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MODULES OVER THE PLANAR GALILEAN CONFORMAL ALGEBRA ARISING FROM FREE MODULES OF RANK ONE

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The planar Galilean conformal algebra $\mathcal{G}$ introduced by Bagchi-Gopakumar and Aizawa is an infinite-dimensional extension of the finite-dimensional Galilean conformal algebra in (2+1)-dimensional space-time. In this paper, we give a complete classification of $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$, where $\mathfrak{h}$ is the Cartan subalgebra (a nilpotent self-normalizing subalgebra) of $\mathcal{G}$, $\mathbb{C}L_0$ is a subalgebra of $\mathfrak{h}$. Also, we determine the necessary and sufficient conditions for these modules to be irreducible, and find the maximal proper submodules when these modules are not irreducible.

1. Introduction

Infinite-dimensional Galilean conformal algebras were introduced by Bagchi and Gopakumar [2009] in order to construct a systematic nonrelativistic limit of the AdS/CFT conjecture (see [Maldacena 1998]). Some physicists believe that AdS/CFT correspondence would be better understood by exploring those algebras (see [Bagchi et al. 2010; Martelli and Tachikawa 2010]). Moreover, those algebras appear in the context of Galilean electrodynamics (see [Bagchi et al. 2014; Festuccia et al. 2016]) and may play an important role in Navier–Stokes equations (see [Bhattacharyya et al. 2009; Fouxon and Oz 2008; Fouxon and Oz 2009; Gusyatnikova and Yumaguzhin 1989]). These reasons make the infinite-dimensional Galilean conformal algebras attract more and more attention from mathematicians and physicists. In particular, the infinite-dimensional Galilean conformal algebra in (1+1)-dimensional space-time is the centerless W-algebra $W(2, 2)$; it has been studied in [Bagchi et al. 2010; Chen and Guo 2017; Zhang and Dong 2009]. This algebra is related to the BMS/GCA correspondence (see [Bagchi 2010]), the tensionless limit of string theory (see [Bagchi 2013]) and statistical mechanics (see [Henkel et al. 2012]).

The infinite-dimensional Galilean conformal algebra $\mathcal{G}$ in (2+1)-dimensional space-time, named the planar Galilean conformal algebra by Aizawa [2013], is an

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infinite-dimensional Lie algebra with a basis $\{L_n, H_n, I_n, J_n \mid n \in \mathbb{Z}\}$ and the Lie brackets are defined by

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n}, \qquad [L_m, H_n] = nH_{m+n},\\
&[L_m, I_n] = (n-m)I_{m+n}, \qquad [L_m, J_n] = (n-m)J_{m+n},\\
&[H_m, I_n] = I_{m+n}, \qquad [H_m, J_n] = -J_{m+n},\\
&[H_m, H_n] = [I_m, I_n] = [J_m, J_n] = [I_m, J_n] = 0 \quad \text{for all } m, n \in \mathbb{Z},
\end{aligned} \tag{1-1}
$$

which is the main object in this paper. This algebra is also the special case of [Martelli and Tachikawa 2010]. As we know, many infinite-dimensional Lie algebras in mathematics and physics are related to finite-dimensional semisimple Lie algebras. For example, the Virasoro algebra contains infinitely many $\mathrm{sl}_2(\mathbb{C})$ as its subalgebras. For the Lie algebra $\mathcal{G}$, there are two interesting features: it contains the Witt algebra as a subalgebra, and it is associated with the Galilean algebra, which is a nonsemisimple Lie algebra. Those would suggest that such an infinite-dimensional algebra is important and its representation theory is different from semisimple counterparts. So far there are a few of results about structures and representations of $\mathcal{G}$. The universal central extension $\bar{\mathcal{G}}$ of $\mathcal{G}$ was determined in [Gao et al. 2016]. The highest weight representations and coadjoint representations of $\mathcal{G}$ were investigated in [Aizawa 2013; Aizawa and Kimura 2011], Whittaker modules and restricted modules over $\mathcal{G}$ were studied in [Chen and Yao 2023; Chen et al. 2022; Gao and Gao 2022].

Recently, a family of nonweight modules over $\mathcal{G}$, called $\mathcal{U}(\mathfrak{h})$-free modules, has attracted more attention from mathematicians, where $\mathfrak{h} = \mathrm{span}\{L_0, H_0\}$ is a nilpotent self-normalizing subalgebra, called the Cartan subalgebra of $\mathcal{G}$. The notion of $\mathcal{U}(\mathfrak{h})$-free modules was first introduced by Nilsson [2015] for the simple Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$. At the same time, these modules were introduced in a very different approach in [Tan and Zhao 2015]. Later, $\mathcal{U}(\mathfrak{h})$-free modules for many important infinite-dimensional Lie algebras were determined, for example, the Virasoro algebra in [Lu and Zhao 2014], the Witt algebra in [Tan and Zhao 2015], affine Kac–Moody algebras in [Cai et al. 2020]. In the present paper, we will study this family of modules over $\mathcal{G}$ and $\bar{\mathcal{G}}$. These lead to many new examples of irreducible modules over $\mathcal{G}$ and $\bar{\mathcal{G}}$.

The paper is organized as follows. In Section 2, we recall the source of the infinite-dimensional Galilean conformal algebras. Then we review the planar Galilean conformal algebra $\mathcal{G}$ and $\bar{\mathcal{G}}$. We show that the $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ coincide with $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\bar{\mathcal{G}}$ respectively; see Corollary 2.3. Lastly, we collect some results about $\mathcal{U}(\mathbb{C}L_0)$-free modules over some Lie algebras related to the Witt algebra for later use. In Section 3, we get all $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 over $\mathcal{G}$, and the necessary and sufficient conditions for these modules to be irreducible are determined; see Theorem 3.2. We also determine the isomorphism classes of these modules; see Theorem 3.3. In Section 4, we obtain the main results of this paper. More precisely, we determine that there are three families of $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$, where $\mathfrak{h} = \mathrm{span}\{L_0, H_0\}$ is the Cartan subalgebra of $\mathcal{G}$; see Theorem 4.12. Also, we give the necessary and sufficient conditions for these modules to be irreducible, and find the maximal proper submodules when these modules are not irreducible; see Theorems 4.13, 4.14 and 4.15. Furthermore, we determine the isomorphism classes of these modules; see Theorem 4.17. Consequently, we give a complete classification of $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ and $\bar{\mathcal{G}}$.

Throughout this paper, we denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{C}$ and $\mathbb{C}^*$ the set of integers, nonnegative integers, positive integers, complex numbers and nonzero complex numbers respectively. All vector spaces and algebras are over $\mathbb{C}$. We denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra for a Lie algebra $\mathfrak{g}$.

2. Notation and preliminaries

In this section, we recall the infinite-dimensional Galilean conformal algebras and collect some known results about $\mathcal{U}(\mathbb{C}L_0)$-free modules over the Lie algebras related to the Witt algebra.

2A. ***From Galilean algebras to infinite-dimensional Galilean conformal algebras.*** In this subsection, we recall the background in which the infinite-dimensional Galilean conformal algebras arise. See [Bagchi and Gopakumar 2009] for more details. First, it is well-known that Galilean algebra $G(d, 1)$ in Galilean space-time $\mathbb{R}^{d,1}$ arises as a contraction of the Poincaré algebra $ISO(d, 1)$. The expressions for the Poincaré generators ($\mu, \nu = 0, 1, \dots, d$)

$$J_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad P_\mu = \partial_\mu,$$

give us the Galilean vector field generators $\{J_{ij}, P_i, B_i, H \mid i, j = 1, 2, \dots, d\}$, where

$$\begin{aligned} J_{ij} &= -(x_i \partial_j - x_j \partial_i), & P_i &= \partial_i, \\ B_i &= J_{0i} = t\partial_i, & H &= P_0 = -\partial_t. \end{aligned} \tag{2-1}$$

and t, x_i are variables. They obey the commutation relations

$$\begin{aligned} [J_{ij}, J_{rs}] &= \delta_{ir} J_{js} + \delta_{is} J_{rj} + \delta_{jr} J_{si} + \delta_{js} J_{ir}, \\ [H, B_i] &= -P_i, \\ [J_{ij}, B_r] &= -(B_i \delta_{jr} - B_j \delta_{ir}), \\ [J_{ij}, P_r] &= -(P_i \delta_{jr} - P_j \delta_{ir}), \\ [J_{ij}, H] &= [H, P_i] = [B_i, P_j] = [B_i, B_j] = [P_i, P_j] = 0. \end{aligned} \tag{2-2}$$

Consequently, we obtain the Galilean algebra

$$G(d,1)=\operatorname{span}\{J_{ij}, P_i, B_i, H \mid i, j=1,2,\dots,d\}$$

with the commutation relations (2-2).

