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We prove sharp pointwise bounds on the complex Green operator and its derivatives on a class of embedded quadric manifolds of high codimension. In particular, we start with the class of quadrics that we previously analyzed (*Trans. Amer. Math. Soc. Ser. B* 10 (2023), 507–541) — ones whose directional Levi forms are nondegenerate, and add in null variables. The null variables do not substantially affect the estimates or analysis at the form levels for which $\Box_b$ is solvable and hypoelliptic. In the nonhypoelliptic degrees, however, the estimates and analysis are substantially different. In the earlier paper, when hypoellipticity of $\Box_b$ failed, so did solvability. Here, however, we show that if there is at least one null variable, $\Box_b$ is always solvable, and the estimates are qualitatively different than in the other cases. Namely, the complex Green operator has blow-ups off of the diagonal. We also characterize when a quadric M whose Levi form vanishes on a complex subspace admits a $\Box_b$-invariant change of coordinates so that M presents with a null variable.

1. Introduction

A quadric submanifold of $\mathbb{C}^n \times \mathbb{C}^m$ is a CR manifold that can be written as a graph of a scalar- or vector-valued Hermitian symmetric quadratic form, ϕ, i.e.,

$$M = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \operatorname{Im} w = \phi(z, z)\}.$$

For a hypersurface ($m = 1$), the analysis of the Kohn Laplacian, $\Box_b$, and the complex Green operator (the relative inverse of $\Box_b$) is well understood and has a long history. The motivating example is the Heisenberg group where $\phi(z, z) = |z|^2$. Its group structure can be exploited to construct explicit convolution kernels to invert the sub-Laplacian as well as the Kohn Laplacian in degree $(0, q)$, $1 \le q \le n - 1$, the cases where $\Box_b$ is invertible [Folland and Stein 1974a; 1974b; Hulanicki 1976;

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Gaveau 1977; Beals et al. 2000; Boggess and Raich 2009]. Estimates of these kernels then show that the Green operator as well as some of its derivatives are continuous operators on $L^p(M)$ as well as in other normed topologies.

For higher codimension quadrics, i.e., $m \geq 2$, much less is understood about the behavior of the Green operator. Part of the difficulty has to do with the fact that the Levi form, ϕ, is vector valued instead of scalar valued as is the case for a hypersurface. Thus, one must consider the *directional Levi form* for each normal direction (see (2) for a precise definition). A breakthrough result came when Peloso and Ricci [2003] characterized the solvability and hypoellipticity for the $\Box_b$-equation on quadrics based on the inertias of the directional Levi forms. This result provided the impetus for much of our research. In [Boggess and Raich 2023], we analyzed the pointwise estimates and L^p regularity of the complex Green operator on $(0, q)$-forms under the assumption that the eigenvalues of each directional Levi form are nonvanishing. In particular, we showed that the complex Green operator in this setting possesses all the same regularity properties as that of the Heisenberg group. On the other hand, there are simple examples of quadrics (see [Boggess and Raich 2021]) where some of the directional Levi forms are degenerate (i.e., have vanishing eigenvalues) and for which the estimates on the complex Green operator have no known parallel with that of any quadric hypersurface. The goal of this paper is to introduce degeneracy into the Levi form in a controlled manner. We do this by adding what we call *null variables* and catalog the effect on the solvability of the $\Box_b$-equation as well as providing sharp estimates for the complex Green operator. As an added dividend, our techniques yield a new result on estimates for the complex Green operator for a hypersurface with null directions in its Levi form.

Analyzing the $\Box_b$-operator on quadrics is a problem that mathematicians have been working on for the past 50 years. Hans Lewy [1957] discovered his famous counterexample of the Cauchy–Kowalevsky theorem in the C^∞ category while investigating the associated $\bar\partial_b$-operator on the Heisenberg group. Regardless of the hypotheses on the Levi form, $\Box_b$ is neither elliptic nor constant coefficient and this makes the function theory difficult. The additional tools provided by the Lie group structure of quadrics permits analysis that is currently unavailable in the general case, especially in the higher codimension setting. For additional background on the $\bar\partial_b$- and $\Box_b$-operators, please see [Boggess 1991; Chen and Shaw 2001; Biard and Straube 2017]. For detailed analysis of the $\Box_b$-operator on quadric manifolds, please see [Boggess 1991; Peloso and Ricci 2003; Boggess and Raich 2011; 2013; 2020; 2022b] and especially [Boggess and Raich 2023].

As mentioned above, in [Boggess and Raich 2023] we analyzed the estimates on the Green operator for a quadric in $\mathbb{C}^n \times \mathbb{C}^m$ where the codimension, m, is at least 2 and where all the directional Levi forms are nondegenerate. As detailed below, this assumption implies that n must be even. In this paper, we add null directions. Therefore, our setting is as follows: let $n' \geq 1$, $n'' \geq 0$, and $n = 2n' + n''$. Let $\phi_0 : \mathbb{C}^{2n'} \times \mathbb{C}^{2n'} \to \mathbb{C}^m$ be a Hermitian symmetric quadratic form; define $\phi : \mathbb{C}^{2n'+n''} \times \mathbb{C}^{2n'+n''}$ by

$$\phi((z', z''), (\tilde{z}', \tilde{z}'')) = \phi_0(z', \tilde{z}').$$

Here, z'' is a *null variable* whereby we mean that ϕ is independent of z''. We let $z = (z', z'')$ so that $z'_k = z_k$ for $1 \leq k \leq 2n'$ and $z''_j = z_j$ for $j = 2n'+1, \ldots, 2n'+n''$. Our main focus is on quadric submanifolds of the form

$$M_\phi = M = \{(z', z'', w) \in \mathbb{C}^{2n'} \times \mathbb{C}^{n''} \times \mathbb{C}^m : \operatorname{Im} w = \phi(z', z')\}. \tag{1}$$

For each unit vector $\nu \in S^{m-1} \subset \mathbb{R}^m$, we define the *directional Levi form* $\phi^\nu : \mathbb{C}^{2n'+n''} \times \mathbb{C}^{2n'+n''} \to \mathbb{C}$ by

$$\phi^\nu(z, \tilde{z}) = \phi(z, \tilde{z}) \cdot \nu = (\tilde{z}')^* A_\nu z', \tag{2}$$

where A_ν is a Hermitian symmetric matrix, depending linearly on the parameter $\nu \in S^{m-1}$. We define the eigenvalues and eigenvectors of the directional Levi forms to be the eigenvalues and eigenvectors of A_ν, and let $n^\pm(\nu)$ be the number of positive/negative eigenvalues of A_ν. When M is a hypersurface, there are directional Levi forms in only two directions: $\nu = 1$ and $\nu = -1$ since S^0 has two points. In codimension $m \geq 2$, ν belongs to the unit sphere S^{m-1}, a connected set. As shown in [Boggess and Raich 2023], the connectivity of S^{m-1}, $m \geq 2$, implies that $n^+(\nu) = n^-(\nu) = n'$ whereas this is not necessarily true for the hypersurface case ($m = 1$).

Peloso and Ricci [2003] found that $\Box_b$ is solvable (resp. hypoelliptic) on $(0, q)$-forms on M_ϕ if and only if there does not exist $\nu \in \mathbb{R}^m \setminus \{0\}$ so that $n^+(\nu) = q$ (resp. $n^+(\nu) \leq q$) and $n^-(\nu) = 2n'+n''-q$ (resp. $n^-(\nu) \leq 2n'+n''-q$). For the $m \geq 2$ and $n'' = 0$ case studied in [Boggess and Raich 2023], $n^+(\nu) = n' = n^-(\nu)$, and hence $\Box_b$ is solvable and hypoelliptic for all $q \neq n'$ and neither solvable nor hypoelliptic when $q = n'$. The lack of solvability is related to the fact that $\ker \Box_b \neq \{0\}$ when $q = n'$. After subtracting the orthogonal projection onto $\ker \Box_b$ in the case $q = n'$, the complex Green operator satisfies estimates analogous to those for the Heisenberg group, that is, estimates that are completely governed by the control metric for M. We know, however, that when the eigenvalues of the directional Levi forms are not bounded away from zero, the control distance does not always suffice to control estimates on $N_{0,q}$. This occurs both for hypersurfaces as well as higher codimension quadrics [Machedon 1988; Nagel and Stein 2006; Boggess and Raich 2021].

As mentioned above, z'' are null variables, and we henceforth assume that $n'' \geq 1$. Given this assumption and the fact that for all $\nu \in \mathbb{R}^m \setminus \{0\}$, $n^+(\nu) = n^-(\nu) = n'$, it follows that $\Box_b$ is solvable on M_ϕ for all $0 \leq q \leq 2n' + n''$. Additionally,

$\Box_b$ fails to be hypoelliptic if q satisfies $n' \le q$ and $n' \le 2n' + n'' - q$, that is, $n' \le q \le n' + n''$. Interestingly, adding in null variables improves the solvability of $\Box_b$ while leaving alone the number of hypoelliptic degrees. The estimate for $N_{0,q}$ in the nonhypoelliptic cases is qualitatively worse than in the hypoelliptic cases. The sharp bound is no longer controlled solely by the control distance and the integral kernel has singularities off of the diagonal. Detailed results are stated in Section 2.

In contrast, the class of hypersurfaces we study are of the form

$$M = \{(z', z'', w) \in \mathbb{C}^{n'} \times \mathbb{C}^{n''} \times \mathbb{C} : \operatorname{Im} w = \phi(z', z')\},$$

where $\phi : \mathbb{C}^{n'} \times \mathbb{C}^{n'} \to \mathbb{C}$ is a Hermitian symmetric, scalar-valued, quadratic form. We write $\phi(z', z') = (z')^* A z'$, where A is a nondegenerate, Hermitian symmetric matrix. Suppose that A has n^+ positive eigenvalues and n^- negative eigenvalues. Here, we are *not* assuming $n^+ = n^-$. Solvability always holds because solvability fails if and only if there is a direction for which the sum of the positive eigenvalues and negative eigenvalues is n. However, this never happens with A or $-A$ as this sum equals $n' < n$. Additionally, hypoellipticity fails if $n^+ \le q \le n - n^-$ or $n^- \le q \le n - n^+$ and holds otherwise. Since $n' = n^+ + n^-$, hypoellipticity fails if and only if $n^+ \le q \le n^+ + n''$ or $n - n^+ - n'' \le q \le n - n^+$. Detailed estimates on the Green operator for a hypersurface with null variables are stated in Section 2.

As with many past researchers (e.g., Folland and Stein [1974a], Nagel et al. [2001], and Nagel and Stein [2006]), our approach to computing a working formula for the Green operator, for $m \ge 1$, involves the integral of the fundamental solution to the heat equation associated to $\Box_b$ in the time variable. However, in the case of a one-dimensional null space ($n'' = 1$), the heat kernel is not integrable in the time variable, and we therefore develop a new technique to obtain the Green operator in this case. The resulting kernel and its estimates are stated in Section 2. Proofs of the theorems stated in Section 2 are given in Sections 3, 4, and 5. In Section 6, we show that the estimates given in our theorems are sharp.

2. Notation and main results

Notation for null variables. Define the projection $\pi : \mathbb{C}^{2n'+n''} \times \mathbb{C}^m \to \mathbb{C}^{2n'+n''} \times \mathbb{R}^m$ by $\pi(z, t+is) = (z, t)$. Given a quadric $M \subset \mathbb{C}^{2n'+n''} \times \mathbb{C}^m$, the projection π induces CR and Lie group structures on $\mathbb{C}^{2n'+n''} \times \mathbb{R}^m$, and we call this Lie group G. Since the projection is a CR isomorphism, we primarily work on G but use the same notation interchangeably for objects on M and their pushfowards/pullbacks on G.

The group structure for G is

$$(z, t) * (\zeta, u) = (z + \zeta, t + u - 2 \operatorname{Im} \phi(z, \zeta)) \quad \text{for } (z, t), (\zeta, u) \in G, \tag{3}$$

and this group operation can easily be lifted to M.

Denote the set of increasing q-tuples by

$$\mathcal{I}_q = \{K = (k_1, \dots, k_q) \in \mathbb{N}^q : 1 \le k_1 < k_2 < \cdots < k_q \le 2n' + n''\}.$$

Definition 2.1. Given an index $K \in \mathcal{I}_q$, we say a current $N_K = \sum_{L\in\mathcal{I}_q} \tilde{N}_{K,L}(z,t)\, d\bar{z}_L$ is a *fundamental solution to* $\Box_b$ on forms spanned by $d\bar{z}_K$ if $\Box_b N_K = \delta_0(z,t)\, d\bar{z}_K$. A *fundamental solution* $N_{0,q}$ to $\Box_b$ acting on $(0,q)$-forms is then given by

$$N_{0,q} f = \sum_{K\in\mathcal{I}_q} N_K\{f_K\, d\bar{z}_K\}.$$

In higher codimension ($m \ge 2$) a fundamental solution to $\Box_b$ on forms spanned by $d\bar{z}_K$ usually involves terms spanned by $d\bar{z}_L$ for $L \neq K$ in addition to $L = K$.

N_K acts on smooth forms with compact support by componentwise convolution with respect to the group structure on G, that is, if $f = f_0\, d\bar{z}^K$, then $N_K * f = \sum_{L\in\mathcal{I}_q} \tilde{N}_{K,L} * f_0\, d\bar{z}_L$. Thus

$$\tilde{N}_{K,L} * f_0(z,t) = \int_{(\zeta,u)\in G} N_{K,L}((z,t)*(\zeta,u)^{-1}) f_0(\zeta,u)\, dv(z)\, dt,$$

where $dv(z)\, dt$ is the usual volume form for G, and (using (3))

$$(z,t)*(\zeta,u)^{-1} = (z-\zeta,\, t-u+2\operatorname{Im}\phi(z,\zeta)).$$

Recall that $\delta_0 * f = f$. Therefore, if N_K is a fundamental solution to $\Box_b$ and $f = f\, d\bar{z}_K$ is a smooth form with compact support, then $\Box_b\{N_K * f\} = f$. As mentioned in the introduction, Peloso and Ricci [2003] showed that solvability in our context is possible in all degrees, i.e., $0 \le q \le n = 2n' + n''$. They also showed that solvability is equivalent to the triviality of the L^2 null space of $\Box_b$. We therefore conclude that if $n'' > 0$, then any two fundamental solutions to $\Box_b$ must differ by a non-L^2 current.

