Pacific Journal of Mathematics

Volume 328 No. 1 January 2024

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 328, No. 1, 2024
https://doi.org/10.2140/pjm.2024.328.1

CATEGORIFICATION OF THE INTERNAL BRAID GROUP ACTION FOR QUANTUM GROUPS I: 2-FUNCTORIALITY

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We define 2-functors on the categorified quantum group of a simply-laced Kac–Moody algebra that induce Lusztig's internal braid group action at the level of the Grothendieck group.

1. Introduction

Geometric representation theory has motivated the study of categorical representation theory. Rather than studying the action of Lie algebras $\mathfrak{g}$, or quantum groups $\boldsymbol{U}_q(\mathfrak{g})$, on $\mathbb{C}(q)$-vector spaces V with weight decompositions $V = \bigoplus_\lambda V_\lambda$, categorical representation theory studies the action of these algebras on graded additive categories $\mathcal{V}$ with decomposition into graded additive subcategories $\mathcal{V} = \bigoplus_\lambda \mathcal{V}_\lambda$. Rather than linear maps between spaces, Chevalley generators act by functors $\mathcal{E}_i \mathbb{1}_\lambda : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+\alpha_i}$, $\mathcal{F}_i \mathbb{1}_\lambda : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-\alpha_i}$ satisfying quantum group relations up to natural isomorphism of functors. The novel and distinguishing feature of higher representation theory is that the natural transformations between such functors contain a wealth of information that is inaccessible within the realm of traditional representation theory.

Indeed, the essence of *categorification* is to uncover this higher level structure and use it to further our understanding of traditional representation theory, as well as related fields. In this article we will focus our attention on the categorical representation theory of the quantum group $\boldsymbol{U}_q(\mathfrak{g})$ associated to a *simply-laced* Kac–Moody algebra $\mathfrak{g}$. Categorified quantum groups are the objects that govern

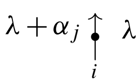
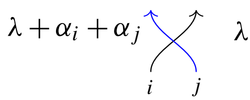
Abram was supported by a USC Graduate School Dissertation Completion Fellowship.
Lauda was partially supported by the NSF grants DMS-1255334, DMS-1664240, DMS-1902092 and Army Research Office W911NF-20-1-0075.
Rose was partially supported by an NSA Young Investigator Grant, a Simons Collaboration Grant, and NSF CAREER grant DMS-2144463.
MSC2020: 17B37, 20F36.
Keywords: categorification, higher representation theory, internal braid group, 2-Kac–Moody algebra, Chuang–Rouquier complexes.

the higher structure and explicitly describe the natural transformations that arise in categorical representations. More precisely, we focus on the higher representation theory of Lusztig's idempotent form $\dot{\boldsymbol{U}} := \dot{\boldsymbol{U}}_q(\mathfrak{g})$. This is a version of the quantum group that arises in geometric representation theory and is most appropriate for studying representations with integral weight decompositions. For the precise definition of $\dot{\boldsymbol{U}}$, see Section 2A.

In most instances when $\dot{\boldsymbol{U}}$ admits a categorical action as described above, the natural transformations between functors arise via the action of a categorified quantum group. The latter is a graded, additive, linear 2-category $\dot{\mathcal{U}}_Q$ associated to $\mathfrak{g}$. The objects in $\dot{\mathcal{U}}_Q$ are elements of the weight lattice $\lambda \in X$ of $\mathfrak{g}$, and the 1-morphisms are generated by Chevalley generators $\mathcal{E}_i \mathbb{1}_\lambda : \lambda \to \lambda + \alpha_i$, $\mathcal{F}_i \mathbb{1}_\lambda : \lambda \to \lambda - \alpha_i$ and identity 1-morphisms $\mathbb{1}_\lambda : \lambda \to \lambda$, i.e., any 1-morphism is given by a finite direct sum of grading shifts of composites of these generators. The 2-morphisms specify maps between composites of Chevalley generators. For example, there are 2-morphisms

$$\mathsf{X}_i : \mathcal{E}_i \mathbb{1}_\lambda \to \mathcal{E}_i \mathbb{1}_\lambda \langle 2 \rangle \quad \text{and} \quad \mathsf{T}_{ij} : \mathcal{E}_i \mathcal{E}_j \mathbb{1}_\lambda \to \mathcal{E}_j \mathcal{E}_i \mathbb{1}_\lambda \langle -\alpha_i \cdot \alpha_j \rangle,$$

where here, and for the duration, $\cdot$ denotes the symmetric bilinear form specified by the Catan datum for $\mathfrak{g}$ (see Section 2A). A novel feature of the categorified quantum group is its diagrammatic generators-and-relations description in which all 2-morphisms are conveniently encoded in a 2-dimensional graphical calculus, e.g., the generating 2-morphisms above have the depictions

$\mathsf{X}_i :=$  and $\mathsf{T}_{ij} :=$  .

Key features are that $\mathcal{F}_i$ and $\mathcal{E}_i$ are biadjoint, and endomorphisms of compositions of $\mathcal{E}_i$'s are given by the so-called *KLR algebras* developed in [18; 24; 25; 54; 55]. Taken together, the relations on 2-morphisms provide explicit isomorphisms lifting relations in $\dot{\boldsymbol{U}}$, and further guarantee that $K_0(\dot{\mathcal{U}}_Q) \cong \dot{\boldsymbol{U}}$, where K_0 denotes taking the split Grothendieck ring to *decategorify*. Otherwise, only shadows of this structure are visible at the decategorified level, e.g., Lusztig's canonical basis of $\dot{\boldsymbol{U}}$ is recovered by taking the classes in $K_0(\dot{\mathcal{U}}_Q)$ of indecomposable 1-morphisms in $\dot{\mathcal{U}}_Q$.

Pioneering work of Chuang and Rouquier demonstrated the importance of the higher structure in categorical representation theory [18]. At the heart of their work is a beautiful categorification of the familiar fact that, in any integrable representation $V = \bigoplus_\lambda V_\lambda$ of $\mathfrak{sl}_2$, the Weyl group action gives rise to an isomorphism

$$\mathsf{t}1_\lambda : V_\lambda \xrightarrow{\cong} V_{-\lambda}$$

between opposite weight spaces. In the quantum setting, the Weyl group for $\mathfrak{sl}_2$ (i.e., the symmetric group $\mathfrak{S}_2$) deforms to the two-strand braid group B_2, and the isomorphism $\mathtt{t}1_\lambda$ can be written in a completion of $\dot{\boldsymbol{U}}(\mathfrak{sl}_2)$ as the infinite sum

$$\mathtt{t}1_\lambda = \begin{cases} \sum_{b\geq 0}(-q)^b F^{(\lambda+b)}E^{(b)}1_\lambda & \text{if } \lambda \geq 0, \\ \sum_{a\geq 0}(-q)^{-\lambda+a}E^{(-\lambda+a)}F^{(a)}1_\lambda & \text{if } \lambda \leq 0, \end{cases} \tag{1-1}$$

where $E^{(a)} = E^a/[a]!$, $F^{(a)} = F^a/[a]!$ are the so-called *divided powers*, and $[a]! = \prod_{m=1}^{a}((q^m - q^{-m})/(q - q^{-1}))$ are quantum factorials. Note that, when acting on an integrable module, only finitely many terms in this infinite sum are nonzero. From the perspective of categorification, the crucial observation about (1-1) is the occurrence of minus signs.

For those initiated in the categorification doctrine, the occurrence of minus signs immediately necessitates the departure from strictly additive categorification. That is, we can no longer work with additive categories $\mathcal{V}_\lambda$, as there is no categorical analogue of subtraction therein. To accommodate such minus signs within a categorical framework, one typically passes to derived, or, more generally, triangulated, categories, where the translation functor gives a categorical notion of multiplying by -1. One manner for doing so is to take the categories of chain complexes $\mathsf{Kom}(\mathcal{V}_\lambda)$ of the weight categories $\mathcal{V}_\lambda$ in an additive categorification, and pass to their homotopy categories of complexes $\mathsf{Com}(\mathcal{V}_\lambda)$. See Section 3D for more details on homotopy categories of additive categories and their Grothendieck groups; we note that we follow [4] in using the nonstandard notation Com to denote the homotopy category, so as not to confuse with our notation K_0 for taking the Grothendieck group/ring. Under decategorification, the classes of such complexes are equal to the alternating sum of the classes of their terms in $K_0(\mathcal{V}_\lambda)$.

The alternating sum in (1-1) suggests that a categorification of $\mathtt{t}1_\lambda$ might be achieved using a chain complex whose differential is built from the 2-morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_2)$. Indeed, Chuang and Rouquier's work determines chain complexes $\tau\mathbb{1}_\lambda$ and $\tau^{-1}\mathbb{1}_\lambda$, the so-called *Rickard complexes*, that lift $\mathtt{t}1_\lambda$ and its inverse $\mathtt{t}^{-1}1_\lambda$ to the categorical setting [18]. The composition of complexes $\tau\tau^{-1}\mathbb{1}_\lambda$ and $\tau^{-1}\tau\mathbb{1}_\lambda$ are both isomorphic to the identity in $\mathsf{Com}(\dot{\mathcal{U}}_Q(\mathfrak{sl}_2))$, i.e., the complexes are homotopy equivalent to (but, in fact, not equal to) $\mathbb{1}_\lambda$ in $\mathsf{Kom}(\dot{\mathcal{U}}_Q(\mathfrak{sl}_2))$. Using this, Chuang and Rouquier lifted the Weyl group action of $\mathfrak{sl}_2$ to define equivalences

$$\tau\mathbb{1}_\lambda : \mathsf{Com}(\mathcal{V}_\lambda) \xrightarrow{\cong} \mathsf{Com}(\mathcal{V}_{-\lambda})$$

lifting $\mathtt{t}1_\lambda$ (to be precise, Chuang and Rouquier originally worked in the nonquantum and abelian/derived setting, with the extension to the quantum and triangulated setting given in work of Rouquier [55] and Cautis and Kamnitzer [13]).

For general $\mathfrak{g}$, the corresponding Weyl group action on integrable representations deforms to an action of the type-$\mathfrak{g}$ braid group $B_\mathfrak{g}$ in the quantum setting; we will follow the standard terminology in referring to this as the *quantum Weyl group* action. Analogous to the $\mathfrak{g} = \mathfrak{sl}_2$ case, this action lifts to highly nontrivial braid group actions in categorical representation theory [13; 55]. To illustrate their far reaching impact in mathematics, we recall just a handful of their many applications:

• Chuang and Rouquier use the equivalence induced by categorical $\mathfrak{sl}_2$ actions on derived categories of modules over the symmetric group in positive characteristic to resolve Broué's abelian defect group conjecture for the symmetric group $\mathfrak{S}_n$ [18].

• Cautis, Kamnitzer, and Licata use categorical $\mathfrak{sl}_2$ actions to resolve a conjecture of Namikawa [48] asserting the existence of a derived equivalence between cotangent bundles of complementary Grassmannians $T^*G(k, N)$ and $T^*G(N-k, N)$ [10; 16]. These varieties are related by a stratified Mukai flop, and the problem of constructing such equivalences had previously only been resolved in the $k = 1$ case [22; 47] and for $G(2, 4)$ in work of Kawamata [23]. More generally, Cautis, Kamnitzer, and Licata construct categorical braid group actions on cotangent bundles to partial flag varieties and Nakajima quiver varieties [11; 13; 15]

• Categorical representations of $\mathfrak{sl}_m$, and the associated braid group actions, can be used to categorify the $\mathfrak{sl}_n$ Reshetikhin–Turaev quantum link invariants via a categorical analogue of the skew Howe duality between $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ [12; 16; 34; 49]. This perspective has led to the solution of a number of conjectures in link homology [50; 53], and provides a framework for connecting link homologies defined using wildly different machinery [12; 34; 39].

At the decategorified level, the braid group action on integrable modules of $\dot{\boldsymbol{U}}_q(\mathfrak{g})$ comes in several flavors:

$$\begin{aligned} \mathrm{t}'_{i,e}1_\lambda &= \sum_{a,b;a-b=\lambda_i} (-q)^{eb} F_i^{(a)} E_i^{(b)} 1_\lambda = \sum_{a,b;a-b=\lambda_i} (-q)^{eb} E_i^{(b)} F_i^{(a)} 1_\lambda, \\ \mathrm{t}''_{i,e}1_\lambda &= \sum_{a,b;-a+b=\lambda_i} (-q)^{eb} E_i^{(a)} F_i^{(b)} 1_\lambda = \sum_{a,b;-a+b=\lambda_i} (-q)^{eb} F_i^{(b)} E_i^{(a)} 1_\lambda, \end{aligned} \tag{1-2}$$

where $e = \pm 1$; see Section 2C for more details. Given the importance of these braid group actions, it is natural to ask how the braid group action $B_\mathfrak{g}$ on an integrable module interacts with the $\dot{\boldsymbol{U}}_q(\mathfrak{g})$ action. This was answered by Lusztig [38, Proposition 37.1.2], who showed that, for each node in the Dynkin diagram $i \in I$ and $e = \pm 1$, there exist algebra automorphisms $T'_{i,e}$ and $T''_{i,e}$ of $\dot{\boldsymbol{U}} = \dot{\boldsymbol{U}}_q(\mathfrak{g})$ uniquely defined by the condition that, for any integrable $\dot{\boldsymbol{U}}$-module V, any $z \in V$, and $u \in 1_\nu \dot{\boldsymbol{U}} 1_\lambda$, we have the equations

$$\begin{aligned} T'_{i,e}(u)\mathrm{t}'_{i,e}1_\lambda(z) &= \mathrm{t}'_{i,e}1_\nu(uz), \\ T''_{i,e}(u)\mathrm{t}''_{i,e}1_\lambda(z) &= \mathrm{t}''_{i,e}1_\nu(uz). \end{aligned} \tag{1-3}$$

Related operators were studied in finite type in [35; 57], then generalized to simply-laced Cartan data in [36] and general Cartan data in [37]. See Section 2D for more details.

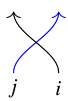
The algebra automorphisms $T'_{i,e}$ and $T''_{i,e}$ each define braid group actions on the algebra $\dot{\boldsymbol{U}}$ itself that we call the internal braid group action. This internal action plays an important role, e.g., in the construction of the PBW basis for $\dot{\boldsymbol{U}}$. Lusztig goes on to give precise formulas for the action of $T'_{i,e}$ and $T''_{i,e}$ on the generators of $\dot{\boldsymbol{U}}$, that unsurprisingly involve minus signs, e.g.,

$$T'_{i,+1}(E_j 1_\lambda) = E_j E_i 1_{s_i(\lambda)} - q E_i E_j 1_{s_i(\lambda)} \quad \text{if } i \cdot j = -1, \tag{1-4}$$

where here s_i are the simple reflections in the Weyl group.

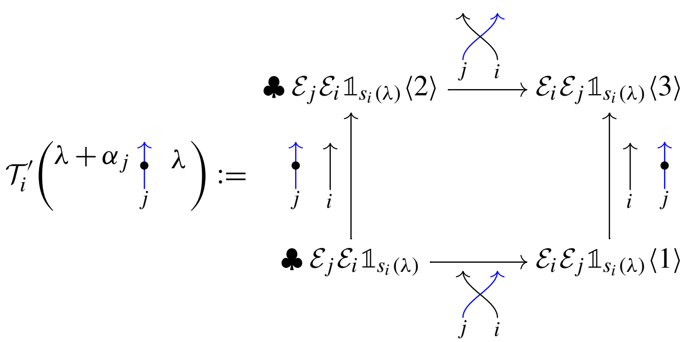
We now describe the results contained in this article. Throughout we let $\mathfrak{g}$ be a simply-laced Kac–Moody algebra.