To obtain the Galilean conformal algebra, we need additional generators

$$\{D, K, K_i \mid i=1,2,\dots,d\},$$

where

$$D=-\Big(\sum_{i=1}^{d} x_i\partial_i+t\partial_t\Big),\quad K=-\Big(\sum_{i=1}^{d} 2tx_i\partial_i+t^2\partial_t\Big),\quad K_i=t^2\partial_i. \tag{2-3}$$

Thus we get that Galilean conformal algebra in $(d+1)$-dimensional space-time is spanned by $\{J_{ij}, P_i, B_i, H, D, K, K_i \mid i, j=1,2,\dots,d\}$ with the commutation relations (2-2) and

$$\begin{aligned}
&[J_{ij}, K_r]=-(K_i\delta_{jr}-K_j\delta_{ir}), \quad [K, B_i]=K_i, \quad [K, P_i]=2B_i,\\
&[H, K_i]=-2B_i, \quad [D, K_i]=-K_i, \quad [D, P_i]=P_i,\\
&[D, H]=H, \quad [H, K]=-2D, \quad [D, K]=-K,\\
&[J_{ij}, D]=[J_{ij}, K]=[D, B_i]=[K, K_i]=[K_i, K_j]=[K_i, B_j]=[K_i, P_j]=0.
\end{aligned}$$

It is clear that Galilean conformal algebra contains Galilean algebra as a subalgebra.

We denote

$$\begin{aligned}
L^{(-1)}&=H, \quad L^{(0)}=D, \quad L^{(+1)}=K,\\
M_i^{(-1)}&=P_i, \quad M_i^{(0)}=B_i, \quad M_i^{(+1)}=K_i.
\end{aligned}$$

Then Galilean conformal algebra in $(d+1)$-dimensional space-time is spanned by $\{J_{ij}, L^{(n)}, M_i^{(n)} \mid i, j=1,2,\dots,d, n=0,\pm1\}$ with the commutation relations

$$\begin{aligned}
&[L^{(m)}, L^{(n)}]=(m-n)L^{(m+n)}, \qquad [L^{(m)}, M_i^{(n)}]=(m-n)M_i^{(m+n)},\\
&[J_{ij}, M_k^{(m)}]=-(M_i^{(m)}\delta_{jk}-M_j^{(m)}\delta_{ik}), \qquad [J_{ij}, L^{(n)}]=[M_i^{(m)}, M_j^{(n)}]=0,
\end{aligned}$$

where $m, n=0,\pm1$, $i, j=1,2,\dots,d$. In fact, we can define the vector fields

$$\begin{aligned}
J_{ij}&=-(x_i\partial_j-x_j\partial_i),\\
L^{(n)}&=-(n+1)t^n\sum_{i=1}^{d} x_i\partial_i-t^{(n+1)}\partial_t,\\
M_i^{(n)}&=t^{(n+1)}\partial_i,
\end{aligned}$$

where $n=0,\pm1$, $i, j=1,2,\dots,d$. These are exactly the vector fields in (2-1) and (2-3), so they generate the Galilean conformal algebra.

Now we have a very natural extension, for arbitrary $n \in \mathbb{Z}$, define

$$J_{ij}^{(n)} = -t^n(x_i\partial_j - x_j\partial_i),$$
$$L^{(n)} = -(n+1)t^n \sum_{i=1}^{d} x_i\partial_i - t^{(n+1)}\partial_t,$$
$$M_i^{(n)} = t^{(n+1)}\partial_i,$$

where $i, j = 1, 2, \dots, d$. Therefore, we obtain the infinite-dimensional Galilean conformal algebra GCA in $(d+1)$-dimensional space-time,

$$\mathrm{GCA} = \mathrm{span}\{J_{ij}^{(n)}, L^{(n)}, M_i^{(n)} \mid n \in \mathbb{Z}, i, j = 1, 2, \dots, d\},$$

satisfying the commutation relations

$$[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)},$$
$$[J_{ij}^{(m)}, J_{rs}^{(n)}] = \delta_{ir}J_{js}^{(m+n)} + \delta_{is}J_{rj}^{(m+n)} + \delta_{jr}J_{si}^{(m+n)} + \delta_{js}J_{ir}^{(m+n)},$$
$$[L^{(m)}, J_{ij}^{(n)}] = -nJ_{ij}^{(m+n)}, \quad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)},$$
$$[J_{ij}^{(m)}, M_k^{(n)}] = -(\delta_{jk}M_i^{(m+n)} - \delta_{ik}M_j^{(m+n)}), \quad [M_i^{(m)}, M_j^{(n)}] = 0.$$

In this paper, we mainly investigate the infinite-dimensional Galilean conformal algebra in (2+1)-dimensional space-time, which is called the planar Galilean conformal algebra by Aizawa [2013].

2B. ***Planar Galilean conformal algebra.*** From Section 2A, the planar Galilean conformal algebra is spanned by $\{J_{12}^{(n)}, L^{(n)}, M_i^{(n)} \mid n \in \mathbb{Z}, i = 1, 2\}$. We denote this algebra by $\mathcal{G}$, then $\mathcal{G}$ is an infinite-dimensional Lie algebra with the commutation relations

$$[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)}, \quad [L^{(m)}, J_{12}^{(n)}] = -nJ_{12}^{(m+n)},$$
$$[L^{(m)}, M_1^{(n)}] = (m-n)M_1^{(m+n)}, \quad [L^{(m)}, M_2^{(n)}] = (m-n)M_2^{(m+n)},$$
$$[J_{12}^{(m)}, M_1^{(n)}] = M_2^{(m+n)}, \quad [J_{12}^{(m)}, M_2^{(n)}] = -M_1^{(m+n)},$$
$$[J_{12}^{(m)}, J_{12}^{(n)}] = [M_1^{(m)}, M_1^{(n)}] = [M_2^{(m)}, M_2^{(n)}] = [M_1^{(m)}, M_2^{(n)}] = 0 \quad \text{for all } m, n \in \mathbb{Z}.$$

For convenience, we would like to simplify the notation (see [Chen et al. 2022]). Let

$$L_n = -L^{(n)}, \qquad H_n = \sqrt{-1}J_{12}^{(n)},$$
$$I_n = M_1^{(n)} + \sqrt{-1}M_2^{(n)}, \quad J_n = M_1^{(n)} - \sqrt{-1}M_2^{(n)} \quad \text{for all } n \in \mathbb{Z}.$$

Then it is easy to check that $\{L_n, H_n, I_n, J_n \mid n \in \mathbb{Z}\}$ satisfy the commutation relations (1-1). Now, we may describe the definition of the planar Galilean conformal algebra as follows.

Definition 2.1. The *planar Galilean conformal algebra* $\mathcal{G}$ is an infinite-dimensional Lie algebra with a basis $\{L_n, H_n, I_n, J_n \mid n \in \mathbb{Z}\}$ subject to the commutation relations (1-1).

Note that the Lie subalgebra $\widetilde{IJ}$ spanned by $\{I_m, J_m \mid m \in \mathbb{Z}\}$ is a commutative ideal of $\mathcal{G}$. Furthermore, $\mathcal{G}$ contains the following interesting subalgebras.

(1) $\mathfrak{h} = \mathrm{span}\{L_0, H_0\}$ is a nilpotent self-normalizing subalgebra, called the Cartan subalgebra of $\mathcal{G}$.

(2) $\mathcal{V} = \mathrm{span}\{L_m \mid m \in \mathbb{Z}\}$ is the centerless Virasoro algebra, i.e., the Witt algebra.

(3) $\mathcal{L} = \mathrm{span}\{L_m, H_m \mid m \in \mathbb{Z}\}$ is the Heisenberg–Virasoro algebra with the one-dimensional center.

(4) $\mathcal{W} = \mathrm{span}\{L_m, I_m \mid m \in \mathbb{Z}\}$ is the centerless $W(2, 2)$ algebra.

(5) $\mathcal{W}' = \mathrm{span}\{L_m, J_m \mid m \in \mathbb{Z}\}$ is the centerless $W(2, 2)$ algebra.

Recall that (see [Gao et al. 2016]) the universal central extension $\bar{\mathcal{G}}$ of the planar Galilean conformal algebra $\mathcal{G}$ is an infinite-dimensional Lie algebra with a basis $\{L_n, H_n, I_n, J_n, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3 \mid n \in \mathbb{Z}\}$ subject to the commutation relations

$$
\begin{aligned}
&[L_m, L_n] = (n-m)L_{m+n} + \tfrac{1}{12}(m^3 - m)\delta_{m+n,0}\boldsymbol{c}_1,\\
&[L_m, H_n] = nH_{m+n} + m^2\delta_{m+n,0}\boldsymbol{c}_2, \qquad [H_m, H_n] = m\delta_{m+n,0}\boldsymbol{c}_3,\\
&[L_m, I_n] = (n-m)I_{m+n}, \qquad [L_m, J_n] = (n-m)J_{m+n},\\
&[H_m, I_n] = I_{m+n}, \qquad [H_m, J_n] = -J_{m+n},\\
&[I_m, I_n] = [J_m, J_n] = [I_m, J_n] = 0 \quad \text{for all } m, n \in \mathbb{Z}.
\end{aligned}
\tag{2-4}
$$

Denote $\mathcal{L}' = \mathrm{span}\{L_m, H_m, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3 \mid m \in \mathbb{Z}\}$, which is a subalgebra of $\bar{\mathcal{G}}$. From Theorem 3 in [Chen and Guo 2017] and Theorem 3.1 in [Han et al. 2017] we get the following lemma.

Lemma 2.2. (1) *Suppose M is an $\mathcal{L}'$-module such that it is a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank* 1. *Then $\boldsymbol{c}_1 M = \boldsymbol{c}_2 M = \boldsymbol{c}_3 M = 0$.*

(2) *Suppose M' is an $\mathcal{L}'$-module such that it is a $\mathcal{U}(\mathfrak{h})$-free module of rank* 1. *Then $\boldsymbol{c}_1 M' = \boldsymbol{c}_2 M' = \boldsymbol{c}_3 M' = 0$.*

So, we have the following corollary.

