carbou and Jizzini 2018] G. Carbou and R. Jizzini, "Very regular solutions for the Landau–Lifschitz equation with electric current", *Chinese Ann. Math. Ser. B* **39**:5 (2018), 889–916. MR Zbl

[Chen and Wang 2021] B. Chen and Y. Wang, "Global weak solutions for Landau–Lifshitz flows and heat flows associated to micromagnetic energy functional", *Commun. Pure Appl. Anal.* **20**:1 (2021), 319–338. MR Zbl

[Chen and Wang 2023a] B. Chen and Y. Wang, "Existence and uniqueness of local regular solution to the Schrödinger flow from a bounded domain in $\mathbb{R}^3$ into $\mathbb{S}^2$", *Comm. Math. Phys.* **402**:1 (2023), 391–428. MR Zbl

[Chen and Wang 2023b] B. Chen and Y. Wang, "Smooth solutions to the Schrödinger flow for maps from smooth bounded domains in Euclidean spaces into $\mathbb{S}^2$", preprint, 2023. arXiv 2111.14835v4

[Chen and Wang 2023c] B. Chen and Y. Wang, "Very regular solution to Landau–Lifshitz–Gilbert system with spin-polarized transport", *Front. Math.* **18**:4 (2023), 751–795. MR Zbl

[Chern et al. 2016] A. Chern, F. Knöppel, U. Pinkall, P. Schröder, and S. Weißmann, "Schrödinger's smoke", *ACM Transactions on Graphics* (*TOG*) **35**:4 (2016), 1–13.

[Ding and Wang 1998] W. Ding and Y. Wang, "Schrödinger flow of maps into symplectic manifolds", *Sci. China*, *Ser. A* **41**:7 (1998), 746–755. Zbl

[Ding and Wang 2001] W. Ding and Y. Wang, "Local Schrödinger flow into Kähler manifolds", *Sci. China Ser. A* **44**:11 (2001), 1446–1464. MR Zbl

[Gilbert 1955] T. L. Gilbert, "A Lagrangian formulation of the gyromagnetic equation of the magnetization field", *Phys. Rev.* **100** (1955), 1243–1255.

[Ionescu and Kenig 2007] A. D. Ionescu and C. E. Kenig, "Low-regularity Schrödinger maps, II: Global well-posedness in dimensions $d \geq 3$", *Comm. Math. Phys.* **271**:2 (2007), 523–559. MR Zbl

[Jia and Wang 2019] Z. Jia and Y. Wang, "Global weak solutions to Landau–Lifshitz equations into compact Lie algebras", *Front. Math. China* **14**:6 (2019), 1163–1196. MR Zbl

[Jia and Wang 2020] Z. Jia and Y. Wang, "Global weak solutions to Landau–Lifshtiz systems with spin-polarized transport", *Discrete Contin. Dyn. Syst.* **40**:3 (2020), 1903–1935. MR Zbl

[Kalousek et al. 2021] M. Kalousek, J. Kortum, and A. Schlömerkemper, "Mathematical analysis of weak and strong solutions to an evolutionary model for magnetoviscoelasticity", *Discrete Contin. Dyn. Syst. Ser. S* **14**:1 (2021), 17–39. MR Zbl

[Landau and Lifshitz 1935] L. Landau and E. Lifshitz, "On the theory of the dispersion of magnetic permeability in ferromagnetic bodies", *Phys. Z. Sowjetunion* **8** (1935), 153–169.

[Pang et al. 2000] P. Y. H. Pang, H. Wang, and Y. Wang, "Local existence for inhomogeneous Schrödinger flow into Kähler manifolds", *Acta Math. Sin.* (*Engl. Ser.*) **16**:3 (2000), 487–504. MR Zbl

[Pang et al. 2001] P. Y. H. Pang, H.-Y. Wang, and Y.-D. Wang, "Schrödinger flow for maps into Kähler manifolds", *Asian J. Math.* **5**:3 (2001), 509–533. MR Zbl

[Pang et al. 2002] P. Y. H. Pang, H.-Y. Wang, and Y.-D. Wang, "Schrödinger flow on Hermitian locally symmetric spaces", *Comm. Anal. Geom.* **10**:4 (2002), 653–681. MR Zbl

[Simon 1987] J. Simon, "Compact sets in the space $L^p(0, T; B)$", *Ann. Mat. Pura Appl.* (4) **146** (1987), 65–96. MR Zbl

[Simon 1996] L. Simon, *Theorems on regularity and singularity of energy minimizing maps*, Birkhäuser, Basel, 1996. MR Zbl

[Sulem et al. 1986] P.-L. Sulem, C. Sulem, and C. Bardos, "On the continuous limit for a system of classical spins", *Comm. Math. Phys.* **107**:3 (1986), 431–454. MR Zbl

[Terng and Uhlenbeck 2006] C.-L. Terng and K. Uhlenbeck, "Schrödinger flows on Grassmannians", pp. 235–256 in *Integrable systems*, *geometry*, *and topology*, edited by C.-L. Terng, AMS/IP Stud. Adv. Math. **36**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Tilioua 2011] M. Tilioua, "Current-induced magnetization dynamics: global existence of weak solutions", *J. Math. Anal. Appl.* **373**:2 (2011), 635–642. MR Zbl

[Visintin 1985] A. Visintin, "On Landau–Lifshitz' equations for ferromagnetism", *Japan J. Appl. Math.* **2**:1 (1985), 69–84. MR Zbl

[Wang 1998] Y.-D. Wang, "Heisenberg chain systems from compact manifolds into S^2", *J. Math. Phys.* **39**:1 (1998), 363–371. MR Zbl

[Wehrheim 2004] K. Wehrheim, *Uhlenbeck compactness*, European Mathematical Society (EMS), Zürich, 2004. MR Zbl

[Zhou et al. 1991] Y. L. Zhou, B. L. Guo, and S. B. Tan, "Existence and uniqueness of smooth solution for system of ferro-magnetic chain", *Sci. China Ser. A* **34**:3 (1991), 257–266. MR

Received January 15, 2023. Revised August 29, 2023.

BO CHEN
SCHOOL OF MATHEMATICS
SOUTH CHINA UNIVERSITY OF TECHNOLOGY
GUANGZHOU
CHINA

cbmath@scut.edu.cn

YOUDE WANG
SCHOOL OF MATHEMATICS AND INFORMATION SCIENCES
GUANGZHOU UNIVERSITY
GUANGZHOU
CHINA

and

HUA LOO-KENG KEY LABORATORY OF MATHEMATICS
UNIVERSITY OF THE CHINESE ACADEMY OF SCIENCES
BEIJING
CHINA

wyd@math.ac.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 326, No. 2, 2023
https://doi.org/10.2140/pjm.2023.326.227

MODULES OVER THE PLANAR GALILEAN CONFORMAL ALGEBRA ARISING FROM FREE MODULES OF RANK ONE

JIN CHENG, DONGFANG GAO AND ZITING ZENG

The planar Galilean conformal algebra $\mathcal{G}$ introduced by Bagchi-Gopakumar and Aizawa is an infinite-dimensional extension of the finite-dimensional Galilean conformal algebra in (2+1)-dimensional space-time. In this paper, we give a complete classification of $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$, where $\mathfrak{h}$ is the Cartan subalgebra (a nilpotent self-normalizing subalgebra) of $\mathcal{G}$, $\mathbb{C}L_0$ is a subalgebra of $\mathfrak{h}$. Also, we determine the necessary and sufficient conditions for these modules to be irreducible, and find the maximal proper submodules when these modules are not irreducible.

1. Introduction

Infinite-dimensional Galilean conformal algebras were introduced by Bagchi and Gopakumar [2009] in order to construct a systematic nonrelativistic limit of the AdS/CFT conjecture (see [Maldacena 1998]). Some physicists believe that AdS/CFT correspondence would be better understood by exploring those algebras (see [Bagchi et al. 2010; Martelli and Tachikawa 2010]). Moreover, those algebras appear in the context of Galilean electrodynamics (see [Bagchi et al. 2014; Festuccia et al. 2016]) and may play an important role in Navier–Stokes equations (see [Bhattacharyya et al. 2009; Fouxon and Oz 2008; Fouxon and Oz 2009; Gusyatnikova and Yumaguzhin 1989]). These reasons make the infinite-dimensional Galilean conformal algebras attract more and more attention from mathematicians and physicists. In particular, the infinite-dimensional Galilean conformal algebra in (1+1)-dimensional space-time is the centerless W-algebra $W(2,2)$; it has been studied in [Bagchi et al. 2010; Chen and Guo 2017; Zhang and Dong 2009]. This algebra is related to the BMS/GCA correspondence (see [Bagchi 2010]), the tensionless limit of string theory (see [Bagchi 2013]) and statistical mechanics (see [Henkel et al. 2012]).

The infinite-dimensional Galilean conformal algebra $\mathcal{G}$ in (2+1)-dimensional space-time, named the planar Galilean conformal algebra by Aizawa [2013], is an

MSC2020: 17B10, 17B65, 17B66, 17B68.
Keywords: planar Galilean conformal algebra, free module, irreducible module, isomorphism.

infinite-dimensional Lie algebra with a basis $\{L_n, H_n, I_n, J_n \mid n \in \mathbb{Z}\}$ and the Lie brackets are defined by

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n}, && [L_m, H_n] = nH_{m+n},\\
&[L_m, I_n] = (n-m)I_{m+n}, && [L_m, J_n] = (n-m)J_{m+n},\\
&[H_m, I_n] = I_{m+n}, && [H_m, J_n] = -J_{m+n},\\
&[H_m, H_n] = [I_m, I_n] = [J_m, J_n] = [I_m, J_n] = 0 \quad \text{for all } m, n \in \mathbb{Z},
\end{aligned}
\tag{1-1}
$$

which is the main object in this paper. This algebra is also the special case of [Martelli and Tachikawa 2010]. As we know, many infinite-dimensional Lie algebras in mathematics and physics are related to finite-dimensional semisimple Lie algebras. For example, the Virasoro algebra contains infinitely many $\mathrm{sl}_2(\mathbb{C})$ as its subalgebras. For the Lie algebra $\mathcal{G}$, there are two interesting features: it contains the Witt algebra as a subalgebra, and it is associated with the Galilean algebra, which is a nonsemisimple Lie algebra. Those would suggest that such an infinite-dimensional algebra is important and its representation theory is different from semisimple counterparts. So far there are a few of results about structures and representations of $\mathcal{G}$. The universal central extension $\bar{\mathcal{G}}$ of $\mathcal{G}$ was determined in [Gao et al. 2016]. The highest weight representations and coadjoint representations of $\mathcal{G}$ were investigated in [Aizawa 2013; Aizawa and Kimura 2011], Whittaker modules and restricted modules over $\mathcal{G}$ were studied in [Chen and Yao 2023; Chen et al. 2022; Gao and Gao 2022].

Recently, a family of nonweight modules over $\mathcal{G}$, called $\mathcal{U}(\mathfrak{h})$-free modules, has attracted more attention from mathematicians, where $\mathfrak{h} = \mathrm{span}\{L_0, H_0\}$ is a nilpotent self-normalizing subalgebra, called the Cartan subalgebra of $\mathcal{G}$. The notion of $\mathcal{U}(\mathfrak{h})$-free modules was first introduced by Nilsson [2015] for the simple Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$. At the same time, these modules were introduced in a very different approach in [Tan and Zhao 2015]. Later, $\mathcal{U}(\mathfrak{h})$-free modules for many important infinite-dimensional Lie algebras were determined, for example, the Virasoro algebra in [Lu and Zhao 2014], the Witt algebra in [Tan and Zhao 2015], affine Kac–Moody algebras in [Cai et al. 2020]. In the present paper, we will study this family of modules over $\mathcal{G}$ and $\bar{\mathcal{G}}$. These lead to many new examples of irreducible modules over $\mathcal{G}$ and $\bar{\mathcal{G}}$.

The paper is organized as follows. In Section 2, we recall the source of the infinite-dimensional Galilean conformal algebras. Then we review the planar Galilean conformal algebra $\mathcal{G}$ and $\bar{\mathcal{G}}$. We show that the $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ coincide with $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\bar{\mathcal{G}}$ respectively; see Corollary 2.3. Lastly, we collect some results about $\mathcal{U}(\mathbb{C}L_0)$-free modules over some Lie algebras related to the Witt algebra for later use. In Section 3, we get all $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 over $\mathcal{G}$, and the necessary and sufficient conditions for these modules to be irreducible are determined; see Theorem 3.2. We also determine the isomorphism classes of these modules; see Theorem 3.3. In Section 4, we obtain the main results of this paper. More precisely, we determine that there are three families of $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$, where $\mathfrak{h} = \text{span}\{L_0, H_0\}$ is the Cartan subalgebra of $\mathcal{G}$; see Theorem 4.12. Also, we give the necessary and sufficient conditions for these modules to be irreducible, and find the maximal proper submodules when these modules are not irreducible; see Theorems 4.13, 4.14 and 4.15. Furthermore, we determine the isomorphism classes of these modules; see Theorem 4.17. Consequently, we give a complete classification of $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ and $\overline{\mathcal{G}}$.

Throughout this paper, we denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{C}$ and $\mathbb{C}^*$ the set of integers, nonnegative integers, positive integers, complex numbers and nonzero complex numbers respectively. All vector spaces and algebras are over $\mathbb{C}$. We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra for a Lie algebra $\mathfrak{g}$.

2. Notation and preliminaries

In this section, we recall the infinite-dimensional Galilean conformal algebras and collect some known results about $\mathcal{U}(\mathbb{C}L_0)$-free modules over the Lie algebras related to the Witt algebra.

2A. ***From Galilean algebras to infinite-dimensional Galilean conformal algebras.*** In this subsection, we recall the background in which the infinite-dimensional Galilean conformal algebras arise. See [Bagchi and Gopakumar 2009] for more details. First, it is well-known that Galilean algebra $G(d, 1)$ in Galilean space-time $\mathbb{R}^{d,1}$ arises as a contraction of the Poincaré algebra $ISO(d, 1)$. The expressions for the Poincaré generators $(\mu, \nu = 0, 1, \ldots, d)$

$$J_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad P_\mu = \partial_\mu,$$

give us the Galilean vector field generators $\{J_{ij}, P_i, B_i, H \mid i, j = 1, 2, \ldots, d\}$, where

$$\begin{aligned} J_{ij} &= -(x_i \partial_j - x_j \partial_i), \quad & P_i &= \partial_i, \\ B_i &= J_{0i} = t\partial_i, & H &= P_0 = -\partial_t. \end{aligned} \tag{2-1}$$

and t, x_i are variables. They obey the commutation relations

$$\begin{aligned} [J_{ij}, J_{rs}] &= \delta_{ir} J_{js} + \delta_{is} J_{rj} + \delta_{jr} J_{si} + \delta_{js} J_{ir}, \\ [H, B_i] &= -P_i, \\ [J_{ij}, B_r] &= -(B_i \delta_{jr} - B_j \delta_{ir}), \\ [J_{ij}, P_r] &= -(P_i \delta_{jr} - P_j \delta_{ir}), \\ [J_{ij}, H] &= [H, P_i] = [B_i, P_j] = [B_i, B_j] = [P_i, P_j] = 0. \end{aligned} \tag{2-2}$$

Consequently, we obtain the Galilean algebra

$$G(d,1)=\operatorname{span}\{J_{ij},\,P_i,\,B_i,\,H \mid i,j=1,2,\dots,d\}$$

with the commutation relations (2-2).

To obtain the Galilean conformal algebra, we need additional generators

$$\{D,\,K,\,K_i \mid i=1,2,\dots,d\},$$

where

$$D=-\Big(\sum_{i=1}^{d}x_i\partial_i+t\partial_t\Big),\quad K=-\Big(\sum_{i=1}^{d}2tx_i\partial_i+t^2\partial_t\Big),\quad K_i=t^2\partial_i. \tag{2-3}$$

Thus we get that Galilean conformal algebra in $(d+1)$-dimensional space-time is spanned by $\{J_{ij},P_i,B_i,H,D,K,K_i \mid i,j=1,2,\dots,d\}$ with the commutation relations (2-2) and

$$\begin{aligned}
&[J_{ij},K_r]=-(K_i\delta_{jr}-K_j\delta_{ir}), && [K,B_i]=K_i, && [K,P_i]=2B_i,\\
&[H,K_i]=-2B_i, && [D,K_i]=-K_i, && [D,P_i]=P_i,\\
&[D,H]=H, && [H,K]=-2D, && [D,K]=-K,
\end{aligned}$$
$$[J_{ij},D]=[J_{ij},K]=[D,B_i]=[K,K_i]=[K_i,K_j]=[K_i,B_j]=[K_i,P_j]=0.$$

It is clear that Galilean conformal algebra contains Galilean algebra as a subalgebra.

We denote

$$\begin{aligned}
L^{(-1)}&=H, & L^{(0)}&=D, & L^{(+1)}&=K,\\
M_i^{(-1)}&=P_i, & M_i^{(0)}&=B_i, & M_i^{(+1)}&=K_i.
\end{aligned}$$

Then Galilean conformal algebra in $(d+1)$-dimensional space-time is spanned by $\{J_{ij},L^{(n)},M_i^{(n)} \mid i,j=1,2,\dots,d,\ n=0,\pm1\}$ with the commutation relations

$$\begin{aligned}
&[L^{(m)},L^{(n)}]=(m-n)L^{(m+n)}, && [L^{(m)},M_i^{(n)}]=(m-n)M_i^{(m+n)},\\
&[J_{ij},M_k^{(m)}]=-(M_i^{(m)}\delta_{jk}-M_j^{(m)}\delta_{ik}), && [J_{ij},L^{(n)}]=[M_i^{(m)},M_j^{(n)}]=0,
\end{aligned}$$

where $m,n=0,\pm1$, $i,j=1,2,\dots,d$. In fact, we can define the vector fields

$$\begin{aligned}
J_{ij}&=-(x_i\partial_j-x_j\partial_i),\\
L^{(n)}&=-(n+1)t^n\sum_{i=1}^{d}x_i\partial_i-t^{(n+1)}\partial_t,\\
M_i^{(n)}&=t^{(n+1)}\partial_i,
\end{aligned}$$

where $n=0,\pm1$, $i,j=1,2,\dots,d$. These are exactly the vector fields in (2-1) and (2-3), so they generate the Galilean conformal algebra.

Now we have a very natural extension, for arbitrary $n \in \mathbb{Z}$, define

$$\begin{aligned}
J_{ij}^{(n)} &= -t^n(x_i\partial_j - x_j\partial_i),\\
L^{(n)} &= -(n+1)t^n\sum_{i=1}^{d} x_i\partial_i - t^{(n+1)}\partial_t,\\
M_i^{(n)} &= t^{(n+1)}\partial_i,
\end{aligned}$$

where $i, j = 1, 2, \ldots, d$. Therefore, we obtain the infinite-dimensional Galilean conformal algebra GCA in $(d+1)$-dimensional space-time,

$$\mathrm{GCA} = \mathrm{span}\{J_{ij}^{(n)}, L^{(n)}, M_i^{(n)} \mid n \in \mathbb{Z}, i, j = 1, 2, \ldots, d\},$$

satisfying the commutation relations

$$\begin{aligned}
&[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)},\\
&[J_{ij}^{(m)}, J_{rs}^{(n)}] = \delta_{ir}J_{js}^{(m+n)} + \delta_{is}J_{rj}^{(m+n)} + \delta_{jr}J_{si}^{(m+n)} + \delta_{js}J_{ir}^{(m+n)},\\
&[L^{(m)}, J_{ij}^{(n)}] = -nJ_{ij}^{(m+n)}, \quad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)},\\
&[J_{ij}^{(m)}, M_k^{(n)}] = -(\delta_{jk}M_i^{(m+n)} - \delta_{ik}M_j^{(m+n)}), \quad [M_i^{(m)}, M_j^{(n)}] = 0.
\end{aligned}$$

In this paper, we mainly investigate the infinite-dimensional Galilean conformal algebra in (2+1)-dimensional space-time, which is called the planar Galilean conformal algebra by Aizawa [2013].

2B. ***Planar Galilean conformal algebra.*** From Section 2A, the planar Galilean conformal algebra is spanned by $\{J_{12}^{(n)}, L^{(n)}, M_i^{(n)} \mid n \in \mathbb{Z}, i = 1, 2\}$. We denote this algebra by $\mathcal{G}$, then $\mathcal{G}$ is an infinite-dimensional Lie algebra with the commutation relations

$$\begin{aligned}
&[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)}, \qquad [L^{(m)}, J_{12}^{(n)}] = -nJ_{12}^{(m+n)},\\
&[L^{(m)}, M_1^{(n)}] = (m-n)M_1^{(m+n)}, \quad [L^{(m)}, M_2^{(n)}] = (m-n)M_2^{(m+n)},\\
&[J_{12}^{(m)}, M_1^{(n)}] = M_2^{(m+n)}, \qquad [J_{12}^{(m)}, M_2^{(n)}] = -M_1^{(m+n)},\\
&[J_{12}^{(m)}, J_{12}^{(n)}] = [M_1^{(m)}, M_1^{(n)}] = [M_2^{(m)}, M_2^{(n)}] = [M_1^{(m)}, M_2^{(n)}] = 0 \quad \text{for all } m, n \in \mathbb{Z}.
\end{aligned}$$

For convenience, we would like to simplify the notation (see [Chen et al. 2022]). Let

$$\begin{aligned}
&L_n = -L^{(n)}, \qquad H_n = \sqrt{-1}J_{12}^{(n)},\\
&I_n = M_1^{(n)} + \sqrt{-1}M_2^{(n)}, \quad J_n = M_1^{(n)} - \sqrt{-1}M_2^{(n)} \quad \text{for all } n \in \mathbb{Z}.
\end{aligned}$$

Then it is easy to check that $\{L_n, H_n, I_n, J_n \mid n \in \mathbb{Z}\}$ satisfy the commutation relations (1-1). Now, we may describe the definition of the planar Galilean conformal algebra as follows.

Definition 2.1. The *planar Galilean conformal algebra* $\mathcal{G}$ is an infinite-dimensional Lie algebra with a basis $\{L_n, H_n, I_n, J_n \mid n \in \mathbb{Z}\}$ subject to the commutation relations (1-1).

Note that the Lie subalgebra $\widetilde{IJ}$ spanned by $\{I_m, J_m \mid m \in \mathbb{Z}\}$ is a commutative ideal of $\mathcal{G}$. Furthermore, $\mathcal{G}$ contains the following interesting subalgebras.

(1) $\mathfrak{h} = \operatorname{span}\{L_0, H_0\}$ is a nilpotent self-normalizing subalgebra, called the Cartan subalgebra of $\mathcal{G}$.

(2) $\mathcal{V} = \operatorname{span}\{L_m \mid m \in \mathbb{Z}\}$ is the centerless Virasoro algebra, i.e., the Witt algebra.

(3) $\mathcal{L} = \operatorname{span}\{L_m, H_m \mid m \in \mathbb{Z}\}$ is the Heisenberg–Virasoro algebra with the one-dimensional center.

(4) $\mathcal{W} = \operatorname{span}\{L_m, I_m \mid m \in \mathbb{Z}\}$ is the centerless $W(2,2)$ algebra.

(5) $\mathcal{W}' = \operatorname{span}\{L_m, J_m \mid m \in \mathbb{Z}\}$ is the centerless $W(2,2)$ algebra.

Recall that (see [Gao et al. 2016]) the universal central extension $\bar{\mathcal{G}}$ of the planar Galilean conformal algebra $\mathcal{G}$ is an infinite-dimensional Lie algebra with a basis $\{L_n, H_n, I_n, J_n, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3 \mid n \in \mathbb{Z}\}$ subject to the commutation relations

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n} + \tfrac{1}{12}(m^3-m)\delta_{m+n,0}\boldsymbol{c}_1,\\
&[L_m, H_n] = nH_{m+n} + m^2\delta_{m+n,0}\boldsymbol{c}_2, \qquad [H_m, H_n] = m\delta_{m+n,0}\boldsymbol{c}_3,\\
&[L_m, I_n] = (n-m)I_{m+n}, \qquad [L_m, J_n] = (n-m)J_{m+n},\\
&[H_m, I_n] = I_{m+n}, \qquad [H_m, J_n] = -J_{m+n},\\
&[I_m, I_n] = [J_m, J_n] = [I_m, J_n] = 0 \quad \text{for all } m, n \in \mathbb{Z}.
\end{aligned} \tag{2-4}
$$

Denote $\mathcal{L}' = \operatorname{span}\{L_m, H_m, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3 \mid m \in \mathbb{Z}\}$, which is a subalgebra of $\bar{\mathcal{G}}$. From Theorem 3 in [Chen and Guo 2017] and Theorem 3.1 in [Han et al. 2017] we get the following lemma.

Lemma 2.2. (1) *Suppose M is an $\mathcal{L}'$-module such that it is a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank* 1. *Then $\boldsymbol{c}_1 M = \boldsymbol{c}_2 M = \boldsymbol{c}_3 M = 0$.*

(2) *Suppose M' is an $\mathcal{L}'$-module such that it is a $\mathcal{U}(\mathfrak{h})$-free module of rank* 1. *Then $\boldsymbol{c}_1 M' = \boldsymbol{c}_2 M' = \boldsymbol{c}_3 M' = 0$.*

So, we have the following corollary.

Corollary 2.3. (1) *Let M be a $\mathcal{U}(\bar{\mathcal{G}})$-module such that M, when considered as a $\mathcal{U}(\mathbb{C}L_0)$-module, is free of rank* 1. *Then $\boldsymbol{c}_1 M = \boldsymbol{c}_2 M = \boldsymbol{c}_3 M = 0$. Thus $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank* 1 *over $\bar{\mathcal{G}}$ coincide with $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank* 1 *over $\mathcal{G}$.*

(2) *Let M' be a $\mathcal{U}(\bar{\mathcal{G}})$-module such that M', when considered as a $\mathcal{U}(\mathfrak{h})$-module, is free of rank* 1. *Then $\boldsymbol{c}_1 M' = \boldsymbol{c}_2 M' = \boldsymbol{c}_3 M' = 0$. Thus $\mathcal{U}(\mathfrak{h})$-free modules of rank* 1 *over $\bar{\mathcal{G}}$ coincide with $\mathcal{U}(\mathfrak{h})$-free modules of rank* 1 *over $\mathcal{G}$.*

Therefore, we mainly classify $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ in the following sections.

Now, we conclude this section by recalling $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 over algebras $\mathcal{V}$, $\mathcal{L}$ and $W(2,2)$, respectively. For any $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, it is not hard to see that the polynomial algebra $\mathbb{C}[L_0]$ has a $\mathcal{V}$-module structure with the following actions

$$L_m(f(L_0)) = \lambda^m (L_0 + m\alpha) f(L_0 - m), \ \forall\, m \in \mathbb{Z},\ f(L_0) \in \mathbb{C}[L_0].$$

Denote this module by $\Omega(\lambda,\alpha)$. Thanks to [Lu and Zhao 2014], we know that $\Omega(\lambda,\alpha)$ is irreducible if and only if $\alpha \neq 0$, and $\Omega(\lambda,0)$ has an irreducible submodule $L_0\Omega(\lambda,0)$ with codimension 1. Note that $\Omega(\lambda,\alpha)$ can be easily viewed as a $\mathcal{W}$ (resp. $\mathcal{W}'$)-module by defining $I_n(\Omega(\lambda,\alpha)) = 0$ (resp. $J_n(\Omega(\lambda,\alpha)) = 0$) for all $n \in \mathbb{Z}$, and the resulting module is denoted by $\Omega(\lambda,\alpha)^{\mathcal{W}}$ (resp. $\Omega(\lambda,\alpha)^{\mathcal{W}'}$). Moreover, we have the following lemmas.

Lemma 2.4 (cf. [Tan and Zhao 2015, Theorem 3]). *Let V be a $\mathcal{V}$-module. Assume that V can be viewed as a $\mathcal{U}(\mathbb{C}L_0)$-module is free of rank* 1. *Then $V \cong \Omega(\lambda,\alpha)$ for some $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$.*

Lemma 2.5 (cf. [Chen and Guo 2017, Theorem 3]). *Let V be a $\mathcal{W}$* (resp. $\mathcal{W}'$)-*module. Assume that V can be viewed as a $\mathcal{U}(\mathbb{C}L_0)$-module is free of rank* 1. *Then $V \cong \Omega(\lambda,\alpha)^{\mathcal{W}}$* (resp. $\Omega(\lambda,\alpha)^{\mathcal{W}'}$) *for some $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$.*

For $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, thanks to [Chen and Guo 2017], we see that the polynomial algebra $\mathbb{C}[L_0]$ is an $\mathcal{L}$-module with the actions

$$\begin{aligned} L_m(f(L_0)) &= \lambda^m (L_0 + m\alpha) f(L_0 - m), \\ H_m(f(L_0)) &= \beta\lambda^m f(L_0 - m) \quad \text{for all } m \in \mathbb{Z},\ f(L_0) \in \mathbb{C}[L_0]. \end{aligned} \tag{2-5}$$

We denote by $\Omega(\lambda,\alpha,\beta)$ this module. From [Chen and Guo 2017], we also know that $\Omega(\lambda,\alpha,\beta)$ is irreducible if and only if $(\alpha,\beta) \neq (0,0)$, and $\Omega(\lambda,0,0)$ has an irreducible submodule $L_0\Omega(\lambda,0,0)$ with codimension 1. Furthermore:

Lemma 2.6 (cf. [Chen and Guo 2017, Theorem 2]). *Let V be an $\mathcal{L}$-module. Assume that V can be viewed as a $\mathcal{U}(\mathbb{C}L_0)$-module is free of rank* 1. *Then $V \cong \Omega(\lambda,\alpha,\beta)$ for some $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$.*

3. $\mathcal{U}(\mathbb{C}L_0)$-free modules over $\mathcal{G}$

In this section, we determine the $\mathcal{G}$-module structures on $\mathcal{U}(\mathbb{C}L_0)$. We give the necessary and sufficient conditions for these modules to be irreducible. Also, we find the maximal proper submodules and get the irreducible quotient modules when these modules are not irreducible. Moreover, we determine the isomorphism classes of these modules.

Note that $\widetilde{IJ}$ is a commutative ideal of $\mathcal{G}$. Thus for any $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, by (2-5) it is easy to see that the polynomial algebra $\mathbb{C}[L_0]$ equips with a $\mathcal{G}$-module structure via the actions

$$\begin{aligned} &L_m(f(L_0)) = \lambda^m (L_0 + m\alpha) f(L_0 - m), \\ &H_m(f(L_0)) = \beta\lambda^m f(L_0 - m), \\ &I_m(f(L_0)) = J_m(f(L_0)) = 0 \quad \text{for all } m \in \mathbb{Z},\ f(L_0) \in \mathbb{C}[L_0]. \end{aligned} \tag{3-1}$$

We denote this module by $\mathcal{A}(\lambda, \alpha, \beta)$.

Now we show that $\{\mathcal{A}(\lambda, \alpha, \beta) \mid \lambda \in \mathbb{C}^*, \alpha, \beta \in \mathbb{C}\}$ exhaust all $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 over $\mathcal{G}$ up to isomorphism.

Lemma 3.1. *Let V be a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank* 1 *over $\mathcal{G}$. We identify V with $\mathbb{C}[L_0]$ as vector spaces.*

(1) $I_m(V) = J_m(V) = 0$ *for all $m \in \mathbb{Z}$.*

(2) *There exist $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ such that*

$$\begin{aligned} &L_m(f(L_0)) = \lambda^m (L_0 + m\alpha) f(L_0 - m), \\ &H_m(f(L_0)) = \beta\lambda^m f(L_0 - m) \quad \textit{for all } f(L_0) \in V,\ m \in \mathbb{Z}. \end{aligned}$$

Proof. (1) It is clear that V may be viewed as a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank 1 over $\mathcal{W}$, since $\mathcal{W}$ is a subalgebra containing $\mathcal{V}$ of $\mathcal{G}$. By Lemma 2.5, we have $I_m(V) = 0$ for all $m \in \mathbb{Z}$. Similarly, we may get $J_m(V) = 0$ for all $m \in \mathbb{Z}$.

(2) We view V as a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank 1 over $\mathcal{L}$. Then the conclusions are clear by Lemma 2.6. □

Theorem 3.2. *Let V be a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank* 1 *over the Lie algebra $\mathcal{G}$.*

(1) *There exist $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ such that $V \cong \mathcal{A}(\lambda, \alpha, \beta)$ as $\mathcal{G}$-modules.*

(2) *V is an irreducible $\mathcal{G}$-module if and only if $V \cong \mathcal{A}(\lambda, \alpha, \beta)$ for some $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ with $(\alpha, \beta) \neq (0, 0)$.*

(3) *If V is isomorphic to $\mathcal{A}(\lambda, 0, 0)$ for some $\lambda \in \mathbb{C}^*$, then V has an irreducible submodule L_0V with codimension* 1.

Proof. (1) is clear from Lemma 3.1 and (3-1).

(2) and (3) follow from the irreducibility of $\mathcal{L}$-module $\Omega(\lambda, \alpha, \beta)$. □

From (3-1) and [Chen and Guo 2017] we can get the following theorem.

Theorem 3.3. *Let $\lambda, \lambda' \in \mathbb{C}^*$, $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$. Then $\mathcal{A}(\lambda, \alpha, \beta)$ and $\mathcal{A}(\lambda', \alpha', \beta')$ are isomorphic as $\mathcal{G}$-modules if and only if $\lambda = \lambda'$, $\alpha = \alpha'$, $\beta = \beta'$.*

4. $\mathcal{U}(\mathfrak{h})$-free modules over $\mathcal{G}$

In this section, we obtain all $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$. The necessary and sufficient conditions for these modules to be irreducible are determined. We also investigate the maximal proper submodules and the irreducible quotient modules when these modules are not irreducible. Furthermore, we determine the isomorphism classes of these modules. These conclusions are the main results of this paper.

4A. ***$\mathcal{U}(\mathfrak{h})$-free modules over $\mathcal{G}$.*** In this subsection, we determine the $\mathcal{G}$-module structures on $\mathcal{U}(\mathfrak{h})$, where $\mathfrak{h} = \mathrm{span}\{L_0, H_0\}$ is the Cartan subalgebra of $\mathcal{G}$.

For any $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$, denote by $T(\lambda, \delta) = \mathbb{C}[H_0, L_0]$ the polynomial algebra over $\mathbb{C}$. It is clear that $T(\lambda, \delta)$ is isomorphic to $\mathcal{U}(\mathfrak{h})$ as vector spaces. First, we consider the $\mathcal{L}$-module structures on $T(\lambda, \delta)$, where $\mathcal{L} = \mathrm{span}\{L_m, H_m \mid m \in \mathbb{Z}\}$. It is not hard to see that we may give $T(\lambda, \delta)$ an $\mathcal{L}$-module structure via the actions

$$\begin{aligned} &L_m(f(H_0, L_0)) = \lambda^m f(H_0, L_0-m)(L_0+m\delta), \\ &H_m(f(H_0, L_0)) = \lambda^m H_0 f(H_0, L_0-m) \quad \text{for all } m \in \mathbb{Z},\ f(H_0, L_0) \in T(\lambda, \delta). \end{aligned} \tag{4-1}$$

Note that $H_0 T(\lambda, \delta)$ is always a proper $\mathcal{L}$-submodule of $T(\lambda, \delta)$. Denote the quotient module $T(\lambda, \bar{\delta}) = T(\lambda, \delta)/H_0 T(\lambda, \delta) = \mathbb{C}[L_0]$, where $\bar{\delta}$ is the constant term of δ. It is easy to see that the actions of $\mathcal{L}$ on $T(\lambda, \bar{\delta})$ are

$$\begin{aligned} &L_m(f(L_0)) = \lambda^m f(L_0 - m)(L_0 + m\bar{\delta}), \\ &H_m(f(L_0)) = 0 \quad \text{for all } m \in \mathbb{Z},\ f(L_0) \in T(\lambda, \bar{\delta}). \end{aligned}$$

Furthermore, we have the following lemma.

Lemma 4.1. (1) *$T(\lambda, \bar{\delta})$ is an irreducible $\mathcal{L}$-module if and only if $\bar{\delta} \neq 0$.*

(2) *If $\bar{\delta} = 0$, then $T(\lambda, \bar{\delta})$ has an irreducible $\mathcal{L}$-submodule $L_0 T(\lambda, \bar{\delta})$ with codimension* 1.

Proof. This directly follows from the irreducibility of $\mathcal{L}$-module $\Omega(\lambda, \bar{\delta}, 0)$, which was introduced in Section 2B. □

By Theorem 3.1 in [Han et al. 2017], we have the following theorem.

Theorem 4.2. *Let M be a $\mathcal{U}(\mathcal{L})$-module such that M, when considered as a $\mathcal{U}(\mathfrak{h})$-module, is free of rank* 1. *Then $M \cong T(\lambda, \delta)$ for some $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$.*

Next, we investigate the $\mathcal{G}$-module structures on $\mathcal{U}(\mathfrak{h})$. We first define three families of $\mathcal{G}$-modules "$\Omega(\lambda, \delta, 0, 0)$, $\Omega(\lambda, \eta_1, \sigma_1, 0)$ and $\Omega(\lambda, \eta_2, 0, \sigma_2)$" as follows:

Definition 4.3. (1) For any $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$, the polynomial algebra $\mathbb{C}[H_0, L_0]$ can be endowed with a $\mathcal{G}$-module structure via the actions

$$
\begin{aligned}
&L_m(f(H_0, L_0)) = \lambda^m f(H_0, L_0-m)(L_0+m\delta),\\
&H_m(f(H_0, L_0)) = \lambda^m H_0 f(H_0, L_0-m), \qquad\qquad (4\text{-}2)\\
&I_m(\mathbb{C}[H_0, L_0]) = J_m(\mathbb{C}[H_0, L_0]) = 0 \ \text{ for all } m \in \mathbb{Z},\ f(H_0, L_0) \in \mathbb{C}[H_0, L_0].
\end{aligned}
$$

We denote this module by $\Omega(\lambda, \delta, 0, 0)$.

(2) For any $\lambda \in \mathbb{C}^*$, $\eta_1 \in \mathbb{C}$, $\sigma_1(\neq 0) \in \mathbb{C}[H_0]$, the polynomial algebra $\mathbb{C}[H_0, L_0]$ has a $\mathcal{G}$-module structure with the actions

$$
\begin{aligned}
L_m(f(H_0, L_0) &= \lambda^m f(H_0, L_0 - m)(L_0 - mH_0 + m\eta_1),\\
H_m(f(H_0, L_0) &= \lambda^m H_0 f(H_0, L_0 - m),\\
I_m(f(H_0, L_0) &= \lambda^m \sigma_1 f(H_0 - 1, L_0 - m),\\
J_m(\mathbb{C}[H_0, L_0) &= 0 \quad \text{for all } m \in \mathbb{Z},\ f(H_0, L_0) \in \mathbb{C}[H_0, L_0].
\end{aligned} \tag{4-3}
$$

This module is denoted by $\Omega(\lambda, \eta_1, \sigma_1, 0)$.

(3) For any $\lambda \in \mathbb{C}^*$, $\eta_2 \in \mathbb{C}$, $\sigma_2(\neq 0) \in \mathbb{C}[H_0]$, the polynomial algebra $\mathbb{C}[H_0, L_0]$ becomes a $\mathcal{G}$-module under the following actions

$$
\begin{aligned}
L_m(f(H_0, L_0) &= \lambda^m f(H_0, L_0 - m)(L_0 + mH_0 + m\eta_2),\\
H_m(f(H_0, L_0) &= \lambda^m H_0 f(H_0, L_0 - m),\\
I_m(\mathbb{C}[H_0, L_0) &= 0,\\
J_m(f(H_0, L_0) &= \lambda^m \sigma_2 f(H_0 + 1, L_0 - m),\\
&\qquad \text{for all } m \in \mathbb{Z},\ f(H_0, L_0) \in \mathbb{C}[H_0, L_0].
\end{aligned} \tag{4-4}
$$

Denote this module by $\Omega(\lambda, \eta_2, 0, \sigma_2)$.

Remark 4.4. (1) It is clear that $\Omega(\lambda, \delta, 0, 0)$ is a $\mathcal{G}$-module by (4-1), since $\widetilde{IJ}$ is an ideal of $\mathcal{G}$. By direct computations we can verify that $\Omega(\lambda, \eta_1, \sigma_1, 0)$ and $\Omega(\lambda, \eta_2, 0, \sigma_2)$ are $\mathcal{G}$-modules.

(2) These three families of $\mathcal{G}$-modules in Definition 4.3, when considered as $\mathcal{U}(\mathfrak{h})$-modules, are all free of rank 1.

In the rest of this subsection, we will show that the three families of $\mathcal{G}$-modules in Definition 4.3 exhaust all $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ up to isomorphism. We break the arguments into the following several lemmas.

From now on, throughout this subsection, N always denotes the $\mathcal{U}(\mathfrak{h})$-free module of rank 1 over $\mathcal{G}$. We identify N with $\mathbb{C}[H_0, L_0]$ as vector spaces. Moreover, it is clear that we can view N as a $\mathcal{U}(\mathfrak{h})$-free module of rank 1 over $\mathcal{L}$. Therefore, by Theorem 4.2 there exist $\lambda \in \mathbb{C}^*$, $\delta(H_0) \in \mathbb{C}[H_0]$ such that

$$\begin{aligned} &L_m(f(H_0, L_0)) = \lambda^m f(H_0, L_0 - m)(L_0 + m\delta(H_0)), \\ &H_m(f(H_0, L_0)) = \lambda^m H_0 f(H_0, L_0 - m) \quad \text{for all } m \in \mathbb{Z},\ f(H_0, L_0) \in N. \end{aligned} \tag{4-5}$$

Lemma 4.5. *The actions of $\mathcal{G}$ on N are completely determined by $L_m(1)$, $H_m(1)$, $I_m(1)$, $J_m(1)$ for all $m \in \mathbb{Z}$.*

Proof. For any $f(H_0, L_0) \in N$, using the commutation relations of $\mathcal{G}$ we see that

$$\begin{aligned} L_m(f(H_0, L_0)) &= f(H_0, L_0 - m)L_m(1), \\ H_m(f(H_0, L_0)) &= f(H_0, L_0 - m)H_m(1), \\ I_m(f(H_0, L_0)) &= f(H_0 - 1, L_0 - m)I_m(1), \\ J_m(f(H_0, L_0)) &= f(H_0 + 1, L_0 - m)J_m(1) \quad \text{for all } m \in \mathbb{Z}. \end{aligned}$$

So Lemma 4.5 is clear. □

From Lemma 4.5, we only need to determine the actions of L_m, H_m, I_m, J_m on 1 for all $m \in \mathbb{Z}$.

Lemma 4.6. *Assume that there exist $k, l \in \mathbb{Z}$ such that $I_k(1) = J_l(1) = 0$. Then $I_m(N) = J_m(N) = 0$ for all $m \in \mathbb{Z}$.*

Proof. For any $i, j \in \mathbb{Z}_+$, we have

$$\begin{aligned} I_k(H_0^i L_0^j) &= (H_0 - 1)^i I_k L_0^j = (H_0 - 1)^i (L_0 - k)^j I_k(1) = 0, \\ J_l(H_0^i L_0^j) &= (H_0 + 1)^i J_l L_0^j = (H_0 + 1)^i (L_0 - l)^j J_l(1) = 0. \end{aligned}$$

Thus $I_k(N) = J_l(N) = 0$. Using the defining relations of $\mathcal{G}$ we see that $I_m(N) = J_m(N) = 0$ for all $m \in \mathbb{Z}$. □

Lemma 4.7. *Suppose that $I_m(1)$ is nonzero for any $m \in \mathbb{Z}$. Denote $I_0(1) = \sum_{i=0}^{q_0} c_{0i}(H_0)L_0^i$, where $q_0 \in \mathbb{Z}_+$, $c_{0i}(H_0) \in \mathbb{C}[H_0]$ for $i = 0, 1, \dots, q_0$.*

(1) *In* (4-5), $\delta(H_0) = \alpha H_0 + \beta$, *for some* $\alpha \in \mathbb{Z}_{\geq -1}$, $\beta \in \mathbb{C}$.

(2) $\deg_{L_0}(I_m(1)) = \alpha + 1 = q_0$ *and*

$$I_m(1) = \lambda^m c_{0(\alpha+1)}(H_0)L_0^{\alpha+1} + (\text{lower} - \text{degree terms in } L_0) \quad \textit{for all } m \in \mathbb{Z}.$$

(3) *If $\alpha \geq 0$, then for any $m \in \mathbb{Z}^*$, the coefficient of L_0^α in $I_m(1)$ is*

$$m\lambda^m(\alpha + 1)c_{0(\alpha+1)}(H_0)\big(\alpha H_0 + \beta - \tfrac{1}{2}\alpha\big).$$

(4) *If $\alpha \geq 0$, then $\alpha = 1$.*

Proof. (1) For any $n \in \mathbb{Z}^*$, denote

$$I_n(1) = \sum_{i=0}^{q_n} c_{ni}(H_0) L_0^i,$$

where $q_n \in \mathbb{Z}_+$, $c_{ni}(H_0) \in \mathbb{C}[H_0]$ and $c_{nq_n}(H_0) \neq 0$. For any $m \in \mathbb{Z}$, we compute

$$\begin{aligned}
&(n-m) I_{m+n}(1) \\
&= [L_m, I_n](1) = L_m I_n(1) - I_n L_m(1) \\
&= \sum_{i=0}^{q_n} L_m c_{ni}(H_0) L_0^i - I_n\big(\lambda^m (L_0 + m\delta(H_0))\big) \\
&= \sum_{i=0}^{q_n} c_{ni}(H_0)(L_0 - m)^i L_m(1) - \big(\lambda^m (L_0 - n + m\delta(H_0 - 1))\big) I_n(1) \\
&= \sum_{i=0}^{q_n} c_{ni}(H_0)(L_0 - m)^i \lambda^m (L_0 + m\delta(H_0)) - \sum_{i=0}^{q_n} \lambda^m (L_0 - n + m\delta(H_0 - 1)) c_{ni}(H_0) L_0^i \\
&= \lambda^m (L_0 + m\delta(H_0)) \sum_{i=0}^{q_n} c_{ni}(H_0)(L_0 - m)^i - \lambda^m (L_0 - n + m\delta(H_0 - 1)) \sum_{i=0}^{q_n} c_{ni}(H_0) L_0^i.
\end{aligned}$$

In the last equality, the coefficients of $L_0^{q_n}$ and $L_0^{q_n - 1}$ are respectively

$$\lambda^m c_{nq_n}(H_0)\big(m\delta(H_0) - mq_n + n - m\delta(H_0 - 1)\big) \tag{4-6}$$

and

$$\begin{aligned}
m^2 q_n \lambda^m c_{nq_n}(H_0)\big(\tfrac{1}{2}(q_n - 1) - \delta(H_0)\big)& \\
+ \lambda^m c_{n(q_n-1)}(H_0)\big(m\delta(H_0) - m\delta(H_0 - 1) - mq_n + m + n\big)&.
\end{aligned} \tag{4-7}$$

Taking $m = n$, from equality (4-6) we deduce

$$n\lambda^n c_{nq_n}(H_0)\big(\delta(H_0) - q_n + 1 - \delta(H_0 - 1)\big) = 0,$$

which implies that $\delta(H_0) = \alpha H_0 + \beta$ and $q_n = \alpha + 1$ for some $\alpha, \beta \in \mathbb{C}$. Note that $q_n \in \mathbb{Z}_+$, thus $\alpha \in \mathbb{Z}_{\geq -1}$, $\beta \in \mathbb{C}$.

(2) From (1), we see that the equality (4-6) becomes

$$\lambda^m c_{nq_n}(H_0)(n - m).$$

Thus for any $m (\neq n) \in \mathbb{Z}$, we have $\deg_{L_0}(I_{m+n}(1)) = q_n = \alpha + 1$ and

$$I_{m+n}(1) = \lambda^m c_{nq_n}(H_0) L_0^{q_n} + (\text{lower} - \text{degree terms in } L_0). \tag{4-8}$$

Taking $m = -n$, we see that $q_0 = q_n = \alpha + 1, c_{0q_0}(H_0) = \lambda^{-n} c_{nq_n}(H_0)$. Using equality (4-8) we see that for any $m(\neq 2n) \in \mathbb{Z}$,

$$\begin{aligned} &\deg_{L_0}(I_m(1)) = \alpha + 1, \\ &I_m(1) = \lambda^m c_{0(\alpha+1)}(H_0) L_0^{\alpha+1} + (\text{lower} - \text{degree terms in } L_0), \end{aligned} \tag{4-9}$$

If we substitute n' for n in the beginning, where $n'(\neq n)$ is nonzero, then we can similarly deduce that (4-9) holds for any $m \neq 2n'$. Therefore, the equality (4-9) holds for any $m \in \mathbb{Z}$.

(3) Using (1) we see that equality (4-7) reads

$$m^2 q_n \lambda^m c_{nq_n}(H_0)\big(\tfrac{1}{2}(q_n - 1) - \delta(H_0)\big) + n\lambda^m c_{n(q_n-1)}(H_0).$$

Taking $m = n(\neq 0)$, we get

$$c_{m\alpha}(H_0) = c_{m(q_m-1)}(H_0) = m q_m c_{mq_m}(H_0)\big(\delta(H_0) - \tfrac{1}{2}(q_m - 1)\big). \tag{4-10}$$

So (3) is clear from (2) and equality (4-10).

(4) For $m, n \in \mathbb{Z}^*$, we may denote

$$I_n(1) = \sum_{j=0}^{\alpha+1} c_{nj}(H_0) L_0^j, \quad I_m(1) = \sum_{l=0}^{\alpha+1} c_{ml}(H_0) L_0^l,$$

where $c_{nj}(H_0), c_{ml}(H_0) \in \mathbb{C}[H_0]$ and $c_{n(\alpha+1)}(H_0), c_{m(\alpha+1)}(H_0) \neq 0$. We compute

$$\begin{aligned} 0 = [I_n, I_m](1) &= I_n I_m(1) - I_m I_n(1) \\ &= \sum_{l=0}^{\alpha+1} I_n c_{ml}(H_0) L_0^l - \sum_{j=0}^{\alpha+1} I_m c_{nj}(H_0) L_0^j \\ &= \sum_{j=0}^{\alpha+1} \sum_{l=0}^{\alpha+1} c_{nj}(H_0) c_{ml}(H_0 - 1) L_0^j (L_0 - n)^l \\ &\qquad - \sum_{j=0}^{\alpha+1} \sum_{l=0}^{\alpha+1} c_{nj}(H_0 - 1) c_{ml}(H_0)(L_0 - m)^j L_0^l. \end{aligned}$$

In the last equality, the coefficients of $L_0^{2\alpha+2}$ and $L_0^{2\alpha+1}$ are respectively

$$c_{n(\alpha+1)}(H_0) c_{m(\alpha+1)}(H_0 - 1) - c_{n(\alpha+1)}(H_0 - 1) c_{m(\alpha+1)}(H_0), \tag{4-11}$$

and

$$\begin{aligned} &c_{n(\alpha+1)}(H_0) c_{m(\alpha+1)}(H_0 - 1)(-n)(\alpha + 1) \\ &\quad + c_{n\alpha}(H_0) c_{m(\alpha+1)}(H_0 - 1) + c_{n(\alpha+1)}(H_0) c_{m\alpha}(H_0 - 1) \\ &\quad - \big(c_{n(\alpha+1)}(H_0 - 1) c_{m(\alpha+1)}(H_0)(-m)(\alpha + 1) \\ &\qquad + c_{n\alpha}(H_0 - 1) c_{m(\alpha+1)}(H_0) + c_{n(\alpha+1)}(H_0 - 1) c_{m\alpha}(H_0)\big). \end{aligned} \tag{4-12}$$

Using (2) and (3) we see that (4-11) and (4-12) read as

$$\lambda^{m+n}c_{0(\alpha+1)}(H_0)c_{0(\alpha+1)}(H_0-1)-\lambda^{m+n}c_{0(\alpha+1)}(H_0-1)c_{0(\alpha+1)}(H_0)$$

and

$$(n-m)(\alpha+1)(\alpha-1)\lambda^{m+n}c_{0(\alpha+1)}(H_0)c_{0(\alpha+1)}(H_0-1),$$

which implies

$$(n-m)(\alpha+1)(\alpha-1)\lambda^{m+n}c_{0(\alpha+1)}(H_0)c_{0(\alpha+1)}(H_0-1)=0$$

for any $m, n \in \mathbb{Z}^*$. Thus $\alpha = 1$. This completes the proof. □

Proposition 4.8. *Suppose that $I_m(1)$ is nonzero for any $m \in \mathbb{Z}$. Then $I_m(1) \in \mathbb{C}[H_0]$ for all $m \in \mathbb{Z}$.*

Proof. It is sufficient to show that $\alpha = -1$ by Lemma 4.7. Now we assume that $\alpha \geq 0$. Then $\alpha = 1$ by Lemma 4.7. Denote

$$I_0(1)=c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0),$$

where $c_{02}(H_0), c_{01}(H_0), c_{00}(H_0) \in \mathbb{C}[H_0]$ with $c_{02}(H_0) \neq 0$. Then by Lemma 4.7 we may write

$$I_1(1)=\lambda c_{02}(H_0)L_0^2+2\lambda\big(H_0+\beta-\tfrac{1}{2}\big)c_{02}(H_0)L_0+c_{10}(H_0)$$

for some $c_{10}(H_0) \in \mathbb{C}[H_0]$. We compute

$$\begin{aligned}
I_1(1)&=[H_1, I_0](1)=H_1I_0(1)-I_0H_1(1)\\
&=H_1\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big)-I_0(\lambda H_0)\\
&=\big(c_{02}(H_0)(L_0-1)^2+c_{01}(H_0)(L_0-1)+c_{00}(H_0)\big)(\lambda H_0)\\
&\qquad-\lambda(H_0-1)\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big)\\
&=\lambda c_{02}(H_0)L_0^2+\lambda\big(-2H_0c_{02}(H_0)+c_{01}(H_0)\big)L_0\\
&\qquad+\lambda\big(H_0c_{02}(H_0)-H_0c_{01}(H_0)+c_{00}(H_0)\big),\\
I_1(1)&=[I_0, L_1](1)=I_0L_1(1)-L_1I_0(1)\\
&=I_0\big(\lambda(L_0+H_0+\beta)\big)-L_1\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big)\\
&=\big(\lambda(L_0+H_0-1+\beta)\big)\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big)\\
&\qquad-\big(c_{02}(H_0)(L_0-1)^2+c_{01}(H_0)(L_0-1)+c_{00}(H_0)\big)\big(\lambda(L_0+H_0+\beta)\big)\\
&=\lambda c_{02}(H_0)L_0^2+2\lambda(H_0+\beta-\tfrac{1}{2})c_{02}(H_0)L_0\\
&\qquad-\lambda\big((c_{02}(H_0)-c_{01}(H_0))(H_0+\beta)+c_{00}(H_0)\big).
\end{aligned}$$

Then by comparing the coefficients of L_0 and the constant terms, we obtain

$$\lambda(-2H_0c_{02}(H_0)+c_{01}(H_0)) = 2\lambda\big(H_0+\beta-\tfrac{1}{2}\big)c_{02}(H_0),$$
$$\lambda\big(H_0c_{02}(H_0)-H_0c_{01}(H_0)+c_{00}(H_0)\big) = -\lambda\big((c_{02}(H_0)-c_{01}(H_0))(H_0+\beta)+c_{00}(H_0)\big).$$

Thus we deduce

$$\begin{aligned} c_{01}(H_0) &= (4H_0+2\beta-1)c_{02}(H_0),\\ c_{00}(H_0) &= (2H_0+\beta)(2H_0+\beta-1)c_{02}(H_0). \end{aligned} \tag{4-13}$$

Finally, we consider

$$\begin{aligned} 0 &= [I_1, I_0](1) = I_1I_0(1)-I_0I_1(1)\\ &= \big(c_{02}(H_0-1)(L_0-1)^2+c_{01}(H_0-1)(L_0-1)+c_{00}(H_0-1)\big)\\ &\qquad\times\big(\lambda c_{02}(H_0)L_0^2+2\lambda\big(H_0+\beta-\tfrac{1}{2}\big)c_{02}(H_0)L_0+c_{10}(H_0)\big)\\ &\quad-\big(\lambda c_{02}(H_0-1)L_0^2+2\lambda\big(H_0-1+\beta-\tfrac{1}{2}\big)c_{02}(H_0-1)L_0+c_{10}(H_0-1)\big)\\ &\qquad\times\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big). \end{aligned} \tag{4-14}$$

In the equality (4-14), the coefficient of L_0^3 is

$$\lambda\big(c_{02}(H_0)c_{01}(H_0-1)-c_{02}(H_0-1)c_{01}(H_0)\big). \tag{4-15}$$

Substituting (4-13) into (4-15), we get

$$-4\lambda c_{02}(H_0)c_{02}(H_0-1) = 0,$$

which implies $c_{02}(H_0) = 0$. This is a contradiction, completing ☐

Proposition 4.9. *Suppose that $J_m(1)$ is nonzero for any $m \in \mathbb{Z}$. Then $J_m(1) \in \mathbb{C}[H_0]$ for all $m \in \mathbb{Z}$.*

Proof. The proof is similar to that of Lemma 4.7 and Proposition 4.8. ☐

Lemma 4.10. *For any $m \in \mathbb{Z}$, $I_m(1) = \lambda^m I_0(1)$, $J_m(1) = \lambda^m J_0(1) \in \mathbb{C}[H_0]$.*

Proof. For any $m, n \in \mathbb{Z}$, using Proposition 4.8 and equality (4-5) we see that

$$\begin{aligned} I_{m+n}(1) &= [H_m, I_n](1) = H_mI_n(1) - I_nH_m(1)\\ &= H_m(1)I_n(1) - I_n(\lambda^m H_0) = \lambda^m H_0I_n(1) - \lambda^m(H_0-1)I_n(1)\\ &= \lambda^m I_n(1). \end{aligned}$$

Taking $n = 0$, we get $I_m(1) = \lambda^m I_0(1)$ for $m \in \mathbb{Z}$. Similarly, we may get $J_m(1) = \lambda^m J_0(1)$ for $m \in \mathbb{Z}$. ☐

Lemma 4.11. (1) *If $I_0(1) \neq 0$, then $\delta = -H_0+\eta'$ for some $\eta' \in \mathbb{C}$.*

(2) *If $J_0(1) \neq 0$, then $\delta = H_0+\eta''$ for some $\eta'' \in \mathbb{C}$.*

Proof. (1) For any $m, n \in \mathbb{Z}$, using Proposition 4.8 and equality (4-5) we compute

$$\begin{aligned}(n-m)I_{m+n}(1) &= [L_m, I_n](1) = L_m I_n(1) - I_n L_m(1)\\ &= L_m(1)I_n(1) - I_n\big(\lambda^m(L_0+m\delta(H_0))\big)\\ &= \lambda^m(L_0+m\delta(H_0))I_n(1) - \lambda^m(L_0-n)I_n(1) - m\lambda^m\delta(H_0-1)I_n(1)\\ &= \lambda^m\big(m(\delta(H_0)-\delta(H_0-1))+n\big)I_n(1),\end{aligned}$$

which yields $(n-m) = \big(m(\delta(H_0)-\delta(H_0-1))+n\big)$ by Lemma 4.10 and $I_0(1) \neq 0$. Thus $\delta(H_0)-\delta(H_0-1) = -1$, which forces $\delta(H_0) = -H_0+\eta'$ for some $\eta' \in \mathbb{C}$.

(2) Proved similarly to (1). □

Now we state the main results of this subsection.

Theorem 4.12. *Let N be a $\mathcal{U}(\mathcal{G})$-module such that N, when considered as a $\mathcal{U}(\mathfrak{h})$-module, is free of rank* 1.

(a) *There exist $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$ such that $L_1(1) = \lambda(L_0+\delta)$, $H_1(1) = \lambda H_0$.*

(b) *If $I_0(1) = J_0(1) = 0$, then $N \cong \Omega(\lambda, \delta, 0, 0)$ as $\mathcal{U}(\mathcal{G})$-modules.*

(c) *If $I_0(1) \neq 0$, $J_0(1) = 0$, then $\delta = -H_0+\eta'$ for some $\eta' \in \mathbb{C}$, and $N \cong \Omega(\lambda, \eta', \sigma_1, 0)$ as $\mathcal{U}(\mathcal{G})$-modules, where $\sigma_1 = I_0(1) \in \mathbb{C}[H_0]$.*

(d) *If $I_0(1) = 0$, $J_0(1) \neq 0$, then $\delta = H_0+\eta''$ for some $\eta'' \in \mathbb{C}$, and $N \cong \Omega(\lambda, \eta'', 0, \sigma_2)$ as $\mathcal{U}(\mathcal{G})$-modules, where $\sigma_2 = J_0(1) \in \mathbb{C}[H_0]$.*

(e) *The case $I_0(1) \neq 0$, $J_0(1) \neq 0$ does not exist.*

Proof. (a) follows from equality (4-5).

(b) follows from Lemmas 4.5, 4.6 and equalities (4-2), (4-5).

(c) and (d) follow from Lemmas 4.5, 4.6, 4.10, 4.11 and equalities (4-3), (4-4), (4-5).

(e) follows from Lemma 4.11. □

4B. *Irreducibility of $\mathcal{U}(\mathfrak{h})$-free modules over $\mathcal{G}$.* In Section 4A, we determined all $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$. These modules have three families $\Omega(\lambda, \delta, 0, 0)$, $\Omega(\lambda, \eta_1, \sigma_1, 0)$ and $\Omega(\lambda, \eta_2, 0, \sigma_2)$ (see Definition 4.3). Here we will give the necessary and sufficient conditions for these modules to be irreducible. Furthermore, we find the maximal proper submodules and obtain irreducible quotient modules when these modules are not irreducible.

Theorem 4.13. *Let $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$. $\bar{\delta}$ denotes the constant term of δ.*

(1) *$\Omega(\lambda, \delta, 0, 0)$ always has a proper $\mathcal{G}$-submodule $H_0\Omega(\lambda, \delta, 0, 0)$. Denote the quotient module $\Omega_1(\lambda, \delta, 0, 0) = \Omega(\lambda, \delta, 0, 0)/H_0\Omega(\lambda, \delta, 0, 0)$.*

(2) *$\Omega_1(\lambda, \delta, 0, 0)$ is an irreducible $\mathcal{G}$-module if and only if $\bar{\delta} \neq 0$.*

(3) $\Omega_1(\lambda, \delta, 0, 0)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_1(\lambda, \delta, 0, 0)$ *with codimension* 1 *when* $\bar{\delta} = 0$. *Consequently, the quotient module*

$$\Omega_1(\lambda, \delta, 0, 0)/L_0\Omega_1(\lambda, \delta, 0, 0)$$

is irreducible.

Proof. These directly follow from the properties of $\mathcal{L}$-module $T(\lambda, \delta)$, which were described in Lemma 4.1. □

Theorem 4.14. *Let* $\lambda \in \mathbb{C}^*,\ \eta_1 \in \mathbb{C},\ \sigma_1(\neq 0) \in \mathbb{C}[H_0]$.

(1) $\Omega(\lambda, \eta_1, \sigma_1, 0)$ *is an irreducible* $\mathcal{G}$*-module if and only if* $\sigma_1 \in \mathbb{C}^*$.

(2) *If* $\sigma_1 = H_0 + \beta$, *where* $\beta \in \mathbb{C}$, *then* $\Omega(\lambda, \eta_1, \sigma_1, 0)$ *has a proper* $\mathcal{G}$*-submodule* $\sigma_1\Omega(\lambda, \eta_1, \sigma_1, 0)$. *Moreover, denote the quotient module*

$$\Omega_1(\lambda, \eta_1, \sigma_1, 0) = \Omega(\lambda, \eta_1, \sigma_1, 0)/\sigma_1\Omega(\lambda, \eta_1, \sigma_1, 0) = \mathbb{C}[L_0].$$

(i) $\Omega_1(\lambda, \eta_1, \sigma_1, 0)$ *is irreducible if and only if* $(\eta_1, \beta) \neq (0, 0)$.

(ii) $\Omega_1(\lambda, \eta_1, \sigma_1, 0)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_1(\lambda, \eta_1, \sigma_1, 0)$ *with codimension* 1 *when* $(\eta_1, \beta) = (0, 0)$. *Consequently,*

$$\Omega_1(\lambda, \eta_1, \sigma_1, 0)/L_0\Omega_1(\lambda, \eta_1, \sigma_1, 0)$$

is irreducible.