For a multiindex $I = (I_1, I_2, I_3) \in \mathbb{N}_0^{4n'+2n''+m}$, the multiindex $I_1 \in \mathbb{N}_0^{4n'}$ records the differentiation in the z' and $\bar{z}'$ variables, $I_2 \in \mathbb{N}_0^{2n''}$ records the differentiation in the z'' and $\bar{z}''$ variables, and $I_3 \in \mathbb{N}_0^m$ records the t-derivatives. Given such a multiindex I, define the *weighted order* of I by $\langle I\rangle = |I_1| + |I_2| + 2|I_3|$ and the *order* of I by $|I| = |I_1| + |I_2| + |I_3|$.

As a consequence of the discussion in Section 1, we assume the following when the codimension, m, is at least 2:

- For each $\nu \in S^{m-1}$, there are n' positive eigenvalues μ_j^ν for j in some index set P^ν of cardinality n from the set $\{1, 2, \dots, 2n'\}$ and n' negative eigenvalues μ_k^ν for $k \in (P^\nu)^c$, the complement of P^ν in $\{1, 2, \dots, 2n'\}$.

Remark 2.2. Given that our nonzero eigenvalues stay bounded away from 0 independently of $\nu \in S^{m-1}$, we may arrange the indices so that $P^\nu = P$ is independent of ν.

Recall the set of increasing q-tuples is denoted by

$$\mathcal{I}_q = \{K = (k_1, \dots, k_q) \in \mathbb{N}^q : 1 \le k_1 < k_2 < \cdots < k_q \le 2n' + n''\}.$$

Also set

$$\mathcal{I}'_{q'} = \{K' = (k_1, \dots, k_{q'}) \in \mathbb{N}^{q'} : 1 \le k_1 < k_2 < \cdots < k_{q'} \le 2n'\},$$
$$\mathcal{I}''_{q''} = \{K'' = (k_1, \dots, k_{q''}) \in \mathbb{N}^{q''} : 2n' + 1 \le k_1 < k_2 < \cdots < k_{q''} \le 2n' + n''\}.$$

Given $K \in I_q$, we can always decompose $K = (K', K'')$ where $K' \in \mathcal{I}'_{q'}$ and $K'' \in \mathcal{I}''_{q''}$ for some q', q'' with $q' + q'' = q$. Our notation follows [Boggess and Raich 2022b]. For $\lambda \in \mathbb{R}^m \setminus \{0\}$, set $\nu = \lambda/|\lambda| \in S^{m-1}$. We write $z' \in \mathbb{C}^{n'}$ in terms of the unit eigenvectors of ϕ_λ which means that $(z')^\lambda_j = (z')^\nu_j$ is given by

$$(z')^\nu := Z(\nu, z') := U(\nu)^* \cdot z',$$

where $U(\nu)$ is the matrix whose columns are the eigenvectors v^ν_k, $1 \le k \le 2n'$, of the directional Levi form ϕ^ν, and $\cdot$ represents matrix multiplication with z' written as a column vector. Note that the corresponding orthonormal basis of (0, 1)-covectors for this basis is

$$d\bar{Z}_j(\nu, z'), \quad 1 \le j \le 2n',$$

where $d\bar{Z}(\nu, z') = U(\nu)^T \cdot d\bar{z}'$, $d\bar{z}'$ is written as a column vector of (0, 1)-forms, and the superscript T stands for transpose. Note that $(z')^\nu = Z(\nu, z')$ depends smoothly on $z' \in \mathbb{C}^{n'}$ but only is *locally integrable* as a function of $\nu \in S^{m-1}$ [Rainer 2011]. The coordinates for the remaining n'' variables, $z'' = (z_{2n'+1}, \dots, z_n)$, do not depend on ν. Denote by $\mathbb{I}_{n''}$ the $n'' \times n''$ identity matrix. We write

$$z^\nu = (z', z'')^\nu = (z^\nu, z'') = Z(\nu, z) = (Z(\nu, z'), z'') = (U(\nu)^* \oplus \mathbb{I}_{n''})(z', z''),$$

where $(A \oplus B)(z', z'') := (A(z'), B(z''))$ for any $n' \times n'$ matrix, A, and any $n'' \times n''$ matrix, B. Also,

$$d\bar{Z}(\nu, z) = (d\bar{Z}(\nu, z'), d\bar{z}'') = (U(\nu)^T \oplus \mathbb{I}_{n''}) \cdot (d\bar{z}', d\bar{z}'').$$

We will need to express $d\bar{z}_K$ in terms of $d\bar{Z}(\nu, z)_L$ for $L \in \mathcal{I}_q$. We have

$$d\bar{z}_K = d\bar{z}'_{K'} \wedge d\bar{z}''_{K''} = \sum_{L' \in \mathcal{I}'_{q'}} \det(\bar{U}(\nu)_{K',L'})\, d\bar{Z}(\nu, z')_{L'} \wedge d\bar{z}''_{K''}, \tag{4}$$

where $\bar{U}(\nu)_{K',L'}$ is the $q' \times q'$ minor of $\bar{U}(\nu)$ comprised of elements in the rows K' and columns L'. Note that if $q = 2n' + n''$, then the above sum only has one term and $\det \bar{U}(\nu)_{K,K} = 1$. In addition, when $q = 0$, $\mathcal{I}_0 = \varnothing$ and the sum (4) does not appear. Similarly,

$$d\bar{Z}(\nu, z')_L = \sum_{J \in \mathcal{I}'_{q'}} \det(U(\nu)^T_{L',J})\, d\bar{z}'_J \wedge d\bar{z}''_{L''}. \tag{5}$$

Throughout the paper, we use the function

$$A(r, \nu, z) = A(r, \nu, z', z'') = \frac{2}{|\log r|}|z''|^2 + \sum_{j=1}^{2n'} |\mu_j^\nu| \left(\frac{1 + r^{|\mu_j^\nu|}}{1 - r^{|\mu_j^\nu|}} \right) |z_j^\nu|^2,$$

where μ_j^ν are the nonzero eigenvalues for A_ν and the dimensional constant is

$$K_{2n'+n'',m} = \frac{4^{2n'+n''}(2n' + n'' + m - 2)!}{2(2\pi)^{2n'+n''+m}}.$$

Main results for codimension ≥ 2. Our first theorem provides a formula for the fundamental solution to $\Box_b$ in the case where the null variable dimension satisfies $n'' \geq 2$.

Theorem 2.3. *Let $M \subset \mathbb{C}^{2n'+n''} \times \mathbb{C}^m$, with $m \geq 2$, $n' \geq 1$, and $n'' \geq 2$, be a quadric submanifold defined by* (1) *with associated projection G, and assume that there exists a Hermitian symmetric quadratic form $\phi_0 : \mathbb{C}^{n'} \times \mathbb{C}^{n'} \to \mathbb{C}^m$ such that*

(1) *$\phi(z, \bar{z}) = \phi_0(z', \bar{z}')$ for all $z \in \mathbb{C}^{2n'+n''}$ and*

(2) *the eigenvalues of the directional Levi forms of ϕ_0 are nonzero.*

For any $0 \leq q \leq 2n' + n''$, there is a fundamental solution $N = N_{0,q}$ to $\Box_b$ on $(0, q)$-forms given by convolution with the kernel

$$N_K(z, t) = K_{2n'+n'',m} \sum_{L \in \mathcal{I}_q} \int_{\nu \in S^{m-1}} \det(\bar{U}(\nu)_{K,L})\, d\bar{Z}(\nu, z)^L \times \int_{r=0}^{1} \frac{1}{|\log r|^{n''}} \Bigg(\prod_{\substack{j \in L^c \cap P \\ j \in L \cap P^c}} \frac{r^{|\mu_j^\nu|}|\mu_j^\nu|}{1 - r^{|\mu_j^\nu|}} \prod_{\substack{k \in L \cap P \\ k \in L^c \cap P^c}} \frac{|\mu_k^\nu|}{1 - r^{|\mu_k^\nu|}} \Bigg) \times \frac{1}{(A(r, \nu, z', z'') - i\nu \cdot t)^{2n+m-1}} \frac{dr\, d\nu}{r}, \tag{6}$$

where $d\nu$ is surface measure on the unit sphere S^{m-1}.

This theorem follows directly from Theorem 2.3 in [Boggess and Raich 2022b], and the formula is similar to the corresponding one in the same work, where $n'' = 0$ (no $\log r$ term appears). The formula for N is the s-integral over $0 \leq s < \infty$ of the partial Fourier transform of the $\Box_b$ heat kernel $\tilde{H}_K(s, z, \hat{\lambda})$; see (16) (where s represents time). For this derivation, we require that this heat kernel is integrable in

s over $0 \le s < \infty$, which, as we shall see below, holds whenever $\Box_b$ is hypoelliptic or $n'' \ge 2$. However, when $n'' = 1$ in the nonhypoelliptic case, this heat kernel fails to be integrable in s and, consequently, the factor $1/(r\,|\log r|^{n''})$ appearing in (6) is *not* integrable in r near $r = 0$ when $n'' = 1$. The numerator is nonvanishing at $r = 0$ when $L = P$. In Theorem 2.4, below, we derive a fundamental solution for $\Box_b$ when $n'' = 1$ and $L = P$ by modifying our earlier construction to ensure greater decay in the time variable s without disturbing the approximation of the identity behavior as $s \to 0$. This kernel requires a genuinely new idea that is not anticipated in [Boggess and Raich 2022b].

Theorem 2.4. *Let $M \subset \mathbb{C}^n \times \mathbb{C}^m$ be a quadric submanifold as in Theorem 2.5 but with $n'' = 1$ (and $n = 2n' + 1$). Let $K \in \mathcal{I}_q$ where $q = n'$ or $q = n' + 1$ and $K' \in \mathcal{I}'_{n'}$. Then $\tilde{H}_K(s, z, \hat{\lambda})$ is not integrable on $(0, n')$- or $(0, n' + 1)$-forms, and a fundamental solution to $\Box_b$ on forms spanned by $d\bar{z}_K$ is given by*

$$
\begin{aligned}
(7)\quad & N_K(z,t) \\
&= K_{2n'+1,m} \sum_{\substack{L\in\mathcal{I}'_{q'}\\ L\neq P}} \int_{\nu\in S^{m-1}} \det(\bar{U}(\nu)_{K',L})\, d\bar{Z}(\nu,z')_L \wedge d\bar{z}''_{K''} \\
&\quad \times \int_{r=0}^{1} \Bigg(\prod_{\substack{j\in(L')^c\cap P\\ j\in L'\cap P^c}} \frac{r^{|\mu_j^\nu|}|\mu_j^\nu|}{1-r^{|\mu_j^\nu|}} \prod_{\substack{k\in L'\cap P\\ k\in(L')^c\cap P^c}} \frac{|\mu_k^\nu|}{1-r^{|\mu_k^\nu|}} \Bigg) \\
&\qquad\qquad\qquad \times \frac{1}{(A(r,\nu,z)-i\nu\cdot t)^{2n'+m}} \frac{dr\,d\nu}{|\log r|r} \\
&\quad + \int_{\nu\in S^{m-1}} \det(\bar{U}(\nu)_{K',P})\, d\bar{Z}(\nu,z')_P \wedge d\bar{z}''_{K''} |\det A_\nu| \\
&\quad \times \int_{r=0}^{\frac{1}{2}} \Bigg[\Bigg(\prod_{j=1}^{2n'} \frac{1}{1-r^{|\mu_j^\nu|}} \Bigg) \frac{1}{(A(r,\nu,z)-i\nu\cdot t)^{2n'+m}} \\
&\qquad\qquad\qquad - \frac{1}{(A(0,\nu,z',0)-i\nu\cdot t)^{2n'+m}} \Bigg] \frac{dr\,d\nu}{|\log r|r} \\
&\quad + \int_{\nu\in S^{m-1}} \det(\bar{U}(\nu)_{K',P})\, d\bar{Z}(\nu,z')_P \wedge d\bar{z}''_{K''} |\det A_\nu| \\
&\quad \times \int_{r=\frac{1}{2}}^{1} \Bigg(\prod_{j=1}^{2n'} \frac{1}{1-r^{|\mu_j^\nu|}} \Bigg) \frac{1}{(A(r,\nu,z)-i\nu\cdot t)^{2n'+m}} \frac{dr\,d\nu}{|\log r|r}.
\end{aligned}
$$

When $L = P$ in the above formula for N, the term inside the large brackets, $[\,\cdot\,]$, in the integrand of (7) vanishes sufficiently quickly at $r = 0$, and thus this term is integrable in r over $0 \le r \le \frac{1}{2}$.

Our main theorem regarding pointwise bounds on the kernel for the fundamental solution of $\Box_b$ is the following:

Theorem 2.5. *Let $M \subset \mathbb{C}^{2n'+n''} \times \mathbb{C}^m$, with $m \geq 2$ and $n', n'' \geq 1$, be a quadric submanifold defined by* (1) *with associated projection G, and assume that there exists a Hermitian symmetric quadratic form $\phi_0 : \mathbb{C}^{n'} \times \mathbb{C}^{n'} \to \mathbb{C}^m$ so that*

(1) *$\phi(z, \bar{z}) = \phi_0(z', \bar{z}')$ for all $z \in \mathbb{C}^{2n'+n''}$ and*

(2) *the eigenvalues of the directional Levi forms of ϕ_0 are nonzero.*

Let $N = N_{0,q}$.