Corollary 2.3. (1) *Let M be a $\mathcal{U}(\bar{\mathcal{G}})$-module such that M, when considered as a $\mathcal{U}(\mathbb{C}L_0)$-module, is free of rank* 1. *Then $\boldsymbol{c}_1 M = \boldsymbol{c}_2 M = \boldsymbol{c}_3 M = 0$. Thus $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank* 1 *over $\bar{\mathcal{G}}$ coincide with $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank* 1 *over $\mathcal{G}$.*

(2) *Let M' be a $\mathcal{U}(\bar{\mathcal{G}})$-module such that M', when considered as a $\mathcal{U}(\mathfrak{h})$-module, is free of rank* 1. *Then $\boldsymbol{c}_1 M' = \boldsymbol{c}_2 M' = \boldsymbol{c}_3 M' = 0$. Thus $\mathcal{U}(\mathfrak{h})$-free modules of rank* 1 *over $\bar{\mathcal{G}}$ coincide with $\mathcal{U}(\mathfrak{h})$-free modules of rank* 1 *over $\mathcal{G}$.*

Therefore, we mainly classify $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ in the following sections.

Now, we conclude this section by recalling $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 over algebras $\mathcal{V}$, $\mathcal{L}$ and $W(2,2)$, respectively. For any $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, it is not hard to see that the polynomial algebra $\mathbb{C}[L_0]$ has a $\mathcal{V}$-module structure with the following actions

$$L_m(f(L_0)) = \lambda^m (L_0 + m\alpha) f(L_0 - m), \ \forall\, m \in \mathbb{Z},\ f(L_0) \in \mathbb{C}[L_0].$$

Denote this module by $\Omega(\lambda, \alpha)$. Thanks to [Lu and Zhao 2014], we know that $\Omega(\lambda, \alpha)$ is irreducible if and only if $\alpha \neq 0$, and $\Omega(\lambda, 0)$ has an irreducible submodule $L_0\Omega(\lambda, 0)$ with codimension 1. Note that $\Omega(\lambda, \alpha)$ can be easily viewed as a $\mathcal{W}$ (resp. $\mathcal{W}'$)-module by defining $I_n(\Omega(\lambda, \alpha)) = 0$ (resp. $J_n(\Omega(\lambda, \alpha)) = 0$) for all $n \in \mathbb{Z}$, and the resulting module is denoted by $\Omega(\lambda, \alpha)^{\mathcal{W}}$ (resp. $\Omega(\lambda, \alpha)^{\mathcal{W}'}$). Moreover, we have the following lemmas.

Lemma 2.4 (cf. [Tan and Zhao 2015, Theorem 3]). *Let V be a $\mathcal{V}$-module. Assume that V can be viewed as a $\mathcal{U}(\mathbb{C}L_0)$-module is free of rank* 1. *Then $V \cong \Omega(\lambda, \alpha)$ for some $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$.*

Lemma 2.5 (cf. [Chen and Guo 2017, Theorem 3]). *Let V be a $\mathcal{W}$* (resp. $\mathcal{W}'$)-*module. Assume that V can be viewed as a $\mathcal{U}(\mathbb{C}L_0)$-module is free of rank* 1. *Then $V \cong \Omega(\lambda, \alpha)^{\mathcal{W}}$* (resp. $\Omega(\lambda, \alpha)^{\mathcal{W}'}$) *for some $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$.*

For $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, thanks to [Chen and Guo 2017], we see that the polynomial algebra $\mathbb{C}[L_0]$ is an $\mathcal{L}$-module with the actions

$$\begin{aligned} L_m(f(L_0)) &= \lambda^m (L_0 + m\alpha) f(L_0 - m), \\ H_m(f(L_0)) &= \beta\lambda^m f(L_0 - m) \quad \text{for all } m \in \mathbb{Z},\ \ f(L_0) \in \mathbb{C}[L_0]. \end{aligned} \tag{2-5}$$

We denote by $\Omega(\lambda, \alpha, \beta)$ this module. From [Chen and Guo 2017], we also know that $\Omega(\lambda, \alpha, \beta)$ is irreducible if and only if $(\alpha, \beta) \neq (0, 0)$, and $\Omega(\lambda, 0, 0)$ has an irreducible submodule $L_0\Omega(\lambda, 0, 0)$ with codimension 1. Furthermore:

Lemma 2.6 (cf. [Chen and Guo 2017, Theorem 2]). *Let V be an $\mathcal{L}$-module. Assume that V can be viewed as a $\mathcal{U}(\mathbb{C}L_0)$-module is free of rank* 1. *Then $V \cong \Omega(\lambda, \alpha, \beta)$ for some $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$.*

3. $\mathcal{U}(\mathbb{C}L_0)$-free modules over $\mathcal{G}$

In this section, we determine the $\mathcal{G}$-module structures on $\mathcal{U}(\mathbb{C}L_0)$. We give the necessary and sufficient conditions for these modules to be irreducible. Also, we find the maximal proper submodules and get the irreducible quotient modules when these modules are not irreducible. Moreover, we determine the isomorphism classes of these modules.

Note that $\widetilde{IJ}$ is a commutative ideal of $\mathcal{G}$. Thus for any $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, by (2-5) it is easy to see that the polynomial algebra $\mathbb{C}[L_0]$ equips with a $\mathcal{G}$-module structure via the actions

$$\begin{aligned} L_m(f(L_0)) &= \lambda^m (L_0 + m\alpha) f(L_0 - m), \\ H_m(f(L_0)) &= \beta\lambda^m f(L_0 - m), \\ I_m(f(L_0)) &= J_m(f(L_0)) = 0 \quad \text{for all } m \in \mathbb{Z},\ f(L_0) \in \mathbb{C}[L_0]. \end{aligned} \tag{3-1}$$

We denote this module by $\mathcal{A}(\lambda, \alpha, \beta)$.

Now we show that $\{\mathcal{A}(\lambda, \alpha, \beta) \mid \lambda \in \mathbb{C}^*, \alpha, \beta \in \mathbb{C}\}$ exhaust all $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 over $\mathcal{G}$ up to isomorphism.

Lemma 3.1. *Let V be a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank 1 over $\mathcal{G}$. We identify V with $\mathbb{C}[L_0]$ as vector spaces.*

(1) $I_m(V) = J_m(V) = 0$ *for all $m \in \mathbb{Z}$.*

(2) *There exist $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ such that*

$$\begin{aligned} L_m(f(L_0)) &= \lambda^m (L_0 + m\alpha) f(L_0 - m), \\ H_m(f(L_0)) &= \beta\lambda^m f(L_0 - m) \quad \text{for all } f(L_0) \in V,\ m \in \mathbb{Z}. \end{aligned}$$

Proof. (1) It is clear that V may be viewed as a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank 1 over $\mathcal{W}$, since $\mathcal{W}$ is a subalgebra containing $\mathcal{V}$ of $\mathcal{G}$. By Lemma 2.5, we have $I_m(V) = 0$ for all $m \in \mathbb{Z}$. Similarly, we may get $J_m(V) = 0$ for all $m \in \mathbb{Z}$.

(2) We view V as a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank 1 over $\mathcal{L}$. Then the conclusions are clear by Lemma 2.6. □

Theorem 3.2. *Let V be a $\mathcal{U}(\mathbb{C}L_0)$-free module of rank 1 over the Lie algebra $\mathcal{G}$.*

(1) *There exist $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ such that $V \cong \mathcal{A}(\lambda, \alpha, \beta)$ as $\mathcal{G}$-modules.*

(2) *V is an irreducible $\mathcal{G}$-module if and only if $V \cong \mathcal{A}(\lambda, \alpha, \beta)$ for some $\lambda \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$ with $(\alpha, \beta) \neq (0, 0)$.*

(3) *If V is isomorphic to $\mathcal{A}(\lambda, 0, 0)$ for some $\lambda \in \mathbb{C}^*$, then V has an irreducible submodule L_0V with codimension 1.*

Proof. (1) is clear from Lemma 3.1 and (3-1).

(2) and (3) follow from the irreducibility of $\mathcal{L}$-module $\Omega(\lambda, \alpha, \beta)$. □

From (3-1) and [Chen and Guo 2017] we can get the following theorem.

Theorem 3.3. *Let $\lambda, \lambda' \in \mathbb{C}^*$, $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$. Then $\mathcal{A}(\lambda, \alpha, \beta)$ and $\mathcal{A}(\lambda', \alpha', \beta')$ are isomorphic as $\mathcal{G}$-modules if and only if $\lambda = \lambda'$, $\alpha = \alpha'$, $\beta = \beta'$.*

4. $\mathcal{U}(\mathfrak{h})$-free modules over $\mathcal{G}$

In this section, we obtain all $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$. The necessary and sufficient conditions for these modules to be irreducible are determined. We also investigate the maximal proper submodules and the irreducible quotient modules when these modules are not irreducible. Furthermore, we determine the isomorphism classes of these modules. These conclusions are the main results of this paper.

4A. $\mathcal{U}(\mathfrak{h})$-*free modules over* $\mathcal{G}$. In this subsection, we determine the $\mathcal{G}$-module structures on $\mathcal{U}(\mathfrak{h})$, where $\mathfrak{h} = \mathrm{span}\{L_0, H_0\}$ is the Cartan subalgebra of $\mathcal{G}$.