(3) *If* $\deg(\sigma_1) = n > 1$, *we may write*

$$\sigma_1 = c\sigma_{11}\sigma_{12}\cdots\sigma_{1n},$$

where $\sigma_{1i} = H_0 + \beta_i$, $\beta_i \in \mathbb{C}$, $c \in \mathbb{C}^*$, *for* $i = 1, 2, \ldots, n$. *Then* $\sigma_{1i}\Omega(\lambda, \eta_1, \sigma_1, 0)$ *is a proper* $\mathcal{G}$*-submodule of* $\Omega(\lambda, \eta_1, \sigma_1, 0)$ *for* $i = 1, 2, \ldots, n$. *Furthermore, denote the quotient module*

$$\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0) = \Omega(\lambda, \eta_1, \sigma_1, 0)/\sigma_{1i}\Omega(\lambda, \eta_1, \sigma_1, 0).$$

(i) $\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)$ *is irreducible if and only if* $(\eta_1, \beta_i) \neq (0, 0)$.

(ii) $\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)$ *with codimension* 1 *when* $(\eta_1, \beta_i) = (0, 0)$. *Consequently,*

$$\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)/L_0\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)$$

is irreducible.

Proof. (1) ($\Rightarrow$). Let $\Omega(\lambda, \eta_1, \sigma_1, 0)$ be an irreducible $\mathcal{G}$-module. Assume that $\deg_{H_0}(\sigma_1) \geq 1$. It is easy to see that $\sigma_1\Omega(\lambda, \eta_1, \sigma_1, 0)$ is a proper $\mathcal{G}$-submodule of $\Omega(\lambda, \eta_1, \sigma_1, 0)$, which contradicts that $\Omega(\lambda, \eta_1, \sigma_1, 0)$ is irreducible.

($\Leftarrow$). Suppose $\sigma_1 \in \mathbb{C}^*$. For arbitrary nonzero $f(H_0, L_0) \in \Omega(\lambda, \eta_1, \sigma_1, 0)$, we write

$$f(H_0, L_0) = \sum_{j=0}^{q} a_j(H_0) L_0^j,$$

where $q \in \mathbb{Z}_+, a_j(H_0) \in \mathbb{C}[H_0], a_q(H_0) \neq 0$. Let $\langle f(H_0, L_0)\rangle$ denote the $\mathcal{G}$-submodule of $\Omega(\lambda, \eta_1, \sigma_1, 0)$ generated by $f(H_0, L_0)$.

If $q > 0$, we compute

$$\begin{aligned} &H_1(f(H_0, L_0)) - \lambda H_0 f(H_0, L_0) \\ &\quad = \lambda H_0 \sum_{j=0}^{q} a_j(H_0)(L_0 - 1)^j - \lambda H_0 \sum_{j=0}^{q} a_j(H_0) L_0^j \\ &\quad = -q\lambda H_0 a_q(H_0) L_0^{q-1} + (\text{lower} - \text{degree terms in } L_0). \end{aligned}$$

Denote

$$f_1(H_0, L_0) = H_1(f(H_0, L_0)) - \lambda H_0 f(H_0, L_0) \in \langle f(H_0, L_0)\rangle$$

with $\deg_{L_0}(f_1(H_0, L_0)) = q - 1$. Therefore, without loss of generality, we may assume that $\deg_{L_0}(f H_0, L_0) = q = 0$. Then we write

$$f(H_0, L_0) = \sum_{i=0}^{p} c_i H_0^i,$$

where $p \in \mathbb{Z}_+, c_i \in \mathbb{C}$ with $c_p \neq 0$.

If $p = 0$, then $\langle f(H_0, L_0)\rangle = \Omega(\lambda, \eta_1, \sigma_1, 0)$ is clear. If $p > 0$, we deduce that

$$\begin{aligned} &I_1(f(H_0, L_0)) - \lambda\sigma_1 f(H_0, L_0) \\ &\quad = \lambda\sigma_1 \sum_{i=0}^{p} c_i (H_0 - 1)^i - \lambda\sigma_1 \sum_{i=0}^{p} c_i H_0^i \\ &\quad = -\lambda\sigma_1 p c_p H_0^{p-1} + (\text{lower} - \text{degree terms in } H_0). \end{aligned}$$

Thus we can get $1 \in \langle f(H_0, L_0)\rangle$, which implies $\langle f(H_0, L_0)\rangle = \Omega(\lambda, \eta_1, \sigma_1, 0)$. Hence $\Omega(\lambda, \eta_1, \sigma_1, 0)$ is irreducible.

(2) First, it is trivial to see that $\sigma_1 \Omega(\lambda, \eta_1, \sigma_1, 0)$ is a proper $\mathcal{G}$-submodule of $\Omega(\lambda, \eta_1, \sigma_1, 0)$. From equality (4-3), we see that the actions of $\mathcal{G}$ on the quotient module $\Omega_1(\lambda, \eta_1, \sigma_1, 0)$ are

$$\begin{aligned} L_m(f(L_0)) &= \lambda^m f(L_0 - m)(L_0 + m\beta + m\eta_1), \\ H_m(f(L_0)) &= -\lambda^m \beta f(L_0 - m), \\ I_m(\Omega_1(\lambda, \eta_1, \sigma_1, 0)) &= J_m(\Omega_1(\lambda, \eta_1, \sigma_1, 0)) = 0 \quad \text{for all } m \in \mathbb{Z}. \end{aligned}$$

Then (i), (ii) follow from irreducibility of $\mathcal{L}$-module $\Omega(\lambda, \beta + \eta_1, -\beta)$, which was introduced in Section 2B.

(3) It is clear that $\sigma_{1i} \Omega(\lambda, \eta_1, \sigma_1, 0)$ is a proper $\mathcal{G}$-submodule of $\Omega(\lambda, \eta_1, \sigma_1, 0)$. The remaining parts are similar to (2). $\square$

Theorem 4.15. *Let* $\lambda \in \mathbb{C}^*$, $\eta_2 \in \mathbb{C}$, $\sigma_2(\neq 0) \in \mathbb{C}[H_0]$.

(1) $\Omega(\lambda, \eta_2, 0, \sigma_2)$ *is an irreducible* $\mathcal{G}$*-module if and only if* $\sigma_2 \in \mathbb{C}^*$.

(2) *If* $\sigma_2 = H_0 + \gamma$, *where* $\gamma \in \mathbb{C}$, *then* $\Omega(\lambda, \eta_2, 0, \sigma_2)$ *has a proper* $\mathcal{G}$*-submodule* $\sigma_2\Omega(\lambda, \eta_2, 0, \sigma_2)$. *Moreover, denote the quotient module*

$$\Omega_2(\lambda, \eta_2, 0, \sigma_2) = \Omega(\lambda, \eta_2, 0, \sigma_2)/\sigma_2\Omega(\lambda, \eta_2, 0, \sigma_2) = \mathbb{C}[L_0].$$

(i) $\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *is irreducible if and only if* $(\eta_2, \gamma) \neq (0, 0)$.

(ii) $\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *with codimension* 1 *when* $(\eta_2, \gamma) = (0, 0)$. *Consequently*,

$$\Omega_2(\lambda, \eta_2, 0, \sigma_2)/L_0\Omega_2(\lambda, \eta_2, 0, \sigma_2)$$

is irreducible.

(3) *If* $\deg(\sigma_2) = n > 1$, *we may write*

$$\sigma_2 = c'\sigma_{21}\sigma_{22}\cdots\sigma_{2n},$$

where $\sigma_{2i} = H_0+\gamma_i$, $\gamma_i \in \mathbb{C}$, $c' \in \mathbb{C}^*$, *for* $i = 1, 2, \ldots, n$. *Then* $\sigma_{2i}\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *is a proper* $\mathcal{G}$*-submodule of* $\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *for* $i = 1, 2, \ldots, n$. *Furthermore, denote* $\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2) = \Omega_2(\lambda, \eta_2, 0, \sigma_2)/\sigma_{2i}\Omega_2(\lambda, \eta_2, 0, \sigma_2)$.

(i) $\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)$ *is irreducible if and only if* $(\eta_2, \gamma_i) \neq (0, 0)$.

(ii) $\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)$ *with codimension* 1 *when* $(\eta_2, \gamma_i) = (0, 0)$. *Consequently*,

$$\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)/L_0\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)$$

is irreducible.

Proof. The proof is similar to that of Theorem 4.14. □

Remark 4.16. By Theorems 3.2, 4.13, 4.14 and 4.15, we may get many new irreducible modules over the planar Galilean conformal algebra $\mathcal{G}$.

4C. ***Isomorphism classes of*** $\mathcal{U}(\mathfrak{h})$***-free modules over*** $\mathcal{G}$**.** In Section 4A, we showed that three families of modules $\Omega(\lambda, \delta, 0, 0)$, $\Omega(\lambda, \eta_1, \sigma_1, 0)$ and $\Omega(\lambda, \eta_2, 0, \sigma_2)$ exhaust all $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$. Now we determine the isomorphism classes of these modules.

Theorem 4.17. *Let* $\lambda, \lambda' \in \mathbb{C}^*$, $\delta, \delta' \in \mathbb{C}[H_0]$, $\eta_1, \eta_1', \eta_2, \eta_2' \in \mathbb{C}$, $\sigma_1, \sigma_1', \sigma_2, \sigma_2' \in \mathbb{C}[H_0] \setminus \{0\}$.

(1) $\Omega(\lambda, \delta, 0, 0) \cong \Omega(\lambda', \delta', 0, 0)$ *if and only if* $\lambda = \lambda'$, $\delta = \delta'$.

(2) $\Omega(\lambda, \eta_1, \sigma_1, 0) \cong \Omega(\lambda', \eta_1', \sigma_1', 0)$ *if and only if* $\lambda = \lambda'$, $\eta_1 = \eta_1'$, $\sigma_1 = \sigma_1'$.

(3) $\Omega(\lambda, \eta_2, 0, \sigma_2) \cong \Omega(\lambda', \eta_2', 0, \sigma_2')$ *if and only if* $\lambda = \lambda'$, $\eta_2 = \eta_2'$, $\sigma_2 = \sigma_2'$.

(4) *Any two of* $\Omega(\lambda, \delta, 0, 0)$, $\Omega(\lambda, \eta_1, \sigma_1, 0)$, $\Omega(\lambda, \eta_2, 0, \sigma_2)$ *are not isomorphic.*

Proof. (1) The "sufficiency" is trivial. We only need to show the "necessity". Suppose

$$\varphi : \Omega(\lambda, \delta, 0, 0) \to \Omega(\lambda', \delta', 0, 0)$$

is a $\mathcal{G}$-module isomorphism.

Claim 1. $\varphi(1) \in \mathbb{C}[H_0]$.

Assume that $\varphi(1) = \sum_{i=0}^{q} a_i(H_0)L_0^i$, where $q > 0, a_i(H_0) \in \mathbb{C}[H_0]$ for $0 \le i \le q$ and $a_q(H_0) \neq 0$. Since φ is a $\mathcal{G}$-module isomorphism, we get $H_1(\varphi(1)) = \varphi(H_1(1)) = \varphi(\lambda H_0) = \lambda H_0(\varphi(1))$. From equality (4-2) we obtain

$$\begin{aligned} H_1(\varphi(1)) &= \lambda' H_0 \sum_{i=0}^{q} a_i(H_0)(L_0-1)^i \\ &= \lambda' H_0 a_q(H_0)L_0^q + \lambda' H_0(-q a_q(H_0) + a_{q-1}(H_0))L_0^{q-1} \\ &\qquad\qquad\qquad\qquad + (\text{lower}-\text{degree terms in } L_0), \\ \lambda H_0(\varphi(1)) &= \lambda H_0 \sum_{i=0}^{q} a_i(H_0)L_0^i \\ &= \lambda H_0 a_q(H_0)L_0^q + \lambda H_0 a_{q-1}(H_0)L_0^{q-1} + (\text{lower}-\text{degree terms in } L_0). \end{aligned}$$

By comparing the coefficients of L_0^q and L_0^{q-1}, we deduce

$$\lambda = \lambda', \quad -\lambda' q H_0 a_q(H_0) = 0.$$

But $-\lambda' q H_0 a_q(H_0) = 0$ is impossible. So $\varphi(1) \in \mathbb{C}[H_0]$. Claim 1 is proved.

Now we may assume $\varphi(1) = \sum_{j=0}^{p} c_j H_0^j$, where $p \in \mathbb{Z}_+, c_j \in \mathbb{C}$ for $0 \le j \le p$ and $c_p \neq 0$. We consider the equality

$$L_1(\varphi(1)) = \varphi(L_1(1)) = \varphi(\lambda(L_0+\delta)) = \lambda(L_0+\delta)(\varphi(1)).$$

It is clear that

$$L_1(\varphi(1)) = \lambda'\varphi(1)(L_0+\delta'), \quad \lambda(L_0+\delta)(\varphi(1)) = \lambda\varphi(1)(L_0+\delta),$$

which imply $\lambda = \lambda', \delta = \delta'$.

(2) The "sufficiency" is clear. We only need to show the "necessity". Suppose that

$$\varphi' : \Omega(\lambda, \eta_1, \sigma_1, 0) \to \Omega(\lambda', \eta_1', \sigma_1', 0)$$

is a $\mathcal{G}$-module isomorphism. Then $\varphi' : \Omega(\lambda, \eta_1, \sigma_1, 0) \to \Omega(\lambda', \eta_1', \sigma_1', 0)$ is an $\mathcal{L}$-module isomorphism. From (1) and equalities (4-2), (4-3) it is not hard to see that $\lambda = \lambda'$, $\eta_1 = \eta_1'$ and $\varphi'(1) \in \mathbb{C}[H_0]$.

Set $\varphi'(1) = \sum_{k=0}^{t} d_k H_0^k$, where $t \in \mathbb{Z}_+$, $d_k \in \mathbb{C}$ for $0 \le k \le t$ and $d_t \ne 0$. Note that $\varphi'(\lambda\sigma_1) = \varphi'(I_1(1))$. We compute

$$\varphi'(\lambda\sigma_1) = \lambda\sigma_1\varphi'(1) = \lambda\sigma_1 \sum_{k=0}^{t} d_k H_0^k,$$
$$\varphi'(I_1(1)) = I_1(\varphi'(1)) = \lambda'\sigma_1' \sum_{k=0}^{t} d_k (H_0 - 1)^k.$$

By comparing the coefficients of H_0^t we obtain $\sigma_1 = \sigma_1'$.

(3) is similar to (2).

(4) is trivial. □

Remark 4.18. We give a complete classification of $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ and $\bar{\mathcal{G}}$ by Theorems 3.2, 3.3, 4.12, 4.17 and Corollary 2.3.

Acknowledgments

This research work is partially supported by NSF of China (Grants 11701340, 11931009, 12071150). J. Cheng is partially supported by Shandong Provincial Natural Science Foundation of China (Grant ZR2022MA072) and undergraduate education reform project of Shandong Normal University (Grant 2021BJ054). D. Gao is partially supported by National Postdoctoral Program for Innovative Talents of China (Grant BX20220158), China Postdoctoral Science Foundation (Grant 2022M711708), CSC of China (No. 201906340096) and Nankai Zhide Foundation. The authors thank the referees for valuable suggestions to improve the paper.

References

[Aizawa 2013] N. Aizawa, “Some properties of planar Galilean conformal algebras”, pp. 301–309 in *Lie theory and its applications in physics*, edited by V. Dobrev, Springer Proc. Math. Stat. **36**, Springer, 2013. MR Zbl

[Aizawa and Kimura 2011] N. Aizawa and Y. Kimura, “Galilean conformal algebras in two spatial dimension”, 2011. arXiv 1112.0634

[Bagchi 2010] A. Bagchi, “Correspondence between asymptotically flat spacetimes and nonrelativistic conformal field theories”, *Phys. Rev. Lett.* **105**:17 (2010), art. id. 171601. MR

[Bagchi 2013] A. Bagchi, “Tensionless strings and Galilean conformal algebra”, *J. High Energy Phys.* 5 (2013), art. id. 141. MR Zbl

[Bagchi and Gopakumar 2009] A. Bagchi and R. Gopakumar, “Galilean conformal algebras and AdS/CFT”, *J. High Energy Phys.* 7 (2009), art. id. 037. MR

[Bagchi et al. 2010] A. Bagchi, R. Gopakumar, I. Mandal, and A. Miwa, “GCA in 2d”, *J. High Energy Phys.* 8 (2010), art. id. 004. MR Zbl

[Bagchi et al. 2014] A. Bagchi, R. Basu, and A. Mehra, “Galilean conformal electrodynamics”, *J. High Energy Phys.* 11 (2014), art. id. 61.

[Bhattacharyya et al. 2009] S. Bhattacharyya, S. Minwalla, and S. R. Wadia, "The incompressible non-relativistic Navier–Stokes equation from gravity", *J. High Energy Phys.* 8 (2009), art. id. 059. MR

[Cai et al. 2020] Y.-A. Cai, H. Tan, and K. Zhao, "New representations of affine Kac–Moody algebras", *J. Algebra* **547** (2020), 95–115. MR Zbl

[Chen and Guo 2017] H. Chen and X. Guo, "Non-weight modules over the Heisenberg–Virasoro algebra and the W algebra $W(2,2)$", *J. Algebra Appl.* **16**:5 (2017), art. id. 1750097. MR Zbl

[Chen and Yao 2023] Q.-F. Chen and Y.-F. Yao, "Simple restricted modules for the universal central extension of the planar Galilean conformal algebra", *J. Algebra* **634** (2023), 698–721. MR Zbl

[Chen et al. 2022] Q.-F. Chen, Y.-F. Yao, and H.-Y. Yang, "Whittaker modules for the planar Galilean conformal algebra and its central extension", *Comm. Algebra* **50**:12 (2022), 5042–5059. MR Zbl

[Festuccia et al. 2016] G. Festuccia, D. Hansen, J. Hartong, and N. A. Obers, "Symmetries and couplings of non-relativistic electrodynamics", *J. High Energy Phys.* 11 (2016), art. id. 037. MR Zbl

[Fouxon and Oz 2008] I. Fouxon and Y. Oz, "Conformal field theory as microscopic dynamics of incompressible Euler and Navier–Stokes equations", *Phys. Rev. Lett.* **101**:26 (2008), art. id. 261602. MR

[Fouxon and Oz 2009] I. Fouxon and Y. Oz, "CFT hydrodynamics: symmetries, exact solutions and gravity", *J. High Energy Phys.* 3 (2009), art. id. 120. MR

[Gao and Gao 2022] D. Gao and Y. Gao, "Representations of the planar Galilean conformal algebra", *Comm. Math. Phys.* **391**:1 (2022), 199–221. MR Zbl

[Gao et al. 2016] S. Gao, D. Liu, and Y. Pei, "Structure of the planar Galilean conformal algebra", *Rep. Math. Phys.* **78**:1 (2016), 107–122. MR Zbl

[Gusyatnikova and Yumaguzhin 1989] V. N. Gusyatnikova and V. A. Yumaguzhin, "Symmetries and conservation laws of Navier–Stokes equations", pp. 65–81 in *Symmetries of partial differential equations, I*, vol. 15, edited by A. M. Vinogradov, 1989. MR Zbl

[Han et al. 2017] J. Han, Q. Chen, and Y. Su, "Modules over the algebra $\mathcal{V}ir(a,b)$", *Linear Algebra Appl.* **515** (2017), 11–23. MR Zbl

[Henkel et al. 2012] M. Henkel, R. Schott, S. Stoimenov, and J. Unterberger, "The Poincaré algebra in the context of ageing systems: Lie structure, representations, Appell systems and coherent states", *Confluentes Math.* **4**:4 (2012), art. id. 1250006. MR

[Lu and Zhao 2014] R. Lu and K. Zhao, "Irreducible Virasoro modules from irreducible Weyl modules", *J. Algebra* **414** (2014), 271–287. MR Zbl

[Maldacena 1998] J. Maldacena, "The large N limit of superconformal field theories and supergravity", *Adv. Theor. Math. Phys.* **2**:2 (1998), 231–252. MR Zbl

[Martelli and Tachikawa 2010] D. Martelli and Y. Tachikawa, "Comments on Galilean conformal field theories and their geometric realization", *J. High Energy Phys.* 5 (2010), art. id. 091. MR Zbl

[Nilsson 2015] J. Nilsson, "Simple $\mathfrak{sl}_{n+1}$-module structures on $\mathcal{U}(\mathfrak{h})$", *J. Algebra* **424** (2015), 294–329. MR Zbl

[Tan and Zhao 2015] H. Tan and K. Zhao, "$\mathcal{W}_n^+$- and $\mathcal{W}_n$-module structures on $U(\mathfrak{h}_n)$", *J. Algebra* **424** (2015), 357–375. MR

[Zhang and Dong 2009] W. Zhang and C. Dong, "W-algebra $W(2,2)$ and the vertex operator algebra $L(\frac{1}{2},0)\otimes L(\frac{1}{2},0)$", *Comm. Math. Phys.* **285**:3 (2009), 991–1004. MR

Received March 26, 2023. Revised November 21, 2023.

JIN CHENG
SCHOOL OF MATHEMATICS AND STATISTICS
SHANDONG NORMAL UNIVERSITY
JINAN
CHINA
cheng934@mail.ustc.edu.cn

DONGFANG GAO
CHERN INSTITUTE OF MATHEMATICS AND LPMC
NANKAI UNIVERSITY
TIANJIN
CHINA
gaodfw@mail.ustc.edu.cn

ZITING ZENG
SCHOOL OF MATHEMATICAL SCIENCES
BEIJING NORMAL UNIVERSITY
BEIJING
CHINA
zengzt@bnu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 326, No. 2, 2023
https://doi.org/10.2140/pjm.2023.326.251

F-ALGEBROIDS AND DEFORMATION QUANTIZATION VIA PRE-LIE ALGEBROIDS

JOHN ALEXANDER CRUZ MORALES, JIEFENG LIU AND YUNHE SHENG

First we introduce the notion of F-algebroids, which is a generalization of F-manifold algebras and F-manifolds, and show that F-algebroids are the corresponding semiclassical limits of pre-Lie formal deformations of commutative associative algebroids. Then we use the deformation cohomology of pre-Lie algebroids to study pre-Lie infinitesimal deformations and extension of pre-Lie n-deformations to pre-Lie $(n+1)$-deformations of a commutative associative algebroid. Next we develop the theory of Dubrovin's dualities of F-algebroids with eventual identities and use Nijenhuis operators on F-algebroids to construct new F-algebroids. Finally we introduce the notion of pre-F-algebroids, which is a generalization of F-manifolds with compatible flat connections. Dubrovin's dualities of pre-F-algebroids with eventual identities, Nijenhuis operators on pre-F-algebroids are discussed.

1. Introduction

The concept of Frobenius manifolds was introduced by Dubrovin [15] as a geometrical manifestation of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) associativity equations in the 2-dimensional topological field theories. Hertling and Manin [17] weakened the conditions of a Frobenius manifold and introduced the notion of an F-manifold. Any Frobenius manifold has an underlying F-manifold structure. F-manifolds appear in many fields of mathematics such as singularity theory [16],

MSC2020: primary 53D17, 53D45, 53D50, 53D55; secondary 14J33.
Keywords: F-algebroids, pre-F-algebroids, eventual identity, Nijenhuis operator.

integrable systems [1; 3; 4; 12; 13; 25; 27], quantum K-theory [21], information geometry [10], operad [30] and so on.

The notion of a Lie algebroid was introduced by Pradines in 1967, which is a generalization of Lie algebras and tangent bundles. Just as Lie algebras are the infinitesimal objects of Lie groups, Lie algebroids are the infinitesimal objects of Lie groupoids. See [28] for the general theory about Lie algebroids. Lie algebroids are now an active domain of research, with applications in various parts of mathematics, such as geometric mechanics, foliation theory, Poisson geometry, differential equations, singularity theory, operad and so on. The notion of a pre-Lie algebroid (also called a left-symmetric algebroid or a Koszul–Vinberg algebroid) is a geometric generalization of a pre-Lie algebra. Pre-Lie algebras arose from the study of convex homogeneous cones, affine manifolds and affine structures on Lie groups, deformation and cohomology theory of associative algebras and then appear in many fields in mathematics and mathematical physics. See the survey article [7] for more details on pre-Lie algebras and [5; 6; 22; 23] for more details on cohomology and applications of pre-Lie algebroids. Dotsenko [14] showed that the graded object of the filtration of the operad encoding pre-Lie algebras is the operad encoding F-manifold algebras, where the notion of an F-manifold algebra is the underlying algebraic structure of an F-manifold. In [24], the notion of pre-Lie formal deformations of commutative associative algebras was introduced and it was shown that F-manifold algebras are the corresponding semiclassical limits. This result is parallel to the fact that the semiclassical limit of an associative formal deformation of a commutative associative algebra is a Poisson algebra.

In this paper, we introduce the notion of F-algebroids, which is a generalization of F-manifold algebras and F-manifolds. There is a slight difference between this F-algebroid and the one introduced in [11]. We introduce the notion of pre-Lie formal deformations of commutative associative algebroids and show that F-algebroids are the corresponding semiclassical limits. Viewing a commutative associative algebroid as a pre-Lie algebroid, we show that pre-Lie infinitesimal deformations and extension of pre-Lie n-deformations to pre-Lie $(n+1)$-deformations of a commutative associative algebroid are classified by the second and third cohomology groups of the pre-Lie algebroid respectively.

F-manifolds with eventual identities were introduced by Manin [29] and then were studied systematically by David and Strachan [13]. We generalize Dubrovin's dualities of F-manifolds with eventual identities to the case of F-algebroids. We introduce the notion of (pseudo)eventual identities on F-algebroids and develop the theory of Dubrovin's dualities of F-algebroids with eventual identities. We introduce the notion of Nijenhuis operators on F-algebroids and use them to construct new F-algebroids. In particular, a pseudoeventual identity naturally gives a Nijenhuis operator on an F-algebroid.

The notion of an F-manifold with a compatible flat connection was introduced by Manin [29]. Applications of F-manifolds with compatible flat connections also appeared in Painlevé equations [2; 3; 18; 25] and integrable systems [1; 4; 19; 26; 27]. We introduce the notion of pre-F-algebroids, which is a generalization of F-manifolds with compatible flat connections. A pre-F-algebroid gives rise to an F-algebroid. We also study pre-F-algebroids with eventual identities and give a characterization of such eventual identities. Furthermore, the theory of Dubrovin's dualities of pre-F-algebroids with eventual identities were developed. We introduce the notion of a Nijenhuis operator on a pre-F-algebroid, and show that a Nijenhuis operator gives rise to a deformed pre-F-algebroid.

Mirror symmetry, roughly speaking, is a duality between symplectic and complex geometry. The theory of Frobenius and F-manifolds plays an important role in this duality. We expect that the notion of F-algebroids might also be relevant in understanding the mirror phenomenon. In particular, the Dubrovin's dual of F-algebroids constructed in this paper should be related to the mirror construction along the way the Dubrovin's dual of Frobenius manifolds is related, at least in some situations, with mirror symmetry. More precisely the question is: Could we consider the construction of Dubrovin's dual of F-algebroids as a kind of mirror construction? In order to answer the question above, we might need to add some extra structures to F-algebroids and include those structures in the construction of the Dubrovin's dual. This would allow us to give a comprehensible interpretation of our construction as a manifestation of a mirror phenomenon. We want to follow this line of thought in future works.

The paper is organized as follows. In Section 2, we introduce the notion of F-algebroids and give some constructions of F-algebroids including the action F-algebroids and direct product F-algebroids. In particular, we show that Poisson manifolds give rise to action F-algebroids naturally. In Section 3, we study pre-Lie formal deformations of a commutative associative algebroid, whose semiclassical limits are F-algebroids. We show that the equivalence classes of pre-Lie infinitesimal deformations of a commutative associative algebroid A are classified by the second cohomology group in the deformation cohomology of A. Furthermore, we study extensions of pre-Lie n-deformations to pre-Lie $(n+1)$-deformations of a commutative associative algebroid A and show that a pre-Lie n-deformation is extendable if and only if its obstruction class in the third cohomology group of the commutative associative algebroid A is trivial. In Section 4, we first study Dubrovin's duality of F-algebroids with eventual identities. Then we use Nijenhuis operators on F-algebroids to construct deformed F-algebroids. In Section 5, first we introduce the notion of a pre-F-algebroid, and show that a pre-F-algebroid gives rise to an F-algebroid. Then we study Dubrovin's duality of pre-F-algebroids with eventual identities. Finally, we introduce the notion of a Nijenhuis operator on a pre-F-algebroid, and show that a Nijenhuis operator on a pre-F-algebroid gives rise to a deformed pre-F-algebroid. At the end, some relations between pre-F-algebroids and F-manifolds with a compatible flat structure are discussed.

2. F-algebroids

We introduce the notion of F-algebroids, which is a generalization of F-manifolds and F-manifold algebras. We give some constructions of F-algebroids including the action F-algebroids and direct product F-algebroids.

Definition 2.1 [14; 17]. An **F-manifold algebra** is a triple $(\mathfrak{g}, [-,-], \cdot)$, where $(\mathfrak{g}, \cdot)$ is a commutative associative algebra and $(\mathfrak{g}, [-,-])$ is a Lie algebra, such that for all $x, y, z, w \in \mathfrak{g}$, the Hertling–Manin relation holds:

$$P_{x\cdot y}(z, w) = x \cdot P_y(z, w) + y \cdot P_x(z, w), \tag{1}$$

where $P_x(y, z)$ is defined by

$$P_x(y, z) = [x, y \cdot z] - [x, y] \cdot z - y \cdot [x, z]. \tag{2}$$

Remark 2.2. Even though Hertling and Manin [17] use the expression F-algebras to refer the objects in the definition above, we will use the terminology introduced in [14] to emphasize that those algebras arise in the study of F-manifolds.

Example 2.3. Any Poisson algebra is an F-manifold algebra.

Definition 2.4 [17]. An **F-manifold** is a pair $(M, \bullet)$, where M is a smooth manifold and $\bullet$ is a $C^\infty(M)$-bilinear, commutative, associative multiplication on the tangent bundle TM such that $(\mathfrak{X}(M), [-,-]_{\mathfrak{X}(M)}, \bullet)$ is an F-manifold algebra, where $[-,-]_{\mathfrak{X}(M)}$ is the Lie bracket of vector fields.

The notion of Lie algebroids was introduced by Pradines in 1967, as a generalization of Lie algebras and tangent bundles. See [28] for the general theory about Lie algebroids.

Definition 2.5. A **Lie algebroid** structure on a vector bundle $A \to M$ is a pair that consists of a Lie algebra structure $[-,-]_A$ on the section space $\Gamma(A)$ and a vector bundle morphism $a_A : A \to TM$, called the anchor, such that

$$[X, fY]_A = f[X, Y]_A + a_A(X)(f)Y \quad \forall\, X, Y \in \Gamma(A),\ f \in C^\infty(M).$$

We denote a Lie algebroid by $(A, [-,-]_A, a_A)$, or A if there is no confusion.

Definition 2.6. A **commutative associative algebroid** is a vector bundle A over M equipped with a $C^\infty(M)$-bilinear, commutative, associative multiplication $\cdot_A$ on the section space $\Gamma(A)$.

We denote a commutative associative algebroid by $(A, \cdot_A)$.

In the following, we give the notion of F-algebroids, which are generalizations of F-manifold algebras and F-manifolds.

Definition 2.7. An **F-algebroid** is a vector bundle A over M equipped with a bilinear operation $\cdot_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$, a skew-symmetric bilinear bracket $[-,-]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$, and a bundle map $a_A : A \to TM$, called the anchor, such that $(A, [-,-]_A, a_A)$ is a Lie algebroid, $(A, \cdot_A)$ is a commutative associative algebroid and $(\Gamma(A), [-,-]_A, \cdot_A)$ is an F-manifold algebra.

We denote an F-algebroid by $(A, [-,-]_A, \cdot_A, a_A)$.

Remark 2.8. Cruz Morales and Torres-Gomez [11] had already defined an F-algebroid. There is a slight difference between the above definition of an F-algebroid and that one. In [11], it is assumed that the base manifold has an F-manifold structure $(M, \bullet)$. An F-algebroid defined in [11] is a vector bundle A over M equipped with a bilinear operation $\cdot_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$, a skew-symmetric bilinear bracket $[-,-]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$, and a bundle map $a_A : A \to TM$, such that $(A, [-,-]_A, a_A)$ is a Lie algebroid, $(A, \cdot_A)$ is a commutative associative algebroid, $(\Gamma(A), [-,-]_A, \cdot_A)$ is an F-manifold algebra and

$$a_A(X \cdot_A Y) = a_A(X) \bullet a_A(Y) \quad \forall\, X, Y \in \Gamma(A). \tag{3}$$

Example 2.9. Any F-manifold algebra is an F-algebroid over a point. Let $(M, \bullet)$ be an F-manifold. Then $(TM, [-,-]_{\mathfrak{X}(M)}, \bullet, \mathrm{Id})$ is an F-algebroid.

Definition 2.10. Let $(A, [-,-]_A, \cdot_A, a_A)$, $(B, [-,-]_B, \cdot_B, a_B)$ be F-algebroids on M. A bundle map $\varphi : A \to B$ is called a **homomorphism** of F-algebroids, if for all $X, Y \in \Gamma(A)$, the following conditions are satisfied:

$$\varphi(X \cdot_A Y) = \varphi(X) \cdot_B \varphi(Y), \quad \varphi([X, Y]_A) = [\varphi(X), \varphi(Y)]_B, \qquad a_B \circ \varphi = a_A.$$

Definition 2.11. Let $(A, [-,-]_A, \cdot_A, a_A)$ be an F-algebroid. A section $e \in \Gamma(A)$ is called the **identity** if $e \cdot_A X = X$ for all $X \in \Gamma(A)$. We denote an F-algebroid $(A, [-,-]_A, \cdot_A, a_A)$ with an identity e by $(A, [-,-]_A, \cdot_A, e, a_A)$.

Proposition 2.12. *Assume that $(A, [-,-]_A, a_A)$ is a Lie algebroid equipped with a $C^\infty(M)$-bilinear, commutative, associative multiplication $\cdot_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$. Define*

$$\begin{aligned}\Phi(X,Y,Z,W)&\\ &:= P_{X\cdot_A Y}(Z,W) - X\cdot_A P_Y(Z,W) - Y\cdot_A P_X(Z,W), \quad \forall X,Y,Z,W \in \Gamma(A),\end{aligned} \tag{4}$$

where P is given by (2). *Then Φ is a tensor field of type* (4, 1) *and*

$$\Phi(X, Y, Z, W) = \Phi(Y, X, Z, W) = \Phi(X, Y, W, Z). \tag{5}$$

Proof. By the commutativity of the associative multiplication $\cdot_A$, we have

$$\Phi(X, Y, Z, W) = \Phi(Y, X, Z, W) = \Phi(X, Y, W, Z).$$

To prove that Φ is a tensor field of type (4, 1), we only need to show

$$\Phi(f X, Y, Z, W) = \Phi(X, Y, f Z, W) = f \Phi(X, Y, Z, W).$$

By a direct calculation, we have

$$\begin{aligned}
&\Phi(f X, Y, Z, W)\\
&= [f(X \cdot_A Y), Z \cdot_A W]_A - Z \cdot_A [f(X \cdot_A Y), W]_A - W \cdot_A [f(X \cdot_A Y), Z]_A\\
&\qquad - f(X \cdot_A P_Y(Z, W)) - Y \cdot_A ([f X, Z \cdot_A W]_A - Z \cdot_A [f X, W]_A - W \cdot_A [f X, Z]_A)\\
&= f P_{X \cdot_A Y}(Z, W) - a_A(Z \cdot_A W)(f)(X \cdot_A Y) + a_A(W)(f)(X \cdot_A Y \cdot_A Z)\\
&\qquad + a_A(Z)(f)(X \cdot_A Y \cdot_A W) - f(X \cdot_A P_Y(Z, W)) - f(Y \cdot_A P_X(Z, W))\\
&\qquad + a_A(Z \cdot_A W)(f)(X \cdot_A Y) - a_A(W)(f)(X \cdot_A Y \cdot_A Z) - a_A(Z)(f)(X \cdot_A Y \cdot_A W)\\
&= f \Phi(X, Y, Z, W).
\end{aligned}$$

Similarly, we also have $\Phi(X, Y, f Z, W) = f \Phi(X, Y, Z, W)$. □

Proposition 2.13. *Let* $(A, [-, -]_A, \cdot_A, a_A)$ *be an* F*-algebroid with an identity* e*. Then*

$$P_e(X, Y) = 0.$$

Proof. It follows from (1) directly. □

Definition 2.14. Let $(\mathfrak{g}, [-, -], \cdot)$ be an F-manifold algebra. An **action** of $\mathfrak{g}$ on a manifold M is a linear map $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ from $\mathfrak{g}$ to the space of vector fields on M, such that

$$\rho([x, y]) = [\rho(x), \rho(y)]_{\mathfrak{X}(M)} \quad \forall x, y \in \mathfrak{g}.$$

Given an action of $\mathfrak{g}$ on M, let $A = M \times \mathfrak{g}$ be the trivial bundle. Define an anchor map $a_\rho : A \to TM$, a multiplication $\cdot_\rho : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ and a bracket $[-, -]_\rho : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ by

$$a_\rho(m, u) = \rho(u)_m \quad \forall m \in M,\ u \in \mathfrak{g}, \tag{6}$$

$$X \cdot_\rho Y = X \cdot Y, \tag{7}$$

$$[X, Y]_\rho = \mathcal{L}_{\rho(X)} Y - \mathcal{L}_{\rho(Y)} X + [X, Y], \quad \forall X, Y \in \Gamma(A), \tag{8}$$

where $X \cdot Y$ and $[X, Y]$ are the pointwise $C^\infty(M)$-bilinear multiplication and bracket, respectively.

Proposition 2.15. *With the above notations,* $(A = M \times \mathfrak{g}, [-, -]_\rho, \cdot_\rho, a_\rho)$ *is an* F*-algebroid, which is called an* **action F-algebroid**, *where* $[-, -]_\rho$, $\cdot_\rho$ *and* a_ρ *are given by* (8), (7) *and* (6), *respectively.*

Proof. Note that the multiplication $\cdot_\rho$ is a $C^\infty(M)$-bilinear, commutative and associative multiplication and $(A, [-,-]_\rho, a_\rho)$ is a Lie algebroid. By Proposition 2.12 and the fact that $\mathfrak{g}$ is an F-manifold algebra, for all $u_1, u_2, u_3, u_4 \in \mathfrak{g}$ and $f_1, f_2, f_3, f_4 \in C^\infty(M)$, we have

$$\Phi(f_1 u_1, f_2 u_2, f_3 u_3, f_4 u_4) = f_1 f_2 f_3 f_4 \Phi(u_1, u_2, u_3, u_4) = 0,$$

which implies that $(\Gamma(A), [-,-]_\rho, \cdot_\rho)$ is an F-manifold algebra. Thus, we obtain $(A, [-,-]_\rho, \cdot_\rho, a_\rho)$ is an F-algebroid. $\square$

Example 2.16. Let $\mathfrak{g}$ be a 2-dimensional vector space with basis $\{e_1, e_2\}$. Then $(\mathfrak{g}, [-,-], \cdot)$ with the nonzero multiplication $\cdot$ and the bracket $[-,-]$

$$e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, \quad [e_1, e_2] = e_2$$

is an F-manifold algebra with the identity e_1. Let (t_1, t_2) be the canonical coordinate systems on $\mathbb{R}^2$. It is straightforward to check that the map $\rho : \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^2)$ defined by

$$\rho(e_1) = t_2 \frac{\partial}{\partial t_2}, \quad \rho(e_2) = t_2 \frac{\partial}{\partial t_1} + t_2^2 \frac{\partial}{\partial t_2}$$

is an action of the F-manifold algebra $\mathfrak{g}$ on $\mathbb{R}^2$. Then $(A = \mathbb{R}^2 \times \mathfrak{g}, [-,-]_\rho, \cdot_\rho, a_\rho)$ is an F-algebroid with an identity $1 \otimes e_1$, where $[-,-]_\rho$, $\cdot_\rho$ and a_ρ are given by

$$a_\rho(m, c_1 e_1 + c_2 e_2) = \left(c_1 t_2 \frac{\partial}{\partial t_2} + c_2 t_2 \frac{\partial}{\partial t_1} + c_2 t_2^2 \frac{\partial}{\partial t_2} \right)\Big|_m \quad \forall m \in \mathbb{R}^2,$$

$$f \otimes (c_1 e_1) \cdot_\rho g \otimes (c_2 e_i) = (fg) \otimes (c_1 c_2 e_i), \quad f \otimes (c_1 e_2) \cdot_\rho g \otimes (c_2 e_2) = 0,$$

$$\begin{aligned}
&[f \otimes (c_1 e_1), g \otimes (c_2 e_2)]_\rho \\
&\quad = f c_1 t_2 \frac{\partial g}{\partial t_2} \otimes (c_2 e_2) - g c_2 \left(t_2 \frac{\partial f}{\partial t_1} + t_2^2 \frac{\partial f}{\partial t_2} \right) \otimes (c_1 e_1) + fg \otimes (c_1 c_2 [e_1, e_2]),
\end{aligned}$$

where $f, g \in C^\infty(\mathbb{R}^2)$, $c_1, c_2 \in \mathbb{R}$, $i \in \{1, 2\}$.

Let A_1 and A_2 be vector bundles over M_1 and M_2 respectively. Denote the projections from $M_1 \times M_2$ to M_1 and M_2 by pr_1 and pr_2 respectively. The product vector bundle $A_1 \times A_2 \to M_1 \times M_2$ can be regarded as the Whitney sum over $M_1 \times M_2$ of the pullback vector bundles $\mathrm{pr}_1^! A_1$ and $\mathrm{pr}_2^! A_2$. Sections of $\mathrm{pr}_1^! A_1$ are of the form $\sum u_i \otimes X_i^1$, where $u_i \in C^\infty(M_1 \times M_2)$ and $X_i^1 \in \Gamma(A_1)$. Similarly, sections of $\mathrm{pr}_2^! A_2$ are of the form $\sum u_i' \otimes X_i^2$, where $u_i' \in C^\infty(M_1 \times M_2)$ and $X_i^2 \in \Gamma(A_2)$. The tangent bundle $T(M_1 \times M_2)$ may in the same way be regarded as the Whitney sum $\mathrm{pr}_1^!(TM_1) \oplus \mathrm{pr}_2^!(TM_2)$. Let $(A_1, [-,-]_{A_1}, a_{A_1})$ and $(A_2, [-,-]_{A_2}, a_{A_2})$ be two Lie algebroids over the base manifolds M_1 and M_2 respectively. We define the anchor $\mathfrak{a} : A_1 \times A_2 \to T(M_1 \times M_2)$ by

$$\mathfrak{a}\Big(\sum (u_i \otimes X_i^1) \oplus \sum (u_j' \otimes X_j^2) \Big) = \sum (u_i \otimes a_{A_1}(X_i^1)) \oplus \sum (u_j' \otimes a_{A_2}(X_j^2)).$$

And the Lie bracket on $A_1 \times A_2$ is determined by the following relations with the Leibniz rule:

$$[\![1\otimes X^1, 1\otimes Y^1]\!] = 1\otimes [X^1, Y^1]_{A_1}, \quad [\![1\otimes X^1, 1\otimes Y^2]\!] = 0,$$
$$[\![1\otimes X^2, 1\otimes Y^2]\!] = 1\otimes [X^2, Y^2]_{A_2}, \quad [\![1\otimes X^2, 1\otimes Y^1]\!] = 0$$

for $X^1, Y^1 \in \Gamma(A_1)$ and $X^2, Y^2 \in \Gamma(A_2)$. See [28] for more details of the direct product Lie algebroids.

Proposition 2.17. *Let $(A_1, [-,-]_{A_1}, \cdot_{A_1}, a_{A_1})$ and $(A_2, [-,-]_{A_2}, \cdot_{A_2}, a_{A_2})$ be two F-algebroids over M_1 and M_2 respectively. Then $(A_1\times A_2, [\![-,-]\!], \diamond, \mathfrak{a})$ is an F-algebroid over $M_1 \times M_2$, where for*

$$X = \sum (u_i \otimes X_i^1) \oplus \sum (u'_j \otimes X_j^2), \qquad Y = \sum (v_k \otimes Y_k^1) \oplus \sum (v'_l \otimes Y_l^2),$$

the associative multiplication $\diamond$ is defined by

$$X \diamond Y = \sum (u_i v_k \otimes (X_i^1 \cdot_{A_1} Y_k^1)) \oplus \sum (u'_j v'_l \otimes (X_j^2 \cdot_{A_2} Y_l^2)).$$

Proof. It follows from straightforward verifications. □

The F-algebroid $(A_1\times A_2, [\![-,-]\!], \diamond, \mathfrak{a})$ is called the direct product F-algebroid.

3. Pre-Lie deformation quantization of commutative associative algebroids

In this section, we study pre-Lie formal deformations of a commutative associative algebroid, whose semiclassical limits are F-algebroids. Viewing the commutative associative algebroid A as a pre-Lie algebroid, we show that the equivalence classes of pre-Lie infinitesimal deformations of a commutative associative algebroid A are classified by the second cohomology group in the deformation cohomology of A and a pre-Lie n-deformation can be extended to a pre-Lie $(n+1)$-deformation if and only if its obstruction class in the third cohomology group is trivial.

Definition 3.1 [9]. A **pre-Lie algebra** is a pair $(\mathfrak{g}, *)$, where $\mathfrak{g}$ is a vector space and $*: \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ is a bilinear multiplication such that for all $x, y, z \in \mathfrak{g}$, the associator

$$(x, y, z) \triangleq x*(y*z) - (x*y)*z \tag{9}$$

is symmetric in x, y, i.e.,

$$(x, y, z) = (y, x, z), \text{ or equivalently, } x*(y*z)-(x*y)*z = y*(x*z)-(y*x)*z.$$

Definition 3.2 [22; 5]. A **pre-Lie algebroid** structure on a vector bundle $A \to M$ is a pair that consists of a pre-Lie algebra structure $*_A$ on the section space $\Gamma(A)$ and a vector bundle morphism $a_A : A \to TM$, called the anchor, such that for all $f \in C^\infty(M)$ and $X, Y \in \Gamma(A)$, the following conditions are satisfied:

(i) $X *_A (fY) = f(X *_A Y) + a_A(X)(f)Y$,

(ii) $(fX) *_A Y = f(X *_A Y)$.

We usually denote a pre-Lie algebroid by $(A, *_A, a_A)$. Any pre-Lie algebra is a pre-Lie algebroid over a point.

A connection ∇ on a manifold M is said to be **flat** if the torsion and the curvature of the connection ∇ vanish identically. A manifold M endowed with a flat connection ∇ is called a **flat manifold**.

Proposition 3.3 [22]. *Let $(A, *_A, a_A)$ be a pre-Lie algebroid. Define a skew-symmetric bilinear bracket operation $[-,-]_A$ on $\Gamma(A)$ by*

$$[X, Y]_A = X *_A Y - Y *_A X \quad \forall\, X, Y \in \Gamma(A). \tag{10}$$

Then $(A, [-,-]_A, a_A)$ is a Lie algebroid, and denoted by A^c, called the **subadjacent Lie algebroid** *of $(A, *_A, a_A)$.*

Example 3.4. Let M be a manifold with a flat connection ∇. Then $(TM, \nabla, \mathrm{Id})$ is a pre-Lie algebroid whose subadjacent Lie algebroid is exactly the tangent Lie algebroid. We denote this pre-Lie algebroid by $T_\nabla M$.

Definition 3.5. Let E be a vector bundle over M. A **multiderivation** of degree n on E is a pair (D, σ_D), where

$$D \in \mathrm{Hom}(\Lambda^{n-1}\Gamma(E) \otimes \Gamma(E), \Gamma(E)) \quad \text{and} \quad \sigma_D \in \Gamma(\mathrm{Hom}(\Lambda^{n-1}E, TM)),$$

such that for all $f \in C^\infty(M)$ and sections $X_i \in \Gamma(E)$, the following conditions are satisfied:

$$D(X_1, \dots, fX_i, \dots, X_{n-1}, X_n) = fD(X_1, \dots, X_i, \dots, X_{n-1}, X_n), \quad i = 1, \dots, n-1,$$
$$D(X_1, \dots, X_{n-1}, fX_n) = fD(X_1, \dots, X_{n-1}, X_n) + \sigma_D(X_1, \dots, X_{n-1})(f)X_n.$$

We will denote by $\mathrm{Der}^n(E)$ the space of multiderivations of degree n, $n \geq 1$.

Let $(A, *_A, a_A)$ be a pre-Lie algebroid. From [22] the deformation complex of A is a cochain complex $(C^*_{\mathrm{def}}(A, A) = \bigoplus_{n\geq 0} \mathrm{Der}^n(A), \mathrm{d}_{\mathrm{def}})$, where for all $X_i \in \Gamma(A)$, $i = 1, 2 \dots, n+1$, the coboundary operator $\mathrm{d}_{\mathrm{def}} : \mathrm{Der}^n(A) \to \mathrm{Der}^{n+1}(A)$ is given by

$$\begin{aligned}
&\mathrm{d}_{\mathrm{def}}\omega(X_1, \dots, X_{n+1}) \\
&= \sum_{i=1}^{n} (-1)^{i+1} X_i *_A \omega(X_1, \dots, \hat{X}_i, \dots, X_{n+1}) \\
&\qquad + \sum_{i=1}^{n} (-1)^{i+1} \omega(X_1, \dots, \hat{X}_i, \dots, X_n, X_i) *_A X_{n+1} \\
&\qquad - \sum_{i=1}^{n} (-1)^{i+1} \omega(X_1, \dots, \hat{X}_i, \dots, X_n, X_i *_A X_{n+1}) \\
&\qquad + \sum_{1\leq i<j\leq n} (-1)^{i+j} \omega([X_i, X_j]_A, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{n+1}),
\end{aligned}$$

in which $\sigma_{\mathrm{d}_{\mathrm{def}}\omega}$ is given by

$$
\begin{aligned}
(11)\quad & \sigma_{\mathrm{d}_{\mathrm{def}}\omega}(X_1,\dots,X_n)\\
&=\sum_{i=1}^{n}(-1)^{i+1}[a_A(X_i),\sigma_\omega(X_1,\dots,\hat{X}_i,\dots,X_n)]_{\mathfrak{X}(M)}\\
&\qquad+\sum_{1\le i<j\le n}(-1)^{i+j}\sigma_\omega([X_i,X_j]_A,X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_n)\\
&\qquad+\sum_{i=1}^{n}(-1)^{i+1}a_A(\omega(X_1,\dots,\hat{X}_i,\dots,X_n,X_i)).
\end{aligned}
$$

The corresponding cohomology, which we denote by $\mathcal{H}^\bullet_{\mathrm{def}}(A,A)$, is called the **deformation cohomology** of the pre-Lie algebroid.

Since any commutative pre-Lie algebra is a commutative associative algebra, we have the following conclusion obviously.

Lemma 3.6. *Any commutative pre-Lie algebroid is a commutative associative algebroid.*

Note that in a commutative pre-Lie algebroid, the anchor must be zero.

Definition 3.7. Assume that $(A,\cdot_A)$ is a commutative associative algebroid. A **pre-Lie formal deformation** of A is a sequence of pairs $(\mu_k,\sigma_{\mu_k})\in\mathrm{Der}^2(A)$ with μ_0 being the commutative associative algebroid multiplication $\cdot_A$ on $\Gamma(A)$ and $\sigma_{\mu_0}=0$ such that the $\mathbb{R}[[\hbar]]$-bilinear product $\cdot_\hbar$ on $\Gamma(A)[[\hbar]]$ and $\mathbb{R}[[\hbar]]$-linear map $\mathfrak{a}_\hbar : A\otimes\mathbb{R}[[\hbar]]\to TM\otimes\mathbb{R}[[\hbar]]$ determined by

$$(12)\qquad X\cdot_\hbar Y=\sum_{k=0}^{\infty}\hbar^k\mu_k(X,Y),$$

$$(13)\qquad \mathfrak{a}_\hbar(X)=\sum_{k=0}^{\infty}\hbar^k\sigma_{\mu_k}(X)\quad \forall\, X,Y\in\Gamma(A)$$

is a pre-Lie algebroid.

One checks directly that $(\Gamma(A)[[\hbar]],\cdot_\hbar)$ is a pre-Lie algebra if and only if

$$
\begin{aligned}
(14)\quad \sum_{i+j=k}\big(\mu_i(\mu_j(X,Y),Z)-\mu_i(X,\mu_j(Y,Z))\big)&\\
=\sum_{i+j=k}\big(\mu_i(\mu_j(Y,X),Z)-\mu_i(Y,\mu_j(X,Z))\big)&
\end{aligned}
$$

for $k\ge 0$.

Theorem 3.8. *Assume that $(A,\cdot_A)$ is a commutative associative algebroid and $(A\otimes\mathbb{R}[[\hbar]],\cdot_\hbar,\mathfrak{a}_\hbar)$ a pre-Lie formal deformation of A. Define a bracket*

$$[-,-]_A:\Gamma(A)\times\Gamma(A)\to\Gamma(A)$$

by

$$[X,Y]_A=\mu_1(X,Y)-\mu_1(Y,X)\quad\forall\, X,Y\in\Gamma(A).$$

Then $(A, [-,-]_A, \cdot_A, \sigma_{\mu_1})$ *is an* F*-algebroid which is called the* **semiclassical limit** *of* $(A\otimes\mathbb{R}[[\hbar]], \cdot_\hbar, \mathfrak{a}_\hbar)$. *The pre-Lie algebroid* $(A\otimes\mathbb{R}[[\hbar]], \cdot_\hbar, \mathfrak{a}_\hbar)$ *is called a* **pre-Lie deformation quantization** *of* $(A, \cdot_A)$.

Proof. Define the bracket $[-,-]_\hbar$ on $\Gamma(A)[[\hbar]]$ by

$$\begin{aligned}[X,Y]_\hbar &= X\cdot_\hbar Y - Y\cdot_\hbar X\\ &= \hbar[X,Y]_A + \hbar^2(\mu_2(X,Y)-\mu_2(Y,X)) + \cdots \quad \forall\, X, Y\in\Gamma(A).\end{aligned}$$

By the fact that $(A\otimes\mathbb{R}[[\hbar]], \cdot_\hbar, \mathfrak{a}_\hbar)$ is a pre-Lie algebroid, $(A[[\hbar]], [-,-]_\hbar, \mathfrak{a}_\hbar)$ is a Lie algebroid. The $\hbar^2$-terms of the Jacobi identity for $[-,-]_\hbar$ gives the Jacobi identity for $[-,-]_A$ and $\hbar$-terms of $[X, fY]_\hbar = f[X,Y]_\hbar + \mathfrak{a}_\hbar(X)(f)Y$ gives

$$[X, fY]_A = f[X,Y]_A + \sigma_{\mu_1}(X)(f)Y.$$

Thus $(A, [-,-]_A, \sigma_{\mu_1})$ is a Lie algebroid.

For $k=1$ in (14), by the commutativity of μ_0, we have

$$\begin{aligned}&\mu_0(\mu_1(X,Y),Z) - \mu_0(X,\mu_1(Y,Z)) - \mu_1(X,\mu_0(Y,Z))\\ &\qquad = \mu_0(\mu_1(Y,X),Z) - \mu_0(Y,\mu_1(X,Z)) - \mu_1(Y,\mu_0(X,Z)).\end{aligned}$$

By a similar proof given by Hertling [16], we can show that the Hertling–Manin relation holds with $X\cdot_A Y = \mu_0(X,Y)$ and $[X,Y]_A = \mu_1(X,Y)-\mu_1(Y,X)$ for $X, Y\in\Gamma(A)$. Thus $(A, [-,-]_A, \cdot_A, \sigma_{\mu_1})$ is an F-algebroid. □

In what follows, we study pre-Lie n-deformations and pre-Lie infinitesimal deformations of commutative associative algebroids.

Definition 3.9. Let $(A, \cdot_A)$ be a commutative associative algebroid. A **pre-Lie** n**-deformation** of A is a sequence of pairs $(\mu_k, \sigma_{\mu_k})\in \mathrm{Der}^2(A)$ for $0\le k\le n$ with μ_0 being the commutative associative algebroid multiplication $\cdot_A$ on $\Gamma(A)$ and $\sigma_{\mu_0}=0$, such that the $\mathbb{R}[[\hbar]]/(\hbar^{n+1})$-bilinear product $\cdot_\hbar$ on $\Gamma(A)[[\hbar]]/(\hbar^{n+1})$ and $\mathbb{R}[[\hbar]]/(\hbar^{n+1})$-linear map $\mathfrak{a}_\hbar : A\otimes\mathbb{R}[[\hbar]]\to TM\otimes\mathbb{R}[[\hbar]]$ determined by

$$X\cdot_\hbar Y = \sum_{k=0}^{n}\hbar^k\mu_k(X,Y), \tag{15}$$

$$\mathfrak{a}_\hbar(X) = \sum_{k=0}^{n}\hbar^k\sigma_{\mu_k}(X) \quad \forall\, X, Y\in\Gamma(A) \tag{16}$$

is a pre-Lie algebroid.

We call a pre-Lie 1-deformation of a commutative associative algebroid $(A, \cdot_A)$ a **pre-Lie infinitesimal deformation** and denote it by $(A, \mu_1, a_A = \sigma_{\mu_1})$.

By direct calculations, $(A, \mu_1, \sigma_{\mu_1})$ is a pre-Lie infinitesimal deformation of a commutative associative algebroid $(A, \cdot_A)$ if and only if for all $X, Y, Z \in \Gamma(A)$

$$\begin{aligned}(17)\quad \mu_1(X, Y)\cdot_A Z - X\cdot_A \mu_1(Y, Z) - \mu_1(X, Y\cdot_A Z)\\ = \mu_1(Y, X)\cdot_A Z - Y\cdot_A \mu_1(X, Z) - \mu_1(Y, X\cdot_A Z).\end{aligned}$$

Equation (17) means that μ_1 is a 2-cocycle, i.e., $\mathrm{d}_{\mathrm{def}}\,\mu_1 = 0$.

Two pre-Lie infinitesimal deformations $A_\hbar = (A, \mu_1, \sigma_{\mu_1})$ and $A'_\hbar = (A, \mu'_1, \sigma_{\mu'_1})$ of a commutative associative algebroid $(A, \cdot_A)$ are said to be **equivalent** if there exist a family of pre-Lie algebroid homomorphisms $\mathrm{Id} + \hbar\varphi : A_\hbar \to A'_\hbar$ modulo $\hbar^2$ for $\varphi \in \mathrm{Der}^1(A)$. A pre-Lie infinitesimal deformation is said to be **trivial** if there exist a family of pre-Lie algebroid homomorphisms $\mathrm{Id}+\hbar\varphi : A_\hbar \to (A, \cdot_A, a_A = 0)$ modulo $\hbar^2$.

By direct calculations, $A_\hbar$ and $A'_\hbar$ are equivalent pre-Lie infinitesimal deformations if and only if

$$\sigma_{\mu_1} = \sigma_{\mu'_1}, \tag{18}$$

$$\mu_1(X, Y) - \mu'_1(X, Y) = X\cdot_A \varphi(Y) + \varphi(X)\cdot_A Y - \varphi(X\cdot_A Y). \tag{19}$$

Equation (19) means that $\mu_1 - \mu'_1 = \mathrm{d}_{\mathrm{def}}\,\varphi$ and (18) can be obtained by (19). Thus we have:

Theorem 3.10. *Let $(A, \cdot_A)$ be a commutative associative algebroid. There is a one-to-one correspondence between the space of equivalence classes of pre-Lie infinitesimal deformations of A and the second cohomology group $\mathcal{H}^2_{\mathrm{def}}(A, A)$.*

It is routine to check that:

Proposition 3.11. *Let $(A, \cdot_A)$ be a commutative associative algebroid such that*

$$\mathcal{H}^2_{\mathrm{def}}(A, A) = 0.$$

Then all pre-Lie infinitesimal deformations of A are trivial.

Definition 3.12. Let $\{(\mu_1, \sigma_{\mu_1}), \dots, (\mu_n, \sigma_{\mu_n})\}$ be a pre-Lie n-deformation of a commutative associative algebroid $(A, \cdot_A)$. If there exists $(\mu_{n+1}, \sigma_{\mu_{n+1}}) \in \mathrm{Der}^2(A)$ such that

$$\{(\mu_1, \sigma_{\mu_1}), \dots, (\mu_n, \sigma_{\mu_n}), (\mu_{n+1}, \sigma_{\mu_{n+1}})\}$$

is a pre-Lie $(n+1)$-deformation of $(A, \cdot_A)$, then

$$\{(\mu_1, \sigma_{\mu_1}), \dots, (\mu_n, \sigma_{\mu_n}), (\mu_{n+1}, \sigma_{\mu_{n+1}})\}$$

is called an **extension** of the pre-Lie n-deformation $\{(\mu_1, \sigma_{\mu_1}), \dots, (\mu_n, \sigma_{\mu_n})\}$.

Theorem 3.13. *For any pre-Lie n-deformation of a commutative associative algebroid* $(A, \cdot_A)$, *the pair* $(\Theta_n, \sigma_{\Theta_n}) \in \mathrm{Der}^3(A)$ *defined by*

$$\Theta_n(X, Y, Z) = \sum_{\substack{i+j=n+1\\ i,j\geq 1}} \big(\mu_i(\mu_j(X, Y), Z) - \mu_i(X, \mu_j(Y, Z)) - \mu_i(\mu_j(Y, X), Z) + \mu_i(Y, \mu_j(X, Z))\big), \tag{20}$$

$$\sigma_{\Theta_n}(X, Y) = \sum_{\substack{i+j=n+1\\ i,j\geq 1}} \big(\sigma_{\mu_i}(\mu_j(X, Y) - \mu_j(Y, X)) - [\sigma_{\mu_i}(X), \sigma_{\mu_j}(Y)]_{\mathfrak{X}(M)}\big) \tag{21}$$

is a cocycle, *i.e.*, $\mathrm{d_{def}}\Theta_n = 0$.

Moreover, *the pre-Lie n-deformation* $\{(\mu_1, \sigma_{\mu_1}), \dots, (\mu_n, \sigma_{\mu_n})\}$ *extends to some pre-Lie* $(n + 1)$*-deformation if and only if* $[\Theta_n] = 0$ *in* $\mathcal{H}^3_{\mathrm{def}}(A, A)$.

Proof. It is obvious that $\Theta_n(X, Y, Z) = -\Theta_n(Y, Z, X)$ for all $X, Y, Z \in \Gamma(A)$. It is straightforward to check that

$$\Theta_n(X, fY, Z) = f\Theta_n(X, Y, Z),$$
$$\Theta_n(X, Y, fZ) = f\Theta_n(X, Y, Z) + \sigma_{\Theta_n}(X, Y)(f)Z.$$

Thus Θ_n is an element of $\mathrm{Der}^3(A)$. By a direct calculation, we have that the cochain $\Theta_n \in \mathrm{Der}^3(A)$ is closed.

Assume that the pre-Lie $(n + 1)$-deformation $\{(\mu_1, \sigma_{\mu_1}), \dots, (\mu_{n+1}, \sigma_{\mu_{n+1}})\}$ of a commutative associative algebroid $(A, \cdot_A)$ is an extension of the pre-Lie n-deformation $\{(\mu_1, \sigma_{\mu_1}), \dots, (\mu_n, \sigma_{\mu_n})\}$. Then we have

$$\begin{aligned}\sum_{\substack{i+j=n+1\\ i,j\geq 1}} \big(\mu_i(\mu_j(X, Y), Z) - \mu_i(X, \mu_j(Y, Z)) - \mu_i(\mu_j(Y, X), Z) + \mu_i(Y, \mu_j(X, Z))\big) \\ = X \cdot_A \mu_{n+1}(Y, Z) - Y \cdot_A \mu_{n+1}(X, Z) + \mu_{n+1}(Y, X) \cdot_A Z - \mu_{n+1}(X, Y) \cdot_A Z \\ + \mu_{n+1}(Y, X) \cdot_A Z - \mu_{n+1}(X, Y) \cdot_A Z.\end{aligned}$$

Note that the left-hand side of the above equality is just $\Theta_n(X, Y, Z)$. We can rewrite the above equality as

$$\Theta_n(X, Y, Z) = \mathrm{d_{def}}\, \mu_{n+1}(X, Y, Z).$$

We conclude that, if a pre-Lie n-deformation of a commutative associative algebroid $(A, \cdot_A)$ extends to a pre-Lie $(n + 1)$-deformation, then Θ_n is a coboundary.