- *Suppose that $0 \leq q < n'$ or $q > n' + n''$. For any multiindex $I \in \mathbb{N}_0^{4n'+2n''+m}$, there exists a constant $C_I > 0$ so that*

$$|D^I N(z, t)| \leq \frac{C_I}{(|z|^2 + |t|)^{2n'+n''+m-1+\frac{1}{2}\langle I\rangle}}. \tag{8}$$

- *Suppose that $n' \leq q \leq n' + n''$ and $n'' \geq 2$. Then there exists a constant $C_I > 0$ so that*

$$|D^I N(z, t)| \leq \frac{C_I}{(|z|^2 + |t|)^{n''-1+\frac{1}{2}|I_2|}(|z'|^2 + |t|)^{2n'+m+\frac{1}{2}|I_1|+|I_3|}}. \tag{9}$$

- *Finally, suppose that $n' \leq q \leq n' + n''$ and $n'' = 1$. Then there exists a constant $C_I > 0$ so that*

$$|D^I N(z, t)| \leq C_I \begin{cases} \dfrac{\log\left(1 + \frac{|z_{n'+1}|^2}{|z'|^2+|t|}\right)}{(|z'|^2 + |t|)^{2n'+m}} & \text{if } I = 0, \\[2ex] \dfrac{1}{(|z|^2 + |t|)^{\frac{1}{2}|I_2|}(|z'|^2 + |t|)^{2n'+m+\frac{1}{2}|I_1|+|I_3|}} & \text{if } I \neq 0. \end{cases} \tag{10}$$

These estimates are sharp.

In this paper, we only provide the proof for the case $I = 0$. The proof in the $I \neq 0$ case provides no additional insights, though we do discuss later how derivatives affect the estimates. Keeping track of higher derivatives requires some bookkeeping, which is thoroughly explained and carried out in [Boggess and Raich 2023].

In the case where $0 \leq q < n'$ or $q > n' + n''$, the estimate in (8) implies that N_q is locally integrable in G and more can be said about the regularity of N_q as an operator using the theory of homogeneous groups. Let $W^{k,p}(M)$ denote the Sobolev space of forms on M with z-, $\bar{z}$- and t-derivatives of order k in $L^p(M)$. Following the approach of [Boggess and Raich 2022a, Section 7.3], we can view G (and hence M) as a homogeneous group with norm function $\rho(z, t) = |z| + |t|^{1/2}$. From (8), it follows that the integration kernel of $N_{0,q}$ and its derivatives have the appropriate pointwise decay (analogous to that in the case of nonzero eigenvalues handled in [Boggess and Raich 2023]). A second consequence of (8) is that $N_{0,q}$ is a tempered distribution, and combining this fact with the natural dilation structure and that $D^I N_{0,q}$ is a convolution operator shows that $D^I N_{0,q}$ is uniformly bounded on normalized bump functions. This is exactly what is required to establish the L^p boundedness, $1 < p < \infty$. The convolution operator $D^I N_{0,q}$ extends to a bounded operator on $W^{k,p}(\mathbb{C}^n \times \mathbb{R}^m)$, and we state this as a corollary to Theorem 2.5.

Corollary 2.6. *Let $M \subset \mathbb{C}^{2n'+n''} \times \mathbb{C}^m$ be a quadric submanifold satisfying the hypothesis of Theorem 2.5. Assume $0 \le q < n'$ or $q > n' + n''$. Given a multiindex $I \in \mathbb{N}_0^{4n+m}$ such that $\langle I \rangle = 2$, the operator $D^I N_{0,q}$ is exactly regular on $W^{k,p}(M)$ for all $k \ge 0$ and all $1 < p < \infty$. In other words, $D^I N_{0,q}$ extends to a bounded operator on $W^{k,p}(M)$. In particular, $D^I N_{0,q}$ is a hypoelliptic operator.*

The regularity properties of $N_{(0,q)}$ are not yet known for $n' \le q \le n' + n''$.

Results for hypersurfaces. Even though our focus is mostly on the higher codimension case, our technique provides a new result in the hypersurface case as well. When M is a hypersurface, M is of the form

$$M = \{(z', z'', w) \in \mathbb{C}^{n'} \times \mathbb{C}^{n''} \times \mathbb{C} : \operatorname{Im} w = \phi_0(z', z')\}, \tag{11}$$

where $\phi_0(z', z') = (z')^* A z'$ and A is a nondegenerate Hermitian matrix. Since A is Hermitian, we can choose coordinates in which A is diagonal. In these coordinates (which we still call (z', z'')),

$$\phi(z, z) = \sum_{j=1}^{n'} \mu_j |z_j|^2,$$

where $\mu_1, \dots, \mu_{n'}$ are the nonzero eigenvalues of A. In the hypersurface case, there is not a requirement that n' is even or $n^+ = n^-$. Also, $\Box_b$ acts diagonally in these coordinates. This means if $f = \sum_{J \in \mathcal{I}_q} f_J \, d\bar{z}_J$, then $\Box_b f = \sum_{J \in \mathcal{I}_q} \Box_J f_J \, d\bar{z}_J$. Consequently, to invert $\Box_b$, we need only to invert the $\Box_J$-operators which is simpler than the higher codimension cases handled in the previous subsection. We continue to let P denote the indices of the positive eigenvalues of A. For the theorems in this section, we need the following notation. Let

$$A(r, z) = \frac{2}{|\log r|} |z''|^2 + \sum_{j=1}^{n'} \frac{1 + r^{|\mu_j|}}{1 - r^{|\mu_j|}} |\mu_j| \, |z_j|^2$$

and

$$\varepsilon_{j,L} = \begin{cases} \operatorname{sgn}(\mu_j), & j \in L, \\ -\operatorname{sgn}(\mu_j), & j \notin L. \end{cases}$$

The proof of Theorem 2.4 is easily adapted to prove the following result.

Theorem 2.7. *Let $M \subset \mathbb{C}^{n'} \times \mathbb{C}^{n''} \times \mathbb{C}$ be a quadric hypersurface described by* (11). *Fix $0 \le q \le n$, where $n = n' + n''$, and let $L \in \mathcal{I}_q$.*

(1) *If $n'' \ge 2$ or $n'' = 1$ and L' is neither P nor P^c, then the fundamental solution to the $\Box_L$-equation given by the inverse Fourier transform in t of $\int_0^\infty e^{-s\Box_L}\, ds$ is*

$$N_L(z,t) = \frac{2^{2n-1}(n-1)!}{(2\pi)^{n+1}} |\det A| \Biggl(\int_0^1 \prod_{j=1}^{n'} \frac{r^{\frac{1}{2}(1-\varepsilon_{j,L})|\mu_j|}}{1-r^{|\mu_j|}} \frac{1}{(A(r,z)-it)^n} \frac{dr}{r\,|\log r|^{n''}} + \int_0^1 \prod_{j=1}^{n'} \frac{r^{\frac{1}{2}(1+\varepsilon_{j,L})|\mu_j|}}{1-r^{|\mu_j|}} \frac{1}{(A(r,z)+it)^n} \frac{dr}{r\,|\log r|^{n''}} \Biggr)$$

(2) *If $n'' = 1$ and $L' = P$, then there is a fundamental solution to the $\Box_L$-equation given by*

$$N_L(z,t) = \frac{2^{2n-1}(n-1)!}{(2\pi)^{n+1}} |\det A| \Biggl(\int_0^1 \prod_{j=1}^{n'} \frac{r^{|\mu_j|}}{1-r^{|\mu_j|}} \frac{1}{(A(r,z)+it)^n} \frac{dr}{r\,|\log r|} + \int_{\frac{1}{2}}^1 \prod_{j=1}^{n'} \frac{1}{1-r^{|\mu_j|}} \frac{1}{(A(r,z)-it)^n} \frac{dr}{r\,|\log r|} + \int_0^{\frac{1}{2}} \Bigl(\prod_{j=1}^{n'} \frac{1}{1-r^{|\mu_j|}} \frac{1}{(A(r,z)-it)^n} - \frac{1}{(A(0,z)-it)^n} \Bigr) \frac{dr}{r\,|\log r|} \Biggr).$$

(3) *If $n'' = 1$ and $L' = P^c$, then*

$$N_P(z,-t) = \overline{N_P(z,t)}$$

is a fundamental solution to the $\Box_L$-equation.

The form of the solutions from Theorem 2.7 are simpler versions than in Theorem 2.4 in the $n'' = 1$ case and (20) in the $n'' \ge 2$ case. The analysis in the higher codimension case shows that the size comes from the r-integral and there is no cancellation in the ν-integral. Consequently, the proof of Theorem 2.5 proves the following theorem as well.

Theorem 2.8. *Let $M \subset \mathbb{C}^{n'} \times \mathbb{C}^{n''} \times \mathbb{C}$ be a quadric hypersurface described by* (11). *Fix $0 \le q \le n$, where $n = n' + n''$, and $L \in \mathcal{I}_q$. For any multiindex $I \in \mathbb{N}_0^{2n+1}$, there exists a constant $C_I > 0$ so that the following hold.*

- *If L' is neither P nor P', then*

$$|D^I N(z,t)| \le \frac{C_I}{(|z|^2 + |t|)^{n+\frac{1}{2}\langle I \rangle}}.$$

This case includes the q for which $\Box_b$ is hypoelliptic.

- *If $n'' \geq 2$ and $L' = P$ or $L' = P^c$, then*

$$|D^I N(z,t)| \leq \frac{C_I}{(|z|^2+|t|)^{n''-1+\frac{1}{2}|I_2|}(|z'|^2+|t|)^{n'+1+\frac{1}{2}|I_1|+|I_3|}}. \tag{12}$$

- *Finally, suppose that $n'' = 1$ and $L' = P$ or $L' = P^c$. Then*

$$|D^I N(z,t)| \leq C_I \begin{cases} \dfrac{\log\left(1+\frac{|z_n|^2}{|z'|^2+|t|}\right)}{(|z'|^2+|t|)^{n'+1}} & \text{if } I = 0, \\ \dfrac{1}{(|z|^2+|t|)^{\frac{1}{2}|I_2|}(|z'|^2+|t|)^{n'+1+\frac{1}{2}|I_1|+|I_3|}} & \text{if } I \neq 0. \end{cases} \tag{13}$$

These estimates are sharp.

Corollary 2.9. *Suppose M is a quadric hypersurface in $\mathbb{C}^n$ satisfying the hypotheses of Theorem 2.8. Fix $0 \leq q \leq n$, where $n = n' + n''$ and $L \in \mathcal{I}_q$. If L' is neither P nor P', then for any multiindex $I \in \mathbb{N}_0^{4n+m}$ with $\langle I \rangle = 2$, the operator $D^I N_L$ extends to a bounded operator on $W^{k,p}(M)$. In particular, $D^I N_L$ is a hypoelliptic operator.*

Remark 2.10. The estimates in (9), (10), (12), and (13) suggest that we investigate N from the point of view of flag kernels, à la Nagel, Ricci, and Stein [2001]. N is the wrong degree to be a flag kernel as it inverts second-order differential operators, just as the Newtonian potential is the wrong degree to be a Calderón–Zygmand operator. The are four types of second-order derivatives (two derivatives in z' variables, two derivatives in z'' variables, one derivative each in z' and z'' variables, and one derivative in a t variable), and only applying two derivatives in z'' variables to N produces a kernel with the correct order of decay. Even in this case, it is currently unclear if the kernel is a flag kernel. It would be an interesting project to understand the complete mapping properties of N and its second-order derivatives.

Vanishing variables. Our above assumption is that z'' is a null variable. There is a more general concept that we call a *vanishing variable* which is defined as follows: z'' is a *vanishing variable* for ϕ if $\phi(z,z) = 0$ whenever $z = (0, z'')$, $z'' \in \mathbb{C}^{n''}$. A null variable is also a vanishing variable but the converse is not true, as illustrated by the example below. We briefly discuss vanishing variables since the techniques in this paper only apply to null variables. We expect that the analysis of estimates for fundamental solutions in the case of vanishing variables will be more complicated.

Here is an example in $\mathbb{C}^3$ where z_3 is a vanishing variable but not a null variable:

$$\begin{aligned} \phi_1(z,z) &= |z_1|^2 - |z_2|^2, \\ \phi_2(z,z) &= \sqrt{2}\,\mathrm{Re}(z_3\bar{z}_1 + z_3\bar{z}_2), \\ \phi_3(z,z) &= \sqrt{2}\,\mathrm{Re}(iz_3\bar{z}_1 - iz_3\bar{z}_2). \end{aligned} \tag{14}$$

Note that z_3 is a vanishing variable but not a null variable for ϕ due to ϕ's dependence on z_3. There is no $\Box_b$-invariant change of coordinates that will make z_3 a null variable for ϕ. Here, a $\Box_b$-invariant change of coordinates between two quadrics M and M' in $\mathbb{C}^n \times \mathbb{C}^m$ is a nonsingular, complex linear map $T : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n \times \mathbb{C}^m$ with $T(M) = M'$ and $T^*(\Box_b f) = \Box_b(T^*(f))$ for all $(0, q)$-forms on M'. As shown in [Boggess and Raich 2020], a $\Box_b$-invariant change of variables requires a unitary change of coordinates in the z variables, i.e., $\hat{z} = U(z)$ where U is a unitary matrix. However, in order to preserve the independence of z_3 for ϕ_1, U must map the copy of $\mathbb{C}^2$ spanned by the z_1 and z_2 axes to itself. Since U is unitary, the orthogonal complement of this set (namely the z_3 axis) must remain invariant under U. Therefore U has the form

$$U = \begin{pmatrix} U_2 & 0 \\ 0 & 1 \end{pmatrix},$$

where U_2 is a 2×2 unitary matrix. A change of variables involving this U cannot remove the dependence of ϕ_2 or ϕ_3 on z_3.

This example illustrates the following point: if z'' is a null variable, then ϕ only depends on the variable z', which is the coordinate for the orthogonal complement of the space spanned by the null variables. This observation and the analysis in the previous paragraph leads to the following theorem.