For any $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$, denote by $T(\lambda, \delta) = \mathbb{C}[H_0, L_0]$ the polynomial algebra over $\mathbb{C}$. It is clear that $T(\lambda, \delta)$ is isomorphic to $\mathcal{U}(\mathfrak{h})$ as vector spaces. First, we consider the $\mathcal{L}$-module structures on $T(\lambda, \delta)$, where $\mathcal{L} = \mathrm{span}\{L_m, H_m \mid m \in \mathbb{Z}\}$. It is not hard to see that we may give $T(\lambda, \delta)$ an $\mathcal{L}$-module structure via the actions

$$\begin{aligned} &L_m(f(H_0, L_0)) = \lambda^m f(H_0, L_0-m)(L_0+m\delta), \\ &H_m(f(H_0, L_0)) = \lambda^m H_0 f(H_0, L_0-m) \quad \text{for all } m \in \mathbb{Z},\ f(H_0, L_0) \in T(\lambda, \delta). \end{aligned} \tag{4-1}$$

Note that $H_0 T(\lambda, \delta)$ is always a proper $\mathcal{L}$-submodule of $T(\lambda, \delta)$. Denote the quotient module $T(\lambda, \bar{\delta}) = T(\lambda, \delta)/H_0 T(\lambda, \delta) = \mathbb{C}[L_0]$, where $\bar{\delta}$ is the constant term of δ. It is easy to see that the actions of $\mathcal{L}$ on $T(\lambda, \bar{\delta})$ are

$$\begin{aligned} &L_m(f(L_0)) = \lambda^m f(L_0 - m)(L_0 + m\bar{\delta}), \\ &H_m(f(L_0)) = 0 \quad \text{for all } m \in \mathbb{Z},\ f(L_0) \in T(\lambda, \bar{\delta}). \end{aligned}$$

Furthermore, we have the following lemma.

Lemma 4.1. (1) *$T(\lambda, \bar{\delta})$ is an irreducible $\mathcal{L}$-module if and only if $\bar{\delta} \neq 0$.*

(2) *If $\bar{\delta} = 0$, then $T(\lambda, \bar{\delta})$ has an irreducible $\mathcal{L}$-submodule $L_0 T(\lambda, \bar{\delta})$ with codimension* 1.

Proof. This directly follows from the irreducibility of $\mathcal{L}$-module $\Omega(\lambda, \bar{\delta}, 0)$, which was introduced in Section 2B. □

By Theorem 3.1 in [Han et al. 2017], we have the following theorem.

Theorem 4.2. *Let M be a $\mathcal{U}(\mathcal{L})$-module such that M, when considered as a $\mathcal{U}(\mathfrak{h})$-module, is free of rank* 1. *Then $M \cong T(\lambda, \delta)$ for some $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$.*

Next, we investigate the $\mathcal{G}$-module structures on $\mathcal{U}(\mathfrak{h})$. We first define three families of $\mathcal{G}$-modules "$\Omega(\lambda, \delta, 0, 0)$, $\Omega(\lambda, \eta_1, \sigma_1, 0)$ and $\Omega(\lambda, \eta_2, 0, \sigma_2)$" as follows:

Definition 4.3. (1) For any $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$, the polynomial algebra $\mathbb{C}[H_0, L_0]$ can be endowed with a $\mathcal{G}$-module structure via the actions

$$\begin{aligned}
&L_m(f(H_0,L_0)) = \lambda^m f(H_0, L_0-m)(L_0+m\delta),\\
&H_m(f(H_0,L_0)) = \lambda^m H_0 f(H_0, L_0-m),\\
&I_m(\mathbb{C}[H_0,L_0]) = J_m(\mathbb{C}[H_0,L_0]) = 0 \ \text{ for all } m\in\mathbb{Z},\ f(H_0,L_0)\in\mathbb{C}[H_0,L_0].
\end{aligned}\tag{4-2}$$

We denote this module by $\Omega(\lambda, \delta, 0, 0)$.

(2) For any $\lambda \in \mathbb{C}^*$, $\eta_1 \in \mathbb{C}$, $\sigma_1(\neq 0) \in \mathbb{C}[H_0]$, the polynomial algebra $\mathbb{C}[H_0, L_0]$ has a $\mathcal{G}$-module structure with the actions

$$\begin{aligned}
L_m(f(H_0,L_0) &= \lambda^m f(H_0, L_0-m)(L_0 - mH_0 + m\eta_1),\\
H_m(f(H_0,L_0) &= \lambda^m H_0 f(H_0, L_0-m),\\
I_m(f(H_0,L_0) &= \lambda^m \sigma_1 f(H_0-1, L_0-m),\\
J_m(\mathbb{C}[H_0,L_0) &= 0 \quad \text{for all } m\in\mathbb{Z},\ \ f(H_0,L_0)\in\mathbb{C}[H_0,L_0].
\end{aligned}\tag{4-3}$$

This module is denoted by $\Omega(\lambda, \eta_1, \sigma_1, 0)$.

(3) For any $\lambda \in \mathbb{C}^*$, $\eta_2 \in \mathbb{C}$, $\sigma_2(\neq 0) \in \mathbb{C}[H_0]$, the polynomial algebra $\mathbb{C}[H_0, L_0]$ becomes a $\mathcal{G}$-module under the following actions

$$\begin{aligned}
L_m(f(H_0,L_0) &= \lambda^m f(H_0, L_0-m)(L_0 + mH_0 + m\eta_2),\\
H_m(f(H_0,L_0) &= \lambda^m H_0 f(H_0, L_0-m),\\
I_m(\mathbb{C}[H_0,L_0) &= 0,\\
J_m(f(H_0,L_0) &= \lambda^m \sigma_2 f(H_0+1, L_0-m),\\
&\qquad \text{for all } m\in\mathbb{Z},\ f(H_0,L_0)\in\mathbb{C}[H_0,L_0].
\end{aligned}\tag{4-4}$$

Denote this module by $\Omega(\lambda, \eta_2, 0, \sigma_2)$.

Remark 4.4. (1) It is clear that $\Omega(\lambda, \delta, 0, 0)$ is a $\mathcal{G}$-module by (4-1), since $\widetilde{IJ}$ is an ideal of $\mathcal{G}$. By direct computations we can verify that $\Omega(\lambda, \eta_1, \sigma_1, 0)$ and $\Omega(\lambda, \eta_2, 0, \sigma_2)$ are $\mathcal{G}$-modules.

(2) These three families of $\mathcal{G}$-modules in Definition 4.3, when considered as $\mathcal{U}(\mathfrak{h})$-modules, are all free of rank 1.

In the rest of this subsection, we will show that the three families of $\mathcal{G}$-modules in Definition 4.3 exhaust all $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ up to isomorphism. We break the arguments into the following several lemmas.

From now on, throughout this subsection, N always denotes the $\mathcal{U}(\mathfrak{h})$-free module of rank 1 over $\mathcal{G}$. We identify N with $\mathbb{C}[H_0, L_0]$ as vector spaces. Moreover, it is clear that we can view N as a $\mathcal{U}(\mathfrak{h})$-free module of rank 1 over $\mathcal{L}$. Therefore, by Theorem 4.2 there exist $\lambda \in \mathbb{C}^*$, $\delta(H_0) \in \mathbb{C}[H_0]$ such that

$$\begin{aligned} L_m(f(H_0, L_0)) &= \lambda^m f(H_0, L_0 - m)(L_0 + m\delta(H_0)), \\ H_m(f(H_0, L_0)) &= \lambda^m H_0 f(H_0, L_0 - m) \quad \text{for all } m \in \mathbb{Z},\ f(H_0, L_0) \in N. \end{aligned} \tag{4-5}$$

Lemma 4.5. *The actions of $\mathcal{G}$ on N are completely determined by $L_m(1)$, $H_m(1)$, $I_m(1)$, $J_m(1)$ for all $m \in \mathbb{Z}$.*

Proof. For any $f(H_0, L_0) \in N$, using the commutation relations of $\mathcal{G}$ we see that

$$\begin{aligned} L_m(f(H_0, L_0)) &= f(H_0, L_0 - m)L_m(1), \\ H_m(f(H_0, L_0)) &= f(H_0, L_0 - m)H_m(1), \\ I_m(f(H_0, L_0)) &= f(H_0 - 1, L_0 - m)I_m(1), \\ J_m(f(H_0, L_0)) &= f(H_0 + 1, L_0 - m)J_m(1) \quad \text{for all } m \in \mathbb{Z}. \end{aligned}$$

So Lemma 4.5 is clear. □

From Lemma 4.5, we only need to determine the actions of L_m, H_m, I_m, J_m on 1 for all $m \in \mathbb{Z}$.