Conversely, if Θ_n is a coboundary, then there exists an element $(\psi, \sigma_\psi) \in \mathrm{Der}^2(A)$ such that

$$\Theta_n(X, Y, Z) = \mathrm{d_{def}}\, \psi(X, Y, Z).$$

It is not hard to check that $\{(\mu_1, \sigma_{\mu_1}), \dots, (\mu_{n+1}, \sigma_{\mu_{n+1}})\}$ with $\mu_{n+1} = \psi$ is a pre-Lie $(n + 1)$-deformation of $(A, \cdot_A)$ and thus this pre-Lie $(n + 1)$-deformation is an extension of the pre-Lie n-deformation $\{(\mu_1, \sigma_{\mu_1}), \dots, (\mu_n, \sigma_{\mu_n})\}$. □

4. Some constructions of F-algebroids

In this section, we use eventual identities and Nijenhuis operators to construct F-algebroids. In particular, a pseudoeventual identity naturally gives a Nijenhuis operator on an F-algebroid.

(Pseudo)eventual identities and Dubrovin's dual of F-algebroids.

Definition 4.1. Let $(A, [-,-]_A, \cdot_A, a_A)$ be an F-algebroid with an identity e. A section $\mathcal{E} \in \Gamma(A)$ is called a **pseudoeventual identity** on the F-algebroid if the following equality holds:

$$P_{\mathcal{E}}(X, Y) = [e, \mathcal{E}]_A \cdot_A X \cdot_A Y \quad \forall X, Y \in \Gamma(A). \tag{22}$$

A pseudoeventual identity $\mathcal{E}$ on the F-algebroid A is called an **eventual identity** if it is invertible, i.e., there is a section $\mathcal{E}^{-1} \in \Gamma(A)$ such that $\mathcal{E}^{-1} \cdot_A \mathcal{E} = \mathcal{E} \cdot_A \mathcal{E}^{-1} = e$.

Denote the set of all pseudoeventual identities on an F-algebroid A by $E(A)$, i.e.,

$$E(A) = \{\mathcal{E} \in \Gamma(A) \mid P_{\mathcal{E}}(X, Y) = [e, \mathcal{E}]_A \cdot_A X \cdot_A Y \ \ \forall X, Y \in \Gamma(A)\}.$$

Proposition 4.2. *Let $(A, [-,-]_A, \cdot_A, a_A)$ be an F-algebroid with an identity e. Then $E(A)$ is an F-manifold subalgebra of $\Gamma(A)$. Moreover, if $\mathcal{E} \in \Gamma(A)$ is an eventual identity on the F-algebroid A, then $\mathcal{E}^{-1}$ is also an eventual identity on A.*

Proof. By a straightforward calculation, $E(A)$ is a subspace of the vector space $\Gamma(A)$.

For any two pseudoeventual identities $\mathcal{E}_1$ and $\mathcal{E}_2$, by (1), we have

$$\begin{aligned} P_{\mathcal{E}_1 \cdot_A \mathcal{E}_2}(X, Y) &= \mathcal{E}_1 \cdot_A P_{\mathcal{E}_2}(X, Y) + \mathcal{E}_2 \cdot_A P_{\mathcal{E}_1}(X, Y) \\ &= (\mathcal{E}_1 \cdot_A [e, \mathcal{E}_2]_A + \mathcal{E}_2 \cdot_A [e, \mathcal{E}_1]_A) \cdot_A X \cdot_A Y = [e, \mathcal{E}_1 \cdot_A \mathcal{E}_2]_A \cdot_A X \cdot_A Y, \end{aligned}$$

where in the last equality we used $P_e(\mathcal{E}_1, \mathcal{E}_2) = 0$. Thus $\mathcal{E}_1 \cdot_A \mathcal{E}_2$ is a pseudoeventual identity.

By (1) and (22), we have

$$\begin{aligned} P_{[\mathcal{E}_2, \mathcal{E}_2]_A}(Z, W) = [\mathcal{E}_1, [e, \mathcal{E}_2]_A \cdot_A Z \cdot_A W]_A &- [e, \mathcal{E}_2]_A \cdot_A [\mathcal{E}_1, Z]_A \cdot_A W \\ &- [e, \mathcal{E}_2]_A \cdot_A Z \cdot_A [\mathcal{E}_1, W]_A - [\mathcal{E}_2, [e, \mathcal{E}_1]_A \cdot_A Z \cdot_A W]_A \\ &+ [e, \mathcal{E}_1]_A \cdot_A [\mathcal{E}_2, Z]_A \cdot_A W + [e, \mathcal{E}_1]_A \cdot_A Z \cdot_A [\mathcal{E}_2, W]_A. \end{aligned}$$

On the other hand, by (22), we have

$$\begin{aligned} [\mathcal{E}_1, [e, \mathcal{E}_2]_A \cdot_A Z \cdot_A W]_A = 2[e, \mathcal{E}_1]_A \cdot [e, \mathcal{E}_2]_A \cdot_A Z \cdot_A W + [\mathcal{E}_1, [e, \mathcal{E}_2]_A]_A \cdot_A Z \cdot_A W \\ + [e, \mathcal{E}_2]_A \cdot_A [\mathcal{E}_1, Z]_A \cdot_A W + [e, \mathcal{E}_2]_A \cdot_A Z \cdot_A [\mathcal{E}_1, W]_A, \end{aligned}$$

$$\begin{aligned} [\mathcal{E}_2, [e, \mathcal{E}_1]_A \cdot_A Z \cdot_A W]_A = 2[e, \mathcal{E}_2]_A \cdot [e, \mathcal{E}_1]_A \cdot_A Z \cdot_A W + [\mathcal{E}_2, [e, \mathcal{E}_1]_A]_A \cdot_A Z \cdot_A W \\ + [e, \mathcal{E}_1]_A \cdot_A [\mathcal{E}_2, Z]_A \cdot_A W + [e, \mathcal{E}_1]_A \cdot_A Z \cdot_A [\mathcal{E}_2, W]_A. \end{aligned}$$

Thus

$$P_{[\mathcal{E}_2,\mathcal{E}_2]_A}(Z,W) = [\mathcal{E}_1,[e,\mathcal{E}_2]_A]_A \cdot_A Z \cdot_A W - [\mathcal{E}_2,[e,\mathcal{E}_1]_A]_A \cdot_A Z \cdot_A W$$
$$= [e,[\mathcal{E}_1,\mathcal{E}_2]_A]_A \cdot_A Z \cdot_A W,$$

which implies that $[\mathcal{E}_1,\mathcal{E}_2]_A$ is a pseudoeventual identity. Therefore, $E(A)$ is an F-manifold subalgebra of $\Gamma(A)$.

Assume that $\mathcal{E}$ is an eventual identity on the F-algebroid A. By Proposition 2.13, we have $P_e(X,Y)=0$. Applying the Hertling–Manin relation with $X=\mathcal{E}$ and $Y=\mathcal{E}^{-1}$, by (22), we obtain

$$P_{\mathcal{E}^{-1}}(X,Y) = -\mathcal{E}^{-2} \cdot_A [e,\mathcal{E}]_A \cdot_A X \cdot_A Y.$$

On the other hand, by $P_e(X,Y)=0$, we have

$$[e,\mathcal{E}]_A \cdot_A \mathcal{E}^{-2} = ([e,\mathcal{E}]_A \cdot_A \mathcal{E}^{-1}) \cdot_A \mathcal{E}^{-1} = (-\mathcal{E} \cdot_A [e,\mathcal{E}^{-1}]_A) \cdot_A \mathcal{E}^{-1} = -[e,\mathcal{E}^{-1}]_A.$$

Thus we have

$$P_{\mathcal{E}^{-1}}(X,Y) = [e,\mathcal{E}^{-1}]_A \cdot_A X \cdot_A Y,$$

which implies that $\mathcal{E}^{-1}$ is also an eventual identity on A. □

A pseudoeventual identity on an F-algebroid gives a new F-algebroid.

Theorem 4.3. *Let $(A,[-,-]_A,\cdot_A,a_A)$ be an F-algebroid with an identity e. Then $\mathcal{E}$ is a pseudoeventual identity on A if and only if $(A,[-,-]_A,\cdot_{\mathcal{E}},a_A)$ is an F-algebroid, where $\cdot_{\mathcal{E}} : \Gamma(A)\times\Gamma(A)\to\Gamma(A)$ is defined by*

$$X \cdot_{\mathcal{E}} Y = X \cdot_A Y \cdot_A \mathcal{E} \quad \forall X,Y \in \Gamma(A). \tag{23}$$

Proof. The proof of this theorem is similar to the proof of Theorem 3 in [13]. We give a sketchy proof here for completeness. Assume that $\mathcal{E}$ is a pseudoeventual identity on A. It is straightforward to check that the multiplication $\cdot_{\mathcal{E}}$ defined by (23) is $C^\infty(M)$-bilinear, commutative and associative.

For $X,Y,Z \in \Gamma(A)$, we set

$$P_X^{\mathcal{E}}(Y,Z) := [X, Y \cdot_{\mathcal{E}} Z]_A - [X,Y]_A \cdot_{\mathcal{E}} Z - Y \cdot_{\mathcal{E}} [X,Z]_A.$$

By a direct calculation, we have

$$P_X^{\mathcal{E}}(Y,Z) = P_X(\mathcal{E}\cdot_A Y, Z) + P_X(\mathcal{E},Y)\cdot_A Z + [X,\mathcal{E}]_A \cdot_A Y \cdot_A Z. \tag{24}$$

Since $\mathcal{E}$ is a pseudoeventual identity on A, by (24), we have

$$\begin{aligned}
&P^{\mathcal{E}}_{X\cdot_{\mathcal{E}}Y}(Z,W) - X\cdot_{\mathcal{E}} P^{\mathcal{E}}_Y(Z,W) - Y\cdot_{\mathcal{E}} P^{\mathcal{E}}_X(Z,W)\\
&= X\cdot_A Y\cdot_A (P_{\mathcal{E}}(\mathcal{E}\cdot_A Z, W) + W\cdot_A P_{\mathcal{E}}(\mathcal{E}, Z))\\
&\qquad\qquad - Z\cdot_A W\cdot_A ([X\cdot_A Y\cdot_A \mathcal{E}, \mathcal{E}]_A + \mathcal{E}\cdot_A X\cdot_A [Y,\mathcal{E}]_A + \mathcal{E}\cdot_A Y\cdot_A [X,\mathcal{E}]_A)\\
&= X\cdot_A Y\cdot_A (P_{\mathcal{E}}(\mathcal{E}\cdot_A Z, W) + W\cdot_A P_{\mathcal{E}}(\mathcal{E}, Z)) - Z\cdot_A W\cdot_A (P_{\mathcal{E}}(\mathcal{E}, X)\cdot_A Y + P_{\mathcal{E}}(\mathcal{E}\cdot_A X, Y))\\
&= X\cdot_A Y\cdot_A ([e,\mathcal{E}]_A\cdot_A \mathcal{E}\cdot_A Z\cdot_A W + [e,\mathcal{E}]_A\cdot_A \mathcal{E}\cdot_A Z\cdot_A W)\\
&\qquad\qquad - Z\cdot_A W\cdot_A ([e,\mathcal{E}]_A\cdot_A \mathcal{E}\cdot_A X\cdot_A Y + [e,\mathcal{E}]_A\cdot_A \mathcal{E}\cdot_A X\cdot_A Y)\\
&= 2[e,\mathcal{E}]_A\cdot_A \mathcal{E}\cdot_A X\cdot_A Y\cdot_A Z\cdot_A W - 2[e,\mathcal{E}]_A\cdot_A \mathcal{E}\cdot_A X\cdot_A Y\cdot_A Z\cdot_A W\\
&= 0,
\end{aligned}$$

which implies that $(A, [-,-]_A, \cdot_{\mathcal{E}}, a_A)$ is an F-algebroid.

The converse can be proved similarly. We omit the details. □

Theorem 4.4. *Let $(A, [-,-]_A, \cdot_A, a_A)$ be an F-algebroid with an identity e. Then $\mathcal{E}$ is an eventual identity on A if and only if $(A, [-,-]_A, \cdot_{\mathcal{E}}, a_A)$ is also an F-algebroid with the identity $\mathcal{E}^{-1}$, which is called the Dubrovin's dual of $(A, [-,-]_A, \cdot_A, a_A)$, where $\cdot_{\mathcal{E}}$ is given by* (23)*. Moreover, e is an eventual identity on the F-algebroid $(A, [-,-]_A, \cdot_{\mathcal{E}}, \mathcal{E}^{-1}, a_A)$ and the map*

$$(A, [-,-]_A, \cdot_A, e, a_A, \mathcal{E}) \to (A, [-,-]_A, \cdot_{\mathcal{E}}, \mathcal{E}^{-1}, a_A, e^{\dagger}) \tag{25}$$

is an involution of the set of F-algebroids with eventual identities, where $e^{\dagger} := \mathcal{E}^{-2}$ is the inverse of e with respect to the multiplication $\cdot_{\mathcal{E}}$.

Proof. By Theorem 4.3, $(A, [-,-]_A, \cdot_{\mathcal{E}}, a_A)$ is an F-algebroid. It is obvious that $\mathcal{E}^{-1}$ is the identity with respect to the multiplication $\cdot_{\mathcal{E}}$ defined by (23).

Next, we show that e is an eventual identity on $(A, [-,-]_A, \cdot_{\mathcal{E}}, \mathcal{E}^{-1}, a_A)$. Since the identity with respective to the multiplication $\cdot_{\mathcal{E}}$ is $\mathcal{E}^{-1}$, we need to show that

$$[e, X\cdot_{\mathcal{E}} Y]_A - [e,X]_A\cdot_{\mathcal{E}} Y - X\cdot_{\mathcal{E}}[e,Y]_A = [\mathcal{E}^{-1}, e]_A\cdot_{\mathcal{E}} X\cdot_{\mathcal{E}} Y \quad \forall X, Y\in\Gamma(A).$$

By a straightforward computation, for any $Z\in\Gamma(A)$, we have

$$\begin{aligned}
&[Z, X\cdot_{\mathcal{E}} Y]_A - [Z,X]_A\cdot_{\mathcal{E}} Y - X\cdot_{\mathcal{E}}[Z,Y]_A\\
&\qquad\qquad = P_Z(\mathcal{E}\cdot_A X, Y) + P_Z(\mathcal{E}, X)\cdot_A Y + [Z,\mathcal{E}]_A\cdot_A X\cdot_A Y.
\end{aligned} \tag{26}$$

Letting $Z=e$ in (26) and using $P_e(X,Y)=0$, we have

$$[e, X\cdot_{\mathcal{E}} Y]_A - [e,X]_A\cdot_{\mathcal{E}} Y - X\cdot_{\mathcal{E}}[e,Y]_A = [e,\mathcal{E}]_A\cdot_A X\cdot_A Y = ([e,\mathcal{E}]_A\cdot_A \mathcal{E}^{-2})\cdot_{\mathcal{E}} X\cdot_{\mathcal{E}} Y.$$

Recall now from the proof of Proposition 4.2 that $[e,\mathcal{E}]_A\cdot_A \mathcal{E}^{-2} = [\mathcal{E}^{-1}, e]_A$. Thus e is an eventual identity on the F-algebroid $(A, [-,-]_A, \cdot_{\mathcal{E}}, \mathcal{E}^{-1}, a_A)$.

Now we show that the map (25) is an involution. Note that $e^{\dagger} := \mathcal{E}^{-2}$ is the inverse of e with respect to the multiplication $\cdot_{\mathcal{E}}$. By Proposition 4.2, $e^{\dagger}$ is also an eventual identity on the F-algebroid $(A, [-,-]_A, \cdot_{\mathcal{E}}, \mathcal{E}^{-1}, a_A)$. Furthermore, for $X, Y \in \Gamma(A)$, we have

$$X \cdot_A Y = X \cdot_{\mathcal{E}} Y \cdot_{\mathcal{E}} \mathcal{E}^{-2} = X \cdot_{\mathcal{E}} Y \cdot_{\mathcal{E}} e^{\dagger},$$

which implies that the map defined by (25) is an involution of the set of F-algebroids with eventual identities. □

Definition 4.5. An F-manifold $(M, \bullet, e)$ is called **semisimple** if there exists canonical local coordinates $(u^1, \dots, u^n)$ on M such that $e = \frac{\partial}{\partial u^1} + \cdots + \frac{\partial}{\partial u^n}$ and

$$\frac{\partial}{\partial u^i} \bullet \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^j}, \quad i, j \in \{1, 2, \dots, n\}$$

Example 4.6. Let $(M, \bullet, e)$ be a semisimple F-manifold. Then e is an identity on the F-algebroid $(TM, [-,-]_{\mathfrak{X}(M)}, \bullet, \mathrm{Id})$. It is straightforward to check that any pseudoeventual identity on $(TM, [-,-]_{\mathfrak{X}(M)}, \bullet, \mathrm{Id})$ is of the form

$$\mathcal{E} = f_1(u^1) \frac{\partial}{\partial u^1} + \cdots + f_n(u^n) \frac{\partial}{\partial x_n},$$

where $f_i(u^i) \in C^\infty(M)$ depends only on u^i for $i = 1, 2, \dots, n$. Furthermore, it was shown in [13] that if all $f_i(u^i)$ are nonvanishing everywhere, then $\mathcal{E} \in \mathfrak{X}(M)$ is an eventual identity.

Nijenhuis operators and deformed F-algebroids. Recall from [8] that a Nijenhuis operator on a commutative associative algebra $(A, \cdot_A)$ is a linear map $N : A \to A$ such that

$$N(x) \cdot_A N(y) = N\big(N(x) \cdot_A y + x \cdot_A N(y) - N(x \cdot_A y)\big) \quad \forall x, y \in A. \tag{27}$$

and a Nijenhuis operator on a Lie algebroid $(A, [-,-]_A, a_A)$ is a bundle map $N : A \to A$ such that

$$\begin{aligned}[N(X), N(Y)]_A \\ = N\big([N(X), Y]_A + [X, N(Y)]_A - N([X, Y]_A)\big) \quad \forall X, Y \in \Gamma(A).\end{aligned} \tag{28}$$

Definition 4.7. Assume that $(A, [-,-]_A, \cdot_A, a_A)$ is an F-algebroid. A bundle map $N : A \to A$ is called a **Nijenhuis operator** on the F-algebroid A if N is both a Nijenhuis operator on the commutative associative algebra $(\Gamma(A), \cdot_A)$ and a Nijenhuis operator on the Lie algebroid $(A, [-,-]_A, a_A)$.

Define the deformed operation $\cdot_N : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ and the deformed bracket $[-,-]_N : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ by

$$X \cdot_N Y = N(X) \cdot_A Y + X \cdot_A N(Y) - N(X \cdot_A Y), \tag{29}$$

$$[X, Y]_N = [N(X), Y]_A + [X, N(Y)]_A - N([X, Y]_A) \quad \forall X, Y \in \Gamma(A). \tag{30}$$

Theorem 4.8. *Assume that $N : A \to A$ is a Nijenhuis operator on an F-algebroid $(A, [-,-]_A, \cdot_A, a_A)$. Then, $(A, [-,-]_N, \cdot_N, a_N = a_A \circ N)$ is an F-algebroid and N is an F-algebroid homomorphism from the F-algebroid*

$$(A, [-,-]_N, \cdot_N, a_N = a_A \circ N)$$

to $(A, [-,-]_A, \cdot_A, a_A)$.

Proof. Since N is a Nijenhuis operator on the commutative associative algebra $(\Gamma(A), \cdot_A)$, it follows that $(\Gamma(A), \cdot_N)$ is a commutative associative algebra [8]. Since N is a Nijenhuis operator on the Lie algebroid $(A, [-,-]_A, a_A)$, we get that $(A, [-,-]_N, a_N)$ is a Lie algebroid [20].

Define

$$\Phi_N(X, Y, Z, W) := P^N_{X \cdot_N Y}(Z, W) - X \cdot_N P^N_Y(Z, W) - Y \cdot_N P^N_X(Z, W), \tag{31}$$

where $X, Y, Z, W \in \Gamma(A)$ and

$$P^N_X(Y, Z) := [X, Y \cdot_N Z]_N - [X, Y]_N \cdot_N Z - Y \cdot_N [X, Z]_N.$$

Since A is an F-algebroid and N is a Nijenhuis operator on A, by a direct calculation, we have

$$\Phi_N(X, Y, Z, W) = 0,$$

which implies that

$$P^N_{X \cdot_N Y}(W, Z) - X \cdot_N P^N_Y(W, Z) - Y \cdot_N P^N_X(W, Z) = 0.$$

Thus $(A, [-,-]_N, \cdot_N, a_N = a_A \circ N)$ is an F-algebroid. It is obvious that N is an F-algebroid homomorphism from the F-algebroid $(A, [-,-]_N, \cdot_N, a_N = a_A \circ N)$ to $(A, [-,-]_A, \cdot_A, a_A)$. □

Lemma 4.9. *Let $(A, [-,-]_A, \cdot_A, a_A)$ be an F-algebroid and N a Nijenhuis operator on A. For all $k, l \in \mathbb{N}$:*

(i) *$(A, [-,-]_{N^k}, \cdot_{N^k}, a_{N^k})$ is an F-algebroid.*

(ii) *N^l is also a Nijenhuis operator on the F-algebroid $(A, [-,-]_{N^k}, \cdot_{N^k}, a_{N^k})$.*

(iii) *The F-algebroids*

$$(A, ([-,-]_{N^k})_{N^l}, (\cdot_{N^k})_{N^l}, a_{N^{k+l}}) \quad \text{and} \quad (A, [-,-]_{N^{k+l}}, \cdot_{N^{k+l}}, a_{N^{k+l}})$$

are the same.

(iv) *N^l is an F-algebroid homomorphism between the F-algebroid*

$$(A, [-,-]_{N^{k+l}}, \cdot_{N^{k+l}}, a_{N^{k+l}}) \quad \text{and} \quad (A, [-,-]_{N^k}, \cdot_{N^k}, a_{N^k}).$$

Proof. Since the above conclusions with respect to Nijenhuis operators on commutative associative algebras [8] and Lie algebroids [20] simultaneously hold, by Theorem 4.8, the conclusions follow immediately. □

We now show that pseudoeventual identities naturally give Nijenhuis operators.

Proposition 4.10. *Let $(A, [-,-]_A, \cdot_A, a_A)$ be an F-algebroid with an identity e and $\mathcal{E}$ a pseudoeventual identity on A. Then the endomorphism $N = \mathcal{E}\cdot_A$ is a Nijenhuis operator on the F-algebroid A. Consequently, $(A, [-,-]_{\mathcal{E}}, \cdot_{\mathcal{E}}, a_{\mathcal{E}})$ is an F-algebroid, where*

$$[X,Y]_{\mathcal{E}} = [\mathcal{E}\cdot_A X, Y]_A + [X, \mathcal{E}\cdot_A Y]_A - \mathcal{E}\cdot_A [X,Y]_A \quad \forall\, X, Y \in \Gamma(A), \tag{32}$$

with $\cdot_{\mathcal{E}}$ given by (23) *and $a_{\mathcal{E}}(X) = a_A(\mathcal{E}\cdot_A X)$.*

Proof. For any $X, Y \in \Gamma(A)$, we have

$$\begin{aligned} N(X)\cdot_A N(Y) - N\big(N(X)\cdot_A Y + X\cdot_A N(Y) - N(X\cdot_A Y)\big) \\ &= X\cdot_A Y\cdot_A \mathcal{E}^2 - \mathcal{E}\cdot_A (X\cdot_A Y\cdot_A \mathcal{E} + X\cdot_A Y\cdot_A \mathcal{E} - X\cdot_A Y\cdot_A \mathcal{E}) \\ &= X\cdot_A Y\cdot_A \mathcal{E}^2 - X\cdot_A Y\cdot_A \mathcal{E}^2 = 0. \end{aligned}$$

Thus $N = \mathcal{E}\cdot_A$ is a Nijenhuis operator on the associative algebra $(\Gamma(A), \cdot_A)$.

Then we show that $N = \mathcal{E}\cdot_A$ is a Nijenhuis operator on the Lie algebroid $(A, [-,-]_A, a_A)$. It is obvious that N is a bundle map. Since $\mathcal{E}$ is a pseudoeventual identity on the F-algebroid A, taking $Y = \mathcal{E}$ in (22), we have

$$[X\cdot_A \mathcal{E}, \mathcal{E}]_A - [X, \mathcal{E}]_A\cdot_A \mathcal{E} = [\mathcal{E}, e]_A\cdot_A X\cdot_A \mathcal{E}. \tag{33}$$

For any $X, Y \in \Gamma(A)$, expanding $[\mathcal{E}\cdot_A X, \mathcal{E}\cdot_A Y]_A$ using the Hertling–Manin relation and by (33), we have

$$[N(X), N(Y)]_A - N\big([N(X), Y]_A + [X, N(Y)]_A - N([X,Y]_A)\big) = 0.$$

Thus $N = \mathcal{E}\cdot_A$ is a Nijenhuis operator on the Lie algebroid $(A, [-,-]_A, a_A)$. Therefore, $N = \mathcal{E}\cdot_A$ is a Nijenhuis operator on the F-algebroid A.

The second claim follows from Theorem 4.8. □

Corollary 4.11. *Let $(M, \bullet)$ be an F-manifold with an identity e and $\mathcal{E}$ a pseudoeventual identity on M. Then there is a new F-algebroid structure on TM given by*

$$X\bullet_{\mathcal{E}} Y = X\bullet Y\bullet \mathcal{E}, \quad [X,Y]_{\mathcal{E}} = [\mathcal{E}\bullet X, Y]_{\mathfrak{X}(M)} + [X, \mathcal{E}\bullet Y]_{\mathfrak{X}(M)} - \mathcal{E}\bullet [X,Y]_{\mathfrak{X}(M)},$$
$$a_{\mathcal{E}}(X) = \mathcal{E}\bullet X \quad \forall\, X, Y \in \mathfrak{X}(M).$$

5. Pre-F-algebroids and eventual identities

In this section, we introduce the notion of a pre-F-algebroid, and show that a pre-F-algebroid gives rise to an F-algebroid. Then we study eventual identities on a pre-F-algebroid, which give new pre-F-algebroids. Finally, we introduce the notion of a Nijenhuis operator on a pre-F-algebroid, and show that a Nijenhuis operator gives rise to a deformed pre-F-algebroid.

Some properties of pre-F-algebroids.

Definition 5.1. Let $(\mathfrak{g}, \cdot)$ be a commutative associative algebra and $(\mathfrak{g}, *)$ a pre-Lie algebra. Define $\Psi : \otimes^3\mathfrak{g} \to \mathfrak{g}$ by

$$\Psi(x, y, z) := x * (y \cdot z) - (x * y) \cdot z - y \cdot (x * z). \tag{34}$$

(i) The triple $(\mathfrak{g}, *, \cdot)$ is called a **pre-F-manifold algebra** if

$$\Psi(x, y, z) = \Psi(y, x, z) \quad \forall\, x, y, z \in \mathfrak{g}, \tag{35}$$

(ii) The triple $(\mathfrak{g}, *, \cdot)$ is called a **pre-Lie commutative algebra** (or **pre-Lie-com algebra**) if

$$\Psi(x, y, z) = 0 \quad \forall\, x, y, z \in \mathfrak{g}. \tag{36}$$

It is obvious that a pre-Lie-com algebra is a pre-F-manifold algebra.

Example 5.2 [24]. Let $(\mathfrak{g}, \cdot)$ be a commutative associative algebra with a derivation D. Then the new product

$$x * y = x \cdot D(y) \quad \forall\, x, y \in \mathfrak{g}$$

makes $(\mathfrak{g}, *, \cdot)$ being a pre-Lie-com algebra. Furthermore, $(\mathfrak{g}, [-, -], \cdot)$ is an F-manifold algebra, where the bracket is given by

$$[x, y] = x * y - y * x = x \cdot D(y) - y \cdot D(x) \quad \forall\, x, y \in \mathfrak{g}.$$

Let $\mathfrak{g} = \mathbb{R}[u^1, x_2, \dots, x_n]$ be the algebra of polynomials in n variables. Denote by $\mathfrak{D}_n = \left\{\sum_{i=1}^n p_i\, \partial_{u^i} \mid p_i \in \mathfrak{g}\right\}$ the space of derivations.

Example 5.3 [24]. Let $\mathfrak{g}$ be the algebra of polynomials in n variables. Define $\cdot : \mathfrak{D}_n \times \mathfrak{D}_n \to \mathfrak{D}_n$ and $* : \mathfrak{D}_n \times \mathfrak{D}_n \to \mathfrak{D}_n$ by

$$(p\partial_{u^i}) \cdot (q\partial_{u^j}) = (pq)\, \delta_{ij}\, \partial_{u^i}, \quad (p\partial_{u^i}) * (q\partial_{u^j}) = p\partial_{u^i}(q)\, \partial_{u^j} \qquad \forall\, p, q \in \mathfrak{g}.$$

Then $(\mathfrak{D}_n, *, \cdot)$ is a pre-Lie-com algebra with the identity $e = \partial_{u^1} + \cdots + \partial_{x_n}$. Furthermore, it follows that $(\mathfrak{D}_n, [-, -], \cdot)$ is an F-manifold algebra with the identity e, where the bracket is given by

$$[p\partial_{u^i}, q\partial_{u^j}] = p\partial_{u^i}(q)\, \partial_{u^j} - q\partial_{u^j}(p)\, \partial_{u^i} \quad \forall\, p, q \in \mathfrak{g}.$$

Definition 5.4. A **pre-F-algebroid** is a vector bundle A over M equipped with bilinear operations $\cdot_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ and $*_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$, and a bundle map $a_A : A \to TM$, called the anchor, such that $(A, *_A, a_A)$ is a pre-Lie algebroid, $(A, \cdot_A)$ is a commutative associative algebroid and $(\Gamma(A), *_A, \cdot_A)$ is a pre-F-manifold algebra. In particular, if $(\Gamma(A), *_A, \cdot_A)$ is a pre-Lie-com algebra, we call this pre-F-algebroid a **pre-Lie-com algebroid**.

We denote a pre-F-algebroid (or pre-Lie-com algebroid) by $(A, *_A, \cdot_A, a_A)$.

Definition 5.5. Let $(A, *_A, \cdot_A, a_A)$ and $(B, *_B, \cdot_B, a_B)$ be pre-F-algebroids over M. A bundle map $\varphi : A \to B$ is called a **homomorphism** of pre-F-algebroids, if the following conditions are satisfied:

$$\varphi(X \cdot_A Y) = \varphi(X) \cdot_B \varphi(Y), \quad \varphi(X *_A Y) = \varphi(X) *_B \varphi(Y), \quad a_B \circ \varphi = a_A$$

for all $X, Y \in \Gamma(A)$.

Proposition 5.6. *Assume that $(A, *_A, \cdot_A, a_A)$ is a pre-F-algebroid. Then we have an F-algebroid $(A, [-,-]_A, \cdot_A, a_A)$, and denoted by A^c, called the* **subadjacent F-algebroid** *of the pre-F-algebroid, where the bracket $[-,-]_A$ is given by*

$$[X, Y]_A = X *_A Y - Y *_A X \quad \forall X, Y \in \Gamma(A). \tag{37}$$

Proof. Since $(A, *_A, a_A)$ is a pre-Lie algebroid, $(A, [-,-]_A, a_A)$ is a Lie algebroid [22]. Since $(\Gamma(A), *_A, \cdot_A)$ is a pre-F-manifold algebra, $(\Gamma(A), [-,-]_A, \cdot_A)$ is an F-manifold algebra [14]. Thus $(A, [-,-]_A, \cdot_A, a_A)$ is an F-algebroid. $\square$

The notion of an F-manifold with a compatible flat connection was introduced by Manin [29]. Recall that an **F-manifold with a compatible flat connection** (**pre-Lie-com manifold**) is a triple $(M, \nabla, \bullet)$, where M is a manifold, ∇ is a flat connection and $\bullet$ is a $C^\infty(M)$-bilinear, commutative and associative multiplication on the tangent bundle TM such that $(TM, \nabla, \bullet, \mathrm{Id})$ is a pre-F-algebroid (pre-Lie-com algebroid). It is obvious that an F-manifold with a compatible flat connection is a special case of pre-F-algebroids. An F-manifold with a compatible flat connection (resp. pre-Lie-com manifold) is called **semisimple** if its subadjacent F-manifold is semisimple.

Proposition 5.7. *Let $(M, \nabla, \bullet, e)$ be a semisimple pre-Lie-com manifold with the canonical local coordinate systems $(u^1, \dots, u^n)$. Then we have*

$$\nabla_{\partial/\partial u^i} \frac{\partial}{\partial u^j} = 0, \quad i, j \in \{1, 2, \dots, n\}.$$

Proof. Set

$$\nabla_{\partial/\partial u^i} \frac{\partial}{\partial u^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

By (36), for any $i, j, k \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} 0 &= \nabla_{\partial/\partial u^i}\left(\frac{\partial}{\partial u^j} \bullet \frac{\partial}{\partial u^k}\right) - \left(\nabla_{\partial/\partial u^i} \frac{\partial}{\partial u^j}\right) \bullet \frac{\partial}{\partial u^k} - \frac{\partial}{\partial u^j} \bullet \left(\nabla_{\partial/\partial u^i} \frac{\partial}{\partial u^k}\right) \\ &= \sum_l \delta_{jk} \Gamma_{ik}^l \frac{\partial}{\partial x_l} - \Gamma_{ij}^k \frac{\partial}{\partial u^k} - \Gamma_{ik}^j \frac{\partial}{\partial u^j}. \end{aligned} \tag{38}$$

For $j \neq k$ in (38), we have $\Gamma_{ij}^k = 0 \ (j \neq k)$. For $j = k$ in (38), we have $\Gamma_{ij}^j = 0$. Thus for any $i, j, k \in \{1, 2, \dots, n\}$, we have $\Gamma_{ij}^k = 0$. $\square$

We give some useful formulas that will be frequently used in what follows.

Lemma 5.8. *Let* $(A, *_A, \cdot_A, a_A)$ *be a pre-*F*-algebroid. Then* $\Psi(X, Y, Z)$ *defined by* (34) *is a tensor field of type* (3, 1) *and symmetric in all arguments. Furthermore,* Ψ *satisfies*

$$\Psi(X\cdot_A Y, Z, W)-\Psi(X, Z, W)\cdot_A Y = \Psi(X\cdot_A Z, Y, W)-\Psi(X, Y, W)\cdot_A Z, \tag{39}$$

$$\Psi(X\cdot_A Y, Z, W)-\Psi(X\cdot_A Z, Y, W) = \Psi(W\cdot_A Y, X, Z)-\Psi(W\cdot_A Z, X, Y) \tag{40}$$

for all $X, Y, Z, W \in \Gamma(A)$.

Proof. It is straightforward to check that $\Psi(X, Y, Z)$ is a tensor field of type (3, 1). The symmetry of $\Psi(X, Y, Z)$ in the first two arguments is the consequence of (35) and in the last two arguments is the consequence of the commutativity of $\cdot_A$.

By the symmetry of Ψ, we have

$$\Psi(X\cdot_A Y, Z, W)-\Psi(X, Z, W)\cdot_A Y = \Psi(X\cdot_A W, Y, Z)-\Psi(X, Y, Z)\cdot_A W. \tag{41}$$

Interchanging Z and W in (41), we have

$$\Psi(X\cdot_A Y, W, Z)-\Psi(X, W, Z)\cdot_A Y = \Psi(X\cdot_A Z, Y, W)-\Psi(X, Y, W)\cdot_A Z.$$

By the symmetry of Ψ, equation (39) follows.

By (39), we have

$$\Psi(X\cdot_A Y, Z, W)-\Psi(X\cdot_A Z, Y, W) = \Psi(X, Z, W)\cdot_A Y-\Psi(X, Y, W)\cdot_A Z,$$

$$\Psi(W\cdot_A Y, X, Z)-\Psi(W\cdot_A Z, X, Y) = \Psi(W, X, Z)\cdot_A Y-\Psi(W, X, Y)\cdot_A Z.$$

By the symmetry of Ψ, we have

$$\Psi(X, Z, W)\cdot_A Y-\Psi(X, Y, W)\cdot_A Z = \Psi(W, X, Z)\cdot_A Y-\Psi(W, X, Y)\cdot_A Z.$$

Thus (40) holds. □

Lemma 5.9. *Let* $(A, *_A, \cdot_A, a_A)$ *be a pre-*F*-algebroid with an identity* e*. Then,*

$$\Psi(e, X, Y) = -(X *_A e)\cdot_A Y, \tag{42}$$

$$(X *_A e)\cdot_A Y = (Y *_A e)\cdot_A X \quad \forall X, Y \in \Gamma(A). \tag{43}$$

Proof. Equation (42) follows by a direct calculation. By the symmetry of Ψ and (42), equation (43) follows. □

Lemma 5.10. *Let* $(A, *_A, \cdot_A, a_A)$ *be a pre-Lie-com algebroid with an identity* e*. Then we have*

$$X *_A e = 0 \quad \forall X \in \Gamma(A). \tag{44}$$

Proof. The conclusion follows from the following relation:

$$X *_A (e\cdot_A e)-(X *_A e)\cdot_A e-(X *_A e)\cdot_A e = 0. \qquad \square$$

Example 5.11. Assume that $\{u\}$ is a coordinate system of $\mathbb{R}$. Define an anchor map $a : T\mathbb{R} \to T\mathbb{R}$, a multiplication $\cdot : \mathfrak{X}(\mathbb{R}) \times \mathfrak{X}(\mathbb{R}) \to \mathfrak{X}(\mathbb{R})$ and a multiplication $* : \mathfrak{X}(\mathbb{R}) \times \mathfrak{X}(\mathbb{R}) \to \mathfrak{X}(\mathbb{R})$ by

$$a\left(f\frac{\partial}{\partial u}\right) = uf\frac{\partial}{\partial u}, \quad f\frac{\partial}{\partial u} \cdot g\frac{\partial}{\partial u} = fg\frac{\partial}{\partial u}, \quad f\frac{\partial}{\partial u} * g\frac{\partial}{\partial u} = uf\frac{\partial g}{\partial u}\frac{\partial}{\partial u}$$

for all $f, g \in C^\infty(\mathbb{R})$. Then $(T\mathbb{R}, *, \cdot, a)$ is a pre-Lie-com algebroid with the identity $\partial/\partial u$. Furthermore, $(T\mathbb{R}, [-,-], \cdot, a)$ is an F-algebroid with the identity $\partial/\partial u$, where $[-,-]$ is given by

$$\left[f\frac{\partial}{\partial u}, g\frac{\partial}{\partial u}\right] = u\left(f\frac{\partial g}{\partial u} - g\frac{\partial f}{\partial u}\right)\frac{\partial}{\partial u}.$$

Definition 5.12. Let $(\mathfrak{g}, *, \cdot)$ be a pre-F-manifold algebra (pre-Lie-com algebra). An **action** of $\mathfrak{g}$ on a manifold M is a linear map $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ from $\mathfrak{g}$ to the space of vector fields on M, such that for all $x, y \in \mathfrak{g}$, we have

$$\rho(x * y - y * x) = [\rho(x), \rho(y)]_{\mathfrak{X}(M)}.$$

Given an action of a pre-F-manifold algebra (pre-Lie-com algebra) $\mathfrak{g}$ on M, let $A = M \times \mathfrak{g}$ be the trivial bundle. Define an anchor map $a_\rho : A \to TM$, a multiplication $\cdot_\rho : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ and a bracket $*_\rho : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ by

$$a_\rho(m, u) = \rho(u)_m \quad \forall\, m \in M,\ u \in \mathfrak{g}, \tag{45}$$

$$X \cdot_\rho Y = X \cdot Y, \tag{46}$$

$$X *_\rho Y = \mathcal{L}_{\rho(X)} Y + X * Y \quad \forall\, X, Y \in \Gamma(A), \tag{47}$$

where $X \cdot Y$ and $X * Y$ are the pointwise $C^\infty(M)$-bilinear multiplication and bracket, respectively.

Proposition 5.13. *With the above notations, we have that* $(A = M \times \mathfrak{g}, *_\rho, \cdot_\rho, a_\rho)$ *is a pre-F-algebroid* (*pre-Lie-com algebroid*), *which we call an* **action pre-F-algebroid** (*action pre-Lie-com algebroid*), *where* $*_\rho$, $\cdot_\rho$ *and* a_ρ *are given by* (47), (46) *and* (45), *respectively.*

Proof. It follows by a similar proof of Proposition 2.15. □

It is obvious that the subadjacent F-algebroid of the action pre-F-algebroid is an action F-algebroid.

Example 5.14. Consider the pre-Lie-com algebra $(\mathfrak{D}_n, \cdot, *)$ given by Example 5.3. Let $(t_1, \dots, t_n)$ be the canonical coordinate systems on $\mathbb{R}^n$. Let $\rho : \mathfrak{D}_n \to \mathfrak{X}(\mathbb{R}^n)$ is a map defined by

$$\rho(p(u^1, \dots, u^n)\,\partial_{u^i}) = p(t_1, \dots, t_n)\frac{\partial}{\partial t_i}, \quad i \in \{1, 2, \dots, n\}.$$

It is straightforward to check that ρ is an action of the pre-Lie-com algebra $\mathfrak{D}_n$ on $\mathbb{R}^n$. Thus $(A=\mathbb{R}^n\times\mathfrak{D}_n, *_\rho, \cdot_\rho, a_\rho)$ is a pre-Lie-com algebroid, where $*_\rho$, $\cdot_\rho$ and a_ρ are given by

$$\begin{aligned}
a_\rho(m, p(u^1,u^2,\dots,u^n)\,\partial_{u^i}) &= p(m)\frac{\partial}{\partial t_i}\Big|_m \quad \forall\, m\in\mathbb{R}^n,\\
(f\otimes(p\partial_{u^i}))\cdot_\rho(g\otimes(q\partial_{u^j})) &= (fg)\otimes(pq\delta_{ij}\,\partial_{u^i}),\\
(f\otimes(p\partial_{u^i}))*_\rho(g\otimes(q\partial_{u^j})) &= fp\frac{\partial g}{\partial t_i}\otimes(q\partial_{u^j})+(fg)\otimes p\partial_{u^i}(q)\,\partial_{u^j},
\end{aligned}$$

where $f,g\in C^\infty(\mathbb{R}^n)$ and $p,q\in\mathbb{R}[u^1,\dots,u^n]$.

Eventual identities of pre-F-algebroids.

Definition 5.15. Assume that $(A, *_A, \cdot_A, a_A)$ is a pre-F-algebroid with an identity e. A section $\mathcal{E}\in\Gamma(A)$ is called a **pseudoeventual identity** on A if the following equalities hold:

$$\Psi(\mathcal{E},X,Y) = -(\mathcal{E}*_A e)\cdot_A X\cdot_A Y, \tag{48}$$

$$(X*_A\mathcal{E})\cdot_A Y = (Y*_A\mathcal{E})\cdot_A X \quad \forall\, X,Y\in\Gamma(A). \tag{49}$$

A pseudoeventual identity $\mathcal{E}$ on the pre-F-algebroid with an identity e is called an **eventual identity** if it is invertible.

Proposition 5.16. *Let $(A, *_A, \cdot_A, e, a_A)$ be a pre-F-algebroid with an identity e. If $\mathcal{E}\in\Gamma(A)$ is a pseudoeventual identity on A, then $\mathcal{E}\in\Gamma(A)$ is a pseudoeventual identity on its subadjacent F-algebroid A^c.*

Proof. By a direct calculation, for $X,Y\in\Gamma(A)$, we have

$$\begin{aligned}
&P_{\mathcal{E}}(X,Y)-[e,\mathcal{E}]_A\cdot_A X\cdot_A Y\\
&\quad=\mathcal{E}*_A(X\cdot_A Y)-(X\cdot_A Y)*_A\mathcal{E}-(\mathcal{E}*_A X)\cdot_A Y+(X*_A\mathcal{E})\cdot_A Y\\
&\qquad\qquad-(\mathcal{E}*_A Y)\cdot_A X+(Y*_A\mathcal{E})\cdot_A X-(e*_A\mathcal{E})\cdot_A X\cdot_A Y+(\mathcal{E}*_A e)\cdot_A X\cdot_A Y\\
&\quad=\Psi(\mathcal{E},X,Y)+(\mathcal{E}*_A e)\cdot_A X\cdot_A Y-(X\cdot_A Y)*_A\mathcal{E}+(X*_A\mathcal{E})\cdot_A Y\\
&\qquad\qquad+(Y*_A\mathcal{E})\cdot_A X-(e*_A\mathcal{E})\cdot_A X\cdot_A Y.
\end{aligned}$$

By (48) and (49), we have

$$P_{\mathcal{E}}(X,Y)-[e,\mathcal{E}]_A\cdot_A X\cdot_A Y=0.$$

Thus $\mathcal{E}\in\Gamma(A)$ is a pseudoeventual identity on its subadjacent F-algebroid A^c. □

By Lemma 5.10, we have:

Proposition 5.17. *Let $(A, *_A, \cdot_A, a_A)$ be a pre-F-algebroid with an identity e and $\mathcal{E}$ an invertible element in $\Gamma(A)$. If $(A, *_A, \cdot_A, a_A)$ is a pre-Lie-com algebroid, then $\mathcal{E}$ is an eventual identity on A if and only if* (49) *holds.*

Lemma 5.18. *Let* $(A, *_A, \cdot_A, e, a_A)$ *be a pre-F-algebroid. Then for* $\mathcal{E} \in \Gamma(A)$, *equation* (48) *holds if and only if*

$$\Psi(X, \mathcal{E} \cdot_A Y, Z) = \Psi(Y, \mathcal{E} \cdot_A X, Z) \quad \forall X, Y, Z \in \Gamma(A). \tag{50}$$

Proof. Assume that (50) holds. By (39), we have

$$\Psi(\mathcal{E}, X, Z) \cdot_A Y - \Psi(\mathcal{E}, Y, Z) \cdot_A X = \Psi(X, \mathcal{E} \cdot_A Y, Z) - \Psi(Y, \mathcal{E} \cdot_A X, Z) = 0. \tag{51}$$

Taking $Y = e$ in (51), we have

$$\Psi(\mathcal{E}, X, Z) = -(\mathcal{E} *_A e) \cdot_A X \cdot_A Z.$$

This implies that (48) holds.

Conversely, if (48) holds, then we have

$$\Psi(\mathcal{E}, X, Z) \cdot_A Y - \Psi(\mathcal{E}, Y, Z) \cdot_A X = -(\mathcal{E} *_A e) \cdot_A X \cdot_A Z \cdot_A Y + (\mathcal{E} *_A e) \cdot_A Y \cdot_A Z \cdot_A X = 0.$$

By (39), we have

$$\Psi(X, \mathcal{E} \cdot_A Y, Z) = \Psi(Y, \mathcal{E} \cdot_A X, Z).$$

This implies that (50) holds. □

Let the set of all pseudoeventual identities on a pre-F-algebroid $(A, *_A, \cdot_A, a_A)$ be $\mathfrak{E}(A)$ with an identity e.

Proposition 5.19. *Let* $(A, *_A, \cdot_A, a_A)$ *be a pre-F-algebroid with an identity e. Then for any* $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{E}(A)$, *we have* $\mathcal{E}_1 \cdot_A \mathcal{E}_2 \in \mathfrak{E}(A)$. *Furthermore, if $\mathcal{E}$ is an eventual identity on A, then $\mathcal{E}^{-1}$ is also an eventual identity on A.*

Proof. Let $\mathcal{E}_1, \mathcal{E}_2$ be two pseudoeventual identities on the pre-F-algebroid A. For all $X, Y, Z \in \Gamma(A)$, by (50), the symmetry of Ψ and Lemma 5.18, we have

$$\Psi(\mathcal{E}_1 \cdot_A \mathcal{E}_2, X, Y) = -((\mathcal{E}_1 \cdot_A \mathcal{E}_2) *_A e) \cdot_A X \cdot_A Y.$$

For all $X, Y \in \Gamma(A)$, by (35), we have

$$\begin{aligned}
&(X *_A (\mathcal{E}_1 \cdot_A \mathcal{E}_2)) \cdot_A Y - (Y *_A (\mathcal{E}_1 \cdot_A \mathcal{E}_2)) \cdot_A X \\
&\quad = \Psi(\mathcal{E}_1, X, \mathcal{E}_2) \cdot_A Y + (X *_A \mathcal{E}_1) \cdot_A \mathcal{E}_2 \cdot_A Y + (X *_A \mathcal{E}_2) \cdot_A \mathcal{E}_1 \cdot_A Y \\
&\qquad\qquad - \Psi(\mathcal{E}_1, Y, \mathcal{E}_2) \cdot_A X - (Y *_A \mathcal{E}_1) \cdot_A \mathcal{E}_2 \cdot_A X - (Y *_A \mathcal{E}_2) \cdot_A \mathcal{E}_1 \cdot_A X.
\end{aligned}$$

By (39) and (50), we have

$$\Psi(\mathcal{E}_1, X, \mathcal{E}_2) \cdot_A Y - \Psi(\mathcal{E}_1, Y, \mathcal{E}_2) \cdot_A X = \Psi(\mathcal{E}_1 \cdot_A Y, X, \mathcal{E}_2) - \Psi(\mathcal{E}_1 \cdot_A X, Y, \mathcal{E}_2) = 0.$$

Using the above relation and by (49), we have

$$(X *_A (\mathcal{E}_1 \cdot_A \mathcal{E}_2)) \cdot_A Y - (Y *_A (\mathcal{E}_1 \cdot_A \mathcal{E}_2)) \cdot_A X = 0.$$

Thus $\mathcal{E}_1 \cdot_A \mathcal{E}_2 \in \mathfrak{E}(A)$.

Using (50) with X and Y replaced by $\mathcal{E}^{-1}\cdot_A X$ and $\mathcal{E}^{-1}\cdot_A Y$ respectively, we get

$$\begin{aligned}0&=\Psi(\mathcal{E}^{-1}\cdot_A X,\mathcal{E}\cdot_A\mathcal{E}^{-1}\cdot_A Y,Z)-\Psi(\mathcal{E}^{-1}\cdot_A Y,\mathcal{E}\cdot_A\mathcal{E}^{-1}\cdot_A X,Z)\\&=\Psi(\mathcal{E}^{-1}\cdot_A X,Y,Z)-\Psi(\mathcal{E}^{-1}\cdot_A Y,X,Z).\end{aligned}$$

By the symmetry of Ψ and Lemma 5.18, we have

$$\Psi(\mathcal{E}^{-1},X,Y)=-(\mathcal{E}^{-1}*_A e)\cdot_A X\cdot_A Y.$$

By (39) and (50), we have

$$\Psi(X,\mathcal{E},\mathcal{E}^{-1})\cdot_A Y=\Psi(Y,\mathcal{E},\mathcal{E}^{-1})\cdot_A X. \tag{52}$$

Furthermore, by a direct calculation, we have

$$\begin{aligned}(X*_A\mathcal{E}^{-1})\cdot_A Y\cdot_A\mathcal{E}&=\Psi(X,\mathcal{E},\mathcal{E}^{-1})\cdot_A Y-(X*_A e)\cdot_A Y+(X*_A\mathcal{E})\cdot_A Y\cdot_A\mathcal{E}^{-1},\\(Y*_A\mathcal{E}^{-1})\cdot_A X\cdot_A\mathcal{E}&=\Psi(Y,\mathcal{E},\mathcal{E}^{-1})\cdot_A X-(Y*_A e)\cdot_A X+(Y*_A\mathcal{E})\cdot_A X\cdot_A\mathcal{E}^{-1}.\end{aligned}$$

By (43), (49) and (52), we have

$$(X*_A\mathcal{E}^{-1})\cdot_A Y\cdot_A\mathcal{E}=(Y*_A\mathcal{E}^{-1})\cdot_A X\cdot_A\mathcal{E}.$$

Because $\mathcal{E}$ is invertible, we have

$$(X*_A\mathcal{E}^{-1})\cdot_A Y=(Y*_A\mathcal{E}^{-1})\cdot_A X.$$

Thus $\mathcal{E}^{-1}$ is an eventual identity on A. □

Proposition 5.20. *Let $(A,*_A,\cdot_A,a_A)$ be a pre-F-algebroid with an identity e. Then $\mathcal{E}$ is a pseudoeventual identity on A if and only if $(A,*_A,\cdot_\mathcal{E},a_A)$ is a pre-F-algebroid, where $\cdot_\mathcal{E}:\Gamma(A)\times\Gamma(A)\to\Gamma(A)$ is given by* (23).

Proof. Define

$$\tilde\Psi(X,Y,Z)=X*_A(Y\cdot_\mathcal{E} Z)-(X*_A Y)\cdot_\mathcal{E} Z-Y\cdot_\mathcal{E}(X*_A Z)\quad\forall X,Y,Z\in\Gamma(A).$$

By a straightforward computation, we have

$$\tilde\Psi(X,Y,Z)=\Psi(X,\mathcal{E}\cdot_A Y,Z)+\Psi(X,\mathcal{E},Y)\cdot_A Z+(X*_A\mathcal{E})\cdot_A Y\cdot_A Z, \tag{53}$$

$$\tilde\Psi(Y,X,Z)=\Psi(Y,\mathcal{E}\cdot_A X,Z)+\Psi(Y,\mathcal{E},X)\cdot_A Z+(Y*_A\mathcal{E})\cdot_A X\cdot_A Z. \tag{54}$$

By the symmetry of Ψ, $(A,*_A,\cdot_\mathcal{E},a_A)$ is a pre-F-algebroid if and only if

$$\Psi(X,\mathcal{E}\cdot_A Y,Z)-\Psi(Y,\mathcal{E}\cdot_A X,Z)=(Y*_A\mathcal{E})\cdot_A X\cdot_A Z-(X*_A\mathcal{E})\cdot_A Y\cdot_A Z. \tag{55}$$

By the symmetry of Ψ and (40), we have

$$\Psi(X,\mathcal{E}\cdot_A Y,e)-\Psi(Y,\mathcal{E}\cdot_A X,e)=\Psi(e\cdot_A Y,\mathcal{E},X)-\Psi(e\cdot_A X,\mathcal{E},Y)=0.$$

Taking $Z = e$ in (55), we have

$$(X *_A \mathcal{E}) \cdot_A Y = (Y *_A \mathcal{E}) \cdot_A X.$$

This implies that (49) holds. Furthermore, by (49), (55) implies that (50) holds. By Lemma 5.18, equation (50) is equivalent to (48). Thus $\mathcal{E}$ is a pseudoeventual identity on $(A, *_A, \cdot_A, e, a_A)$.

On the other hand, if $\mathcal{E}$ is a pseudoeventual identity on $(A, *_A, \cdot_A, e, a_A)$, by Lemma 5.18, we have

$$\Psi(X, \mathcal{E} \cdot_A Y, Z) = \Psi(Y, \mathcal{E} \cdot_A X, Z).$$

Furthermore, (55) follows by (49). Thus $(A, *_A, \cdot_{\mathcal{E}}, a_A)$ is a pre-F-algebroid. □

Corollary 5.21. *Let $(M, \nabla, \bullet)$ be an F-manifold with a compatible flat connection and $\mathcal{E}$ a pseudoeventual identity on M. Then $(M, \nabla, \bullet_{\mathcal{E}})$ is also an F-manifold with a compatible flat connection, where $\bullet_{\mathcal{E}}$ is given by*

$$X \bullet_{\mathcal{E}} Y = X \bullet Y \bullet \mathcal{E} \quad \forall X, Y \in \mathfrak{X}(M). \tag{56}$$

Theorem 5.22. *Let $(A, *_A, \cdot_A, a_A)$ be a pre-F-algebroid with an identity e. Then $\mathcal{E}$ is an eventual identity on A if and only if $(A, *_A, \cdot_{\mathcal{E}}, a_A)$ is a pre-F-algebroid with the identity $\mathcal{E}^{-1}$, which is called the Dubrovin's dual of $(A, *_A, \cdot_A, a_A)$, where $\cdot_{\mathcal{E}}$ is given by* (23)*. Moreover, on the pre-F-algebroid $(A, *_A, \cdot_{\mathcal{E}}, \mathcal{E}^{-1}, a_A)$, e is an eventual identity and the map*

$$(A, *_A, \cdot_A, e, a_A, \mathcal{E}) \to (A, *_A, \cdot_{\mathcal{E}}, \mathcal{E}^{-1}, a_A, e^{\dagger}) \tag{57}$$

is an involution of the set of pre-F-algebroids with eventual identities, where $e^{\dagger} = \mathcal{E}^{-2}$ is the inverse of e with respect to the multiplication $\cdot_{\mathcal{E}}$.

Proof. By Proposition 5.20, the first claim follows immediately. For the second claim, assume that $\mathcal{E}$ is an eventual identity on $(A, *_A, \cdot_A, e, a_A)$. We need to show that e is an eventual identity on the pre-F-algebroid $(A, *_A, \cdot_{\mathcal{E}}, \mathcal{E}^{-1}, a_A)$, i.e.,

$$\tilde{\Psi}(e, X, Y) = -(e *_A \mathcal{E}^{-1}) \cdot_{\mathcal{E}} X \cdot_{\mathcal{E}} Y, \tag{58}$$

$$(X *_A e) \cdot_{\mathcal{E}} Y = (Y *_A e) \cdot_{\mathcal{E}} X. \tag{59}$$

By (43), we have

$$(X *_A e) \cdot_{\mathcal{E}} Y - (Y *_A e) \cdot_{\mathcal{E}} X = ((X *_A e) \cdot_A Y - (Y *_A e) \cdot_A X) \cdot_A \mathcal{E} = 0,$$

which implies that (59) holds.

On the one hand, by (48) and (50), we have

$$\begin{aligned}\tilde{\Psi}(e, X, Y) &= \Psi(\mathcal{E}, X, Y) + \Psi(\mathcal{E}, e, X) \cdot_A Y + (e *_A \mathcal{E}) \cdot_A X \cdot_A Y \\ &= -2(\mathcal{E} *_A e) \cdot_A X \cdot_A Y + (e *_A \mathcal{E}) \cdot_A X \cdot_A Y.\end{aligned}$$

On the other hand, taking $X=\mathcal{E}$ and $Y=\mathcal{E}^{-1}$ in (48), by the symmetry of Ψ, we have

$$e *_A e-(e *_A \mathcal{E}) \cdot_A \mathcal{E}^{-1}-(e *_A \mathcal{E}^{-1}) \cdot_A \mathcal{E}=-(\mathcal{E} *_A e) \cdot_A \mathcal{E}^{-1}.$$

Furthermore, by (43), we have

$$(e *_A \mathcal{E}^{-1}) \cdot_A \mathcal{E}^2=(e *_A e) \cdot_A \mathcal{E}-e *_A \mathcal{E}+\mathcal{E} *_A e=2\mathcal{E} *_A e-e *_A \mathcal{E}.$$

Thus we have

$$\tilde{\Psi}(e, X, Y)=-(e *_A \mathcal{E}^{-1}) \cdot_A \mathcal{E}^2 \cdot_A X \cdot_A Y=-(e *_A \mathcal{E}^{-1}) \cdot_{\mathcal{E}} X \cdot_{\mathcal{E}} Y.$$

which implies that (58) holds.

By Proposition 5.19, we have that $e^\dagger=\mathcal{E}^{-2}$ is an eventual identity on the pre-F-algebroid $(A, *_A, \cdot_{\mathcal{E}}, \mathcal{E}^{-1}, a_A)$. Then similar to the proof of Theorem 4.4, the map given by (57) is an involution of the set of pre-F-algebroids with eventual identities. □

Example 5.23. Consider the pre-Lie-com algebra $(\mathfrak{g}, *, \cdot)$ with an identity e given by Example 5.2. By a direct calculation, for any $\mathcal{E} \in \mathfrak{g}$, we have

$$(x * \mathcal{E}) \cdot y-(y * \mathcal{E}) \cdot x=x \cdot D(\mathcal{E}) \cdot y-y \cdot D(\mathcal{E}) \cdot x=0 \quad \forall x, y \in \mathfrak{g}.$$

By Proposition 5.17, $\mathcal{E}$ is a pseudoeventual identity on $\mathfrak{g}$. Thus any element of $\mathfrak{g}$ is a pseudoeventual identity on $\mathfrak{g}$. Furthermore, for any $\mathcal{E} \in \mathfrak{g}$, there is a new pre-$F$-manifold algebra structure on $\mathfrak{g}$ given by

$$x \cdot_{\mathcal{E}} y=x \cdot y \cdot \mathcal{E}, \quad x * y=x \cdot D(y) \qquad \forall x, y \in \mathfrak{g}.$$

Example 5.24. Let $(M, \nabla, \bullet, e)$ be a semisimple pre-Lie-com manifold with local coordinate systems $(u^1, \ldots, u^n)$. Then any pseudoeventual identity on TM is

$$\mathcal{E}=f_1(u^1) \frac{\partial}{\partial u^1}+\cdots+f_n(u^n) \frac{\partial}{\partial u^n},$$

where $f_i(u^i) \in C^\infty(M)$ depends only on u^i for $i=1,2, \ldots, n$. Furthermore, if all $f_i(u^i)$ are nonvanishing everywhere, then $\mathcal{E} \in \mathfrak{X}(M)$ is an eventual identity.

Example 5.25. Let (u^1, u^2) be a local coordinate systems on $\mathbb{R}^2$. Define

$$\frac{\partial}{\partial u^1} \bullet \frac{\partial}{\partial u^i}=\frac{\partial}{\partial u^i}, \quad \frac{\partial}{\partial u^2} \bullet \frac{\partial}{\partial u^2}=0, \quad \frac{\partial}{\partial u^i} * \frac{\partial}{\partial u^j}=0, \qquad i, j \in \{1,2\}.$$

Then $(T\mathbb{R}^2, *, \bullet, \mathrm{Id})$ is a pre-Lie-com algebroid with the identity $\partial/\partial u^1$ and thus $(T\mathbb{R}^2, *, \bullet, \mathrm{Id})$ is a pre-F-algebroid with the identity $\partial/\partial u^1$.

Furthermore, any pseudoeventual identity on $(T\mathbb{R}^2, *, \bullet, \mathrm{Id})$ is of the form

$$\mathcal{E}=f_1(u^1) \frac{\partial}{\partial u^1}+f_2(u^1, u^2) \frac{\partial}{\partial u^2},$$

with $\partial f_1/\partial u^1 = \partial f_2/\partial u^2$, where $f_1 \in C^\infty(\mathbb{R}^2)$ depends only on u^1 and f_2 is any smooth function. Furthermore, any pseudoeventual identity on the subadjacent F-algebroid of $(T\mathbb{R}^2, *, \bullet, \mathrm{Id})$ is of the form

$$\mathcal{E} = f_1(u^1)\frac{\partial}{\partial u^1} + f_2(u^1, u^2)\frac{\partial}{\partial u^2}.$$

In particular, if $f_1(u^1)$ is nonvanishing everywhere, then $\mathcal{E}$ is an eventual identity on the subadjacent F-algebroid of $(T\mathbb{R}^2, *, \bullet, \mathrm{Id})$.