Theorem 2.11. *Suppose L is a complex subspace of $\mathbb{C}$-dimension n'' in $\mathbb{C}^n$ $(n'' \le n)$, and suppose $\phi(z, z) = 0$ for all $z \in L$. Then there exists a $\Box_b$-invariant change of variables so that $z'' \in \mathbb{C}^{n''}$ is a null variable for ϕ if and only if for each $1 \le j \le n$, the map $z \in \mathbb{C}^n \to A_j z$ preserves $L^\perp$ (the orthogonal complement of L in $\mathbb{C}^n$), where A_j are the Hermitian matrices corresponding to the directional Levi forms of the standard basis vectors, E_j, $1 \le j \le m$, in $\mathbb{R}^m$, that is, $\phi_j(z, z) = z^* A_j z$.*

Proof. The proof is clear — if there is a unitary change of variables mapping L to a space spanned by the null variable z'', then the matrices A_j, $1 \le j \le n$, in the new variables must preserve the directions spanned by the z' variables. Since U is unitary, in the original coordinates, A_j must map $L^\perp$ to itself. The converse is similar. $\Box$

From a practical point of view, finding a null variable or vanishing variable for a given ϕ can proceed as follows. First, establish whether all the A_j have a common kernel. If the common kernel is trivial, then there are no vanishing or null variables. If there is a nontrivial common kernel, then diagonalize the matrix representing one of the coordinate functions, say A_1. At least one of the variables, say z_n, is a vanishing variable (representing an eigenvector corresponding to the zero eigenvalue of A_1). Next, see if the other component functions are independent of z_n. If so, then z_n is also a null variable. If not, then z_n is a vanishing variable but not a null variable. There may be additional vanishing and/or null variables depending on the dimension of the common kernel.

3. The $\Box_b$-heat equation and the proof of Theorem 2.4

$\Box_b$ and the partial Fourier transform. The operator $\Box_b$ is translation invariant in t, and so we introduce the *partial Fourier transform* of a function $f(z,t)$ by

$$f(z,\hat{\lambda}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f(t) e^{-i\lambda\cdot t}\, dt$$

with ˆ appearing over the transform variables. As is shown in [Peloso and Ricci 2003], for a fixed $\lambda \in \mathbb{R}^m$ (with $\nu = \lambda/|\lambda|$), the coordinates $Z(\nu, z')$ that diagonalize A_ν also diagonalize $\widehat{\Box}_b$. On the transform side, we treat λ as a parameter and write the transformed operator as $\widehat{\Box}_b^\lambda$. Fix $K \in \mathcal{I}_q$. Note that if $f(z,t) = f_K d\bar{z}_K$ and $q' = |K'|$, then

$$f(z,\hat{\lambda}) = f_K(z,\hat{\lambda})\, d\bar{z}'_{K'} \wedge d\bar{z}''_{K''} = \sum_{L\in\mathcal{I}'_{q'}} f_K(z,\hat{\lambda}) \det(\bar{U}(\nu)_{K',L})\, d\bar{Z}(\nu,z')_L \wedge d\bar{z}''_{K''}.$$

One of the reasons for using the $Z(\nu, z')$ coordinates is that $\widehat{\Box}_b^\lambda$ acts diagonally in these coordinates (see [Boggess and Raich 2022b]). Specifically,

$$(\Box_b f)(z,\hat{\lambda}) = \widehat{\Box}_b^\lambda\{f(z,\hat{\lambda})\} = \sum_{L\in\mathcal{I}'_{q'}} \widehat{\Box}_L^\lambda\{f_K(z,\hat{\lambda}) \det(\bar{U}(\nu)_{K',L})\}\, d\bar{Z}(\nu,z')_L \wedge d\bar{z}''_{K''}$$

where

$$\widehat{\Box}_L^\lambda = -\tfrac{1}{4}\Delta_z + 2i\sum_{k=1}^n \mu_k^\lambda \operatorname{Im}\{z_k^\nu \partial_{z_k^\nu}\} + \sum_{k=1}^n (\mu_k^\lambda)^2 |z_k^\nu|^2 - \Bigl(\sum_{k\in L}\mu_k^\lambda - \sum_{k\notin L}\mu_k^\lambda\Bigr)$$

and Δ_z is the ordinary Laplacian in the indicated variables. Our approach to solving the $\Box_b$-equation is via the $\Box_b$-heat equation. Given the diagonalization of $\widehat{\Box}_b$, it is enough to solve the $\widehat{\Box}_L^\lambda$ equations

$$\begin{gathered}\Bigl(\frac{\partial}{\partial s} + \widehat{\Box}_L^\lambda\Bigr)\{\tilde{H}_L(s,z,\hat{\lambda})\} = 0 \quad \text{for } s > 0,\\ \tilde{H}_L(s=0,z,\hat{\lambda}) = (2\pi)^{-m/2}\delta_0(z)\otimes 1_\lambda,\end{gathered} \tag{15}$$

where $\delta_0(z)$ is the Dirac-delta function centered at the origin in the z variables and 1_λ is the function which is identically 1 for all $\lambda\in\mathbb{R}^m$. The function $\tilde{H}_L(s,z,\hat{\lambda})$ is called the heat kernel and is given by (see [Boggess and Raich 2011])

$$\tilde{H}_L(s,z,\hat{\lambda}) = \frac{2^n}{(2\pi)^{m/2+n}} \frac{e^{-|z''|^2/s}}{s^{n''}} \prod_{j=1}^{2n'} \frac{e^{s\varepsilon^\nu_{j,L}|\mu_j^\lambda|}|\mu_j^\lambda|}{\sinh(s|\mu_j^\lambda|)} e^{-|\mu_j^\lambda|\coth(s|\mu_j^\lambda|)|Z_j(\nu,z')|^2} \tag{16}$$

where

$$\varepsilon^{\alpha}_{j,L} = \begin{cases} \operatorname{sgn}(\mu^{\lambda}_j) & \text{if } j \in L, \\ -\operatorname{sgn}(\mu^{\lambda}_j) & \text{if } j \notin L. \end{cases}$$

Integrability in s over $0 \le s < \infty$ holds when $n'' \ge 2$ or when $L \neq P$. However, integrability fails when $L = P$ and $n'' = 1$ since

$$\tilde{H}_P(s, z, \hat{\lambda}) = \frac{2^n}{(2\pi)^{m/2+n}} \frac{e^{-|z''|^2/s}}{s} \prod_{j=1}^{2n'} \frac{e^{s|\mu^{\lambda}_j|}|\mu^{\lambda}_j|}{\sinh(s|\mu^{\lambda}_j|)} e^{-|\mu^{\lambda}_j| \coth(s|\mu^{\lambda}_j|)|Z_j(\nu,z')|^2}$$

and so $\tilde{H}_P(s, z, \hat{\lambda})$ decays like $\frac{1}{s}$ as $s \to \infty$. Consequently, the harmonic projection onto $\ker \widehat{\Box}^{\lambda}_L$ is 0 yet the “formula"

$$(\widehat{\Box}^{\lambda}_P)^{-1} = \int_0^{\infty} e^{-s\widehat{\Box}^{\lambda}_P}\, ds$$

fails to hold because the integral on the right-hand side diverges.

Proof of Theorem 2.4. Set $\delta_{L,P} = 1$ if $L = P$ and $\delta_{L,P} = 0$ otherwise. Define

$$\begin{aligned} \tilde{S}_{L,P}(z', \hat{\lambda}) &= \lim_{s\to\infty} \frac{2^n}{(2\pi)^{m/2+n}} \prod_{j=1}^{2n'} \frac{e^{s|\mu^{\lambda}_j|}|\mu^{\lambda}_j|}{\sinh(s|\mu^{\lambda}_j|)} e^{-|\mu^{\lambda}_j| \coth(s|\mu^{\lambda}_j|)|Z_j(\nu,z')|^2} \delta_{L,P} \\ &= \frac{2^{n+2n'}}{(2\pi)^{m/2+n}} |\det A_\lambda| \prod_{j=1}^{2n'} e^{-|\mu^{\lambda}_j||Z_j(\nu,z')|^2} \delta_{L,P}. \end{aligned}$$

Let χ be an indicator function on the ray $[b, \infty)$ where $b > 0$ is to be determined later. Set

$$\tilde{N}_L(z, \hat{\lambda}) = \int_0^{\infty} \tilde{H}_L(s, z, \hat{\lambda}) - \frac{\chi(s|\lambda|)}{s} \tilde{S}_{L,P}(z', \hat{\lambda})\, ds. \tag{17}$$

The integral defining $\tilde{N}_L$ converges because

$$\frac{e^{s|\mu^{\lambda}_j|}}{\sinh(s|\mu^{\lambda}_j|)} e^{-|\mu^{\lambda}_j| \coth(s|\mu^{\lambda}_j|)|Z_j(\nu,z')|^2} - 2e^{-|\mu^{\lambda}_j||Z_j(\nu,z')|^2}$$

decays exponentially in s (and the integral kernel is $\partial \tilde{H}_L/\partial s$ near 0). Not coincidentally, $\tilde{S}_P(z', \hat{\lambda})$ is the integral kernel of the harmonic projection onto $\ker\{\widehat{\Box}^{\lambda,M_0}_P\}$ on the quadric M_0. Since $\widehat{\Box}^{\lambda}_P = -\triangle_{z''} + \widehat{\Box}^{\lambda,M_0}_P$, it follows that $\widehat{\Box}^{\lambda}_L \tilde{S}_{L,P} = 0$ for all L.

Consequently,

$$\begin{aligned}\widehat{\Box}_L^{\lambda}\tilde{N}_L(z,\hat{\lambda}) &= \int_0^{\infty}\widehat{\Box}_L^{\lambda}\tilde{H}_L(s,z,\hat{\lambda}) - \frac{\chi(s|\lambda|)}{s}\widehat{\Box}_L^{\lambda}\tilde{S}_{L,P}(z',\hat{\lambda})\,ds \\ &= -\int_0^{\infty}\frac{\partial\tilde{H}_L(s,z,\hat{\lambda})}{\partial s}\,ds = \delta_0(z)\otimes 1_{\lambda} \quad \text{by (15)},\end{aligned}$$

as desired. The latter integral converges as $s\to\infty$ because $\partial\tilde{H}_L(s,z,\hat{\lambda})/\partial s$ decays at least as fast as s^{-2}. We can now construct a solution to invert $\Box_b$ using the modified $\tilde{N}_L(z,\hat{\lambda})$ functions. Following the argument of [Boggess and Raich 2022b, Proposition 3.2], we have the following solution. In the following statement $\mathcal{F}_{\lambda}^{-1}$ denotes the inverse partial Fourier transform in λ.

Proposition 3.1. *For given indices $K\in\mathcal{I}_q$ and $L\in\mathcal{I}'_{q'}$, define*

$$N_{K,L}(z,\hat{\lambda}) = \det(\bar{U}(\nu)_{K',L})\tilde{N}_L((z',z''),\hat{\lambda})\,d\bar{Z}(z',\nu)_L\wedge d\bar{z}_{K''}, \tag{18}$$

where $\tilde{N}_L(z',z'',\hat{\lambda})$ is defined by (17). *Then there is a fundamental solution to $\Box_b$ on M applied to a form spanned by $d\bar{z}_K$ given by*

$$N_K(z,t) = \mathcal{F}_{\lambda}^{-1}\Bigg\{\sum_{L\in\mathcal{I}'_{q'}} N_{K,L}(z,\hat{\lambda})\Bigg\}(t). \tag{19}$$

We now continue with the proof of Theorem 2.4. If $L\neq P$, then $\tilde{S}_{L,P}(z',\hat{\lambda})=0$ in (17). Recalling that $n''=1$, the calculation in Section 4 of [Boggess and Raich 2022b] shows

$$\begin{aligned}&\mathcal{F}_{\lambda}^{-1}\{N_{K,L}(z,\hat{\lambda})\}(t) \\ &\quad = K_{n,m}\int_{\nu\in S^{m-1}}\det(\bar{U}(\nu)_{K',L})\,d\bar{Z}(\nu,z')_L\wedge d\bar{z}''_{K''} \\ &\qquad\times\int_{r=0}^{1}\Bigg(\prod_{\substack{j\in(L')^c\cap P\\ j\in L'\cap P^c}}\frac{r^{|\mu_j^{\nu}|}|\mu_j^{\nu}|}{1-r^{|\mu_j^{\nu}|}}\prod_{\substack{k\in L'\cap P\\ k\in(L')^c\cap P^c}}\frac{|\mu_k^{\nu}|}{1-r^{|\mu_k^{\nu}|}}\Bigg) \\ &\qquad\qquad\times\frac{1}{(A(r,\nu,z)-i\nu\cdot t)^{2n'+m}}\,\frac{dr\,d\nu}{|\log r|\,r}.\end{aligned} \tag{20}$$

This establishes the terms in (7) where $L\neq P$.