Lemma 4.6. *Assume that there exist $k, l \in \mathbb{Z}$ such that $I_k(1) = J_l(1) = 0$. Then $I_m(N) = J_m(N) = 0$ for all $m \in \mathbb{Z}$.*

Proof. For any $i, j \in \mathbb{Z}_+$, we have

$$\begin{aligned} I_k(H_0^i L_0^j) &= (H_0 - 1)^i I_k L_0^j = (H_0 - 1)^i (L_0 - k)^j I_k(1) = 0, \\ J_l(H_0^i L_0^j) &= (H_0 + 1)^i J_l L_0^j = (H_0 + 1)^i (L_0 - l)^j J_l(1) = 0. \end{aligned}$$

Thus $I_k(N) = J_l(N) = 0$. Using the defining relations of $\mathcal{G}$ we see that $I_m(N) = J_m(N) = 0$ for all $m \in \mathbb{Z}$. □

Lemma 4.7. *Suppose that $I_m(1)$ is nonzero for any $m \in \mathbb{Z}$. Denote $I_0(1) = \sum_{i=0}^{q_0} c_{0i}(H_0)L_0^i$, where $q_0 \in \mathbb{Z}_+$, $c_{0i}(H_0) \in \mathbb{C}[H_0]$ for $i = 0, 1, \dots, q_0$.*

(1) *In (4-5), $\delta(H_0) = \alpha H_0 + \beta$, for some $\alpha \in \mathbb{Z}_{\geq -1}$, $\beta \in \mathbb{C}$.*

(2) $\deg_{L_0}(I_m(1)) = \alpha + 1 = q_0$ *and*

$$I_m(1) = \lambda^m c_{0(\alpha+1)}(H_0)L_0^{\alpha+1} + (\text{lower} - \text{degree terms in } L_0) \quad \textit{for all } m \in \mathbb{Z}.$$

(3) *If $\alpha \geq 0$, then for any $m \in \mathbb{Z}^*$, the coefficient of L_0^α in $I_m(1)$ is*

$$m\lambda^m(\alpha + 1)c_{0(\alpha+1)}(H_0)\big(\alpha H_0 + \beta - \tfrac{1}{2}\alpha\big).$$

(4) *If $\alpha \geq 0$, then $\alpha = 1$.*

Proof. (1) For any $n \in \mathbb{Z}^*$, denote

$$I_n(1) = \sum_{i=0}^{q_n} c_{ni}(H_0) L_0^i,$$

where $q_n \in \mathbb{Z}_+$, $c_{ni}(H_0) \in \mathbb{C}[H_0]$ and $c_{nq_n}(H_0) \neq 0$. For any $m \in \mathbb{Z}$, we compute

$$\begin{aligned}
&(n-m) I_{m+n}(1) \\
&= [L_m, I_n](1) = L_m I_n(1) - I_n L_m(1) \\
&= \sum_{i=0}^{q_n} L_m c_{ni}(H_0) L_0^i - I_n\big(\lambda^m (L_0 + m\delta(H_0))\big) \\
&= \sum_{i=0}^{q_n} c_{ni}(H_0)(L_0 - m)^i L_m(1) - \big(\lambda^m (L_0 - n + m\delta(H_0 - 1))\big) I_n(1) \\
&= \sum_{i=0}^{q_n} c_{ni}(H_0)(L_0 - m)^i \lambda^m (L_0 + m\delta(H_0)) - \sum_{i=0}^{q_n} \lambda^m (L_0 - n + m\delta(H_0 - 1)) c_{ni}(H_0) L_0^i \\
&= \lambda^m (L_0 + m\delta(H_0)) \sum_{i=0}^{q_n} c_{ni}(H_0)(L_0 - m)^i - \lambda^m (L_0 - n + m\delta(H_0 - 1)) \sum_{i=0}^{q_n} c_{ni}(H_0) L_0^i.
\end{aligned}$$

In the last equality, the coefficients of $L_0^{q_n}$ and $L_0^{q_n - 1}$ are respectively

$$\lambda^m c_{nq_n}(H_0)\big(m\delta(H_0) - mq_n + n - m\delta(H_0 - 1)\big) \tag{4-6}$$

and

$$\begin{aligned} m^2 q_n \lambda^m c_{nq_n}(H_0)\big(\tfrac{1}{2}(q_n - 1) - \delta(H_0)\big) \\ + \lambda^m c_{n(q_n - 1)}(H_0)\big(m\delta(H_0) - m\delta(H_0 - 1) - mq_n + m + n\big). \end{aligned} \tag{4-7}$$

Taking $m = n$, from equality (4-6) we deduce

$$n\lambda^n c_{nq_n}(H_0)\big(\delta(H_0) - q_n + 1 - \delta(H_0 - 1)\big) = 0,$$

which implies that $\delta(H_0) = \alpha H_0 + \beta$ and $q_n = \alpha + 1$ for some $\alpha, \beta \in \mathbb{C}$. Note that $q_n \in \mathbb{Z}_+$, thus $\alpha \in \mathbb{Z}_{\geq -1}$, $\beta \in \mathbb{C}$.

(2) From (1), we see that the equality (4-6) becomes

$$\lambda^m c_{nq_n}(H_0)(n - m).$$

Thus for any $m(\neq n) \in \mathbb{Z}$, we have $\deg_{L_0}(I_{m+n}(1)) = q_n = \alpha + 1$ and

$$I_{m+n}(1) = \lambda^m c_{nq_n}(H_0) L_0^{q_n} + (\text{lower} - \text{degree terms in } L_0). \tag{4-8}$$

Taking $m=-n$, we see that $q_0=q_n=\alpha+1$, $c_{0q_0}(H_0)=\lambda^{-n}c_{nq_n}(H_0)$. Using equality (4-8) we see that for any $m(\neq 2n)\in\mathbb{Z}$,

$$\begin{aligned}&\deg_{L_0}(I_m(1))=\alpha+1,\\ &I_m(1)=\lambda^m c_{0(\alpha+1)}(H_0)L_0^{\alpha+1}+(\text{lower}-\text{degree terms in } L_0),\end{aligned}\tag{4-9}$$

If we substitute n' for n in the beginning, where $n'(\neq n)$ is nonzero, then we can similarly deduce that (4-9) holds for any $m\neq 2n'$. Therefore, the equality (4-9) holds for any $m\in\mathbb{Z}$.

(3) Using (1) we see that equality (4-7) reads

$$m^2q_n\lambda^m c_{nq_n}(H_0)\big(\tfrac{1}{2}(q_n-1)-\delta(H_0)\big)+n\lambda^m c_{n(q_n-1)}(H_0).$$

Taking $m=n(\neq 0)$, we get

$$c_{m\alpha}(H_0)=c_{m(q_m-1)}(H_0)=mq_mc_{mq_m}(H_0)\big(\delta(H_0)-\tfrac{1}{2}(q_m-1)\big).\tag{4-10}$$

So (3) is clear from (2) and equality (4-10).

(4) For $m,n\in\mathbb{Z}^*$, we may denote

$$I_n(1)=\sum_{j=0}^{\alpha+1}c_{nj}(H_0)L_0^j,\quad I_m(1)=\sum_{l=0}^{\alpha+1}c_{ml}(H_0)L_0^l,$$

where $c_{nj}(H_0),c_{ml}(H_0)\in\mathbb{C}[H_0]$ and $c_{n(\alpha+1)}(H_0),c_{m(\alpha+1)}(H_0)\neq 0$. We compute

$$\begin{aligned}0&=[I_n,I_m](1)=I_nI_m(1)-I_mI_n(1)\\ &=\sum_{l=0}^{\alpha+1}I_nc_{ml}(H_0)L_0^l-\sum_{j=0}^{\alpha+1}I_mc_{nj}(H_0)L_0^j\\ &=\sum_{j=0}^{\alpha+1}\sum_{l=0}^{\alpha+1}c_{nj}(H_0)c_{ml}(H_0-1)L_0^j(L_0-n)^l\\ &\qquad\qquad-\sum_{j=0}^{\alpha+1}\sum_{l=0}^{\alpha+1}c_{nj}(H_0-1)c_{ml}(H_0)(L_0-m)^jL_0^l.\end{aligned}$$

In the last equality, the coefficients of $L_0^{2\alpha+2}$ and $L_0^{2\alpha+1}$ are respectively

$$c_{n(\alpha+1)}(H_0)c_{m(\alpha+1)}(H_0-1)-c_{n(\alpha+1)}(H_0-1)c_{m(\alpha+1)}(H_0),\tag{4-11}$$

and

$$\begin{aligned}&c_{n(\alpha+1)}(H_0)c_{m(\alpha+1)}(H_0-1)(-n)(\alpha+1)\\ &\quad+c_{n\alpha}(H_0)c_{m(\alpha+1)}(H_0-1)+c_{n(\alpha+1)}(H_0)c_{m\alpha}(H_0-1)\\ &\quad-\big(c_{n(\alpha+1)}(H_0-1)c_{m(\alpha+1)}(H_0)(-m)(\alpha+1)\\ &\qquad+c_{n\alpha}(H_0-1)c_{m(\alpha+1)}(H_0)+c_{n(\alpha+1)}(H_0-1)c_{m\alpha}(H_0)\big).\end{aligned}\tag{4-12}$$

Using (2) and (3) we see that (4-11) and (4-12) read as

$$\lambda^{m+n}c_{0(\alpha+1)}(H_0)c_{0(\alpha+1)}(H_0-1)-\lambda^{m+n}c_{0(\alpha+1)}(H_0-1)c_{0(\alpha+1)}(H_0)$$

and

$$(n-m)(\alpha+1)(\alpha-1)\lambda^{m+n}c_{0(\alpha+1)}(H_0)c_{0(\alpha+1)}(H_0-1),$$

which implies

$$(n-m)(\alpha+1)(\alpha-1)\lambda^{m+n}c_{0(\alpha+1)}(H_0)c_{0(\alpha+1)}(H_0-1)=0$$

for any $m, n\in\mathbb{Z}^*$. Thus $\alpha=1$. This completes the proof. □

Proposition 4.8. *Suppose that $I_m(1)$ is nonzero for any $m\in\mathbb{Z}$. Then $I_m(1)\in\mathbb{C}[H_0]$ for all $m\in\mathbb{Z}$.*