Theorem 5.26 [27]. *Let $(M, \nabla, \bullet)$ be an F-manifold with a compatible flat connection. Let $(u^1, u^2, \dots, u^n)$ be the canonical coordinate systems on M. If X and Y in $\mathfrak{X}(M)$ satisfy*

$$(\nabla_Z X)\bullet W = (\nabla_W X)\bullet Z, \quad (\nabla_Z Y)\bullet W = (\nabla_W Y)\bullet Z \qquad \forall\, W, Z \in \mathfrak{X}(M),$$

then the associated flows

$$u^i_t = c^i_{jk} X^k u^i_x \quad \text{and} \quad u^i_\tau = c^i_{jk} Y^k u^j_x \tag{60}$$

commute, where

$$\frac{\partial}{\partial u^i}\bullet\frac{\partial}{\partial u^j} = c^k_{ij}\frac{\partial}{\partial u^k}, \quad X = X^i\frac{\partial}{\partial u^i} \quad \text{and} \quad Y = Y^i\frac{\partial}{\partial u^i}.$$

Proposition 5.27. *Let $(M, \nabla, \bullet)$ be an F-manifold with a compatible flat connection and an identity e. Assume that $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{X}(M)$ are pseudoeventual identities. Then the flows*

$$u^i_t = c^i_{jk} X^k u^i_x, \quad u^i_\tau = c^i_{jk} Y^k u^j_x, \quad u^i_s = X^p Y^q c^i_{jk} c^k_{pq} u^i_x \tag{61}$$

commute, where

$$\frac{\partial}{\partial u^i}\bullet\frac{\partial}{\partial u^j} = c^k_{ij}\frac{\partial}{\partial u^k}, \quad \mathcal{E}_1 = X^i\frac{\partial}{\partial u^i} \quad \text{and} \quad \mathcal{E}_2 = Y^i\frac{\partial}{\partial u^i}.$$

Proof. Since $\mathcal{E}_1 \in \mathfrak{X}(M)$ and $\mathcal{E}_2 \in \mathfrak{X}(M)$ are pseudoeventual identities on $(M, \nabla, \bullet)$, by Proposition 5.19, $\mathcal{E}_1 \bullet \mathcal{E}_2$ is also a pseudoeventual identity. Thus $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_1 \bullet \mathcal{E}_2$ satisfy (49). Furthermore, we have

$$\mathcal{E}_1 \bullet \mathcal{E}_2 = X^p Y^q c^k_{pq}\frac{\partial}{\partial u^k}.$$

By Theorem 5.26, the claim follows. □

Theorem 5.28 [27]. *Let $(M, \nabla, \bullet)$ be an F-manifold with a compatible flat connection. Let $(u^1, u^2, \dots, u^n)$ be the canonical coordinate systems on M and $(X_{(1,0)}, \dots, X_{(n,0)})$ a basis of flat vector fields. Define the primary flows by*

$$u^i_{t_{(p,0)}} = c^i_{jk} X^k_{(p,0)} u^j_x. \tag{62}$$

Then there is a well-defined higher flows of the hierarchy defined by

$$u^i_{t_{(p,\alpha)}} = c^i_{jk}\, X^k_{(p,\alpha)}\, u^j_x, \tag{63}$$

by means of the following recursive relations:

$$\nabla_{\partial/\partial u^j} X^i_{(p,\alpha)} = c^i_{jk}\, X^k_{(p,\alpha-1)}\, u^k_x. \tag{64}$$

Furthermore, *the flows of the principal hierarchy* (63) *commute.*

Proposition 5.29. *Let $(M, \nabla, \bullet)$ be an F-manifold with a compatible flat connection and an identity e. Let $(X_{(1,0)}, \dots, X_{(n,0)})$ be a basis of flat vector fields. Assume that $\mathcal{E} \in \mathfrak{X}(M)$ is a pseudoeventual identity. Define the primary flows by*

$$u^i_{t_{(p,0)}} = c^m_{jk}\, c^i_{ml}\, \mathcal{E}^l X^k_{(p,0)}\, u^j_x, \tag{65}$$

where $\mathcal{E} = \mathcal{E}^i(\partial/\partial u^i)$. Then there is a well-defined higher flows of the hierarchy defined by

$$u^i_{t_{(p,\alpha)}} = c^m_{jk}\, c^i_{ml}\, \mathcal{E}^l X^k_{(p,\alpha)}\, u^j_x, \tag{66}$$

by means of the following recursive relations:

$$\nabla_{\partial/\partial u^j} X^i_{(p,\alpha)} = c^m_{jk}\, c^i_{ml}\, \mathcal{E}^l X^k_{(p,\alpha-1)}\, u^k_x. \tag{67}$$

Furthermore, *the flows of the principal hierarchy* (66) *commute.*

Proof. Since $\mathcal{E} \in \mathfrak{X}(M)$ is a pseudoeventual identity on $(M, \nabla, \bullet)$, we have by Proposition 5.20 that $(M, \nabla, \bullet_{\mathcal{E}})$ is also an F-manifold with a compatible flat connection, where

$$X \bullet_{\mathcal{E}} Y = X \bullet Y \bullet \mathcal{E} \quad \forall X, Y \in \mathfrak{X}(M).$$

Furthermore, we have

$$\frac{\partial}{\partial u^i} \bullet_{\mathcal{E}} \frac{\partial}{\partial u^j} = c^m_{ij}\, c^k_{ml}\, \mathcal{E}^l \frac{\partial}{\partial u^k}.$$

By Theorem 5.28, the claim follows. □

Nijenhuis operators and deformed pre-F-algebroids. From [22] a Nijenhuis operator on a pre-Lie algebroid $(A, *_A, a_A)$ is a bundle map $N : A \to A$ such that

$$N(X) *_A N(Y) = N\big(N(X) *_A Y + X *_A N(Y) - N(X *_A Y)\big) \quad \forall X, Y \in \Gamma(A). \tag{68}$$

Definition 5.30. Let $(A, *_A, \cdot_A, a_A)$ be a pre-F-algebroid. A bundle map $N : A \to A$ is called a **Nijenhuis operator** on $(A, *_A, \cdot_A, a_A)$ if N is both a Nijenhuis operator on the commutative associative algebra $(\Gamma(A), \cdot_A)$ and a Nijenhuis operator on the pre-Lie algebroid $(A, *_A, a_A)$.

Theorem 5.31. *Assume that $N : A \to A$ is a Nijenhuis operator on a pre-F-algebroid $(A, *_A, \cdot_A, a_A)$. Then $(A, *_N, \cdot_N, a_N = a_A \circ N)$ is a pre-F-algebroid and N is a homomorphism from the pre-F-algebroid $(A, *_N, \cdot_N, a_N = a_A \circ N)$ to $(A, *_A, \cdot_A, a_A)$, where the operation $\cdot_N$ is given by equation* (29) *and the operation $*_N : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is given by*

$$X *_N Y = N(X) *_A Y + X *_A N(Y) - N(X *_A Y) \quad \forall X, Y \in \Gamma(A). \tag{69}$$

Proof. Since N is a Nijenhuis operator on the commutative associative algebra $(\Gamma(A), \cdot_A)$, it follows that $(\Gamma(A), \cdot_N)$ is a commutative associative algebra. Since N is a Nijenhuis operator on the pre-Lie algebroid $(A, *_A, a_A)$, $(A, *_N, a_N)$ is a pre-Lie algebroid [22].

Define

$$\begin{aligned}\Psi_N(X, Y, Z)& \\ := X *_N (Y \cdot_N Z) &- (X *_N Y) \cdot_N Z - (X *_N Z) \cdot_N Y \quad \forall X, Y, Z \in \Gamma(A).\end{aligned} \tag{70}$$

By a direct calculation, we have

$$\begin{aligned}\Psi_N(X, Y, Z) = \Psi(NX, NY, Z) + \Psi(NX, Y, NZ) + \Psi(X, NY, NZ)& \\ - N\big(\Psi(NX, Y, Z) + \Psi(X, NY, Z) + \Psi(X, Y, NZ)\big)& \\ + N^2(\Psi(X, Y, Z)).&\end{aligned}$$

Thus by (35), we have

$$\Psi_N(X, Y, Z) = \Psi_N(Y, X, Z).$$

This implies that $(A, *_N, \cdot_N, a_N = a_A \circ N)$ is a pre-F-algebroid. It is not hard to see that N is a homomorphism from the pre-F-algebroid $(A, *_N, \cdot_N, a_N = a_A \circ N)$ to $(A, *_A, \cdot_A, a_A)$. □

Proposition 5.32. *Let $(A, *_A, \cdot_A, a_A)$ be a pre-F-algebroid with an identity e and $\mathcal{E}$ a pseudoeventual identity on A. Then the endomorphism $N = \mathcal{E} \cdot_A$ is a Nijenhuis operator on the pre-F-algebroid $(A, *_A, \cdot_A, a_A)$. Furthermore, $(A, *_\mathcal{E}, \cdot_\mathcal{E}, a_\mathcal{E})$ is a pre-F-algebroid, where the multiplication $*_\mathcal{E}$ is given by*

$$X *_\mathcal{E} Y = (\mathcal{E} \cdot_A X) *_A Y + X *_A (\mathcal{E} \cdot_A Y) - \mathcal{E} \cdot_A (X *_A Y) \quad \forall X, Y \in \Gamma(A), \tag{71}$$

the multiplication $\cdot_\mathcal{E}$ is given by (23) *and $a_\mathcal{E}(X) = a_A(\mathcal{E} \cdot_A X)$.*

Proof. By (35), we have

$$\Psi(\mathcal{E} \cdot_A X, \mathcal{E}, Y) = \Psi(Y, \mathcal{E} \cdot_A X, \mathcal{E}) \quad \forall X, Y \in \Gamma(A),$$

which implies that

$$(\mathcal{E} \cdot_A X) *_A (\mathcal{E} \cdot_A Y) = Y *_A (X \cdot_A \mathcal{E} \cdot_A \mathcal{E}) - (Y *_A (\mathcal{E} \cdot_A X)) \cdot_A \mathcal{E} + ((\mathcal{E} \cdot X) *_A Y) \cdot_A \mathcal{E}. \tag{72}$$

Since $\mathcal{E}$ is a pseudoeventual identity on A, by (48) and the symmetry of Ψ, we have

$$\Psi(X, \mathcal{E}, Y) = -(\mathcal{E} *_A e) \cdot_A X \cdot_A Y.$$

which implies that

$$X *_A (\mathcal{E} \cdot_A Y) = -(\mathcal{E} *_A e) \cdot_A X \cdot_A Y - (X *_A \mathcal{E}) \cdot_A Y - (X *_A Y) \cdot_A \mathcal{E}. \tag{73}$$

By (48), (49), (72), (73) and the symmetry of Ψ, we have

$$N(X) *_A N(Y) - N\big(N(X) *_A Y + X *_A N(Y) - N(X *_A Y)\big) = 0.$$

Thus $N = \mathcal{E} \cdot_A$ is a Nijenhuis operator on the pre-Lie algebroid $(A, *_A, a_A)$.

Also, $N = \mathcal{E} \cdot_A$ is a Nijenhuis operator on the commutative associative algebra $(\Gamma(A), \cdot_A)$. Therefore, $N = \mathcal{E} \cdot_A$ is a Nijenhuis operator on the pre-F-algebroid $(A, *_A, \cdot_A, a_A)$. The second claim follows. □

Corollary 5.33. *Let $(M, \nabla, \bullet)$ be an F-manifold with a compatible flat connection and $\mathcal{E}$ a pseudoeventual identity on M. Then there is a new pre-F-algebroid structure on TM given by*

$$X \bullet_{\mathcal{E}} Y = X \bullet Y \bullet \mathcal{E}, \quad X *_{\mathcal{E}} Y = \nabla_{\mathcal{E} \bullet X} Y + \nabla_{\mathcal{E} \bullet Y} X - \mathcal{E} \bullet (\nabla_X Y),$$
$$a_{\mathcal{E}}(X) = \mathcal{E} \bullet X \quad \forall\, X, Y \in \mathfrak{X}(M).$$

Acknowledgements

This research was supported by NSFC (grants 11922110, 11901501). Liu was also supported by the National Key Research and Development Program of China (grant 2021YFA1002000). Cruz Morales wants to thank Max Planck Institute for Mathematics — where the last part of this work was written — for its hospitality and financial support. The authors thank the referees for valuable comments that improved the presentation of the paper.

References

[1] A. Arsie and P. Lorenzoni, "F-manifolds with eventual identities, bidifferential calculus and twisted Lenard–Magri chains", *Int. Math. Res. Not.* **2013**:17 (2013), 3931–3976. MR Zbl

[2] A. Arsie and P. Lorenzoni, "From the Darboux–Egorov system to bi-flat F-manifolds", *J. Geom. Phys.* **70** (2013), 98–116. MR Zbl

[3] A. Arsie and P. Lorenzoni, "F-manifolds, multi-flat structures and Painlevé transcendents", *Asian J. Math.* **23**:5 (2019), 877–904. MR Zbl

[4] A. Arsie, A. Buryak, P. Lorenzoni, and P. Rossi, "Flat F-manifolds, F-CohFTs, and integrable hierarchies", *Comm. Math. Phys.* **388**:1 (2021), 291–328. MR Zbl

[5] M. N. Boyom, "Cohomology of Koszul–Vinberg algebroids and Poisson manifolds, I", pp. 99–110 in *Lie algebroids and related topics in differential geometry* (Warsaw, 2000), Banach Center Publ. **54**, Polish Acad. Sci. Inst. Math., Warsaw, 2001. MR Zbl

[6] M. N. Boyom, "KV-cohomology of Koszul–Vinberg algebroids and Poisson manifolds", *Internat. J. Math.* **16**:9 (2005), 1033–1061. MR Zbl

[7] D. Burde, "Left-symmetric algebras, or pre-Lie algebras in geometry and physics", *Cent. Eur. J. Math.* **4**:3 (2006), 323–357. MR Zbl

[8] J. F. Cariñena, J. Grabowski, and G. Marmo, "Quantum bi-Hamiltonian systems", *Internat. J. Modern Phys. A* **15**:30 (2000), 4797–4810. MR Zbl

[9] A. Cayley, "On the theory of the analytical forms called trees", pp. 242–246 in *The collected mathematical papers*, Cambridge University Press, Cambridge, 1890.

[10] N. Combe and Y. I. Manin, "F-manifolds and geometry of information", *Bull. Lond. Math. Soc.* **52**:5 (2020), 777–792. MR Zbl

[11] J. A. Cruz Morales and A. Torres-Gomez, "On F-algebroids and Dubrovin's duality", *Arch. Math.* (*Brno*) **55**:2 (2019), 109–122. MR

[12] L. David and I. A. B. Strachan, "Compatible metrics on a manifold and nonlocal bi-Hamiltonian structures", *Int. Math. Res. Not.* **2004**:66 (2004), 3533–3557. MR Zbl

[13] L. David and I. A. B. Strachan, "Dubrovin's duality for F-manifolds with eventual identities", *Adv. Math.* **226**:5 (2011), 4031–4060. MR Zbl

[14] V. Dotsenko, "Algebraic structures of F-manifolds via pre-Lie algebras", *Ann. Mat. Pura Appl.* (4) **198**:2 (2019), 517–527. MR Zbl

[15] B. Dubrovin, "Geometry of 2D topological field theories", pp. 120–348 in *Integrable systems and quantum groups* (Terme, 1993), Springer, Berlin, 1996. Zbl

[16] C. Hertling, *Frobenius manifolds and moduli spaces for singularities*, Cambridge Tracts in Mathematics **151**, Cambridge University Press, Cambridge, 2002. MR Zbl

[17] C. Hertling and Y. Manin, "Weak Frobenius manifolds", *Internat. Math. Res. Notices* **1999**:6 (1999), 277–286. MR Zbl

[18] M. Kato, T. Mano, and J. Sekiguchi, "Flat structure on the space of isomonodromic deformations", *SIGMA Symmetry Integrability Geom. Methods Appl.* **16** (2020), art. id. 110. MR Zbl

[19] H. Kawakami and T. Mano, "Regular flat structure and generalized Okubo system", *Comm. Math. Phys.* **369**:2 (2019), 403–431. MR Zbl

[20] Y. Kosmann-Schwarzbach and F. Magri, "Poisson–Nijenhuis structures", *Ann. Inst. H. Poincaré Phys. Théor.* **53**:1 (1990), 35–81. MR Zbl

[21] Y.-P. Lee, "Quantum K-theory, I: Foundations", *Duke Math. J.* **121**:3 (2004), 389–424. MR Zbl

[22] J. Liu, Y. Sheng, C. Bai, and Z. Chen, "Left-symmetric algebroids", *Math. Nachr.* **289**:14-15 (2016), 1893–1908. MR Zbl

[23] J. Liu, Y. Sheng, and C. Bai, "Pre-symplectic algebroids and their applications", *Lett. Math. Phys.* **108**:3 (2018), 779–804. MR Zbl

[24] J. Liu, Y. Sheng, and C. Bai, "F-manifold algebras and deformation quantization via pre-Lie algebras", *J. Algebra* **559** (2020), 467–495. MR Zbl

[25] P. Lorenzoni, "Darboux–Egorov system, bi-flat F-manifolds and Painlevé, VI", *Int. Math. Res. Not.* **2014**:12 (2014), 3279–3302. MR Zbl

[26] P. Lorenzoni and M. Pedroni, "Natural connections for semi-Hamiltonian systems: the case of the ϵ-system", *Lett. Math. Phys.* **97**:1 (2011), 85–108. MR

[27] P. Lorenzoni, M. Pedroni, and A. Raimondo, "F-manifolds and integrable systems of hydrodynamic type", *Arch. Math.* (*Brno*) **47**:3 (2011), 163–180. MR Zbl

[28] K. C. H. Mackenzie, *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series **213**, Cambridge University Press, Cambridge, 2005. MR Zbl

[29] Y. I. Manin, "F-manifolds with flat structure and Dubrovin's duality", *Adv. Math.* **198**:1 (2005), 5–26. Zbl

[30] S. A. Merkulov, "Operads, deformation theory and F-manifolds", pp. 213–251 in *Frobenius manifolds*, Aspects Math. **E36**, Friedr. Vieweg, Wiesbaden, 2004. MR Zbl

Received August 11, 2022. Revised November 20, 2023.

JOHN ALEXANDER CRUZ MORALES
MAX PLANCK INSTITUTE FOR MATHEMATICS
BONN
GERMANY
Current address:
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD NACIONAL DE COLOMBIA
BOGOTÁ
COLOMBIA
jacruzmo@unal.edu.co

JIEFENG LIU
SCHOOL OF MATHEMATICS AND STATISTICS
NORTHEAST NORMAL UNIVERSITY
JILIN
CHINA
liujf534@nenu.edu.cn

YUNHE SHENG
DEPARTMENT OF MATHEMATICS
JILIN UNIVERSITY
JILIN
CHINA
shengyh@jlu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 326, No. 2, 2023
https://doi.org/10.2140/pjm.2023.326.285

EXISTENCE OF PRINCIPAL VALUES OF SOME SINGULAR INTEGRALS ON CANTOR SETS, AND HAUSDORFF DIMENSION

JULIÀ CUFÍ, JUAN JESÚS DONAIRE, PERTTI MATTILA AND JOAN VERDERA

Consider a standard Cantor set in the plane of Hausdorff dimension 1. If the linear density of the associated measure μ vanishes, then the set of points where the principal value of the Cauchy singular integral of μ exists has Hausdorff dimension 1. The result is extended to Cantor sets in $\mathbb{R}^d$ of Hausdorff dimension α and Riesz singular integrals of homogeneity $-\alpha$, $0 < \alpha < d$: the set of points where the principal value of the Riesz singular integral of μ exists has Hausdorff dimension α. A martingale associated with the singular integral is introduced to support the proof.

1. Introduction

Our main result deals with the Cauchy singular integral on Cantor sets in the plane and the proof extends with minor variations to the Riesz transforms in $\mathbb{R}^d$. We first proceed to formulate the result for the Cauchy integral and then we take care of the Riesz transforms.

The appropriate Cantor sets for the Cauchy integral are defined as follows. Let $(\lambda_n)_{n=1}^{\infty}$ a sequence of real numbers satisfying $\frac{1}{4} \leq \lambda_n \leq \lambda < \frac{1}{2}$. Let $Q_0 := [0, 1] \times [0, 1]$ be the unit square. Take the 4 squares contained in Q_0 with sides of length λ_1 parallel to the coordinate axis having a vertex in common with Q_0 (the 4 "corner squares" of side length λ_1). Repeat in each of these 4 squares the same procedure with the dilation factor λ_1 replaced by λ_2 to get 16 squares of side length $\lambda_1\lambda_2$. Proceeding inductively we obtain at the n-th step 4^n squares Q_j^n, $1 \leq j \leq 4^n$, of side length $s_n = \lambda_1 \cdots \lambda_n$. Our Cantor set is

$$K = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{4^n} Q_j^n.$$

MSC2020: primary 42B20; secondary 30E20.
Keywords: Cauchy singular integral, Riesz singular integral, Cantor set, Hausdorff dimension, martingale.

Let μ be the Borel probability measure on K with $\mu(Q_j^n) = 4^{-n}$ and denote by a_n the linear density at generation n, that is,

$$a_n = \frac{1}{4^n s_n} = \frac{\mu(Q_j^n)}{s_n} \leq 1.$$

Set $\mathcal{D}_n = \{Q_j^n : j = 1, \ldots, 4^n\}$ and $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n$.

Theorem 1.1. *If* $\lim_{n\to\infty} a_n = 0$, *then the set of points* $z \in K$ *for which the principal value*

$$\lim_{\varepsilon\to 0} \int_{|w-z|>\varepsilon} \frac{1}{w-z}\, d\mu w \tag{1-1}$$

exists has Hausdorff dimension greater than or equal to 1.

This solves a problem posed in [Cufí et al. 2022, Open problem 5.5, p. 1621].

If $a_n = 1$ for all n, then K is the famous Garnett–Ivanov Cantor set, which has positive and finite one-dimensional Hausdorff measure but zero analytic capacity. In this case it was noticed in [Cufí et al. 2022] that the principal value does not exist at any point of K. If $a_n \to 0$, then the Hausdorff dimension of K is greater than or equal to 1 and it has non-sigma finite one-dimensional Hausdorff measure. If in addition $\sum_n a_n^2 < \infty$, then the principal value exists μ almost everywhere. So Theorem 1.1 is relevant only when $a_n \to 0$ slowly. That the condition $\sum_n a_n^2 < \infty$ implies the almost everywhere existence of principal values can be seen in two ways. First, we introduce a martingale $(S_n)_{n=0}^{\infty}$ (see (2-1)) and show that the increments $|S_{n+1}(x) - S_n(x)|$ are bounded by $C a_n$, with the constant C independent of n and x. In Lemma 2.4 we prove that for any point x the principal value exists at x if and only if $(S_n(x))_{n=0}^{\infty}$ converges. If $\sum_n a_n^2 < \infty$, then S_n is an L^2 martingale and consequently it converges almost everywhere. Alternatively, the condition $\sum_n a_n^2 < \infty$ implies that the Cauchy singular integral operator is bounded in $L^2(\mu)$. In [Mattila and Verdera 2009] it was shown in a very general setting that L^2 boundedness together with zero density of the measure yields the almost everywhere existence of principal values.

The main argument in the proof of Theorem 1.1 deals with case where $\sum_n a_n^2 = \infty$. It is a variation of a line of reasoning used in other situations (see [Donaire et al. 2014]). We use a stopping time argument to show that $(S_n(x))_{n=0}^{\infty}$ converges to 0 in a set of Hausdorff dimension 1 (indeed, given any complex number z_0 the martingale $(S_n(x))_{n=0}^{\infty}$ converges to z_0 in a set of Hausdorff dimension 1). We get the dimension 1 conclusion by applying a lemma of Hungerford [1988]. For the sake of the reader we present a proof of Hungerford's lemma in our context in Appendix A.

Our proof extends with only technical modifications to cover the case of other odd kernels, for instance,

$$\frac{\bar{z}^m}{z^{m+1}}, \quad m = 1, 2, \ldots$$

But one of the ingredients of our method fails for the odd kernel $(z+\bar{z})/z^2$ and we do not know whether Theorem 1.1 holds in this case. The difficulty is indicated at the fifth line after the statement of Lemma 3.1.

In $\mathbb{R}^d$ our proof works for the Riesz transforms of any homogeneity $-\alpha,\ 0<\alpha<d$. These are the vector valued singular integrals with kernel

$$R^\alpha(x)=\frac{x}{|x|^{1+\alpha}},\quad 0<\alpha<d.$$

The appropriate Cantor sets for the α-Riesz transform are those of Hausdorff dimension α. They are constructed by the procedure outlined before in the planar case with dilation factors that satisfy $2^{-d/\alpha}\le\lambda_n\le\lambda<2^{-1}$. At generation n one has 2^{dn} cubes Q_j^n of side length $s_n=\lambda_1\cdots\lambda_n$. The Cantor set is defined by

$$K=\bigcap_{n=1}^{\infty}\bigcup_{j=1}^{2^{dn}}Q_j^n$$

and the canonical measure on K by $\mu(Q_j^n)=2^{-dn},\ 1\le j\le 2^{dn}$. The α density is $a_n=2^{-dn}s_n^{-\alpha}=\mu(Q_j^n)s_n^{-\alpha}\le 1$. For $\lambda_n=2^{-d/\alpha},\ n=1,2\ldots,$ one gets the self similar Cantor set of dimension α. If $a_n\to 0$ then our Cantor set has Hausdorff dimension $\ge\alpha$ and non σ finite Hausdorff α-dimensional measure.

Theorem 1.2. *If* $\lim_{n\to\infty}a_n=0$, *then the set of points* $x\in K$ *for which the principal value*

$$\lim_{\varepsilon\to 0}\int_{|y-x|>\varepsilon}R^\alpha(y-x)\,d\mu y \tag{1-2}$$

exists has Hausdorff dimension greater than or equal to α.

In Appendix B we give some indications on how to adapt the proof for the Cauchy kernel to the Riesz transforms in higher dimensions.

We let $\operatorname{diam}(A)$ denote the diameter and $\dim A$ the Hausdorff dimension of a set A. We use the notation $a\lesssim b$ to mean that $a\le Cb$ for some constant C which may depend on λ and d, and $a\sim b$ for $a\lesssim b$ and $b\lesssim a$.

2. Martingales

Let C be the Cauchy kernel, $C(x)=1/x$ for $x\in\mathbb{C},x\neq 0$. For each $x\in K$ let $Q_n(x)$ be the square in $\mathcal{D}_n$ containing x. Define the truncated Cauchy integral at generation n as

$$T_n(x)=\int_{K\setminus Q_n(x)}C(x-y)\,d\mu y,\quad x\in K,$$

and a martingale $(S_n(x))_{n=0}^\infty$ by

$$S_n(x)=S_{Q_n(x)}=⨍_{Q_n(x)}T_n\,d\mu,\quad x\in K. \tag{2-1}$$

Remark 2.1. That S_n is a martingale is easily checked. The reader will realise that the martingale condition also holds for kernels $K(x, y)$ satisfying the antisymmetry condition $K(x, y) = -K(y, x)$.

We shall prove:

Theorem 2.2. *If* $\lim_{n\to\infty} a_n = 0$, *then the set of points* $x \in K$ *for which* $(S_n(x))_{n=0}^{\infty}$ *converges has Hausdorff dimension greater than or equal to* 1.

We first show that the martingale (2-1) has uniformly bounded increments.

Lemma 2.3. *There exists a positive constant* $C = C(\lambda)$ *such that*

$$|S_{n+1}(x) - S_n(x)| \le C a_n, \quad n = 0, 1, \dots \text{ and } x \in K. \tag{2-2}$$

Thus if $\sum_n a_n$ converges, $(S_n(x))_{n=0}^{\infty}$ converges for all $x \in K$. As mentioned in the introduction, even the weaker condition $\sum_n a_n^2 < \infty$ implies that $(S_n(x))_{n=0}^{\infty}$ converges for μ almost all $x \in K$. Hence we shall assume that $\sum_n a_n^2 = \infty$. Under this assumption one proves in [Cufí et al. 2022] that the set where the principal values fail to exist has full μ measure. In Lemma 2.4 below we show that principal values exist if and only if the martingale converges. Hence $(S_n(x))_{n=0}^{\infty}$ is not convergent for μ almost all $x \in K$. By a standard result in martingale theory (see, for example, [Shiryaev 1996, Corollary 6, p. 561]) we get

$$\limsup_{n\to\infty} |S_n(x) - S_m(x)| = \infty, \quad \text{for all } m = 0, 1, \dots \text{ and } \mu \text{ a.e.} \tag{2-3}$$

Proof of Lemma 2.3. Set $Q_n = Q_n(x),\ x \in K,\ n = 1, 2, \dots$ Then

$$\begin{aligned} S_{n+1}(x) - S_n(x) &= ⨍_{Q_{n+1}} \int_{K\setminus Q_{n+1}} C(z-y)\, d\mu y\, d\mu z - ⨍_{Q_n} \int_{K\setminus Q_n} C(w-y)\, d\mu y\, d\mu w \\ &= \int_{K\setminus Q_n} \Big(⨍_{Q_{n+1}} C(z-y)\, d\mu z - ⨍_{Q_n} C(w-y)\, d\mu w \Big)\, d\mu y \\ &\qquad + \int_{Q_n\setminus Q_{n+1}} ⨍_{Q_{n+1}} C(z-y)\, d\mu z\, d\mu y. \end{aligned}$$

The last double integral is $\lesssim a_n$, where the implicit constant depends on λ here and for the rest of the proof.

To estimate the first summand above we remark that for each $z' \in Q_{n+1}$ and $w' \in Q_n$ we have

$$\begin{aligned} &⨍_{Q_{n+1}} C(z-y)\, d\mu z - ⨍_{Q_n} C(w-y)\, d\mu w \\ &\quad = ⨍_{Q_{n+1}} \big(C(z-y) - C(z'-y)\big)\, d\mu z - ⨍_{Q_n} \big(C(w-y) - C(w'-y)\big)\, d\mu w \\ &\qquad + C(z'-y) - C(w'-y). \end{aligned}$$

Clearly

$$\left|C(z'-y)-C(w'-y)\right| \lesssim s_n\,|x-y|^{-2}, \quad y\in K\setminus Q_n,\ \ x\in Q_n.$$

Hence

$$\left|⨍_{Q_{n+1}}\big(C(z-y)-C(z'-y)\big)\,d\mu z\right| \lesssim s_n\,|x-y|^{-2}$$

and

$$\left|⨍_{Q_{n}}\big(C(w-y)-C(w'-y)\big)\,d\mu z\right| \lesssim s_n\,|x-y|^{-2}.$$

Setting

$$R_j = Q_j\setminus Q_{j+1},$$

the absolute value of the first summand of $S_{n+1}(x)-S_n(x)$ is

$$\begin{aligned}&\lesssim s_n\int_{K\setminus Q_n}|x-y|^{-2}\,d\mu y \sim s_n\sum_{j=0}^{n-1}s_j^{-2}\mu(R_j)\\ &= s_n\sum_{j=0}^{n-1}s_j^{-2}4^{-j}\lesssim s_n s_n^{-2}4^{-n}=a_n,\end{aligned}$$

because $s_j^{-2}4^{-j}\le(s_{j+1}^{-2}\lambda^2)4^{-j}=(4\lambda^2)s_{j+1}^{-2}4^{-j-1}$, so $s_j^{-2}4^{-j}\lesssim(4\lambda^2)^{n-j}s_n^{-2}4^{-n}$. Hence $|S_{n+1}(x)-S_n(x)|\lesssim a_n$. □

By the following lemma Theorem 2.2 is equivalent to Theorem 1.1.

Lemma 2.4. *If* $\lim_{n\to\infty}a_n=0$, *then for each* $x\in K$ *the principal value* (1-1) *exists if and only if the sequence* $(S_n(x))_{n=0}^{\infty}$ *converges.*

Proof. Set $Q_n=Q_n(x)$ for $x\in K$ and $n=1,2,\ldots$ Then by the proof of Lemma 2.3

$$\begin{aligned}\left|S_n(x)-\int_{K\setminus Q_n}\frac{1}{x-y}\,d\mu y\right| &= \left|⨍_{Q_n}\int_{K\setminus Q_n}\left(\frac{1}{x'-y}-\frac{1}{x-y}\right)d\mu y\,d\mu x'\right|\\ &\le C\,a_n,\end{aligned}$$

where the constant depends on λ. Compare now a given truncation $\int_{K\setminus B(x,\varepsilon)}\frac{1}{x-y}\,d\mu y$, $0<\varepsilon<1$, with $\int_{K\setminus Q_n}\frac{1}{x-y}\,d\mu y$ where n is chosen so that $\operatorname{diam}(Q_n)\le\varepsilon<\operatorname{diam}(Q_{n-1})$. Since $Q_n\subset B(x,\varepsilon)$ we have

$$\begin{aligned}\left|\int_{K\setminus Q_n}\frac{1}{x-y}\,d\mu y-\int_{K\setminus B(x,\varepsilon)}\frac{1}{x-y}\,d\mu y\right| &= \left|\int_{B(x,\varepsilon)\setminus Q_n}\frac{1}{x-y}\,d\mu y\right|\\ &\le C\,\frac{\mu B(x,\varepsilon)}{s_n},\end{aligned}$$

with $C = C(\lambda)$. To complete the proof just remark that, since $\varepsilon < \operatorname{diam}(Q_{n-1})$, $B(x, \varepsilon)$ can intersect at most N squares in $\mathcal{D}_n$, with N an absolute constant. Hence $\mu B(x, \varepsilon) \le C\,\mu(Q_n)$. $\square$

We proceed now to discuss relative martingales.

For $x \in R \subset Q,\ Q \in \mathcal{D}_m,\ R \in \mathcal{D}_n,\ m < n$, we define the relative martingale starting at Q as

$$S_{Q,R}(x) = S_{Q,R} = \fint_R \int_{Q\setminus R} C(z-y)\, d\mu y\, d\mu z.$$

Then for some constant C,

$$|S_R - S_Q - S_{Q,R}| \le C\, a_m. \tag{2-4}$$

Indeed, we have

$$\begin{aligned} S_R - S_Q &= \fint_R \int_{K\setminus R} C(z-y)\, d\mu y\, d\mu z - \fint_Q \int_{K\setminus Q} C(w-y)\, d\mu y\, d\mu w \\ &= \int_{K\setminus Q} \left(\fint_R C(z-y)\, d\mu z - \fint_Q C(w-y)\, d\mu w \right) d\mu y \\ &\qquad\qquad + \int_{Q\setminus R} \fint_R C(z-y)\, d\mu z\, d\mu y \\ &= \int_{K\setminus Q} \left(\fint_R C(z-y)\, d\mu z - \fint_Q C(w-y)\, d\mu w \right) d\mu y + S_{Q,R}. \end{aligned}$$

The first summand above is bounded in absolute value by a constant times a_m by the same argument as in the proof of (2-2).

As for (2-2) we have for $R \subset \tilde{R} \subset Q$, $Q \in \mathcal{D}_m$, $\tilde{R} \in \mathcal{D}_n$, $R \in \mathcal{D}_{n+1}$,

$$|S_{Q,R} - S_{Q,\tilde{R}}| \le C\, a_n. \tag{2-5}$$

3. The stopping time argument

The proof of Theorem 2.2 is based on a stopping time argument for which we need some preliminary facts.

Given a nonzero complex number z consider the sector $\sigma(z, \theta),\ 0 < \theta < \pi$, with vertex at z and aperture θ whose axis is the semiline emanating from z and passing through 0. That is, $w \in \sigma(z, \theta)$ if and only if

$$\left\langle \frac{w-z}{|w-z|}, \frac{-z}{|z|} \right\rangle \ge \cos\left(\frac{\theta}{2}\right)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in the plane.

The octants with vertex 0 are the eight sectors

$$\sigma_j = \Big\{ w \in \mathbb{C} : w = |w| e^{i\phi},\ (j-1)\frac{\pi}{4} \le \phi \le j\frac{\pi}{4} \Big\}, \quad 1 \le j \le 8.$$

These are the sectors with vertex the origin of amplitude 45° degrees and having an edge over a coordinate axis. It will be convenient to expand these octants so that they have the same axis and amplitude of 75°. In other words, we are adding 15° in each direction. Denote the expanded sectors by $\tilde{\sigma}_j$. The octants with vertex z are the sectors $\sigma_j(z) = z + \sigma_j$, $1 \le j \le 8$, and the expanded octants $\tilde{\sigma}_j(z) = z + \tilde{\sigma}_j$.

We have the following obvious lemma.

Lemma 3.1. *Given any sector σ of vertex z and amplitude* 120° *there exists an octant with vertex z, say $\sigma_j(z)$ for some index j between* 1 *and* 8, *such that $\tilde{\sigma}_j(z) \subset \sigma$.*

Consider the symmetries with respect to the coordinate axis and the main diagonal. That is, $f_1(x+iy) = -x+iy$, $f_2(x+iy) = x-iy$ and $f_3(x+iy) = y+ix$ for $x+iy \in \mathbb{C}$. For any $j, k = 1, \dots, 8$, by composing two such symmetries we obtain a linear mapping $f_{j,k}$ that maps the octant σ_j onto the octant σ_k. Observe that $C(f_j(z)) = f_j(C(z))$ for $j = 1, 2$, and $C(f_3(z)) = -f_3(C(z))$. It is precisely this last identity that fails for the kernel $(z+\bar{z})/z^2$.

Let $Q \in \mathcal{D}$ and let c_Q be its centre. Define

$$f_{Q,j,k}(x) = f_{j,k}(x - c_Q) + c_Q, \quad x \in Q, \quad j, k = 1, \dots, 8,$$

so that

$$f_{Q,j,k}(x) - f_{Q,j,k}(y) = f_{j,k}(x-y), \quad x, y \in Q, \quad j, k = 1, \dots, 8.$$

We claim that

$$S_{Q, f_{Q,j,k}(R)} = \varepsilon_{j,k}\, f_{j,k}(S_{Q,R}), \quad R \subset Q, \quad Q, R \in \mathcal{D}, \tag{3-1}$$

where $\varepsilon_{j,k} = \pm 1$. We check (3-1) by the general formula for the image (push-forward) $\nu^{\sharp,f}$ of a measure ν under a Borel map f (see, for example, [Mattila 1995, Theorem 1.19])

$$\int_{f(A)} g\, d\nu^{\sharp,f} = \int_A (g \circ f)\, d\nu.$$

The restriction of μ to Q is invariant under the maps $f_{Q,j,k}$, i.e., $(\mu|Q)^{\sharp, f_{Q,j,k}} = \mu|Q$. Hence, since $Q \setminus f_{Q,j,k}(R) = f_{Q,j,k}(Q \setminus R)$ and

$$C\big(f_{Q,j,k}(z) - f_{Q,j,k}(w)\big) = \varepsilon_{j,k}\, f_{j,k}(C(z-w)),$$

we obtain

$$\int_{Q \setminus f_{Q,j,k}(R)} \int_{f_{Q,j,k}(R)} C(z-w)\, d\mu z\, d\mu w = \varepsilon_{j,k}\, f_{j,k}\bigg(\int_{Q \setminus R} \int_R C(z-w)\, d\mu z\, d\mu w \bigg),$$

from which (3-1) follows.

Assume that we have fixed an octant σ_j and that for some square $R \in \mathcal{D}$ contained in Q we have $S_{Q,R} \in \sigma_k$ with $k \neq j$. We claim that we can find a square $R' \in \mathcal{D}$ contained in Q, of the same size as R, such that $|S_{Q,R'}| = |S_{Q,R}|$ and $S_{Q,R'} \in \sigma_k$.

If $\varepsilon_{j,k} = 1$ then the value of the relative martingale at the square $f_{Q,j,k}(R)$ is $f_{j,k}(S_{Q,R}) \in \sigma_j$. Note that the size of $f_{Q,j,k}(R)$ is exactly the size of R and $|S_{Q,f_{Q,j,k}(R)}| = |S_{Q,R}|$.

To treat the case $\varepsilon_{j,k} = -1$ let us introduce the mapping $\gamma : Q \to Q$ defined by $\gamma(x) = -(x - c_Q) + c_Q$. Then γ^2 is the identity mapping on Q and $S_{Q,\gamma(R)} = -S_{Q,R}$ for each square $R \in \mathcal{D}$ contained in Q. Setting $R' = (\gamma \circ f_{Q,j,k})(R)$ we get

$$f_{jk}(S_{Q,R}) = -S_{Q,f_{Q,j,k}(R)} = S_{Q,(\gamma \circ f_{Q,j,k})(R)} = S_{Q,R'}.$$

We shall need the following elementary lemma.

Lemma 3.2. *If* $z \in \mathbb{C}$, $w \in \sigma(z, 120°)$ *and* $0 < |w - z| < |z|/2$, *then* $|w| \leq |z| - |w - z|/4$.

Proof. Let $R = |z|$, $r = |w - z|$ and let v be the third vertex, in addition to 0 and z, of the equilateral triangle containing w. Under the assumptions of the lemma $|w|$ is maximized when w lies on the side connecting z and v. Assuming that w is on that side, project w on the side connecting 0 and z and apply Pythagoras to obtain

$$|w|^2 = (R - r/2)^2 + (\sqrt{3}r/2)^2 = r^2 + R^2 - rR \leq (R - r/4)^2 = (|z| - |w - z|/4)^2$$

because of the assumption $r < R/2$. □

Proof of Theorem 2.2. We assume, as we may, that $\sum_n a_n^2 = \infty$. Then for μ almost all x the sequence $(S_n(x))_{n=0}^{\infty}$ diverges and (2-3) holds.

Let M be a big positive integer to be chosen later. We replace $(a_n)_{n=0}^{\infty}$ by the nonincreasing sequence $b_n = C \max_{m \geq n} a_m$, where C is as in inequalities (2-2), (2-4) and (2-5), which now read

(3-2) $\quad |S_{n+1}(x) - S_n(x)| \leq b_n, \quad n = 0, 1, \ldots$ and $x \in K$,

(3-3) $\quad |S_R - S_Q - S_{Q,R}| \leq b_m, \quad Q \in \mathcal{D}_m,\ R \in \mathcal{D}_n,\ R \subset Q$,

(3-4) $\quad |S_{Q,R} - S_{Q,\tilde{R}}| \leq b_n, \quad Q \in \mathcal{D}_m,\ R \in \mathcal{D}_{n+1},\ \tilde{R} \in \mathcal{D}_n,\ R \subset \tilde{R} \subset Q$.

We plan to define a sequence of stopping time conditions. At each step a family of stopping time squares will arise, which is going to be the family $\mathcal{F}_n$ in Lemma A.1 (Hungerford's lemma). The first stopping time is special and its goal is to have a family of squares with relatively large $|S_Q|$ for each square Q in the family.

The first stopping time condition is

(3-5) $$|S_Q| > M b_0.$$

Declare Q a stopping time square of first generation if Q is a square in $\mathcal{D}$ for which $|S_Q| > M b_0$ and $|S_{Q'}| \leq M b_0$, $Q \subsetneq Q'$. We call $\mathcal{F}_1$ the set of stopping time squares of first generation. One may think of this as a process as follows. One takes a point $x \in K$ and looks at the squares in $\mathcal{D}$ containing x. One examines all those squares, starting at Q_0 and checks whether condition (3-5) is satisfied. If it is not, then one proceeds to the square containing x in the next generation. The process stops when one finds a square Q containing x for which (3-5) holds. Note that the set of x for which the process never stops has vanishing μ measure by (2-3). Hence $\sum_{Q \in \mathcal{F}_1} \mu(Q) = 1$. Since $S_{Q_0} = 0$, it follows from (3-2) that it is necessary to descend at least $M+1$ generations to find the first stopping time square.

The second stopping time condition is slightly different. Let $Q \in \mathcal{F}_1$. The second stopping time is performed on the relative martingale associated with Q and its condition is

$$|S_{Q,R}| > M b_M. \tag{3-6}$$

A stopping time square R of second generation satisfies $|S_{Q,R}| > M b_M$ and

$$|S_{Q,R'}| \le M b_M, \quad R' \in \mathcal{D}, \quad R \subsetneq R' \subset Q.$$

By (2-3) and (3-3) the stopping time squares of second generation cover almost all Q. Again, by (3-4) and the fact that $S_{Q,Q} = 0$ one has to descend through at least $M+1$ generations to find a stopping time square of second generation. Hence if R is a stopping time square of second generation and $R \in \mathcal{D}_n$ then $n \ge 2(M+1)$. We do not put all stopping time squares of second generation in $\mathcal{F}_2(Q)$. We put a stopping time square of second generation R in $\mathcal{F}_2(Q)$ provided $S_R \in \sigma(S_Q, 120°)$. That there are many such stopping time squares can be shown as follows.

Let R be a stopping time square of second generation. Let α denote the angle between the vectors $S_R - S_Q$ and $S_{Q,R}$. Then by (3-3),

$$|S_R - S_Q| \ge |S_{Q,R}| - b_M \ge (M-1) b_M$$

and

$$0 \le |\sin\alpha| \le \frac{|S_R - S_Q - S_{Q,R}|}{|S_R - S_Q|} \le \frac{b_M}{(M-1) b_M} = \frac{1}{M-1} < \sin 15°,$$

provided $M - 1 > 1/\sin 15°$, which we assume. Since $|S_R - S_Q - S_{Q,R}| < |S_R - S_Q|$ and $S_{R,Q} = S_R - S_Q + (S_{Q,R} - S_R - S_Q)$, we see that $\cos\alpha > 0$. Thus $|\alpha| < 15°$.

By Lemma 3.1 there is j with $1 \le j \le 8$ such that $\tilde{\sigma_j}(S_Q) \subset \sigma(S_Q, 120°)$. If we are lucky enough that we have $S_{Q,R} \in \sigma_j$ and so $S_R - S_Q \in \tilde{\sigma_j}$, which yields $S_R \in \tilde{\sigma_j}(S_Q) \subset \sigma(S_Q, 120°)$.

But it may occur that $S_{Q,R} \in \sigma_k$, $k \ne j$. Applying two symmetries $f_{Q,j,k}$ of Q, or a symmetry of the form $\gamma \circ f_{Q,j,k}$ in the worst case, as we discussed before Lemma 3.2, we obtain a stopping time square R' of second generation and of the same size as R such that $S_{Q,R'} \in \sigma_j$ and so $S_{R'} \in \tilde{\sigma_j}(S(Q)) \subset \sigma(S_Q, 120°)$, as desired.

Therefore, subdividing the stopping time squares of second generation in eight classes, according to the octant to which $S_{Q,R}$ belongs, we get

$$\sum_{R\in\mathcal{F}_2(Q)} \mu(R) \geq \tfrac{1}{8}\,\mu(Q). \tag{3-7}$$

Define $\mathcal{F}_2 = \bigcup_{Q\in\mathcal{F}_1} \mathcal{F}_2(Q)$.

Let us obtain some properties of stopping time squares R in $\mathcal{F}_2(Q)$. Let $\tilde{R}$ be the father of R. Then $|S_{Q,\tilde{R}}| \leq M\,b_M$ and so

$$|S_{\tilde{R}} - S_Q| \leq |S_{Q,\tilde{R}}| + |S_{\tilde{R}} - S_Q - S_{Q,\tilde{R}}| \leq (M+1)b_M$$

and

$$|S_R - S_Q| \leq |S_R - S_{\tilde{R}}| + |S_{\tilde{R}} - S_Q| \leq b_M + (M+1)b_M = (M+2)b_M.$$

Now two possibilities appear.

If $|S_Q| \leq 2|S_R - S_Q| \leq 2(M+2)b_M$, then

$$|S_R| \leq |S_R - S_Q| + |S_Q| \leq 3(M+2)b_M.$$

If $|S_Q| > 2|S_R - S_Q|$, since $S_R \in \sigma(S_Q, 120°)$ we can apply Lemma 3.2 to get

$$|S_R| \leq |S_Q| - |S_R - S_Q|/4 \leq |S_Q| - (M-1)b_M/4 \leq |S_Q| - b_M$$

provided $M \geq 5$.

Therefore at least one of the following two inequalities holds: either

$$|S_R| \leq 3(M+2)b_M, \tag{3-8}$$

or

$$|S_R| \leq |S_Q| - b_M. \tag{3-9}$$

We can proceed to define inductively $\mathcal{F}_n$ for $n \geq 3$, in a way analogous to what we did to define $\mathcal{F}_2$ from $\mathcal{F}_1$. Assume that we have defined $\mathcal{F}_{n-1} = \bigcup_{Q\in\mathcal{F}_{n-2}} \mathcal{F}_{n-1}(Q)$. Given $Q \in \mathcal{F}_{n-1}$ we set the n generation stopping time in the relative martingale associated with Q as

$$|S_{Q,R}| > Mb_{(n-1)M}$$

If R is a stopping time square of n-th generation then besides the previous inequality one has

$$|S_{Q,R'}| \leq Mb_{(n-1)M}, \quad R' \in \mathcal{D}, \quad R \subsetneq R' \subset Q,$$

whence

$$|S_{R'} - S_Q| \leq |S_{Q,R'}| + b_{(n-1)M} \leq (M+1)b_{(n-1)M}. \tag{3-10}$$

Note that if R is a stopping time square of generation n, we can take advantage of the symmetries of Q, as before, to find another one, say R', of the same size with the additional property that $S_{R'} \in \sigma(S_Q, 120°)$. Define $\mathcal{F}_n(Q)$ as the stopping time squares R of generation n such that $S_R \in \sigma(S_Q, 120°)$ and $\mathcal{F}_n = \bigcup_{Q \in \mathcal{F}_{n-1}} \mathcal{F}_n(Q)$. We then have

$$\sum_{R \in \mathcal{F}_n(Q)} \mu(R) \geq \tfrac{1}{8}\, \mu(Q). \tag{3-11}$$

Given $R \in \mathcal{F}_n(Q)$, we have as before that at least one of the following two inequalities holds: either

$$|S_R| \leq 3(M+2) b_{(n-1)M} \tag{3-12}$$

or

$$|S_R| \leq |S_Q| - b_{(n-1)M}. \tag{3-13}$$

Set $F = \bigcap_{n=1}^{\infty} \bigcup_{Q \in \mathcal{F}_n} Q$. To complete the proof we shall show that the hypotheses of Hungerford's Lemma A.1 are fulfilled and that

$$\lim_{m \to \infty} S_m(x) = 0, \qquad x \in F. \tag{3-14}$$

For (b) in Hungerford's Lemma A.1 recall that each stopping time square has descended at least $M+1$ generations from the generating square in the previous family. Then one has (b) with ε replaced by $1/4^M$ and taking M big enough one has $1/4^M < \varepsilon$. Condition (c) with $c = 1/8$ is (3-11).

To prove (3-14), take $x \in F$. For every $n = 1, 2, \ldots$, there is a unique $Q_n \in \mathcal{F}_n$ such that $x \in Q_n$. Let m_n be the unique positive integer satisfying $Q_n \in \mathcal{D}_{m_n}$. Clearly the sequence m_n is increasing and $m_n > Mn$. Since $S_{Q_n} = S_{m_n}(x)$ we have by (3-12) and (3-13) that either

$$|S_{m_n}(x)| \leq 3(M+2) b_{(n-1)M} \tag{3-15}$$

or

$$|S_{m_n}(x)| \leq |S_{m_{n-1}}(x)| - b_{(n-1)M}, \qquad n = 1, 2, \ldots \tag{3-16}$$

For $m_{n-1} < m < m_n$ we have by (3-10)

$$|S_m(x) - S_{m_{n-1}}(x)| \leq (M+1) b_{(n-1)M}. \tag{3-17}$$

To conclude that $\lim_{m \to \infty} S_m(x) = 0$ it is enough to show that $\lim_{n \to \infty} S_{m_n}(x) = 0$.

We say that $n \in \mathcal{N}_1$, if (3-16) holds and $n \in \mathcal{N}_2$, if (3-15) holds and (3-16) fails. Because $\sum_n b_n$ diverges and $(b_n)_{n=1}^{\infty}$ is nonincreasing, $\sum_n b_{(n-1)M}$ also diverges. It follows that (3-16) cannot hold for infinitely many consecutive n, whence $\mathcal{N}_2$ is infinite.

Let $n \in \mathcal{N}_2$ and let $N > n$ be such that $k \in \mathcal{N}_1$ for all $n < k < N$. Then by (3-16) and (3-15) for $n < k < N$,

$$|S_{m_k}(x)| \le |S_{m_n}(x)| \le 3(M+2)b_{(n-1)M}.$$

It follows that $\lim_{m\to\infty} S_m(x) = 0$. □

Appendix A. A lemma on Hausdorff dimension

Let μ be the canonical measure associated with a Cantor set in $\mathbb{R}^d$, as defined in the Introduction before the statement of Theorem 1.2. Denote by $\mathcal{D}_n$ the set of all cubes Q_j^n, $1 \le j \le 2^{dn}$, appearing at the n-th generation of the construction and $\mathcal{D} = \bigcup_n \mathcal{D}_n$.

The following lemma is due to Hungerford [1988], who worked in a one-dimensional context.

Lemma A.1. *Let* $0 < \varepsilon < c < 1$ *and let* $\mathcal{F}_n$ *be a disjoint family of cubes in* $\mathcal{D}$, *for* $n = 0, 1, 2, \ldots,$ *satisfying the following.*

(a) $\mathcal{F}_0 = \{Q_0\}$.

(b) *If* $Q \in \mathcal{F}_{n+1}$, *then there exists* $\tilde{Q} \in \mathcal{F}_n$ *with* $Q \subset \tilde{Q}$ *and* $\mu(Q) \le \varepsilon\mu(\tilde{Q})$.

(c) *If* $Q \in \mathcal{F}_n$, *then*

$$\sum_{R\subset Q,\, R\in\mathcal{F}_{n+1}} \mu(R) \ge c\,\mu(Q).$$

Let $E = \bigcap_n \bigcup_{Q\in\mathcal{F}_n}$. *Then*

$$\dim E \ge \alpha(1 - \log c/\log\varepsilon).$$

Proof. Set $\beta = \alpha(1 - \log c/\log\varepsilon)$. We will construct a Borel probability measure ν with $\nu(E) = 1$ such that for some constant C and for all balls $B(x,r)$ centred at x of radius r one has

$$\nu(B(x,r)) \le Cr^\beta \quad \text{for } x \in E,\quad 0 < r \le 1. \tag{A-1}$$

Then Frostman's lemma will give the result.

Let us define the functions $\nu_n : \mathcal{F}_n \to \mathbb{R}$, $n = 0, 1, 2\ldots$, setting first $\nu_0(Q_0) = 1$. Suppose that $\nu_1, \ldots, \nu_{n-1}$ are defined and let for $Q \in \mathcal{F}_n$, with $\tilde{Q}$ as in (b),

$$\nu_n(Q) = \frac{\nu_{n-1}(\tilde{Q})}{\sum_{R\in\mathcal{F}_n, R\subset\tilde{Q}} \mu(R)}\mu(Q).$$

Then we define the Borel measures ν_n setting

$$\nu_n(A) = \sum_{Q\in\mathcal{F}_n} \frac{\nu_n(Q)}{\mu(Q)}\mu(A\cap Q) \quad \text{for } A \subset \mathbb{R}^d.$$

Then for $Q \in \mathcal{F}_n$,

$$\begin{aligned}\nu_{n+1}(Q) &= \sum_{R\in\mathcal{F}_{n+1}, R\subset Q} \nu_{n+1}(R)\\ &= \sum_{R\in\mathcal{F}_{n+1}, R\subset Q} \frac{\nu_n(Q)}{\sum_{P\in\mathcal{F}_{n+1}, P\subset Q}\mu(P)}\mu(R)\\ &= \nu_n(Q).\end{aligned}$$

Iterating this we have

$$\nu_m(Q) = \nu_n(Q) \quad \text{for } Q \in \mathcal{F}_n,\ m > n. \tag{A-2}$$

In particular, each ν_n is a probability measure and some subsequence of (ν_n) converges weakly to a probability measure ν such that $\nu(Q) = \nu_n(Q)$ for $Q \in \mathcal{D}_n$.

Since

$$\nu\Big(\bigcup_{Q\in\mathcal{F}_n} Q\Big) = \sum_{Q\in\mathcal{F}_n} \nu(Q) = \sum_{Q\in\mathcal{F}_n} \nu_n(Q) = 1,$$

we have $\nu(E) = 1$. Therefore $\nu(E \setminus \bigcup_{Q\in\mathcal{F}_n} Q) = 0$ for every n, so

$$\nu(Q) = \sum_{R\subset Q, R\in\mathcal{F}_{n+1}} \nu(R), \quad Q \in \mathcal{F}_n. \tag{A-3}$$

It remains to verify (A-1). First of all we have by condition (c) for $Q \in \mathcal{F}_n$, $n \geq 2$,

$$\frac{\nu(Q)}{\mu(Q)} = \frac{\nu_n(Q)}{\mu(Q)} = \frac{\nu_{n-1}(\tilde{Q})}{\sum_{R\subset\tilde{Q}, R\in\mathcal{F}_n}\mu(R)} \leq \frac{\nu(\tilde{Q})}{c\mu(\tilde{Q})},$$

and by induction,

$$\frac{\nu(Q)}{\mu(Q)} \leq c^{-n} \quad \text{for } Q \in \mathcal{F}_n,\ n = 1, 2\dots. \tag{A-4}$$

Now let us prove that

$$\nu(Q) \leq Cd(Q)^\beta \quad \text{for } Q \in \mathcal{D}. \tag{A-5}$$

Take n such that $\varepsilon^{n+1} \leq \mu(Q) < \varepsilon^n$. We may assume that $\nu(Q) > 0$. Then Q intersects a square R in the family $\mathcal{F}_{n+1}$. Since by (b) $\mu(R) \leq \varepsilon^{n+1} \leq \mu(Q)$, one has $R \subset Q$. We have, by (A-3) and (A-4),

$$\nu(Q) = \sum_{R\subset Q, R\in\mathcal{F}_{n+1}} \nu(R) \leq c^{-n-1} \sum_{R\subset Q, R\in\mathcal{F}_{n+1}} \mu(R) \leq c^{-n-1}\mu(Q).$$

Since $\mu(Q) \le d(Q)^\alpha$ it is enough to show that $c^{-n}\mu(Q) \le \mu(Q)^{\beta/\alpha}$ which is

$$c^{-n} \le \mu(Q)^{-\log c/\log\varepsilon},$$

that is,

$$-n\log c \le -(\log c/\log\varepsilon)\log\mu(Q),$$

or $n \le \log\mu(Q)/\log\varepsilon$, which is a consequence of $\mu(Q) < \varepsilon^n$.

To finish, let $x \in E$ and $0 < r \le 1$. For some n, x belongs to a square $Q \in \mathcal{D}_n$ with $d(Q)/4 \le r \le d(Q)$. Then $B(x,r)$ can meet at most 4^d squares of $\mathcal{D}_n$, and so by (A-5), $\nu(B(x,r)) \le 4^d\,\nu(Q) \le 4^d\,C\,d(Q)^\beta \le 4^{\beta+d}\,C\,r^\beta$ and (A-1) follows. $\square$

Appendix B. The Riesz transforms in $\mathbb{R}^d$

We first slightly modify the argument in [Cufí et al. 2022] to show that $\sum_{n=1}^\infty a_n^2 = \infty$ yields divergence a.e. of the martingale. If the martingale converges in a set of positive measure, then also the principal values of the Riesz transform exist in a set E of positive measure, by the analog of Lemma 2.4. By a result of Tolsa [2014, Theorem 8.13] we find a set $F \subset E$ of positive measure on which the singular Riesz transform operator is bounded on $L^2(\mu_{|F})$. In particular, the capacity of F associated with the Riesz kernel is positive and so also that of the Cantor set. The main result of [Mateu and Tolsa 2004] (see Theorem 1.2, p. 678 and its extension in the last formula in p. 696) states that the α-Riesz capacity of the Cantor set is comparable to $\left(\sum_{n=1}^\infty a_n^2\right)^{-1/2}$, so that positive capacity yields a convergent series. We remark that the previous argument uses very strong results, in particular the nonhomogeneous $T(1)$-Theorem of Nazarov, Treil and Volberg, to extract the subset F on which the singular Riesz transform is $L^2(\mu_{|F})$ bounded. In [Cufí et al. 2022] one resorts to Menger curvature, which is not available for kernels of homogeneity $-\alpha$ with $1 < \alpha < d$, and the proof is slightly simpler. It would be desirable to have a direct argument relating the series to the convergence of the martingale.

The part of the stopping time argument of Section 3 that does not obviously extend to higher dimensions is related to the sector $\sigma(z, 120°)$. In particular, one should replace the 45° degrees sectors centred at the origin with one edge on a coordinate axis with other regions. We proceed as follows. Divide $\mathbb{R}^d$ into 2^d regions (which in $\mathbb{R}^3$ are the usual octants) by requiring that each coordinate has a definite sign. For example,

$$O = \{x \in \mathbb{R}^d : x_1 \ge 0, x_2 \ge 0, \dots x_d \ge 0\}$$

or

$$O' = \{x \in \mathbb{R}^d : x_1 \le 0, x_2 \ge 0, \dots x_d \ge 0\}$$

are such regions. Let us concentrate in the region O. Divide O in the $d!$ subregions determined by a permutation σ of the d variables

$$O_\sigma = \{x \in \mathbb{R}^d : 0 \le x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(d)}\}.$$

Note that the maximal angle between two vectors lying in a subregion O_σ is precisely $\arccos(d^{-1/2})$, which approaches 90° as $d \to \infty$. Given a cone Γ with vertex at the origin and aperture θ, we would like to find a region O_σ contained in the cone Γ. This can be done as follows. The axis of the cone is a ray emanating from the origin contained in O_σ for some σ. Taking $\theta = \theta(d) < \pi$ close enough to π one can achieve $O_\sigma \subset \Gamma$. Indeed, something stronger can be obtained: there exists a sufficiently small angle $\gamma = \gamma(d)$ such that expanding O_σ in all directions by at most γ degrees one still remains in the cone Γ.

The planar argument now works with θ in place of 120°.

One also needs to have enough linear isometries to transport one region O_σ into another $O_{\sigma'}$. Consider the following kinds of linear isometries. Fix a variable x_i and take the mapping that leaves the other variables invariant and changes the sign to the x_i variable. Given two variables x_i and x_j with $i \neq j$ consider the mapping that leaves the other variables invariant and interchanges x_i and x_j. Finally take the mapping $x \to -x$. Let $\mathcal{S}$ be the set of such linear isometries. One can easily check that given two regions O_σ and $O_{\sigma'}$ one can map one into the other by composing finitely many isometries in $\mathcal{S}$.

All these elements lead to a stopping time argument that proves Theorem 1.2.

Acknowledgements

The authors are grateful to X. Tolsa for various conversations on the subject and to the referee for many suggestions that have improved considerably the exposition. Pertti Mattila is grateful for the invitation to visit the Department of Mathematics at Universitat Autònoma de Barcelona. J. Verdera acknowledges support from the grants 2021-SGR-00071 (Generalitat de Catalunya), PID2020-112881GB-I00 and Severo Ochoa and Maria de Maeztu CEX2020-001084-M (Ministerio de Ciencia e Innovación). J. J. Donaire acknowledges support from the grant MTM2017-85666-P (Ministerio de Ciencia e Innovación)

References

[Cufí et al. 2022] J. Cufí, A. C. Ponce, and J. Verdera, "The precise representative for the gradient of the Riesz potential of a finite measure", *J. Lond. Math. Soc.* (2) **106**:2 (2022), 1603–1627. MR Zbl

[Donaire et al. 2014] J. J. Donaire, J. G. Llorente, and A. Nicolau, "Differentiability of functions in the Zygmund class", *Proc. Lond. Math. Soc.* (3) **108**:1 (2014), 133–158. MR Zbl

[Hungerford 1988] G. J. Hungerford, *Boundaries of smooth sets and singular sets of Blaschke products in the little Bloch class*, Ph.D. thesis, California Institute of Technology, 1988.

[Mateu and Tolsa 2004] J. Mateu and X. Tolsa, "Riesz transforms and harmonic Lip_1-capacity in Cantor sets", *Proc. London Math. Soc.* (3) **89**:3 (2004), 676–696. MR Zbl

[Mattila 1995] P. Mattila, *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, 1995. MR Zbl

[Mattila and Verdera 2009] P. Mattila and J. Verdera, "Convergence of singular integrals with general measures", *J. Eur. Math. Soc.* **11**:2 (2009), 257–271. MR Zbl

[Shiryaev 1996] A. N. Shiryaev, *Probability*, 2nd ed., Graduate Texts in Mathematics **95**, Springer, 1996. MR Zbl

[Tolsa 2014] X. Tolsa, *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón–Zygmund theory*, Progress in Mathematics **307**, Springer, 2014. MR Zbl

Received June 8, 2023. Revised September 26, 2023.

JULIÀ CUFÍ
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
BARCELONA
CATALONIA
SPAIN
julia.cufi@uab.cat

JUAN JESÚS DONAIRE
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
BARCELONA
CATALONIA
SPAIN
juanjesus.donaire@uab.cat

PERTTI MATTILA
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF HELSINKI
HELSINKI
FINLAND
pertti.mattila@helsinki.fi

JOAN VERDERA
DEPARTMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
BARCELONA
CATALONIA
SPAIN
and
CENTRE DE RECERCA MATEMÀTICA
BARCELONA
CATALONIA
SPAIN
joan.verdera@uab.cat

PACIFIC JOURNAL OF MATHEMATICS
Vol. 326, No. 2, 2023
https://doi.org/10.2140/pjm.2023.326.301

CERTAIN FOURIER OPERATORS AND THEIR ASSOCIATED POISSON SUMMATION FORMULAE ON GL_1

DIHUA JIANG AND ZHILIN LUO

We explore the possibility of using harmonic analysis on GL_1 to understand Langlands automorphic L-functions in general, as a vast generalization of the PhD Thesis of J. Tate in 1950. For a split reductive group G over a number field k, let $G^\vee(\mathbb{C})$ be its complex dual group and ρ be an n-dimensional complex representation of $G^\vee(\mathbb{C})$. For any irreducible cuspidal automorphic representation σ of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of k, we introduce the space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ of (σ,ρ)-Schwartz functions on $\mathbb{A}^\times$ and (σ,ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ that takes $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$, where $\widetilde{\sigma}$ is the contragredient of σ. By assuming the local Langlands functoriality for the pair (G,ρ), we show that the (σ,ρ)-theta functions $\Theta_{\sigma,\rho}(x,\phi) := \sum_{\alpha\in k^\times}\phi(\alpha x)$ converge absolutely for all $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$. We state conjectures on the (σ,ρ)-Poisson summation formula on GL_1, and prove them in the case where $G = \mathrm{GL}_n$ and ρ is the standard representation of $\mathrm{GL}_n(\mathbb{C})$. This is done with the help of results of Godement and Jacquet (1972). As an application, we provide a spectral interpretation of the critical zeros of the standard L-functions $L(s,\pi\times\chi)$ for any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ and idele class character χ of k, extending theorems of C. Soulé (2001) and A. Connes (1999). Other applications are in the introduction.