When $L=P$, the $S_{P,P}$ term is present in $\tilde{N}_P$ (see (17)) and we compute the inverse Fourier transform in λ by switching to polar coordinates, $\lambda=\tau\nu$, $\tau\geq 0$,

$\nu \in S^{m-1}$. We have

$$\begin{aligned}(21)\quad &\mathcal{F}_\lambda^{-1}\{N_{K,P}(z,\hat{\lambda})\}(t)\\ &= \frac{1}{(2\pi)^{m/2}} \int_{\lambda\in\mathbb{R}^m} e^{i\lambda\cdot t}\{\det(\bar{U}(\nu)_{K',P})\tilde{N}_P(z,\hat{\lambda})\, d\bar{Z}(z,\nu)_P \wedge d\bar{z}''_{K''}\}\, d\lambda\\ &= \frac{1}{(2\pi)^{m/2}} \int_{\nu\in S^{m-1}} \det(\bar{U}(\nu)_{K',P})\, d\bar{Z}(z,\nu)_P \wedge d\bar{z}''_{K''}\\ &\quad\times \int_{\tau=0}^{\infty} e^{i\tau\nu\cdot t} \int_{s=0}^{\infty} \left(\tilde{H}_P(s,z,\widehat{\tau\nu}) - \frac{\chi(s\tau)}{s} S_{P,P}(z',\widehat{\tau\nu})\right)\tau^{m-1}\, ds\, d\tau\, d\nu,\end{aligned}$$

where $d\nu$ is surface measure on the unit sphere S^{m-1}. Now we insert the heat kernel, $\tilde{H}_P$, from (16) and focus on the above s,τ-integral in (21), denoted by I_ν. Note that

$$\mu_j^\lambda = \tau\mu_j^\nu \quad \text{and} \quad \det A_\lambda = \tau^{2n'} \det A_\nu.$$

We scale in s by replacing $s\tau$ by s and then integrate we in τ. With $C_{m,n} = 2^n/(2\pi)^{m/2+n}$, we have

$$\begin{aligned} I_\nu &= C_{m,n}|\det A_\nu| \int_{s=0}^{\infty}\int_{\tau=0}^{\infty} \Bigg(e^{-\tau|z''|^2/s} \prod_{j=1}^{2n'} \frac{e^{s|\mu_j^\nu|}}{\sinh(s|\mu_j^\nu|)} e^{-|\mu_j^\nu|\coth(s|\mu_j^\nu|)\tau|z_j^\nu|^2}\\ &\qquad\qquad - 2^{2n'}\chi(s)\prod_{j=1}^{2n'} e^{-|\mu_j^\nu|\tau|z_j^\nu|^2}\Bigg) e^{it\cdot\nu\tau}\,\tau^{2n'+m-2}\, d\tau\,\frac{ds}{s}\\ &= C_{m,n}|\det A_\nu| \int_{s=0}^{\infty}\int_{\tau=0}^{\infty} \Bigg(\prod_{j=1}^{2n'} \frac{e^{s|\mu_j^\nu|}}{\sinh(s|\mu_j^\nu|)} e^{-\tau\left(|z''|^2/s + \sum_{j=1}^{2n'} |\mu_j^\nu|\coth(s|\mu_j^\nu|)|z_j^\nu|^2 - i\nu\cdot t\right)}\\ &\qquad\qquad - 2^{2n'}\chi(s) e^{-\tau\left(\sum_{j=1}^{2n'}|\mu_j^\nu||z_j^\nu|^2 - it\cdot\nu\right)}\Bigg)\tau^{2n'+m-1}\, d\tau\,\frac{ds}{s}\\ &= (2n'+m-1)!\, C_{m,n}|\det A_\nu|\\ &\quad\times \int_{s=0}^{\infty}\Bigg(\bigg(\prod_{j=1}^{2n'}\frac{e^{s|\mu_j^\nu|}}{\sinh(s|\mu_j^\nu|)}\bigg) \frac{1}{\left(\frac{|z''|^2}{s} + \sum_{j=1}^{2n'}|\mu_j^\nu|\coth(s|\mu_j^\nu|)|z_j^\nu|^2 - i\nu\cdot t\right)^{2n'+m}}\\ &\qquad\qquad - 2^{2n'}\chi(s)\frac{1}{\left(\sum_{j=1}^{2n'}|\mu_j^\nu||z_j^\nu|^2 - it\cdot\nu\right)^{2n'+m}}\Bigg)\frac{ds}{s}, \end{aligned}$$

where the last equality uses the formula

$$\int_0^\infty \tau^p e^{-\alpha\tau}\, d\tau = \frac{p!}{\alpha^{p+1}} \quad \text{for } \operatorname{Re}\alpha > 0.$$

We use the substitution $r = e^{-2s}$ in the remaining s-integral (and so $ds/s = -dr/(r\,|\log r|)$ and the oriented r-limits of integration become 1 to 0) to obtain

$$I_\nu = K_{m,n}|\det A_\nu| \int_{r=0}^{1} \left(\left(\prod_{j=1}^{2n'} \frac{1}{1 - r^{|\mu_j^\nu|}} \right) \frac{1}{(A(r,\nu,z',z'') - i\nu\cdot t)^{2n'+m}} - \chi\left(\tfrac{1}{2}|\log r|\right) \frac{1}{(A(0,\nu,z',0) - i\nu\cdot t)^{2n'+m}} \right) \frac{dr}{r\,|\log r|}.$$

We choose $b = \frac{1}{2}\log 2$ so that $\chi(\frac{1}{2}|\log r|)$ is the characteristic function of $\left[0, \frac{1}{2}\right]$. From (21), observe that

$$\mathcal{F}_\lambda^{-1}\{N_{K,P}(z,\hat{\lambda})\}(t) = \int_{\nu\in S^{m-1}} I_\nu \det(\bar{U}(\nu)_{K',P})\, d\bar{Z}(z,\nu)_P \wedge d\bar{z}''_{K''}\, d\nu,$$

which equals the term in (7) with $L = P$. Therefore, the proof of Theorem 2.4 is complete.

4. Proof of Theorem 2.5, $|t| \geq |z|^2$

In [Boggess and Raich 2023], the case when $|t| \geq |z|^2$ is the most delicate for the proof of the estimates. In our current manuscript, when $n'' \geq 2$, the case $|t| \geq |z|^2$ is handled by adapting the argument from the corresponding argument in [Boggess and Raich 2023]. Here we only sketch this argument with details on the modifications needed to handle the null variables (z''). We then provide complete details when $n'' = 1$ since new ideas are involved.

The primary new term is $(2\,|z''|^2)/|\log r|$ that appears in $A(r,\nu,z',z'')$. However, the series expansion for $1/|\log r|$ around $r = 1$ has leading term $1/(1-r)$, so the effect of the null directions on the estimates near $r = 1$ is the same as for the nonnull directions. Some bookkeeping is required but the estimates in our context here are very similar to the estimates presented in detail in [Boggess and Raich 2023].

The first step of the analysis is to factor out $|t|^{2n'+n''+m-1}$ from the denominator and rotate in ν via an orthogonal matrix M_t chosen so that $M_t(t/|t|)$ is the unit vector in the ν_1 direction (so in the new coordinates, $\nu\cdot t = \nu_1|t|$). We also set $\nu^t = M_t^{-1}\nu$,

$$p = (p', p'') = \frac{z}{|t|^{1/2}} \in \mathbb{C}^{2n'+n''},$$

$$Q(\nu^t, p) = \frac{Z(\nu^t, z)}{|t|^{1/2}} = \frac{(Z(\nu^t, z'), z'')}{|t|^{1/2}} = \frac{(U(\nu^t)^* \cdot z', z'')}{|t|^{1/2}}.$$

Note that $|Q(\nu^t, p)|^2 = |p|^2$ since U_{ν^t} is unitary.

We obtain

$$N_K(z,t) = |t|^{-(2n'+n''+m-1)} \sum_{L' \in \mathcal{I}'_{q'}} N_{KL'}(p)$$

where

$$\text{(22)}\quad N_{K,L'}(p) = \int_{\nu^t \in S^{m-1}} \int_{r=0}^{1} \frac{\det(\bar{U}(\nu^t)_{K',L'}) B_{L'}(r,\nu^t)\, d\bar{Z}(\nu^t, z')_{L'} \wedge d\bar{z}''_{K''}}{(A(r,\nu^t,p) - i\nu_1)^{2n'+n''+m-1}} \frac{d\nu\, dr}{r\,|\log r|^{n''}}$$

if $L' \neq P$ or $n'' \geq 2$ and where

$$\text{(23)}\qquad B_{L'}(r,\nu) = \prod_{\substack{j \in (L')^c \cap P \\ j \in L' \cap P^c}} \frac{r^{|\mu_j^\nu|} |\mu_j^\nu|}{1 - r^{|\mu_j^\nu|}} \prod_{\substack{k \in L' \cap P \\ k \in (L')^c \cap P^c}} \frac{|\mu_k^\nu|}{1 - r^{|\mu_k^\nu|}},$$

$$\text{(24)}\qquad A(r,\nu,p) = \frac{2}{|\log r|} |p''|^2 + \sum_{j=1}^{2n'} |\mu_j^\nu| \left(\frac{1 + r^{|\mu_j^\nu|}}{1 - r^{|\mu_j^\nu|}} \right) |Q_j(\nu, p')|^2.$$

If $L' = P$ and $n'' = 1$, then

$$\begin{aligned} N_{K,P}(p) = \int_{\nu^t \in S^{m-1}} & \det(\bar{U}(\nu^t)_{K',P})\, d\bar{Z}(\nu^t, z')_P \wedge d\bar{z}''_{K''} |\det A_{\nu^t}| \\ & \times \int_{r=0}^{\frac{1}{2}} \left(\left(\prod_{j=1}^{n-1} \frac{1}{1 - r^{|\mu_j^{\nu^t}|}} \right) \frac{1}{(A(r,\nu^t,p) - i\nu_1)^{2n'+m}} \right. \\ & \qquad \left. - \frac{1}{(A(0,\nu^t,p',0) - i\nu_1)^{2n'+m}} \right) \frac{dr\, d\nu^t}{|\log r| r} \\ & + \int_{\nu^t \in S^{m-1}} \det(\bar{U}(\nu^t)_{K',P})\, d\bar{Z}(\nu^t, z')_P \wedge d\bar{z}''_{K''} |\det A_{\nu^t}| \\ & \times \int_{r=\frac{1}{2}}^{1} \left(\prod_{j=1}^{n-1} \frac{1}{1 - r^{|\mu_j^{\nu^t}|}} \right) \frac{1}{(A(r,\nu^t,p) - i\nu_1)^{2n'+m}} \frac{dr\, d\nu}{|\log r| r}. \end{aligned}$$

To prove Theorem 2.5 in the case that $|t| \geq |z|^2$ and $0 \leq q \leq 2n' + n''$, it suffices to prove the following theorem.

Theorem 4.1. *There is a uniform constant $C > 0$ so that $|N_{K,L}(p)| \leq C$ for all $p \in \mathbb{C}^{2n'+n''}$ with $|p| \leq 1$ and all $K, L \in I_q$ with $0 \leq q \leq 2n' + n''$.*

We first sketch the estimate of the kernel near $r = 1$ using the ideas from [Boggess and Raich 2023].

Subcase: $|t| \geq |z|^2$ and $\frac{1}{2} < r < 1$. We prove Theorem 4.1. We start with a key result — Lemma 5.2 in [Boggess and Raich 2023], which we restate here.

Lemma 4.2. *Let*

$$B(r,\nu)=B_{\varnothing}(r,\nu)=\prod_{j\in P}\frac{r^{|\mu_j^\nu|}|\mu_j^\nu|}{1-r^{|\mu_j^\nu|}}\prod_{k\in P^c}\frac{|\mu_k^\nu|}{1-r^{|\mu_k^\nu|}}. \tag{25}$$

Then

$$\sum_{L'\in\mathcal{I}'_{q'}}\det(\bar{U}(\nu)_{K',L'})\,d\bar{Z}(\nu,z')_{L'}B_{L'}(r,\nu)=\sum_{J'\in\mathcal{I}'_{q'}}\det([r^{-\bar{A}_\nu}]_{K',J'})B(r,\nu)\,d\bar{z}_{J'} \tag{26}$$

is real analytic in $\nu\in S^{m-1}$ *and* $0<r<1$.

Remark 4.3. The real content of this lemma is the real analyticity in ν of the expression in (26), especially in view of the fact that the eigenvalues μ_j^ν are not necessarily real analytic or even smooth in the parameter ν. As shown in [Boggess and Raich 2023], the expression $B(r,\nu)$ is real analytic in ν due to the fact that the positive eigenvalues are bounded away from the negative eigenvalues. In addition, $r^{-\bar{A}_\nu}$ is real analytic in ν since A_ν depends linearly on ν.

Using Lemma 4.2, a typical term for $N_{K,L}(p)$ in (22) — with $\frac{1}{2}\le r<1$ for the domain of integration — is

$$N^u_{K,J}(p)=\int_{\nu\in S^{m-1}}\int_{r=\frac{1}{2}}^{1}\frac{\det([r^{-\bar{A}_\nu}]_{K',J'})B(r,\nu)}{(A(r,\nu^t,p)-i\nu_1)^{2n'+n''+m-1}}\frac{d\nu\,dr}{r\,|\log r|^{n''}}. \tag{27}$$

The superscript u refers to the fact that the integral is over the "upper" piece of the r-interval. Our goal in this section is to establish the following lemma.

Lemma 4.4. *There is a uniform constant C such that*

$$|N^u_{K,J}(p)|\le C$$

for all $p\in\mathbb{C}^{2n'+n''}$ *with* $|p|\le 1$.

As in [Boggess and Raich 2023], we use the change of variable

$$r=r(s)=\frac{s-1}{s+1}\qquad\text{or equivalently}\qquad s=\frac{r+1}{1-r}\quad\text{with}\quad\frac{dr}{r}=\frac{2\,ds}{s^2-1} \tag{28}$$

and observe that $\frac{1}{2}\le r<1$ transforms to $s\ge 3$. We obtain

$$\begin{aligned} N^u_{K,J}(p) &\\ &=2\int_{\nu\in S^{m-1}}\int_{s=3}^{\infty}\frac{\det[r(s)^{-\bar{A}_\nu}]_{K',J'}}{(A(r(s),\nu^t,p)-i\nu_1)^{2n'+n''+m-1}}\frac{B(r(s),\nu)r'(s)}{r(s)}\frac{ds\,d\nu}{|\log r(s)|^{n''}}. \end{aligned} \tag{29}$$

We then expand the various components of the integrand defining $N^u_{K,J}(p)$ on the last line of (29) about $s=\infty$. We briefly outline the main steps in Sections 5, 6, and 7 in [Boggess and Raich 2023] and point out the differences needed to deal with the factor of $|\log r(s)|^{n''}$ in the denominator. From Proposition 5.4 in [Boggess and Raich 2023], we have

$$\frac{B(r(s),\nu)r'(s)}{r(s)} = \frac{2}{2^{2n'}\left(1-\frac{1}{s^2}\right)}\left(\sum_{\ell=0}^{2n'-1} P_\ell(\nu)s^{2n'-\ell-2} + \frac{O(s,\nu)}{s^2}\right), \tag{30}$$

(31) a typical monomial in $P_\ell(\nu) = \nu^{\ell-e}$, where e is even with $0 \le e \le \ell$.