1A. ***Categorifying $T'_{i,e}$ and $T''_{i,e}$.*** In Section 4 we define graded, additive 2-functors $\mathcal{T}'_{i,e}, \mathcal{T}''_{i,e} : \dot{\mathcal{U}}_Q \to \mathsf{Com}(\dot{\mathcal{U}}_Q)$. To do so, we first assign explicit chain complexes to generating 1-morphisms in $\dot{\mathcal{U}}_Q$ that lift the formulae defining $T'_{i,e}$ and $T''_{i,e}$, e.g., (1-4) lifts to the assignment



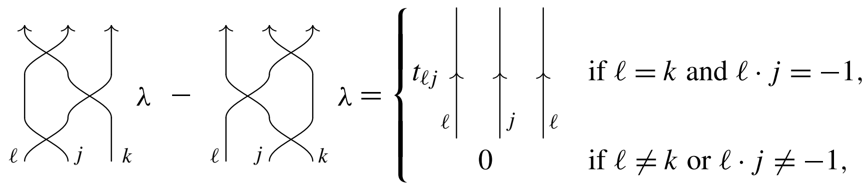
$$\mathcal{T}'_{i,+1}(\mathcal{E}_j \mathbb{1}_\lambda) = \clubsuit\, \mathcal{E}_j\mathcal{E}_i \mathbb{1}_{s_i(\lambda)} \longrightarrow \mathcal{E}_i\mathcal{E}_j \mathbb{1}_{s_i(\lambda)}\langle 1\rangle \quad \text{if } i \cdot j = -1,$$

where (here, and throughout) $\clubsuit$ denotes the term in homological degree zero. Functoriality then requires that the composite $xy\mathbb{1}_\lambda$ of composable 1-morphisms $y\mathbb{1}_{\lambda'}$ and $x\mathbb{1}_\lambda$ is sent to the composition of chain complexes $\mathcal{T}'_{i,+1}(y\mathbb{1}_{\lambda'})\mathcal{T}'_{i,+1}(x\mathbb{1}_\lambda)$, defined using composition of 1-morphisms in $\dot{\mathcal{U}}_Q$ in a manner similar to taking the tensor product of chain complexes. To complete the definition of $\mathcal{T}'_{i,e}$ and $\mathcal{T}''_{i,e}$, we then assign an explicit chain map $\mathcal{T}'_{i,+1}(\alpha) : \mathcal{T}'_{i,+1}(x\mathbb{1}_\lambda) \to \mathcal{T}'_{i,+1}(x'\mathbb{1}_\lambda)$ to each generating 2-morphism $\alpha : x\mathbb{1}_\lambda \to x'\mathbb{1}_\lambda$ in $\dot{\mathcal{U}}$, e.g., the 2-morphism $X_j : \mathcal{E}_j\mathbb{1}_\lambda \to \mathcal{E}_j\mathbb{1}_\lambda\langle 2\rangle$ is sent by $T'_{i,+1}$ to



which is a chain map by the $i \neq j$ dot sliding relation; see (5) in Definition 3.3. Finally, we show that the images of relations in $\dot{\mathcal{U}}_Q$ are satisfied in $\mathsf{Com}(\dot{\mathcal{U}}_Q)$, up to homotopy.

Proving that $\mathcal{T}'_{i,+1}$ is a well-defined 2-functor requires an immense number of verifications. The diagrammatic relations defining $\dot{\mathcal{U}}_Q$ involve strands colored by the Dynkin nodes of $\mathfrak{g}$, and depend on the adjacency of the colors involved. For example, the relation involving the greatest number of strands is



where $t_{\ell j}$ is a scalar defined in Section 3A. Showing that $\mathcal{T}'_{i,+1}$ preserves this relation for all i and all triples j, k, ℓ requires considering all possible types of adjacency relations between the nodes corresponding to i, j, k, ℓ, requiring 27 essentially distinct case that need to be verified. The complexity is further exacerbated by the fact that $\mathcal{T}'_{i,+1}$ often only preserves a relation up to homotopy.

Unfortunately, we are not aware of a means to define the 2-functors lifting Lusztig's formulae without explicitly constructing the chain homotopies for each relation and each possible coloring by nodes $i \in I$. We have made every attempt to provide sufficient detail in this work to aid in any future applications of these 2-functors, and in particular provide sufficient detail so that the relevant homotopies can be easily extracted.

Our main result in this article is the following theorem.

Theorem 1.1. *Let $\mathfrak{g}$ be a simply-laced Kac–Moody algebra. Then there is an explicitly defined 2-functor*

$$\mathcal{T}'_{i,+1} : \dot{\mathcal{U}}_Q(\mathfrak{g}) \to \mathsf{Com}(\dot{\mathcal{U}}_Q(\mathfrak{g}))$$

so that the induced map

$$[\mathcal{T}'_{i,+1}] : \dot{\boldsymbol{U}}_q(\mathfrak{g}) \cong K_0(\dot{\mathcal{U}}_Q(\mathfrak{g})) \to K_0(\mathsf{Com}(\dot{\mathcal{U}}_Q(\mathfrak{g}))) \cong \dot{\boldsymbol{U}}_q(\mathfrak{g})$$

agrees with $T'_{i,+1}$.

At the level of 1-morphisms, such functors have already appeared at the categorical level in [12; 13] and were given a geometric interpretation in [20; 21; 62; 63]; however, to our knowledge, no information about extending these maps to 2-morphisms has appeared previously. As such, Theorem 1.1 initiates the study of Lusztig's operators at the 2-categorical level. In fact, we conjecture much more. At the decategorified level, Lusztig's operators are invertible and satisfy the braid relations. These properties, combined with our forthcoming work, stated in Theorem 1.4 below, suggest the following:

Conjecture 1.2. Let $\mathfrak{g}$ be a (simply-laced) Kac–Moody algebra. Then $\mathcal{T}'_{i,+1}$ extends to an autoequivalence of $\mathsf{Com}(\dot{\mathcal{U}}_Q(\mathfrak{g}))$ so that the induced automorphism $[\mathcal{T}'_{i,+1}]$ of $\dot{\boldsymbol{U}}_q(\mathfrak{g}) \cong K_0(\mathsf{Com}(\dot{\mathcal{U}}_Q(\mathfrak{g})))$ agrees with $T'_{i,+1}$, and the $\mathcal{T}'_{i,+1}$ satisfy the braid relations.

The extension (of domain) to the homotopy category is a problem in obstruction theory that we plan to attack in future work. Having done so, the proof of braid relations will be a straightforward (but tedious) check.

1B. ***Symmetries and the internal braid group action.*** There are a number of other (anti)linear (anti)automorphisms $\underline{\sigma}$, $\underline{\omega}$, $\underline{\psi}$ defined on $\dot{\boldsymbol{U}}$; see Section 2B for their definitions. These (anti)involutions allow one to pass between the variants $T'_{i,e}$ and $T''_{i,e}$ of the internal braid group generators via conjugation, i.e.,

$$\underline{\sigma} T'_{i,e} \underline{\sigma} = T''_{i,-e}, \quad \underline{\omega} T'_{i,e} \underline{\omega} = T''_{i,e}, \quad \underline{\psi} T'_{i,e} \underline{\psi} = T'_{i,-e}, \quad \underline{\psi} T''_{i,e} \underline{\psi} = T''_{i,-e}. \tag{1-5}$$

In [26] these symmetries were lifted to define 2-functors σ, ω, ψ on a certain version of the categorified quantum group. Each has a natural interpretation in terms of symmetries of the graphical calculus for $\dot{\mathcal{U}}_Q$, and, in the $\mathfrak{sl}_2$ case, were extended to the homotopy category of complexes in [4].

Recall (or see Section 3A below) that the definition of $\dot{\mathcal{U}}_Q$ requires a choice of scalar parameters Q; it was recently shown that there is a natural normalization for the categorified quantum group associated to an arbitrary KLR algebra and choice of Q [5]. This so-called *cyclic version* of $\dot{\mathcal{U}}_Q$ satisfies the property that diagrams that are planar isotopic relative to their boundaries specify the same 2-morphism in $\dot{\mathcal{U}}_Q$, a property that only holds up to scalars in previous formulations. Given the utility of the cyclic version, we also prove the following result, which defines these symmetries in this setting.

Theorem 1.3. *There are invertible 2-functors σ, ω, ψ defined on the cyclic version of the categorified quantum group $\dot{\mathcal{U}}_Q$ that categorify the symmetries $\underline{\sigma}$, $\underline{\omega}$, $\underline{\psi}$, i.e.,*

$$[\sigma] = \underline{\sigma}, \quad [\omega] = \underline{\omega}, \quad [\psi] = \underline{\psi}$$

in $K_0(\dot{\mathcal{U}}_Q(\mathfrak{g})) \cong \dot{\boldsymbol{U}}_q(\mathfrak{g})$.

Defining these 2-functors requires several subtle aspects involving the choice of scalars Q, so we include the details below in Section 3E. Using these symmetries, we use the categorical analogue of (1-5) to define the variants $\mathcal{T}'_{i,-1}$ and $\mathcal{T}''_{i,e}$ of the internal braid group action.

1C. ***Compatibility with Rickard complexes.*** As noted above, the defining feature of the internal braid group action at the decategorified level is its compatibility with the quantum Weyl group action, given in (1-3). In a sequel to this paper [1], we show that our 2-functors $\mathcal{T}'_{i,+1}$ satisfy an analogous compatibility with the Rickard complexes.

To be precise, note that the first equality in (1-3) asserts that the actions of the elements $T'_{i,e}(u)\mathrm{t}'_{i,e}1_\lambda$ and $\mathrm{t}'_{i,e}1_\nu u$ on the λ weight space of any integrable representation agree for all $u \in 1_\nu \dot{\boldsymbol{U}} 1_\lambda$. Equivalently, for any integrable representation $V = \bigoplus_\lambda V_\lambda$ there is an equality between the corresponding linear maps $1_\nu \dot{\boldsymbol{U}} 1_\lambda \to \mathrm{Hom}(V_\lambda, V_{s_i(\nu)})$. At the categorical level, the operation of composing with the complex $\tau'_{i,+1}$ defines a functor

$$\tau'_{i,+1}\mathbb{1}_\nu(-)\mathbb{1}_\lambda : \mathrm{Hom}_{\mathcal{U}_Q}(\lambda, \nu) \to \mathrm{Hom}_{\mathsf{Com}(\mathcal{U}_Q)}(\lambda, s_i(\nu)), \tag{1-6}$$

and we can similarly consider the functor $\mathcal{T}'_{i,e}(-)\tau'_{i,e}\mathbb{1}_\lambda$, which maps between the same Hom-categories. The main result of [1] is the following:

Theorem 1.4. *For all objects λ, ν in $\mathcal{U}_Q$, there is an isomorphism of functors*

$$\sqsupset : \tau'_{i,+1}\mathbb{1}_\nu(-)\mathbb{1}_\lambda \cong \mathcal{T}'_{i,e}(-)\tau'_{i,e}\mathbb{1}_\lambda \tag{1-7}$$

between Hom-*categories* $\mathrm{Hom}_{\mathcal{U}_Q}(\lambda, \nu) \to \mathrm{Hom}_{\mathsf{Com}(\mathcal{U}_Q)}(\lambda, s_i(\nu))$.

1D. ***Applications of the internal braid group action.***

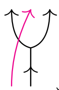
1D1. *PBW basis and their categorifications.* In finite type, Lusztig's internal braid group action can be used to deduce the quantum PBW theorem for $\dot{\boldsymbol{U}}^+(\mathfrak{g})$, providing a basis of monomials that are useful in many applications. The KLR algebra provides a categorification of $\dot{\boldsymbol{U}}^+(\mathfrak{g})$ via its category of projective/finitely generated modules [24; 25; 55]. Therein, the indecomposable projective modules correspond to the canonical basis of $\dot{\boldsymbol{U}}^+(\mathfrak{g})$ [59], while the simple modules correspond to the dual canonical basis [8; 60]. At the categorical level, the analogues of PBW monomials lead to a rich theory of standard modules for KLR algebras. In finite type, standard modules were first described in [31] (see also [6; 9; 19; 20; 41; 42]), and in affine type they were studied in [29; 30; 43; 58]; in these studies, the focus has been on finding specific modules over KLR algebras that lift a given PBW monomial. In forthcoming work [44], McNamara plans to use our 2-functors $\mathcal{T}'_{i,+1}$ to build projective resolutions of standard KLR modules, producing a categorical lift of Lusztig's internal braid group construction of the PBW basis, and giving a strengthening of Kato's results on reflection functors for KLR algebras [20].

1D2. *Quantum affine algebras.* There is no obstruction to defining the 2-functors $\mathcal{T}'_{i,+1}$ in arbitrary symmetrizable type, except that the check of well definedness is much more involved. For example, Lusztig provides the explicit formula

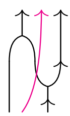
$$T'_{i,+1}(E_\ell 1_\lambda) = \sum_{j=0}^{-i\cdot\ell} (-q)^j E_i^{(j)} E_\ell E_i^{(-i\cdot\ell-j)} 1_{s_i(\lambda)} \tag{1-8}$$

in arbitrary type (compare to (1-4) above), which suggests that the categorified Lusztig operator $\mathcal{T}'_{i,1}$ should send $\mathcal{E}_\ell \mathbb{1}_\lambda$ to a complex of length $1 - i\cdot\ell$. It is not difficult to specify a complex lifting (1-8), for example, we could set

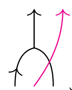
$$\mathcal{T}'_{i,1}(\mathcal{E}_\ell \mathbb{1}_\lambda) := \clubsuit \mathcal{E}_\ell \mathcal{E}_i^{(-i\cdot\ell)} \mathbb{1}_{s_i(\lambda)}$$



$$\longrightarrow \mathcal{E}_i \mathcal{E}_\ell \mathcal{E}_i^{(-i\cdot\ell-1)} \mathbb{1}_{s_i(\lambda)} \langle 1 \rangle$$



$$\longrightarrow \cdots$$

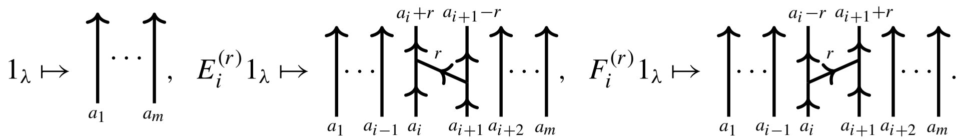


$$\longrightarrow \mathcal{E}_i^{(-i\cdot\ell)} \mathcal{E}_\ell \mathbb{1}_{s_i(\lambda)} \langle -i\cdot\ell \rangle.$$

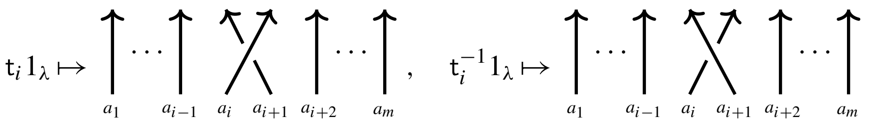
Here, the terms in the differential are given using the *thick calculus* from [27], and an easy computation therein verifies that they square to zero. The appearance of complexes containing more than two nonzero terms suggests that even more of the defining relations in $\dot{\mathcal{U}}_Q$ may be preserved by $\mathcal{T}'_{i,1}$ only up to homotopy, exacerbating the difficulty of checking that these 2-functors are well defined. Despite this, we note one interesting application of an extension of our 2-functors to non-simply-laced type: it may be possible to promote Beck's description [3] of the loop presentation of affine algebras in terms of the internal braid group action to the categorical level, giving a categorification of affine algebras in their loop realization.

1D3. *Link invariants and skew Howe duality.* As referenced above, one can study the $\mathfrak{sl}_n$ quantum link invariants via $\dot{\boldsymbol{U}}(\mathfrak{sl}_m)$ representation theory using quantum skew Howe duality. The latter is the quantum analogue of the duality arising from the commuting actions of $\dot{\boldsymbol{U}}(\mathfrak{sl}_n)$ and $\dot{\boldsymbol{U}}(\mathfrak{sl}_m)$ on the quantum exterior power $\bigwedge^N(\mathbb{C}_q^m \otimes \mathbb{C}_q^n)$. The $\mathfrak{sl}_n$ link invariants admit a formulation in terms of MOY calculus [46] and $\mathfrak{sl}_n$ webs [28; 32; 45], certain trivalent graphs which specify the morphisms in a diagrammatic description of the category of $\dot{\boldsymbol{U}}(\mathfrak{sl}_n)$ representations.