The research of this paper is supported in part by the NSF Grants DMS-1901802 and DMS-2200890.
MSC2020: primary 11F66, 43A32, 46S10; secondary 11F70, 22E50, 43A80.
Keywords: invariant distribution, Fourier operator, Poisson summation formula, automorphic representation, automorphic L-function, representation of real and p-adic reductive groups, spectral interpretation of critical zeros of automorphic L-functions.

1. Introduction

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. It is well known that $\mathbb{A}$ is a locally compact abelian group and the diagonal embedding of k into $\mathbb{A}$ is a lattice, i.e., the image, which is still denoted by k, is discrete and the quotient $k\backslash\mathbb{A}$ is compact. The classical theory of harmonic analysis on the quotient $k\backslash\mathbb{A}$ — in particular, the famous 1950 Princeton thesis of J. Tate [44] — has had a great impact on the modern development of number theory, especially on the theory of automorphic L-functions.

In Tate's thesis, the classical Fourier transform and the associated Poisson summation formula are responsible for the meromorphic continuation and global functional equation of the Hecke L-function $L(s, \chi)$ attached to an automorphic character χ of $k^\times\backslash\mathbb{A}^\times$.

In their pioneering work in 1972, R. Godement and H. Jacquet extended the work of Tate on $L(s, \chi)$ (and also the work of T. Tamagawa in [43]) to the standard automorphic L-function $L(s, \pi)$ attached to any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ [16]. In their work, the Fourier transform and the associated Poisson summation formula for $M_n(k)\backslash M_n(\mathbb{A})$ are responsible for the meromorphic continuation and global functional equation of $L(s, \pi)$. Here M_n denotes the space of all $n\times n$ matrices.

In 2000, A. Braverman and D. Kazhdan [6] proposed that there should exist a generalized Fourier transform $\mathcal{F}_{\rho,\psi}$ on $G(\mathbb{A})$ for any reductive group G defined over k and any finite-dimensional complex representation ρ of the L-group ${}^LG$; and if the associated Poisson summation formula could be established, then there is a hope to prove the Langlands conjecture [29] on meromorphic continuation and global functional equation for automorphic L-function $L(s, \pi, \rho)$ attached to the pair (π, ρ), where π is any irreducible cuspidal automorphic representation of $G(\mathbb{A})$. In [33; 34], one may find careful discussions on the spherical case of and a helpful introduction to the proposal. In his 2020 paper [37], B. C. Ngô suggests that such generalized Fourier transforms could be put in a framework that generalizes the classical Hankel transform for harmonic analysis on GL_1 and might be more useful in the trace formula approach to establish the Langlands conjecture of functoriality in general.

1A. GL_1-*theory.* We develop GL_1-theory to explore a possibility of using harmonic analysis on GL_1 to understand Langlands automorphic L-functions in general, which would be a vast generalization of the classical work of Tate in [44] or of the more systematical treatment by A. Weil in [48]. The development goes in two steps. The first step is to establish it for the standard automorphic L-function $L(s, \pi)$ associated with an irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$. When $n = 1$ and π is an automorphic character χ, it is the theory developed in Tate's thesis. The second step is to formulate the framework for the general automorphic L-function $L(s,\pi,\rho)$ associated with a pair (π,ρ) as introduced above.

The GL_1-theory for a standard L-function $L(s,\pi)$ is a reformulation and refinement of the Godement–Jacquet theory [16] for $L(s,\pi)$ of GL_n. It is based on the determinant morphism

$$\det : M_n \to \mathbb{G}_a; \quad \mathrm{GL}_n \to \mathbb{G}_m, \tag{1-1}$$

where $\mathbb{G}_a(k) = k$ and $\mathbb{G}_m(k) = \mathrm{GL}_1(k) = k^\times$. We write $\pi = \bigotimes_{\nu\in|k|} \pi_\nu$ where $|k|$ is the set of local places of k and π_ν is an irreducible admissible representation of $\mathrm{GL}_n(k_\nu)$, which is of Casselman–Wallach type if k_ν is an Archimedean local field. For each π_ν, by taking the fiber integration along det as defined in (3-6), we define in Definition 3.3 the π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$. It is important to understand the structure of the space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ of π_ν-Schwartz functions on $k_\nu^\times$, whose properties are discussed intensively in Section 3. In particular, by Proposition 3.2 and Corollary 3.8, we have that

$$\mathcal{C}_c^\infty(k_\nu^\times) \subset \mathcal{S}_{\pi_\nu}(k_\nu^\times) \subset \mathcal{C}^\infty(k_\nu^\times).$$

It is important to mention that Theorem 7.1 provides a new characterization of $\mathcal{C}_c^\infty(k_\nu^\times)$ as a subspace of $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ by means of the fiber integration along det in (3-6). Through diagram (3-16), we define the π_ν-Fourier operator (or transform) $\mathcal{F}_{\pi_\nu,\psi_\nu}$, where ψ_ν is the ν-component of a fixed nontrivial character ψ of $k\backslash\mathbb{A}$. By the local GL_1-theory (Theorems 3.4 and 3.10), there exists a so-called *basic function* $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ when $\nu < \infty$ and π_ν is unramified, and the π_ν-Fourier operator maps the π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ to the $\widetilde{\pi}_\nu$-Schwartz space $\mathcal{S}_{\widetilde{\pi}_\nu}(k_\nu^\times)$ with $\mathcal{F}_{\pi_\nu,\psi_\nu}(\mathbb{L}_{\pi_\nu}) = \mathbb{L}_{\widetilde{\pi}_\nu}$. The global π-Schwartz space $\mathcal{S}_\pi(\mathbb{A}^\times)$ is defined to be the restricted tensor product

$$\mathcal{S}_\pi(\mathbb{A}^\times) := \bigotimes_{\nu\in|k|} \mathcal{S}_{\pi_\nu}(k_\nu^\times)$$

with respect to the basic functions $\mathbb{L}_{\pi_\nu}$ for almost all finite local places, and the global π-Fourier operator $\mathcal{F}_{\pi,\psi}$ is defined by

$$\mathcal{F}_{\pi,\psi}(\phi) := \bigotimes_{\nu\in|k|} \mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)$$

for any factorizable functions $\phi = \bigotimes_{\nu\in|k|} \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$. One of the main results in the global GL_1-theory is the π-Poisson summation formula on GL_1.

Theorem 1.1 (π-Poisson summation formula, Theorem 4.7). *Let π be an irreducible cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A})$. *For any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, the π-theta function*

$$\Theta_\pi(x,\phi) := \sum_{\alpha\in k^\times} \phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$, *and we have the identity*

$$\Theta_\pi(x, \phi) = \Theta_{\widetilde{\pi}}(x^{-1}, \mathcal{F}_{\pi,\psi}(\phi)) \quad \textit{for } x \in \mathbb{A}^\times. \tag{1-2}$$

According to the tradition in literature, the π-Poisson summation formula in (1-2) may also be called the π-theta inversion formula. Our proof of Theorem 1.1 (Theorem 4.7) is based on the work of Godement–Jacquet in [16].

The GL_1-theory for general L-functions $L(s, \sigma, \rho)$ is formulated by means of the local Langlands functorial conjecture associated with ρ, which is the major conjecture in the local theory of the Langlands program.

For a k-split reductive group G, let $G^\vee(\mathbb{C})$ be its complex dual group and ρ be an n-dimensional complex representation of $G^\vee(\mathbb{C})$. For any irreducible cuspidal automorphic representation $\sigma = \bigotimes_{\nu\in|k|} \sigma_\nu$ of $G(\mathbb{A})$, we assume that the local Langlands functorial transfer $\pi_\nu = \pi_\nu(\sigma_\nu, \rho)$ exists and is an irreducible admissible representation of $\mathrm{GL}_n(k_\nu)$, which is of the Casselman–Wallach type if k_ν is Archimedean. We define as in (6-5) the (σ_ν, ρ)-*Schwartz space* on $k_\nu^\times$ to be

$$\mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times) := \mathcal{S}_{\pi_\nu}(k_\nu^\times),$$

and at unramified local places, the (σ_ν, ρ)-basic function $\mathbb{L}_{\sigma_\nu,\rho}$ is taken to be the π_ν-basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$. Then we can define as in (6-6) the (σ, ρ)-*Schwartz space* on $\mathbb{A}^\times$ to be

$$\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times) := \bigotimes_\nu \mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times),$$

which is the restricted tensor product with respect to the basic function $\mathbb{L}_{\sigma_\nu,\rho}$ at almost all finite local places, and define, as in (6-8), the (σ, ρ)-Fourier operator (or transform) $\mathcal{F}_{\sigma,\rho,\psi}$ that takes $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$, where $\widetilde{\sigma}$ is the contragredient of σ. The first result in the global GL_1-theory for $L(s, \sigma, \rho)$ is the following.

Theorem 1.2. *With notations as introduced above, for all* $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, *the* (σ, ρ)-*theta function*

$$\Theta_{\sigma,\rho}(x, \phi) := \sum_{\alpha\in k^\times} \phi(\alpha x) \tag{1-3}$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$.

It is clear that Theorem 1.2 is a special case of Theorem 6.2, which asserts the same result as in Theorem 1.2 for much more general σ. The proof of Theorem 6.2 is deduced from the technical result (Theorem 5.4), which can be stated as follows.

Theorem 1.3 (Theorem 5.4). *Let* $\pi = \bigotimes_{\nu\in|k|} \pi_\nu$ *be an irreducible admissible representation of* $\mathrm{GL}_n(\mathbb{A})$ *with Assumption 5.1. Then for any* $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times) :=$

$\bigotimes_{\nu\in|k|}\mathcal{S}_{\pi_\nu}(k_\nu^\times)$, *the* π*-theta function*

$$\Theta_\pi(x,\phi):=\sum_{\alpha\in k^\times}\phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x\in\mathbb{A}^\times$.

We refer to Section 5 for notation not given here. Section 5 is devoted to develop the basic properties of such general theta functions. Then we show that for any irreducible admissible automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$, Assumption 5.1 holds (Proposition 5.5). As a consequence, we obtain the following general assertion.

Corollary 1.4 (Corollary 5.6). *Let* π *be any irreducible admissible automorphic representation of* $\mathrm{GL}_n(\mathbb{A})$. *For any* $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$, *the* π*-theta function*

$$\Theta_\pi(x,\phi)=\sum_{\alpha\in k^\times}\phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x\in\mathbb{A}^\times$.

It remains to be an interesting problem to establish the π-Poisson summation formula for such general π-theta functions as in Corollary 1.4, although Theorem 7.3 obtains the π-Poisson summation formula as in Theorem 1.1 for $\Theta_\pi(x,\phi)$ when π is any irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ and ϕ has restrictions at two local places (see Theorem 7.3 for details).

The following is the main statement in the global GL_1-theory for $L(s,\sigma,\rho)$.

Conjecture 1.5 ((σ,ρ)-Poisson summation formula). *Let* $\rho: G^\vee(\mathbb{C})\to\mathrm{GL}_n(\mathbb{C})$ *be any finite-dimensional representation of the complex dual group* $G^\vee(\mathbb{C})$ *and* σ *be an irreducible cuspidal automorphic representation of* $G(\mathbb{A})$. *Then there exist nontrivial* $k^\times$*-invariant linear functionals* $\mathcal{E}_{\sigma,\rho}$ *and* $\mathcal{E}_{\widetilde{\sigma},\rho}$ *on* $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ *and* $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$, *respectively, such that the* (σ,ρ)*-Poisson summation formula*

$$\mathcal{E}_{\sigma,\rho}(\phi)=\mathcal{E}_{\widetilde{\sigma},\rho}(\mathcal{F}_{\sigma,\rho,\psi}(\phi))$$

holds for $\phi\in\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, *where* $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ *and* $\mathcal{F}_{\sigma,\rho,\psi}$ *are defined in Section 6B.*

It is expected that such Poisson summation formulae on GL_1 should be responsible for the Langlands conjecture on the global functional equation of automorphic L-functions associated with the pairs (σ,ρ). Variants of Conjecture 1.5 will be discussed in Section 7C and see Conjecture 7.4 for details. It is clear that Theorem 1.1 proves Conjecture 1.5 for the case when σ is an irreducible cuspidal automorphic representation π of $G(\mathbb{A})=\mathrm{GL}_n(\mathbb{A})$ and ρ is the standard representation of $G^\vee(\mathbb{C})=\mathrm{GL}_n(\mathbb{C})$ (Theorem 4.7). A variant of Theorem 4.7 (Theorem 1.1) is established in Theorem 7.3 when π is an irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, based on the characterization in Theorem 7.1 of the subspace $\mathcal{C}_c^\infty(k_\nu^\times)$ in $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ through the fiber integration.

It is important to mention that according to the definition of $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and $\mathcal{F}_{\sigma,\rho,\psi}$ in (6-6) and (6-8), respectively, if the image of σ under the Langlands functorial transfer associated with ρ (if it exists) is an irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$, then the nontrivial $k^\times$-invariant linear functionals $\mathcal{E}_{\sigma,\rho}$ and $\mathcal{E}_{\widetilde{\sigma},\rho}$ in Conjecture 1.5 can be taken to be

$$\mathcal{E}_{\sigma,\rho}(\phi) = \Theta_{\sigma,\rho}(1,\phi) \quad \text{and} \quad \mathcal{E}_{\widetilde{\sigma},\rho}(\phi) = \Theta_{\widetilde{\sigma},\rho}(1,\phi)$$

for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ (see Corollary 6.3 for details). In this case, Conjecture 1.5 follows from Theorem 1.1 (Theorem 4.7). Therefore, Conjecture 1.5 is supported by various known cases of the global Langlands functoriality conjecture associated with $\rho : G^\vee(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$.

From the point of view of the global Langlands functoriality conjecture, it is important to extend Theorem 1.1 (Theorem 4.7) to more general irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, which may yield new understanding of the nature of the both nontrivial $k^\times$-invariant linear functionals $\mathcal{E}_{\sigma,\rho}$ and $\mathcal{E}_{\widetilde{\sigma},\rho}$ in Conjecture 1.5. At this point, we would also like to bring the attention of the reader to the work of L. Lafforgue [27; 28] on the relations between the global Langlands functoriality conjecture and a certain nonlinear Poisson formula conjecture.

The ultimate goal in the global theory for $L(s,\sigma,\rho)$ is to prove Conjecture 1.5 without using the global Langlands functoriality. It is expected that Conjecture 1.5 can be proved directly for a split classical group G and the standard representation ρ of the complex dual group $G^\vee(\mathbb{C})$, by using the doubling method of I. Piatetski-Shapiro and S. Rallis in [14] and the recent work of L. Zhang and the authors in [26] and of J. Getz and B. Liu in [15].

As applications of the GL_1-theory for automorphic L-functions and the π-Poisson summation formulas, we are able to provide in Theorem 8.1 a spectral interpretation of the critical zeros of the standard L-functions $L(s,\pi\times\chi)$ for any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ and idele class character χ of k. Theorem 8.1 is a reformulation of [40, Theorem 2] in the adelic framework of A. Connes in [11] and is an extension of [11, Theorem III.1] from the Hecke L-functions $L(s,\chi)$ to the automorphic L-functions $L(s,\pi\times\chi)$. In [24], Zhaolin Li and Dihua Jiang provide a new proof of the Voronoi summation formula for GL_n [20, Theorem 1] by means of Theorem 4.7 (Theorem 1.1), in other words, by means of the GL_1-reformulation of the Godement–Jacquet theory for the standard L-functions of GL_n. This GL_1-theory also proves in [24] the (GL_n,π)-version with the Godement–Jacquet kernels of the Clozel theorem [10, Theorem 1.1], which was proved by L. Clozel for $n = 1$ and with the Tate kernels. In their upcoming work [35], Ngô and Luo use the ideas and the methods of this paper and of [25] to treat the local theory of the Braverman–Kazhdan–Ngô proposal for the torus case.

1B. ***Brief explanation of each section.*** In Section 2, we reformulate the local theory of Godement–Jacquet [16] in terms of the framework of the Braverman–Kazhdan–Ngô proposal. We take $F = k_\nu$ for every $\nu \in |k|$ and recall the local theory of the Mellin transforms, mainly from [21, Chapter I]. In general, it could be highly nontrivial to reformulate the known Rankin–Selberg theory for certain automorphic L-functions in terms of the framework of the Braverman–Kazhdan–Ngô proposal as indicated in [26]. The key point is that one has to figure out the invariant distribution Φ_ν on $G(k_\nu)$, which controls the local theory proposed by Braverman–Kazhdan in [6] and by Ngô in [37]. Even in the case of Godement–Jacquet, the candidate of such an invariant distribution $\Phi_{\mathrm{GJ},\nu}$ is expected to the experts, but there is no written document available. We provide the details in Section 2C and the results are given in Proposition 2.8.

In Section 3, we fully develop the local theory of harmonic analysis on GL_1 for the Langlands local L-factors $L(s,\pi)$ and γ-factors $\gamma(s,\pi,\psi)$, attached to any irreducible admissible representations π of $\mathrm{GL}_n(F)$. When F is non-Archimedean, we take π to be irreducible smooth representations of $\mathrm{GL}_n(F)$; and when F is Archimedean, we take π to be irreducible Casselman–Wallach representations of $\mathrm{GL}_n(F)$ [4; 9; 41; 46]. The set of equivalence classes of all such representations of $\mathrm{GL}_n(F)$ is denoted by $\Pi_F(\mathrm{GL}_n)$.

By Theorem 2.3, via the Mellin inversion, the local Godement–Jacquet L-functions (or L-factors) (or even general local Langlands L-functions) could be a GL_1-object, i.e., there exists a subspace of smooth functions $\mathcal{C}^\infty(F^\times)$, whose Mellin transform sees the corresponding local L-functions. One of the goals in this section is to recover such a subspace associated to a local Godement–Jacquet L-function $L(s,\pi)$ by means of the matrix coefficients of π. More precisely, we introduce the space of π-Schwartz functions on $F^\times$ for any $\pi \in \Pi_F(\mathrm{GL}_n)$, which is denoted by $\mathcal{S}_\pi(F^\times)$ (Definition 3.3). By Proposition 3.2, we have that $\mathcal{S}_\pi(F^\times) \subset \mathcal{C}^\infty(F^\times)$. The first local result is Theorem 3.4, which establishes the local theory of zeta integrals on GL_1 for the Langlands local L-function $L(s,\pi)$ for any $\pi \in \Pi_F(\mathrm{GL}_n)$. The relevant local functional equation and the properties of the π-Fourier operator (transform) $\mathcal{F}_{\pi,\psi}$ as defined in (3-17) is established in Theorem 3.10, the second local result.

We note that in [25], a further local theory has been developed so that the π-Fourier operator $\mathcal{F}_{\pi,\psi}$ can be expressed as a convolution operator with kernel functions $k_{\pi,\psi}$ for any $\pi \in \Pi_F(\mathrm{GL}_n)$ [25, Theorem 5.1]. In [24], such kernel functions are proved to be the normalized Bessel functions associated with π and a certain Weyl group element of GL_n. Hence, the π-Fourier operator $\mathcal{F}_{\pi,\psi}$ is a natural generalization of the classical Hankel transform.

In Section 4, we develop the global theory of harmonic analysis on GL_1 for the standard automorphic L-functions $L(s,\pi)$ associated with any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$. To do this, we consider any irreducible admissible representation $\pi = \bigotimes_{\nu\in|k|} \pi_\nu$ of $\mathrm{GL}_n(\mathbb{A})$, with $\pi_\nu \in \Pi_{k_\nu}(\mathrm{GL}_n)$, and introduce, for more general π, the π-Schwartz space $\mathcal{S}_\pi(\mathbb{A}^\times) = \bigotimes_{\nu\in|k|} \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ in (4-1), where the restricted tensor product with respect to the basic function $\mathbb{L}_{\pi_\nu}$ (as defined in Theorem 3.4) is taken at almost all finite local places ν. The π-Fourier operator $\mathcal{F}_{\pi,\psi}(\phi) = \bigotimes_{\nu\in|k|} \mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)$ is defined in (4-3), with $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$. The main global result in this section is Theorem 4.7, which is a restatement of Theorem 1.1 and establishes the π-Poisson summation formula on GL_1 for any irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$.

To understand the Poisson summation formulae in Conjecture 1.5, it is desirable to explore variants of Theorem 4.7 when the automorphic representation π may not be cuspidal, from the point of view of the global Langlands functoriality. In Section 5, we first show that for any irreducible admissible representation π of $\mathrm{GL}_n(\mathbb{A})$, which may not be automorphic, but satisfies Assumption 5.1, the π-theta functions

$$\Theta_\pi(x,\phi) = \sum_{\gamma\in k^\times} \phi(\gamma x) \quad \text{for } \phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$$

converge absolutely and locally uniformly as functions in $x \in \mathbb{A}^\times$ (Theorem 5.4). Then we show that Assumption 5.1 holds for any automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ (Proposition 5.5). With Theorem 5.4, we are ready to explore a more general situation in order to formulate Conjecture 1.5 and its variant (Conjecture 7.4).

In Section 6, we consider any k-split reductive group G. In Section 6B, for any finite-dimensional representation ρ of the complex dual group $G^\vee(\mathbb{C})$, we define the relevant Schwartz spaces $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, called the (σ,ρ)-Schwartz space, in (6-6), and (σ,ρ)-Fourier operators $\mathcal{F}_{\sigma,\rho,\psi}$ in (6-8) for any irreducible cuspidal automorphic representation σ of $G(\mathbb{A})$, under the assumption (Assumption 6.1) that the local Langlands reciprocity map exists for G over all finite local places ν of k. We prove in such a generality the convergence properties of the (σ,ρ)-theta function $\Theta_{\sigma,\rho}(x,\phi)$ as defined in (1-3) for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and any $x \in \mathbb{A}^\times$ (Theorem 6.2, which contains Theorem 1.2 as a special case).

In Section 7, after we establish a new characterization of $\mathcal{C}_c^\infty(k_\nu^\times)$ as a subspace of $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ in Theorem 7.1 at all local places of k, we prove a variant of Theorem 4.7 when π is an irreducible square-integrable automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ (Theorem 7.3). Finally we write down a variant of Conjecture 1.5 with more details in Conjecture 7.4.

In order to understand the Poisson summation formulae in Conjectures 1.5 and 7.4, we have to explore and develop harmonic analysis on GL_1 initiated by the (σ,ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ and the (σ,ρ)-Schwartz space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, both locally and globally. We refer to [24; 25] for a further discussion of the local theory, while a further global theory remains to be developed in our future work.

In Section 8, as an application of the GL_1-harmonic analysis we developed beforehand, we provide a spectral interpretation of the critical zeros of the automorphic L-functions $L(s, \pi \times \chi)$ (Theorem 8.1) for any irreducible cuspidal automorphic representation π of $GL_n(\mathbb{A})$ and any character χ of the idele class group of k. It can be viewed as a reformulation of [40, Theorem 2] in the adelic framework of A. Connes in [11] and an extension of [11, Theorem III.1] from Hecke L-functions $L(s, \chi)$ to automorphic L-functions $L(s, \pi \times \chi)$. The proof uses a combination of arguments in [40], and those in [11], together with the results developed before Section 8. Further results along the line of [11] will be written in our forthcoming work.

2. Godement–Jacquet theory and reformulation

2A. ***Mellin transforms.*** We recall the local theory of Mellin transforms from the book of Igusa [21, Chapter I] and state them in a slightly more general situation in order to treat the case that meromorphic functions may have poles that are not real numbers. Since the proofs are almost the same, we omit the details.

Let F be a local field of characteristic zero. This means that it is either the complex field $\mathbb{C}$, the real field $\mathbb{R}$, or a finite extension of the p-adic field $\mathbb{Q}_p$ for some prime p.

When F is non-Archimedean, let $\mathfrak{o}_F$ be the ring of integers with maximal ideal $\mathfrak{p}_F$ and fix a uniformizer ϖ_F of $\mathfrak{p}_F$. Let $\mathfrak{o}_F/\mathfrak{p}_F = \kappa_F \simeq \mathbb{F}_q$. Fix the norm $|x|_F = q^{-\mathrm{ord}_F(x)}$ where $\mathrm{ord}_F : F \to \mathbb{Z}$ is the valuation on F such that $\mathrm{ord}_F(\varpi_F) = 1$. Fix the Haar measure d^+x on F so that $\mathrm{vol}(\mathrm{d}^+x, \mathfrak{o}_F) = 1$. Let $\psi = \psi_F$ be an additive character of F which is trivial on $\mathfrak{o}_F$ but nontrivial on $\varpi_F^{-1} \cdot \mathfrak{o}_F$. In particular the standard Fourier transform defined via ψ_F is self-dual w.r.t. d^+x. Similarly, fix a multiplicative Haar measure $\mathrm{d}^\times x$ on $F^\times$, which is normalized so that $\mathrm{vol}(\mathrm{d}^\times x, \mathfrak{o}_F^\times) = 1$. In particular $\mathrm{d}^\times x = (1/\zeta_F(1)) \cdot (\mathrm{d}^+x/|x|_F)$, where $\zeta_F(s)$ is the local Dedekind zeta factor attached to F.

When F is Archimedean, define on F the norm

$$|z|_F = \begin{cases} \text{absolute value of } z, & F = \mathbb{R}, \\ z\bar{z}, & F = \mathbb{C}. \end{cases}$$

Take the Haar measure d^+x on F that is the usual Lebesgue measure on F, and set

$$\mathrm{d}^\times x = \begin{cases} \frac{\mathrm{d}^+x}{2|x|_F}, & F = \mathbb{R}, \\ \frac{\mathrm{d}^+x}{2\pi|x|_F}, & F = \mathbb{C}, \end{cases}$$

the multiplicative Haar measures on $F^\times$. The additive character $\psi = \psi_F$ of F is chosen as

$$\psi_F(x) = \begin{cases} \exp(2\pi i x), & F = \mathbb{R}, \\ \exp(2\pi i (x + \bar{x})), & F = \mathbb{C}. \end{cases}$$

For convenience, define on F the norm

$$|\cdot| = \begin{cases} |\cdot|_F, & F \neq \mathbb{C}, \\ |\cdot|_F^{1/2}, & F = \mathbb{C}. \end{cases}$$

We denote by $\mathfrak{X}(F^\times)$ the set of all quasicharacters of $F^\times$. Define the topological group Ω_F to be $\{\pm 1\}$ if $F = \mathbb{R}$, $\mathbb{C}_1^\times$ if $F = \mathbb{C}$, and the unit group $\mathfrak{o}_F^\times$ if F is non-Archimedean. It is clear that any $\chi \in \mathfrak{X}(F^\times)$ can be written as

$$\chi(x) = \chi_u(x) = \chi_{u,\omega}(x) = |x|_F^u\, \omega(\mathrm{ac}(x)), \tag{2-1}$$

for any $x \in F^\times$, with $u \in \mathbb{C}$ and $\omega \in \Omega_F^\wedge$, the Pontryagin dual of Ω_F. Here $\mathrm{ac}(x) = x/|x|_F \in \mathfrak{o}_F^\times$ if F is non-Archimedean, and

$$\mathrm{ac}(x) = \begin{cases} \frac{x}{|x|_F} \in \{\pm 1\}, & F = \mathbb{R}, \\ \frac{x}{|x|} = \frac{x}{|x|_F^{1/2}} \in \mathbb{C}_1^\times, & F = \mathbb{C}. \end{cases} \tag{2-2}$$

It is clear that the unitary character ω of Ω_F is uniquely determined by $\chi \in \mathfrak{X}(F^\times)$, in particular, we have

$$\omega(\mathrm{ac}(x)) = \mathrm{ac}(x)^p, \tag{2-3}$$

with $p \in \{0, 1\}$ if $F = \mathbb{R}$ and $p \in \mathbb{Z}$ if $F = \mathbb{C}$. Hence, we may sometimes write $\chi = (u, \omega)$ and $\omega(x) = \omega(\mathrm{ac}(x))$ for $x \in F^\times$.

For any local field F of characteristic zero, following [21, Sections I.4 and I.5], we define the following two spaces of functions associated to the local field F.

Definition 2.1. Let $\mathfrak{F}(F^\times)$ be the space of complex-valued functions $\mathfrak{f}$ such that:

(1) $\mathfrak{f} \in \mathcal{C}^\infty(F^\times)$, the space of all smooth functions on $F^\times$.

(2) When F is non-Archimedean, $\mathfrak{f}(x) = 0$ for $|x|_F$ sufficiently large. When F is Archimedean, we define $\mathfrak{f}^{(n)} := \mathrm{d}^n\mathfrak{f}/\mathrm{d}x^n$ if $F = \mathbb{R}$, and $\mathfrak{f}^{(n)} = \mathfrak{f}^{(a+b)} := \partial^{a+b}\mathfrak{f}/(\partial^a x \partial^b \bar{x})$ if $F = \mathbb{C}$ and $n = a + b$. Then we have

$$\mathfrak{f}^{(n)}(x) = \mathrm{o}(|x|_F^\rho)$$

as $|x|_F \to \infty$ for any ρ and any $n = a + b \in \mathbb{Z}_{\geq 0}$ with $a, b \in \mathbb{Z}_{\geq 0}$.

(3) When F is Archimedean, there exists

- a sequence $\{m_k\}_{k=0}^\infty$ of positive integers,
- a sequence of smooth functions $\{a_{k,m}\}$ on $\{\pm 1\}$ if $F = \mathbb{R}$ and on $\mathbb{C}_1^\times$ if $F = \mathbb{C}$, parameterized by $m = 1, 2, \ldots, m_k$ and $k \in \mathbb{Z}_{\geq 0}$,
- a sequence $\{\lambda_k\}_{k=0}^\infty$ of complex numbers with $\{\mathrm{Re}(\lambda_k)\}_{k=0}^\infty$ a strictly increasing sequence of real numbers with no finite accumulation point and $\mathrm{Re}(\lambda_0) \geq \lambda \in \mathbb{R}$,

such that

$$\lim_{|x|_F\to 0}\left\{\mathfrak{f}(x)-\sum_{k=0}^{\infty}\sum_{m=1}^{m_k} a_{k,m}(\mathrm{ac}(x))|x|_F^{\lambda_k}(\ln|x|_F)^{m-1}\right\}=0.$$

The limit is termwise differentiable and uniform (even after termwise differentiation) in $\mathrm{ac}(x)$.

When F is non-Archimedean, one can take the sequence $\{\lambda_k\}$ to be a finite set Λ and the sequence $\{m_k\}$ to be a finite subset of $\mathbb{Z}_{\geq 0}$. The smooth functions $\{a_{k,m}(\mathrm{ac}(x))\}$ are on the unit group $\mathfrak{o}_F^\times$.

Since the topological group Ω is compact and abelian, we have the following Fourier expansion for the smooth functions $\{a_{k,m}(\mathrm{ac}(x))\}$ on Ω:

$$a_{k,m}(\mathrm{ac}(x))=\sum_{\omega\in\Omega^\wedge} a_{k,m,\omega}\,\omega(\mathrm{ac}(x)).$$

In the Archimedean case, we may write $a_{k,m,\omega}=a_{k,m,p}$ with $p\in\{0,1\}$ if $F=\mathbb{R}$ and $p\in\mathbb{Z}$ if $F=\mathbb{C}$.

Definition 2.2. With the same notation as in Definition 2.1, let $\mathcal{Z}(\mathfrak{X}(F^\times))$ be the space of complex-valued functions $\mathfrak{z}(\chi_{s,\omega})=\mathfrak{z}\big(|\cdot|_F^s\,\omega(\mathrm{ac}(\cdot))\big)$ on $\mathfrak{X}(F^\times)$ such that:

(1) $\mathfrak{z}(\chi_{s,\omega})$ is meromorphic on $\mathfrak{X}(F^\times)$ with poles at most for $s=-\lambda_j$ with λ_j belonging to the given set $\{\lambda_k\}_{k=0}^{\infty}$ if F is Archimedean; and belonging to the given finite set Λ if F is non-Archimedean.

(2) For any $k\geq 0$, the difference

$$\mathfrak{z}(\chi_{s,\omega})-\sum_{m=1}^{m_k}\frac{b_{k,m,\omega}}{(s+\lambda_k)^m}$$

is holomorphic for s in a small neighborhood of $-\lambda_k$ if F is Archimedean; and is a polynomial in $\mathbb{C}[q^s,q^{-s}]$ if F is non-Archimedean.

(3) When F is non-Archimedean, the function $\mathfrak{z}(\chi_{s,\omega})$ is identically zero for almost all characters $\omega\in\Omega^\wedge$ with $\Omega=\mathfrak{o}_F^\times$. When F is Archimedean, for every polynomial $P(s,p)$ in s, p with coefficients in $\mathbb{C}$, and every pair of real numbers $a<b$, the function $P(s,p)\mathfrak{z}(\chi_{s,\omega})$ is bounded when s belongs to the vertical strip

$$S_{a,b}=\{s\in\mathbb{C}\mid a\leq\mathrm{Re}(s)\leq b\},\tag{2-4}$$

with neighborhoods of $-\lambda_0,-\lambda_1,\ldots$ removed therefrom. More precisely, there exists a constant c depending only on P, $\mathfrak{z}$, a, b, but neither on s nor on p, such that

$$|P(s,p)\mathfrak{z}(\chi_{s,\omega})|\leq c$$

when s runs in the vertical strip $S_{a,b}$ with small neighborhoods of $-\lambda_0, -\lambda_1, \dots$ removed.

The main results on the local theory of Mellin transforms established in [21, Chapter I] are as follows.

Theorem 2.3 (Mellin transforms). *There is a bijective linear correspondence $\mathcal{M} = \mathcal{M}_F$ between the space $\mathfrak{F}(F^\times)$ and the space $\mathcal{Z}(\mathfrak{X}(F^\times))$. More precisely, for $\mathfrak{f} \in \mathfrak{F}(F^\times)$,*

$$\mathcal{M}(\mathfrak{f})(\chi_{s,\omega}) = \int_{F^\times} \mathfrak{f}(x)\chi_{s,\omega}(x)\, \mathrm{d}^\times x$$

defines a holomorphic function on

$$\mathfrak{X}_{-\sigma_0}(F^\times) = \{\chi_{s,\omega}(\,\cdot\,) = |\cdot|_F^s \omega(\mathrm{ac}(\,\cdot\,)) \in \mathfrak{X}(F^\times) \mid \mathrm{Re}(s) > -\sigma_0\}$$

for some $\sigma_0 \in \mathbb{R}$, which has a meromorphic continuation to all characters $\chi_{s,\omega} \in \mathfrak{X}(F^\times)$ and belongs to $\mathcal{Z}(\mathfrak{X}(F^\times))$ after meromorphic continuation. Conversely, for $\mathfrak{z} \in \mathcal{Z}(\mathfrak{X}(F^\times))$ and $x \in F^\times$, the Mellin inverse transform $\mathcal{M}_F^{-1}(\mathfrak{z})(x)$ belongs to the space $\mathfrak{F}(F^\times)$. We have the identities

$$\mathcal{M}(\mathcal{M}^{-1}(\mathfrak{z})) = \mathfrak{z} \quad \text{and} \quad \mathcal{M}^{-1}(\mathcal{M}(\mathfrak{f})) = \mathfrak{f}$$

for any $\mathfrak{f} \in \mathfrak{F}(F^\times)$ and $\mathfrak{z} \in \mathcal{Z}(\mathfrak{X}(F^\times))$. Here the Mellin inverse transform is explicitly given as follows.

When F is Archimedean, the Mellin inverse transform $\mathcal{M}_F^{-1}(\mathfrak{z})(x)$ is given by

$$\mathcal{M}^{-1}(\mathfrak{z})(x) := \sum_{\omega \in \Omega_F^\wedge} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathfrak{z}(\chi_{s,\omega})\chi_{s,\omega}(x)^{-1}\, \mathrm{d}s \tag{2-5}$$

with $\omega(\mathrm{ac}(x)) = \mathrm{ac}(x)^p$, which defines a function $\mathfrak{f}$ in $\mathfrak{F}(F^\times)$ independent of $\sigma > -\sigma_0$, and the coefficients $a_{k,m,p}$ and $b_{k,m,p}$ satisfy the relations

$$b_{k,m,p} = (-1)^{m-1}(m-1)! \cdot a_{k,m,-p}$$

for every $k \geq 0$, $m \geq 1$ with $p \in \{0, 1\}$ if $F = \mathbb{R}$ and $p \in \mathbb{Z}$ if $F = \mathbb{C}$. The coefficients $a_{k,m,p}$ and $b_{k,m,p}$ satisfy the relations

$$b_{\lambda,m,\omega} = \sum_{j=m}^{m_\lambda} e_{j,m}(-\ln q)^{j-1} a_{\lambda,j,\omega^{-1}}$$

with $e_{j,m}$ defined by the following identity of polynomials in a formal unknown t:

$$t^{n-1} = \sum_{\ell=1}^{n} e_{n,\ell} \binom{t+\ell-1}{\ell-1}.$$

If F is non-Archimedean, the Mellin inverse transform $\mathcal{M}_F^{-1}(\mathfrak{z})(x)$ is given by

$$\mathcal{M}_F^{-1}(\mathfrak{z})(x) := \sum_{\omega\in\Omega^\wedge} \big(\mathrm{Res}_{z=0}(\mathfrak{z}(\chi_{s,\omega})|x|_F^{-s}q^s)\big)\omega(\mathrm{ac}(x))^{-1}, \tag{2-6}$$

which defines a function $\mathfrak{f}$ in $\mathfrak{F}(F^\times)$. Here $z = q^{-s}$ for abbreviation.

2B. ***Local theory of Godement–Jacquet.*** Let $G_n := \mathrm{GL}_n$ be the general linear group defined over F. Fix the following maximal (open if F is non-Archimedean) compact subgroup K of $G_n(F) = \mathrm{GL}_n(F)$:

$$K = \begin{cases} \mathrm{GL}_n(\mathfrak{o}_F), & F \text{ is non-Archimedean},\\ O(n), & F = \mathbb{R},\\ U(n), & F = \mathbb{C}. \end{cases} \tag{2-7}$$

Fix the Haar measure $\mathrm{d}g = \mathrm{d}^+g/|\det g|_F^n$ on $G_n(F)$ where d^+g is the measure induced from the standard additive measure on $M_n(F)$, the F-vector space of $n\times n$-matrices. In particular, $G_n(F)$ embeds into $M_n(F)$ in a standard way.

Let $\Pi_F(G_n)$ be the set of equivalence classes of irreducible smooth representations of $G_n(F)$ when F is non-Archimedean; and of irreducible Casselman–Wallach representations of $G_n(F)$ when F is Archimedean. Let $\mathcal{C}(\pi)$ be the space of smooth matrix coefficients attached to π.

Let $\mathcal{S}(M_n(F))$ be the space of the standard Schwartz–Bruhat functions on $M_n(F)$. The standard Fourier transform $\mathcal{F}_\psi$ acting on $\mathcal{S}(M_n(F))$ is defined as

$$\mathcal{F}_\psi(f)(x) = \int_{M_n(F)} \psi(\mathrm{tr}(xy)) f(y)\,\mathrm{d}^+y, \tag{2-8}$$

where ψ is a nontrivial additive character of F. The standard Fourier transform $\mathcal{F}_\psi$ extends to a unitary operator on the space $L^2(M(F), \mathrm{d}^+x)$ and satisfies the identity

$$\mathcal{F}_\psi \circ \mathcal{F}_{\psi^{-1}} = \mathrm{Id}. \tag{2-9}$$

For any $\pi \in \Pi_F(G_n)$ and any quasicharacter $\chi \in \mathfrak{X}(F^\times)$, the local zeta integral of Godement and Jacquet is defined by

$$\mathcal{Z}(s, f, \varphi_\pi, \chi) = \int_{G_n(F)} f(g)\varphi_\pi(g)\chi(\det g)|\det g|_F^{s+(n-1)/2}\,\mathrm{d}g, \tag{2-10}$$

for any $f \in \mathcal{S}(M_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$. The following theorem contains the main results in the local theory of the Godement and Jacquet zeta integrals [16, Chapter I].

Theorem 2.4. *With the notation introduced above, the following statements hold for any $f \in \mathcal{S}(M_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$:*

(1) *The zeta integral* $\mathcal{Z}(s, f, \varphi_\pi, \chi)$ *defined in* (2-10) *is absolutely convergent for* $\mathrm{Re}(s)$ *sufficiently large and admits a meromorphic continuation to* $s \in \mathbb{C}$.

(2) $\mathcal{Z}(s, f, \varphi_\pi, \chi)$ *is a holomorphic multiple of the Langlands local L-function* $L(s, \pi \times \chi)$ *associated to* (π, χ) *and the standard embedding*

$$\mathrm{std} : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}).$$

When F *is non-Archimedean, the fractional ideal* $\mathcal{I}_{\pi,\chi}$ *that is generated by the local zeta integrals* $\mathcal{Z}(s, f, \varphi_\pi, \chi)$ *is of the form*

$$\mathcal{I}_{\pi,\chi} = \{\mathcal{Z}(s, f, \varphi_\pi, \chi) \mid f \in \mathcal{S}(M_n(F)),\ \varphi_\pi \in \mathcal{C}(\pi)\} = L(s, \pi \times \chi) \cdot \mathbb{C}[q^s, q^{-s}];$$

and when F *is Archimedean, the local zeta integrals* $\mathcal{Z}(s, f, \varphi_\pi, \chi)$, *with unitary characters* χ, *have the following property. Let* $S_{a,b}$ *be the vertical strip for any* $a < b$, *defined in* (2-4). *If* $P_\chi(s)$ *is a polynomial in* s *such that the product* $P_\chi(s)L(s, \pi \times \chi)$ *is bounded in the vertical strip* $S_{a,b}$, *then the product* $P_\chi(s)\mathcal{Z}(s, f, \varphi_\pi, \chi)$ *must be bounded in the same vertical strip* $S_{a,b}$.

(3) *The local functional equation*

$$\mathcal{Z}(1 - s, \mathcal{F}_\psi(f), \varphi_\pi^\vee, \chi^{-1}) = \gamma(s, \pi \times \chi, \psi) \cdot \mathcal{Z}(s, f, \varphi_\pi, \chi)$$

holds after meromorphic continuation, where the function $\varphi_\pi^\vee(g)$ *is defined as* $\varphi_\pi(g^{-1}) \in \mathcal{C}(\widetilde{\pi})$, *and* $\gamma(s, \pi \times \chi, \psi)$ *is the Langlands local gamma function associated to* (π, χ) *and* std.

(4) *When* F *is non-Archimedean and* π *is unramified, take* $f^\circ(g) = \mathbb{1}_{M_n(\mathfrak{o}_F)}(g)$ *to be the characteristic function of* $M_n(\mathfrak{o}_F)$ *and* $\varphi_\pi^\circ(g)$ *to be the zonal spherical function associated to* π. *Then the identity*

$$\mathcal{Z}(s, f^\circ, \varphi_\pi^\circ, \chi) = L(s, \pi \times \chi)$$

holds for any unramified characters χ *and all* $s \in \mathbb{C}$ *as meromorphic functions in* s.

For the statements of the current version of Theorem 2.4, we have some comments in order. When F is non-Archimedean, the theorem is [16, Theorem 3.3]. When F is Archimedean, the statements were established in [16] only for K-finite vectors f in $\mathcal{S}(M_n(F))$ and φ_π in $\mathcal{C}(\pi)$, and were extended to general smooth vectors in [23, Section 4.7] and also in [32, Theorem 3.10]. About the boundedness on vertical strips, we refer to [23, Section 4].

2C. ***Reformulation of Godement–Jacquet theory.*** The local theory of Godement–Jacquet zeta integrals can be reformulated within harmonic analysis and L^2-theory.

For $f \in \mathcal{S}(M_n(F))$, we define

$$\xi_f(g) := |\det g|_F^{n/2} \cdot f(g) \tag{2-11}$$

for $g \in G_n(F)$. Then we define the Schwartz space on $G_n(F)$ to be

$$\mathcal{S}_{\mathrm{std}}(G_n(F)) := \{\xi \in \mathcal{C}^\infty(G_n(F)) \mid |\det g|^{-n/2} \cdot \xi(g) \in \mathcal{S}(M_n(F))\}. \tag{2-12}$$

Proposition 2.5. *The Schwartz space $\mathcal{S}_{\mathrm{std}}(G_n(F))$ is a subspace of $L^2(G_n(F), \mathrm{d}g)$, which is the space of square-integrable functions on $G_n(F)$.*

Proof. For $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$, write $\xi(g) = |\det g|_F^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$. We deduce the square-integrability of ξ by the computation

$$\int_{G_n(F)} \xi(g)\overline{\xi(g)}\,\mathrm{d}g = \int_{G_n(F)} f(g)\overline{f(g)}\,\mathrm{d}^+g = \int_{M_n(F)} f(g)\overline{f(g)}\,\mathrm{d}^+g < \infty. \qquad \square$$

Define the distribution kernel in the local theory of Godement–Jacquet to be

$$\Phi_{\mathrm{GJ}}(g) := \psi(\mathrm{tr}\, g) \cdot |\det g|_F^{n/2}, \tag{2-13}$$

where ψ is a nontrivial additive character of F. We compute the convolution $\Phi_{\mathrm{GJ}} * \xi^\vee$ for any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ with $\xi(g) = |\det g|_F^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$:

$$\begin{aligned}
\Phi_{\mathrm{GJ}} * \xi^\vee(g) &= \int_{G_n(F)} \Phi_{\mathrm{GJ}}(h)\xi(g^{-1}h)\,\mathrm{d}h \\
&= \int_{G_n(F)} \psi(\mathrm{tr}\, h) \cdot |\det h|_F^{n/2} \cdot |\det g^{-1}h|_F^{n/2} \cdot f(g^{-1}h)\,\mathrm{d}h \\
&= \int_{G_n(F)} f(h)\psi(\mathrm{tr}\, gh) \cdot |\det gh|_F^{n/2} \cdot |\det h|_F^{n/2}\,\mathrm{d}h \\
&= |\det g|_F^{n/2} \int_{M_n(F)} f(h)\psi(\mathrm{tr}\, gh)\,\mathrm{d}^+h \\
&= |\det g|_F^{n/2} \cdot \mathcal{F}_\psi(f)(g).
\end{aligned}$$

Since $\mathcal{F}_\psi(f)(g)$ belongs to $\mathcal{S}(M_n(F))$, by definition, we must have that $\Phi_{\mathrm{GJ}} * \xi^\vee(g)$ belongs to $\mathcal{S}_{\mathrm{std}}(G_n(F))$. We define the Fourier operator $\mathcal{F}_{\mathrm{GJ}}$ in the Godement–Jacquet theory to be

$$\mathcal{F}_{\mathrm{GJ}}(\xi)(g) := (\Phi_{\mathrm{GJ}} * \xi^\vee)(g) \tag{2-14}$$

for any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$.

Proposition 2.6. *For any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ with $\xi(g) = |\det g|_F^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$, the Fourier operator $\mathcal{F}_{\mathrm{GJ}}$ on $\mathcal{S}_{\mathrm{std}}(G_n(F))$ and the classical Fourier transform $\mathcal{F}_\psi$ on $\mathcal{S}(M_n(F))$ are related by the identity*

$$\mathcal{F}_{\mathrm{GJ}}(\xi)(g) = (\Phi_{\mathrm{GJ}} * \xi^\vee)(g) = |\det g|_F^{n/2} \cdot \mathcal{F}_\psi(f)(g) = |\det g|_F^{n/2} \cdot \mathcal{F}_\psi\big(|\det(\cdot)|^{-n/2}\xi\big)(g).$$

For any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ with $\xi(g) = |\det g|_F^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$, the zeta integral can be renormalized as

$$\begin{aligned} \mathcal{Z}(s, f, \varphi_\pi, \chi) &= \int_{G_n(F)} |\det g|_F^{n/2} f(g) \varphi_\pi(g) \chi(\det g) |\det g|_F^{s-1/2}\, \mathrm{d}g \\ &= \mathcal{Z}(s, \xi, \varphi_\pi, \chi). \end{aligned} \tag{2-15}$$

We compute the other side of the functional equation of the Godement–Jacquet zeta integrals:

$$\begin{aligned} \mathcal{Z}(1-s, \mathcal{F}_\psi(f), \varphi_\pi^\vee, \chi^{-1}) &= \int_{G_n(F)} |\det g|_F^{n/2} \mathcal{F}_\psi(f)(g) \varphi_\pi^\vee(g) \chi^{-1}(g) |\det g|_F^{1/2-s} \mathrm{d}g \\ &= \int_{G_n(F)} \mathcal{F}_{\mathrm{GJ}}(\xi)(g) \varphi_\pi^\vee(g) \chi^{-1}(g) |\det g|_F^{1/2-s} \mathrm{d}g \\ &= \mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi), \varphi_\pi^\vee, \chi^{-1}). \end{aligned}$$

Proposition 2.7. *For any $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$, $\varphi_\pi \in \mathcal{C}(\pi)$, and $\chi \in \mathfrak{X}(F^\times)$, the zeta integral defined by*

$$\mathcal{Z}(s, \xi, \varphi_\pi, \chi) = \int_{G_n(F)} \xi(g) \varphi_\pi(g) \chi(\det g) |\det g|_F^{s-1/2}\, \mathrm{d}g$$

satisfies the functional equation

$$\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi), \varphi_\pi^\vee, \chi^{-1}) = \gamma(s, \pi \times \chi, \psi) \cdot \mathcal{Z}(s, \xi, \varphi_\pi, \chi),$$

which holds as meromorphic functions in s.

We are going to understand the Godement–Jacquet distribution Φ_{GJ} in terms of the Bernstein center of $G_n(F)$, when F is non-Archimedean. Recall from [3] that the Bernstein center $\mathfrak{Z}(G(F))$ of a reductive group $G(F)$ over a non-Archimedean local field F is defined to be the endomorphism ring of the identity functor on the category of smooth representations of $G(F)$. It turns out that the Bernstein center $\mathfrak{Z}(G(F))$ can be identified with the space of invariant and essentially compactly supported distributions on $G(F)$, where an invariant distribution Φ on $G(F)$ is called essentially compactly supported if $\Phi * \mathcal{C}_c^\infty(G(F)) \subset \mathcal{C}_c^\infty(G(F))$. It was proved in [3] that through the Plancherel transform, the Bernstein center $\mathfrak{Z}(G(F))$ can also be identified with the space of regular functions on the Bernstein variety $\Omega(G(F))$ attached to $G(F)$, where $\Omega(G(F))$ is an infinite disjoint union of finite-dimensional complex algebraic varieties.

Proposition 2.8. *Let F be a non-Archimedean local field of characteristic zero. For any $m \in \mathbb{Z}$, define*

$$G_n(F)_m = \{g \in G_n(F) \mid |\det g|_F = q_F^{-m}\}.$$

Let $\mathbb{1}_m := \mathbb{1}_{G_n(F)_m}$ be the characteristic function of $G_n(F)_m \subset G_n(F)$. Then the following statements hold:

(1) *The invariant distribution*

$$\Phi_{\mathrm{GJ},m}(g) := \Phi_{\mathrm{GJ}}(g)\mathbb{1}_{G_n(F)_m}(g) = \Phi_{\mathrm{GJ}}(g)\mathbb{1}_m(g) \tag{2-16}$$

lies in the Bernstein center $\mathfrak{Z}(G_n(F))$ of $G_n(F)$.

(2) *Let $f_{\mathrm{GJ},m}$ be the regular function on $\Omega(G_n(F))$ attached to $\Phi_{\mathrm{GJ},m} \in \mathfrak{Z}(G_n(F))$. For every $\pi \in \Pi_F(G_n)$, $\chi \in \mathfrak{X}(F^\times)$, and $s \in \mathbb{C}$, define*

$$\pi_{\chi_s} := \pi \otimes \chi_s = \pi \otimes \chi(\det)|\det|_F^s.$$

Then the Laurent series

$$f_{\mathrm{GJ}}(\pi_{\chi_s}) = \sum_{m\in\mathbb{Z}} f_{\mathrm{GJ},m}(\pi_{\chi_s})$$

is convergent for $\mathrm{Re}(s)$ *sufficiently large, with a meromorphic continuation to $s \in \mathbb{C}$, and*

$$f_{\mathrm{GJ}}(\pi_{\chi_s}) = \gamma\left(\tfrac{1}{2}, \widetilde{\pi_{\chi_s}}, \psi\right) = \gamma\left(\tfrac{1}{2} - s, \widetilde{\pi} \times \chi^{-1}, \psi\right).$$

Proof. For part (1), we have to show that the invariant distribution $\Phi_{\mathrm{GJ},m}(g)$ is essentially compact on $G_n(F)$. By a simple reduction, it suffices to show that, for any open compact subgroup $\mathcal{K}$ of $G_n(\mathfrak{o}_F)$, we have

$$\Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}} \in \mathcal{C}_c^\infty(G_n(F)).$$

Since $\mathbb{1}_{\mathcal{K}}(g) = \mathbb{1}_{\mathcal{K}}(g^{-1}) = \mathbb{1}_{\mathcal{K}}^\vee(g)$, the convolution $\Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}} = \Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}}^\vee$ can be written as

$$\begin{aligned}\Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}}^\vee(g) &= \int_{G_n(F)} \Phi_{\mathrm{GJ},m}(h)\mathbb{1}_{\mathcal{K}}(g^{-1}h)\,\mathrm{d}h \\ &= \int_{G_n(F)} \Phi_{\mathrm{GJ},m}(gh)\mathbb{1}_{\mathcal{K}}(h)\,\mathrm{d}h \\ &= \int_{G_n(F)} \psi(\mathrm{tr}\, gh)|\det gh|^{n/2}\mathbb{1}_m(gh)\mathbb{1}_{\mathcal{K}}(h)\,\mathrm{d}h.\end{aligned}$$

By definition, $\mathbb{1}_{\mathcal{K}}(h) \neq 0$ if and only if $|\det h|_F = 1$, and $\mathbb{1}_m(gh) \neq 0$ if and only if $|\det g|_F = q_F^{-m}$, i.e., $g \in G_n(F)_m$. This implies that $\mathbb{1}_m(gh) = \mathbb{1}_m(g)$. The last integral can be written as

$$q_F^{-(mn)/2}\mathbb{1}_m(g)\int_{G_n(F)} \psi(\mathrm{tr}(gh))\mathbb{1}_{\mathcal{K}}(h)\,\mathrm{d}h,$$

which can be written as

$$q_F^{-(mn)/2}\mathbb{1}_m(g)\int_{M_n(F)} \psi(\mathrm{tr}(gh))\mathbb{1}_{\mathcal{K}}(h)\,\mathrm{d}^+h = q_F^{-(mn)/2}\mathbb{1}_m(g)\mathcal{F}_\psi(\mathbb{1}_{\mathcal{K}})(g).$$

Hence, we obtain that

$$\Phi_{\mathrm{GJ},m} * \mathbb{1}_{\mathcal{K}}(g) = |\det g|_F^{n/2} \mathbb{1}_m(g) \mathcal{F}_\psi(\mathbb{1}_{\mathcal{K}})(g).$$

Since $\mathcal{F}_\psi(\mathbb{1}_{\mathcal{K}})(g) \in \mathcal{S}(M_n(F))$ and $|\det g|_F^{n/2} \mathbb{1}_m(g)$ is smooth on $M_n(F)$, we obtain that the convolution $\Phi_{m,\psi} * \mathbb{1}_{\mathcal{K}}(g)$ belongs to $\mathcal{C}_c^\infty(G_n(F))$ and the invariant distribution $\Phi_{\mathrm{GJ},m}(g)$ is essentially compact on $G_n(F)$.

For part (2), recall from [3] that the regular function $f_{\mathrm{GJ},m}$ attached to $\Phi_{\mathrm{GJ},m}$ is defined as follows. For any $\pi \in \Pi_F(G_n)$ and $v \in \pi$, there exists an open compact subgroup $\mathcal{K}$ of $G_n(F)$, such that $v \in \pi^{\mathcal{K}}$, the subspace of $\mathcal{K}$-fixed vectors in π. We may define an action of $\Phi_{\mathrm{GJ},m}$ on π via

$$\pi(\Phi_{\mathrm{GJ},m})(v) := \pi(\Phi_{\mathrm{GJ},m} * \mathfrak{c}_{\mathcal{K}})(v), \tag{2-17}$$

where $\mathfrak{c}_{\mathcal{K}} := \mathrm{vol}(\mathcal{K})^{-1} \mathbb{1}_{\mathcal{K}}$ is the normalized characteristic function of $\mathcal{K}$. Since $\Phi_{\mathrm{GJ},m} * \mathfrak{c}_{\mathcal{K}}$ lies in $\mathcal{C}_c^\infty(G_n(F))$, the right-hand side is well defined, and so is the left-hand side. It is clear that the action defined in (2-17) does not depend on the choice of such an open compact subgroup $\mathcal{K}$. By Schur's lemma, there exists a constant $f_{\mathrm{GJ},m}(\pi)$, depending on π, such that

$$\pi(\Phi_{\mathrm{GJ},m}) = f_{\mathrm{GJ},m}(\pi) \cdot \mathrm{Id}_\pi. \tag{2-18}$$

For each $m \in \mathbb{Z}$, we define, for any $\xi \in \mathcal{C}_c^\infty(G_n(F))$,

$$\mathcal{F}_{\mathrm{GJ},m}(\xi)(g) := (\Phi_{\mathrm{GJ},m} * \xi^\vee)(g) = \int_{G_n(F)} \Phi_{\mathrm{GJ},m}(h) \xi(g^{-1}h)\, \mathrm{d}h. \tag{2-19}$$

In order to include the quasicharacters $\chi \in \mathfrak{X}(F^\times)$ in the gamma function, we write

$$\varphi_{\pi[\chi]}(g) := \varphi_\pi[\chi](g) := \varphi_\pi(g)\chi(\det g) = (\chi(g)\pi(g)v, \tilde{v}), \tag{2-20}$$

with $v \in V_\pi$ and $\tilde{v} \in V_{\tilde{\pi}}$, which is a matrix coefficient of π twisted by χ. We may denote the space of such twisted matrix coefficients of π by $\mathcal{C}(\pi[\chi])$. It is clear that we have

$$\mathcal{Z}(s, \xi, \varphi_{\pi[\chi]}) = \mathcal{Z}(s, \xi, \varphi_\pi, \chi).$$

For each $m \in \mathbb{Z}$, $\varphi_{\pi[\chi]} \in \mathcal{C}(\pi[\chi])$, and $\chi \in \mathfrak{X}(F^\times)$, consider the zeta function of Godement–Jacquet, with $\mathcal{F}_{\mathrm{GJ},m}(\xi)$ defined as in (2-19),

$$\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ},m}(\xi), \varphi^\vee_{\pi[\chi]}) = \mathcal{Z}(1-s, \Phi_{\mathrm{GJ},m} * \xi^\vee, \varphi^\vee_{\pi[\chi]}). \tag{2-21}$$

By part (1) as proved above, we obtain that $\Phi_{\mathrm{GJ},m} * \xi^\vee \in \mathcal{C}_c^\infty(G_n(F))$ for any $\xi \in \mathcal{C}_c^\infty(G_n(F))$. Hence, the integral in (2-21) is absolutely convergent for any $s \in \mathbb{C}$ when $\xi \in \mathcal{C}_c^\infty(G_n(F))$.

We write the right-hand side of (2-21) as

$$\int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)\varphi_\pi(g^{-1})\chi^{-1}(\det g)|\det g|_F^{1/2-s}\,\mathrm{d}g, \tag{2-22}$$

which is equal to

$$\begin{aligned}\int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)(v, \widetilde{\pi}(g)\tilde{v})\chi_{s-1/2}(\det g)^{-1}\,\mathrm{d}g \\ = \left(v, \int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)\widetilde{\pi_{\chi_{s-1/2}}}(g)\tilde{v}\,\mathrm{d}g\right).\end{aligned} \tag{2-23}$$

It is clear that

$$\begin{aligned}\int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)\widetilde{\pi_{\chi_{s-1/2}}}(g)\tilde{v}\,\mathrm{d}g &= \widetilde{\pi_{\chi_{s-1/2}}}(\Phi_{\mathrm{GJ},m} * \xi^\vee)\tilde{v} \\ &= \widetilde{\pi_{\chi_{s-1/2}}}(\Phi_{\mathrm{GJ},m})(\widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v}).\end{aligned}$$

Since $\xi^\vee$ belongs to $\mathcal{C}_c^\infty(G_n(F))$, the vector $\widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v}$ belongs to the space of $\widetilde{\pi_{\chi_{s-1/2}}}$. By definition, we have

$$\widetilde{\pi_{\chi_{s-1/2}}}(\Phi_{\mathrm{GJ},m}) = f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}) \cdot I_{\widetilde{\pi_{\chi_{s-1/2}}}}. \tag{2-24}$$

Hence, we can write the right-hand side of (2-23) as

$$\left(v, \int_{G_n(F)} \Phi_{\mathrm{GJ},m} * \xi^\vee(g)\widetilde{\pi_{\chi_{s-1/2}}}(g)\tilde{v}\,\mathrm{d}g\right) = f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}) \cdot (v, \widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v}).$$

Next we compute the twisted coefficient $(v, \widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v})$ on the right-hand side of the above equation as

$$\begin{aligned}&(v, \widetilde{\pi_{\chi_{s-1/2}}}(\xi^\vee)\tilde{v}) \\ &\qquad = \int_{G_n(F)} \xi^\vee(h)(v, \widetilde{\pi_{\chi_{s-1/2}}}(h)\tilde{v})\,\mathrm{d}h = \int_{G_n(F)} \xi(h^{-1})(\pi_{\chi_{s-1/2}}(h^{-1})v, \tilde{v})\,\mathrm{d}h \\ &\qquad = \int_{G_n(F)} \xi(h)(\pi_{\chi_{s-1/2}}(h)v, \tilde{v})\,\mathrm{d}h = \int_{G_n(F)} \xi(h)\varphi_{\pi[\chi]}(h)|\det h|^{s-1/2}\,\mathrm{d}h \\ &\qquad = \mathcal{Z}(s, \xi, \varphi_{\pi[\chi]}).\end{aligned}$$

Hence, we obtain the functional equation

$$\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ},m}(\xi), \varphi^\vee_{\pi[\chi]}) = f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}) \cdot \mathcal{Z}(s, \xi, \varphi_{\pi[\chi]}) \tag{2-25}$$

for any $\xi \in \mathcal{C}_c^\infty(G_n(F))$, $\varphi_\pi \in \mathcal{C}(\pi)$ and $\chi \in \mathfrak{X}(F^\times)$.

Theorem 2.4 implies that, when $\mathrm{Re}(s)$ is sufficiently small, the zeta integral $\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi), \varphi^\vee_{\pi[\chi]})$ converges absolutely for any $\xi \in \mathcal{C}_c^\infty(G_n(F))$, any $\varphi_\pi \in$

$\mathcal{C}(\pi)$ and any unitary character $\chi \in \mathfrak{X}(F^\times)$. We write it as

$$\begin{aligned}\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi), \varphi^\vee_{\pi[\chi]}) \\ &= \sum_{m\in\mathbb{Z}} \mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ},m}(\xi), \varphi^\vee_{\pi[\chi]}) = \mathcal{Z}(s, \xi, \varphi_{\pi[\chi]}) \cdot \sum_{m\in\mathbb{Z}} f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}).\end{aligned}$$

By comparing with the right-hand side of the functional equation in Theorem 2.4, we obtain that, whenever $\mathrm{Re}(s)$ is sufficiently small,

$$f_{\mathrm{GJ}}(\widetilde{\pi_{\chi_{s-1/2}}}) = \sum_{m\in\mathbb{Z}} f_{\mathrm{GJ},m}(\widetilde{\pi_{\chi_{s-1/2}}}) = \gamma(s, \pi\otimes\chi, \psi) = \gamma(s, \pi_\chi, \psi). \tag{2-26}$$

By changing $s \to s + \frac{1}{2}$, we get

$$f_{\mathrm{GJ}}(\widetilde{\pi_{\chi_s}}) = \gamma\big(s+\tfrac{1}{2}, \pi_\chi, \psi\big) = \gamma\big(\tfrac{1}{2}, \pi_{\chi_s}, \psi\big).$$

By taking the contragredient of π_{χ_s}, we obtain that

$$f_{\mathrm{GJ}}(\pi_{\chi_s}) = \gamma\big(\tfrac{1}{2}, \widetilde{\pi_{\chi_s}}, \psi\big) = \gamma\big(\tfrac{1}{2}-s, \widetilde{\pi}\times\chi^{-1}, \psi\big).$$

This finishes the proof of part (2). □

3. π-Schwartz functions and Fourier operators

3A. ***Two spaces associated to π.*** For any $\pi \in \Pi_F(G_n)$, we are going to define two spaces associated to π: $\mathcal{L}_\pi(\mathfrak{X}(F^\times))$ and $\mathcal{S}_\pi(F^\times)$.