Here, $P_\ell(\nu)$ is a polynomial in $\nu = (\nu_1, \dots \nu_m) \in S^{m-1}$ of total degree ℓ. By an abuse of notation, the term $\nu^{\ell-e}$ in (31) stands for a monomial in the coordinates of ν of total degree $\ell - e$. Also note that the term $(1 - s^{-2})^{-1}$ on the right-hand side of (30) only has even powers of $1/s$ in its expansion about $s = \infty$.

Next, we use the second part of Proposition 5.4 in [Boggess and Raich 2023] to expand $\det[r(s)^{-\bar{A}_\nu}]_{K,J}$ around $s = \infty$. The result is a sum of terms of the form

$$\frac{\nu^{\ell'-e'}}{s^{\ell'}}, \tag{32}$$

where $\ell' \ge 1$, e' is an even integer with $0 \le e' \le \ell'$, and $\nu^{\ell'-e'}$ is a monomial of degree $\ell' - e'$ in the coordinates of $\nu \in S^{m-1}$.

Now, we expand $|\log r(s)|^{-n''}$ about $s = \infty$ and obtain

$$\frac{1}{|\log r(s)|^{n''}} = \frac{s^{n''}}{2^{n''}} + \sum_{k=1}^{\infty} c_{k,n''} s^{n''-2k}. \tag{33}$$

Finally, we have the following expansion of the terms involving $A(r(s), \nu^t, p)$ from equation (36) in [Boggess and Raich 2023] (with $I_1, I_2, I_3 = \varnothing$):

$$\begin{aligned} &\frac{1}{(A(r(s),\nu^t,p) - i\nu_1)^{2n'+n''+m-1}} \\ &= \frac{1}{(s|p|^2 - i\nu_1)^{2n'+n''+m-1}}\left(1 + \sum_{j=1}^{\infty} \frac{\alpha_j\left(\sum_{k=1}^{\infty} \frac{p^*\cdot\left(p_{2k}(A_{\nu^t})\oplus\frac{1}{2}c_{k,1}\mathbb{I}_{n''}\right)\cdot p}{s^{2k-1}}\right)^j}{(s|p|^2 - i\nu_1)^j}\right). \end{aligned} \tag{34}$$

Now we assemble a typical term in the expansion of the integrand in (29) by multiplying the typical terms from (31), (32), (33), and (34). We summarize a typical term from each of the components that comprise (29) in the following chart:

term	typical term	notes
$\det[r(s)^{-\bar{A}_\nu}]_{K,J}$	$\frac{\nu^{\ell'-e_1}}{s^{\ell'}}$	$\ell' \ge 1$, e_1 is even, and $0 \le e_1 \le \ell'$
$\frac{B(r(s),\nu)r'(s)}{r(s)}$	$\nu^{\ell-e_2}s^{2n'-\ell-e_3-2}$	e_2 and e_3 are even, $0 \le e_2 \le \ell$
$\frac{1}{\lvert\log r(s)\rvert^{n''}}$	$s^{n''-e_4}$	e_4 is even
(34)	$\frac{1}{(s\lvert p\rvert^2-i\nu_1)^{2n'+n''+m-1+j}}\frac{\nu^{j(2k-e_5)}}{s^{j(2k-1)}}$	$j, k \ge 1$, e_5 is even, and $0 \le e_5 \le k$

The typical terms of (29) that require the most care are those involving powers of s which are greater than -2. The remaining terms comprise the “remainder term” and will be handled later. From the above chart, we see that a typical term from the integrand of (29) is of the form

$$C(p,\bar{p})^{2j}\frac{s^{N_j-2-\ell_j-2k_j}\nu^{\ell_j-e_j}}{(s|p|^2-i\nu_1)^{N_j+m-1}}, \tag{35}$$

where the integers N_j, ℓ_j, e_j, k_j satisfy

$$N_j=2n'+n''+j,\quad e_j>0 \text{ is even},\quad 0<e_j\le\ell_j,\quad \text{and}\quad k_j\ge 0. \tag{36}$$

What is relevant for the proof of Lemma 4.5 below is that a typical term in the expansion satisfies

$$\text{exponent(denominator)} - \text{exponent}(s) - \text{exponent}(\nu) = m+1+E, \tag{37}$$

where E is an even, nonnegative integer.

In view of Lemma 4.2, the remainder term is analytic in $\nu\in S^{m-1}$ and $s>3$. In addition, the typical term is

$$\frac{O(p^{\alpha'})O(\nu,s)}{(s|p|^2-i\nu_1)^{\alpha}s^{\beta}}, \tag{38}$$

where $O(\nu,s)$ is real analytic in $\nu\in S^{m-1}$ and $s\ge 3$, bounded in s, and $\beta\ge 2$.

Analysis of typical term in (35). We will now show that the integral (over $\nu\in S^{m-1}$ and $s\ge 1$) of the typical term in (35) is bounded in p. We will also show the same for the remainder term in (38).

As to the first task, let $\hat{r}=|p|^2>0$ and define

$$H_{N,\ell,m,e,k}(\hat{r},s,\nu)=\frac{s^{N-2-\ell-2k}\nu^{\ell-e}}{(s\hat{r}-i\nu_1)^{N+m-1}}.$$

To establish Lemma 4.4 over the region $\frac{1}{2}\le r<1$, we need to show that for each $\ell\ge 0$, there is a uniform constant C such that

$$\left|\int_{\nu\in S^{m-1}}\int_{s=3}^{\infty}H_{N,\ell,m,e,I_3,k}(\hat{r},s,\nu)\,ds\,d\nu\right|\le C \tag{39}$$

for all $\hat{r}>0$ near zero.

As discussed at the end of Section 7 in [Boggess and Raich 2023], we can assume the monomial $\nu^{\ell-e}$ depends on ν_1 only (by writing $\nu=(\nu_1,\nu')$ and noting that integrals of odd powers of monomials in ν' over $\nu'\in S^{m-2}$ are zero). We let $x=\nu_1$, and then the surface measure on the unit sphere in S^{m-1} can be written as

$$d\nu=(1-x^2)^{(m-3)/2}\,dx\,d\nu'$$

where $d\nu'$ is the surface measure on S^{m-2}.

The desired estimate in (39) will follow from the next lemma.

Lemma 4.5. *For any nonnegative integers N, m and ℓ with $m \geq 2$ and any even integer E with $0 \leq E \leq |\ell|$, let*

$$A^{\ell,E}_{N,m,k}(\hat{r}) = \int_{x=-1}^{1} \int_{s=3}^{\infty} \frac{(1-x^2)^{(m-3)/2} s^{N-2-\ell-2k} x^{\ell-E}\, ds\, dx}{(s\hat{r} - ix)^{N+m-1}}.$$

Then $A^{\ell,E}_{N,m,k}(\hat{r})$ is a smooth function of $\hat{r} > 0$ up to $\hat{r} = 0$.

This lemma is almost identical to Lemma 8.1 in [Boggess and Raich 2023] (the difference is in the exponent of s). Below, we give a short argument to reduce our lemma to Lemma 8.1 in [Boggess and Raich 2023].

Proof of Lemma 4.5. First write

$$A^{\ell,E}_{N,m,k}(\hat{r}) = C_{N,\ell} D_{\hat{r}}^{N-(2+\ell+2k)} \{B^{\ell,E}_{m}(\hat{r})\},$$

where $C_{N,\ell}$ is a constant and

$$B^{\ell,E,2k}_{m}(\hat{r}) = \int_{x=-1}^{1} \int_{s=3}^{\infty} \frac{(1-x^2)^{(m-3)/2} x^{\ell-E}\, ds\, dx}{(s\hat{r} - ix)^{m+\ell+2k+1}}.$$

Here, $D^j_{\hat{r}}$ indicates the j-th derivative with respect to $\hat{r}$. The index j is allowed to be negative in which case this means the $|j|$-th antiderivative with respect to $\hat{r}$ (with a particular initial condition specified at a fixed value of $\hat{r} = \hat{r}_0 > 0$).

Note, $B^{\ell,E,2k}_{m}(\hat{r})$ is identical to the corresponding expression in the proof of [Boggess and Raich 2023, Lemma 8.1] except that the exponent in the denominator differs by the even integer $2k \geq 0$. The rest of the proof proceeds exactly as the proof of Lemma 8.1 to show that $B^{\ell,E,2k}_{m}(\hat{r})$ is smooth for $\hat{r} > 0$ up to $\hat{r} = 0$. □

Analysis of Remainder Term in (38). The remainder term in (38) is

$$\frac{O(\nu, s)}{(s|p|^2 - i\nu_1)^{\alpha} s^{\beta}} \quad \text{with } \beta \geq 2 \text{ and } \alpha \geq 2.$$

As above, we set $x = \nu_1$. Since $s^{-\beta}$ is integrable over $\{s \geq 3\}$ and since $O(\nu', \nu_1, x)$ is real analytic (and hence uniformly bounded) in $\nu' \in \sqrt{1-x^2} S^{m-2}$ ($(m-2)$-dimensional sphere of radius $\sqrt{1-x^2}$), the following lemma will finish the proof of Theorem 4.1 for the integral over the region $\frac{1}{2} \leq r < 1$ (and in the case $|t| \geq |z|^2$ and $0 \leq q \leq 2n' + n''$).

Lemma 4.6. *For $m \geq 2$, let*

$$R(s, \hat{r}, \nu') = \int_{x=-1}^{1} \frac{(1-x^2)^{(m-3)/2} O(\nu', x, s)\, dx}{(s\hat{r} - ix)^{\alpha}}.$$

Then $R(s, \hat{r}, \nu')$ is uniformly bounded for $s \geq 3$, $\hat{r} \geq 0$, and $\nu' \in \sqrt{1-x^2} S^{m-2}$.

This lemma is identical to Lemma 9.1 in [Boggess and Raich 2023]. The basic idea is to use Cauchy's theorem to deform the contour of integration into the upper half plane and away from $x = 0$.

***Subcase:* $|t| \geq |z|^2$ *and* $0 < r < \frac{1}{2}$.** We first assume that $n'' \geq 2$ or $n'' = 1$ and $J' \neq P$. We start with the lower r version of (27). In this case, however, we stick with the r variable, $0 \leq r \leq \frac{1}{2}$ (instead of changing to s). We rewrite this term here:

$$N^{\ell}_{K,J}(p) = \int_{\nu \in S^{m-1}} \int_{r=0}^{\frac{1}{2}} \frac{\det([r^{-\bar{A}_\nu}]_{K',J'}) B(r, \nu)}{(A(r, \nu^t, p) - i\nu_1)^{2n+m-1}} \frac{d\nu\, dr}{r\,|\log r|^{n''}}. \tag{40}$$

The ℓ superscript indicates that we are working on the lower half of the r-interval. $N^{\ell}_{K,J}(p)$ is the coefficient of the $d\bar{z}_{J'}$ component of

$$\int_{\nu^t \in S^{m-1}} \int_{r=0}^{\frac{1}{2}} \frac{\det(\bar{U}(\nu^t)_{K',J'})\, d\bar{Z}(\nu^t, z)_{L'} \wedge d\bar{z}''_{K''} B_{L'}(r, \nu^t)}{(A(r, \nu^t, p) - i\nu_1)^{2n+m-1}} \frac{d\nu\, dr}{r\,|\log r|^{n''}} \tag{41}$$

Our goal is to prove the following:

Lemma 4.7. *We have*

$$|N^{\ell}_{K,J}(p)| \leq C \quad \textit{for all } p = \frac{z}{|t|^{1/2}} \in \mathbb{C}^{2n'+n''}, \tag{42}$$

where C is a uniform constant.

Proof. The proof is nearly identical to the proof of Lemma 10.1 in [Boggess and Raich 2023] with the only difference being the presence of the log-terms. We give a quick outline. We are in a case where at least one of $L \cap P^c$ or $L^c \cap P$ is nonempty. In view of (23), there must be a positive power of r in the numerator of $B_{L'}(r, \nu)$. Therefore

$$\frac{|B_{L'}(r, \nu)|}{r\,|\log r|^{n''}} \leq \frac{C r^{c_0}}{r\,|\log r|^{n''}}, \tag{43}$$

where C and c_0 are uniform positive constants. Having a positive power of r in the numerator turns out to be one of the most useful terms for offsetting enough of the blow-up of $1/r$ as $r \to 0$ to guarantee integrability in r near 0. We repeatedly use this fact in both the $|t|$ large and $|z|$ large cases. In fact, as soon as there is a factor of r^{c_0} for some $c_0 > 0$ in the numerator, we can use a straightforward size argument to bound the integrand.

For $|t| \geq |z|^2$, the presence of a positive power of r allows for the following. First, the integrand of $N^{\ell}_{K,J}$ is integrable over the interval $0 < r < \frac{1}{2}$. Therefore, the integral on the right-hand side of (41) over the set $\{0 \leq r \leq \frac{1}{2}\} \times \{|\nu_1| \geq \frac{1}{2}\}$ is uniformly bounded for $p \in C^{2n'+n''}$. Thus, we turn our attention to the integral over $\{0 \leq r \leq \frac{1}{2}\} \times \{|\nu_1| \leq \frac{1}{2}\}$.

The idea is to integrate by parts in ν_1 over the integral in (40) over the interval $\left\{|\nu_1| \le \frac{1}{2}\right\}$ to reduce the power of $(A(r, \nu_t, p) - i\nu_1)$ in the denominator where $A(r, \nu_t, p)$ is defined in (24). As shown in Section 10 in [Boggess and Raich 2023], $A(r, \nu_t, p)$ is analytic in $\nu \in S^{m-1}$.