Proof. It is sufficient to show that $\alpha=-1$ by Lemma 4.7. Now we assume that $\alpha\geq 0$. Then $\alpha=1$ by Lemma 4.7. Denote

$$I_0(1)=c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0),$$

where $c_{02}(H_0), c_{01}(H_0), c_{00}(H_0)\in\mathbb{C}[H_0]$ with $c_{02}(H_0)\neq 0$. Then by Lemma 4.7 we may write

$$I_1(1)=\lambda c_{02}(H_0)L_0^2+2\lambda\big(H_0+\beta-\tfrac{1}{2}\big)c_{02}(H_0)L_0+c_{10}(H_0)$$

for some $c_{10}(H_0)\in\mathbb{C}[H_0]$. We compute

$$\begin{aligned}
I_1(1)&=[H_1,I_0](1)=H_1I_0(1)-I_0H_1(1)\\
&=H_1\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big)-I_0(\lambda H_0)\\
&=\big(c_{02}(H_0)(L_0-1)^2+c_{01}(H_0)(L_0-1)+c_{00}(H_0)\big)(\lambda H_0)\\
&\qquad-\lambda(H_0-1)\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big)\\
&=\lambda c_{02}(H_0)L_0^2+\lambda\big(-2H_0c_{02}(H_0)+c_{01}(H_0)\big)L_0\\
&\qquad+\lambda\big(H_0c_{02}(H_0)-H_0c_{01}(H_0)+c_{00}(H_0)\big),\\
I_1(1)&=[I_0,L_1](1)=I_0L_1(1)-L_1I_0(1)\\
&=I_0\big(\lambda(L_0+H_0+\beta)\big)-L_1\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big)\\
&=\big(\lambda(L_0+H_0-1+\beta)\big)\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big)\\
&\qquad-\big(c_{02}(H_0)(L_0-1)^2+c_{01}(H_0)(L_0-1)+c_{00}(H_0)\big)\big(\lambda(L_0+H_0+\beta)\big)\\
&=\lambda c_{02}(H_0)L_0^2+2\lambda(H_0+\beta-\tfrac{1}{2})c_{02}(H_0)L_0\\
&\qquad-\lambda\big((c_{02}(H_0)-c_{01}(H_0))(H_0+\beta)+c_{00}(H_0)\big).
\end{aligned}$$

Then by comparing the coefficients of L_0 and the constant terms, we obtain

$$\lambda(-2H_0c_{02}(H_0)+c_{01}(H_0)) = 2\lambda\big(H_0+\beta-\tfrac{1}{2}\big)c_{02}(H_0),$$
$$\lambda\big(H_0c_{02}(H_0)-H_0c_{01}(H_0)+c_{00}(H_0)\big) = -\lambda\big((c_{02}(H_0)-c_{01}(H_0))(H_0+\beta)+c_{00}(H_0)\big).$$

Thus we deduce

$$\begin{aligned} c_{01}(H_0) &= (4H_0+2\beta-1)c_{02}(H_0), \\ c_{00}(H_0) &= (2H_0+\beta)(2H_0+\beta-1)c_{02}(H_0). \end{aligned} \tag{4-13}$$

Finally, we consider

$$\begin{aligned} 0 &= [I_1, I_0](1) = I_1I_0(1)-I_0I_1(1) \\ &= \big(c_{02}(H_0-1)(L_0-1)^2+c_{01}(H_0-1)(L_0-1)+c_{00}(H_0-1)\big) \\ &\qquad \times\big(\lambda c_{02}(H_0)L_0^2+2\lambda\big(H_0+\beta-\tfrac{1}{2}\big)c_{02}(H_0)L_0+c_{10}(H_0)\big) \\ &\quad -\big(\lambda c_{02}(H_0-1)L_0^2+2\lambda\big(H_0-1+\beta-\tfrac{1}{2}\big)c_{02}(H_0-1)L_0+c_{10}(H_0-1)\big) \\ &\qquad \times\big(c_{02}(H_0)L_0^2+c_{01}(H_0)L_0+c_{00}(H_0)\big). \end{aligned} \tag{4-14}$$

In the equality (4-14), the coefficient of L_0^3 is

$$\lambda\big(c_{02}(H_0)c_{01}(H_0-1)-c_{02}(H_0-1)c_{01}(H_0)\big). \tag{4-15}$$

Substituting (4-13) into (4-15), we get

$$-4\lambda c_{02}(H_0)c_{02}(H_0-1) = 0,$$

which implies $c_{02}(H_0) = 0$. This is a contradiction, completing □

Proposition 4.9. *Suppose that $J_m(1)$ is nonzero for any $m \in \mathbb{Z}$. Then $J_m(1) \in \mathbb{C}[H_0]$ for all $m \in \mathbb{Z}$.*

Proof. The proof is similar to that of Lemma 4.7 and Proposition 4.8. □

Lemma 4.10. *For any $m \in \mathbb{Z}$, $I_m(1) = \lambda^m I_0(1)$, $J_m(1) = \lambda^m J_0(1) \in \mathbb{C}[H_0]$.*

Proof. For any $m, n \in \mathbb{Z}$, using Proposition 4.8 and equality (4-5) we see that

$$\begin{aligned} I_{m+n}(1) &= [H_m, I_n](1) = H_mI_n(1) - I_nH_m(1) \\ &= H_m(1)I_n(1) - I_n(\lambda^m H_0) = \lambda^m H_0 I_n(1) - \lambda^m(H_0-1)I_n(1) \\ &= \lambda^m I_n(1). \end{aligned}$$

Taking $n = 0$, we get $I_m(1) = \lambda^m I_0(1)$ for $m \in \mathbb{Z}$. Similarly, we may get $J_m(1) = \lambda^m J_0(1)$ for $m \in \mathbb{Z}$. □

Lemma 4.11. (1) *If $I_0(1) \neq 0$, then $\delta = -H_0+\eta'$ for some $\eta' \in \mathbb{C}$.*

(2) *If $J_0(1) \neq 0$, then $\delta = H_0+\eta''$ for some $\eta'' \in \mathbb{C}$.*

Proof. (1) For any $m, n \in \mathbb{Z}$, using Proposition 4.8 and equality (4-5) we compute

$$\begin{aligned}(n-m)I_{m+n}(1) &= [L_m, I_n](1) = L_m I_n(1) - I_n L_m(1)\\ &= L_m(1)I_n(1) - I_n\big(\lambda^m(L_0+m\delta(H_0))\big)\\ &= \lambda^m(L_0+m\delta(H_0))I_n(1) - \lambda^m(L_0-n)I_n(1) - m\lambda^m\delta(H_0-1)I_n(1)\\ &= \lambda^m\big(m(\delta(H_0)-\delta(H_0-1))+n\big)I_n(1),\end{aligned}$$

which yields $(n-m) = \big(m(\delta(H_0)-\delta(H_0-1))+n\big)$ by Lemma 4.10 and $I_0(1) \neq 0$. Thus $\delta(H_0) - \delta(H_0-1) = -1$, which forces $\delta(H_0) = -H_0 + \eta'$ for some $\eta' \in \mathbb{C}$.

(2) Proved similarly to (1). □

Now we state the main results of this subsection.

Theorem 4.12. *Let N be a $\mathcal{U}(\mathcal{G})$-module such that N, when considered as a $\mathcal{U}(\mathfrak{h})$-module, is free of rank* 1.

(a) *There exist $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$ such that $L_1(1) = \lambda(L_0+\delta)$, $H_1(1) = \lambda H_0$.*

(b) *If $I_0(1) = J_0(1) = 0$, then $N \cong \Omega(\lambda, \delta, 0, 0)$ as $\mathcal{U}(\mathcal{G})$-modules.*

(c) *If $I_0(1) \neq 0$, $J_0(1) = 0$, then $\delta = -H_0 + \eta'$ for some $\eta' \in \mathbb{C}$, and $N \cong \Omega(\lambda, \eta', \sigma_1, 0)$ as $\mathcal{U}(\mathcal{G})$-modules, where $\sigma_1 = I_0(1) \in \mathbb{C}[H_0]$.*

(d) *If $I_0(1) = 0$, $J_0(1) \neq 0$, then $\delta = H_0 + \eta''$ for some $\eta'' \in \mathbb{C}$, and $N \cong \Omega(\lambda, \eta'', 0, \sigma_2)$ as $\mathcal{U}(\mathcal{G})$-modules, where $\sigma_2 = J_0(1) \in \mathbb{C}[H_0]$.*

(e) *The case $I_0(1) \neq 0$, $J_0(1) \neq 0$ does not exist.*

Proof. (a) follows from equality (4-5).

(b) follows from Lemmas 4.5, 4.6 and equalities (4-2), (4-5).

(c) and (d) follow from Lemmas 4.5, 4.6, 4.10, 4.11 and equalities (4-3), (4-4), (4-5).