The space $\mathcal{L}_\pi = \mathcal{L}_\pi(\mathfrak{X}(F^\times))$ consists of $\mathbb{C}$-valued meromorphic functions $\mathfrak{z}(\chi)$ on $\mathfrak{X}(F^\times)$ that satisfy the following conditions:

(1) $\mathfrak{z}(\chi_{s,\omega})$ is a holomorphic multiple of the standard local L-function $L(s, \pi\times\omega)$ with $\chi_{s,\omega}(x) = |x|_F^s\,\omega(\mathrm{ac}(x))$.

(2) If F is non-Archimedean, $\mathfrak{z}(\chi_{s,\omega})$ is nonzero for finitely many $\omega \in \Omega^\wedge$, and for each $\omega \in \Omega^\wedge$, $\mathfrak{z}(\chi_{s,\omega}) \in L(s, \pi\times\omega)\cdot\mathbb{C}[q^s, q^{-s}]$.

(3) If F is Archimedean, for any polynomial $P(\chi_{s,\omega}) = P_\omega(s)$, if the function $P(\chi_{s,\omega})L(s, \pi\times\omega)$ is holomorphic in any vertical strip $S_{a,b}$ as in (2-4), with small neighborhoods at the possible poles of the L-function $L(s, \pi\times\omega)$ removed, then for any $\mathfrak{z}(\chi_{s,\omega}) \in \mathcal{L}_\pi$, the product $P(\chi_{s,\omega})\mathfrak{z}(\chi_{s,\omega})$ is bounded in the same strip $S_{a,b}$, with small neighborhoods at the possible poles of the L-function $L(s, \pi\times\omega)$ removed.

From part (3), we define a seminorm to be

$$\mu_{a,b:P}(\mathfrak{z}) := \sup_{a\le\mathrm{Re}(s)\le b} |P(\chi_{s,\omega})\cdot\mathfrak{z}(\chi_{s,\omega})|.$$

Then the space $\mathcal{L}_\pi$ is complete under the topology that is defined by the family of seminorms $\mu_{a,b:P}$ for all possible choice of data a, b; P as in part (3) [23, Section 4].

Proposition 3.1. *For any $\pi \in \Pi_F(G_n)$, the space $\mathcal{L}_\pi$ is a subspace of $\mathcal{Z}(\mathfrak{X}(F^\times))$ as defined in Definition 2.2.*

Proof. When F is non-Archimedean, the statement is a consequence of Theorem 2.4. We would like to focus on the case when F is Archimedean. In this case, it suffices to estimate the boundedness condition. To do so, recall the classical Stirling formula (see [23, p. 81], for instance)

$$\Gamma(x+iy) \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-(\pi/2)|y|} \tag{3-1}$$

for x fixed and $|y| \to \infty$.

Consider the Archimedean local L-functions $L(s, \pi \times \omega) = L(s, \pi \times \mathrm{ac}(\,\cdot\,)^p)$, which can be explicitly expressed in terms of classical Γ-functions with the local Langlands parameter of π. For instance, from [16, Section 8], there exists a finite family of pairs $\{(l_i, u_i)\}_{i=1}^t$ with

$$u_i \in \mathbb{C}, \quad l_i \in \begin{cases} \mathbb{Z}/2\mathbb{Z} \simeq \{0,1\}, & F = \mathbb{R}, \\ \mathbb{Z}, & F = \mathbb{C}, \end{cases}$$

such that in the fixed bounded vertical strip

$$S_{a,b} = \{s \in \mathbb{C} \mid a \le \mathrm{Re}(s) \le b\},$$

up to a bounded factor in $S_{a,b}$, we have

$$L(s, \pi \times \mathrm{ac}(\,\cdot\,)^p) \sim \begin{cases} \prod_{i=1}^t \Gamma\bigl(\frac{s+u_i+l_i+p}{2}\bigr), & F = \mathbb{R}, \\ \prod_{i=1}^t \Gamma\bigl(s+u_i+\frac{|l_i+p|}{2}\bigr), & F = \mathbb{C}, \end{cases}$$

with $p \in \mathbb{Z}/2\mathbb{Z} \simeq \{0,1\}$ if $F = \mathbb{R}$; and $p \in \mathbb{Z}$ if $F = \mathbb{C}$. Here $l_i + p$ is understood to be zero if both l_i and p are equal to 1 when $F = \mathbb{R}$.

It follows from the classical Stirling formula in (3-1), in particular the exponential decay of $\Gamma(x+iy)$ along the imaginary axis, for any polynomial $P_\omega(s) = P(s) \in \mathbb{C}[s]$ when $F = \mathbb{R}$, and $P_\omega(s) = P(s, p) \in \mathbb{C}[s, p]$, the product $P(s, p)L(s, \pi \times \mathrm{ac}(\,\cdot\,)^p)$ is bounded in vertical strip $S_{a,b}$ with small neighborhoods at the possible poles removed. Hence, from the definition of the space $\mathcal{L}_\pi(\mathfrak{X}(F^\times))$, for any $\mathfrak{z}(\chi_{s,\omega}) \in \mathcal{L}_\pi(\mathfrak{X}(F^\times))$, the product $P(s, p)\mathfrak{z}(\chi)$, with $\chi(x) = |x|_F^s \mathrm{ac}(x)^p$ is bounded in vertical strip $S_{a,b}$ with small neighborhoods at the possible poles of the L-function $L(s, \pi \times \mathrm{ac}(\,\cdot\,)^p)$ removed. Therefore, we obtain that the space $\mathcal{L}_\pi = \mathcal{L}_\pi(\mathfrak{X}(F^\times))$ is contained in the space $\mathcal{Z}(\mathfrak{X}(F^\times))$, as defined in Definition 2.2. □

For any $\pi \in \Pi_F(G_n)$, we define (Definition 3.3) the π-Schwartz space $\mathcal{S}_\pi(F^\times) \subset \mathcal{C}^\infty(F^\times)$ attached to π, by using the theory of local zeta integrals of Godement–Jacquet, and prove that

$$\mathcal{S}_\pi(F^\times) = \mathcal{M}^{-1}(\mathcal{L}_\pi) \subset \mathcal{C}^\infty(F^\times) \tag{3-2}$$

by Theorems 2.3 and 2.4.

Consider the determinant map

$$\det = \det_F : G_n(F) = \mathrm{GL}_n(F) \to F^\times. \tag{3-3}$$

It is clear that the kernel ker(det) equals $\mathrm{SL}_n(F)$. For each $x \in F^\times$, the fiber of the determinant map det is

$$G_n(F)_x := \{g \in G_n(F) \mid \det g = x\}. \tag{3-4}$$

It is clear that each fiber $G_n(F)_x$ is an $\mathrm{SL}_n(F)$-torsor. Hence, one has the $\mathrm{SL}_n(F)$-invariant measure $\mathrm{d}_x g$ that is induced from the (normalized) Haar measure $\mathrm{d}_1 g$ on $\mathrm{SL}_n(F)$.

For $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ as defined in (2-12), $\varphi_\pi \in \mathcal{C}(\pi)$, and $\chi \in \mathfrak{X}(F^\times)$, the local zeta integral of Godement and Jacquet, as normalized in (2-15), can be written as

$$\mathcal{Z}(s, \xi, \varphi_\pi, \chi) = \int_{F^\times} \Big(\int_{G_n(F)_x} \xi(g)\varphi_\pi(g)\, \mathrm{d}_x g \Big) \chi(x) |x|_F^{s-1/2}\, \mathrm{d}^\times x. \tag{3-5}$$

By part (1) of Theorem 2.4, the local zeta integral converges absolutely for $\mathrm{Re}(s)$ large. Hence, the inner integral of (3-5) satisfies

$$\phi_{\xi,\varphi_\pi}(x) := \int_{G_n(F)_x} \xi(g)\varphi_\pi(g)\, \mathrm{d}_x g = |x|_F^{n/2} \int_{G_n(F)_x} f(g)\varphi_\pi(g)\, \mathrm{d}_x g, \tag{3-6}$$

if $\xi(g) = |\det g|^{n/2} \cdot f(g)$ for some $f \in \mathcal{S}(M_n(F))$, is absolutely convergent for almost all $x \in F^\times$ and defines the fiber integration along the fibration in (3-3).

Proposition 3.2. *For $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, the fiber integration in* (3-6) *that defines the function $\phi_{\xi,\varphi_\pi}(x)$ is absolutely convergent for all $x \in F^\times$, and the function $\phi_{\xi,\varphi_\pi}(x)$ is smooth over $F^\times$.*

Proof. It is enough to show the proposition for the integral

$$\int_{G_n(F)_x} f(g)\varphi_\pi(g)\, \mathrm{d}_x g \tag{3-7}$$

with any $f \in \mathcal{S}(M_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$. In this case, the product $f \cdot \varphi_\pi$ is smooth on $G_n(F)$. Since the fiber $G_n(F)_x$ for any $x \in F^\times$ is closed in $G_n(F)$ and in $M_n(F)$, the restriction of f to the fiber $G_n(F)_x$ is a Schwartz function on $G_n(F)_x$ (see [5] for F non-Archimedean and [1, Theorem 4.6.1] for F Archimedean).

When F is non-Archimedean, any $\varphi_\pi(g) \in \mathcal{C}(\pi)$ is locally constant (smooth) on $G_n(F)$, and hence is smooth on the fiber $G_n(F)_x$. This implies that the restriction of $f \cdot \varphi_\pi$ is locally constant and compactly supported on the fiber $G_n(F)_x$. Hence, the integral in (3-7) is absolutely convergent for all $x \in F^\times$, and defines a smooth function in x over $F^\times$.

When F is Archimedean, since π is a Casselman–Wallach representation of $G_n(F)$, the matrix coefficient φ_π has at most polynomial growth on $G_n(F)$ [45, Theorem 4.3.5], as well as on the fiber $G_n(F)_x$. This implies that the restriction of

$f \cdot \varphi_\pi$ is a Schwartz function on the fiber $G_n(F)_x$ ([1, Definition 4.1.1]). Thus the integral in (3-7) is absolutely convergent for all $x \in F^\times$. Now we write the integral in (3-7) as

$$\int_{G_n(F)_x} f(g)\varphi_\pi(g)\,\mathrm{d}_x g = \int_{\mathrm{SL}_n(F)} f(t_1(x)g)\varphi_\pi(t_1(x)g)\,\mathrm{d}_1 g, \tag{3-8}$$

where $t_1(x) = \mathrm{diag}(x, 1, \ldots, 1) \in G_n(F)$ and $\mathrm{d}_1 g$ is the Haar measure of $\mathrm{SL}_n(F)$. It is clear that the absolute convergence of the integral in (3-8) is uniform when x runs in any compact subset of $F^\times$. Hence, the integral in (3-7) defines a smooth function in x over $F^\times$. □

For $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, the function $\phi_{\xi,\varphi_\pi}(x)$ given in Proposition 3.2 via the fiber integration (3-6) is called a π*-Schwartz function* on $F^\times$ associated to the pair (ξ, φ_π). Here is the definition of π-Schwartz space.

Definition 3.3 (π-Schwartz space). For any $\pi \in \Pi_F(G_n)$, the space of π-Schwartz functions is defined by

$$\mathcal{S}_\pi(F^\times) = \mathrm{Span}\{\phi_{\xi,\varphi_\pi} \in \mathcal{C}^\infty(F^\times) \mid \xi \in \mathcal{S}_{\mathrm{std}}(G_n(F)),\ \varphi_\pi \in \mathcal{C}(\pi)\},$$

where the π-Schwartz function ϕ_{ξ,φ_π} associated to a pair (ξ, φ_π) is defined in (3-6).

For any $\phi \in \mathcal{S}_\pi(F^\times)$ and a quasicharacter $\chi \in \mathfrak{X}(F^\times)$, define a GL_1 zeta integral $\mathcal{Z}(s, \phi, \chi)$ associated to the pair (ϕ, χ) to be

$$\mathcal{Z}(s, \phi, \chi) = \int_{F^\times} \phi(x)\chi(x)|x|_F^{s-1/2}\,\mathrm{d}^\times x. \tag{3-9}$$

When $\phi = \phi_{\xi,\varphi_\pi}$ for some $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, from Theorem 2.4, we have the identity of local zeta integrals

$$\mathcal{Z}(s, \phi, \chi) = \mathcal{Z}(s, \xi, \varphi_\pi, \chi), \tag{3-10}$$

which holds for $\mathrm{Re}(s)$ sufficiently large and then for all $s \in \mathbb{C}$ by meromorphic continuation. Therefore, Theorem 2.4 can be restated for the GL_1 zeta integrals $\mathcal{Z}(s, \phi, \chi)$.

Theorem 3.4 (GL_1 zeta integrals). *The* GL_1 *zeta integral* $\mathcal{Z}(s, \phi, \chi)$ *as defined in* (3-9) *for any* $\phi \in \mathcal{S}_\pi(F^\times)$ *and any quasicharacter* $\chi \in \mathfrak{X}(F^\times)$ *enjoys the properties*:

(1) *The zeta integral* $\mathcal{Z}(s, \phi, \chi)$ *is absolutely convergent for* $\mathrm{Re}(s)$ *sufficiently large, and admits a meromorphic continuation to* $s \in \mathbb{C}$.

(2) *The zeta integral* $\mathcal{Z}(s, \phi, \chi)$ *is a holomorphic multiple of the Langlands local* L*-function* $L(s, \pi \times \chi)$ *associated to* (π, χ) *and the standard embedding*

$$\mathrm{std} : \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C}).$$

When F is non-Archimedean, the fractional ideal generated by the local zeta integrals $\mathcal{Z}(s,\phi,\chi)$ is of the form

$$\{\mathcal{Z}(s,\phi,\chi)\mid \phi\in\mathcal{S}_\pi(F^\times)\}=L(s,\pi\times\chi)\cdot\mathbb{C}[q^s,q^{-s}];$$

and when F is Archimedean, the GL_1 *zeta integrals $\mathcal{Z}(s,\phi,\chi)$, with unitary characters χ, have the following property. Let $S_{a,b}$ be the vertical strip for any $a<b$, as defined in* (2-4). *If $P_\chi(s)$ is a polynomial in s such that the product $P_\chi(s)L(s,\pi\times\chi)$ is bounded in the vertical strip $S_{a,b}$, with small neighborhoods at the possible poles of the L-function $L(s,\pi\times\chi)$ removed, then the product $P_\chi(s)\mathcal{Z}(s,\phi,\chi)$ must be bounded in the same vertical strip $S_{a,b}$, with small neighborhoods at the possible poles of the L-function $L(s,\pi\times\chi)$ removed.*

(3) *When F is non-Archimedean, and π is unramified, define*

$$\mathbb{L}_\pi(x):=\phi_{\xi^\circ,\varphi_\pi^\circ}(x),$$

where $\xi^\circ(g)=|\det g|^{n/2}\mathbb{1}_{M_n(\mathfrak{o}_F)}(g)$, with $\mathbb{1}_{M_n(\mathfrak{o}_F)}(g)$ being the characteristic function of $M_n(\mathfrak{o}_F)$, and $\varphi_\pi^\circ(g)$ is the zonal spherical function associated to π. Then the identity

$$\mathcal{Z}(s,\mathbb{L}_\pi,\chi)=L(s,\pi\times\chi)$$

holds for any unramified characters χ and all $s\in\mathbb{C}$ as meromorphic functions in s.

We are going to discuss the relation between the π-Schwartz functions and the square-integrable functions in $L^2(F^\times,\mathrm{d}^\times x)$.

Proposition 3.5. *For any $\pi\in\Pi_F(G_n)$, there exists a real number α_π such that for any $\phi\in\mathcal{S}_\pi(F^\times)$ and for any $\kappa\geq\alpha_\pi+n/2$, the function $|x|_F^\kappa\phi(x)$ belongs to the space $L^2(F^\times,\mathrm{d}^\times x)$ of square-integrable functions on $F^\times$.*

Proof. For any $\alpha_0\in\mathbb{R}$, we consider the following inner product of the function $|x|^{\alpha_0/2}\phi(x)$ for any $\phi(x)\in\mathcal{S}_\pi(F^\times)$. We write $\phi=\phi_{\xi,\varphi_\pi}$ for some $\xi\in\mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi\in\mathcal{C}(\pi)$ and write $\xi(g)=|\det g|_F^{n/2}f(g)$ with $f\in\mathcal{S}(M_n(F))$. Then

$$\begin{aligned}(3\text{-}11)\quad &\int_{F^\times}\phi(x)\overline{\phi(x)}|x|_F^{\alpha_0}\,\mathrm{d}^\times x\\ &=\int_{F^\times}|x|_F^{\alpha_0+n}\,\mathrm{d}^\times x\int_{\det g_1=\det g_2=x}f(g_1)\varphi_\pi(g_1)\overline{f(g_2)}\overline{\varphi_\pi(g_2)}\,\mathrm{d}_x g_1\,\mathrm{d}_x g_2\\ &=\int_{(G_n(F)\times G_n(F))^\circ}f(g_1)\varphi_\pi(g_1)\overline{f(g_2)}\overline{\varphi_\pi(g_2)}|\det g_1|_F^{\alpha_0+n}\,\mathrm{d}(g_1,g_2)^\circ,\end{aligned}$$

where $(G_n(F)\times G_n(F))^\circ:=\{(g_1,g_2)\in G_n(F)\times G_n(F)\mid\det g_1=\det g_2\}$ and $\mathrm{d}(g_1,g_2)^\circ$ is a Haar measure on $(G_n(F)\times G_n(F))^\circ$, which makes the above fiber integration factorization hold.

We consider the natural embedding

$$(G_n(F)\times G_n(F))^\circ \hookrightarrow (M_n(F)\times M_n(F))^\circ$$

with an open dense image, where

$$(M_n(F)\times M_n(F))^\circ := \{(X,Y)\in M_n(F)\times M_n(F) \mid \det X = \det Y\},$$

which is the fiber product with respect to the determinant map $X\mapsto \det X$, and is a closed subvariety of the affine space $M_n(F)\times M_n(F)$. The natural group action of $G_n\times G_n$ on $M_n\times M_n$ via

$$(g,h)((X,Y)) = (gX, hY)$$

for $(g,h)\in G_n\times G_n$ and $(X,Y)\in M_n\times M_n$ yields the action of $(G_n(F)\times G_n(F))^\circ$ on $(M_n(F)\times M_n(F))^\circ$ by restriction. Take $\mathrm{d}^+X\wedge \mathrm{d}^+Y$ to be an additive Haar measure on $M_n(F)\times M_n(F)$ with $|\det gh|_F^n$ the modulus function of the action of $G_n\times G_n$ on $M_n\times M_n$. Take the measure $\mathrm{d}^+(X,Y)^\circ$ on $(M_n(F)\times M_n(F))^\circ$, which is the pullback of the measure $\mathrm{d}^+X\wedge\mathrm{d}^+Y$ through the fiber product embedding. Then the modulus function of the action of $(G_n(F)\times G_n(F))^\circ$ on $(M_n(F)\times M_n(F))^\circ$ is

$$|\det gh|_F^n = |\det g|_F^{2n} = |\det h|_F^{2n}$$

for any $(g,h)\in (G_n(F)\times G_n(F))^\circ$. It is easy to check that $\mathrm{d}^+(g,h)^\circ/|\det gh|_F^n$ is a Haar measure on $(G_n(F)\times G_n(F))^\circ$. Hence, there is a constant $c>0$, such that

$$\mathrm{d}(g,h)^\circ = c\cdot \frac{\mathrm{d}^+(g,h)^\circ}{|\det gh|_F^n}.$$

The integral in (3-11) can be written as

$$\int_{(M_n(F)\times M_n(F))^\circ} f(X)\varphi_\pi(X)\overline{f(Y)}\overline{\varphi_\pi(Y)}|\det X|_F^{\alpha_0-n}\,\mathrm{d}^+(X,Y)^\circ. \tag{3-12}$$

Here we assume that $\alpha_0\ge n$ and both $\varphi_\pi(g_1)$ and $\varphi_\pi(g_2)$ are viewed as measurable functions on $M_n(F)$ that extend by zero to the boundary $M_n(F)\backslash \mathrm{GL}_n(F)$.

Since the F-analytical manifold $(M_n(F)\times M_n(F))^\circ$ is closed in $M_n(F)\times M_n(F)$, the restriction of the Schwartz function $f(g_1)\times\overline{f(g_2)}$ to $(M_n(F)\times M_n(F))^\circ$ is still a Schwartz function, which is smooth and compactly supported when F is non-Archimedean, and is in the sense of [1] when F is Archimedean. By Theorem 2.4, the zeta integral of Godement–Jacquet $\mathcal{Z}(s,f,\varphi_\pi,\chi)$ converges absolutely for $\mathrm{Re}(s)$ sufficiently large. It follows that for any $\pi\in\Pi_F(G_n)$, there exists a real number α_π such that for any $\varphi_\pi\in\mathcal{C}(\pi)$ and any $\mathrm{Re}(s)\ge\alpha_\pi$, the product $|\det(g)|_F^s\varphi_\pi(g)$ is bounded when $\det g$ tends to zero.

We write the F-analytical closed submanifold $(M_n(F)\times M_n(F))^\circ$ as a union of two closed submanifolds:

$$(M_n(F)\times M_n(F))^\circ=(M_n(F)\times M_n(F))^\circ_{\geq\varepsilon}\cup(M_n(F)\times M_n(F))^\circ_{\leq\varepsilon},$$

where

$$(M_n(F)\times M_n(F))^\circ_{\geq\varepsilon}=\{(g_1,g_2)\in M_n(F)^{\Delta\det}\mid |\det g_1|_F\geq\varepsilon\}$$

and

$$(M_n(F)\times M_n(F))^\circ_{\leq\varepsilon}=\{(g_1,g_2)\in M_n(F)^{\Delta\det}\mid |\det g_1|_F\leq\varepsilon\}.$$

For any $\pi\in\Pi_F(G_n)$, the restriction of the product $\varphi_\pi(g_1)\overline{\varphi_\pi(g_2)}\cdot|\det g_1|_F^{s-n}$ to the closed submanifold $(M_n(F)\times M_n(F))^\circ_{\geq\varepsilon}$ is of moderate growth and its restriction to the closed submanifold $(M_n(F)\times M_n(F))^\circ_{\leq\varepsilon}$ is bounded whenever $\mathrm{Re}(s)\geq 2\alpha_\pi+n$. It is also clear the Schwartz function $f(g_1)\times\overline{f(g_2)}$ on $(M_n(F)\times M_n(F))^\circ$ remains a Schwartz function when restricted to either the closed submanifold $(M_n(F)\times M_n(F))^\circ_{\geq\varepsilon}$ or the closed submanifold $(M_n(F)\times M_n(F))^\circ_{\leq\varepsilon}$. Hence, for any $\alpha_0\in\mathbb{R}$ with $\alpha_0\geq 2\alpha_\pi+n$, the integral

$$\int_{(M_n(F)\times M_n(F))^\circ} f(X)\varphi_\pi(X)\overline{f(Y)}\overline{\varphi_\pi(Y)}|\det X|_F^{\alpha_0-n}\,\mathrm{d}^+(X,Y)^\circ$$

converges absolutely, and so does the integral

$$\int_{F^\times}\phi(x)\overline{\phi(x)}|x|_F^{\alpha_0}\,\mathrm{d}^\times x.$$

It follows that the product $\phi(x)|x|_F^{\kappa}$ is square integrable on $F^\times$ for $\kappa=\alpha_0/2\geq\alpha_\pi+n/2$. □

Corollary 3.6. *If $\pi\in\Pi_F(G_n)$ is unitarizable, then for any $\phi\in\mathcal{S}_\pi(F^\times)$, the function $|x|_F^{n/2}\cdot\phi(x)$ belongs to the space $L^2(F^\times,\mathrm{d}^\times x)$ of square-integrable functions on $F^\times$.*

Proof. If $\pi\in\Pi_F(G_n)$ is unitarizable, then the matrix coefficient $\varphi_\pi(g)$ is bounded above over $G_n(F)$. For $\phi\in\mathcal{S}_\pi(F^\times)$, we write $\phi=\phi_{\xi,\varphi_\pi}$ with $\xi\in\mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi\in\mathcal{C}(\pi)$, and write $\xi(g)=|\det g|_F^{n/2}\cdot f(g)$ with $f\in\mathcal{S}(M_n(F))$. We compute the inner product of $|x|_F^{n/2}\cdot\phi(x)$ as

$$\begin{aligned}(3\text{-}13)\quad &\int_{F^\times}\phi(x)\overline{\phi(x)}|x|_F^{n}\,\mathrm{d}^\times x\\ &\leq\int_{F^\times}|x|_F^{2n}\int_{G_n(F)_x}|f(g_1)\varphi_\pi(g_1)|\,\mathrm{d}_x g_1\int_{G_n(F)_x}|f(g_2)\varphi_\pi(g_2)|\,\mathrm{d}_x g_2\,\mathrm{d}^\times x\\ &\leq c(\varphi_\pi)\cdot\int_{F^\times}|x|_F^{2n}\int_{G_n(F)_x}|f(g_1)|\,\mathrm{d}_x g_1\int_{G_n(F)_x}|f(g_2)|\,\mathrm{d}_x g_2\,\mathrm{d}^\times x\end{aligned}$$

for some positive constant $c(\varphi_\pi)$ depending on φ_π. By following the proof of Proposition 3.5, we obtain that

$$\int_{F^\times}\phi(x)\overline{\phi(x)}|x|_F^n\,\mathrm{d}^\times x \le c\cdot c(\varphi_\pi)\int_{(M_n(F)\times M_n(F))^\circ}|f(X)|\cdot|\overline{f(Y)}|\,\mathrm{d}(X,Y)^\circ. \tag{3-14}$$

The integral on the right-hand side of (3-14) comes from the integral in (3-12) with $\alpha_0 = n$. As explained in the proof of Proposition 3.5, the product $f(X)\times\overline{f(Y)}$ is a Schwartz function on $(M_n(F)\times M_n(F))^\circ$. Hence, the integral on the right-hand side of (3-14) converges. □

By using Proposition 3.5 and Theorem 3.4, together with Theorem 2.3, we are able to understand the π-Schwartz space $\mathcal{S}_\pi(F^\times)$ by means of the L-functions $L(s,\pi\times\chi)$ for any $\pi\in\Pi_F(n)$.

Proposition 3.7. *For any $\pi\in\Pi_F(G_n)$, the π-Schwartz space $\mathcal{S}_\pi(F^\times)$ is contained in the space $\mathfrak{F}(F^\times)$ as defined in Definition 2.1*

Proof. Note first that the GL_1 zeta integral attached to $\phi\in\mathcal{S}_\pi(F^\times)$ is the same as the Mellin transform of ϕ up to a shift in s by the unramified part of χ. By Theorem 3.4 and Proposition 3.1, the image of $\mathcal{S}_\pi(F^\times)$ under Mellin transform is contained in the space $\mathcal{L}_\pi(\mathfrak{X}(F^\times))$ and hence in the space $\mathcal{Z}(\mathfrak{X}(F^\times))$. By Theorem 2.3, for any $\phi\in\mathcal{S}_\pi(F^\times)$, there exists $\phi_0\in\mathfrak{F}(F^\times)$, such that

$$\mathcal{M}(\phi-\phi_0)(\chi)=0 \tag{3-15}$$

holds identically for any quasicharacter $\chi\in\mathfrak{X}(F^\times)$. It remains to show that $\phi-\phi_0=0$ holds identically. By smoothness of ϕ and ϕ_0, it suffices to show that after unramified twist, both ϕ and ϕ_0 are square integrable on $F^\times$.

For $\phi_0\in\mathfrak{F}(F^\times)$, there exists $s_0\in\mathbb{R}$ such that, for any $\mathrm{Re}(s)>s_0$,

$$\lim_{x\to 0}\phi_0(x)|x|_F^{s+1}=0,$$

and the limit is preserved by differentiation on both sides when F is Archimedean. It follows that $\phi_0(x)|x|_F^s$ is indeed square integrable on $F^\times$ for $\mathrm{Re}(s)>s_0$, via the asymptotic formula appearing in the definition of $\mathfrak{F}(F^\times)$.

For any $\phi\in\mathcal{S}_\pi(F^\times)$, by Proposition 3.5, there exists $\alpha_\pi\in\mathbb{R}_{>0}$ such that the function $|x|_F^s\phi(x)$ is square integrable if $\mathrm{Re}(s)\ge\alpha_\pi+n/2$. By taking $\kappa>\max\{s_0,\alpha_\pi+n/2\}$, we obtain that both $\phi_0(x)|x|_F^\kappa$ and $\phi(x)|x|_F^\kappa$ are square integrable over $F^\times$. From (3-15), we obtain that the Mellin transform

$$\mathcal{M}\big(\phi(x)|x|_F^\kappa-\phi_0(x)|x|_F^\kappa\big)(\chi)=0$$

for all quasicharacters $\chi\in\mathfrak{X}(F^\times)$, in particular, for all unitary characters χ of $F^\times$. Therefore, by the Mellin inversion formula (Theorem 2.3), we obtain that

$$\phi(x)|x|_F^\kappa-\phi_0(x)|x|_F^\kappa=0$$

as functions in the space $L^2(F^\times, \mathrm{d}^\times x)$. Since both $\phi(x)$ and $\phi_0(x)$ are smooth, we must have that $\phi(x) = \phi_0(x) \in \mathfrak{F}(F^\times)$. $\square$

Finally we are ready to characterize the Mellin inversion $\mathcal{M}^{-1}(\mathcal{L}_\pi)$ in terms of the π-Schwartz space $\mathcal{S}_\pi(F^\times)$ as in (3-2).

Corollary 3.8. *For any $\pi \in \Pi_F(G_n)$, the Mellin inversion $\mathcal{M}^{-1}(\mathcal{L}_\pi)$ coincides with the space $\mathcal{S}_\pi(F^\times)$ defined by*

$$\mathcal{S}_\pi(F^\times) = \mathcal{M}^{-1}(\mathcal{L}_\pi) \subset \mathcal{C}^\infty(F^\times).$$

In particular, the space $\mathcal{C}_c^\infty(F^\times)$ of smooth compactly supported functions on $F^\times$ is contained in the π-Schwartz space $\mathcal{S}_\pi(F^\times)$.

Proof. By Proposition 3.7, we have that the space $\mathcal{S}_\pi(F^\times)$ is contained in the space $\mathfrak{F}(F^\times)$. By Theorem 3.4, the Mellin transform (GL_1 zeta integral) of the space $\mathcal{S}_\pi(F^\times)$ is equal to the space $\mathcal{L}_\pi = \mathcal{L}_\pi(\mathfrak{X}(F^\times))$. Hence, we obtain that $\mathcal{S}_\pi(F^\times) = \mathcal{M}^{-1}(\mathcal{L}_\pi)$, because the Mellin transform is a bijective correspondence between the space $\mathfrak{F}(F^\times)$ and the space $\mathcal{Z}(\mathfrak{X}(F^\times))$ (Theorem 2.3). Finally, since the space $\mathcal{L}_\pi$ contains the space of holomorphic functions on $\mathfrak{X}(F^\times)$ that are of Paley–Wiener type along the vertical strips, it is clear from Theorem 2.3 again that $\mathcal{C}_c^\infty(F^\times)$ is contained in the π-Schwartz space $\mathcal{S}_\pi(F^\times)$. $\square$

The relevant functional equation for GL_1 zeta integrals will be discussed in the next section.

3B. *Fourier operators.* We define a Fourier operator $\mathcal{F}_{\pi,\psi}$ from the π-Schwartz space $\mathcal{S}_\pi(F^\times)$ to the $\widetilde{\pi}$-Schwartz space $\mathcal{S}_{\widetilde{\pi}}(F^\times)$ for any $\pi \in \Pi_F(G_n)$ with smooth contragredient $\widetilde{\pi}$ and prove the functional equation for GL_1 zeta integrals $\mathcal{Z}(s, \phi, \chi)$.

The Fourier operator (transform) $\mathcal{F}_{\pi,\psi}$ is defined by the diagram

$$\begin{array}{ccc} \mathcal{S}_{\mathrm{std}}(G_n(F)) \otimes \mathcal{C}(\pi) & \xrightarrow{(\mathcal{F}_{\mathrm{GJ}}, (\cdot)^\vee)} & \mathcal{S}_{\mathrm{std}}(G_n(F)) \otimes \mathcal{C}(\widetilde{\pi}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{S}_\pi(F^\times) & \xrightarrow{\mathcal{F}_{\pi,\psi}} & \mathcal{S}_{\widetilde{\pi}}(F^\times) \end{array} \tag{3-16}$$

where ψ is a nontrivial additive character of F. More precisely, for $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(F^\times)$ with a $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and a $\varphi_\pi \in \mathcal{C}(\pi)$, we define

$$\mathcal{F}_{\pi,\psi}(\phi) = \mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi}) := \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi),\varphi_\pi^\vee}, \tag{3-17}$$

where $\varphi_\pi^\vee(g) = \varphi_\pi(g^{-1}) \in \mathcal{C}(\widetilde{\pi})$. Hence, we obtain that

$$\mathcal{F}_{\pi,\psi}(\phi) = \mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi}) \in \mathcal{S}_{\widetilde{\pi}}(F^\times). \tag{3-18}$$

It remains to check that the definition of the Fourier operator in (3-17) is independent of the choice of $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$.

Proposition 3.9. *The Fourier operator* $\mathcal{F}_{\pi,\psi}$ *as in* (3-17) *is independent of the choice of* $\xi \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ *and* $\varphi_\pi \in \mathcal{C}(\pi)$.

Proof. Assume that $\phi_{\xi_1,\varphi_{\pi,1}} = \phi_{\xi_2,\varphi_{\pi,2}}$ for some $\xi_1, \xi_2 \in \mathcal{S}_{\mathrm{std}}(G_n(F))$ and $\varphi_{\pi,1}, \varphi_{\pi,2} \in \mathcal{C}(\pi)$. We want to show that $\mathcal{F}_{\pi,\psi}(\phi_{\xi_1,\varphi_{\pi,1}}) = \mathcal{F}_{\pi,\psi}(\phi_{\xi_2,\varphi_{\pi,2}})$.

From (3-10), we must have that

$$\mathcal{Z}(s, \xi_1, \varphi_{\pi,1}, \chi) = \mathcal{Z}(s, \xi_2, \varphi_{\pi,2}, \chi)$$

for all quasicharacters $\chi \in \mathfrak{X}(F^\times)$ and all $s \in \mathbb{C}$. Of course, the identity holds for $\mathrm{Re}(s)$ large and then for all $s \in \mathbb{C}$ by meromorphic continuation. By the functional equation in Proposition 2.7, we obtain the identity

$$\mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi_1), \varphi^\vee_{\pi,1}, \chi^{-1}) = \mathcal{Z}(1-s, \mathcal{F}_{\mathrm{GJ}}(\xi_2), \varphi^\vee_{\pi,2}, \chi^{-1})$$

for all $\chi \in \mathfrak{X}(F^\times)$ with $\mathrm{Re}(s)$ sufficiently small first and then all $s \in \mathbb{C}$ by meromorphic continuation. It follows by the identity in (3-10) again that, for all $\chi \in \mathfrak{X}(F^\times)$ and for $\mathrm{Re}(s) + \mathrm{Re}(\chi)$ sufficiently large, the identity

$$\int_{F^\times} (\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_1),\varphi^\vee_{\pi,1}}(x) - \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_2),\varphi^\vee_{\pi,2}}(x))\chi(x)|x|_F^{s-1/2}\,\mathrm{d}^\times x = 0$$

holds. By Proposition 3.7, we have that $\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_1),\varphi^\vee_{\pi,1}}(x) - \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_2),\varphi^\vee_{\pi,2}}(x)$ belongs to $\mathfrak{F}(F^\times)$. Finally, by Theorem 2.3, we must have that

$$\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_1),\varphi^\vee_{\pi,1}}(x) - \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_2),\varphi^\vee_{\pi,2}}(x) = 0$$

as functions on $F^\times$. Therefore, we proved that

$$\phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_1),\varphi^\vee_{\pi,1}}(x) = \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi_2),\varphi^\vee_{\pi,2}}(x)$$

as functions on $F^\times$, and $\mathcal{F}_{\pi,\psi}(\phi_{\xi_1,\varphi_{\pi,1}}) = \mathcal{F}_{\pi,\psi}(\phi_{\xi_2,\varphi_{\pi,2}})$. □

The following theorem on the local functional equation for the GL_1 zeta integrals $\mathcal{Z}(s, \phi, \chi)$ is a direct consequence of Theorem 2.4 and Proposition 3.9.

Theorem 3.10 (GL_1 functional equation). *For any* $\pi \in \Pi_F(G_n)$ *and its contragredient* $\widetilde{\pi} \in \Pi_F(G_n)$, *there exists a Fourier operator* $\mathcal{F}_{\pi,\psi}$, *which takes* $\phi \in \mathcal{S}_\pi(F^\times)$ *to* $\mathcal{F}_{\pi,\psi}(\phi) \in \mathcal{S}_{\widetilde{\pi}}(F^\times)$, *such that, after meromorphic continuation, the functional equation*

$$\mathcal{Z}(1-s, \mathcal{F}_{\pi,\psi}(\phi), \chi^{-1}) = \gamma(s, \pi\times\chi, \psi)\cdot\mathcal{Z}(s, \phi, \chi),$$

holds for any $\phi \in \mathcal{S}_\pi(F^\times)$. *The identities*

$$\mathcal{F}_{\widetilde{\pi},\psi^{-1}} \circ \mathcal{F}_{\pi,\psi} = \mathrm{Id} \quad \text{and} \quad \mathcal{F}_{\pi,\psi} \circ \mathcal{F}_{\widetilde{\pi},\psi^{-1}} = \mathrm{Id}$$

hold. When F is non-Archimedean, and π is unramified, the Fourier operator $\mathcal{F}_{\pi,\psi}$ takes the basic function $\mathbb{L}_\pi \in \mathcal{S}_\pi(F^\times)$ to the basic function $\mathbb{L}_{\widetilde{\pi}} \in \mathcal{S}_{\widetilde{\pi}}(F^\times)$:

$$\mathcal{F}_{\pi,\psi}(\mathbb{L}_\pi) = \mathbb{L}_{\widetilde{\pi}},$$

where the basic function $\mathbb{L}_\pi$ is defined in Theorem 3.4.

4. π-Poisson summation formula on GL_1

Let k be a number field and $\mathbb{A}$ be the ring of adeles of k. Denote by $|k|$ the set of all local places of k and by $|k|_\infty$ the set of all Archimedean local places of k. We may write

$$|k| = |k|_\infty \cup |k|_f,$$

where $|k|_f$ is the set of non-Archimedean local places of k. For each $\nu \in |k|$, we write $F = k_\nu$. Let $\Pi_{\mathbb{A}}(G_n)$ be the set of equivalence classes of irreducible admissible representations of $G_n(\mathbb{A})$. If we write $\pi = \bigotimes_{\nu\in|k|} \pi_\nu$, then we assume that $\pi_\nu \in \Pi_{k_\nu}(G_n)$, where at almost all finite local places ν, the local representations π_ν are unramified. When ν is a finite local place, π_ν is an irreducible admissible representation of $G_n(k_\nu)$, and when ν is an infinite local place, we assume that π_ν is of Casselman–Wallach type as representation of $G_n(k_\nu)$. Let $\mathcal{A}(G_n) \subset \Pi_{\mathbb{A}}(G_n)$ be the subset consisting of equivalence classes of irreducible admissible automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, and $\mathcal{A}_{\mathrm{cusp}}(G_n)$ be the subset of cuspidal members of $\mathcal{A}(G_n)$.

4A. *π-Schwartz space and Fourier operator.* Take any $\pi = \bigotimes_{\nu\in|k|} \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$. At each $\nu \in |k|$, the π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ is defined in Definition 3.3. Recall from Theorems 3.4 and 3.10 the basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ of π_ν when the local component π_ν of π is unramified. It is clear from the definition that $\mathbb{L}_{\pi_\nu}(1) = 1$ (We have to normalize various local measures in the computations. Actually it follows from the fact that the Laurent expansion of the unramified local L-factor has constant term 1.)

For the given $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, we define the π-Schwartz space on $\mathbb{A}^\times$ to be

$$\mathcal{S}_\pi(\mathbb{A}^\times) := \bigotimes_{\nu\in|k|} \mathcal{S}_{\pi_\nu}(k_\nu^\times), \tag{4-1}$$

which is the restricted tensor product of the local π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ with respect to the family of the basic functions $\mathbb{L}_{\pi_\nu}$ for the local places ν at which π_ν are unramified. The factorizable vectors $\phi = \bigotimes_\nu \phi_\nu$ in $\mathcal{S}_\pi(\mathbb{A}^\times)$ can be written as

$$\phi(x) = \prod_{\nu\in|k|} \phi_\nu(x_\nu). \tag{4-2}$$

Here for almost all finite local places ν, $\phi_\nu(x_\nu) = \mathbb{L}_{\pi_\nu}(x_\nu)$. According to our normalization, we have $\mathbb{L}_{\pi_\nu}(x_\nu) = 1$ when $x_\nu \in \mathfrak{o}_\nu^\times$, the unit group of the ring $\mathfrak{o}_\nu$ of integers at ν. Hence, for any given $x \in \mathbb{A}^\times$, the product in (4-2) is a finite product over Archimedean local places and finitely many non-Archimedean local places containing all ramified local places of π.

For any factorizable vectors $\phi = \bigotimes_\nu \phi_\nu$ in $\mathcal{S}_\pi(\mathbb{A}^\times)$, we define the π-Fourier operator

$$\mathcal{F}_{\pi,\psi}(\phi) := \bigotimes_{\nu \in |k|} \mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu), \tag{4-3}$$

where for each $\nu \in |k|$, $\mathcal{F}_{\pi_\nu,\psi_\nu}$ is the local Fourier operator as defined in (3-16) and (3-17). It is clear that $\mathcal{F}_{\pi_\nu,\psi_\nu}$ takes the π_ν-Schwartz space $\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ to the $\widetilde{\pi}_\nu$-Schwartz space $\mathcal{S}_{\widetilde{\pi}_\nu}(k_\nu^\times)$ and enjoys the property

$$\mathcal{F}_{\pi_\nu,\psi}(\mathbb{L}_{\pi_\nu}) = \mathbb{L}_{\widetilde{\pi}_\nu}$$

when the data are unramified at ν. Hence, the Fourier operator $\mathcal{F}_{\pi,\psi}$ as defined in (4-3) maps the π-Schwartz space $\mathcal{S}_\pi(\mathbb{A}^\times)$ to the $\widetilde{\pi}$-Schwartz space $\mathcal{S}_{\widetilde{\pi}}(\mathbb{A}^\times)$.

4B. *Global zeta integral.* For any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, we define the global zeta integrals to be

$$\mathcal{Z}(s,\phi,\chi) := \int_{\mathbb{A}^\times} \phi(x)\chi(x)|x|_{\mathbb{A}}^{s-1/2}\,\mathrm{d}^\times x \tag{4-4}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and characters χ of $k^\times\backslash\mathbb{A}^\times$. When $\phi = \bigotimes_\nu \phi_\nu$, we have

$$\mathcal{Z}(s,\phi,\chi) = \prod_{\nu \in |k|} \mathcal{Z}(s,\phi_\nu,\chi_\nu).$$

Let S be a finite subset of $|k|$, which contains all Archimedean local places and all the finite local places ν at which π_ν or χ_ν is ramified. Then we write

$$\mathcal{Z}(s,\phi,\chi) = L^S(s,\pi\times\chi)\cdot\prod_{\nu\in S}\mathcal{Z}(s,\phi_\nu,\chi_\nu),$$

according to Theorem 3.4. If π is unitarizable, the partial L-function $L^S(s,\pi\times\chi)$ converges absolutely for $\mathrm{Re}(s)$ large. By Theorem 3.4 again, the finite Euler product $\prod_{\nu\in S}\mathcal{Z}(s,\phi_\nu,\chi_\nu)$ converges absolutely for $\mathrm{Re}(s)$ large. We deduce the following proposition.

Proposition 4.1. *Let $\pi \in \Pi_{\mathbb{A}}(G_n)$ be unitarizable. Then for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and any character χ of $k^\times\backslash\mathbb{A}^\times$, the zeta integral $\mathcal{Z}(s,\phi,\chi)$ as defined in* (4-4) *converges absolutely for* $\mathrm{Re}(s)$ *sufficiently large.*

We apply Proposition 4.1 to the case that $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n) \subset \mathcal{A}(G_n) \subset \Pi_{\mathbb{A}}(G_n)$. If $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, then it is unitary. In this case, the zeta integral $\mathcal{Z}(s,\phi,\chi)$ can be identified with the Godement–Jacquet global zeta integral. For any $f = \bigotimes_\nu f_\nu \in$

$\mathcal{S}(M_n(\mathbb{A}))$ and any $\varphi_\pi \in \mathcal{C}(\pi)$, the Godement–Jacquet global zeta integral is defined to be

$$\mathcal{Z}(s, f, \varphi_\pi, \chi) := \int_{\mathrm{GL}_n(\mathbb{A})} f(g)\varphi_\pi(g)\chi(\det g)|\det g|_F^{s+(n-1)/2}\,\mathrm{d}g, \tag{4-5}$$

Theorem 4.2 [16, Theorem 13.8]. *For $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ and any unitary automorphic character χ of $k^\times\backslash\mathbb{A}^\times$, the global zeta integral $\mathcal{Z}(s, f, \varphi_\pi, \chi)$ converges absolutely for* $\mathrm{Re}(s) > (n+1)/2$, *admits analytic continuation to an entire function in $s \in \mathbb{C}$, and satisfies the global functional equation*

$$\mathcal{Z}(s, f, \varphi_\pi, \chi) = \mathcal{Z}(1-s, \mathcal{F}_\psi(f), \varphi_\pi^\vee, \chi^{-1}), \tag{4-6}$$

where $\mathcal{F}_\psi$ is the global Fourier transform from $\mathcal{S}(M_n(\mathbb{A}))$ to $\mathcal{S}(M_n(\mathbb{A}))$ associated to the additive character ψ of $k\backslash\mathbb{A}$.

For $\mathrm{Re}(s) > (n+1)/2$, we write

$$\mathcal{Z}(s, f, \varphi_\pi, \chi) = \int_{\mathbb{A}^\times}\left(|x|_{\mathbb{A}}^{n/2}\int_{G_n(\mathbb{A})_x} f(g)\varphi_\pi(g)\,\mathrm{d}_x g\right)\chi(x)|x|_{\mathbb{A}}^{s-1/2}\,\mathrm{d}^\times x, \tag{4-7}$$

where $G_n(\mathbb{A})_x := \{g \in G_n(\mathbb{A}) \mid \det g = x\}$ is an $\mathrm{SL}_n(\mathbb{A})$-torsor, and the measure $\mathrm{d}_x g$ is $\mathrm{SL}_n(\mathbb{A})$-invariant. As in the local situations, we define, for any $x \in \mathbb{A}^\times$,

$$\phi_{\xi,\varphi_\pi}(x) := \int_{G_n(\mathbb{A})_x} \xi(g)\varphi_\pi(g)\,\mathrm{d}_x g = |x|_{\mathbb{A}}^{n/2}\int_{G_n(\mathbb{A})_x} f(g)\varphi_\pi(g)\,\mathrm{d}_x g, \tag{4-8}$$

where $\xi(g) := |\det g|_{\mathbb{A}}^{n/2}\cdot f(g)$ belongs to the space

$$\mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A})) = \{\xi \in \mathcal{C}^\infty(G_n(\mathbb{A})) \mid \xi(g)\cdot|\det g|_{\mathbb{A}}^{-n/2} \in \mathcal{S}(M_n(\mathbb{A}))\}. \tag{4-9}$$

It is clear that

$$\mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A})) = \bigotimes_{\nu\in|k|}\mathcal{S}_{\mathrm{std}}(G_n(k_\nu)). \tag{4-10}$$

Write $G_n(\mathbb{A})$ as a direct product decomposition:

$$G_n(\mathbb{A}) = A_n(\mathbb{R})^+\cdot G_n(\mathbb{A})^1, \tag{4-11}$$

where $G_n(\mathbb{A})^1 := \{g \in G_n(\mathbb{A}) \mid |\det g|_{\mathbb{A}} = 1\}$ and $A_n(\mathbb{R})^+$ is the identity connected component of the center $Z_{G_n}(\mathbb{R})$ of $G_n(\mathbb{R})$. As in [16, Section 13], any matrix coefficient φ_π of $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ can be written as

$$\varphi_\pi(g) = \int_{A_n(\mathbb{R})^+G_n(k)\backslash G_n(\mathbb{A})}\alpha_\pi(hg)\alpha_{\widetilde{\pi}}(h)\,\mathrm{d}h = \int_{G_n(k)\backslash G_n(\mathbb{A})^1}\alpha_\pi(hg)\alpha_{\widetilde{\pi}}(h)\,\mathrm{d}h \tag{4-12}$$

for some $\alpha_\pi \in V_\pi$ and $\alpha_{\widetilde{\pi}} \in V_{\widetilde{\pi}}$, where V_π is the cuspidal automorphic realization of π in $L^2(G_n(k)\backslash G_n(\mathbb{A}), \omega)$ with central character $\omega_\pi = \omega$. In this case, we have $\omega_{\widetilde{\pi}} = \omega^{-1}$. In the integral in (4-8), the coefficient $\varphi_\pi(g)$ is bounded over $G_n(\mathbb{A})$. Since $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $G_n(\mathbb{A})_x$ is a closed submanifold in $M_n(\mathbb{A})$, the restriction to $G_n(\mathbb{A})_x$ of the Schwartz function f is still a Schwartz function on $G_n(\mathbb{A})_x$. Hence, the integral in (4-8) converges absolutely for any $x \in \mathbb{A}^\times$, and the convergence is uniform when x runs in any given compact neighborhood of $\mathbb{A}^\times$.

Proposition 4.3. *For $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, the function $\phi_{\xi,\varphi_\pi}(x)$ as defined in* (4-8) *is smooth on $\mathbb{A}^\times$. If $\xi(g) = \bigotimes_\nu \xi_\nu = |\det g|^{n/2} \cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ with $f = \bigotimes_\nu f_\nu \in \mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi = \bigotimes_\nu \varphi_{\pi_\nu}$, then the function defined by*

$$\phi_{\xi,\varphi_\pi}(x) = \prod_{\nu \in |k|} \phi_{\xi_\nu,\varphi_{\pi_\nu}}(x_\nu)$$

for any $x \in \mathbb{A}^\times$ belongs to $\mathcal{S}_\pi(\mathbb{A}^\times)$.

Proof. Since the integral in (4-8) converges absolutely for any $x \in \mathbb{A}^\times$, and the convergence is uniform when x runs in any given compact neighborhood of $\mathbb{A}^\times$, the function $\phi_{\xi,\varphi_\pi}(x)$ is smooth on $\mathbb{A}^\times$.

To prove the second statement, we take $f = \bigotimes_\nu f_\nu \in \mathcal{S}(M_n(\mathbb{A}))$. Since $\mathcal{C}(\pi) = \bigotimes_\nu \mathcal{C}(\pi_\nu)$, we take $\varphi_\pi = \bigotimes_\nu \varphi_{\pi_\nu}$ with $\varphi_{\pi_\nu} \in \mathcal{C}(\pi_\nu)$. Then there exists a finite subset S_0 which contains all Archimedean local places of k such that for any finite local place ν of k, if $\nu \notin S_0$, then $f_\nu = f_\nu^\circ = \mathbb{1}_{M_n(\mathfrak{o}_\nu)}$, π_ν is unramified and $\varphi_{\pi_\nu} = \varphi_{\pi_\nu}^\circ$, which is the zonal spherical function on $G_n(k_\nu)$ associated to π_ν. For any $x \in \mathbb{A}^\times$, and for any finite subset S of $|k|$ that contains S_0 and $x_\nu \in \mathfrak{o}_\nu^\times$ if $\nu \notin S$, we have

$$\phi_{\xi,\varphi_\pi}(x) = \int_{\det g = x} \xi(g)\varphi_\pi(g)\,\mathrm{d}_x g = \lim_S \prod_{\nu \in S} \int_{\det g_\nu = x_\nu} \xi_\nu(g_\nu)\varphi_{\pi_\nu}(g_\nu)\,\mathrm{d}_{x_\nu} g_\nu \tag{4-13}$$

with $\xi(g) = |\det g|_{\mathbb{A}}^{n/2} \cdot f(g)$ and $\xi = \bigotimes_\nu \xi_\nu$, where $\xi_\nu(g) = |\det g|_\nu^{n/2} \cdot f_\nu(g)$. At $\nu \notin S$, we have $|x_\nu|_\nu = 1$ and the local integral identity

$$\int_{\det g_\nu = x_\nu} \xi_\nu(g_\nu)\varphi_{\pi_\nu}(g_\nu)\,\mathrm{d}_{x_\nu} g_\nu = \int_{\det g_\nu = x_\nu} \mathbb{1}_{M_n(\mathfrak{o}_\nu)}(g_\nu)\varphi_{\pi_\nu}^\circ(g_\nu)\,\mathrm{d}_{x_\nu} g_\nu = \mathrm{vol}(G_n(\mathfrak{o}_\nu)_{x_\nu}) = 1.$$

Hence, we obtain that $\phi_{\xi,\varphi_\pi}(x) = \prod_\nu \phi_{\xi_\nu,\varphi_{\pi_\nu}}(x_\nu)$. □

Hence, we obtain the relation between the global GL_1 zeta integrals defined in (4-4) and the global Godement–Jacquet zeta integrals defined in (4-5).

Corollary 4.4. *If $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, then for any $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(\mathbb{A}^\times)$ with $\xi(g) = |\det g|_{\mathbb{A}}^{n/2} \cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, the identity*

$$\mathcal{Z}(s, \phi, \chi) = \mathcal{Z}(s, f, \varphi_\pi, \chi)$$

holds for any character χ of $k^\times \backslash \mathbb{A}^\times$ and $\mathrm{Re}(s)$ *sufficiently large.*

Proposition 4.5. *If $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, then for any $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(\mathbb{A}^\times)$ with $\xi(g) = |\det g|_{\mathbb{A}}^{n/2} \cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, the identity*

$$\mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi})(x) = \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi),\varphi_\pi^\vee}(x)$$

holds for any $x \in \mathbb{A}^\times$. For any $x \in \mathbb{A}^\times$, the $\mathbb{A}^\times$-equivariant property

$$\mathcal{F}_{\pi,\psi}(\phi^x)(y) = \mathcal{F}_{\pi,\psi}(\phi)(yx^{-1})$$

holds, where $\phi^x(y) := \phi(yx)$.

Proof. Assume that $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(\mathbb{A}^\times)$ with $\xi(g) = |\det g|_{\mathbb{A}}^{n/2} \cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi \in \mathcal{C}(\pi)$ is factorizable: $\phi = \bigotimes_\nu \phi_\nu$. By definition (4-3), we have

$$\mathcal{F}_{\pi,\psi}(\phi)(x) = \prod_{\nu\in|k|} \mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)(x_\nu).$$

Write $\phi_\nu(x_\nu) = \phi_{\xi_\nu,\varphi_{\pi_\nu}}(x_\nu)$. Then we have

$$\mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)(x_\nu) = \phi_{\mathcal{F}_{\mathrm{GJ},\nu}(\xi_\nu),\varphi_{\pi_\nu}^\vee}(x_\nu).$$

When the data involved are unramified, we have from the simple calculation below (4-13) that $\mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)(x_\nu) = 1$. Hence, we obtain

$$\mathcal{F}_{\pi,\psi}(\phi)(x) = \prod_\nu \mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)(x_\nu) = \prod_\nu \phi_{\mathcal{F}_{\mathrm{GJ},\nu}(\xi_\nu),\varphi_{\pi_\nu}^\vee}(x_\nu) = \phi_{\mathcal{F}_{\mathrm{GJ}}(\xi),\varphi_\pi^\vee}(x)$$

as in (4-13).

In order to verify the $\mathbb{A}^\times$-equivariant property $\mathcal{F}_{\pi,\psi}(\phi^x)(y) = \mathcal{F}_{\pi,\psi}(\phi)(yx^{-1})$ for any $x, y \in \mathbb{A}^\times$, it is enough to verify that the local Fourier operators $\mathcal{F}_{\pi_\nu,\psi_\nu}$ for all local place $\nu \in |k|$ enjoy the same equivariant property. This local equivariant property for the Fourier operators $\mathcal{F}_{\pi_\nu,\psi_\nu}$ can be deduced from the local functional equation for the zeta integral $\mathcal{Z}(s, \phi, \chi)$ in Theorem 3.10 through a simple computation. □

We can deduce the following result from Theorem 4.2.

Theorem 4.6. *Let π be an irreducible unitary cuspidal automorphic representation of $G_n(\mathbb{A})$ with the local component π_ν being of Casselman–Wallach type at all $\nu \in |k|_\infty$. For any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and any unitary character χ of $k^\times\backslash\mathbb{A}^\times$, the global zeta integral $\mathcal{Z}(s, \phi, \chi)$ converges absolutely for* $\mathrm{Re}(s) > (n+1)/2$, *admits analytic continuation to an entire function in $s \in \mathbb{C}$, and satisfies the functional equation*

$$\mathcal{Z}(s, \phi, \chi) = \mathcal{Z}(1-s, \mathcal{F}_{\pi,\psi}(\phi), \chi^{-1}),$$

where $\mathcal{F}_{\pi,\psi}$ is the Fourier operator as defined in (4-3) *that takes $\mathcal{S}_\pi(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\pi}}(\mathbb{A}^\times)$.*

4C. π-*Poisson summation formula.* We establish here the Poisson summation formula on GL_1 for the Fourier operator $\mathcal{F}_{\pi,\psi}$, which is associated to any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, and takes $\mathcal{S}_\pi(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\pi}}(\mathbb{A}^\times)$. Technically, it is possible to establish such a summation formula from the global functional equation in Theorem 4.6. However, we are going to take a slightly different way below.

Theorem 4.7 (π-Poisson summation formula). *For any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, take $\widetilde{\pi}$ to be the contragredient of π. For any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, the π-theta function*

$$\Theta_\pi(x,\phi) := \sum_{\alpha\in k^\times} \phi(\alpha x)$$

converges absolutely and locally uniformly for any $x \in \mathbb{A}^\times$, and we have the identity

$$\Theta_\pi(x,\phi) = \Theta_{\widetilde{\pi}}(x^{-1}, \mathcal{F}_{\pi,\psi}(\phi)),$$

as functions in $x \in \mathbb{A}^\times$, where $\mathcal{F}_{\pi,\psi}$ is the Fourier operator as defined in (4-3) *that takes $\mathcal{S}_\pi(\mathbb{A}^\times)$ to $\mathcal{S}_{\widetilde{\pi}}(\mathbb{A}^\times)$.*

Proof. It is clear that $\Theta_\pi(x,\phi)=\Theta_\pi(1,\phi^x)$ with $\phi^x(y)=\phi(xy)$. By Proposition 4.5, we have $\Theta_{\widetilde{\pi}}(x^{-1},\mathcal{F}_{\pi,\psi}(\phi)) = \Theta_{\widetilde{\pi}}(1,\mathcal{F}_{\pi,\psi}(\phi^x))$. Since $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ is arbitrary, it is enough to show that

$$\Theta_\pi(1,\phi) := \sum_{\alpha\in k^\times} \phi(\alpha)$$

converges absolutely and the identity

$$\Theta_\pi(1,\phi) = \Theta_{\widetilde{\pi}}(1,\mathcal{F}_{\pi,\psi}(\phi))$$

holds.

In order to prove that the summation $\Theta_\pi(1,\phi)$ is absolutely convergent, we write $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(\mathbb{A}^\times)$ with $\xi(g) = |\det g|_{\mathbb{A}}^{n/2} \cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f \in \mathcal{S}(M_n(\mathbb{A}))$ and $\varphi_\pi \in \mathcal{C}(\pi)$. From (4-12) we have

$$\varphi_\pi(g) = \int_{A_n(\mathbb{R})^+G_n(k)\backslash G_n(\mathbb{A})} \beta_1(hg)\beta_2(h)\,\mathrm{d}h = \int_{G_n(k)\backslash G_n(\mathbb{A})^1} \beta_1(hg)\beta_2(h)\,\mathrm{d}h \tag{4-14}$$

for some $\beta_1 \in V_\pi$ and $\beta_2 \in V_{\widetilde{\pi}}$, where V_π is the cuspidal automorphic realization of π in $L^2(G_n(k)\backslash G_n(\mathbb{A}),\omega)$ and so is $V_{\widetilde{\pi}}$.

First, we have that

$$\begin{aligned}\Theta_\pi(1,\phi) &= \sum_{\alpha\in k^\times} \phi_{\xi,\varphi_\pi}(\alpha) \\ &= \sum_{\alpha\in k^\times} \int_{G_n(\mathbb{A})_\alpha} \xi(g) \int_{G_n(k)\backslash G_n(\mathbb{A})^1} \beta_1(hg)\beta_2(h)\,\mathrm{d}h\,\mathrm{d}_\alpha g.\end{aligned} \tag{4-15}$$

By changing variable $g \to h^{-1}g$, we have that $\det g = \alpha \cdot \det h$ and the last expression in (4-15) becomes

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1} \sum_{\alpha\in k^\times} \int_{G_n(\mathbb{A})_{\alpha\cdot\det h}} \xi(h^{-1}g)\beta_1(g)\beta_2(h)\, \mathrm{d}_{\alpha\cdot\det h} g\, \mathrm{d}h. \tag{4-16}$$

For $g \in G_n(\mathbb{A})_{\alpha\cdot\det h}$, we change g to $t_1(\alpha)\cdot y$ with $\det y = \det h$, where $t_1(\alpha) = \mathrm{diag}(\alpha, I_{n-1}) \in G_n(k)$. Then (4-16) can be written as

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1} \sum_{\alpha\in k^\times} \int_{\mathrm{GL}_n(\mathbb{A})_{\det h}} \xi(h^{-1}t_1(\alpha)g)\beta_1(g)\beta_2(h)\, \mathrm{d}_{\det h} g\, \mathrm{d}h, \tag{4-17}$$

since β_1 is automorphic. For any $h \in G_n(\mathbb{A})^1$, we have $|\det h|_{\mathbb{A}} = 1$. Hence, we must have that $G_n(\mathbb{A})_{\det h} \subset G_n(\mathbb{A})^1$. It is clear that $G_n(\mathbb{A})_{\det h}$ is an $\mathrm{SL}_n(\mathbb{A})$-torsor and the measure $\mathrm{d}_{\det h} g$ is left-$\mathrm{SL}_n(k)$-invariant. Hence, (4-17) can be written as

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1} \sum_{\alpha\in k^\times} \sum_{\epsilon\in \mathrm{SL}_n(k)} \int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}} \xi(h^{-1}t_1(\alpha)\epsilon g)\beta_1(g)\beta_2(h)\, \mathrm{d}_{\det h} g\, \mathrm{d}h.$$

Since any element $\gamma \in G_n(k)$ can be written as a product of $t_1(\alpha)$ and ϵ in a unique way, we obtain that the above expression is equal to

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1} \int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}} \Bigg(\sum_{\gamma\in G_n(k)} \xi(h^{-1}\gamma g)\Bigg)\beta_1(g)\beta_2(h)\, \mathrm{d}_{\det h} g\, \mathrm{d}h. \tag{4-18}$$

Since $\xi(g) = |\det g|_{\mathbb{A}}^{n/2} \cdot f(g) \in \mathcal{S}_{\mathrm{std}}(G_n(\mathbb{A}))$ for some $f \in \mathcal{S}(M_n(\mathbb{A}))$, and $h \in G_n(\mathbb{A})^1$ and $g \in G_n(\mathbb{A})_{\det h}$, we must have that

$$\xi(h^{-1}\gamma g) = |\det(h^{-1}\gamma g)|_{\mathbb{A}}^{n/2} \cdot f(h^{-1}\gamma g) = f(h^{-1}\gamma g). \tag{4-19}$$

Hence, we obtain that

$$\sum_{\gamma\in G_n(k)} \xi(h^{-1}\gamma g) = \sum_{\gamma\in G_n(k)} f(h^{-1}\gamma g). \tag{4-20}$$

By [16, Lemma 11.7], for any $f \in \mathcal{S}(M_n(\mathbb{A}))$, the summation $\sum_{\gamma\in G_n(k)} f(h^{-1}\gamma g)$ is of moderate growth in $g, h \in G_n(k)\backslash G_n(\mathbb{A})$ as an automorphic function on $G_n(k)\backslash G_n(\mathbb{A}) \times G_n(k)\backslash G_n(\mathbb{A})$, and so is the summation $\sum_{\gamma\in G_n(k)} \xi(h^{-1}\gamma g)$ as an automorphic function in $g, h \in G_n(k)\backslash G_n(\mathbb{A})^1$. Since both $\beta_1(g)$ and $\beta_2(h)$ are cuspidal, we obtain that the integral in (4-18) converges absolutely, and so does the π-theta function $\Theta_\pi(1, \phi)$ at $x = 1$.