Cautis, Kamnitzer, and Morrison show that skew Howe duality admits a graphical description in terms of so-called ladder webs, and use this to give an entirely diagrammatic description of the full subcategory of quantum $\mathfrak{sl}_n$ representations tensor generated by the fundamental representations [17]. In this formulation, skew Howe duality specifies a representation of $\dot{\boldsymbol{U}}(\mathfrak{sl}_m)$ in which an $\mathfrak{sl}_m$ weight $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m-1})$ is sent to the to the m-tuple $(a_1, a_2, \ldots, a_m)$ that satisfies $0 \le a_i \le n$, $\sum_{i=1}^m a_i = N$ and $\lambda_i = a_i - a_{i+1}$, and weights not satisfying these conditions are sent to zero. This representation maps the generators of $\dot{\boldsymbol{U}}(\mathfrak{sl}_m)$ as follows:

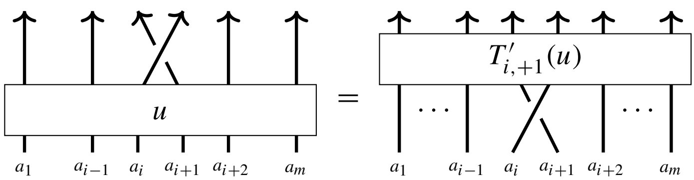


Under this representation, the braiding on the category of $\dot{\boldsymbol{U}}(\mathfrak{sl}_n)$ is given by the quantum Weyl group action, that is, diagrammatically, we have

(1-9) 

In this way, these link invariants can be computed and studied via the elements in $\dot{\boldsymbol{U}}(\mathfrak{sl}_m)$ corresponding to a given link diagram.

Under this correspondence, the internal braid group action plays an interesting role in the diagrammatic description of quantum $\mathfrak{sl}_n$ link invariants, as (1-3) shows how to slide the image of an arbitrary element $u \in \dot{\boldsymbol{U}}(\mathfrak{sl}_m)$ through a crossing, i.e., it gives the equality



where we abuse notation in denoting elements in $\dot{\boldsymbol{U}}(\mathfrak{sl}_m)$ and their images under the skew Howe representation via the same symbols.

This entire story lifts to the categorical level, allowing for the study of Khovanov homology [34] and Khovanov–Rozansky homology [12; 49] following Cautis, Kamnitzer, and Licata's pioneering work in using categorical skew Howe duality to study algebro-geometric categorifications of the $\mathfrak{sl}_n$ link polynomials [16]. The crucial point is that (1-9) lifts to map the Rickard complexes to the chain complexes assigned to crossings in $\mathfrak{sl}_n$ link homology.

In the foam-based description of link homology [2; 40; 49], categorical skew Howe duality maps generators in $\dot{\mathcal{U}}_Q(\mathfrak{sl}_m)$ to explicit $\mathfrak{sl}_n$ foams, certain singular surfaces that categorify $\mathfrak{sl}_n$ webs. Theorem 1.4 then explicitly shows how to slide not only webs, but also foams mapping between them, through crossings in $\mathfrak{sl}_n$ link homology. At the level of 1-morphisms (webs), this interaction is key to the stability results used to define $\mathfrak{sl}_n$ analogues of Jones–Wenzl projectors [12; 51; 56], and we anticipate that our extension to the level of 2-morphisms will prove useful for future arguments in link homology.

2. The quantum group and Lusztig symmetries

2A. *The quantum group $U_q(\mathfrak{g})$.*

2A1. *Root datum.* For the remainder of this article we restrict our attention to simply-laced Kac–Moody algebras. These algebras are associated to a choice of simply-laced Cartan datum consisting of

- a finite set I, and
- a $\mathbb{Z}$-valued symmetric bilinear form $\cdot$ on $\mathbb{Z}I$ satisfying $i \cdot i = 2$ for all $i \in I$ and $i \cdot j \in \{0, -1\}$ for $i \neq j$,

and root datum given by

- a free $\mathbb{Z}$-module X, called the *weight lattice*, and
- a choice of *simple roots* $\{\alpha_i\}_{i\in I} \subset X$ and *simple coroots* $\{h_i\}_{i\in I} \subset X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ that satisfy $\langle h_i, \alpha_j\rangle = 2\frac{i\cdot j}{i\cdot i}$, where here $\langle\,\cdot\,,\cdot\,\rangle : X^\vee \times X \to \mathbb{Z}$ is the canonical pairing.

In this case, $a_{ij} := \langle h_i, \alpha_j\rangle = i \cdot j$, so $(a_{ij})_{i,j\in I}$ is a symmetric generalized Cartan matrix. Given an arbitrary *weight* $\lambda \in X$, we will often abbreviate $\langle h_i, \lambda\rangle$ by either $\langle i, \lambda\rangle$ or simply by λ_i. We let $\{\Lambda_i\}_{i\in I} \subset X$ denote the *fundamental weights*, which are characterized by the property that $\langle h_i, \Lambda_j\rangle = \delta_{ij}$ for all $i, j \in I$.

We let $X^+ \subset X$ denote the *dominant weights*, which are those of the form $\sum_i \lambda_i\Lambda_i$ for $\lambda_i \geq 0$. Associated to a simply-laced Cartan datum is a graph Γ without loops or multiple edges, with a vertex for each $i \in I$ and an edge from vertex i to vertex j if and only if $i \cdot j = -1$.

2A2. *The simply-laced quantum group.* The quantum group $\boldsymbol{U} = \boldsymbol{U}_q(\mathfrak{g})$ associated to a simply-laced root datum is the unital, associative $\mathbb{Q}(q)$-algebra given by generators E_i, F_i, K_μ for $i \in I$ and $\mu \in X^\vee$, subject to the relations

(a) $K_0 = 1$ and $K_\mu K_{\mu'} = K_{\mu+\mu'}$ for all $\mu, \mu' \in X^\vee$,

(b) $K_\mu E_i = q^{\langle\mu,\alpha_i\rangle} E_i K_\mu$ for all $i \in I$, $\mu \in X^\vee$,

(c) $K_\mu F_i = q^{-\langle\mu,\alpha_i\rangle} F_i K_\mu$ for all $i \in I$, $\mu \in X^\vee$,

(d) $E_iF_j - F_jE_i = \delta_{ij}((K_{h_i} - K_{h_i}^{-1})/(q - q^{-1}))$, where we set $K_i := K_{h_i}$, and

(e) for all $i \neq j$,

$$\sum_{a+b=-\langle i,\alpha_j\rangle+1} (-1)^a E_i^{(a)} E_j E_i^{(b)} = 0 \quad \text{and} \quad \sum_{a+b=-\langle i,\alpha_j\rangle+1} (-1)^a F_i^{(a)} F_j F_i^{(b)} = 0,$$

where $E_i^{(a)} = E_i^a/[a]!$, $F_i^{(a)} = F_i^a/[a]!$, and $[a]! = \prod_{m=1}^a ((q^m - q^{-m})/(q - q^{-1}))$.

2A3. *The integral idempotented form of quantum group.* We will work with the idempotent form of $\boldsymbol{U}$, which is adapted to the study of $\boldsymbol{U}$-modules with weight space decompositions. This nonunital algebra is equipped with a collection of orthogonal idempotents, and hence can be described as a $\mathbb{Q}(q)$-linear category $\dot{\boldsymbol{U}} = \dot{\boldsymbol{U}}_q(\mathfrak{g})$, defined as follows. The objects of $\dot{\boldsymbol{U}}$ are elements of X, and the Hom-space between $\lambda, \nu \in X$ is defined to be

$$\dot{\boldsymbol{U}}(\lambda,\nu) := \boldsymbol{U} \Big/ \Bigg(\sum_{\mu \in X^\vee} \boldsymbol{U}(K_\mu - q^{\langle \mu,\lambda\rangle}) + \sum_{\mu\in X^\vee} (K_\mu - q^{\langle\mu,\nu\rangle})\boldsymbol{U} \Bigg).$$

The identity morphism of $\lambda \in X$ is denoted by 1_λ and we will typically abbreviate the element $1_\mu x 1_\lambda \in \dot{\boldsymbol{U}}(\lambda,\mu)$ determined by $x \in \boldsymbol{U}$ by either $1_\mu x$ or $x 1_\lambda$, e.g., we have $E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i$ and $F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i$. Composition in $\dot{\boldsymbol{U}}$ is induced by multiplication in $\boldsymbol{U}$, that is,

$$(1_\mu x 1_\nu)(1_\nu y 1_\lambda) = 1_\mu x y 1_\lambda$$

for $x, y \in \boldsymbol{U}$, $\lambda, \mu, \nu \in X$. The idempotent form $\dot{\boldsymbol{U}}$ admits an integral form, defined as the $\mathbb{Z}[q,q^{-1}]$-lattice ${}_{\mathcal{A}}\dot{\boldsymbol{U}} \subset \dot{\boldsymbol{U}}$ spanned by products of divided powers $E_i^{(a)} 1_\lambda$ and $F_i^{(a)} 1_\lambda$.

2B. ***(Anti)linear (anti)automorphisms of*** $\dot{\boldsymbol{U}}$**.** We use several $\mathbb{Z}[q,q^{-1}]$-(anti)linear (anti)automorphisms in this paper. For $f \in \mathbb{Q}(q)$, let $f \mapsto \bar{f}$ be the $\mathbb{Q}$-linear involution of $\mathbb{Q}(q)$ that sends q to q^{-1}.

- The $\mathbb{Q}(q)$-linear algebra anti-involution $\underline{\sigma} : \boldsymbol{U} \to \boldsymbol{U}$ is given by

$$\begin{aligned} &\underline{\sigma}(E_i) = E_i, \quad \underline{\sigma}(F_i) = F_i, \quad \underline{\sigma}(K_i) = K_i^{-1}, \\ &\underline{\sigma}(fx) = f\underline{\sigma}(x) \quad \text{for } f \in \mathbb{Q}(q) \text{ and } x \in \boldsymbol{U}, \\ &\underline{\sigma}(xy) = \underline{\sigma}(y)\underline{\sigma}(x) \quad \text{for } x, y \in \boldsymbol{U}. \end{aligned}$$

- The $\mathbb{Q}(q)$-linear algebra involution $\underline{\omega} : \boldsymbol{U} \to \boldsymbol{U}$ is given by

$$\begin{aligned} &\underline{\omega}(E_i) = F_i, \quad \underline{\omega}(F_i) = E_i, \quad \underline{\omega}(K_i) = K_i^{-1}, \\ &\underline{\omega}(fx) = f\underline{\omega}(x) \quad \text{for } f \in \mathbb{Q}(q) \text{ and } x \in \boldsymbol{U}, \\ &\underline{\omega}(xy) = \underline{\omega}(x)\underline{\omega}(y) \quad \text{for } x, y \in \boldsymbol{U}. \end{aligned}$$

- The $\mathbb{Q}(q)$-antilinear algebra involution $\underline{\psi} : \boldsymbol{U} \to \boldsymbol{U}$ is given by

$$\begin{aligned} &\underline{\psi}(E_i) = E_i, \quad \underline{\psi}(F_i) = F_i, \quad \underline{\psi}(K_i) = K_i^{-1}, \\ &\underline{\psi}(fx) = \bar{f}\underline{\psi}(x) \quad \text{for } f \in \mathbb{Q}(q) \text{ and } x \in \boldsymbol{U}, \\ &\underline{\psi}(xy) = \underline{\psi}(x)\underline{\psi}(y) \quad \text{for } x, y \in \boldsymbol{U}. \end{aligned}$$

These (anti)linear (anti)involutions pairwise commute and generate the group $G = (\mathbb{Z}_2)^3$ of (anti)linear (anti)automorphisms acting on $\boldsymbol{U}$. The (anti)involutions $\underline{\sigma}$, $\underline{\omega}$, and $\underline{\psi}$ all extend to $\dot{\boldsymbol{U}}$ and ${}_{\mathcal{A}}\dot{\boldsymbol{U}}$ by setting

$$\underline{\sigma}(1_\lambda) = 1_{-\lambda}, \quad \underline{\omega}(1_\lambda) = 1_{-\lambda}, \quad \underline{\psi}(1_\lambda) = 1_\lambda,$$

and taking the induced maps on each summand $1_{\lambda'}\dot{\boldsymbol{U}}1_\lambda$.

2C. ***Quantum Weyl group action on integrable $\dot{\boldsymbol{U}}$-modules.*** Let $V = \bigoplus_\lambda V_\lambda$ be an integrable $\dot{\boldsymbol{U}}$-module. Then, for $e = \pm 1$, Lusztig [38, 5.2.1] defines linear maps $\mathrm{t}'_{i,e}, \mathrm{t}''_{i,e} : V \to V$ by

$$\mathrm{t}'_{i,e}(z) = \sum_{a,b,c;\,a-b+c=\lambda_i} (-1)^b q^{e(-ac+b)} F_i^{(a)} E_i^{(b)} F_i^{(c)} z,$$
$$\mathrm{t}''_{i,e}(z) = \sum_{a,b,c;\,-a+b-c=\lambda_i} (-1)^b q^{e(-ac+b)} E_i^{(a)} F_i^{(b)} E_i^{(c)} z,$$

for $z \in V_\lambda$ that are commonly called the *quantum Weyl group* elements. They are mutually inverse automorphisms (specifically, they satisfy $\mathrm{t}'_{i,e}\mathrm{t}''_{i,-e} = \mathrm{Id} = \mathrm{t}''_{i,-e}\mathrm{t}'_{i,e}$) that satisfy the relations

$$\mathrm{t}'_{i,e}\mathrm{t}'_{j,e}\mathrm{t}'_{i,e} = \mathrm{t}'_{j,e}\mathrm{t}'_{i,e}\mathrm{t}'_{j,e} \quad \text{and} \quad \mathrm{t}''_{i,e}\mathrm{t}''_{j,e}\mathrm{t}''_{i,e} = \mathrm{t}''_{j,e}\mathrm{t}''_{i,e}\mathrm{t}''_{j,e} \qquad \text{if } i \cdot j = -1,$$
$$\mathrm{t}'_{i,e}\mathrm{t}'_{j,e} = \mathrm{t}'_{j,e}\mathrm{t}'_{i,e} \quad \text{and} \quad \mathrm{t}''_{i,e}\mathrm{t}''_{j,e} = \mathrm{t}''_{j,e}\mathrm{t}''_{i,e} \qquad \text{if } i \cdot j = 0,$$

and thus define an action of the type-$\mathfrak{g}$ braid group on any integrable module [38, Theorem 39.4.3]. This action on a particular weight space can be conveniently described by the infinite sums

$$\mathrm{t}'_{i,e}1_\lambda = \sum_{a,b,c;\,a-b+c=\lambda_i} (-1)^b q^{e(-ac+b)} F_i^{(a)} E_i^{(b)} F_i^{(c)} 1_\lambda,$$
$$\mathrm{t}''_{i,e}1_\lambda = \sum_{a,b,c;\,-a+b-c=\lambda_i} (-1)^b q^{e(-ac+b)} E_i^{(a)} F_i^{(b)} E_i^{(c)} 1_\lambda$$

of elements in $\dot{\boldsymbol{U}}$, from which the maps $\mathrm{t}'_{i,e}$,$\mathrm{t}''_{i,e}$ can be recovered by taking the sum over all $\lambda \in X$. It was shown in [17, Lemma 6.1.1] that these elements admit the simpler form given in (1-2) above, that is, all terms with $c \neq 0$ cancel.