Let

$$X(r, \nu, p) := \left(\frac{\partial}{\partial \nu_1}\{A(r, \nu^t, p)\} - i\right)^{-1}$$

and note that

$$X(r, \nu, p) D_{\nu_1}\left\{\frac{-i(2n' + n'' - 2)^{-1}}{(A(r, \nu^t, p) - i\nu_1)^{2n'+n''-2}}\right\} = \frac{1}{(A(r, \nu^t, p) - i\nu_1)^{2n'+n''-1}}.$$

When integrating by parts with $X(r, \nu, p)D_{\nu_1}$ over $\left\{|\nu_1| \le \frac{1}{2}\right\}$, there will be terms involving the ν_1-derivatives of $X(r, \nu, p)$, $r^{-\bar{A}^\nu}$ and $B(r, \nu)$ that occur in the integrand of (40). These derivatives produce additional powers of $|\log r|$ which do not affect the integrability in r over $0 \le r \le \frac{1}{2}$. In addition, there are boundary terms at $|\nu_1| = \frac{1}{2}$ and these terms are uniformly integrable on $\left\{0 \le r \le \frac{1}{2}\right\} \times \left\{|\nu_1| = \frac{1}{2}\right\}$.

This process of integration by parts with $X(r, \nu, p)D_{\nu_1}$ can be repeated until the integrand in (40) involves only $\log(A(r, \nu^t, p) - i\nu_1)$ (using the principle branch of log since the A term is positive). This log-term is uniformly integrable on $\left\{0 \le r \le \frac{1}{2}\right\} \times \left\{|\nu_1| \le \frac{1}{2}\right\}$, and thus Lemma 4.7 is proved. For more details, see Section 10 of [Boggess and Raich 2023] (where z-, $\bar{z}$-, and t-derivatives are also handled in full generality).

The remaining case is $n'' = 1$ and $J' = P$ where the relevant term to estimate is given by (7) with the r-interval of integration restricted to $0 \le r \le \frac{1}{2}$. We first recall [Boggess and Raich 2023, Lemma 12.3].

Lemma 4.8. *The following functions are analytic as a function of $\nu \in S^{m-1}$:*

- $\nu \to |\det A_\nu|$.
- $\nu \to A(0, \nu, p) = \sum_{j=1}^{2n'} |\mu_j^\nu||p_j^\nu|^2$.
- $\nu \to \det(\bar{U}(\nu)_{K,P})\, d\bar{Z}(p, \nu)^P = \sum_{J \in \mathcal{I}_n} \det(\bar{U}(\nu)_{K,P}) \det[U(\nu)_{P,J}]^T\, d\bar{z}^J$.

Therefore, the functions to estimate in (7) with the r-interval of integration restricted to $0 \le r \le \frac{1}{2}$ are of the form

$$\begin{aligned} N_{K,J}^{\ell}(p) = \int_{\nu^t \in S^{m-1}} & \det(\bar{U}(\nu)_{K,P}) \det[U(\nu)_{P,J}]^T |\det A_{\nu^t}| \\ & \times \int_{r=0}^{\frac{1}{2}} \left(\left(\prod_{j=1}^{2n'} \frac{1}{1 - r^{|\mu_j^{\nu^t}|}} \right) \frac{1}{(A(r, \nu^t, p) - i\nu_1)^{2n'+m}} \right. \\ & \qquad \left. - \frac{1}{(A(0, \nu^t, p', 0) - i\nu_1)^{2n'+m}} \right) \frac{dr\, d\nu^t}{|\log r| r}. \end{aligned}$$

By writing

$$\frac{1}{1-r^{|\mu_j^{\nu^t}|}} = 1 + \frac{r^{|\mu_j^{\nu^t}|}}{1-r^{|\mu_j^{\nu^t}|}} \quad \text{and} \quad \frac{1+r^{|\mu_j^{\nu^t}|}}{1-r^{|\mu_j^{\nu^t}|}} = 1 + \frac{2r^{|\mu_j^{\nu^t}|}}{1-r^{|\mu_j^{\nu^t}|}}, \tag{44}$$

we can write

$$\begin{aligned} N^{\ell}_{K,J}(p) = \int_{\nu^t\in S^{m-1}} & \det(\bar{U}(\nu)_{K,P}) \det[U(\nu)_{P,J}]^T |\det A_{\nu^t}| \\ & \times \int_{r=0}^{\frac{1}{2}} \Biggl(\frac{1}{\bigl(\frac{2}{|\log r|}|q''|^2 + A(0,\nu^t,p) - i\nu_1\bigr)^{2n'+m}} \\ & \qquad - \frac{1}{(A(0,\nu^t,p',0)-i\nu_1)^{2n'+m}} \Biggr) \frac{dr\, d\nu^t}{|\log r| r} + \text{OK}, \end{aligned} \tag{45}$$

where the OK term is comprised of terms with r^{c_0} in the numerator for values $c_0 > 0$ and the discussion after (43) applies. Focusing on the integral in r, we let $s = -2/\log r$ so that $ds/s = dr/(|\log r| r)$ so that

$$\begin{aligned} &\int_{r=0}^{\frac{1}{2}} \Biggl(\frac{1}{\bigl(\frac{2}{|\log r|}|p''|^2 + A(0,\nu^t,p',0) - i\nu_1\bigr)^{2n'+m}} - \frac{1}{(A(0,\nu^t,p',0)-i\nu_1)^{2n'+m}} \Biggr) \frac{dr}{r\,|\log r|} \\ &\quad = \int_{s=0}^{\frac{2}{\log 2}} \Biggl(\frac{1}{(s|p''|^2 + A(0,\nu^t,p',0) - i\nu_1)^{2n'+m}} - \frac{1}{(A(0,\nu^t,p',0)-i\nu_1)^{2n'+m}} \Biggr) \frac{ds}{s}. \end{aligned}$$

By Lemma A.1, with $a = |p''|^2$, $b = A(0,\nu^t,p',0) - i\nu_1$, and $\gamma = 2/\log 2$,

$$\begin{aligned} &\int_{s=0}^{\frac{2}{\log 2}} \Biggl(\frac{1}{(s|p''|^2 + A(0,\nu^t,p',0) - i\nu_1)^{2n'+m}} - \frac{1}{(A(0,\nu^t,p',0)-i\nu_1)^{2n'+m}} \Biggr) \frac{ds}{s} \\ &\quad = \frac{1}{(A(0,\nu^t,p',0)-i\nu_1)^{2n'+m}} \log\Biggl(1 + \frac{2}{\log 2} \frac{|z''|^2}{A(0,\nu^t,z',0) - i\nu_1|t|}\Biggr) \\ &\qquad + E_{2n'+m}(|p''|^2, A(0,p',0) - i\alpha_1). \end{aligned} \tag{46}$$

To complete the proof of Lemma 4.7, we use Lemma 4.8 and shift the contour in ν_1 to avoid $\nu_1 = 0$. By doing this,

$$|A(0,\nu^t,p',0) - i\nu_1| \sim |p'|^2 + 1$$

on the new contour and basic size estimates now suffice. □

5. Proof of Theorem 2.5, $|z|^2 \geq |t|$

***Subcase:* $|z|^2 \geq |t|$ *and* $0 < r < \frac{1}{2}$.** Analogous to the case when $|t| \geq |z|^2$, we investigate the terms in (41) but with the term $|t|^{2n'+n''+m-1}$ inserted back into the denominator of the integrand. Using (5) we are led to estimate the term

$$N^{\ell}_{K,L,J}(z,t) = \int_{\nu\in S^{m-1}} \det(\bar{U}(\nu)_{K',L}) \det(U(\nu)_{L,J'})|\det A_\nu| \times \int_{r=0}^{\frac{1}{2}} \Bigg(\prod_{\substack{j\in(L')^c\cap P\\ j\in L'\cap P^c}} \frac{r^{|\mu^\nu_j|}}{1-r^{|\mu^\nu_j|}} \prod_{\substack{k\in L'\cap P\\ k\in(L')^c\cap P^c}} \frac{1}{1-r^{|\mu^\nu_k|}} \Bigg) \times \frac{1}{(A(r,\nu,z)-i\nu\cdot t)^{2n'+n''+m-1}} \frac{dr\,d\nu}{|\log r|^{n''} r}, \tag{47}$$

when $n'' \geq 2$ or $L' \neq P$, and

$$N^{\ell}_{K,P,J}(z,t) = \int_{\nu^t\in S^{m-1}} \det(\bar{U}(\nu)_{K,P}) \det[U(\nu)_{P,J}]^T |\det A_{\nu^t}| \times \int_{r=0}^{\frac{1}{2}} \Bigg(\Bigg(\prod_{j=1}^{2n'} \frac{1}{1-r^{|\mu^{\nu^t}_j|}} \Bigg) \frac{1}{(A(r,\nu^t,z)-i\nu\cdot t)^{2n'+m}} - \frac{1}{(A(0,\nu^t,z',0)-i\nu\cdot t)^{2n'+m}} \Bigg) \frac{dr\,d\nu^t}{|\log r|\,r}, \tag{48}$$

when $n''=1$ and $L'=P$.

We start with the case $n''=1$ and $L'=P$ because the analysis of (48) is virtually identical to that of (45). The same reductions and equalities hold, and factoring $|t|$ back into (46) is the calculation that we need. The size estimates are more straightforward than the $|t|$ large case because we do not have to shift the contour.

We now focus on (47). We first assume $|z'|^2 \geq |z''|^2$. The upper bound estimates in this case will follow directly from size estimates. Since $|A(r,\nu,z)-i\nu\cdot t| \geq c|z'|^2$ and either $1/(r|\log r|^{n''})$ is integrable near $r=0$ ($n''\geq 2$) or there is an r^{c_0} term in the numerator ($n''=1$ and $J'\neq P$), we use size estimates to establish

$$|N^{\ell}_{K,L,J}(z,t)| \leq \frac{C}{|z'|^{2(2n'+n''+m-1)}}.$$

The $|z''| \geq |z'|$ estimate requires more care. In the case that there is a factor of r^{c_0} in the numerator, the estimate is straightforward with size estimates, as bounding $(1+r^\mu)/(1-r^\mu)$ by $|\log r|$ shows that

$$\frac{r^{c_0}}{|A(r,\nu,z)-i\nu\cdot t|^{2n'+n''+m-1}|\log r|^{n''} r} \leq \frac{r^{c_0}}{\big(\frac{|z|^2}{|\log r|}\big)^{2n'+n''+m-1}|\log r|^{n''} r} = \frac{1}{|z|^{2(2n'+n''+m-1)}} r^{c_0-1}|\log r|^{2n'+m-1}$$

is integrable at 0, and the estimate

$$|N^{\ell}_{K,L,J}(z,t)| \le C|z|^{-2(2n'+n''+m-1)} \tag{49}$$

holds. A factor r^{c_0} will always be present whenever N is hypoelliptic, that is, when $0 \le q < n'$ or $n'+n'' < q \le n$. Additionally, it will also be present when $n' \le q \le n'+n''$ as long as $L \ne P$ and (49) holds, a better estimate than (9).

It remains to analyze

$$N^{\ell}_{K,P,J}(z,t) = \int_{\nu\in S^{m-1}} \det(\bar{U}(\nu)_{K',P}) \det(U(\nu)_{P,J'})|\det A_\nu| \times \int_{r=0}^{\frac12} \prod_{k=1}^{2n'} \frac{1}{1-r^{|\mu_k^\nu|}} \frac{1}{(A(r,\nu,z)-i\nu\cdot t)^{2n'+n''+m-1}} \frac{dr\,d\nu}{|\log r|^{n''} r},$$

when $n'' \ge 2$. As we have seen, once we have a positive power of r in the numerator, we can use size estimates to obtain the estimates in (8). This is relevant for the error estimates when $|z'|^2 \ge |t|$ in two ways. First, we can apply (44) to replace $\prod_{k=1}^{2n'} 1/(1-r^{|\mu_k^\nu|})$ by 1 and an OK term. Second, since

$$A(r,\nu,z) = \frac{2}{|\log r|}|z''|^2 + \sum_{j=1}^{2n'} |\mu_j^\nu||z_j^\nu|^2 + \sum_{j=1}^{2n'} \frac{2r^{|\mu_j^\nu|}}{1-r^{|\mu_j^\nu|}}|z_j^\nu|^2,$$

we can write

$$\frac{1}{(A(r,\nu,z)-i\nu\cdot t)^{2n'+n''+m-1}} = \frac{1}{(A_0(r,\nu,z)+i\nu\cdot t)^{2n'+n''+m-1}} + \frac{O(r^{c_0})}{(A_0(r,\nu,z)+i\nu\cdot t)^{2n'+n''+m-1}} + \frac{O(r^{c_0}|z'|^2)}{(A_0(r,\nu,z)+i\nu\cdot t)^{2n'+n''+m}},$$

where

$$A_0(r,\nu,z) = \frac{2}{|\log r|}|z''|^2 + \sum_{j=1}^{2n'} |\mu_j^\nu||z_j^\nu|^2$$

and $c_0 > 0$. The first error term arises from estimating $B_P(r,\nu)$ by $|\det(A_\nu)|$. The second error term uses the expansion

$$\frac{1}{(V+\zeta)^{2n'+n''+m-1}} = \frac{1}{V^{2n'+n''+m-1}} + \sum_{j=1}^{\infty} \alpha_j \frac{\zeta^j}{V^{2n'+n''+m-1+j}}$$

and therefore has $A_0(r,\nu,z)$ raised to one higher power than in the main term. When integrated, however, the estimate from the extra degree in the denominator is offset by the additional factor of $|z|^2$ in the numerator.