Now we continue with the integral in (4-18) to prove the identity

$$\Theta_\pi(1, \phi) = \Theta_{\widetilde{\pi}}(1, \mathcal{F}_{\pi,\psi}(\phi)). \tag{4-21}$$

Recall from [16, Section 11; 34, Theorem 4.0.1] the classical Poisson summation formula

$$\sum_{\gamma\in M_n(k)} f(h^{-1}\gamma g)=\sum_{\gamma\in M_n(k)} |\det gh^{-1}|_{\mathbb{A}}^{-n}\mathcal{F}_\psi(f)(g^{-1}\gamma h) \tag{4-22}$$

for any $f\in\mathcal{S}(M_n(\mathbb{A}))$ and $h,g\in G_n(\mathbb{A})$. When $g,h\in G_n(\mathbb{A})^1$, it can be rewritten according to the rank of $\gamma\in M_n(k)$ as

$$\sum_{\gamma\in G_n(k)} f(h^{-1}\gamma g) \\ =\sum_{\gamma\in G_n(k)}\mathcal{F}_\psi(f)(g^{-1}\gamma h)+\sum_{\substack{\gamma\in M_n(k)\\ \mathrm{rank}(\gamma)<n}}\mathcal{F}_\psi(f)(g^{-1}\gamma h)-\sum_{\substack{\gamma\in M_n(k)\\ \mathrm{rank}(\gamma)<n}} f(h^{-1}\gamma g).$$

We denote the boundary terms by

$$B_f(h,g):=\sum_{\substack{\gamma\in M_n(k)\\ \mathrm{rank}(\gamma)<n}}\mathcal{F}_\psi(f)(g^{-1}\gamma h)-\sum_{\substack{\gamma\in M_n(k)\\ \mathrm{rank}(\gamma)<n}} f(h^{-1}\gamma g). \tag{4-23}$$

Then (4-18) can be written as a sum of the two terms

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}}\Bigg(\sum_{\gamma\in G_n(k)}\mathcal{F}_\psi(f)(g^{-1}\gamma h)\Bigg)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h, \tag{4-24}$$

and

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}} B_f(h,g)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h. \tag{4-25}$$

From the proofs of [16, Lemma 12.13; 34, Lemma 4.1.4], we must have that the term in (4-25) is zero, because of the cuspidality of both $\beta_1(g)$ and $\beta_2(h)$. Hence, we obtain that $\Theta_\pi(1,\phi)=\Theta_\pi(1,\phi_{\xi,\varphi_\pi})$ is equal to the term in (4-24).

Now we write (4-24) as

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{\mathrm{SL}_n(k)\backslash G_n(\mathbb{A})_{\det h}}\Bigg(\sum_{\gamma\in G_n(k)}\mathcal{F}_\psi(f)((\gamma g)^{-1}h)\Bigg)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h.$$

By writing back that $\gamma=t_1(\alpha)\cdot\epsilon$ with $\alpha\in k^\times$ and $\epsilon\in\mathrm{SL}_n(k)$, we obtain that the above expression is equal to

$$\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{G_n(\mathbb{A})_{\det h}}\Bigg(\sum_{\alpha\in k^\times}\mathcal{F}_\psi(f)((t_1(\alpha)g)^{-1}h)\Bigg)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\det h}g\,\mathrm{d}h. \tag{4-26}$$

By changing $t_1(\alpha)g$ to g, we write (4-26) as

$$\sum_{\alpha\in k^\times}\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\int_{G_n(\mathbb{A})_{\alpha\cdot\det h}}\mathcal{F}_\psi(f)(g^{-1}h)\beta_1(g)\beta_2(h)\,\mathrm{d}_{\alpha\cdot\det h}g\,\mathrm{d}h. \tag{4-27}$$

After changing variable $g \to hg$, (4-27) can be written as

$$\sum_{\alpha\in k^\times}\int_{G_n(\mathbb{A})_\alpha}\mathcal{F}_\psi(f)(g^{-1})\int_{G_n(k)\backslash G_n(\mathbb{A})^1}\beta_1(hg)\beta_2(h)\,\mathrm{d}h\,\mathrm{d}_\alpha g, \tag{4-28}$$

which is equal to

$$\sum_{\alpha\in k^\times}\int_{G_n(\mathbb{A})_\alpha}\mathcal{F}_\psi(f)(g^{-1})\varphi_\pi(g)\,\mathrm{d}_\alpha g. \tag{4-29}$$

Finally, by changing g to g^{-1}, we obtain that (4-18) is equal to

$$\sum_{\alpha\in k^\times}\int_{G_n(\mathbb{A})_\alpha}\mathcal{F}_\psi(f)(g)\varphi_\pi(g^{-1})\,\mathrm{d}_\alpha g. \tag{4-30}$$

By Proposition 2.6, when $\det g = \alpha \in k^\times$, we have

$$\mathcal{F}_\psi(f)(g) = \mathcal{F}_{\mathrm{GJ}}(\xi)(g)$$

for $\xi(g) = |\det g|^{n/2}\cdot f(g)$. Hence, the summation in (4-30) is equal to

$$\sum_{\alpha\in k^\times}\mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi})(1) = \Theta_{\widetilde{\pi}}(1, \mathcal{F}_{\pi,\psi}(\phi_{\xi,\varphi_\pi})).$$

This proves the π-Poisson summation formula

$$\Theta_\pi(1,\phi) = \Theta_{\widetilde{\pi}}(1, \mathcal{F}_{\pi,\psi}(\phi))$$

for all $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$.

For the locally uniform convergence of the π-theta function $\Theta_\pi(x,\phi)$, since $\Theta_\pi(x,\phi) = \Theta_\pi(1,\phi^x)$, it is enough to prove the locally uniform convergence of the π-theta function $\Theta_\pi(x,\phi)$ around $x = 1$. One may verify this directly from the discussion in the proof given above. It also follows directly from Proposition 4.8 below in this case. We are done. □

Similar to the work of [40], we obtain the following uniform estimate of the π-theta function $\Theta_\pi(x,\phi)$, which is important to the application in Section 8.

Proposition 4.8. *For any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, take any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$. For any $\kappa > 0$, there exists a positive constant $c_{\kappa,\phi}$ such that the π-theta function $\Theta_\pi(x,\phi)$ enjoys the property*

$$|\Theta_\pi(x,\phi)| \le c_{\kappa,\phi}\cdot\min\{|x|_{\mathbb{A}}, |x|_{\mathbb{A}}^{-1}\}^\kappa.$$

Proof. This is a reformulation of part (ii) of [40, Theorem 1] and can be proved accordingly. We omit the details. □

Remark 4.9. The proof of the π-Poisson summation formula in Theorem 4.7 uses the Poisson summation formula associated to the classical Fourier transform $\mathcal{F}_\psi$ over the affine space $M_n(\mathbb{A})$, without using the global functional equation for the global zeta integrals $\mathcal{Z}(s, \phi, \chi)$ in Theorem 4.6. Hence, we are able to obtain the global functional equation for the global zeta integrals $\mathcal{Z}(s, \phi, \chi)$ as in Theorem 4.6 by using the π-Poisson summation formula in Theorem 4.7. Of course, this is essentially the same proof as the one that uses the global functional equation of Godement–Jacquet zeta functions in Theorem 4.2. However, it seems still meaningful to point out the contribution of the π-Poisson summation formulae on GL_1 in the theory of the global functional equation for the standard automorphic L-function $L(s, \pi \times \chi)$ for any automorphic characters χ of $\mathbb{A}^\times$ and any irreducible cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A})$, as an extension in a different perspective of Tate's thesis to the study of higher degree automorphic L-functions.

5. Convergence of generalized theta functions

In order to prove Theorem 1.2 and explore other possible cases of Conjecture 1.5, beyond Theorem 4.7 (or Theorem 1.1), we study the convergence issue of general π-theta functions associated with $\pi \in \Pi_{\mathbb{A}}(G_n)$, which may not be automorphic.

5A. ***Convergence of π-theta functions.*** Recall from Section 4, if $\pi = \bigotimes_{\nu\in|k|} \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, then for every $\nu \in |k|$, $\pi_\nu \in \Pi_{k_\nu}(G_n)$, the set of equivalence classes of irreducible admissible representations of $G_n(k_\nu)$, where at almost all finite local places ν, π_ν is unramified and at any infinite local place ν, π_ν is of Casselman–Wallach type as representation of $G_n(k_\nu)$. As in (4-1), for any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, we have that

$$\mathcal{S}_\pi(\mathbb{A}^\times) = \bigotimes_{\nu\in|k|} \mathcal{S}_{\pi_\nu}(k_\nu^\times).$$

For $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, we are going to show that the π-theta function

$$\Theta_\pi(x, \phi) = \sum_{\alpha\in k^\times} \phi(\alpha x) \tag{5-1}$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$, under an assumption (Assumption 5.1) on the unramified local components π_ν of π.

For any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, let S_π be a finite subset of local places of k containing $|k|_\infty$ such that for any finite local place $\nu \notin S_\pi$, the local component π_ν is unramified. For any π_ν with $\nu \notin S_\pi$, via the Satake isomorphism, one has the

Frobenius–Hecke conjugacy class $c(\pi_\nu)$ in $G_n^\vee(\mathbb{C})$ associated to π_ν. We write

$$c(\pi_\nu) := \mathrm{diag}(q_\nu^{s_1(\pi_\nu)}, \ldots, q_\nu^{s_n(\pi_\nu)}) \in \mathrm{GL}_n(\mathbb{C}) = G_n^\vee(\mathbb{C}), \tag{5-2}$$

up to the adjoint action of $G_n^\vee(\mathbb{C})$, with $s_j(\pi_\nu) \in \mathbb{C}$ for $j = 1, 2, \ldots, n$, where q_ν is the cardinality of the residue field $\kappa_\nu = \mathfrak{o}_\nu/\mathfrak{p}_\nu$. The following is the assumption we take on the unramified local components π_ν of π.

Assumption 5.1 (uniform bound). *Let $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ be an irreducible admissible representation of $G_n(\mathbb{A})$. There exists a positive real number κ_π, which depends only on π, such that*

$$\max_{1\le j\le n} \{\mathrm{Re}(s_j(\pi_\nu))\} < \kappa_\pi$$

for every $\nu \notin S_\pi$.

Then we need to prove some technical local results.

Lemma 5.2. *For any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ with Assumption 5.1, there exists a positive real number $s_\pi \ge \kappa_\pi$ such that, for any real number $a_0 > s_\pi$, the limit*

$$\lim_{|x|_\nu \to 0} \phi_\nu(x)|x|_\nu^{a_0} = 0$$

holds, as a function in $x \in k_\nu^\times$, for any $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ and any local place $\nu \in |k|$. In particular, $\phi_\nu(x)|x|_\nu^{a_0}$ extends to a continuous function on k_ν, which is compactly supported on k_ν if $\nu \in |k|_f$ and is of Schwartz type at ∞ of k_ν if $\nu \in |k|_\infty$.

Proof. By Proposition 3.7, at any $\nu \in |k|$, we have that $\mathcal{S}_{\pi_\nu}(k_\nu^\times) \subset \mathfrak{F}(k_\nu^\times)$, which is defined in Definition 2.1. In the following we discuss separately for $\nu \in |k|_\infty$ and for $\nu \in |k|_f$.

When $\nu \in |k|_\infty$, the asymptotic of $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ near $x = 0$ is characterized in Definition 2.1. In particular, following the notation in Definition 2.1, the fixed sequence $\{\lambda_k\}_{k=0}^\infty$ has strictly increasing real part $\{\mathrm{Re}(\lambda_k)\}_{k=0}^\infty$. Hence, for any positive real number $s_0 \in \mathbb{R}$ satisfying the inequality

$$s_0 + \mathrm{Re}(\lambda_0) > 0,$$

the limit

$$\lim_{|x|_\nu \to 0} \phi_\nu(x) \cdot |x|_\nu^{s_0} = 0 \tag{5-3}$$

holds, because the limit formula in Definition 2.1 is termwise differentiable and uniform (even after termwise differentiation). Hence, the function $\phi_\nu(x) \cdot |x|_\nu^{s_0}$ is continuous over k_ν for any positive real number s_0 satisfying $s_0 + \mathrm{Re}(\lambda_0) > 0$. It is clear that the function $\phi_\nu(x) \cdot |x|_\nu^{s_0}$ is still of Schwartz type at ∞. Since the set $|k|_\infty$ is finite, it is possible to choose a sufficiently positive $s_\infty \in \mathbb{R}$ such that the prescribed property holds for all functions $\phi_\nu(x)\cdot|x|_\nu^{a_0}$ with $\phi_\nu\in\mathcal{S}_{\pi_\nu}(k_\nu^\times)$ at all $\nu\in|k|_\infty$, as long as $a_0\geq s_\infty$.

It remains to treat the case when $\nu\in|k|_f$, the finite local places of k. We consider the local zeta integrals $\mathcal{Z}(s,\phi_\nu,\omega_\nu)$ for any $\phi_\nu\in\mathcal{S}_{\pi_\nu}(k_\nu^\times)$, and any unitary character $\omega_\nu\in\Omega_\nu^\wedge$. By Theorem 3.4, it converges absolutely for $\mathrm{Re}(s)$ sufficiently positive and admits a meromorphic continuation to $s\in\mathbb{C}$. For each $\nu\in|k|_f$, we take c_{π_ν} to be a sufficiently positive real number, such that $\mathcal{Z}(s,\phi_\nu,\omega_\nu)$ converges absolutely for $\mathrm{Re}(s)>c_{\pi_\nu}$. If $\nu\notin S_\pi$, then π_ν is unramified. In this case, the zeta integral $\mathcal{Z}(s,\phi_\nu,\omega_\nu)$ converges absolutely for $\mathrm{Re}(s)>\kappa_\pi$, where the positive real number κ_π depends on π only, according to Assumption 5.1. Hence, if we take a positive real number c_π with

$$c_\pi:=\max\{\kappa_\pi,c_{\pi_\nu}\mid\nu\in S_\pi\cap|k|_f\},\tag{5-4}$$

then for any $\phi_\nu\in\mathcal{S}_{\pi_\nu}(k_\nu^\times)$, and any unitary character $\omega_\nu\in\Omega_\nu^\wedge$, the local zeta integral $\mathcal{Z}(s,\phi_\nu,\omega_\nu)$ converges absolutely for $\mathrm{Re}(s)>c_\pi$ at all finite local places $\nu\in|k|_f$.

By the Mellin inversion formula as displayed in (2-6), we have

$$\phi_\nu(x)\cdot|x|_\nu^d=\sum_{\omega_\nu\in\Omega_\nu^\wedge}\big(\mathrm{Res}_{z=0}(\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)|x|_\nu^{-s}q_\nu^s)\big)\omega_\nu(\mathrm{ac}(x))^{-1},\tag{5-5}$$

where $z=q_\nu^{-s}$ and $d>c_\pi$. Since the summation on the right-hand side is finite, it suffices to show that the limit formula

$$\lim_{|x|_\nu\to0}\mathrm{Res}_{z=0}(\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)|x|_\nu^{-s}q_\nu^s)=0\tag{5-6}$$

holds for each $\omega_\nu\in\Omega_\nu^\wedge$.

It is clear that $\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)$ is holomorphic for $\mathrm{Re}(s)>-(d-c_\pi)$. By Theorem 3.4, we have

$$\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)=p_\nu(s)\cdot L(s+d,\pi_\nu\times\omega_\nu),$$

where $p_\nu(s)\in\mathbb{C}[q_\nu^s,q_\nu^{-s}]$, depending on ϕ_ν. By the supercuspidal support of $\pi_\nu\otimes\omega_\nu$, we obtain that the representation $\pi_\nu\otimes\omega_\nu$ can be embedded, as an irreducible subrepresentation, into the induced representation

$$\pi_\nu\otimes\omega_\nu\hookrightarrow\Pi_\nu:=\mathrm{Ind}_{P(k_\nu)}^{G_n(k_\nu)}\tau_{\nu,1}\otimes\cdots\otimes\tau_{\nu,t_\nu},$$

where $\tau_{\nu,j}$ is an irreducible supercuspidal representation of $G_{a_{\nu,j}}(k_\nu)$ with $n=a_{\nu,1}+\cdots+a_{\nu,t_\nu}$ (see [22]). By [16, Theorem 3.4], we have

$$L(s,\Pi_\nu)=L(s,\tau_{\nu,1})\cdots L(s,\tau_{\nu,t_\nu}).$$

By [16, Corollary 3.6], we have

$$\frac{L(s,\pi_\nu\times\omega_\nu)}{L(s,\Pi_\nu)}$$

is a polynomial in q_ν^{-s}. Hence, we obtain that for the given $\phi_\nu\in\mathcal{S}_{\pi_\nu}(k_\nu^\times)$, there exists a polynomial $\mathcal{P}_\nu(s)$ in q_ν^s and q_ν^{-s}, depending on $\pi_\nu\otimes\omega_\nu$ and ϕ_ν, such that

$$\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)=\mathcal{P}_\nu(s)L(s+d,\Pi_\nu). \tag{5-7}$$

By applying [16, Proposition 5.11] to the local L-functions $L(s,\tau_{\nu,j})$, we obtain that $L(s,\tau_{\nu,j})=1$ when $\tau_{\nu,j}$ is either supercuspidal ($a_{\nu,j}\ge 2$) or a ramified character ($a_{\nu,j}=1$). Hence, there exists an integer $1\le r_\nu\le t_\nu\le n$, such that

$$\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)=\mathcal{P}_\nu(s)\prod_{j=1}^{r_\nu}\frac{1}{1-q_\nu^{-s-d+s_{\nu,j}}}=\prod_{j=1}^{r_\nu}\Bigl(\sum_{\ell_j=0}^{\infty}q_\nu^{-(s+d-s_{\nu,j})\ell_j}\Bigr) \tag{5-8}$$

for some $s_{\nu,j}\in\mathbb{C}$, with $j=1,2,\dots,r_\nu$.

Now we are ready to discuss the limit in (5-6). For $z=q_\nu^{-s}$, we have

$$\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)|x|_\nu^{-s}q_\nu^s=\mathcal{P}_\nu(z)\cdot\frac{\prod_{j=1}^{r_\nu}\bigl(\sum_{\ell_j=0}^{\infty}q_\nu^{-\ell_j(d-s_{\nu,j})}\cdot z^{\ell_j}\bigr)}{z^{\mathrm{ord}_\nu(x)+1}}, \tag{5-9}$$

where $\mathcal{P}_\nu(z)$ is a polynomial function in z,z^{-1}. By taking the residue at $z=0$, we obtain that

$$\mathrm{Res}_{z=0}(\mathcal{Z}(s+d,\phi_\nu,\omega_\nu)|x|_\nu^{-s}q_\nu^s)=\mathfrak{C}_0(x), \tag{5-10}$$

where $\mathfrak{C}_0(x)$ is the coefficient of the constant term of

$$\mathcal{P}_\nu(z)\cdot\frac{\prod_{j=1}^{r_\nu}\bigl(\sum_{\ell_j=0}^{\infty}q_\nu^{-\ell_j(d-s_{\nu,j})}\cdot z^{\ell_j}\bigr)}{z^{\mathrm{ord}_\nu(x)}}. \tag{5-11}$$

Since $\mathcal{P}_\nu(z)$ is a polynomial function in z,z^{-1} with degree depending on π, without loss of generality, we may assume that $\mathcal{P}_\nu(z)\equiv 1$ when we compute $\mathfrak{C}_0(x)$. In this case, the constant term of (5-11) with $\mathcal{P}_\nu(z)\equiv 1$ is equal to

$$\sum_{\substack{\ell_1+\cdots+\ell_{r_\nu}=\mathrm{ord}_\nu(x)\\ \ell_1,\dots,\ell_{r_\nu}\ge 0}} q_\nu^{-\ell_1(d-s_{\nu,j})-\cdots-\ell_{t_0}(d-s_{\nu,j})}. \tag{5-12}$$

When $\nu\notin S_\pi$, π_ν is unramified,

$$\mathrm{diag}(q_\nu^{s_{\nu,1}},\dots,q_\nu^{s_{\nu,n}})=c(\pi_\nu)$$

is the Frobenius–Hecke conjugacy class associated to π_ν in $G_n^\vee(\mathbb{C})$ with $s_{\nu,j}=s_j(\pi_\nu)$ for $j=1,2,\dots,n$. By Assumption 5.1 and the definition of the positive real number

c_π as in (5-4), we take $d_0 = 0$ and have

$$d - \mathrm{Re}(s_{\nu,j}) > c_\pi - \mathrm{Re}(s_{\nu,j}) \geq 0 \tag{5-13}$$

for all $j = 1, 2, \ldots, n$. For the remaining finite local places ν, we may choose a positive real number d_0 such that

$$d + d_0 - \mathrm{Re}(s_{\nu,j}) > c_\pi + d_0 - \mathrm{Re}(s_{\nu,j}) \geq 0 \tag{5-14}$$

for all $j = 1, 2, \ldots, r_\nu$ and all $\nu \in S_\pi \cap |k|_f$. Hence, with the choice of d_0, we have

$$\begin{aligned} &\left|\mathrm{Res}_{z=0}(\mathcal{Z}(s+d+d_0, \phi_\nu, \omega_\nu)|x|_\nu^{-s} q_\nu^s)\right| \\ &\qquad \leq \sum_{\substack{\ell_1+\cdots+\ell_{r_\nu}=\mathrm{ord}_\nu(x) \\ \ell_1,\ldots,\ell_{r_\nu}\geq 0}} q_\nu^{-\sum_{j=1}^{r_\nu} \ell_j (d+d_0-\mathrm{Re}(s_{\nu,j}))} \\ &\qquad \leq \sum_{\substack{\ell_1+\cdots+\ell_{r_\nu}=\mathrm{ord}_\nu(x) \\ \ell_1,\ldots,\ell_{r_\nu}\geq 0}} q_\nu^{-\mathrm{ord}_\nu(x)(d+d_0-\max_j\{\mathrm{Re}(s_{\nu,j})\})} \\ &\qquad = \binom{\mathrm{ord}_\nu(x)+r_\nu-1}{r_\nu-1} \cdot q_\nu^{-\mathrm{ord}_\nu(x)(d+d_0-\max_j\{\mathrm{Re}(s_{\nu,j})\})}. \end{aligned} \tag{5-15}$$

Since $d + d_0 - \max_j\{\mathrm{Re}(s_{\nu,j})\} > 0$, and the function $\binom{\mathrm{ord}_\nu(x)+r_\nu-1}{r_\nu-1}$ is a polynomial in $\mathrm{ord}_\nu(x)$, we must have that

$$\lim_{\mathrm{ord}_\nu(x)\to+\infty} \binom{\mathrm{ord}_\nu(x)+r_\nu-1}{r_\nu-1} \cdot q_\nu^{-\mathrm{ord}_\nu(x)(d+d_0-\max_j\{\mathrm{Re}(s_{\nu,j})\})} = 0.$$

By (5-5), if $d > c_\pi + d_0$, then we must have that

$$\lim_{x_\nu\to 0} \phi_\nu(x) \cdot |x|_\nu^d = 0$$

for all $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ and at all $\nu \in |k|_f$. It is clear that the function $\phi_\nu(x) \cdot |x|_\nu^d$ is continuous over k_ν and has compact support.

Finally, by taking a positive real number $s_\pi = \max\{s_\infty, c_\pi + d_0\}$, we obtain that for any $a_0 > s_\pi$, the function $\phi_\nu(x)|x|_\nu^{a_0}$ is continuous over k_ν and has the limit

$$\lim_{|x|_\nu\to 0} \phi_\nu(x)|x|_\nu^{a_0} = 0,$$

for any $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ and at any local place $\nu \in |k|$. We are done. $\square$

Lemma 5.3. *Let $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ satisfy Assumption 5.1. For any $\nu \notin S_\pi$, the basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ is supported on $\mathfrak{o}_\nu - \{0\}$ with*

$$\mathbb{L}_{\pi_\nu}(\mathfrak{o}_\nu^\times) = 1.$$

There exists a positive real number $b_\pi \geq s_\pi$, which is independent of ν, such that, for any $b_0 > b_\pi$,

$$\left|\mathbb{L}_{\pi_\nu}(x)\cdot|x|_\nu^{b_0}\right| \leq 1$$

holds, as a function in $x \in k_\nu^\times$, for all $\nu \notin S_\pi$.

Proof. We continue with the proof of Lemma 5.2 for the non-Archimedean case, and specialize it to the unramified situation. Note that the basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ is the Mellin inversion of the local unramified L-factor

$$\mathcal{Z}(s, \mathbb{L}_{\pi_\nu}) = L(s, \pi_\nu),$$

whose Mellin inversion can be calculated by (5-5) after setting $\mathcal{P}_\nu(s) = 1$. In other words, taking the constant s_π as in Lemma 5.2, we have, for any $a_0 > s_\pi$,

$$\mathbb{L}_{\pi_\nu}(x)\cdot|x|^{a_0} = \mathrm{Res}_{z=0}(\mathcal{Z}(s+a_0, \mathbb{L}_{\pi_\nu})|x|_\nu^{-s}q_\nu^s).$$

As in (5-9), we write

$$\text{(5-16)}\quad \mathcal{Z}(s+a_0, \mathbb{L}_{\pi_\nu}) = \frac{1}{\prod_{j=1}^n (1-q_\nu^{-s-a_0+s_j(\pi_\nu)})} = \prod_{j=1}^n \biggl(\sum_{\ell_j \geq 0} q_\nu^{\ell_j(s_j(\pi_\nu)-a_0)} z^{\ell_j}\biggr),$$

where we write $z = q_\nu^{-s}$ and $c_j(\pi_\nu) = q_\nu^{s_j(\pi_\nu)}$. From the Laurent expansion on the right-hand side, we obtain that the function

$$\mathcal{Z}(s+a_0, \mathbb{L}_{\pi_\nu})|x|_\nu^{-s}q_\nu^s$$

is holomorphic in $z = q_\nu^{-s}$ whenever $x \notin \mathfrak{o}_\nu$. By taking the residue at $z = 0$, we obtain that

$$\mathbb{L}_{\pi_\nu}(x)\cdot|x|^{a_0} = 0 \quad \text{for } x \notin \mathfrak{o}_\nu.$$

Hence, the basic function $\mathbb{L}_{\pi_\nu}(x)$ has support included in $\mathfrak{o}_\nu$. Similarly, we apply the Mellin inversion, as calculated by (5-5), to the case $x \in \mathfrak{o}^\times$, and obtain that the residue picks up the constant term of the right-hand side of (5-16) as a function of $z = q^{-s}$, which is equal to 1. Therefore, we obtain

$$\mathbb{L}_{\pi_\nu}(\mathfrak{o}_\nu^\times) = 1.$$

Finally, whenever $x \in \mathfrak{o}_F \setminus \{0\}$, we apply (5-15) to the unramified case, and obtain that

$$\left|\mathbb{L}_{\pi_\nu}(x)\cdot|x|^b\right| \leq \binom{\mathrm{ord}_\nu(x)+n-1}{n-1}\cdot q_\nu^{-\mathrm{ord}_\nu(x)\cdot\min_j\{b-\mathrm{Re}(s_j(\pi_\nu))\}},$$

as long as $b > s_\pi$. By Assumption 5.1, we have

$$\min_{1\leq j\leq n}\{b-s_j(\pi_\nu)\} > \min_{1\leq j\leq n}\{\kappa_\pi - s_j(\pi_\nu)\} > 0.$$

Therefore, whenever $\mathrm{ord}_\nu(x) \geq 1$,

$$\binom{\mathrm{ord}_\nu(x)+n-1}{n-1} \cdot q_\nu^{-\mathrm{ord}_\nu(x)\cdot \min_j\{b-\mathrm{Re}(s_j(\pi_\nu))\}} \leq \binom{\mathrm{ord}_\nu(x)+n-1}{n-1} \cdot 2^{-\mathrm{ord}_\nu(x)\cdot \min_j\{b-\mathrm{Re}(s_j(\pi_\nu))\}}$$

since $q_\nu \geq 2$ for any $\nu \notin S_\pi$. It turns out that we only need to find a positive integer $b_\pi \geq s_\pi \in \mathbb{R}$ such that, for any $b > b_\pi$,

$$\binom{\mathrm{ord}_\nu(x)+n-1}{n-1} \cdot 2^{-\mathrm{ord}_\nu(x)\cdot \min_j\{b-\mathrm{Re}(s_j(\pi_\nu))\}} \leq 1$$

holds for any $\nu \notin S_\pi$ and $\mathrm{ord}_\nu(x) \geq 1$. Equivalently, after applying the function $\log_2$ on both sides, the above inequality becomes

$$\log_2 \binom{\mathrm{ord}_\nu(x)+n-1}{n-1} - \mathrm{ord}_\nu(x) \cdot \min_j\{b - \mathrm{Re}(s_j(\pi_\nu))\} \leq 0.$$

Hence, it suffices to show the existence of $b_\pi \in \mathbb{R}$ so that

$$\min_j\{b - \mathrm{Re}(s_j(\pi_\nu))\} = b - \max_j\{\mathrm{Re}(s_j(\pi_\nu))\} > b_\pi - \kappa_\pi \geq \frac{\log_2 \binom{\mathrm{ord}_\nu(x)+n-1}{n-1}}{\mathrm{ord}_\nu(x)}$$

for any $\mathrm{ord}_\nu(x) \geq 1$, i.e.,

$$b_\pi \geq \kappa_\pi + \frac{\log_2 \binom{\mathrm{ord}_\nu(x)+n-1}{n-1}}{\mathrm{ord}_\nu(x)} \tag{5-17}$$

for any $\mathrm{ord}_\nu(x) \geq 1$. As a function of $t \geq 1$,

$$\log_2 \binom{t+n-1}{n-1} = \log_2 \frac{\prod_{k=1}^{n-1}(t+k)}{(n-1)!} \geq \log_2 \frac{\prod_{k=1}^{n-1}(1+k)}{(n-1)!} \geq \log_2 n \geq 0.$$

Thus we obtain that

$$\frac{\log_2 \binom{t+n-1}{n-1}}{t} \geq 0$$

for any $t \geq 1$. On the other hand, by L'Hôspital's rule, one must have that

$$\lim_{t\to\infty} \frac{\log_2 \binom{t+n-1}{n-1}}{t} = 0.$$

It follows, as a continuous function in $t \geq 1$, there exists a constant $c_0 \in \mathbb{R}$ such that

$$\frac{\log_2 \binom{t+n-1}{n-1}}{t} < c_0$$

for any $t \geq 1$. It is clear now that the inequality in (5-17) holds for any

$$b_\pi \geq \kappa_\pi + c_0.$$

Therefore it suffices to take $b_\pi = \max\{s_\pi, \kappa_\pi + c_0\}$. We are done. □

We are ready to establish the first property for the π-theta functions $\Theta_\pi(x, \phi)$ in such generality.

Theorem 5.4 (convergence of π-theta functions). *Fix any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ with Assumption 5.1. Then, for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, the π-theta function*

$$\Theta_\pi(x, \phi) := \sum_{\alpha \in k^\times} \phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$.

Proof. For any $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, let S_π be a finite subset of local places of k containing $|k|_\infty$ and for any finite local place $\nu \notin S_\pi$, the local component π_ν is unramified. We may assume that $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ is a pure restricted tensor of the form

$$\phi = \left(\bigotimes_{\nu \notin S_\pi} \mathbb{L}_{\pi_\nu}\right) \otimes \left(\bigotimes_{\nu \in S_\pi} \phi_\nu\right) = \phi_\infty \otimes \phi_f \tag{5-18}$$

with $\phi_\nu \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$ for all $\nu \in S_\pi$, $\phi_\infty = \bigotimes_{\nu \in |k|_\infty} \phi_\nu$ and $\phi_f = \bigotimes_{\nu \in |k|_f} \phi_\nu$.

Fix a positive real number $s_0 > b_\pi \geq s_\pi \geq \kappa_\pi$ where the constants κ_π, s_π, and b_π are as given in Assumption 5.1, Lemma 5.2, and Lemma 5.3, respectively. By Lemma 5.3, for any $\nu \notin S_\pi$, we have the function $\mathbb{L}_{\pi_\nu}(x)\,|x|_\nu^{s_0}$ is continuous on k_ν and supported on $\mathfrak{o}_\nu$. We have

$$\left|\mathbb{L}_{\pi_\nu}(x)\,|x|_\nu^{s_0}\right| \leq 1 \tag{5-19}$$

for every $\nu \notin S_\pi$. Similarly, for any finite $\nu \in S_\pi \cap |k|_f$, the function $\phi_\nu(x)\,|x|_\nu^{s_0}$ is continuous on k_ν with compact support. We may assume that the support of $\phi_\nu(x)\,|x|_\nu^{s_0}$ is contained in a fractional ideal $\mathfrak{p}_\nu^{m_\nu}$ for some integer $m_\nu \in \mathbb{Z}$. Write $\mathfrak{o}_\phi := \prod_{\nu \notin S_\pi} \mathfrak{o}_\nu$ and $\mathfrak{m}_\phi := \prod_{\nu \in S_\pi \cap |k|_f} \mathfrak{p}^{m_\nu}$. Then, by the weak approximation theorem [48], the product

$$\mathfrak{m}(\phi) := \mathfrak{o}_\phi \cdot \mathfrak{m}_\phi \tag{5-20}$$

is a fractional ideal of $\mathfrak{o} = \mathfrak{o}_k$, the ring of integers in k.

For any $\alpha \in k^\times$, the Artin product formula shows that $|\alpha|_{\mathbb{A}} = 1$ [48]. Hence, we obtain that

$$\Theta_\pi(1, \phi) = \sum_{\alpha \in k^\times} \phi(\alpha) = \sum_{\alpha \in k^\times} \phi(\alpha) \cdot |\alpha|_{\mathbb{A}}^{s_0}. \tag{5-21}$$

From the support of the functions $\phi_\nu \cdot |\cdot|^{s_0}$ for all $\nu \in |k|_f$, we write

$$\Theta_\pi(1,\phi) = \sum_{\alpha\in k^\times\cap\mathfrak{m}(\phi)} (\phi_\infty(\alpha)\cdot|\alpha|_\infty^{s_0})\cdot(\phi_f(\alpha)\cdot|\alpha|_f^{s_0}). \tag{5-22}$$

It is clear that for $\alpha \in k^\times \cap \mathfrak{m}(\phi)$, we have that

$$\begin{aligned}\left|\phi_f(\alpha)\cdot|\alpha|_f^{s_0}\right| &= \Bigl(\prod_{\nu\notin S_\pi}\left|\mathbb{L}_{\pi_\nu}(\alpha)\cdot|\alpha|_\nu^{s_0}\right|\Bigr)\cdot\Bigl(\prod_{\nu\in S_\pi\cap|k|_f}\left|\phi_\nu(\alpha)\cdot|\alpha|_\nu^{s_0}\right|\Bigr)\\ &\le \prod_{\nu\in S_\pi\cap|k|_f}\left|\phi_\nu(\alpha)\cdot|\alpha|_\nu^{s_0}\right|,\end{aligned}$$

because of (5-19). By Lemma 5.2, there exists a real constant c_ϕ, such that

$$\prod_{\nu\in S_\pi\cap|k|_f}\left|\phi_\nu(\alpha)\cdot|\alpha|_\nu^{s_0}\right| \le c_\phi. \tag{5-23}$$

Hence, we obtain that

$$\left|\Theta_\pi(1,\phi)\right| \le c_\phi\cdot\sum_{\alpha\in k^\times\cap\mathfrak{m}(\phi)}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0}. \tag{5-24}$$

Since the fractional ideal $\mathfrak{m}(\phi)$ of k is a lattice in $\mathbb{A}_\infty = \prod_{\nu\in|k|_\infty} k_\nu$, it suffices to show that the summation

$$\sum_{\alpha\in\mathfrak{m}(\phi)}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0} \tag{5-25}$$

is absolutely convergent.

Consider the compact set

$$\mathcal{B}_\infty(1) := \{(\alpha_\nu)\in\mathbb{A}_\infty \mid |\alpha_\nu|_\nu \le 1,\ \forall\nu\in|k|_\infty\}. \tag{5-26}$$

We write (5-25) as

$$\sum_{\alpha\in\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1)}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0} + \sum_{\alpha\in\mathfrak{m}(\phi)\smallsetminus(\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1))}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0}. \tag{5-27}$$

It is clear that the intersection of $\mathfrak{m}(\phi)$ with $\mathcal{B}_\infty(1)$ is a finite set. By Lemma 5.2, the function $\phi_\infty(x)|x|_\infty^{s_0}$ is continuous over $\mathbb{A}_\infty$, and hence is bounded over $\mathcal{B}_\infty(1)$. Thus, in (5-27), the first summation

$$\sum_{\alpha\in\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1)}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0}$$

is bounded. The second summation in (5-27), which is

$$\sum_{\alpha\in\mathfrak{m}(\phi)\smallsetminus(\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1))}|\phi_\infty(\alpha)|\cdot|\alpha|_\infty^{s_0},$$

where the function $\phi_\infty(x)\cdot|x|_\infty^{s_0}$ is of Schwartz type over $\mathbb{A}_\infty \smallsetminus \mathcal{B}_\infty(1)$, is bounded by the same proof for the absolute convergence of the classical Poisson summation formula [21, Chapter 4; 47]. This proves the absolute convergence of $\Theta_\pi(x,\phi)$ for any $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$.

For any $x\in\mathbb{A}^\times$, we have $\Theta_\pi(x,\phi)=\Theta_\pi(1,\phi^x)$ with $\phi^x(y)=\phi(yx)$. Hence, $\Theta_\pi(x,\phi)$ converges absolutely for any $\phi\in\mathcal{S}_\pi(\mathbb{A}^\times)$.

For the locally uniform convergence of the π-theta function $\Theta_\pi(x,\phi)$ at any $x\in\mathbb{A}^\times$, by using $\Theta_\pi(x,\phi)=\Theta_\pi(1,\phi^x)$ again, it is enough to show the locally uniform convergence of $\Theta_\pi(x,\phi)$ at $x=1$ for any given factorizable function ϕ as in (5-18). As in (5-21), we may write

$$\Theta_\pi(x,\phi)=\sum_{\alpha\in k^\times}\phi(\alpha x)\cdot|\alpha|_{\mathbb{A}}^{s_0}. \tag{5-28}$$

Since $\phi=\left(\bigotimes_{\nu\notin S_\pi}\mathbb{L}_{\pi_\nu}\right)\otimes\left(\bigotimes_{\nu\in S_\pi}\phi_\nu\right)$ as in (5-18), we have $\mathfrak{m}(\phi)=\prod_{\nu\in|k|_f}\mathfrak{m}(\phi)_\nu$ as in (5-20), where $\mathfrak{m}(\phi)_\nu$ is a fractional ideal of k_ν containing the support of the function $\phi_\nu(x)\cdot|x|_\nu^{s_0}$. As in (5-22), we write

$$\Theta_\pi(x,\phi)=\sum_{\alpha\in k^\times\cap\mathfrak{m}(\phi)}(\phi_\infty(\alpha x_\infty)\cdot|\alpha|_\infty^{s_0})\cdot(\phi_f(\alpha x_f)\cdot|\alpha|_f^{s_0}). \tag{5-29}$$

Take a compact open neighborhood $\Omega_f(\phi)$ of $x_f=1$ in $\mathbb{A}_f^\times$ to be

$$\Omega_f(\phi)=\Big(\prod_{\nu\notin S_\pi}\mathfrak{o}_\nu^\times\Big)\cdot\Big(\prod_{\nu\in|k|_f\cap S_\pi}(1+\mathfrak{p}_\nu^{d_\nu})\Big),$$

where d_ν is a positive integer for $\nu\in|k|_f\cap S_\pi$. For any $x_f\in\Omega_f(\phi)$, if $\nu\notin S_\pi$, then $x_\nu\in\mathfrak{o}_\nu^\times$ and $\alpha\neq 0$ and $\alpha\in\mathfrak{o}_\nu$. Hence, $\alpha x_\nu\neq 0$ and $\alpha x_\nu\in\mathfrak{o}_\nu$. In this case, we have that

$$\left|\phi_\nu(\alpha x_\nu)\cdot|\alpha|_\nu^{s_0}\right|=\left|\mathbb{L}_{\pi_\nu}(\alpha x_\nu)\cdot|\alpha x_\nu|_\nu^{s_0}\right|\le 1$$

by (5-19). If $\nu\in S_\pi\cap|k|_f$, then $\alpha\in\mathfrak{p}_\nu^{m_\nu}$ and $x_\nu\in 1+\mathfrak{p}_\nu^{d_\nu}$, and hence we have that $\alpha x_\nu\in\mathfrak{p}_\nu^{m_\nu}$. In this case, we have that

$$\left|\phi_\nu(\alpha x_\nu)\cdot|\alpha|_\nu^{s_0}\right|=\left|\phi_\nu(\alpha x_\nu)\cdot|\alpha x_\nu|_\nu^{s_0}\right|.$$

As in (5-23), there exists a real constant c_ϕ, which is independent of $x_f\in\Omega_f(\phi)$, such that

$$\left|\phi_f(\alpha x_f)\cdot|\alpha|_f^{s_0}\right|\le\prod_{\nu\in S_\pi\cap|k|_f}\left|\phi_\nu(\alpha x_\nu)\cdot|\alpha|_\nu^{s_0}\right|\le c_\phi.$$

Hence, we obtain that

$$|\Theta_\pi(x,\phi)|\le c_\phi\cdot\sum_{\alpha\in k^\times\cap\mathfrak{m}(\phi)}\left|\phi_\infty(\alpha x_\infty)\cdot|\alpha|_\infty^{s_0}\right|\le c_\phi\cdot\sum_{\alpha\in\mathfrak{m}(\phi)}\left|\phi_\infty(\alpha x_\infty)\cdot|\alpha|_\infty^{s_0}\right|. \tag{5-30}$$

When x_∞ runs over a compact neighborhood Ω_∞ of 1 in $\mathbb{A}_\infty$, by the same argument, we are reduced to showing that

$$\sum_{\alpha\in\mathfrak{m}(\phi)\smallsetminus(\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1))} |\phi_\infty(\alpha x_\infty)|\cdot|\alpha|_\infty^{s_0} = |x_\infty|_\infty^{-s_0}\cdot \sum_{\alpha\in\mathfrak{m}(\phi)\smallsetminus(\mathfrak{m}(\phi)\cap\mathcal{B}_\infty(1))} |\phi_\infty(\alpha x_\infty)|\cdot|\alpha x_\infty|_\infty^{s_0}$$

converges uniformly. Since the function $\phi_\infty(x)\cdot|x|_\infty^{s_0}$ is of Schwartz type over $\mathbb{A}_\infty\smallsetminus\mathcal{B}_\infty(1)$, the uniform convergence of the last summation with $x_\infty\in\Omega_\infty$ follows from the same proof of that for the classical theta functions. We omit the details and finish the proof. $\square$

5B. ***Justification of Assumption 5.1.*** We prove Assumption 5.1 when $\pi\in\mathcal{A}(G_n)$ is any irreducible admissible automorphic representation of $G_n(\mathbb{A})$, which is contained in $\Pi_{\mathbb{A}}(G_n)$.

Proposition 5.5. *For any $\pi\in\mathcal{A}(G_n)$, Assumption 5.1 holds.*

Proof. A cuspidal datum (P,ε) of G_n consists of a standard parabolic subgroup P of G_n with Levi decomposition $P=M\cdot N$ with the Levi subgroup M and the unipotent radical N, and an irreducible cuspidal automorphic representation ε of $M(\mathbb{A})$, which is square integrable up to a twist of automorphic character of $M(\mathbb{A})$. For any $\pi=\bigotimes_{\nu\in|k|}\pi_\nu\in\mathcal{A}(G_n)$, by [30], there exists a cuspidal datum (P,ε) of G_n, such that π can be realized as an irreducible subquotient of the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})}(\varepsilon)$ of $G_n(\mathbb{A})$. It follows that for any $\nu\in|k|$, the ν-component π_ν can be realized as an irreducible subquotient of the induced representation $\mathrm{Ind}_{P(k_\nu)}^{G_n(k_\nu)}(\varepsilon_\nu)$ of $G_n(k_\nu)$, where ε_ν is the ν-component of $\varepsilon=\bigotimes_\nu\varepsilon_\nu$.

Let T be the maximal torus of G_n, consisting of all diagonal matrices, and $B=T\cdot U$ be the Borel subgroup of G_n, consisting of all upper-triangular matrices. Take S to be a finite subset of $|k|$, such that S contains $|k|_\infty$ and for any $\nu\notin S$, π_ν and ε_ν are unramified. It is well known (see [7], for instance) that there exists an unramified character η_ν of the maximal torus $T(k_\nu)$, such that ε_ν embeds as a subrepresentation into the unramified induced representation $\mathrm{Ind}_{(M\cap B)(k_\nu)}^{M(k_n u)}(\eta_\nu)$. By induction in stages, we have $\mathrm{Ind}_{P(k_\nu)}^{G_n(k_\nu)}(\varepsilon_\nu)$ embeds as a subrepresentation into the spherical induced representation $\mathrm{Ind}_{B(k_\nu)}^{G_n(k_\nu)}(\eta_\nu)$ of $G_n(k_\nu)$. Hence, the irreducible spherical representation π_ν is the unique spherical subquotient of $\mathrm{Ind}_{B(k_\nu)}^{G_n(k_\nu)}(\eta_\nu)$. Via the Satake isomorphism, the Frobenius–Hecke conjugacy class of π_ν in $G_n(\mathbb{C})$ is

$$c(\pi_\nu)=\mathrm{diag}(\eta_\nu^1(\varpi_\nu),\dots,\eta_\nu^n(\varpi_\nu)).$$

Here ϖ_ν is the uniformizer of the prime ideal $\mathfrak{p}_\nu$, and for any $t=\mathrm{diag}(t_1,\dots,t_n)\in T(k_\nu)$, the unramified character η_ν is given by

$$\eta_\nu(t)=\eta_\nu^1(t_1)\cdots\eta_\nu^n(t_n).$$

It is clear that the conjugacy class of the semisimple element $c(\pi_\nu)$ in the complex dual group $M^\vee(\mathbb{C})$ of the Levi subgroup M is the Frobenius–Hecke conjugacy class $c(\varepsilon_\nu)$ of ε_ν. In other words, both π_ν and ε_ν share the same Satake parameter in $T^\vee(\mathbb{C})^{W_n}$, where W_n is the Weyl group of $G_n^\vee(\mathbb{C})$.

Take δ_ε to be an automorphic character of $M(\mathbb{A})$ such that $\delta_\varepsilon \otimes \varepsilon$ is square integrable modulo the center of M. Then for $\nu \notin S$, the ν-component $(\delta_\varepsilon \otimes \varepsilon)_\nu$ is spherical and unitary. By the classification of the spherical unitary dual of GL_n over a non-Archimedean local field k_ν [42], we obtain

$$\left|\log_{q_\nu} \max_{1\le j\le n} \left\{|(\delta_\varepsilon)_\nu^j(\varpi_\nu)\eta_\nu^j(\varpi_\nu)|\right\}\right| \le \frac{n-1}{2}.$$

Since the unramified part of the automorphic character δ_ε is completely determined by ε and the cuspidal datum (P, ε) of π is uniquely determined by π, up to conjugation, we obtain that there exists a positive real number κ_π, depending only on $\pi \in \mathcal{A}(G_n)$, such that

$$\left|\log_{q_\nu} \max_{1\le j\le n} \left\{|\eta_\nu^j(\varpi_\nu)|\right\}\right| < \kappa_\pi.$$

This justifies the assumption. $\square$

By Theorem 5.4 and Proposition 5.5, we obtain the following absolute convergence.

Corollary 5.6. *For any $\pi \in \mathcal{A}(G_n)$ and for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, the π-theta function*

$$\Theta_\pi(x, \phi) = \sum_{\alpha\in k^\times} \phi(\alpha x)$$

converges absolutely and locally uniformly as a function in $x \in \mathbb{A}^\times$.

Another consequence of Proposition 5.5 is the absolute convergence of the global zeta integral of Godement–Jacquet type for any $\pi \in \mathcal{A}(G_n)$.

Corollary 5.7. *For any $\pi \in \mathcal{A}(G_n)$, there exists a positive real number $r_\pi \in \mathbb{R}$, such that the global zeta integral*

$$\mathcal{Z}(s, f, \varphi_\pi) = \int_{\mathrm{GL}_n(\mathbb{A})} f(g)\varphi_\pi(g)|\det g|_{\mathbb{A}}^{s+(n-1)/2}\, \mathrm{d}g, \quad f \in \mathcal{S}(M_n(\mathbb{A})),\ \varphi_\pi \in \mathcal{C}(\pi)$$

is absolutely convergent for any $\mathrm{Re}(s) > r_\pi$.

Proof. There is no harm to assume that $f = \bigotimes_\nu f_\nu$ is a pure restricted tensor. Similarly, one can write $\varphi_\pi = \prod_\nu \varphi_{\pi_\nu}$. For the given $\pi \in \mathcal{A}(G_n)$, take the finite subset S of $|k|$ as in the proof of Proposition 5.5. Then for $\nu \notin S$, the function f_ν

is the characteristic function of $M_n(\mathfrak{o}_\nu)$, and φ_{π_ν} is the zonal spherical function attached to the unramified representation π_ν. From [16, Chapter I, §7], we have

$$\mathcal{Z}(s, f_\nu, \varphi_{\pi_\nu}) = \frac{1}{\det(I_n - \alpha(\pi_\nu)q_\nu^{-s})} = L(s, \pi_\nu),$$

where the left-hand side is absolutely convergent whenever $\mathrm{Re}(s) > \kappa_\pi$, where κ_π is determined in the proof of Proposition 5.5. It follows that

$$\prod_{\nu \notin S} \mathcal{Z}(s, f_\nu, \varphi_{\pi_\nu}) = \prod_{\nu \notin S} \frac{1}{\det(I_n - \alpha(\pi_\nu)q_\nu^{-s})} = L^S(s, \pi)$$

is absolutely convergent for $\mathrm{Re}(s) > \kappa_\pi + 1$. As S is a finite set, it is clear that one can choose a real number r_π to be sufficiently positive (depending on π only) such that the global zeta integral

$$\mathcal{Z}(s, f, \varphi_\pi) = L^S(s, \pi) \cdot \prod_{\nu \in S} \mathcal{Z}(s, f_\nu, \varphi_{\pi_\nu})$$

converges absolutely for $\mathrm{Re}(s) > r_\pi$. We are done. □

6. (σ, ρ)-theta functions on GL_1

For any k-split reductive group G, as in Section 4, we denote by $\Pi_{\mathbb{A}}(G)$ the set of irreducible admissible representations of $G(\mathbb{A})$. If we write $\sigma = \bigotimes_{\nu \in |k|} \sigma_\nu$, then we assume that $\sigma_\nu \in \Pi_{k_\nu}(G)$, where at almost all finite local places ν, the local representations σ_ν are unramified. When ν is a finite local place, σ_ν is an irreducible admissible representation of $G(k_\nu)$, and when ν is an infinite local place, we assume that σ_ν is of Casselman–Wallach type as a representation of $G(k_\nu)$. Let $\mathcal{A}(G) \subset \Pi_{\mathbb{A}}(G_n)$ be the subset consisting of equivalence classes of irreducible admissible automorphic representations of $G(\mathbb{A})$, and $\mathcal{A}_{\mathrm{cusp}}(G)$ be the subset of cuspidal members of $\mathcal{A}(G)$.

For any $\sigma \in \Pi_{\mathbb{A}}(G)$ and $\rho : G^\vee \to \mathrm{GL}_n(\mathbb{C})$, we are going to introduce the (σ, ρ)-Schwartz space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$, the (σ, ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ and (σ, ρ)-theta functions $\Theta_{\sigma,\rho}(x, \phi)$ by means of the existence of the local Langlands reciprocity map as in the local Langlands conjecture for G. The idea is to use the local Langlands conjecture for the pair (G, ρ) as input and to formulate the global statements, such as the (σ, ρ)-Poisson summation formula, which is expected to be responsible for the global functional equation for the Langlands L-function $L(s, \sigma, \rho)$ as predicted by the Langlands conjecture, as output. The goal in this section is to prove Theorem 6.2, which contains Theorem 1.2 as a special case and serves a base for the discussion on Conjecture 1.5 and its refinement in Section 7.

6A. ***On the local Langlands conjecture.*** We briefly review the local Langlands conjecture for G over any local field $F = k_\nu$ for any local place $\nu \in |k|$.

For any Archimedean local field, the local Langlands conjecture for G is a theorem of Langlands, which follows from the Langlands classification theory [31]. At any non-Archimedean local places, for unramified representations, their local Langlands parameters are uniquely determined by the Satake isomorphism [7; 38]. In the following we review the local Langlands conjecture for an F-split reductive group G over a non-Archimedean local field F of characteristic zero.

Let $\mathcal{W}_F$ be the Weil group attached to F. The set of local Langlands parameters is denoted by $\Phi_F(G)$, which consists of continuous, Frobenius semisimple homomorphisms

$$\varsigma : \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee, \tag{6-1}$$

up to conjugation by $G^\vee$. The local Langlands conjecture asserts that there exists a reciprocity map

$$\mathfrak{R}_{F,G} : \mathrm{Rep}(G(F)) \to \Phi_F(G), \tag{6-2}$$

where $\mathrm{Rep}(G(F))$ is the set of equivalence classes of smooth representations of $G(F)$ of finite length. $\mathfrak{R}_{F,G}$ is expected to be surjective with finite fibers, and to satisfy a series of compatibility conditions. Beyond the existence, one has to formulate and prove the uniqueness of such a local Langlands reciprocity map.

When $G = \mathrm{GL}_n$, it is a theorem of Harris–Taylor [17], of G. Henniart [19] and of P. Scholze [39] that the local Langlands reciprocity map exists and is unique with compatibility of local factors, plus other conditions. Note that in this case, the uniqueness of such a local Langlands reciprocity map is proved by Henniart using the special case of the local converse theorem [18]. However, such a uniqueness is not known in general. When G is an F-quasisplit classical group, then such a local Langlands reciprocity map exists due to the endoscopic classification of J. Arthur [2].

In their recent work [13], L. Fargues and P. Scholze use the geometrization method to understand the local Langlands conjecture. In particular, they establish a local Langlands reciprocity map for any F-split reductive groups considered in this paper. More precisely, Theorem I.9.6 of [13] asserts that for any F-split reductive group G, there exists a local Langlands reciprocity map $\mathfrak{R}_{F,G}$ from $\mathrm{Rep}(G(F))$ to $\Phi_F(G)$, satisfying nine compatibility conditions. In particular when $G = \mathrm{GL}_n$, the reciprocity map of Fargues and Scholze coincides with the unique one for GL_n. When G is an F-quasisplit classical group, the reciprocity map of Fargues and Scholze coincides with the one by Arthur. Although it is still not known (as far as the authors know) if the reciprocity map of Fargues and Scholze is unique, it is the most promising one towards the local Langlands conjecture in great generality.

From now on, we are going to take the following assumption.

Assumption 6.1. *Over any non-Archimedean local field F of characteristic zero, for any F-split reductive group G, the reciprocity map $\mathfrak{R}_{F,G}$ exists for the local Langlands conjecture for G.*

We may simply take the reciprocity map $\mathfrak{R}_{F,G}$ as defined in [13, Theorem I.9.6] for the local Langlands conjecture. In fact, the relevant discussions in the rest of this paper make no essential difference on which reciprocity map $\mathfrak{R}_{F,G}$ we are going to take. Of course, the difference may occur if one discuss the definition of local L-functions $L(s, \sigma, \rho)$ or γ-functions $\gamma(s, \sigma, \rho, \psi)$. but we are not going to discuss those objects in the rest of this paper.

6B. ***Convergence of (σ, ρ)-theta functions.*** Let G be a k-split reductive group. Take $\rho : G^\vee(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ to be any finite-dimensional representation of the complex dual group $G^\vee(\mathbb{C})$. For any $\sigma \in \Pi_{\mathbb{A}}(G)$, we write $\sigma = \bigotimes_\nu \sigma_\nu$ with $\sigma_\nu \in \Pi_{k_\nu}(G)$. By Assumption 6.1, for any local place $\nu \in |k|$, there exists a local L-parameter $\varsigma_\nu = \varsigma_\nu(\sigma_\nu)$ such that the composition $\rho \circ \varsigma_\nu$ is a local L-parameter for $G_n(k_\nu) = \mathrm{GL}_n(k_\nu)$. By the local Langlands conjecture for GL_n [17; 19; 31; 39], there exists a unique irreducible admissible representation

$$\pi_\nu = \pi_\nu(\sigma, \rho, \mathfrak{R}_{k_\nu,G}) \tag{6-3}$$

belonging to $\Pi_F(G_n)$, which we may simply denote, if there is no confusion, by

$$\pi_\nu = \pi_\nu(\sigma_\nu, \rho). \tag{6-4}$$

According to the Langlands functoriality conjecture, it makes sense to define the (σ_ν, ρ)-*Schwartz space* on $k_\nu^\times$ to be

$$\mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times) := \mathcal{S}_{\pi_\nu}(k_\nu^\times). \tag{6-5}$$

At unramified local places, the (σ_ν, ρ)-basic function $\mathbb{L}_{\sigma_\nu,\rho}$ is taken to be the π_ν-basic function $\mathbb{L}_{\pi_\nu} \in \mathcal{S}_{\pi_\nu}(k_\nu^\times)$. Then we can define the (σ, ρ)-*Schwartz space* on $\mathbb{A}^\times$ to be the restricted tensor product

$$\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times) := \bigotimes_\nu \mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times) \tag{6-6}$$

with respect to the basic function $\mathbb{L}_{\sigma_\nu,\rho}$ at almost all finite local places. Note that the definition of the (σ, ρ)-Schwartz space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ may rely on the assumption of the local Langlands reciprocity map (Assumption 6.1) at the ramified finite local places of σ, when G is a general k-split reductive group.

Let $\psi = \bigotimes_\nu \psi_\nu$ be a nontrivial additive character of $\mathbb{A}$ with $\psi(a) = 1$ for any $a \in k$. Define the (σ_ν, ρ)-*Fourier operator* $\mathcal{F}_{\sigma_\nu,\rho,\psi_\nu}$ on $k_\nu^\times$ to be

$$\mathcal{F}_{\sigma_\nu,\rho,\psi_\nu} := \mathcal{F}_{\pi_\nu,\psi_\nu}, \tag{6-7}$$

which is a linear transformation from the (σ_ν, ρ)-Schwartz space $\mathcal{S}_{\sigma_\nu,\rho}(k_\nu^\times)$ to the $(\widetilde{\sigma_\nu}, \rho)$-Schwartz space $\mathcal{S}_{\widetilde{\sigma_\nu},\rho}(k_\nu^\times)$. Then we define the (σ, ρ)-*Fourier operator*

$$\mathcal{F}_{\sigma,\rho,\psi} := \bigotimes_\nu \mathcal{F}_{\sigma_\nu,\rho,\psi_\nu}, \tag{6-8}$$

which is a linear transformation from the (σ, ρ)-Schwartz space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ to the $(\widetilde{\sigma}, \rho)$-Schwartz space $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$. Again, the definition of the (σ, ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ may rely on the assumption of the local Langlands reciprocity map (Assumption 6.1) at the ramified finite local places of σ, when G is a general k-split reductive group.

Theorem 6.2 (convergence of (σ, ρ)-theta functions). *Let $\rho : G^\vee(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ be any finite-dimensional representation of the complex dual group $G^\vee(\mathbb{C})$. Take Assumption 6.1 for G. Then, for any given unitary $\sigma \in \Pi_{\mathbb{A}}(G)$, the (σ, ρ)-theta function*

$$\Theta_{\sigma,\rho}(x, \phi) := \sum_{\alpha \in k^\times} \phi(\alpha x)$$

converges absolutely and locally uniformly for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and $x \in \mathbb{A}^\times$.

Proof. As discussed above, under Assumption 6.1 for G, for any $\sigma = \bigotimes_\nu \sigma_\nu \in \Pi_{\mathbb{A}}(G)$, we obtain $\pi_\nu = \pi_\nu(\sigma_\nu, \rho)$ of $\mathrm{GL}_n(k_\nu)$ for all $\nu \in |k|$. Note that at $\nu \in |k|_\infty$, π_ν is taken to be of Casselman–Wallach type. Hence, $\pi := \bigotimes_\nu \pi_\nu$ is an irreducible admissible representation of $G_n(\mathbb{A})$ and belongs to $\Pi_{\mathbb{A}}(G_n)$. From (6-5) and (6-6), we have that

$$\Theta_{\sigma,\rho}(x, \phi) = \Theta_\pi(x, \phi)$$

for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times) = \mathcal{S}_\pi(\mathbb{A}^\times)$. By Theorem 5.4, it is sufficient to verify Assumption 5.1 for this π.

Since $\sigma = \bigotimes_\nu \sigma_\nu$ is unitary as a representation of $G(\mathbb{A})$, we must have that σ_ν is an irreducible admissible unitary representation of $G(k_\nu)$ at every $\nu \in |k|$, and is unramified for almost all $\nu \in |k|$. Since G is k-split, we can fix a Borel pair (B, T) of G defined over k, with a fixed maximal k-split torus T of G. Let ϱ be the half-sum of positive roots with respect to the given pair (B, T) and let δ_B be the modular character of $B(k_\nu)$. Then, for any coweight $\omega^\vee \in \mathrm{Hom}(\mathbb{G}_m, T)$,

$$\delta_B(\omega^\vee(\varpi_\nu))^{1/2} = q_\nu^{\langle \varrho, \omega^\vee \rangle},$$

where ϖ_ν is a fixed uniformizer in $\mathfrak{o}_\nu$ and $\omega^\vee$ is viewed as a cocharacter from $k_\nu^\times$ to $T(k_\nu)$.

Let S be a finite subset of $|k|$ containing $|k|_\infty$, such that for any $\nu \notin S$, both σ_ν and π_ν are unramified. For any $\nu \notin S$, σ_ν is unitary and unramified. Then the zonal spherical function attached to σ_ν, which is the normalized matrix coefficient of σ_ν

attached to spherical vectors in σ_ν, is bounded by 1 — see [7, p. 151, (40)], for instance. Now let

$$c(\sigma_\nu) = (q_\nu^{s_1(\sigma_\nu)}, \ldots, q_\nu^{s_r(\sigma_\nu)})$$

be the Frobenius–Hecke conjugacy class of σ_ν inside $T^\vee(\mathbb{C}) \simeq (\mathbb{C}^\times)^r$, where r is the k-rational rank of G. Then, by [36, Theorem 4.7.1],

$$\max_{1\le j\le r}\{|s_j(\sigma_\nu)|\} \le \max_{1\le j\le r}\{|\langle \varrho, \omega_j^\vee\rangle|\},$$

where $\{\omega_j^\vee\}_{j=1}^r$ is a fixed set of fundamental coweights. Note that the result of [36] assumes G to be simple-connected. But if we go over the proof of [36, Theorem 4.7.1], the only result used is the explicit formula for zonal spherical functions when the Frobenius–Hecke conjugacy class $c(\sigma_\nu)$ of σ_ν is nonsingular. Hence, it suffices to apply the general formula appearing in [8, Theorem 4.2] to the proof of [36, Theorem 4.7.1]. Therefore $\max_{1\le j\le r}\{|s_j(\sigma_\nu)|\}$ has an upper bound which is independent of the local places ν.