2D. ***Lusztig's internal braid group action.*** For each $i \in I$ and $e = \pm 1$, Lusztig defines algebra automorphisms $T'_{i,e}$ and $T''_{i,e}$ of $\dot{\boldsymbol{U}} = \dot{\boldsymbol{U}}_q(\mathfrak{g})$ defined uniquely by the compatibility with the quantum Weyl group action given in (1-3) above. They are given explicitly in [38, 41.1.2] by

$$T'_{i,e}(1_\lambda) = 1_{s_i(\lambda)},$$

(2-1)
$$T'_{i,e}(E_\ell 1_\lambda) = \begin{cases} -q^{-e(2+\lambda_i)} F_i 1_{s_i(\lambda)} & \text{if } i = \ell, \\ E_\ell E_i 1_{s_i(\lambda)} - q^e E_i E_\ell 1_{s_i(\lambda)} & \text{if } i \cdot \ell = -1, \\ E_\ell 1_{s_i(\lambda)} & \text{if } i \cdot \ell = 0, \end{cases}$$

$$T'_{i,e}(F_\ell 1_\lambda) = \begin{cases} -q^{e(\lambda_i)} E_i 1_{s_i(\lambda)} & \text{if } i = \ell, \\ F_i F_\ell 1_{s_i(\lambda)} - q^{-e} F_\ell F_i 1_{s_i(\lambda)} & \text{if } i \cdot \ell = -1, \\ F_\ell 1_{s_i(\lambda)} & \text{if } i \cdot \ell = 0, \end{cases}$$

and

$$T''_{i,e}(1_\lambda) = 1_{s_i(\lambda)},$$

$$T''_{i,e}(E_\ell 1_\lambda) = \begin{cases} -q^{-e(\lambda_i)} F_i 1_{s_i(\lambda)} & \text{if } i = \ell, \\ E_i E_\ell 1_{s_i(\lambda)} - q^{-e} E_\ell E_i 1_{s_i(\lambda)} & \text{if } i \cdot \ell = -1, \\ E_\ell 1_{s_i(\lambda)} & \text{if } i \cdot \ell = 0, \end{cases}$$

$$T''_{i,e}(F_\ell 1_\lambda) = \begin{cases} -q^{e(\lambda_i-2)} E_i 1_{s_i(\lambda)} & \text{if } i = \ell, \\ F_\ell F_i 1_{s_i(\lambda)} - q^e F_i F_\ell 1_{s_i(\lambda)} & \text{if } i \cdot \ell = -1, \\ F_\ell 1_{s_i(\lambda)} & \text{if } i \cdot \ell = 0, \end{cases}$$

where s_i is the Weyl group element corresponding to the simple root α_i, i.e., $s_i(\lambda) = \lambda - \langle i, \lambda \rangle \alpha_i$. Lusztig further shows [38, 41.1.1] that $(T'_{i,e})^{-1} = T''_{i,-e}$, and that these automorphisms interact with the automorphisms from Section 2B as in (1-5) above. As a consequence, we see that both $T'_{i,e}$ and $T''_{i,e}$ are invariant under conjugation by the triple composite $\underline{\sigma}\,\underline{\omega}\,\underline{\psi}$.

In what follows, we focus our attention on the automorphisms $T'_{i,1}$, since similar results can be deduced for $T''_{i,-1}$, $T''_{i,1}$, and $T'_{i,-1}$ using (1-5). When the context is clear, we will abbreviate $T'_{i,1}$ by T'_i. In [38, 39.2.4 and 39.2.5], Lusztig shows that the T'_i satisfy

$$T'_i T'_\ell T'_i = T'_\ell T'_i T'_\ell \quad \text{if } i \cdot \ell = -1,$$
$$T'_i T'_\ell = T'_\ell T'_i \qquad \text{if } i \cdot \ell = 0,$$

and hence defined a type-$\mathfrak{g}$ braid group action on $\dot{\boldsymbol{U}}$.

3. The categorified quantum group

We recall the definition of the categorified quantum group $\mathcal{U}_Q(\mathfrak{g})$, specifically the cyclic version from [5], and establish a number of additional properties needed for our arguments.

3A. ***Choice of scalars Q.*** Let $\Bbbk$ be a field, not necessarily algebraically closed or characteristic zero.

Definition 3.1. A *choice of scalars* Q associated to a simply-laced Cartan datum, consist of elements $\{t_{ij}\}_{i,j\in I}$ satisfying

- $t_{ii} = 1$ for all $i \in I$ and $t_{ij} \in \Bbbk^\times$ for $i \neq j$,
- $t_{ij} = t_{ji}$ when $a_{ij} = 0$.

We say that a choice of scalars Q is *integral* if $t_{ij} = \pm 1$ for all $i, j \in I$.

The choice of scalars Q controls the form of the KLR algebra R_Q that categorifies the positive half of the quantum group $\dot{\boldsymbol{U}}$, and the 2-category $\mathcal{U}_Q(\mathfrak{g})$ is governed by the products $v_{ij} = t_{ij}^{-1} t_{ji}$ taken over all pairs $i, j \in I$, which can be viewed as a $\Bbbk^\times$-valued 1-cocycle on the graph Γ associated to the Cartan datum.

Definition 3.2. A choice of *bubble parameters* C consists of elements $c_{i,\lambda} \in \Bbbk^\times$ for $i \in I$ and $\lambda \in X$. We say that they are *compatible* with the scalars Q if

$$c_{i,\lambda+\alpha_j}/c_{i,\lambda} = t_{ij}. \tag{3-1}$$

Given any choice of scalars Q, we obtain a compatible choice of bubble parameters by fixing $c_{i,\lambda}$ for a representative in every coset of the root lattice in the weight lattice, and then extending to the entire weight lattice using (3-1). For a compatible choice, note that the bubble parameters remain constant along an $\mathfrak{sl}_2$-string since

$$c_{i,\lambda+n\alpha_i} = t_{ii}^n c_{i,\lambda} = c_{i,\lambda}.$$

3B. *Definition of the 2-category $\mathcal{U}_Q(\mathfrak{g})$.* Recall that a *graded linear* category is an additive category equipped with an auto-equivalence $\langle 1\rangle$ called the *shift* (see, for example, [2]), and a graded additive 2-category is a category enriched in graded linear categories. Throughout, we will use $\langle t\rangle$ to denote the auto-equivalence given by applying $\langle 1\rangle$ t times, and $\langle -t\rangle$ to denote the auto-equivalence obtained by applying the inverse of $\langle 1\rangle$ t times.

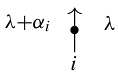
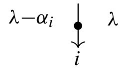
Definition 3.3. Fix a choice of scalars Q and compatible bubble parameters C. Then the 2-category $\mathcal{U}_Q := \mathcal{U}_Q^{\mathrm{cyc}}(\mathfrak{g})$ is the graded linear 2-category with:

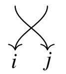
- Objects: $\lambda \in X$.
- 1-morphisms: formal direct sums of shifts of compositions of the generating 1-morphisms:

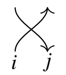
$$\mathbb{1}_\lambda, \quad \mathbb{1}_{\lambda+\alpha_i}\mathcal{E}_i = \mathbb{1}_{\lambda+\alpha_i}\mathcal{E}_i\mathbb{1}_\lambda = \mathcal{E}_i\mathbb{1}_\lambda, \quad \mathbb{1}_{\lambda-\alpha_i}\mathcal{F}_i = \mathbb{1}_{\lambda-\alpha_i}\mathcal{F}_i\mathbb{1}_\lambda = \mathcal{F}_i\mathbb{1}_\lambda$$

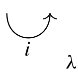
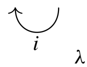
for $i \in I$ and $\lambda \in X$.

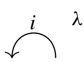
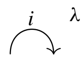
• 2-morphisms: Hom-spaces are $\Bbbk$-vector spaces spanned by (horizontal and vertical) compositions of the decorated tangle-like diagrams

 $: \mathcal{E}_i\mathbb{1}_\lambda \to \mathcal{E}_i\mathbb{1}_\lambda\langle 2\rangle,$  $: \mathcal{F}_i\mathbb{1}_\lambda \to \mathcal{F}_i\mathbb{1}_\lambda\langle 2\rangle,$

 $\lambda : \mathcal{E}_i\mathcal{E}_j\mathbb{1}_\lambda \to \mathcal{E}_j\mathcal{E}_i\mathbb{1}_\lambda\langle -i\cdot j\rangle,$  $\lambda : \mathcal{F}_i\mathcal{F}_j\mathbb{1}_\lambda \to \mathcal{F}_j\mathcal{F}_i\mathbb{1}_\lambda\langle -i\cdot j\rangle,$

 $\lambda : \mathcal{F}_i\mathcal{E}_j\mathbb{1}_\lambda \to \mathcal{E}_j\mathcal{F}_i\mathbb{1}_\lambda,$  $\lambda : \mathcal{E}_i\mathcal{F}_j\mathbb{1}_\lambda \to \mathcal{F}_j\mathcal{E}_i\mathbb{1}_\lambda,$

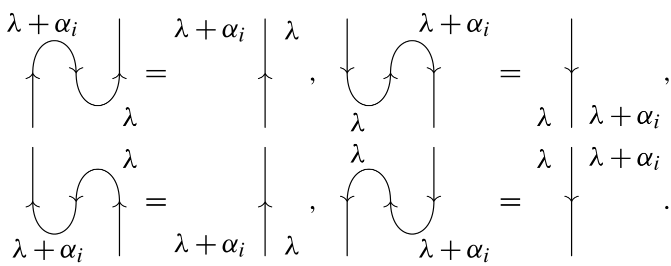
 $: \mathbb{1}_\lambda \to \mathcal{F}_i\mathcal{E}_i\mathbb{1}_\lambda\langle 1+\lambda_i\rangle,$  $: \mathbb{1}_\lambda \to \mathcal{E}_i\mathcal{F}_i\mathbb{1}_\lambda\langle 1-\lambda_i\rangle,$

 $: \mathcal{F}_i\mathcal{E}_i\mathbb{1}_\lambda \to \mathbb{1}_\lambda\langle 1+\lambda_i\rangle,$  $: \mathcal{E}_i\mathcal{F}_i\mathbb{1}_\lambda \to \mathbb{1}_\lambda\langle 1-\lambda_i\rangle.$

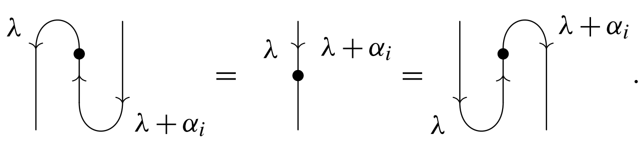
Note that we follow the grading conventions in [14; 34], which are opposite to those from [26]. We read such diagrams from right to left and bottom to top, and the identity 2-morphisms of the 1-morphisms $\mathcal{E}_i\mathbb{1}_\lambda$ and $\mathcal{F}_i\mathbb{1}_\lambda$ are depicted by upward and downward oriented segments labeled by i, respectively.

The following local relations are imposed on the 2-morphisms:

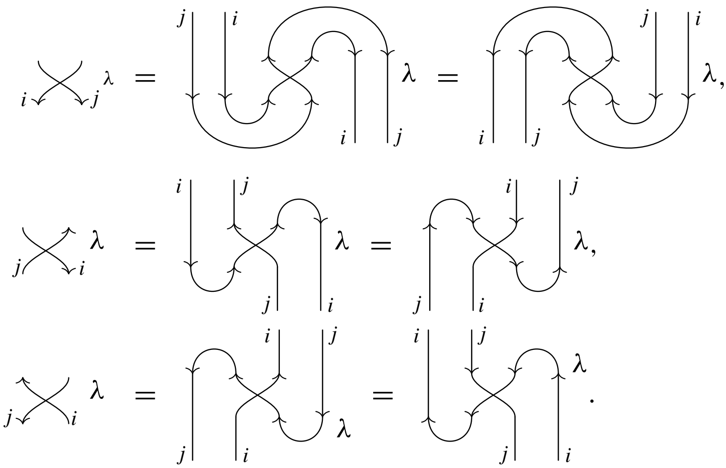
(1) Right and left adjunction:



(2) Dot cyclicity:

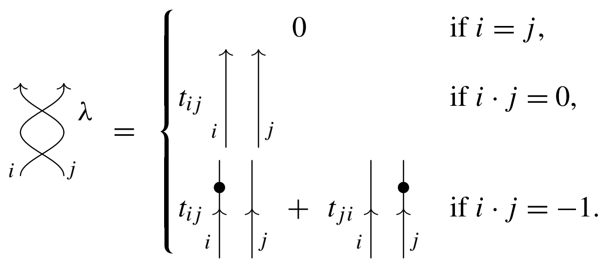


(3) Crossing cyclicity:

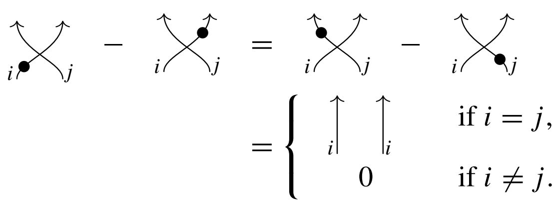


The next three relations imply that the $\mathcal{E}$'s (and $\mathcal{F}'s$) carry an action of the KLR algebra associated to Q:

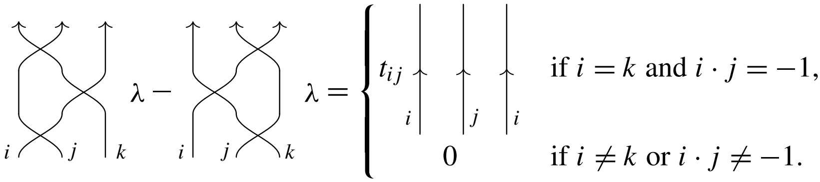
(4) Quadratic KLR:



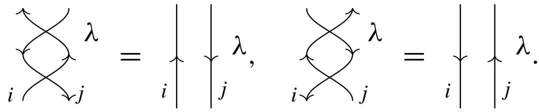
(5) Dot slide:



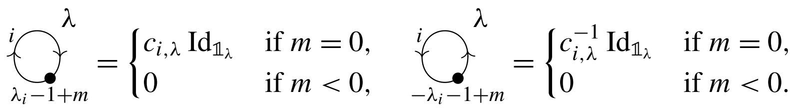
(6) Cubic KLR:



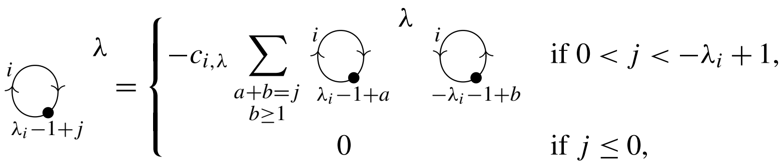
(7) Mixed EF: for $i \neq j$,



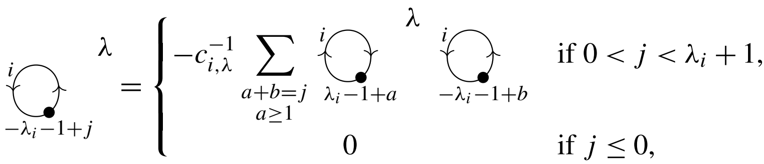
(8) Bubble relations:



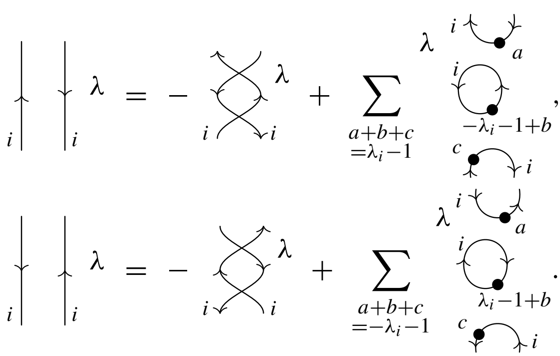
(9) Extended $\mathfrak{sl}_2$ relations: These final relations are the most involved, and require the introduction of *fake bubbles* — positive degree endomorphisms of $\mathbb{1}_\lambda$ that are denoted by a bubble carrying a formal label by a negative number of dots. They are defined by



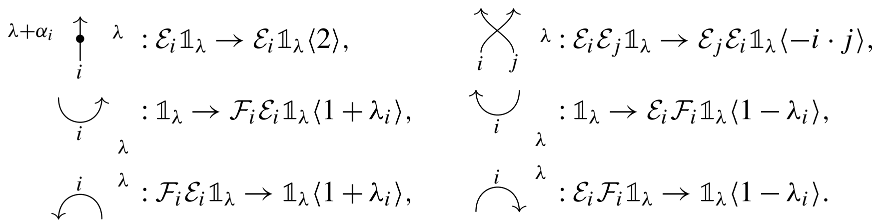
when $\lambda_i < 0$, and by



when $\lambda_i > 0$. The extended $\mathfrak{sl}_2$ relations are then as follows, where we employ the convention here (and throughout) that all summations are "increasing", that is, $\sum_{a+b+c=\mu} X_{a,b,c}$ is zero if $\mu < 0$:



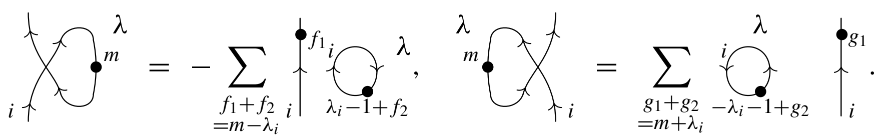
Remark 3.4. We will find it helpful to work with the reduced presentation for $\mathcal{U}_Q$ where we restrict to the generating 2-morphisms



Indeed, the downward dot 2-morphism and sideways and downward crossings can be defined in various ways by composing the upward versions with caps and cups, and the cyclicity relations show that they do not depend on the choices made in doing so. Further, Brundan [7] has shown that this presentation can be further simplified to agree with the one given by Rouquier [54] that requires a smaller set of axioms, together with the requirement that certain 2-morphisms are (abstractly) invertible. Although this further reduced presentation is helpful in checking that biadjointness and cyclicity hold in various 2-representations, it is not useful in our present work, as showing that the required maps are invertible essentially requires verifying the omitted axioms in $\mathcal{U}_Q$.