(e) follows from Lemma 4.11. □

4B. ***Irreducibility of $\mathcal{U}(\mathfrak{h})$-free modules over $\mathcal{G}$.*** In Section 4A, we determined all $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$. These modules have three families $\Omega(\lambda, \delta, 0, 0)$, $\Omega(\lambda, \eta_1, \sigma_1, 0)$ and $\Omega(\lambda, \eta_2, 0, \sigma_2)$ (see Definition 4.3). Here we will give the necessary and sufficient conditions for these modules to be irreducible. Furthermore, we find the maximal proper submodules and obtain irreducible quotient modules when these modules are not irreducible.

Theorem 4.13. *Let $\lambda \in \mathbb{C}^*$, $\delta \in \mathbb{C}[H_0]$. $\bar{\delta}$ denotes the constant term of δ.*

(1) *$\Omega(\lambda, \delta, 0, 0)$ always has a proper $\mathcal{G}$-submodule $H_0\Omega(\lambda, \delta, 0, 0)$. Denote the quotient module $\Omega_1(\lambda, \delta, 0, 0) = \Omega(\lambda, \delta, 0, 0)/H_0\Omega(\lambda, \delta, 0, 0)$.*

(2) *$\Omega_1(\lambda, \delta, 0, 0)$ is an irreducible $\mathcal{G}$-module if and only if $\bar{\delta} \neq 0$.*

(3) $\Omega_1(\lambda, \delta, 0, 0)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_1(\lambda, \delta, 0, 0)$ *with codimension* 1 *when* $\bar{\delta} = 0$. *Consequently, the quotient module*

$$\Omega_1(\lambda, \delta, 0, 0)/L_0\Omega_1(\lambda, \delta, 0, 0)$$

is irreducible.

Proof. These directly follow from the properties of $\mathcal{L}$-module $T(\lambda, \delta)$, which were described in Lemma 4.1. □

Theorem 4.14. *Let* $\lambda \in \mathbb{C}^*,\ \eta_1 \in \mathbb{C},\ \sigma_1(\neq 0) \in \mathbb{C}[H_0]$.

(1) $\Omega(\lambda, \eta_1, \sigma_1, 0)$ *is an irreducible* $\mathcal{G}$*-module if and only if* $\sigma_1 \in \mathbb{C}^*$.

(2) *If* $\sigma_1 = H_0 + \beta$, *where* $\beta \in \mathbb{C}$, *then* $\Omega(\lambda, \eta_1, \sigma_1, 0)$ *has a proper* $\mathcal{G}$*-submodule* $\sigma_1\Omega(\lambda, \eta_1, \sigma_1, 0)$. *Moreover, denote the quotient module*

$$\Omega_1(\lambda, \eta_1, \sigma_1, 0) = \Omega(\lambda, \eta_1, \sigma_1, 0)/\sigma_1\Omega(\lambda, \eta_1, \sigma_1, 0) = \mathbb{C}[L_0].$$

(i) $\Omega_1(\lambda, \eta_1, \sigma_1, 0)$ *is irreducible if and only if* $(\eta_1, \beta) \neq (0, 0)$.
(ii) $\Omega_1(\lambda, \eta_1, \sigma_1, 0)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_1(\lambda, \eta_1, \sigma_1, 0)$ *with codimension* 1 *when* $(\eta_1, \beta) = (0, 0)$. *Consequently,*

$$\Omega_1(\lambda, \eta_1, \sigma_1, 0)/L_0\Omega_1(\lambda, \eta_1, \sigma_1, 0)$$

is irreducible.

(3) *If* $\deg(\sigma_1) = n > 1$, *we may write*

$$\sigma_1 = c\sigma_{11}\sigma_{12}\cdots\sigma_{1n},$$

where $\sigma_{1i} = H_0 + \beta_i$, $\beta_i \in \mathbb{C}$, $c \in \mathbb{C}^*$, *for* $i = 1, 2, \ldots, n$. *Then* $\sigma_{1i}\Omega(\lambda, \eta_1, \sigma_1, 0)$ *is a proper* $\mathcal{G}$*-submodule of* $\Omega(\lambda, \eta_1, \sigma_1, 0)$ *for* $i = 1, 2, \ldots, n$. *Furthermore, denote the quotient module*

$$\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0) = \Omega(\lambda, \eta_1, \sigma_1, 0)/\sigma_{1i}\Omega(\lambda, \eta_1, \sigma_1, 0).$$

(i) $\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)$ *is irreducible if and only if* $(\eta_1, \beta_i) \neq (0, 0)$.
(ii) $\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)$ *with codimension* 1 *when* $(\eta_1, \beta_i) = (0, 0)$. *Consequently,*

$$\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)/L_0\Omega_{1i}(\lambda, \eta_1, \sigma_1, 0)$$

is irreducible.

Proof. (1) ($\Rightarrow$). Let $\Omega(\lambda, \eta_1, \sigma_1, 0)$ be an irreducible $\mathcal{G}$-module. Assume that $\deg_{H_0}(\sigma_1) \geq 1$. It is easy to see that $\sigma_1\Omega(\lambda, \eta_1, \sigma_1, 0)$ is a proper $\mathcal{G}$-submodule of $\Omega(\lambda, \eta_1, \sigma_1, 0)$, which contradicts that $\Omega(\lambda, \eta_1, \sigma_1, 0)$ is irreducible.

($\Leftarrow$). Suppose $\sigma_1 \in \mathbb{C}^*$. For arbitrary nonzero $f(H_0, L_0) \in \Omega(\lambda, \eta_1, \sigma_1, 0)$, we write

$$f(H_0, L_0) = \sum_{j=0}^{q} a_j(H_0) L_0^j,$$

where $q \in \mathbb{Z}_+, a_j(H_0) \in \mathbb{C}[H_0], a_q(H_0) \neq 0$. Let $\langle f(H_0, L_0)\rangle$ denote the $\mathcal{G}$-submodule of $\Omega(\lambda, \eta_1, \sigma_1, 0)$ generated by $f(H_0, L_0)$.

If $q > 0$, we compute

$$\begin{aligned} &H_1(f(H_0, L_0)) - \lambda H_0 f(H_0, L_0) \\ &\quad = \lambda H_0 \sum_{j=0}^{q} a_j(H_0)(L_0 - 1)^j - \lambda H_0 \sum_{j=0}^{q} a_j(H_0) L_0^j \\ &\quad = -q\lambda H_0 a_q(H_0) L_0^{q-1} + (\text{lower} - \text{degree terms in } L_0). \end{aligned}$$

Denote

$$f_1(H_0, L_0) = H_1(f(H_0, L_0)) - \lambda H_0 f(H_0, L_0) \in \langle f(H_0, L_0)\rangle$$

with $\deg_{L_0}(f_1(H_0, L_0)) = q - 1$. Therefore, without loss of generality, we may assume that $\deg_{L_0}(f H_0, L_0) = q = 0$. Then we write

$$f(H_0, L_0) = \sum_{i=0}^{p} c_i H_0^i,$$

where $p \in \mathbb{Z}_+$, $c_i \in \mathbb{C}$ with $c_p \neq 0$.

If $p = 0$, then $\langle f(H_0, L_0)\rangle = \Omega(\lambda, \eta_1, \sigma_1, 0)$ is clear. If $p > 0$, we deduce that

$$\begin{aligned} &I_1(f(H_0, L_0)) - \lambda\sigma_1 f(H_0, L_0) \\ &\quad = \lambda\sigma_1 \sum_{i=0}^{p} c_i (H_0 - 1)^i - \lambda\sigma_1 \sum_{i=0}^{p} c_i H_0^i \\ &\quad = -\lambda\sigma_1 p c_p H_0^{p-1} + (\text{lower} - \text{degree terms in } H_0). \end{aligned}$$

Thus we can get $1 \in \langle f(H_0, L_0)\rangle$, which implies $\langle f(H_0, L_0)\rangle = \Omega(\lambda, \eta_1, \sigma_1, 0)$. Hence $\Omega(\lambda, \eta_1, \sigma_1, 0)$ is irreducible.

(2) First, it is trivial to see that $\sigma_1\Omega(\lambda, \eta_1, \sigma_1, 0)$ is a proper $\mathcal{G}$-submodule of $\Omega(\lambda, \eta_1, \sigma_1, 0)$. From equality (4-3), we see that the actions of $\mathcal{G}$ on the quotient module $\Omega_1(\lambda, \eta_1, \sigma_1, 0)$ are

$$\begin{aligned} L_m(f(L_0)) &= \lambda^m f(L_0 - m)(L_0 + m\beta + m\eta_1), \\ H_m(f(L_0)) &= -\lambda^m \beta f(L_0 - m), \\ I_m(\Omega_1(\lambda, \eta_1, \sigma_1, 0)) &= J_m(\Omega_1(\lambda, \eta_1, \sigma_1, 0)) = 0 \quad \text{for all } m \in \mathbb{Z}. \end{aligned}$$

Then (i), (ii) follow from irreducibility of $\mathcal{L}$-module $\Omega(\lambda, \beta + \eta_1, -\beta)$, which was introduced in Section 2B.

(3) It is clear that $\sigma_{1i}\Omega(\lambda, \eta_1, \sigma_1, 0)$ is a proper $\mathcal{G}$-submodule of $\Omega(\lambda, \eta_1, \sigma_1, 0)$. The remaining parts are similar to (2). $\square$

Theorem 4.15. *Let* $\lambda \in \mathbb{C}^*$, $\eta_2 \in \mathbb{C}$, $\sigma_2(\neq 0) \in \mathbb{C}[H_0]$.