This means that the remaining term to analyze is

$$\int_{\nu\in S^{m-1}} \det(\overline{U(\nu)_{K',P}})\det(U(\nu)^T_{P,J'})|\det A_\nu| \\ \times\int_0^{\frac{1}{2}} \frac{1}{(A_0(r,\nu,z)-i\nu\cdot t)^{2n'+n''+m-1}}\frac{dr\,d\nu}{r\,|\log r|^{n''}}.$$

We factor out $2|z''|^2$ from the denominator and let

$$a=\sum_{j=1}^{2n'}|\mu_j^\nu|\frac{|z_j^\nu|^2}{|z''|^2}-i\nu\cdot\frac{t}{|z''|^2}.$$

Note that $1/\log 2+a=O(1)$. By (53), we compute

$$\begin{aligned}\int_0^{\frac{1}{2}} &\frac{1}{(A_0(r,\nu,z)-i\nu\cdot t)^{2n'+n''+m-1}}\frac{dr\,d\nu}{r\,|\log r|^{n''}}\\ &=\frac{1}{|z''|^{2(2n'+n''+m-1)}}\int_0^{\frac{1}{\log 2}}\frac{s^{n''-2}}{(s+a)^{2n'+n''+m-1}}\,ds\\ &=\frac{1}{|z''|^{2(2n'+n''+m-1)}}\sum_{\ell=0}^{n''-2}\binom{n''-2}{\ell}\frac{(-1)^{n''-\ell}}{2n'+n''+m-1-\ell-1}\Big(\frac{1}{a^{2n'+m}}+O(1)\Big)\\ &=\frac{C}{|z''|^{2(n''-1)}}\frac{1}{\big(\sum_{j=1}^{2n'}|\mu_j^\nu||z_j^\nu|^2-i\nu\cdot t\big)^{2n'+m}}+O(|z''|^{-2(2n'+n''+m-1)}).\end{aligned}$$

If $|z'|^2\ge|t|$, then $\sum_{j=1}^{2n'}|\mu_j^\nu||z_j^\nu|^2-i\nu\cdot t=O(|z'|^2)$, and size estimates produce $O(|z|^{-2(n''-1)}|z'|^{-2(2n'+m)})$, the desired estimate. If, on the other hand, $|t|\ge|z'|^2$, then we treat the integral similarly to the large $|t|$ case, rotating in ν and factoring out $|t|$ to produce the integral

$$\frac{C}{|z''|^{2(n''-1)}|t|^{2n'+m}}\int_{\nu\in S^{m-1}}\det(\overline{U(\nu)_{K',P}})\det(U(\nu)^T_{P,J'})\frac{|\det A_\nu|}{\big(\sum_{j=1}^{2n'}|\mu_j^\nu||q_j^\nu|^2-i\nu_1\big)^{2n'+m}}\,d\nu$$

where $q_j^\nu=z_j^\nu/|t|^{1/2}$. The integrand in the above integral is $O(1)$ when $|\nu_1|\ge\frac{1}{2}$. In the case $|\nu_1|\le\frac{1}{2}$, we handled this exact type of integral in [Boggess and Raich 2023, (68)] and showed that the above integral is bounded by $C/(|z''|^{2(n''-1)}|t|^{2n'+m})$ (in fact, this bound is sharp).

***Subcase:* $|z|^2\ge|t|$ *and* $\frac{1}{2}<r<1$.** We are finally in a position to finish the proof of the estimates in Theorem 2.5. As with the previous subsection, we include the term $|t|^{-(2n'+n''+m-1)}$ in the integrand. Define $N^u_{K,J}(z,t)$ analogously to $N^u_{K,J}(p)$ in (27), with the r-integral over $\big[\frac{1}{2},1\big]$ and including the term $|t|^{-(2n'+n''+m-1)}$ in the integrand. We follow the analysis of the $|t|$ large case through (35) to obtain

$$\begin{aligned} N^u_{K,J}(z,t) &:= \int_{s=3}^{\infty}\int_{\nu\in S^{m-1}} \frac{\text{typical term in } N_{K,J}(p)}{|t|^{2n'+n''+m-1}}\,d\nu\,ds \\ &= \int_{s=3}^{\infty}\int_{\nu\in S^{m-1}} C(z,\bar z)^{2j}\,\frac{s^{N_j-2-\ell_j-K_j}\nu^{\ell_j-e_j}}{(s|z|^2-i\nu_1|t|)^{N_j+m-1}}\,d\nu\,ds, \end{aligned}$$

where $N_j = 2n'+n''+j$ and $\ell_j, K_j \ge 0$, $m \ge 2$. Since $|z|^2 \ge |t|$ and $|\nu| = 1$, we use size estimates and drop the t-term in the denominator to obtain

$$\begin{aligned} |N^u_{K,J}(z,t)| &\le \int_{s=3}^{\infty}\int_{\nu\in S^{m-1}} \frac{C|z|^{2j}s^{N_j-1-\ell_j-K_j}}{(s|z|^2)^{N_j+m-1}}\,d\nu\,ds \\ &\le \int_{s=3}^{\infty}\int_{\nu\in S^{m-1}} \frac{C}{|z|^{2(2n'+n''+m-1)}}\cdot\frac{1}{s^3}\,d\nu\,ds \end{aligned}$$

after taking into account the constraints on ℓ_j, K_j, m. Therefore

$$|N^u_{K,J}(z,t)| \le \frac{C}{|z|^{2(2n'+n''+m-1)}}, \tag{50}$$

and we have established the estimates in Theorem 2.5.

Higher derivatives. As mentioned in the introduction, we will refer the reader to [Boggess and Raich 2023] for details on how to handle the estimates for higher derivatives. Here is the basic idea on how to obtain the estimates for derivatives. Note that z and $\bar z$ appear quadratically in $A(r,\nu,z)$ and t only appears in the $\nu\cdot t$ term. Thus, differentiating (20) once with a z' or $\bar z'$ derivative adds one more factor of $A(r,\nu,z) - i\nu\cdot t$ to the denominator along with a linear z' or $\bar z'$ term in the numerator. The overall estimate in (8) changes by a factor of $(|z|^2+|t|)^{-1/2}$. By contrast, a t-derivative of (20) also adds a factor of $A(r,\nu,z) - i\nu\cdot t$ to the denominator but with no compensating factor of z', $\bar z'$ or t in the numerator. Thus the overall estimate in (8) changes by a factor of $(|z|^2+|t|)^{-1}$. The z''- and $\bar z''$-derivatives behave similarly. This is the basic idea behind why there is a $\frac{1}{2}$ in front of the exponents $|I_1|$ and $|I_2|$, which represent z- or $\bar z$-derivatives, and not in front of $|I_3|$, which represents t-derivatives.

6. Conclusion of the proof Theorem 2.5 — sharpness of the estimates

We will show the dominant term in (9) is nonzero for the index $K = P$ provided the eigenvectors of A_ν depend continuously on ν.

We focus on the $d\bar z'_P$ component of N_P (here, the value of n'' is not important because we are focusing on the integral in ν). Ignoring the power of $|z'|$ out front, this term is

$$N_P = \int_{\nu\in S^{m-1}} |\det U(\nu)_{P,P}|^2 \frac{|\det A_\nu|}{\left(\frac{2}{|\log r|}|z''|^2 + \sum_{j=1}^{2n'} |\mu_j^\nu|\,|z_j^\nu|^2 - i\nu\cdot t\right)^{2n'+m}}\,d\nu.$$

Consider the case when $|t|$ is smaller than $|z'|^2 < |z''|^2$. We factor out $|z'|^{2(2n'+m)}$ from the denominator and obtain $N_P = |z'|^{-2(2n'+m)}\tilde N_P$ where

$$\tilde N_P = \int_{\nu\in S^{m-1}} |\det U(\nu)_{P,P}|^2 \frac{|\det A_\nu|}{\left(\frac{2}{|\log r|}|q''|^2 + \sum_{j=1}^{2n'} |\mu_j^\nu|\,|q_j^\nu|^2 - i\nu\cdot q_t\right)^{2n'+m}}\,d\nu$$

with $q'' = z''/|z'|$, $q_j^\nu = |z_j^\nu|^2/|z'|^2$ and $q_t = t/|z'|^2$.

Now take a limit as $q_t \to 0$ and we obtain

$$\lim_{q_t\to 0}\tilde N_P = \int_{\nu\in S^{m-1}} |\det U(\nu)_{P,P}|^2 \frac{|\det A_\nu|}{\left(\frac{2}{|\log r|}|q''|^2 + \sum_{j=1}^{2n'} |\mu_j^\nu|\,|q_j^\nu|^2\right)^{2n'+m}}\,d\nu. \tag{51}$$

Now $\mu_j^\nu \neq 0$ for $j=1,\dots,2n'$; and $\sum_{j=1}^{2n'} q_j^\nu = 1$; and $\det A_\nu \neq 0$ for all $\nu\in S^{m-1}$. So if the integral on the right-hand side of (51) vanishes, then we conclude that $\det U(\nu)_{P,P} = 0$ for all $\nu\in S^{m-1}$ except for a set of zero measure in ν. Thus, to conclude the proof of Theorem 2.5, we have only to show

$$\int_{\nu\in S^{m-1}} |\det U(\nu)_{[P,P]}|^2\,d\nu > 0, \tag{52}$$

where $U(\nu)$ is the unitary matrix which diagonalizes A_ν and where P is the set of indices corresponding to the positive eigenvalues of A_ν and $U(\nu)_{[P,P]}$ is the $P\times P$ minor matrix of $U(\nu)$. We may assume $P=\{1,2,\dots,n'\}$ and $P^c=\{n'+1,\dots,2n'\}$ where here, the eigenvalues are counted with multiplicity. We also let $N = 2n'$.

Define

- $N_0 =$ essential sup{the number of distinct eigenvalues of $A_\nu : \nu\in S^{m-1}$},
- $S_0 = \{\nu\in S^{m-1}$: the number of distinct eigenvalues of $A_\nu = N_0\}$,
- $\lambda_j(\nu)$, $1\le j\le N_0$, are the distinct eigenvalues of A_ν for $\nu\in S_0$,
- $E_j(\nu)$ equals the eigenspace of $\lambda_j(\nu)$ in $\mathbb{C}^N$ for $\nu\in S_0$.

Note that N_0 is an even number between 1 and $N = 2n'$. The set S_0 has positive measure by the definition of essential sup. Since there are only a finite number of choices for $\dim_{\mathbb{C}}\{E_j(\nu)\}$, we can shrink S_0, but still with positive measure, so that $\dim_{\mathbb{C}}\{E_j(\nu)\}$ is constant in $\nu\in S_0$ for each $1\le j\le N_0$.

Although we are not assuming the eigenvalues are continuous in $\nu\in S^{m-1}$, the $\lambda_j(\cdot)$ are measurable functions that are locally integrable on S^{m-1}. Using the usual row and column operations together with Gram–Schmidt, we can find an orthonormal set of eigenvectors for the eigenspace $E_j(\nu)$ of the form

$$U_j^k(\nu), \quad 1 \le j \le N_0,\ 1 \le k \le \dim_{\mathbb{C}}\{E_j(\nu)\},$$

where these C^N-valued functions are measurable and integrable in $\nu \in S^{m-1}$. Now let $U(\nu)$ be the unitary matrix with column vectors $U_j^k(\nu)$.

By removing a set of measure zero from S_0, we can assume that every point in S_0 lies in the Lebesgue set of each $\lambda_j(\cdot)$ and $U_j^k(\cdot)$ as well as all n-fold products of the component entries of $U_j^k(\cdot)$. Now fix any $\nu_0 \in S_0$ and choose coordinates for $\mathbb{C}^N$ which diagonalize A_{ν_0} where the first n diagonal entries correspond to the positive eigenvalues of A_{ν_0}. Note that in these coordinates, $U_{[P,P]}(\nu_0)$ is the identity matrix.

Now, for $\varepsilon > 0$, define

$$B(\nu_0, \varepsilon) = \{\nu \in S^{m-1} : |\nu - \nu_0| < \varepsilon\}.$$

From the Lebesgue differentiation theorem,

$$\lim_{\varepsilon \to 0} \frac{1}{|B(\nu_0, \varepsilon)|} \int_{\nu \in B(\nu_0,\varepsilon)} |\det U_{[P,P]}(\nu)|^2 \, d\nu \to |\det U_{[P,P]}(\nu_0)|^2 = 1,$$

where $|B(\nu_0, \varepsilon)|$ is the Lebesgue measure of $B(\nu_0, \varepsilon)$ relative to S^{m-1}. We conclude that, for small enough $\varepsilon > 0$,

$$\int_{\nu \in B(\nu_0,\varepsilon)} |\det U_{[P,P]}(\nu)|^2 \, d\nu > 0,$$

and this implies (52).

Appendix: Calculus computations

Lemma A.1. *Suppose that* $a, \gamma > 0$, $b \neq 0$, *and* $k \in \mathbb{N}$. *Then*

$$\int_0^\gamma \frac{1}{s(as+b)^k} - \frac{1}{sb^k} \, ds = \frac{1}{b^k} \log\left(1 + \gamma \frac{a}{b}\right) + E(a, b),$$

where $E_k(a, b, \gamma)$ *is comprised of a sum of terms of the form*

$$E_k(a, b, \gamma) = \sum_{\ell=0}^{k} \frac{c_\ell}{b^\ell (a\gamma + b)^{k-\ell}}$$

for some constants c_ℓ.

Proof. The proof is a computation using a partial fraction decomposition, recognizing that the $1/s$ terms cancel (so that the integral converges). □

In that vein, we also have the following. We compute

$$\begin{aligned}\int_0^{\frac{1}{\log 2}} \frac{s^{k-2}}{(s+a)^\ell}\,ds &= \int_0^{\frac{1}{\log 2}} \frac{(s+a-a)^{k-2}}{(s+a)^\ell}\,ds \\ &= \sum_{i=0}^{k-2}\binom{k-2}{i}(-1)^{k-2-i}\int_0^{\frac{1}{\log 2}} \frac{a^{k-2-i}}{(s+a)^{\ell-i+2}}\,ds \\ &= \sum_{i=0}^{k-2}\binom{k-2}{i}\frac{(-1)^{k-1-i}}{(-\ell+i+1)}\left(\frac{1}{a^{\ell-k+1}} - \frac{a^{k-2-i}}{\left(\frac{1}{\log 2}+a\right)^{\ell-i-1}}\right). \qquad (53)\end{aligned}$$

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