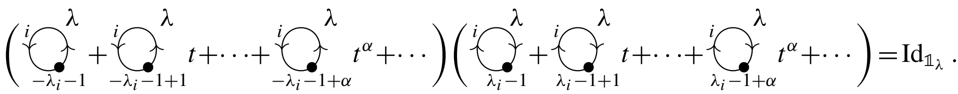
3C. *Additional relations in $\mathcal{U}_Q$.* Here, we collect additional useful relations that will be used in later sections.

3C1. *Curl relations.* Dotted curls can be reduced to simpler diagrams using



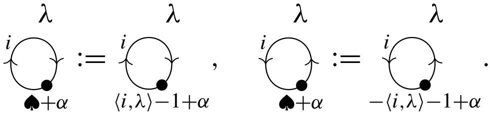
Note that in [14; 33] the $m = 0$ cases of these relations were included in the defining list of relations, but it was shown in [5, Lemma 3.2] that these relations (for arbitrary m) follow from the relations presented above.

3C2. *Infinite Grassmannian relations.* These relations are obtained by equating the terms homogeneous in t in the expression below:



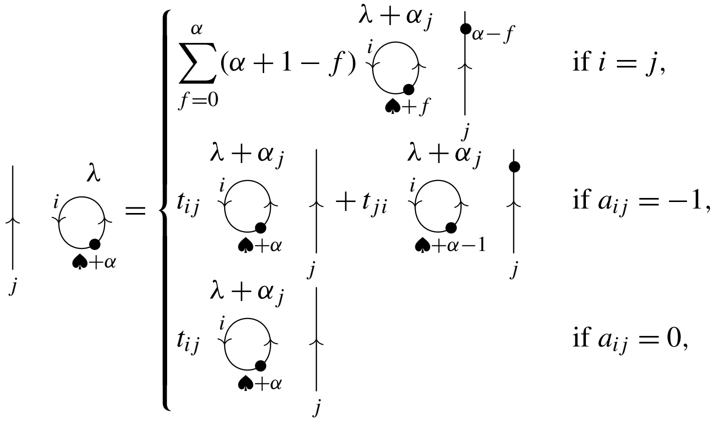
For low powers of t, these relations encode the definition of fake bubbles in terms of (real) bubbles, and, for higher powers of t, they follow from the curl and extended $\mathfrak{sl}_2$ relations.

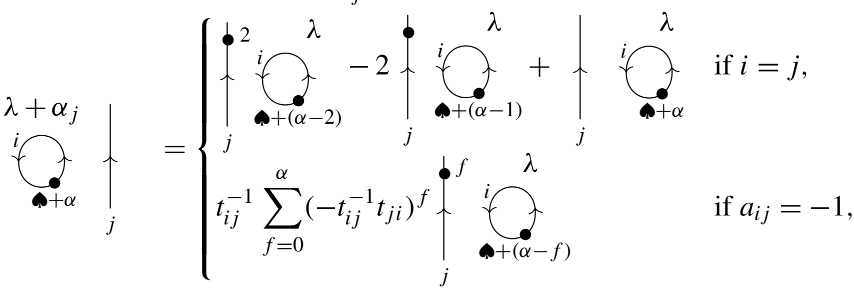
3C3. *Bubble slides.* In what follows, we make use of the shorthand notation [27]



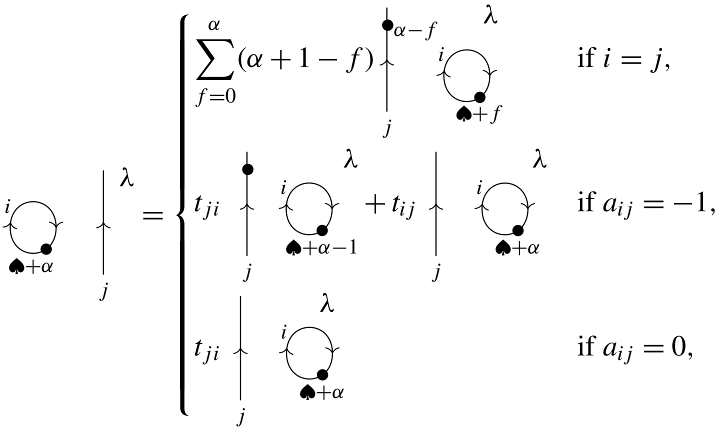
As long as $\alpha \geq 0$, this notation makes sense even when $\spadesuit + \alpha < 0$, in which case these are the fake bubbles defined in the previous section.

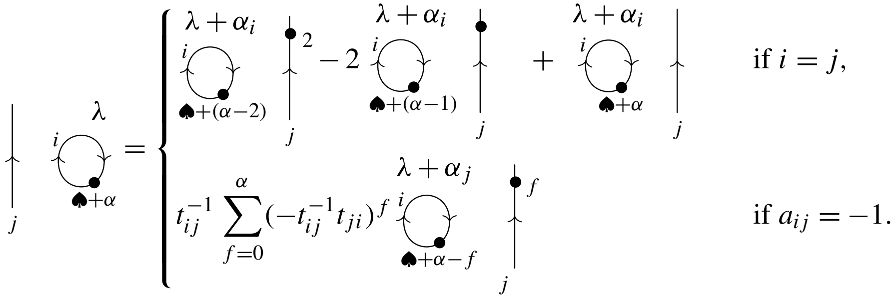
Counterclockwise bubbles can be slid through upward oriented lines via





and we have similar relations involving clockwise bubbles:

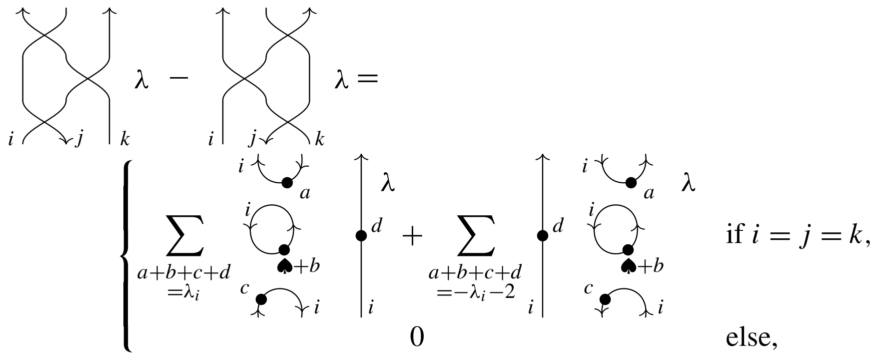




Both types of bubbles can then be slid through downward oriented lines using these relations and the cyclicity of $\mathcal{U}_Q(\mathfrak{g})$.

3C4. *Triple intersections.* We have

(3-2)



which is [33, Proposition 5.8] when $i = j = k$, and follows from cyclicity, the mixed EF relation, and the cubic KLR relation in the other case.

3D. *The 2-categories $\dot{\mathcal{U}}_Q$, $\mathsf{Kom}(\dot{\mathcal{U}}_Q)$, and $\mathsf{Com}(\dot{\mathcal{U}}_Q)$.*

3D1. *Categories of complexes.* Given an additive category $\mathcal{M}$, we let $\mathsf{Kom}(\mathcal{M})$ denote the category of bounded complexes in $\mathcal{M}$. By convention, we work with *cochain* complexes, so an object (X, d) of $\mathsf{Kom}(\mathcal{M})$ is a collection of objects X^i in $\mathcal{M}$ together with maps

$$\cdots \xrightarrow{d_{i-2}} X^{i-1} \xrightarrow{d_{i-1}} X^i \xrightarrow{d_i} X^{i+1} \xrightarrow{d_{i+1}} \cdots$$

such that $d_{i+1}d_i = 0$ and only finitely many of the X^i's are nonzero. A morphism $f : (X, d) \to (Y, d')$ in $\mathsf{Kom}(\mathcal{M})$ consists of a collection of morphisms $f_i : X^i \to Y^i$ in $\mathcal{M}$ such that $f_{i+1}d_i = d'_i f_i$.

Recall that morphisms $f, g : (X, d) \to (Y, d')$ in $\mathsf{Kom}(\mathcal{M})$ are called (*chain*) *homotopic*, denoted by $f \sim g$, if there exist morphisms $h^i : X^i \to Y^{i-1}$ such that $f_i - g_i = h^{i+1}d_i + d'_{i-1}h^i$ for all i. A morphism of complexes is said to be null-homotopic if it is homotopic to the zero map.

Definition 3.5. The homotopy category $\mathsf{Com}(\mathcal{M})$ is the additive category with the same objects as $\mathsf{Kom}(\mathcal{M})$ with morphisms given by morphisms in $\mathsf{Kom}(\mathcal{M})$ modulo null-homotopic morphisms.

We say that two complexes (X, d_X) and (Y, d_Y) are homotopy equivalent provided they are isomorphic in $\mathsf{Com}(\mathcal{M})$, and denote this by $X \simeq Y$.

If $\mathcal{M}$ is monoidal, then $\mathsf{Kom}(\mathcal{M})$ is also monoidal, with the tensor product (XY, d) of (X, d_X) and (Y, d_Y) defined as

$$(XY)^i = \bigoplus_{r+s=i} X^r Y^s, \quad d_i := \sum_{r+s=i} (d_X)_r \operatorname{Id}_{Y^s} + (-1)^r \operatorname{Id}_{X^r} (d_Y)_s. \tag{3-3}$$

Here, we denote the tensor product of objects and morphisms in $\mathcal{M}$ by juxtaposition. Given chain maps $f : (X, d_X) \to (X', d_{X'})$ and $g : (Y, d_Y) \to (Y', d_{Y'})$ define the tensor product $fg : (XY, d) \to (X'Y', d')$ of chain maps by setting

$$f_i = \bigoplus_{r+s=i} f_r g_s. \tag{3-4}$$

It is straightforward to check that if $f \sim f'$ and $g \sim g'$, then $fg \sim f'g'$, so $\mathsf{Com}(\mathcal{M})$ inherits a monoidal structure from $\mathsf{Kom}(\mathcal{M})$.

Remark 3.6. More generally[1], if $\mathcal{C}$ is an additive 2-category, we can consider the 2-categories $\mathsf{Kom}(\mathcal{C})$ and $\mathsf{Com}(\mathcal{C})$ obtained by taking complexes in each Hom-category. The above description of tensor product of complexes specifies how to take horizontal composition in $\mathsf{Kom}(\mathcal{C})$ and $\mathsf{Com}(\mathcal{C})$.

3D2. *Karoubi envelope.* The Karoubi envelope $\mathsf{Kar}(\mathcal{M})$ of a category $\mathcal{M}$ is the universal enlargement of $\mathcal{M}$ in which all idempotents split. Recall that we say an idempotent $e : b \to b$ in a category $\mathcal{M}$ splits if there exist morphisms $b \xrightarrow{g} b' \xrightarrow{h} b$ such that $e = hg$ and $gh = \operatorname{Id}_{b'}$. The Karoubi envelope $\mathsf{Kar}(\mathcal{M})$ admits an explicit description as the category whose objects are pairs (b, e), where $e : b \to b$ is an idempotent of $\mathcal{M}$, and where morphisms are triples of the form

$$(e, f, e') : (b, e) \to (b', e')$$

for $f : b \to b'$ in $\mathcal{M}$ satisfying $f = e' f = f e$. Composition is induced from composition in $\mathcal{M}$, and the identity morphism is $(e, e, e) : (b, e) \to (b, e)$.

The identity map $\operatorname{Id}_b : b \to b$ is an idempotent, and the assignment $b \mapsto (b, \operatorname{Id}_b)$ defines a fully faithful functor $\mathcal{M} \hookrightarrow \mathsf{Kar}(\mathcal{M})$, and this functor is universal among functors from $\mathcal{M}$ to idempotent split categories. If $\mathcal{M}$ is additive then so is $\mathsf{Kar}(\mathcal{M})$ and this embedding is additive; in this case, for $(b, e) \in \mathsf{Kar}(\mathcal{M})$, we have that $b \cong \operatorname{im} e \oplus \operatorname{im}(\operatorname{Id}_b - e)$ where $\operatorname{im} e := (b, e)$. See [33, Section 9] for more details.

[1]Recall that a monoidal category can be interpreted as a 2-category with only one object.

The following result shows that the Karoubi envelope interacts nicely with passage to (homotopy) categories of complexes.

Proposition 3.7 [4, Propositions 3.6 and 3.7]. *For any additive category* $\mathcal{M}$ *there is a canonical equivalence* $\mathsf{Kom}(\mathsf{Kar}(\mathcal{M})) \cong \mathsf{Kar}(\mathsf{Kom}(\mathcal{M}))$. *If* $\mathcal{M}$ *is* $\Bbbk$*-linear with finite-dimensional* Hom*-spaces, then there is a canonical equivalence* $\mathsf{Com}(\mathsf{Kar}(\mathcal{M})) \cong \mathsf{Kar}(\mathsf{Com}(\mathcal{M}))$.

3D3. *Karoubi envelope of* $\mathcal{U}_Q$.

Definition 3.8. The additive 2-category $\dot{\mathcal{U}}_Q$ has the same objects as $\mathcal{U}_Q$ and has Hom-categories given by $\dot{\mathcal{U}}_Q(\lambda, \lambda') = \mathsf{Kar}(\mathcal{U}_Q(\lambda, \lambda'))$.

Horizontal composition in $\dot{\mathcal{U}}_Q$ is induced from composition in $\mathcal{U}_Q$ using the universal property of the Karoubi envelope, and we similarly obtain an additive, fully-faithful 2-functor $\mathcal{U}_Q \to \dot{\mathcal{U}}_Q$ that is universal with respect to splitting idempotents in the Hom-categories $\dot{\mathcal{U}}_Q(\lambda, \lambda')$. The significance of the 2-category $\dot{\mathcal{U}}_Q(\mathfrak{g})$ is given by the following theorem.