At unramified local places, we obtain the Frobenius–Hecke conjugacy class $c(\pi_\nu)$ of π_ν to be

$$c(\pi_\nu) = \rho(c(\sigma_\nu))$$

for all $\nu \notin S$. It is clear that for this $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$, the family of the Frobenius–Hecke conjugacy classes

$$\{c(\pi_\nu) \mid \forall \nu \notin S\}$$

associated to the irreducible admissible representation π satisfies Assumption 5.1. We are done. □

Note that the definition of the (σ, ρ)-theta function $\Theta_{\sigma,\rho}(x, \phi)$ may depend on the existence of the local Langlands reciprocity map $\mathfrak{R}_{F,G}$ for general G (Assumption 6.1), However, the absolute convergence of $\Theta_{\sigma,\rho}(x, \phi)$ in Theorem 6.2 only depends on the unramified data, and hence is independent of Assumption 6.1. As a consequence of Theorem 4.7, we have:

Corollary 6.3. *Assume the global Langlands functoriality is valid for* (G, ρ). *For* $\sigma \in \mathcal{A}_{\mathrm{cusp}}(G)$, *if its functorial transfer* π *is cuspidal on* $G_n(\mathbb{A})$, *then Conjecture 1.5 holds with* $\mathcal{E}_{\sigma,\rho}(\phi) = \Theta_{\sigma,\rho}(1, \phi)$ *for any* $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$.

7. Variants of Conjecture 1.5

In Theorem 4.7, we established a π-Poisson summation formula (Conjecture 1.5) for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ and $\rho = \mathrm{std}$. We explore the possibilities when π is not cuspidal.

7A. ***Certain special Schwartz functions.*** As before, we take F to be any local field of characteristic zero. For any $\pi \in \Pi_F(G_n)$, recall from Definition 3.3 the space of π-Schwartz functions

$$\mathcal{S}_\pi(F^\times) = \mathrm{Span}\{\phi_{\xi,\varphi_\pi} \in \mathcal{C}^\infty(F^\times) \mid \xi \in \mathcal{S}_{\mathrm{std}}(G_n(F)),\ \varphi_\pi \in \mathcal{C}(\pi)\},$$

where the π-Schwartz function ϕ_{ξ,φ_π} associated to a pair (ξ, φ_π) is defined in (3-6). We introduce here a subspace of $\mathcal{S}_\pi(F^\times)$:

$$\mathcal{S}^\circ_\pi(F^\times) := \mathrm{Span}\{\phi_{\xi,\varphi_\pi} \mid \xi \in \mathcal{C}^\infty_c(G_n(F)),\ \varphi_\pi \in \mathcal{C}(\pi)\}. \tag{7-1}$$

We prove the following result, which provides a new description of the test functions in $\mathcal{C}^\infty_c(F^\times)$, the space of all smooth, compactly supported functions on $F^\times$.

Theorem 7.1. *Let F be any local field of characteristic zero. For any $\pi \in \Pi_F(G_n)$, the subspace $\mathcal{S}^\circ_\pi(F^\times)$ of $\mathcal{S}_\pi(F^\times)$ as defined in* (7-1) *is equal to the space $\mathcal{C}^\infty_c(F^\times)$, i.e.,*

$$\mathcal{S}^\circ_\pi(F^\times) = \mathcal{C}^\infty_c(F^\times).$$

First of all, via the determinant morphism $\det : G_n \to \mathbb{G}_m$, it is not hard to verify that the fiber integration

$$\xi \mapsto \int_{\det g = x} \xi(g)\, \mathrm{d}_x g$$

yields a surjective homomorphism from $\mathcal{C}^\infty_c(G_n(F))$ to $\mathcal{C}^\infty_c(F^\times)$. For any $\xi \in \mathcal{C}^\infty_c(G_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, the product $\xi(g)\varphi_\pi(g)$ belongs to $\mathcal{C}^\infty_c(G_n(F))$. With the fiber integration through det, the function $\phi_{\xi,\varphi_\pi}(x)$ belongs to $\mathcal{C}^\infty_c(F^\times)$. Hence, we obtain that

$$\mathcal{S}^\circ_\pi(F^\times) \subset \mathcal{C}^\infty_c(F^\times) \tag{7-2}$$

for any $\pi \in \Pi_F(G_n)$. To prove the converse of (7-2), we are going to use different arguments for the non-Archimedean case and the Archimedean case.

We first consider the *non-Archimedean case*. For any quasicharacter $\chi \in \mathfrak{X}(F^\times)$, it can be written in a unique way as $\chi(x) = |x|_F^s \cdot \omega(x)$ with $s \in \mathbb{C}$ and $\omega \in \Omega^\wedge$. We may write $\chi = \chi_{s,\omega}$ and $\mathfrak{X}(F^\times) = \mathbb{C} \times \Omega^\wedge$. Furthermore, we write the space $\mathcal{Z}(\mathfrak{X}(F^\times))$ defined in Definition 2.2 as $\mathcal{Z}(\mathbb{C} \times \Omega^\wedge)$. We denote by $\mathcal{L}_{\mathrm{cpt}}$ the subspace of $\mathcal{Z}(\mathbb{C} \times \Omega^\wedge)$ consisting of functions $\mathfrak{z}(\chi_{s,\omega}) = \mathfrak{z}(s,\omega) \in \mathcal{Z}(\mathbb{C} \times \Omega^\wedge)$ with the two properties

(1) $\mathfrak{z}(s,\omega)$ is nonzero at finitely many $\omega \in \Omega^\wedge$;

(2) for any $\omega \in \Omega^\wedge$, $\mathfrak{z}(s,\omega) \in \mathbb{C}[q^s, q^{-s}]$.

By Theorem 2.3, the subspace $\mathcal{L}_{\mathrm{cpt}}$ is in one-to-one correspondence with $\mathcal{C}^\infty_c(F^\times)$ via the Mellin transform and its inversion. Denote by $\mathcal{L}^\circ_\pi$ the subspace of $\mathcal{L}_{\mathrm{cpt}}$ that is in one-to-one correspondence with the subspace $\mathcal{S}^\circ_\pi(F^\times)$. From the discussion right after [34, Theorem 3.1.1], for any given $\omega \in \Omega^\wedge$, the subspace

$$\mathcal{I}^\circ_{\pi,\omega} := \{\mathcal{Z}(s, \xi, \varphi_\pi, \omega) \mid \xi \in \mathcal{C}^\infty_c(G_n(F)),\ \varphi_\pi \in \mathcal{C}(\pi)\}$$

of the fractional ideal $\mathcal{I}_{\pi,\omega}$ as in Theorem 2.4 is equal to $\mathbb{C}[q^s, q^{-s}]$. For the fixed $\omega \in \Omega^\wedge$, the space $\mathcal{I}^\circ_{\pi,\omega}$ consists of the restriction of functions in $\mathcal{L}_{\mathrm{cpt}}$ to the slice $\mathbb{C} \times \{\omega\}$, according to the definition of the space $\mathcal{L}_{\mathrm{cpt}}$. In other words, for any fixed $\omega \in \Omega^\wedge$ and $\mathfrak{z}(s, \omega) \in \mathcal{L}_{\mathrm{cpt}}$, there exists finitely many $\xi^j_\omega \in \mathcal{C}^\infty_c(G_n(F))$ and $\varphi^j_{\pi,\omega} \in \mathcal{C}(\pi)$, such that

$$\mathfrak{z}(s, \omega) = \sum_j \mathcal{Z}(s, \xi^j_\omega, \varphi^j_{\pi,\omega}, \omega) = \sum_j \mathcal{Z}(s, \phi_{\xi^j_\omega, \varphi^j_{\pi,\omega}}, \omega)$$

for any $s \in \mathbb{C}$. Hence, with any fixed $\omega \in \Omega^\wedge$, for any $\mathfrak{z}(s, \omega) \in \mathcal{L}_{\mathrm{cpt}}$, there exists $\mathfrak{z}^\circ(s, \omega) \in \mathcal{L}^\circ_\pi$ such that

$$\mathfrak{z}(s, \omega) = \mathfrak{z}^\circ(s, \omega) \tag{7-3}$$

as functions in $s \in \mathbb{C}$.

Define, for each $\omega_0 \in \Omega^\wedge$, a function $\mathfrak{z}_{\omega_0}(s, \omega)$ with the property

$$\mathfrak{z}_{\omega_0}(s, \omega) = \begin{cases} 1, & \text{if } \omega = \omega_0, \\ 0, & \text{if } \omega \neq \omega_0. \end{cases}$$

By definition, the function $\mathfrak{z}_{\omega_0}(s, \omega)$ belongs to $\mathcal{L}_{\mathrm{cpt}}$ for each $\omega_0 \in \Omega^\wedge$. Hence, from (7-3), we have

$$\mathfrak{z}(s, \omega) = \sum_{\omega_0 \in \Omega^\wedge} \mathfrak{z}_{\omega_0}(s, \omega) \cdot \mathfrak{z}(s, \omega) = \sum_{\omega_0 \in \Omega^\wedge} \mathfrak{z}_{\omega_0}(s, \omega) \cdot \mathfrak{z}^\circ(s, \omega_0), \tag{7-4}$$

for any $\mathfrak{z}(s, \omega) \in \mathcal{L}_{\mathrm{cpt}}$. Note here that the summations only take finitely many $\omega_0 \in \Omega^\wedge$. Hence, to prove the converse of (7-2), it is enough to show that

$$\mathfrak{z}_{\omega_0}(s, \omega) \cdot \mathfrak{z}^\circ(s, \omega_0) \in \mathcal{L}^\circ_\pi \tag{7-5}$$

for every $\omega_0 \in \Omega^\wedge$. It is clear that (7-5) is an easy consequence of the following proposition.

Proposition 7.2. *The space $\mathcal{L}_{\mathrm{cpt}}$ is an associative algebra without identity, and the space $\mathcal{L}^\circ_\pi$ is an $\mathcal{L}_{\mathrm{cpt}}$-module under multiplication.*

Proof. From the definition of $\mathcal{L}_{\mathrm{cpt}}$, it is clear that $\mathcal{L}_{\mathrm{cpt}}$ is an associative algebra under the multiplication and has no identity.

To prove that $\mathcal{L}^\circ_\pi$ is an $\mathcal{L}_{\mathrm{cpt}}$-module, we take $\mathfrak{z}(s, \omega) \in \mathcal{L}_{\mathrm{cpt}}$ and write ϕ as the Mellin inversion of $\mathfrak{z}(s, \omega)$. Via the determinant morphism $\det : G_n(F) \to F^\times$, we write

$$\phi(x) = \int_{\det g = x} f(g)\, \mathrm{d}_x g$$

for some $f \in \mathcal{C}_c^\infty(G_n(F))$. For any $\xi \in \mathcal{C}_c^\infty(G_n(F))$ and $\varphi_\pi \in \mathcal{C}(\pi)$, we write $\mathfrak{z}^\circ(s,\omega) \in \mathcal{L}_\pi^\circ$ to be the Mellin transform of the function $\phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi^\circ(F^\times)$. It is enough to show that

$$\mathfrak{z}(s,\omega)\cdot\mathfrak{z}^\circ(s,\omega) \in \mathcal{L}_\pi^\circ. \tag{7-6}$$

It is clear that

$$\mathfrak{z}(s,\omega)\cdot\mathfrak{z}^\circ(s,\omega) = \mathcal{Z}(s, \phi * \phi_{\xi,\varphi_\pi}, \omega). \tag{7-7}$$

Now we compute the right-hand side of (7-7) with a fixed $\omega \in \Omega^\wedge$:

$$\begin{aligned} \mathcal{Z}(s, \phi * \phi_{\xi,\varphi_\pi}, \omega) &= \int_{x\in F^\times} \omega(x)|x|_F^{s-1/2}\, \mathrm{d}^\times x \int_{y\in F^\times} \phi(y)\phi_{\xi,\varphi_\pi}(y^{-1}x)\, \mathrm{d}^\times y \\ &= \int_{F^\times} \omega(x)|x|_F^{s-1/2}\, \mathrm{d}^\times x \int_{F^\times} \mathrm{d}^\times y \int_{\det g = y} f(g)\, \mathrm{d}_y g \\ &\qquad \cdot \int_{\det h = y^{-1}x} \xi(h)\varphi_\pi(h)\, \mathrm{d}_{y^{-1}x} h. \end{aligned} \tag{7-8}$$

After changing variable $g \to gh^{-1}$, the right-hand side of (7-8) is equal to

$$\int_{F^\times} \omega(x)|x|_F^{s-1/2}\, \mathrm{d}^\times x \int_{F^\times} \mathrm{d}^\times y \int_{\det g = x} f(gh^{-1})\, \mathrm{d}_x g \cdot \int_{\det h = y^{-1}x} \xi(h)\varphi_\pi(h)\, \mathrm{d}_{y^{-1}x} h. \tag{7-9}$$

In (7-9), the integration in $y \in F^\times$ yields the identity

$$\int_{y\in F^\times} \mathrm{d}^\times y \int_{\det h = y^{-1}x} f(gh^{-1})\xi(h)\varphi_\pi(h)\, \mathrm{d}_{y^{-1}x} h = \int_{G_n(F)} f(gh^{-1})\xi(h)\varphi_\pi(h)\, \mathrm{d}h. \tag{7-10}$$

By applying (7-10) to (7-9), we can write (7-9) as

$$\int_{F^\times} \omega(x)|x|_F^{s-1/2}\, \mathrm{d}^\times x \int_{\det g = x} \int_{G_n(F)} f(gh^{-1})\xi(h)\varphi_\pi(h)\, \mathrm{d}h\, \mathrm{d}_x g,$$

which is equal to

$$\int_{g\in G_n(F)} \int_{h\in G_n(F)} f(gh^{-1})\xi(h)\varphi_\pi(h)\omega(\det g)|\det g|_F^{s-1/2}\, \mathrm{d}h\, \mathrm{d}g. \tag{7-11}$$

By taking a change of variable $h \to h^{-1}g$, (7-11) can be written as

$$\int_{g\in G_n(F)} \int_{h\in G_n(F)} f(h)\xi(h^{-1}g)\varphi_\pi(h^{-1}g)\omega(\det g)|\det g|_F^{s-1/2}\, \mathrm{d}h\, \mathrm{d}g. \tag{7-12}$$

Since $f, \xi \in \mathcal{C}_c^\infty(G_n(F))$, the function

$$(g, h) \mapsto f(h)\xi(h^{-1}g)$$

belongs to the space $\mathcal{C}_c^\infty(G_n(k_\nu) \times G_n(k_\nu))$. By [5, 1.22], we have

$$\mathcal{C}_c^\infty(G_n(k_\nu) \times G_n(k_\nu)) \simeq \mathcal{C}_c^\infty(G_n(k_\nu)) \otimes \mathcal{C}_c^\infty(G_n(k_\nu)).$$

We may write

$$f(h)\xi(h^{-1}g) = \sum_{j=1}^{r} \xi_j(g)\xi^j(h)$$

for some $\xi_j(g)$ and $\xi^j(h)$ in $\mathcal{C}_c^\infty(G_n(F))$. Meanwhile, we write

$$\varphi_\pi(h^{-1}g) = \langle \pi(h^{-1}g)v, \tilde{v} \rangle = \langle \pi(g)v, \widetilde{\pi}(h)\tilde{v} \rangle, \quad v \in \pi,\ \tilde{v} \in \widetilde{\pi}. \tag{7-13}$$

By applying those explicit expressions to the integral in (7-12), we obtain that (7-12) is equal to

$$\begin{aligned}
&\sum_{j=1}^{r} \int_{g \in G_n(F)} \int_{h \in G_n(F)} \xi_i(g)\xi^i(h)\langle \pi(g)v, \widetilde{\pi}(h)\tilde{v}\rangle \omega(\det g)|\det g|_F^{s-1/2}\,\mathrm{d}h\,\mathrm{d}g \\
&\quad = \sum_{j=1}^{r} \int_{g \in G_n(F)} \xi_i(g)\omega(\det g)|\det g|_F^{s-1/2}\,\mathrm{d}g \int_{h \in G_n(F)} \xi^i(h)\langle \pi(g)v, \widetilde{\pi}(h)\tilde{v}\rangle\,\mathrm{d}h \\
&\quad = \sum_{j=1}^{r} \int_{G_n(F)} \xi_i(g)\langle \pi(g)v, \widetilde{\pi}(\xi^j)\tilde{v}\rangle \omega(\det g)|\det g|_F^{s-1/2}\,\mathrm{d}g.
\end{aligned}$$

By writing $\varphi_{\pi,j}(g) := \langle \pi(g)v, \widetilde{\pi}(\xi^j)\tilde{v}\rangle$, we obtain that

$$\begin{aligned}
\mathcal{Z}(s, \phi * \phi_{\xi,\varphi_\pi}, \omega) &= \sum_{j=1}^{r} \int_{G_n(F)} \xi_i(g)\varphi_{\pi,j}(g)\omega(\det g)|\det g|_F^{s-1/2}\,\mathrm{d}g \\
&= \sum_{j=1}^{r} \mathcal{Z}(s, \phi_{\xi_j,\varphi_{\pi,j}}, \omega).
\end{aligned} \tag{7-14}$$

By definition of $\mathcal{L}_\pi^\circ$, we obtain that the right-hand side of (7-14) belongs to the space $\mathcal{L}_\pi^\circ$, and so does the function $\mathcal{Z}(s, \phi * \phi_{\xi,\varphi_\pi}, \omega)$. Therefore we have established (7-6). We are done. □

We have finished the proof of Theorem 7.1 for the non-Archimedean case. Now we turn to the proof the converse of (7-2), and hence Theorem 7.1 for the Archimedean case.

We first treat the case when $F \simeq \mathbb{C}$. It is clear that the multiplication map

$$\begin{aligned} \mathfrak{m} : \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) &\to G_n(\mathbb{C}) \\ (a, h) &\mapsto a \cdot h \end{aligned} \tag{7-15}$$

provides a surjective group homomorphism with finite kernel, which in particular is a smooth (covering) map. The push-forward map along $\mathfrak{m}$, which we denote by

$$\mathfrak{m}_* : \mathcal{C}_c^\infty(\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})) \to \mathcal{C}_c^\infty(G_n(\mathbb{C})) \tag{7-16}$$

is surjective. In fact, the surjectivity can be easily verified as follows. For any $f \in \mathcal{C}_c^\infty(G_n(\mathbb{C}))$, let $\mathfrak{m}^*(f)$ be the pull-back of f along $\mathfrak{m}$, which is given by

$$\mathfrak{m}^*(f)(a, h) = f(a \cdot h), \quad (a, h) \in \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}).$$

Then we obtain that

$$\mathfrak{m}_*(\mathfrak{m}^*(f))(h) = \sum_{\substack{(a,h)\in\mathbb{C}^\times\times\mathrm{SL}_n(\mathbb{C}) \\ a\cdot h = g}} f(a \cdot h) = |\ker(\mathfrak{m})| \cdot f(g), \quad g \in G_n(\mathbb{C}).$$

It is clear now that the subspace $\mathcal{S}_\pi^\circ(\mathbb{C}^\times)$ of $\mathcal{S}_\pi(\mathbb{C}^\times)$ is equal to the space spanned by the functions

$$\begin{aligned} \phi_{\mathfrak{m}_*(f),\varphi_\pi}(x) &= \int_{\det g = x} \mathfrak{m}_*(f)(g)\varphi_\pi(g)\, \mathrm{d}_x g \\ &= \int_{\det g = x} \sum_{\substack{(a,h)\in\mathbb{C}^\times\times\mathrm{SL}_n(\mathbb{C}) \\ a\cdot h = g}} f(a, h)\varphi_\pi(g)\, \mathrm{d}_x g \end{aligned} \tag{7-17}$$

with all $f \in \mathcal{C}_c^\infty(\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))$ and $\varphi_\pi \in \mathcal{C}(\pi)$. Thus, in order to show the converse of (7-2), it suffices to show that any function in $\mathcal{C}_c^\infty(\mathbb{C}^\times)$ is of the form as in the last line of (7-17).

Let χ_π be the central character of π. By a change of variable, we write (7-17) as

$$\phi_{\mathfrak{m}_*(f),\varphi_\pi}(x) = \int_{\mathrm{SL}_n(\mathbb{C})} \sum_{a^n = x} f(a, h) \cdot \chi_\pi(a) \cdot \varphi_\pi(h)\, \mathrm{d}_1 h. \tag{7-18}$$

Assume that $f \in \mathcal{C}_c^\infty(\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))$ is given by a pure tensor

$$f(a, h) = f_1(a) \cdot f_2(h)$$

with $f_1 \in \mathcal{C}_c^\infty(\mathbb{C}^\times)$ and $f_2 \in \mathcal{C}_c^\infty(\mathrm{SL}_n(\mathbb{C}))$. Then (7-18) can be written as

$$\phi_{\mathfrak{m}_*(f),\varphi_\pi}(x) = \Big(\sum_{a^n = x} f_1(a)\chi_\pi(a)\Big) \cdot \int_{\mathrm{SL}_n(\mathbb{C})} f_2(h)\varphi_\pi(h)\, \mathrm{d}_1 h. \tag{7-19}$$

It is clear that multiplying by the character $\chi_{\pi_{\mathbb{C}}}$ stabilizes the space $\mathcal{C}_c^\infty(\mathbb{C}^\times)$, which means that $f_1(a)\chi_\pi(a) \in \mathcal{C}_c^\infty(\mathbb{C}^\times)$ for any $f_1 \in \mathcal{C}_c^\infty(\mathbb{C}^\times)$. The map

$$\mathcal{C}_c^\infty(\mathbb{C}^\times) \to \mathcal{C}_c^\infty(\mathbb{C}^\times) \quad \text{with } f \mapsto \Bigl(x \mapsto \sum_{a^n = x} f(a)\Bigr)$$

is surjective, since it is the push-forward map along the surjective covering map

$$\mathbb{C}^\times \to \mathbb{C}^\times \quad \text{with } a \mapsto a^n.$$

Therefore, any function in $\mathcal{C}_c^\infty(\mathbb{C}^\times)$ can be written as $\phi_{\mathfrak{m}_*(f),\varphi_\pi}(x)$ for some $\varphi_\pi \in \mathcal{C}(\pi)$ and $f \in \mathcal{C}_c^\infty(\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}))$. This finishes the proof of the converse of (7-2).

We now turn to the case when $F = \mathbb{R}$ and treat the cases of n odd and of n even separately.

When n is odd, the multiplication map

$$\begin{aligned} \mathfrak{m} : \mathbb{R}^\times \times \mathrm{SL}_n(\mathbb{R}) &\to G_n(\mathbb{R}) \\ (a, g) &\mapsto a \cdot g \end{aligned}$$

is surjective, the proof in the complex case is applicable and yields a proof for this case. We omit the details here.

When n is even, we write $G_n(\mathbb{R})$ as a disjoint union two real smooth manifolds:

$$G_n(\mathbb{R}) = G_n^+(\mathbb{R}) \sqcup G_n^-(\mathbb{R}),$$

where $G_n^+(\mathbb{R})$ (resp. $G_n^-(\mathbb{R})$) consists of elements in $G_n(\mathbb{R})$ with positive (resp. negative) determinant.

By the surjectivity of the map

$$\mathbb{R}_{>0} \times \mathrm{SL}_n(\mathbb{R}) \to G_n^+(\mathbb{R}) \quad \text{with } (a, g) \mapsto a \cdot g,$$

the proof in the complex case shows that the space $\mathcal{S}_\pi^\circ(\mathbb{R}^\times)$ contains the space $\mathcal{C}_c^\infty(\mathbb{R}_{>0})$. Since $\mathbb{R}^\times = \mathbb{R}_{>0} \sqcup \mathbb{R}_{<0}$, we have that

$$\mathcal{C}_c^\infty(\mathbb{R}^\times) = \mathcal{C}_c^\infty(\mathbb{R}_{>0}) \oplus \mathcal{C}_c^\infty(\mathbb{R}_{<0}).$$

It remains to show that

$$\mathcal{C}_c^\infty(\mathbb{R}_{<0}) \subset \mathcal{S}_\pi^\circ(\mathbb{R}^\times). \tag{7-20}$$

Take $\theta = \mathrm{diag}(-1, 1, \ldots, 1) \in G_n(\mathbb{R})$ and consider the map

$$\mathfrak{m}^- : \mathbb{R}_{>0} \times \mathrm{SL}_n(\mathbb{R}) \to G_n^-(\mathbb{R}) \quad \text{with } (a, h) \mapsto a \cdot h \cdot \theta.$$

As the complex situation, for any $f\in\mathcal{C}_c^\infty(\mathbb{R}_{>0}\times\mathrm{SL}_n(\mathbb{R}))$, we set

$$\phi_{\mathfrak{m}_*^-(f),\varphi_\pi}(x)=\int_{\det g=x}\sum_{\substack{(a,h)\in\mathbb{R}_{>0}\times\mathrm{SL}_n(\mathbb{R})\\ a\cdot h\cdot\theta=g}} f(a,h)\cdot\varphi_\pi(g)\,\mathrm{d}_x g, \tag{7-21}$$

for $x\in\mathbb{R}_{<0}$. We only need to show the space spanned by the functions of the form

$$\{x\mapsto\phi_{\mathfrak{m}_*^-(f),\varphi_\pi}(x)\mid f\in\mathcal{C}_c^\infty(\mathbb{R}_{>0}\times\mathrm{SL}_n(\mathbb{R})),\ \varphi_\pi\in\mathcal{C}(\pi)\} \tag{7-22}$$

with $x\in\mathbb{R}_{<0}$ contains (and hence is equal to) the space $\mathcal{C}_c^\infty(\mathbb{R}_{<0})$.

By a simple change of variable, we obtain that

$$\phi_{\mathfrak{m}_*^-(f),\varphi_\pi}(x)=\int_{\mathrm{SL}_n(\mathbb{R})}\sum_{a^n=-x} f(a,h)\cdot\chi_\pi(a)\varphi_\pi(h\cdot\theta)\,\mathrm{d}_1 h, \tag{7-23}$$

where χ_π is the central character of $\pi\in\Pi_{\mathbb{R}}(n)$. Assume that $f(a,h)=f_1(a)\cdot f_2(h)$ is a pure tensor with $f_1\in\mathcal{C}_c^\infty(\mathbb{R}_{>0})$ and $f_2\in\mathcal{C}_c^\infty(\mathrm{SL}_n(\mathbb{R}))$. Then (7-23) can be written as

$$\phi_{\mathfrak{m}_*^-(f),\varphi_\pi}(x)=\sum_{a^n=-x} f_1(a)\chi_{\pi_{\mathbb{R}}}(a)\cdot\int_{\mathrm{SL}_n(\mathbb{R})} f_2(h)\varphi_\pi(h\cdot\theta)\,\mathrm{d}_1 h, \tag{7-24}$$

with $x\in\mathbb{R}_{<0}$. For $y=-x>0$, the functions of the form

$$\sum_{a^n=y} f_1(a)\chi_{\pi_{\mathbb{R}}}(a)\cdot\int_{\mathrm{SL}_n(\mathbb{R})} f_2(h)\varphi_\pi(h\cdot\theta)\,\mathrm{d}_1 h$$

recover the space $\mathcal{C}_c^\infty(\mathbb{R}_{>0})$, as treated in the previous case. Thus, the functions of the form in (7-24) recover the space $\mathcal{C}_c^\infty(\mathbb{R}_{<0})$. This completes the proof for the converse of (7-2) for the Archimedean case. Therefore, we finish the proof of Theorem 7.1.

7B. ***A variant of π-Poisson summation formulae.*** For any $\pi=\bigotimes_{\nu\in|k|}\in\Pi_{\mathbb{A}}(G_n)$, we define in (4-1) the space of π-Schwartz functions on $\mathbb{A}^\times$:

$$\mathcal{S}_\pi(\mathbb{A}^\times)=\bigotimes_\nu\mathcal{S}_{\pi_\nu}(k_\nu^\times).$$

We define $\mathcal{S}_\pi^\circ(\mathbb{A}^\times)$ to be the subspace of $\mathcal{S}_\pi(\mathbb{A}^\times)$ that is spanned by the functions of the form $\phi=\bigotimes_\nu\phi_\nu\in\mathcal{S}_\pi(\mathbb{A}^\times)$ with at least one local component ϕ_ν belonging to $\mathcal{C}_c^\infty(k_\nu^\times)$. Note that for any $\phi=\bigotimes_\nu\phi_\nu\in\mathcal{S}_\pi(\mathbb{A}^\times)$, there are at most finitely many local components from $\mathcal{C}_c^\infty(k_\nu^\times)$. It is also easy to verify from the definition of the π-Fourier operator $\mathcal{F}_{\pi,\psi}$ as in (4-3) and Theorem 7.1 that there exist functions $\phi=\bigotimes_\nu\phi_\nu\in\mathcal{S}_\pi(\mathbb{A}^\times)$, such that

$$\mathcal{F}_{\pi,\psi}(\phi)=\bigotimes_{\nu\in|k|}\mathcal{F}_{\pi_\nu,\psi_\nu}(\phi_\nu)\in\mathcal{S}_{\widetilde{\pi}}^\circ(\mathbb{A}^\times).$$

We define $\mathcal{S}^{\circ\circ}_\pi(\mathbb{A}^\times)$ to be the subspace of $\mathcal{S}^{\circ}_\pi(\mathbb{A}^\times)$ that is spanned by the functions of the form $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}^{\circ}_\pi(\mathbb{A}^\times)$ with the property that $\mathcal{F}_{\pi,\psi}(\phi) \in \mathcal{S}^{\circ}_{\widetilde{\pi}}(\mathbb{A}^\times)$.

Theorem 7.3. *Assume that $\pi \in \mathcal{A}(G_n)$ is square integrable. For any $\phi \in \mathcal{S}^{\circ\circ}_\pi(\mathbb{A}^\times)$, the π-Poisson summation formula*

$$\Theta_\pi(x, \phi) = \Theta_{\widetilde{\pi}}(x^{-1}, \mathcal{F}_{\pi,\psi}(\phi))$$

holds as functions in $x \in \mathbb{A}^\times$.

Proof. By Corollary 5.6, both $\Theta_\pi(x, \phi)$ and $\Theta_{\widetilde{\pi}}(x^{-1}, \mathcal{F}_{\pi,\psi}(\phi))$ are absolutely convergent. It suffices to show the equality. The proof goes in the same way as Theorem 4.7 when $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$. The first key point is that when π is square integrable, its matrix coefficients can also be realized as the integrals in (4-14), with $\beta_1, \beta_2 \in V_\pi$ being not necessarily cuspidal.

The second key point is to prove that the boundary terms defined in (4-23) vanish automatically by the local assumption on ϕ at the two local places ν_1 and ν_2. More precisely, take $\phi = \phi_{\xi,\varphi_\pi} \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and assume that

$$\phi = \bigotimes_\nu \phi_\nu = \bigotimes_\nu \phi_{\xi_\nu,\varphi_{\pi_\nu}}$$

with $\xi_\nu(g) = |\det g|_\nu^{n/2} f_\nu(g)$ for some $f_\nu \in \mathcal{S}(M_n(k_\nu))$, and $\varphi_{\pi_\nu} \in \mathcal{C}(\pi_\nu)$. The assumption at the two local places ν_1 and ν_2 is the same as that $f_{\nu_1} \in \mathcal{C}_c^\infty(G_n(k_{\nu_1}))$ and $\mathcal{F}_{\psi_{\nu_2}}(f_{\nu_2}) \in \mathcal{C}_c^\infty(G_n(k_{\nu_2}))$. For $f = \bigotimes_\nu f_\nu$ and $\mathcal{F}_\psi(f) = \bigotimes_\nu \mathcal{F}_{\psi_\nu}(f_\nu)$ with the above f_{ν_1} at the given local place ν_1 and $\mathcal{F}_{\psi_{\nu_2}}(f_{\nu_2})$ at the given local place ν_2, the boundary terms $B_f(h, g)$ in (4-23) must vanish automatically. Therefore, the summation identity is established. We refer the other details of the proof to the proof of Theorem 4.7. □

7C. *Refinement of Conjecture 1.5.* We are going to state our conjecture on (σ, ρ)-Poisson summation formula on GL_1 with more details, which refines Conjecture 1.5. We will continue with the discussions in Section 6B. By Assumption 6.1, for $\sigma \in \mathcal{A}_{\mathrm{cusp}}(G)$, there exists a $\pi = \bigotimes_\nu \pi_\nu \in \Pi_{\mathbb{A}}(G_n)$ with $\pi_\nu = \pi_\nu(\sigma_\nu, \rho)$ for all ν. We define the space $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ of (σ, ρ)-Schwartz functions as in (6-5) and (6-6); and the (σ, ρ)-Fourier operator $\mathcal{F}_{\sigma,\rho,\psi}$ as in (6-7) and (6-8). Finally we define the space $\mathcal{S}^{\circ\circ}_{\sigma,\rho}(\mathbb{A}^\times)$ to be equal to the space $\mathcal{S}^{\circ\circ}_\pi(\mathbb{A}^\times)$, which is defined in Section 7B.

Conjecture 7.4 (refinement of Conjecture 1.5). *Let G be a k-split reductive group, and $\rho : G^\vee(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ be a representation of the complex dual group $G^\vee(\mathbb{C})$. With Assumption 6.1, for any $\sigma \in \mathcal{A}_{\mathrm{cusp}}(G)$, there exist $k^\times$-invariant linear functionals $\mathcal{E}_{\sigma,\rho}$ and $\mathcal{E}_{\widetilde{\sigma},\rho}$ on $\mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and $\mathcal{S}_{\widetilde{\sigma},\rho}(\mathbb{A}^\times)$, respectively, such that the (σ, ρ)-Poisson summation formula*

$$\mathcal{E}_{\sigma,\rho}(\phi) = \mathcal{E}_{\widetilde{\sigma},\rho}(\mathcal{F}_{\sigma,\rho,\psi}(\phi)) \tag{7-25}$$

holds for $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$. *If* $\phi \in \mathcal{S}^{\circ\circ}_{\sigma,\rho}(\mathbb{A}^\times)$, *then the identity in* (7-25) *holds for*

$$\mathcal{E}_{\sigma,\rho}(\phi)(x) = \Theta_{\sigma,\rho}(x,\phi) = \sum_{\alpha\in k^\times} \phi(\alpha x)$$

with $x \in \mathbb{A}^\times$.

We make remarks on Conjecture 1.5 and its refinement Conjecture 7.4.

Remark 7.5. In Corollary 6.3, we have proved that if the global Langlands functoriality is valid for (G,ρ) and the image of σ under the functorial transfer is cuspidal on $G_n(\mathbb{A})$, then Conjectures 1.5 and 7.4 hold with

$$\mathcal{E}_{\sigma,\rho}(\phi)(x) = \Theta_{\sigma,\rho}(x,\phi) = \sum_{\alpha\in k^\times} \phi(\alpha x)$$

for any $\phi \in \mathcal{S}_{\sigma,\rho}(\mathbb{A}^\times)$ and any $x \in \mathbb{A}^\times$. If the global Langlands functoriality is valid for (G,ρ) and the image of σ under the functorial transfer is square integrable on $G_n(\mathbb{A})$, then by Theorem 7.3, a similar (σ,ρ)-Poisson summation formula in Conjecture 7.4 holds for $\phi \in \mathcal{S}^{\circ\circ}_{\pi}(\mathbb{A}^\times)$.

8. Critical zeros of $L(s,\pi\times\chi)$

We provide a spectral interpretation of critical zeros of the automorphic $L(s,\pi\times\chi)$ for any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ and character χ of the idele class group $\mathcal{C}_k = k^\times\backslash\mathbb{A}^\times$ for a number field k. This can be viewed as a reformulation of [40, Theorem 2] (see also [12]) in the adelic formulation of A. Connes [11], and is a extension of [11, Theorem III.1] from the Hecke L-functions $L(s,\chi)$ to the standard automorphic L-functions $L(s,\pi\times\chi)$.

8A. ***Pólya–Hilbert–Connes pairs.*** For a number field k, denote by $\mathbb{A}^1 = \mathbb{A}^1_k := \ker(|\cdot|_{\mathbb{A}})$ the norm one ideles of k. Denote by $\mathcal{C}_k := k^\times\backslash\mathbb{A}^\times$ the idele class group of k, and define $\mathcal{C}^1_k := k^\times\backslash\mathbb{A}^1$. Then $\mathbb{A}^\times$ has a noncanonical decomposition

$$\mathbb{A}^\times = \mathbb{A}^1 \times \mathbb{R}^\times_+, \tag{8-1}$$

where $\mathbb{R}^\times_+ = |\mathbb{A}^\times|_{\mathbb{A}}$ is the connected component of 1. In the following, we choose and fix a section of the short exact sequence

$$1 \to \mathbb{A}^1 \to \mathbb{A}^\times \to \mathbb{R}^\times_+ \to 1.$$

This gives a fixed noncanonical decomposition

$$\mathcal{C}_k = \mathcal{C}^1_k \times \mathbb{R}^\times_+. \tag{8-2}$$

For any $\delta > 0$, define $L^2_\delta(\mathcal{C}_k)$ to the space consisting of measurable functions

$$\theta : \mathcal{C}_k \to \mathbb{C}$$

with a finite Sobolev norm $\|\cdot\|_\delta$ as defined by

$$\|\theta\|_\delta^2 := \int_{\mathcal{C}_k} |\theta(x)|^2 (1+(\log|x|_{\mathbb{A}})^2)^{\delta/2}\,\mathrm{d}^\times x. \tag{8-3}$$

It is clear that the space $L^2_\delta(\mathcal{C}_k)$ is a $\mathcal{C}_k$-module via the right translation $\mathfrak{r}_\delta$ defined by

$$\mathfrak{r}_\delta(a)(\theta)(x) := \theta(xa) \tag{8-4}$$

for any $\theta \in L^2_\delta(\mathcal{C}_k)$ and $a, x \in \mathcal{C}_k$. Note that the $\mathcal{C}_k$-module $L^2_\delta(\mathcal{C}_k)$ is not unitary, but has the property

$$\|\mathfrak{r}_\delta(x)\| = o(\log|x|_{\mathbb{A}})^{\delta/2}, \quad |x|_{\mathbb{A}} \to \infty. \tag{8-5}$$

For any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$, take any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$. By Proposition 4.8, for any $\kappa > 0$, there exists a positive constant $c_{\kappa,\phi}$ such that the π-theta function $\Theta_\pi(x,\phi)$ enjoys the property

$$|\Theta_\pi(x,\phi)| \le c_{\kappa,\phi}\cdot \min\{|x|_{\mathbb{A}}, |x|_{\mathbb{A}}^{-1}\}^\kappa,$$

in particular, $\Theta_\pi(x,\phi)$ decays rapidly when $|x|_{\mathbb{A}} \to 0$ or $|x|_{\mathbb{A}} \to \infty$, and hence belongs to $L^2_\delta(\mathcal{C}_k)$. Define

$$\|\phi\|_\delta^2 := \int_{\mathcal{C}_k} |\Theta_\pi(x,\phi)|^2 (1+(\log|x|_{\mathbb{A}})^2)^{\delta/2}\,\mathrm{d}^\times x \tag{8-6}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$. Then we have the embedding

$$\Theta_\pi : \phi \in \mathcal{S}_\pi(\mathbb{A}^\times) \mapsto \Theta_\pi(\,\cdot\,,\phi) \in L^2_\delta(\mathcal{C}_k) \tag{8-7}$$

with respect to the Sobolev norms defined in (8-3) and (8-6), respectively.

Denote by $\overline{\Theta_\pi}$ the completion of the image $\Theta_\pi(\mathcal{S}_\pi(\mathbb{A}^\times))$ in $L^2_\delta(\mathcal{C}_k)$. Since

$$\mathfrak{r}_\delta(y)(\Theta_\pi(\,\cdot\,,\phi))(x) = \Theta_\pi(x, \mathfrak{r}_\delta(y)\phi)$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$, with $x, y \in \mathcal{C}_k$, the closed subspace $\overline{\Theta_\pi}$ is also a $\mathcal{C}_k$-module. Define the quotient space

$$\mathcal{H}_{\pi,\delta} := L^2_\delta(\mathcal{C}_k)/\overline{\Theta_\pi}, \tag{8-8}$$

which is also a $\mathcal{C}_k$-module. The associated representation is denoted by $\mathfrak{r}_{\pi,\delta}$. It is clear that the restriction of the $\mathcal{C}_k$-module to $\mathcal{C}_k^1$ is unitary and has the decomposition

$$\mathcal{H}_{\pi,\delta}|_{\mathcal{C}_k^1} = \bigoplus_{\chi \in \widehat{\mathcal{C}_k^1}} \mathcal{H}_{\pi,\delta,\chi}. \tag{8-9}$$

By the fixed (noncanonical) decomposition in (8-2), each eigenspace $\mathcal{H}_{\pi,\delta,\chi}$ is a module of $\mathbb{R}_+^\times$. The associated representation is denoted by $\mathfrak{r}_{\pi,\delta,\chi}$. Note that $\mathfrak{r}_{\pi,\delta,\chi}$

is also a representation of $\mathcal{C}_k = \mathcal{C}_k^1 \times \mathbb{R}_+^\times$ on $\mathcal{H}_{\pi,\delta,\chi}$. The action of $\mathbb{R}_+^\times$ on $\mathcal{H}_{\pi,\delta,\chi}$ generates a flow with the infinitesimal generator

$$\mathfrak{D}_{\pi,\delta,\chi}(\theta) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\big(\mathfrak{r}_{\pi,\delta,\chi}(\exp(\epsilon)-1)\big)\theta \tag{8-10}$$

for any $\theta \in \mathcal{H}_{\pi,\delta,\chi}$. As in [11], one should take the pair

$$(\mathcal{H}_{\pi,\delta,\chi}, \mathfrak{D}_{\pi,\delta,\chi}) \tag{8-11}$$

to be a candidate of the Pólya–Hilbert space. We call it a *Pólya–Hilbert–Connes pair*.

For any $\chi \in \widehat{\mathcal{C}_k^1}$, by the fixed noncanonical decomposition $\mathcal{C}_k = \mathcal{C}_k^1 \times \mathbb{R}_+^\times$ as in (8-2), the character χ has a unique extension to $\mathcal{C}_k$ by defining that it is trivial on $\mathbb{R}_+^\times$. We may still denote the extended character by χ.

Theorem 8.1 (critical zeros of $L(s,\pi\times\chi)$). *Given any $\pi \in \mathcal{A}_{\mathrm{cusp}}(G_n)$ and any character $\chi \in \widehat{\mathcal{C}_k^1}$, take $\mathfrak{D}_{\pi,\delta,\chi}$ as in* (8-10) *with* $\delta > 1$.

(1) *The spectrum $\mathrm{Sp}(\mathfrak{D}_{\pi,\delta,\chi})$ is discrete and is contained in $i\cdot\mathbb{R}$ with $i=\sqrt{-1}$.*

(2) *$\mu \in \mathrm{Sp}(\mathfrak{D}_{\pi,\delta,\chi})$ if and only if $L\big(\frac12+\mu, \pi\times\chi\big)=0$.*

(3) *The multiplicity $m_{\mathrm{Sp}(\mathfrak{D}_{\pi,\delta,\chi})}(\mu)$ is equal to the largest integer $m < \frac12(1+\delta)$ with $m \le m_{L(s,\pi\times\chi)}\big(\frac12+\mu\big)$, the multiplicity of $\frac12+\mu$ as a zero of the automorphic L-function $L(s,\pi\times\chi)$.*

Note Theorem 8.1 can be viewed as a reformulation of [40, Theorem 2] in the adelic framework of [11] and is an extension of [11, Theorem III.1] from the Hecke L-functions $L(s,\chi)$ to the standard automorphic L-functions $L(s,\pi\times\chi)$. See also [12] for relevant discussion.

8B. *Proof of Theorem 8.1.* We are going to prove Theorem 8.1 by using an argument that combines the approach of [11] and that of [40].

Consider the pairing

$$L^2_\delta(\mathcal{C}_k)\times L^2_{-\delta}(\mathcal{C}_k) \to \mathbb{C} \quad \text{with } (\theta,\eta)\mapsto \langle\theta,\eta\rangle, \tag{8-12}$$

where the pairing is defined by the integral

$$\langle\theta,\eta\rangle := \int_{\mathcal{C}_k} \theta(x)\eta(x)\,\mathrm{d}^\times x.$$

For any $y \in \mathcal{C}_k$, we have

$$\langle \mathfrak{r}_\delta(y)\theta, \eta\rangle = \langle\theta, \mathfrak{r}_{-\delta}(y^{-1})\eta\rangle$$

for any $\theta \in L^2_\delta(\mathcal{C}_k)$ and $\eta \in L^2_{-\delta}(\mathcal{C}_k)$.

Consider a function $\eta \in L^2_{-\delta}(\mathcal{C}_k)$ as a distribution on the eigenspace $\mathcal{H}_{\pi,\delta,\chi}$. Then

$$\langle \theta, \eta \rangle = 0 \tag{8-13}$$

for any $\theta \in \overline{\Theta_\pi}$, and, for any $t \in \mathcal{C}_k^1$,

$$\mathfrak{r}_{-\delta}(t)\eta = \chi^{-1}(t)\eta$$

as a distribution on $\mathcal{H}_{\pi,\delta,\chi}$. Hence, we may write, for $x = ta \in \mathcal{C}_k = \mathcal{C}_k^1 \times \mathbb{R}_+^\times$, the fixed noncanonical decomposition, that

$$\eta(x) = \chi^{-1}(t)\beta(a), \tag{8-14}$$

where $\beta(a)$ is a measurable function on $\mathbb{R}_+^\times$ with

$$\|\beta\|_\delta = \int_{\mathbb{R}_+^\times} |\beta(a)|^2 (1 + (\log|a|)^2)^{-\delta/2} \, \mathrm{d}^\times a < \infty.$$

The orthogonality in (8-13) can be written as

$$\int_{\mathcal{C}_k} \Theta_\pi(x, \phi)\eta(x) \, \mathrm{d}^\times x = 0 \tag{8-15}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$. As in [40], we prove the following lemma, which is a reformulation of Lemma 1 of [40].

Lemma 8.2. *The subspace of $\overline{\Theta_\pi}$ generated by functions of type*

$$(b * \Theta_\pi(\,\cdot\,, \phi))(t) = \int_{\mathcal{C}_k} b(x)\Theta_\pi(x^{-1}t, \phi) \, \mathrm{d}^\times x$$

with all $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$ is dense in $\overline{\Theta_\pi}$.

Proof. We reformulate the proof of [40, Lemma 1]. For any $\theta \in \overline{\Theta_\pi}$, we have

$$(b * \theta)(t) = \int_{\mathcal{C}_k} b(x)\theta(x^{-1}t) \, \mathrm{d}^\times x = \int_{\mathcal{C}_k} b(x)\theta^\vee(t^{-1}x) \, \mathrm{d}^\times x = \mathfrak{r}_\delta(b)(\theta^\vee)(t^{-1})$$

for any $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$. Since $\overline{\Theta_\pi}$ is a closed subspace of $L^2_\delta(\mathcal{C}_k)$ and is a $\mathcal{C}_k$-module, it is clear that $b * \theta$ belongs to $\overline{\Theta_\pi}$. In particular, we have that $b * \Theta_\pi(\,\cdot\,, \phi)$ belongs to $\overline{\Theta_\pi}$ for all $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$ and all $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$.

Next, by [11, Lemma 5], there exists a sequence of functions $\{f_n\}$ with f_n belonging to the space $\mathcal{S}(\mathcal{C}_k)$ of the Bruhat–Schwartz functions on $\mathcal{C}_k$, such that $\mathfrak{r}_\delta(f_n)$ tends strongly to 1 in $L^2_\delta(\mathcal{C}_k)$ and the norm of $\mathfrak{r}_\delta(f_n)$ are bounded. Now following the same argument as in the proof of [40, Lemma 1], we obtain that there exists a sequence of functions $b_n \in \mathcal{C}_c^\infty(\mathcal{C}_k)$ with the properties

(1) $\mathfrak{r}_\delta(b_n)$ converges strongly to 1;

(2) the norm of $\mathfrak{r}_\delta(b_n)$ is bounded;

(3) $b_n * \Theta_\pi(\,\cdot\,, \phi)$ converges to $\Theta_\pi(\,\cdot\,, \phi)$ for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$.

Therefore the linear span of $b * \Theta_\pi(\cdot, \phi)$ with $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$ and $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ is dense in $\overline{\Theta_\pi}$. We are done. $\square$

By Lemma 8.2, it is enough to consider the orthogonality

$$\int_{\mathcal{C}_k} (b * \Theta_\pi(\cdot, \phi))(x)\eta(x)\, \mathrm{d}^\times x = 0 \tag{8-16}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$.

Lemma 8.3. *For any $\eta \in L^2_{-\delta}(\mathcal{C}_k)$, the integral*

$$\int_{\mathcal{C}_k} (b * \Theta_\pi(\cdot, \phi))(x)\eta(x)\, \mathrm{d}^\times x$$

is zero for any $b \in \mathcal{C}_c^\infty(\mathcal{C}_k)$ and any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ if and only if

$$L\big(\tfrac{1}{2} + i\mu, \pi \times \chi\big) \cdot \mathcal{M}(\eta)(\chi_{i\mu})$$

is zero as a function in $\chi_{i\mu}$, where $\chi_{i\mu}$ is any unitary character of $\mathcal{C}_k$ that can be written as $\chi_{i\mu}(x) = \chi(t)a^{i\mu}$ for $x = ta \in \mathcal{C}_k = \mathcal{C}_k^1 \times \mathbb{R}_+^\times$, the fixed noncanonical decomposition.

Proof. We are going to apply the Parseval formula for the Fourier transform from $\mathcal{C}_k$ to its unitary dual $\widehat{\mathcal{C}_k}$ to (8-16). Since $\chi_{i\mu}(x) = \chi(t)a^{i\mu}$, the Fourier transform for $\mathcal{C}_k$ is

$$\mathcal{M}(\theta)(\chi_{i\mu}) = \int_{\mathcal{C}_k} \theta(x)\chi_{i\mu}^{-1}(x)\, \mathrm{d}^\times x.$$

By applying the Parseval formula to the integral

$$\int_{\mathcal{C}_k} (b * \Theta_\pi(\cdot, \phi))(x)\eta(x)\, \mathrm{d}^\times x,$$

we obtain that (8-16) is equivalent to

$$\int_{\widehat{\mathcal{C}_k}} \mathcal{M}(b)(\chi_{i\mu})\mathcal{M}(\Theta_\pi(\cdot, \phi))(\chi_{i\mu})\mathcal{M}(\eta)(\chi_{i\mu})\, \mathrm{d}\chi_{i\mu} = 0 \tag{8-17}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$. It is easy to verify from definition that

$$\mathcal{M}(\Theta_\pi(\cdot, \phi))(\chi_{i\mu}) = \mathcal{Z}\big(\tfrac{1}{2} + i\mu, \phi, \chi\big),$$

where the right-hand side is the global (GL_1) zeta integral as defined in (4-4). From Corollary 4.4 and [16, Proposition 13.9], the global zeta integral $\mathcal{Z}\big(\tfrac{1}{2} + i\mu, \phi, \chi\big)$ is a bounded function in μ. Hence, the product

$$\mathcal{T}_{\phi,\eta}(\chi_{i\mu}) := \mathcal{Z}\big(\tfrac{1}{2} + i\mu, \phi, \chi\big) \cdot \mathcal{M}(\eta)(\chi_{i\mu})$$

is a tempered distribution on $\widehat{\mathcal{C}_k}$. It follows that (8-17) is the same as

$$\int_{\widehat{\mathcal{C}_k}} \mathcal{M}(b)(\chi_{i\mu})\mathcal{T}_{\phi,\eta}(\chi_{i\mu})\, \mathrm{d}\chi_{i\mu} = 0 \tag{8-18}$$

for any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$. Denote by $\widehat{\mathcal{T}}_{\phi,\eta}(x)$ the (inverse) Fourier transform of $\mathcal{T}_{\phi,\eta}(\chi_{i\mu})$. By using the Parseval formula for the (inverse) Fourier transform, we obtain that (8-18) is equivalent to

$$\int_{\mathcal{C}_k} b(x)\widehat{\mathcal{T}}_{\phi,\eta}(x)\,\mathrm{d}^\times x = 0 \tag{8-19}$$

for all $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $b(x) \in \mathcal{C}_c^\infty(\mathcal{C}_k)$. Hence, we must have that (8-19) holds if and only if $\widehat{\mathcal{T}}_{\phi,\eta}(x) = 0$ as distribution on $\mathcal{C}_k$, which is equivalent to $\mathcal{T}_{\phi,\eta}(\chi_{i\mu}) = 0$ as distribution on $\widehat{\mathcal{C}_k}$. In other words, we obtain that for any $\eta \in L^2_{-\delta}(\mathcal{C}_k)$, the integral

$$\int_{\mathcal{C}_k} (b * \Theta_\pi(\,\cdot\,,\phi))(x)\eta(x)\,\mathrm{d}^\times x$$

is zero for any $b \in \mathcal{C}_c^\infty(\mathcal{C}_k)$ and any $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ if and only if

$$\mathcal{Z}\bigl(\tfrac{1}{2} + i\mu, \phi, \chi\bigr) \cdot \mathcal{M}(\eta)(\chi_{i\mu}) = 0 \tag{8-20}$$

for all $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$. By Corollary 4.4 and [16, Theorem 13.8], there exist finitely many $\phi_1, \dots, \phi_\ell \in \mathcal{S}_\pi(\mathbb{A}^\times)$ such that

$$\mathcal{Z}\bigl(\tfrac{1}{2} + i\mu, \phi_1, \chi\bigr) + \cdots + \mathcal{Z}\bigl(\tfrac{1}{2} + i\mu, \phi_\ell, \chi\bigr) = L\bigl(\tfrac{1}{2} + i\mu, \pi \times \chi\bigr).$$

Thus we obtain that (8-20) implies

$$L\bigl(\tfrac{1}{2} + i\mu, \pi \times \chi\bigr) \cdot \mathcal{M}(\eta)(\chi_{i\mu}) = 0 \tag{8-21}$$

as a function in $\chi_{i\mu}$.

To prove the converse, we consider factorizable data $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $\chi = \bigotimes_\nu \chi_\nu$. The global zeta integral factorizes into an Euler product

$$\mathcal{Z}(s, \phi, \chi) = \prod_\nu \mathcal{Z}(s, \phi_\nu, \chi_\nu).$$

By Theorem 3.4, we obtain that

$$\mathcal{Z}(s, \phi, \chi) = L(s, \pi \times \chi) \cdot \prod_{\nu \in S} \frac{\mathcal{Z}(s, \phi_\nu, \chi_\nu)}{L(s, \pi_\nu \times \chi_\nu)},$$

where S is the finite set of local places, including all Archimedean local places of k, such that for any $\nu \notin S$, the data π_ν and χ_ν are unramified, and the quotient $\mathcal{Z}(s, \phi_\nu, \chi_\nu)/L(s, \pi_\nu \times \chi_\nu)$ is holomorphic in $s \in \mathbb{C}$. Hence, if $\eta \in L^2_{-\delta}(\mathbb{A}^\times)$ satisfies

$$L\bigl(\tfrac{1}{2} + i\mu, \pi \times \chi\bigr) \cdot \mathcal{M}(\eta)(\chi_{i\mu}) = 0$$

as a function in $\chi_{i\mu}$, i.e., (8-21) holds, then (8-20) holds for factorizable data $\phi = \bigotimes_\nu \phi_\nu \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and $\chi = \bigotimes_\nu \chi_\nu$. Hence, it holds for all $\phi \in \mathcal{S}_\pi(\mathbb{A}^\times)$ and all χ. We are done. $\square$

The rest of the proof of Theorem 8.1 is exactly the same as the proof of [40, Theorem 2, page 178], which follows from the same argument of Connes (in the proof of [11, Theorem III.1, pp. 86–87]). We omit the details.

Acknowledgement

We would like to thank the referee for helpful comments, which made the paper more readable.

References

[1] A. Aizenbud and D. Gourevitch, "Schwartz functions on Nash manifolds", *Int. Math. Res. Not.* **2008**:5 (2008), art. id. 155. MR Zbl

[2] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications **61**, American Mathematical Society, Providence, RI, 2013. MR Zbl

[3] J. N. Bernstein, "Le 'centre' de Bernstein", pp. 1–32 in *Representations of reductive groups over a local field*, edited by P. Deligne, Hermann, Paris, 1984. MR Zbl

[4] J. Bernstein and B. Krötz, "Smooth Fréchet globalizations of Harish-Chandra modules", *Israel J. Math.* **199**:1 (2014), 45–111. MR Zbl

[5] J. N. Bernstein and A. V. Zelevinsky, "Representations of the group $\mathrm{GL}(n, F)$, where F is a local non-Archimedean field", *Uspehi Mat. Nauk* **31**:3(189) (1976), 5–70. In Russian; translated in *Russian Math. Surveys* **31**:3 (1976), 1–68. MR Zbl

[6] A. Braverman and D. Kazhdan, "γ-functions of representations and lifting", pp. 237–278 in *Visions in mathematics* (Tel Aviv, 1999), vol. 1, edited by N. Alon et al., Birkhäuser, 2000. MR Zbl

[7] P. Cartier, "Representations of p-adic groups: a survey", pp. 111–155 in *Automorphic forms, representations and L-functions*, *I* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979. MR Zbl

[8] W. Casselman, "The unramified principal series of $\mathfrak{p}$-adic groups, I: The spherical function", *Compositio Math.* **40**:3 (1980), 387–406. MR Zbl

[9] W. Casselman, "Canonical extensions of Harish-Chandra modules to representations of G", *Canad. J. Math.* **41**:3 (1989), 385–438. MR Zbl

[10] L. Clozel and P. Sarnak, "A universal lower bound for certain quadratic integrals of automorphic L-functions", preprint, 2022. arXiv 2203.12475

[11] A. Connes, "Trace formula in noncommutative geometry and the zeros of the Riemann zeta function", *Selecta Math.* (*N.S.*) **5**:1 (1999), 29–106. MR Zbl

[12] A. Deitmar, "A Polya–Hilbert operator for automorphic L-functions", *Indag. Math.* (*N.S.*) **12**:2 (2001), 157–175. MR Zbl

[13] L. Fargues and P. Scholze, "Geometrization of the local Langlands correspondence", preprint, 2021. arXiv 2102.13459

[14] S. Gelbart, I. Piatetski-Shapiro, and S. Rallis, *Explicit constructions of automorphic L-functions*, Lecture Notes in Mathematics **1254**, Springer, 1987. MR Zbl

[15] J. R. Getz and B. Liu, "A refined Poisson summation formula for certain Braverman–Kazhdan spaces", *Sci. China Math.* **64**:6 (2021), 1127–1156. MR Zbl

[16] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics **260**, Springer, 1972. MR Zbl

[17] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies **151**, Princeton University Press, 2001. MR Zbl

[18] G. Henniart, "Caractérisation de la correspondance de Langlands locale par les facteurs ϵ de paires", *Invent. Math.* **113**:2 (1993), 339–350. MR Zbl

[19] G. Henniart, "Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique", *Invent. Math.* **139**:2 (2000), 439–455. MR Zbl

[20] A. Ichino and N. Templier, "On the Voronoi formula for $\mathrm{GL}(n)$", *Amer. J. Math.* **135**:1 (2013), 65–101. MR

[21] J.-i. Igusa, *Forms of higher degree*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics **59**, Tata Institute of Fundamental Research, Bombay, 1978. MR Zbl

[22] H. Jacquet, "Représentations des groupes linéaires p-adiques", pp. 119–220 in *Theory of group representations and Fourier analysis* (Montecatini Terme, Italy, 1970), edited by F. Gherardelli, Ed. Cremonese, Rome, 1971. MR Zbl

[23] H. Jacquet, "Archimedean Rankin–Selberg integrals", pp. 57–172 in *Automorphic forms and L-functions, II: local aspects*, edited by D. Ginzburg et al., Contemp. Math. **489**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

[24] D. Jiang and Z. Li, "The Voronoi summation formula for GL_n and the Godement–Jacquet kernels", 2023. arXiv 2306.02554

[25] D. Jiang and Z. Luo, "Certain Fourier operators on GL_1 and local Langlands gamma functions", *Pacific J. Math.* **318**:2 (2022), 339–374. MR Zbl

[26] D. Jiang, Z. Luo, and L. Zhang, "Harmonic analysis and gamma functions on symplectic groups", preprint, 2021. To appear in *Mem. Amer. Math. Soc.* arXiv 2006.08126v2

[27] L. Lafforgue, "Noyaux du transfert automorphe de Langlands et formules de Poisson non linéaires", *Jpn. J. Math.* **9**:1 (2014), 1–68. MR Zbl

[28] L. Lafforgue, "Du transfert automorphe de Langlands aux formules de Poisson non linéaires", *Ann. Inst. Fourier* (*Grenoble*) **66**:3 (2016), 899–1012. MR Zbl

[29] R. P. Langlands, "Problems in the theory of automorphic forms", pp. 18–61 in *Lectures in modern analysis and applications, III*, edited by C. T. Taam, Lecture Notes in Math. **170**, Springer, 1970. MR

[30] R. Langlands, "On the notion of an automorphic representation. A supplement to the preceding paper", pp. 203–207 in *Automorphic forms, representations and L-functions* (Corvallis, OR, 1977), vol. 2, edited by A. Borel and W. Casselman, Proc. Symp. Pure Math. **33**, Am. Math. Soc., Providence, RI, 1979. Zbl

[31] R. P. Langlands, "On the classification of irreducible representations of real algebraic groups", pp. 101–170 in *Representation theory and harmonic analysis on semisimple Lie groups*, edited by P. J. Sally, Jr., Math. Surveys Monogr. **31**, Amer. Math. Soc., Providence, RI, 1989. MR

[32] W.-W. Li, "Generalized zeta integrals on certain real prehomogeneous vector spaces", *Nagoya Math. J.* **249** (2023), 50–87. MR Zbl

[33] Z. Luo, "On the Braverman–Kazhdan proposal for local factors: spherical case", *Pacific J. Math.* **300**:2 (2019), 431–471. MR Zbl

[34] Z. Luo, "An introduction to the proposal of Braverman and Kazhdan", pp. Chapter 7 in *On the Langlands program: endoscopy and beyond*, edited by W. T. Gan et al., Lect. Notes Inst. Math. Sci. Natl. Univ. Singap. **43**, World Sci., Hackensack, NJ, 2023.

[35] Z. Luo and B. C. Ngo, "Non-abelinan Fourier kernels on SL_2 and GL_2", preprint, 2023.

[36] I. G. Macdonald, *Spherical functions on a group of p-adic type*, Publications of the Ramanujan Institute **2**, University of Madras, Centre for Advanced Study in Mathematics, Ramanujan Institute, Madras, 1971. MR Zbl

[37] B. C. Ngô, "Hankel transform, Langlands functoriality and functional equation of automorphic L-functions", *Jpn. J. Math.* **15**:1 (2020), 121–167. MR

[38] I. Satake, "Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields", *Inst. Hautes Études Sci. Publ. Math.* 18 (1963), 5–69. MR Zbl

[39] P. Scholze, "The local Langlands correspondence for GL_n over p-adic fields", *Invent. Math.* **192**:3 (2013), 663–715. MR Zbl

[40] C. Soulé, "On zeroes of automorphic L-functions", pp. 167–179 in *Dynamical*, *spectral*, *and arithmetic zeta functions* (San Antonio, TX, 1999), edited by M. L. Lapidus and M. van Frankenhuysen, Contemp. Math. **290**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[41] B. Sun and C.-B. Zhu, "A general form of Gelfand–Kazhdan criterion", *Manuscripta Math.* **136**:1-2 (2011), 185–197. MR Zbl

[42] M. Tadić, "Spherical unitary dual of general linear group over non-Archimedean local field", *Ann. Inst. Fourier* (*Grenoble*) **36**:2 (1986), 47–55. MR Zbl

[43] T. Tamagawa, "On the ζ-functions of a division algebra", *Ann. of Math.* (2) **77** (1963), 387–405. MR Zbl

[44] J. T. Tate, "Fourier analysis in number fields, and Hecke's zeta-function", pp. 305–347 in *Algebraic number theory* (Brighton, 1965), edited by J. W. S. Cassels and A. Fröhlich, Thompson, Washington, DC, 1967. MR

[45] N. R. Wallach, *Real reductive groups*, *I*, Pure and Applied Mathematics **132**, Academic Press, Boston, 1988. MR Zbl

[46] N. R. Wallach, *Real reductive groups*, *II*, Pure and Applied Mathematics **132**, Academic Press, Boston, 1992. MR

[47] A. Weil, "Sur la formule de Siegel dans la théorie des groupes classiques", *Acta Math.* **113** (1965), 1–87. MR Zbl

[48] A. Weil, *Basic number theory*, Springer, 1995. MR Zbl

Received May 10, 2023. Revised November 18, 2023.

DIHUA JIANG
SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MN
UNITED STATES

jiang034@umn.edu

ZHILIN LUO
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
CHICAGO, IL
UNITED STATES

zhilinchicago@uchicago.edu

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 326 No. 2 October 2023