(1) $\Omega(\lambda, \eta_2, 0, \sigma_2)$ *is an irreducible* $\mathcal{G}$*-module if and only if* $\sigma_2 \in \mathbb{C}^*$.

(2) *If* $\sigma_2 = H_0 + \gamma$, *where* $\gamma \in \mathbb{C}$, *then* $\Omega(\lambda, \eta_2, 0, \sigma_2)$ *has a proper* $\mathcal{G}$*-submodule* $\sigma_2\Omega(\lambda, \eta_2, 0, \sigma_2)$. *Moreover, denote the quotient module*

$$\Omega_2(\lambda, \eta_2, 0, \sigma_2) = \Omega(\lambda, \eta_2, 0, \sigma_2)/\sigma_2\Omega(\lambda, \eta_2, 0, \sigma_2) = \mathbb{C}[L_0].$$

(i) $\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *is irreducible if and only if* $(\eta_2, \gamma) \neq (0, 0)$.

(ii) $\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *with codimension* 1 *when* $(\eta_2, \gamma) = (0, 0)$. *Consequently,*

$$\Omega_2(\lambda, \eta_2, 0, \sigma_2)/L_0\Omega_2(\lambda, \eta_2, 0, \sigma_2)$$

is irreducible.

(3) *If* $\deg(\sigma_2) = n > 1$, *we may write*

$$\sigma_2 = c'\sigma_{21}\sigma_{22} \cdots \sigma_{2n},$$

where $\sigma_{2i} = H_0+\gamma_i$, $\gamma_i \in \mathbb{C}$, $c' \in \mathbb{C}^*$, *for* $i = 1, 2, \ldots, n$. *Then* $\sigma_{2i}\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *is a proper* $\mathcal{G}$*-submodule of* $\Omega_2(\lambda, \eta_2, 0, \sigma_2)$ *for* $i = 1, 2, \ldots, n$. *Furthermore, denote* $\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2) = \Omega_2(\lambda, \eta_2, 0, \sigma_2)/\sigma_{2i}\Omega_2(\lambda, \eta_2, 0, \sigma_2)$.

(i) $\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)$ *is irreducible if and only if* $(\eta_2, \gamma_i) \neq (0, 0)$.

(ii) $\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)$ *has an irreducible* $\mathcal{G}$*-submodule* $L_0\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)$ *with codimension* 1 *when* $(\eta_2, \gamma_i) = (0, 0)$. *Consequently,*

$$\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)/L_0\Omega_{2i}(\lambda, \eta_2, 0, \sigma_2)$$

is irreducible.

Proof. The proof is similar to that of Theorem 4.14. □

Remark 4.16. By Theorems 3.2, 4.13, 4.14 and 4.15, we may get many new irreducible modules over the planar Galilean conformal algebra $\mathcal{G}$.

4C. ***Isomorphism classes of*** $\mathcal{U}(\mathfrak{h})$***-free modules over*** $\mathcal{G}$***.*** In Section 4A, we showed that three families of modules $\Omega(\lambda, \delta, 0, 0)$, $\Omega(\lambda, \eta_1, \sigma_1, 0)$ and $\Omega(\lambda, \eta_2, 0, \sigma_2)$ exhaust all $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$. Now we determine the isomorphism classes of these modules.

Theorem 4.17. *Let* $\lambda, \lambda' \in \mathbb{C}^*$, $\delta, \delta' \in \mathbb{C}[H_0]$, $\eta_1, \eta_1', \eta_2, \eta_2' \in \mathbb{C}$, $\sigma_1, \sigma_1', \sigma_2, \sigma_2' \in \mathbb{C}[H_0] \setminus \{0\}$.

(1) $\Omega(\lambda, \delta, 0, 0) \cong \Omega(\lambda', \delta', 0, 0)$ *if and only if* $\lambda = \lambda'$, $\delta = \delta'$.

(2) $\Omega(\lambda, \eta_1, \sigma_1, 0) \cong \Omega(\lambda', \eta_1', \sigma_1', 0)$ *if and only if* $\lambda = \lambda'$, $\eta_1 = \eta_1'$, $\sigma_1 = \sigma_1'$.

(3) $\Omega(\lambda, \eta_2, 0, \sigma_2) \cong \Omega(\lambda', \eta_2', 0, \sigma_2')$ *if and only if* $\lambda = \lambda'$, $\eta_2 = \eta_2'$, $\sigma_2 = \sigma_2'$.

(4) *Any two of* $\Omega(\lambda, \delta, 0, 0)$, $\Omega(\lambda, \eta_1, \sigma_1, 0)$, $\Omega(\lambda, \eta_2, 0, \sigma_2)$ *are not isomorphic.*

Proof. (1) The "sufficiency" is trivial. We only need to show the "necessity". Suppose

$$\varphi : \Omega(\lambda, \delta, 0, 0) \to \Omega(\lambda', \delta', 0, 0)$$

is a $\mathcal{G}$-module isomorphism.

Claim 1. $\varphi(1) \in \mathbb{C}[H_0]$.

Assume that $\varphi(1) = \sum_{i=0}^{q} a_i(H_0)L_0^i$, where $q > 0, a_i(H_0) \in \mathbb{C}[H_0]$ for $0 \le i \le q$ and $a_q(H_0) \neq 0$. Since φ is a $\mathcal{G}$-module isomorphism, we get $H_1(\varphi(1)) = \varphi(H_1(1)) = \varphi(\lambda H_0) = \lambda H_0(\varphi(1))$. From equality (4-2) we obtain

$$\begin{aligned} H_1(\varphi(1)) &= \lambda' H_0 \sum_{i=0}^{q} a_i(H_0)(L_0-1)^i \\ &= \lambda' H_0 a_q(H_0)L_0^q + \lambda' H_0(-q a_q(H_0) + a_{q-1}(H_0))L_0^{q-1} \\ &\qquad + (\text{lower}-\text{degree terms in } L_0), \end{aligned}$$

$$\begin{aligned} \lambda H_0(\varphi(1)) &= \lambda H_0 \sum_{i=0}^{q} a_i(H_0)L_0^i \\ &= \lambda H_0 a_q(H_0)L_0^q + \lambda H_0 a_{q-1}(H_0)L_0^{q-1} + (\text{lower}-\text{degree terms in } L_0). \end{aligned}$$

By comparing the coefficients of L_0^q and L_0^{q-1}, we deduce

$$\lambda = \lambda', \quad -\lambda' q H_0 a_q(H_0) = 0.$$

But $-\lambda' q H_0 a_q(H_0) = 0$ is impossible. So $\varphi(1) \in \mathbb{C}[H_0]$. Claim 1 is proved.

Now we may assume $\varphi(1) = \sum_{j=0}^{p} c_j H_0^j$, where $p \in \mathbb{Z}_+$, $c_j \in \mathbb{C}$ for $0 \le j \le p$ and $c_p \neq 0$. We consider the equality

$$L_1(\varphi(1)) = \varphi(L_1(1)) = \varphi(\lambda(L_0+\delta)) = \lambda(L_0+\delta)(\varphi(1)).$$

It is clear that

$$L_1(\varphi(1)) = \lambda'\varphi(1)(L_0+\delta'), \quad \lambda(L_0+\delta)(\varphi(1)) = \lambda\varphi(1)(L_0+\delta),$$

which imply $\lambda = \lambda'$, $\delta = \delta'$.

(2) The "sufficiency" is clear. We only need to show the "necessity". Suppose that

$$\varphi' : \Omega(\lambda, \eta_1, \sigma_1, 0) \to \Omega(\lambda', \eta_1', \sigma_1', 0)$$

is a $\mathcal{G}$-module isomorphism. Then $\varphi' : \Omega(\lambda, \eta_1, \sigma_1, 0) \to \Omega(\lambda', \eta_1', \sigma_1', 0)$ is an $\mathcal{L}$-module isomorphism. From (1) and equalities (4-2), (4-3) it is not hard to see that $\lambda = \lambda'$, $\eta_1 = \eta_1'$ and $\varphi'(1) \in \mathbb{C}[H_0]$.

Set $\varphi'(1)=\sum_{k=0}^{t} d_k H_0^k$, where $t\in\mathbb{Z}_+$, $d_k\in\mathbb{C}$ for $0\le k\le t$ and $d_t\ne 0$. Note that $\varphi'(\lambda\sigma_1)=\varphi'(I_1(1))$. We compute

$$\varphi'(\lambda\sigma_1)=\lambda\sigma_1\varphi'(1)=\lambda\sigma_1\sum_{k=0}^{t} d_k H_0^k,$$
$$\varphi'(I_1(1))=I_1(\varphi'(1))=\lambda'\sigma_1'\sum_{k=0}^{t} d_k (H_0-1)^k.$$

By comparing the coefficients of H_0^t we obtain $\sigma_1=\sigma_1'$.

(3) is similar to (2).

(4) is trivial. □

Remark 4.18. We give a complete classification of $\mathcal{U}(\mathbb{C}L_0)$-free modules of rank 1 and $\mathcal{U}(\mathfrak{h})$-free modules of rank 1 over $\mathcal{G}$ and $\bar{\mathcal{G}}$ by Theorems 3.2, 3.3, 4.12, 4.17 and Corollary 2.3.

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