Theorem 3.9 [26; 33; 61]. *There is an isomorphism* $\gamma : {}_{\mathcal{A}}\dot{\boldsymbol{U}} \xrightarrow{\cong} K_0(\dot{\mathcal{U}}_Q(\mathfrak{g}))$ *where* $K_0(\dot{\mathcal{U}}_Q)$ *denotes the split Grothendieck ring of* $\dot{\mathcal{U}}_Q$.

For $\mathfrak{g} = \mathfrak{sl}_2$, this theorem also holds over $\mathbb{Z}$ by the results in [27].

3D4. *Karoubian envelopes of* $\mathsf{Kom}(\mathcal{U})$ *and* $\mathsf{Com}(\mathcal{U})$. Following Remark 3.6 above, we consider the 2-categories $\mathsf{Kom}(\mathcal{U}_Q)$ and $\mathsf{Com}(\mathcal{U}_Q)$. Noting that the 2-Hom-spaces $\mathcal{U}_Q(x, y\langle t\rangle)$ are finite-dimensional $\Bbbk$-vector spaces for each $t \in \mathbb{Z}$, Proposition 3.7 gives equivalences

$$\mathsf{Kar}(\mathsf{Kom}(\mathcal{U}_Q)) \cong \mathsf{Kom}(\dot{\mathcal{U}}_Q), \qquad \mathsf{Kar}(\mathsf{Com}(\mathcal{U}_Q)) \cong \mathsf{Com}(\dot{\mathcal{U}}_Q).$$

We arrange the various 2-categories built from $\mathcal{U}_Q$ into the following diagram, wherein the horizontal arrows denote passage to the Karoubian envelope, and vertical arrows denote the canonical maps between the various categories of complexes:

$$\begin{array}{ccc}
\mathcal{U}_Q & \hookrightarrow & \dot{\mathcal{U}} = \mathsf{Kar}(\mathcal{U}_Q) \\
\downarrow & & \downarrow \\
\mathsf{Kom}(\mathcal{U}_Q) & \hookrightarrow & \mathsf{Kom}(\dot{\mathcal{U}}_Q) \cong \mathsf{Kar}(\mathsf{Kom}(\mathcal{U})) \\
\twoheaddownarrow & & \twoheaddownarrow \\
\mathsf{Com}(\mathcal{U}_Q) & \hookrightarrow & \mathsf{Com}(\dot{\mathcal{U}}_Q) \cong \mathsf{Kar}(\mathsf{Com}(\mathcal{U}))
\end{array}$$

Theorem 3.9 and the main result of [52] imply that

$$K_0(\mathsf{Kar}(\mathsf{Com}(\mathcal{U}_Q))) \cong K_0(\mathsf{Com}(\mathsf{Kar}(\mathcal{U}_Q))) \cong K_0(\mathsf{Kar}(\mathcal{U}_Q)) \cong K_0(\dot{\mathcal{U}}_Q) \cong {}_{\mathcal{A}}\dot{\boldsymbol{U}},$$

where we employ the triangulated Grothendieck group for the categories of complexes. We can hence view the Karoubi envelope of the homotopy category $\mathsf{Com}(\mathcal{U}_Q)$ as a categorification of the integral idempotent form ${}_{\mathcal{A}}\dot{\boldsymbol{U}}$ of the quantum group.

3E. ***Symmetries of categorified quantum groups.*** We now use symmetries of the diagrammatic relations in $\mathcal{U}_Q$ to define 2-functors σ, ω, and ψ (for a general choice of scalars Q and bubble parameters C) that lift the symmetries of quantum groups from Section 2B. This extends the work of Khovanov and Lauda in [26], who defined such functors in the specific case where $t_{ij} = 1 = c_{i,\lambda}$ for all $i, j \in I$ and $\lambda \in X$. These 2-functors extend naturally to 2-functors on $\dot{\mathcal{U}}_Q$, $\mathsf{Kom}(\dot{\mathcal{U}}_Q)$, and $\mathsf{Com}(\dot{\mathcal{U}}_Q)$ [4], and induce the corresponding quantum group symmetries $\underline{\sigma}$, $\underline{\omega}$, and $\underline{\psi}$ on ${}_{\mathcal{A}}\dot{\boldsymbol{U}}$ upon passing to K_0. For this reason, we refer to them as symmetry 2-functors.

Rather than being 2-endofunctors of $\mathcal{U}_Q$, some of these symmetries map between versions $\mathcal{U}_Q$ and $\mathcal{U}'_Q$ of the categorified quantum group corresponding to *different* bubble parameters. (Caveat lector: $\mathcal{U}'_Q$ should *not* be confused with $\mathcal{U}_{Q'}$ from [14] which instead corresponds to a different choice of scalars Q.) We define $\mathcal{U}'_Q$ to be the 2-category given in Definition 3.3, but with the bubble parameters for $\mathcal{U}_Q$ replaced by primed bubble parameters $(c_{i,\lambda})' := c_{i,-\lambda}^{-1}$. The primed bubble parameters are still compatible with the choice of scalars Q (used for both $\mathcal{U}_Q$ and $\mathcal{U}'_Q$), since

$$\frac{(c_{i,\lambda+\alpha_j})'}{(c_{i,\lambda})'} = \frac{c_{i,-(\lambda+\alpha_j)}^{-1}}{c_{i,-\lambda}^{-1}} = \frac{c_{i,-\lambda}}{c_{i,-\lambda-\alpha_j}} = t_{ij}.$$

In addition to mapping between versions of the categorified quantum group corresponding to different bubble parameters, the symmetry 2-functors possess various flavors of contravariance. Nevertheless, they are morally pairwise-commuting involutions, as the double application of a given symmetry is the identity and the result of a composition does not depend on the order, despite the domain and codomain being different versions of the categorified quantum group. Given this, we will slightly abuse notation and refer to the symmetry and its inverse by the same symbol.

3E1. *Forms of 2-categorical contravariance.* Recall that a contravariant functor $\mathbf{C} \to \mathbf{D}$ can be rephrased in terms of a (covariant) functor $\mathbf{C} \to \mathbf{D}^{\mathrm{op}}$, where $\mathbf{D}^{\mathrm{op}}$ is the opposite category, defined to have the same objects as in $\mathbf{D}$, but with $\mathbf{D}^{\mathrm{op}}(x, y) := \mathbf{D}(y, x)$, i.e., the direction of the morphisms is opposite to that in $\mathbf{D}$. For a 2-category $\mathcal{C}$, we can take the opposite 2-category in various ways, depending on whether we take the opposite at the 1-morphism or 2-morphism level (or both). Denote by $\mathcal{C}^{\mathrm{op}}$ the 2-category with the same objects as $\mathcal{C}$, and where we've taken the opposite with respect to 1-morphisms, that is, for objects x, y in $\mathcal{C}$, we let the Hom-categories be given by $\mathcal{C}^{\mathrm{op}}(x, y) := \mathcal{C}(y, x)$. Let $\mathcal{C}^{\mathrm{co}}$ denote the 2-category with the same objects and 1-morphisms as $\mathcal{C}$, but with opposite 2-morphisms, i.e., for objects x, y in $\mathcal{C}$, we let the Hom-categories be given by $\mathcal{C}^{\mathrm{co}}(x, y) := \mathcal{C}(x, y)^{\mathrm{op}}$.

Finally, $\mathcal{C}^{\mathrm{coop}}$ is the 2-category in which we've taken opposite 1-morphisms and 2-morphisms, in other words, $\mathcal{C}^{\mathrm{coop}}(x, y) := \mathcal{C}(y, x)^{\mathrm{op}}$.

In the case of the 2-category $\mathcal{U}_Q$, functors between these opposite 2-categories correspond to $\mathbb{Z}[q, q^{-1}]$-(anti)linear algebra (anti)automorphisms of ${}_{\mathcal{A}}\dot{\boldsymbol{U}}$ upon taking the Grothendieck group, as summarized in the following table:

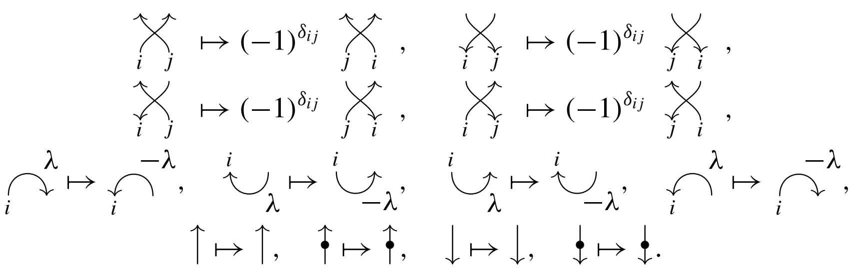
2-functor	induced map on ${}_{\mathcal{A}}\dot{\boldsymbol{U}}$
$\dot{\mathcal{U}}_Q \to \dot{\mathcal{U}}_Q$	$\mathbb{Z}[q, q^{-1}]$-linear homomorphism
$\dot{\mathcal{U}}_Q \to \dot{\mathcal{U}}_Q^{\mathrm{op}}$	$\mathbb{Z}[q, q^{-1}]$-linear antihomomorphism
$\dot{\mathcal{U}}_Q \to \dot{\mathcal{U}}_Q^{\mathrm{co}}$	$\mathbb{Z}[q, q^{-1}]$-antilinear homomorphism
$\dot{\mathcal{U}}_Q \to \dot{\mathcal{U}}_Q^{\mathrm{coop}}$	$\mathbb{Z}[q, q^{-1}]$-antilinear antihomomorphism

In the following sections, we will explicitly describe σ, ω, and ψ. To do so, we will use the notational convention from [26] that $\mathcal{E}_{-i} := \mathcal{F}_i$.

3E2. *The 2-functor* $\sigma : \mathcal{U}_Q \to (\mathcal{U}'_Q)^{\mathrm{op}}$. Consider the operation on the diagrammatic calculus or $\mathcal{U}_Q$ that reflects a diagram across a vertical axis, replaces $\lambda \leftrightarrow -\lambda$, and scales all ii-crossings by -1. This operation is contravariant for composition of 1-morphisms, covariant for composition of 2-morphisms, preserves the degree of a diagram, and takes relations in $\mathcal{U}_Q$ to those in $\mathcal{U}'_Q$. As such, it defines an invertible 2-functor given explicitly as

$$\sigma : \mathcal{U}_Q \to (\mathcal{U}'_Q)^{\mathrm{op}},$$

$$\lambda \mapsto -\lambda, \quad \mathbb{1}_\mu \mathcal{E}_{\pm i_1} \mathcal{E}_{\pm i_2} \cdots \mathcal{E}_{\pm i_m} \mathbb{1}_\lambda \langle t \rangle \mapsto \mathbb{1}_{-\lambda} \mathcal{E}_{\pm i_m} \cdots \mathcal{E}_{\pm i_2} \mathcal{E}_{\pm i_1} \mathbb{1}_{-\mu} \langle t \rangle,$$



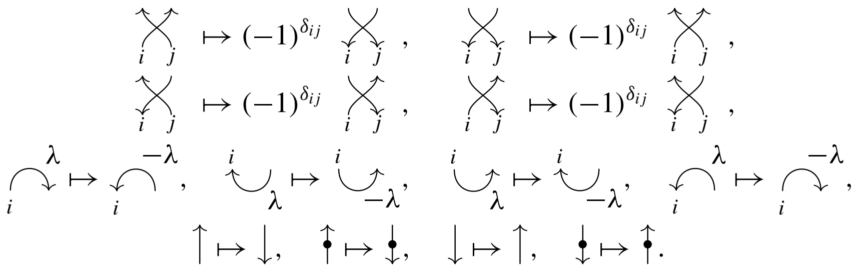
This extends to a 2-functor $\sigma : \mathsf{Kom}(\mathcal{U}_Q) \to \mathsf{Kom}(\mathcal{U}'_Q)^{\mathrm{op}}$ defined on 1-morphisms via

$$(X, d) \mapsto (\cdots \to \sigma(X^{i-1}) \xrightarrow{(-1)^{i-1}\sigma(d_{i-1})} \sigma(X^i) \xrightarrow{(-1)^i \sigma(d_i)} \sigma(X^{i+1}) \to \cdots)$$

and on 2-morphisms by applying σ componentwise. The alternating differential is essential here to preserve composition of 1-morphisms (contravariantly), due to the sign conventions in taking horizontal composition of complexes.

3E3. *The 2-functor* $\omega : \mathcal{U}_Q \to \mathcal{U}'_Q$. Consider the operation on the diagrammatic calculus for $\mathcal{U}_Q$ that reverses the orientation of each strand, replaces $\lambda \leftrightarrow -\lambda$, and scales all ii-crossings by -1. This operation is covariant for composition of both 1-morphisms and 2-morphisms, preserves the degree of a diagram, and takes relations in $\mathcal{U}_Q$ to those in $\mathcal{U}'_Q$. This defines an invertible 2-functor given explicitly as

$$\omega : \mathcal{U}_Q \to \mathcal{U}'_Q,$$
$$\lambda \mapsto -\lambda, \quad \mathbb{1}_\mu \mathcal{E}_{\pm i_1}\mathcal{E}_{\pm i_2}\cdots\mathcal{E}_{\pm i_m}\mathbb{1}_\lambda\langle t\rangle \mapsto \mathbb{1}_{-\mu}\mathcal{E}_{\mp i_1}\mathcal{E}_{\mp i_2}\cdots\mathcal{E}_{\mp i_m}\mathbb{1}_{-\lambda}\langle t\rangle,$$



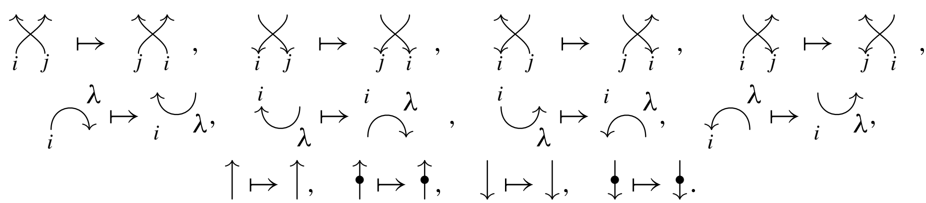
This extends to a 2-functor $\omega : \mathsf{Kom}(\mathcal{U}_Q) \to \mathsf{Kom}(\mathcal{U}'_Q)$ defined on 1-morphisms via

$$(X, d) \mapsto (\cdots \to \omega(X^{i-1}) \xrightarrow{\omega(d_{i-1})} \omega(X^i) \xrightarrow{\omega(d_i)} \omega(X^{i+1}) \to \cdots)$$

and on 2-morphisms by applying ω componentwise.

3E4. *The 2-functor* $\psi : \mathcal{U}_Q \to (\mathcal{U}_Q)^{\mathrm{co}}$. Consider the operation on the diagrammatic calculus for $\mathcal{U}_Q$ that reflects a diagram across a horizontal axis, and reverses the orientation. This operation is covariant for composition of 1-morphisms, contravariant for composition of 2-morphisms, and preserves the relations in $\mathcal{U}_Q$. It determines an invertible 2-functor given explicitly as

$$\psi : \mathcal{U}_Q \to (\mathcal{U}_Q)^{\mathrm{co}},$$
$$\lambda \mapsto \lambda, \quad \mathbb{1}_\mu \mathcal{E}_{\pm i_1}\mathcal{E}_{\pm i_2}\cdots\mathcal{E}_{\pm i_m}\mathbb{1}_\lambda\langle t\rangle \mapsto \mathbb{1}_{\mu}\mathcal{E}_{\pm i_1}\mathcal{E}_{\pm i_2}\cdots\mathcal{E}_{\pm i_m}\mathbb{1}_{\lambda}\langle -t\rangle,$$



Note that ψ must negate grading shift in order to be degree-preserving, due to 2-morphism contravariance. As such, it descends to an antilinear map on the Grothendieck group. This extends to a 2-functor $\psi : \mathsf{Kom}(\mathcal{U}_Q) \to \mathsf{Kom}(\mathcal{U}_Q)^{\mathrm{co}}$ given on 1-morphisms by

$$(X, d) \mapsto (\cdots \to \psi(X^{i+1}) \xrightarrow{\psi(d_i)} \psi(X^i) \xrightarrow{\psi(d_{i-1})} \psi(X^{i-1}) \to \cdots)$$

and on 2-morphisms by applying ψ componentwise. Implicit in this formula is that ψ negates the homological degree, i.e., for (X, d) in $\mathsf{Kom}(\mathcal{U})$ we have $\psi(X)^i = \psi(X^{-i})$.

3E5. *Properties of symmetries of categorified quantum groups.* The symmetries σ, ω, and ψ are graded, additive, $\Bbbk$-linear 2-functors, and it is immediate from their definitions that each squares to the identity. The induced 2-functors on categories of complexes descend to homotopy categories. The following result is immediate from the above definitions.

Theorem 1.3. *Under the isomorphism*

$$K_0(\dot{\mathcal{U}}_Q) \cong {}_{\mathcal{A}}\dot{\boldsymbol{U}} \cong K_0(\dot{\mathcal{U}}'_Q)$$

(see Theorem 3.9), the 2-functors defined above descend to the corresponding symmetries: $[\sigma] = \underline{d}\sigma$, $[\omega] = \underline{d}\omega$, $[\psi] = \underline{d}\psi$.

Remark 3.10. The symmetry $\omega\psi$ (which reflects a diagram across a horizontal axis, sends λ to $-\lambda$, and scales all ii-crossings by -1) is closely related to the Chevalley involution introduced in [7]. There, Brundan uses this to move between the 2-categories $\mathcal{U}_Q^{\mathrm{co}}$ and $\mathcal{U}_{Q'}$ associated to different choices of scalars. In the cyclic setting, changing the choice of scalars from Q to Q' is no longer necessary, provided we change the choice of bubble parameters from C to C' as above.

4. Defining the categorical Lusztig operator $\mathcal{T}'_{i,1}$

We now proceed to explicitly define additive, $\Bbbk$-linear 2-functors $\mathcal{T}'_{i,1} : \mathcal{U}_Q \to \mathsf{Com}(\mathcal{U}_Q)$ for each $i \in I$. In Section 4A we define $\mathcal{T}'_{i,1}$ on objects and generating 1-morphisms, and extend via additive 2-functoriality to all 1-morphisms, i.e., we send the horizontal composition of generators to the appropriate horizontal composition of the complexes giving their images, via (3-3), and map direct sums to the corresponding direct sums. In Section 4B, we extend this definition to the 2-morphisms in $\mathcal{U}_Q$, assigning explicit chain maps to generating 2-morphisms, again extending to all 2-morphisms as required by additive 2-functoriality.

Section 5 is then devoted to showing that $\mathcal{T}'_{i,1}$ is well defined, that is, showing that it preserves all defining relations on 2-morphisms of $\mathcal{U}$, up to chain homotopy. We also explicitly compute the chain homotopies involved. We note that this check is considerably lengthened due to the many relations that must be checked, and the piecewise nature of the definition of the (categorified) Lusztig operator, specifically, its dependency on the value of the bilinear form on I.

Theorem 1.1. *Let $\mathfrak{g}$ be a simply-laced Kac–Moody algebra. Then the data given below defines a 2-functor*

$$\mathcal{T}'_{i,+1} : \dot{\mathcal{U}}_Q(\mathfrak{g}) \to \mathsf{Com}(\dot{\mathcal{U}}_Q(\mathfrak{g}))$$

so that the induced map on $K_0(\dot{\mathcal{U}}_Q(\mathfrak{g})) \cong \dot{\boldsymbol{U}}_q(\mathfrak{g})$ *satisfies*

$$[\mathcal{T}'_{i,+1}] = T'_{i,+1} : \dot{\boldsymbol{U}}_q(\mathfrak{g}) \to \dot{\boldsymbol{U}}_q(\mathfrak{g}).$$

Given this, we then define the other versions of the categorified Lusztig operators using the symmetries of categorified quantum groups from Section 3E.

Definition 4.1. Let

$$\mathcal{T}''_{i,-1} := \sigma \mathcal{T}'_{i,1} \sigma, \quad \mathcal{T}''_{i,1} := \omega \mathcal{T}'_{i,1} \omega, \quad \mathcal{T}'_{i,-1} := \psi \mathcal{T}'_{i,1} \psi,$$

where in each case we apply $\mathcal{T}'_{i,1}$ on the appropriate version of the categorified quantum group, as determined by the codomain of the categorified symmetry.

The following result now follows from Theorems 1.1 and 1.3.

Corollary 4.2. *Upon passing to* $K_0(\dot{\mathcal{U}}_Q(\mathfrak{g}))$, *we have*

$$\begin{aligned}
[\mathcal{T}''_{i,-1}] &= [\sigma \mathcal{T}'_{i,1} \sigma] = [\sigma][\mathcal{T}'_{i,1}][\sigma] = \underline{\sigma} T'_{i,1} \underline{\sigma} = T''_{i,-1}, \\
[\mathcal{T}''_{i,1}] &= [\omega \mathcal{T}'_{i,1} \omega] = [\omega][\mathcal{T}'_{i,1}][\omega] = \underline{\omega} T'_{i,1} \underline{\omega} = T''_{i,1}, \\
[\mathcal{T}'_{i,-1}] &= [\psi \mathcal{T}'_{i,1} \psi] = [\psi][\mathcal{T}'_{i,1}][\psi] = \underline{\psi} T'_{i,1} \underline{\psi} = T'_{i,-1}.
\end{aligned}$$

Recall from the introduction that, while a similar categorification has previously been defined on 1-morphisms [12], our definition extends to the 2-morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{g})$, meaning that our categorified Lusztig operators help illuminate the higher structure of categorified quantum groups.

We now proceed with the definition, regularly abbreviating $\mathcal{T}'_{i,1}$ simply by $\mathcal{T}'_i$. In addition, we will make use of color in the diagrammatic calculus for $\dot{\mathcal{U}}_Q$ in specifying $\mathcal{T}'_i$ as follows: strands which are i-labeled (their label agrees with subscript on $\mathcal{T}'_i$) will be black, those whose labels j and j' satisfy $i \cdot j = -1 = i \cdot j'$ will be blue and magenta (respectively), and those with label k satisfying $i \cdot k = 0$ will be green, unless stated otherwise.

4A. ***$\mathcal{T}'_{i,1}$ on objects and 1-morphisms.*** On objects, we define the 2-functor $\mathcal{T}'_{i,1}$ by

$$\mathcal{T}'_i(\lambda) = s_i(\lambda),$$

where s_i is the corresponding Weyl group element, defined by $s_i(\lambda) = \lambda - \lambda_i \alpha_i$. On generating 1-morphisms, we define

$$\mathcal{T}'_i(\mathbb{1}_\lambda) = \clubsuit \mathbb{1}_{s_i(\lambda)},$$

$$\mathcal{T}'_i(\mathcal{E}_\ell \mathbb{1}_\lambda) = \begin{cases} \mathcal{F}_i \mathbb{1}_{s_i(\lambda)} \langle -2-\lambda_i \rangle \to \clubsuit\, 0 & \text{if } i = \ell, \\ \clubsuit \mathcal{E}_\ell \mathcal{E}_i \mathbb{1}_{s_i(\lambda)} \xrightarrow{\;X\;} \mathcal{E}_i \mathcal{E}_\ell \mathbb{1}_{s_i(\lambda)} \langle 1 \rangle & \text{if } i \cdot \ell = -1, \\ \clubsuit \mathcal{E}_\ell \mathbb{1}_{s_i(\lambda)} & \text{if } i \cdot \ell = 0, \end{cases}$$

$$\mathcal{T}_i'(\mathcal{F}_\ell \mathbb{1}_\lambda) = \begin{cases} \clubsuit\, 0 \to \mathcal{E}_i \mathbb{1}_{s_i(\lambda)}\langle \lambda_i \rangle & \text{if } i = \ell, \\ \mathcal{F}_\ell \mathcal{F}_i \mathbb{1}_{s_i(\lambda)}\langle -1 \rangle \xrightarrow{\ \ell \quad i\ } \clubsuit \mathcal{F}_i \mathcal{F}_\ell \mathbb{1}_{s_i(\lambda)} & \text{if } i \cdot \ell = -1, \\ \clubsuit \mathcal{F}_\ell \mathbb{1}_{s_i(\lambda)} & \text{if } i \cdot \ell = 0, \end{cases}$$

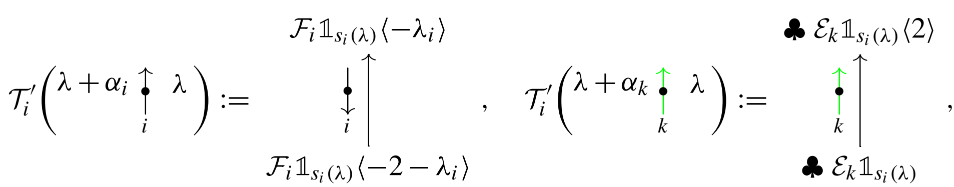
where we have omitted all nonzero terms in these complexes, and we follow our convention in denoting homological degree zero with a ♣. Since each of these complexes has at most two nonzero terms, it is trivial that the square of the differential is zero.

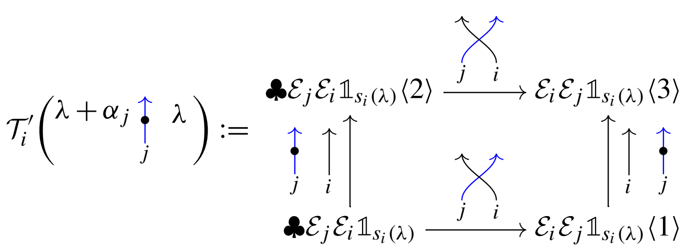
4B. *Definition of $\mathcal{T}_{i,1}'$ on 2-morphisms.* The 2-functor $\mathcal{T}_{i,1}'$ is given on generating 2-morphisms as follows. In these equations, we let our strand labels satisfy $i \cdot j = -1 = i \cdot j'$ and $i \cdot k = 0$, and follow the color conventions specified above. We will omit labeling the weight $s_i(\lambda)$ in the far-right region of the diagrams in the codomain, and in most cases will also only show the nonzero terms in our complexes. Additionally, we will depict complexes of the form

$$W \xrightarrow{\binom{\alpha}{\beta}} X \oplus Y \xrightarrow{(\gamma\ \delta)} Z$$

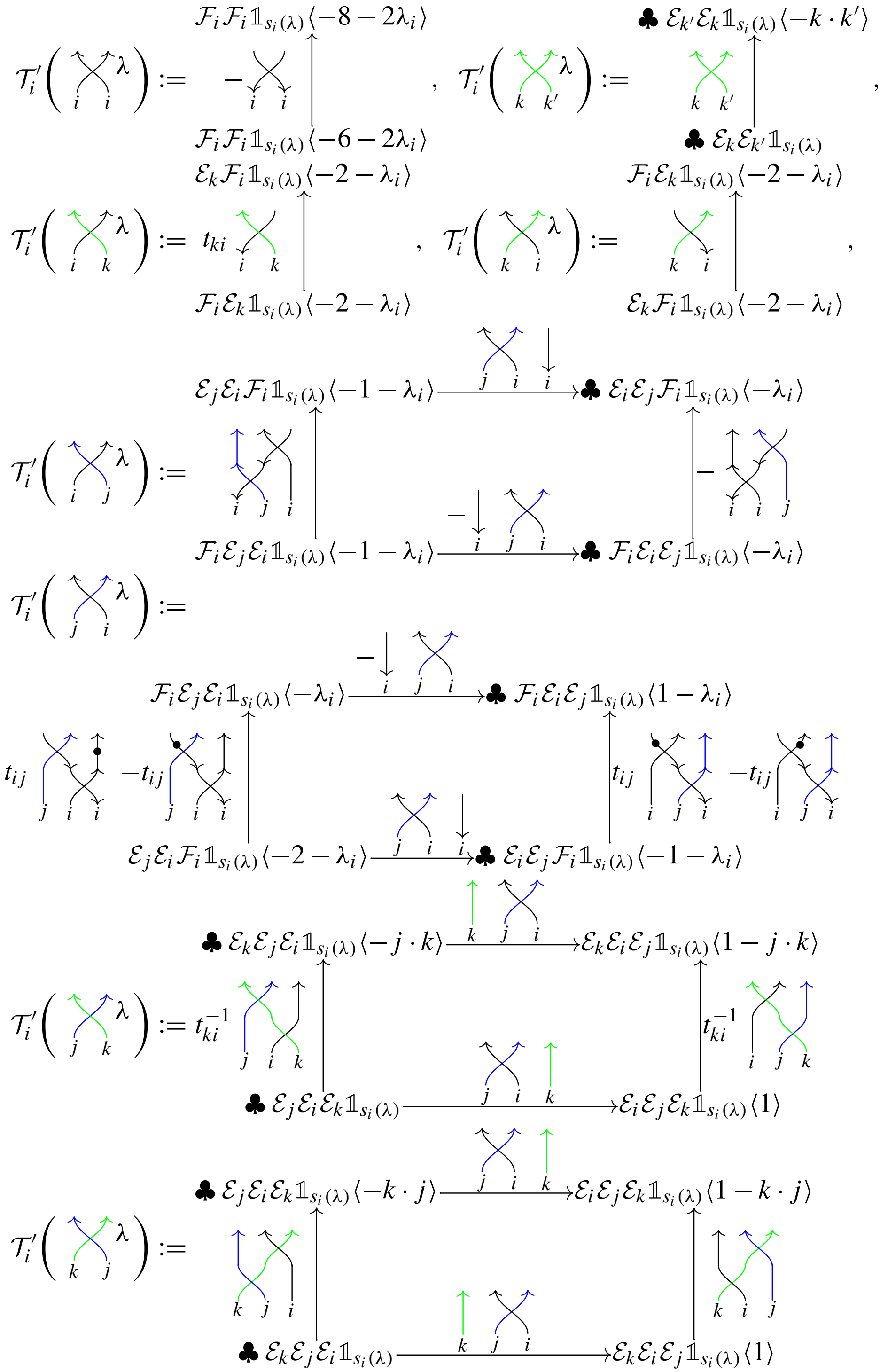
as anticommutative squares with arrows labeled by the corresponding maps, for example, the last diagram in Section 4B2 depicts a chain map between such complexes. In all cases, the chain map condition follows from the defining relations in $\dot{\mathcal{U}}_Q$.

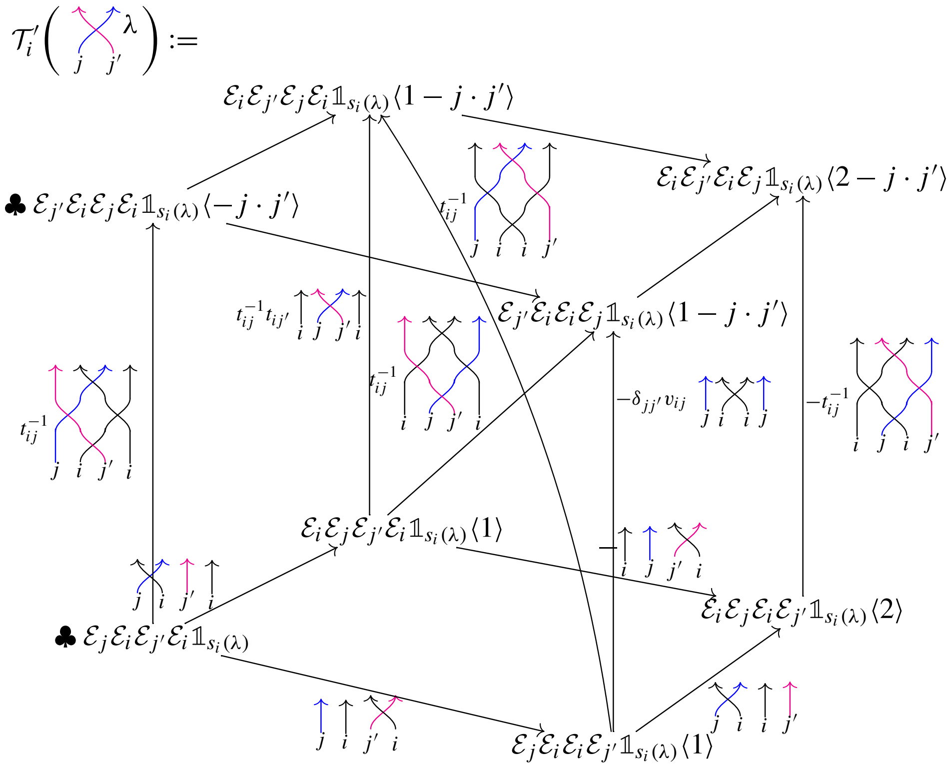
4B1. *Definition of $\mathcal{T}_{i,1}'$ on upwards dot 2-morphisms.* We have





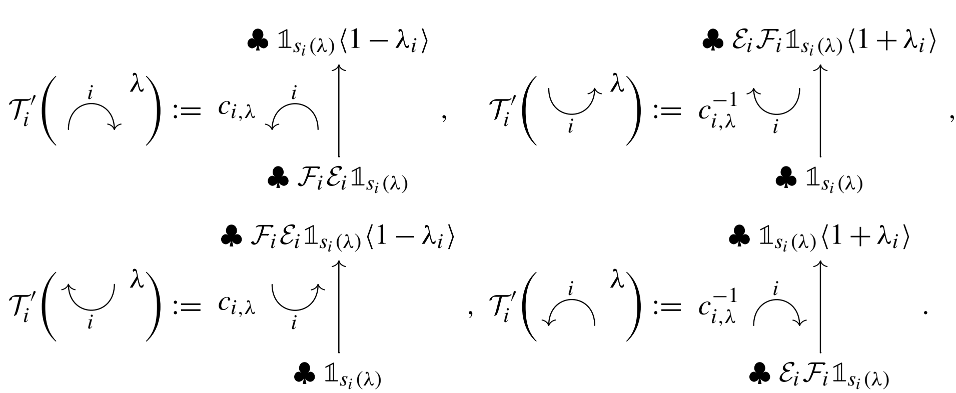
4B2. *Definition of $\mathcal{T}_{i,1}'$ on upwards crossing 2-morphisms.* We have


$\mathcal{F}_i\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -8-2\lambda_i\rangle$
$\mathcal{F}_i\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -6-2\lambda_i\rangle$
$\clubsuit\ \mathcal{E}_{k'}\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}\langle -k\cdot k'\rangle$
$\clubsuit\ \mathcal{E}_k\mathcal{E}_{k'}\mathbb{1}_{s_i(\lambda)}$
$\mathcal{E}_k\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -2-\lambda_i\rangle$
t_{ki}
$\mathcal{F}_i\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}\langle -2-\lambda_i\rangle$
$\mathcal{F}_i\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}\langle -2-\lambda_i\rangle$
$\mathcal{E}_k\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -2-\lambda_i\rangle$
$\mathcal{E}_j\mathcal{E}_i\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1-\lambda_i\rangle$
$\clubsuit\ \mathcal{E}_i\mathcal{E}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -\lambda_i\rangle$
$\mathcal{F}_i\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle -1-\lambda_i\rangle$
$\clubsuit\ \mathcal{F}_i\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle -\lambda_i\rangle$
$\mathcal{F}_i\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle -\lambda_i\rangle$
$\clubsuit\ \mathcal{F}_i\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle 1-\lambda_i\rangle$
t_{ij}
$-t_{ij}$
$\mathcal{E}_j\mathcal{E}_i\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -2-\lambda_i\rangle$
$\clubsuit\ \mathcal{E}_i\mathcal{E}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1-\lambda_i\rangle$
$\clubsuit\ \mathcal{E}_k\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle -j\cdot k\rangle$
$\mathcal{E}_k\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle 1-j\cdot k\rangle$
t_{ki}^{-1}
$\clubsuit\ \mathcal{E}_j\mathcal{E}_i\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}$
$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}\langle 1\rangle$
$\clubsuit\ \mathcal{E}_j\mathcal{E}_i\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}\langle -k\cdot j\rangle$
$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}\langle 1-k\cdot j\rangle$
$\clubsuit\ \mathcal{E}_k\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}$
$\mathcal{E}_k\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle$



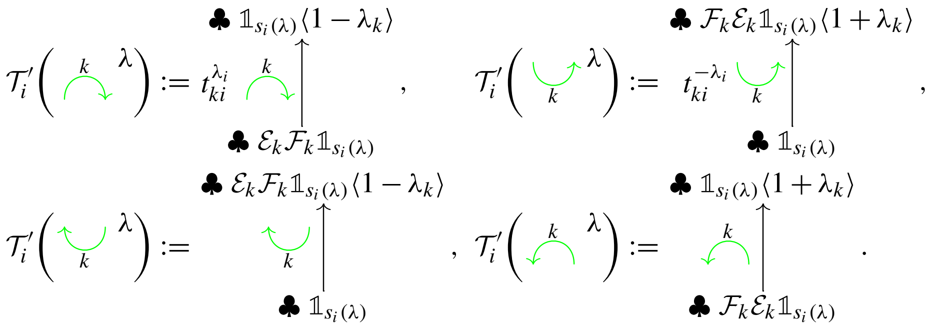
In this last diagram, we have omitted the differentials on the codomain, so as not to overcrowd it; they are given analogously to those in the domain, with $j \leftrightarrow j'$. Recall also that $v_{ij} := t_{ij}^{-1}t_{ji}$.

4B3. *Definition of $\mathcal{T}_{i,1}'$ on cap and cup 2-morphisms.* We have



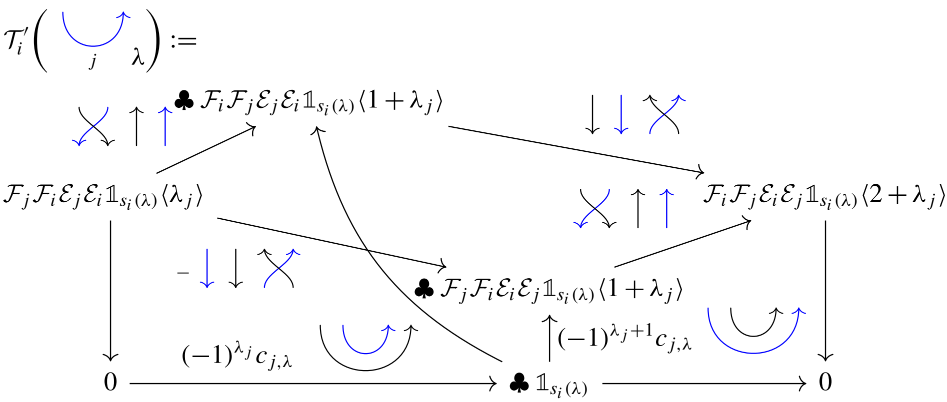
Note that the maps have the correct degree since the rightmost region in all the images is labeled by $s_i(\lambda)$, and $1\pm\langle i, s_i(\lambda)\rangle = 1\pm\langle i, \lambda-\lambda_i\alpha_i\rangle = 1\pm\lambda_i\mp 2\lambda_i = 1\mp\lambda_i$.

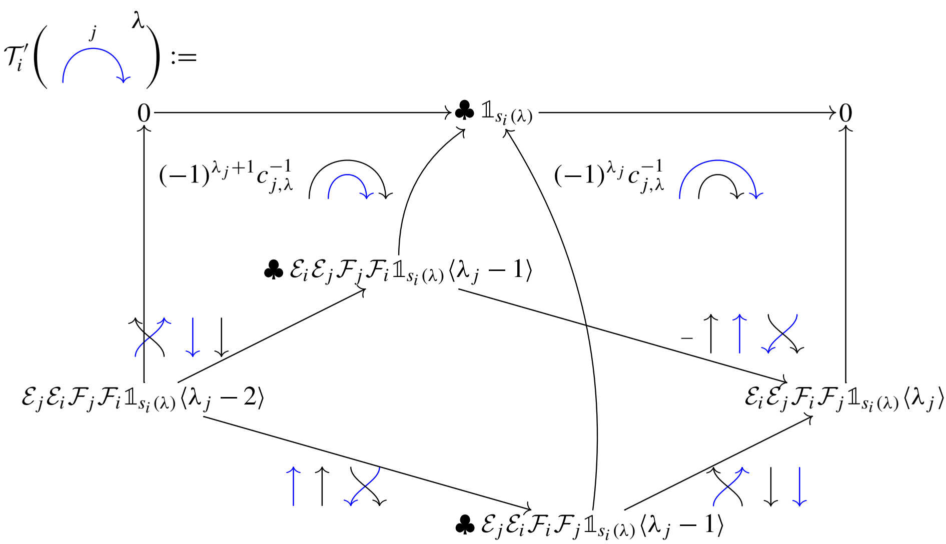
We have

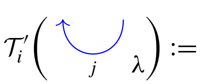


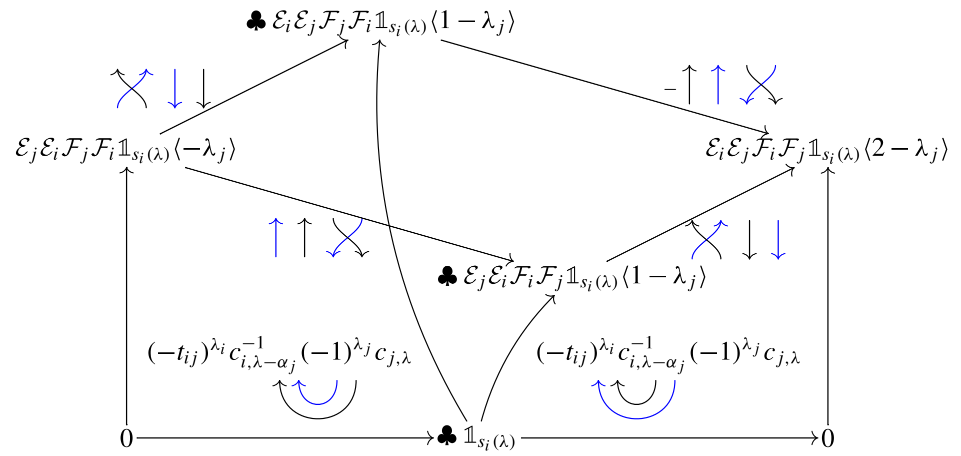
The maps have the correct degree since $1 \pm \langle k, s_i(\lambda)\rangle = 1 \pm \langle k, \lambda - \lambda_i\alpha_i\rangle = 1 \pm \lambda_k$.

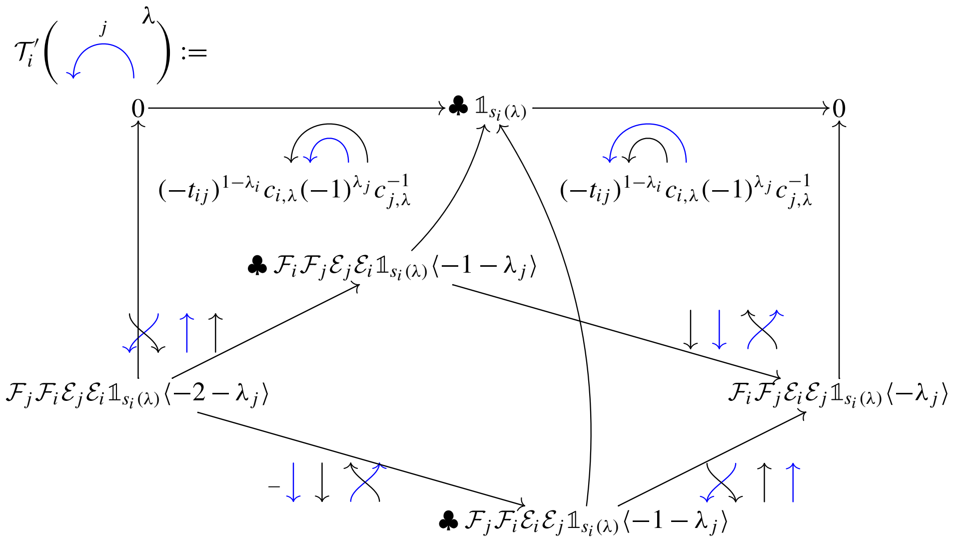
We have



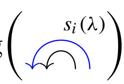








As above, a simple computation shows that the maps have the correct degree, e.g.,



$$\deg\Big(\cdots\Big) = 1 + \langle j, s_i(\lambda)\rangle + 1 + \langle i, s_i(\lambda) + \alpha_j$$
$$= 2 + \lambda_j - \lambda_i(j\cdot i) + \lambda_i - \lambda_i(i\cdot i) + i\cdot j = 1 + \lambda_j.$$

5. Proof that categorified Lusztig operators are well defined

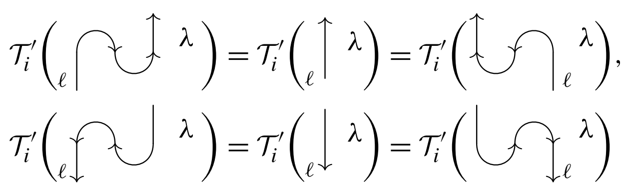
We show that $\mathcal{T}'_{i,1}$ is well defined, i.e., that $\mathcal{T}'_{i,1}$ preserves the defining relations in $\mathcal{U}_Q$, up to chain homotopy. We'll see, however, that many cases do not require a chain homotopy. For example, $\mathcal{T}'_{i,1}$ preserves on the nose any relation that does not involve j-labeled strands (for $i\cdot j=-1$), since here the complexes involved have only one nonzero term (in the same homological degree), precluding the existence of nontrivial chain homotopies. A complete proof consists of checking many cases for each relation, since $\mathcal{T}'_{i,1}$ is defined in a piecewise manner that depends on the connectivity of the graph associated to the simply-laced root datum.

To simplify this task, we will work with the presentation of $\mathcal{U}_Q$ implicit in Remark 3.4. Specifically, we view downward dot and sideways and downward crossing 2-morphisms as defined in terms of cap/cup 2-morphisms and their upward analogues (in the case of downward dots, we choose the presentation in terms of right-oriented caps/cups). It follows that $\mathcal{T}'_{i,1}$ is already fixed on these 2-morphisms (by 2-functoriality), and we record its value on these composite 2-morphisms in the Appendix. We make extensive use of these computations in the sections that follow.

Throughout, we will continue with our convention that the labels $j, j' \in I$ satisfy $i \cdot j = -1 = i \cdot j'$ and correspond to blue and magenta strands, while the labels $k, k' \in I$ satisfy $i \cdot k = 0 = i \cdot k'$ and correspond to green strands. We also let $\ell \in I$ denote an arbitrary label.

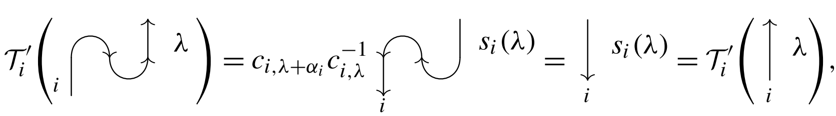
5A. *Adjunction relations.* We verify the right and left adjunction relations given in Definition 3.3(1).

Proposition 5.1. *For all $\ell \in I$ the equalities*

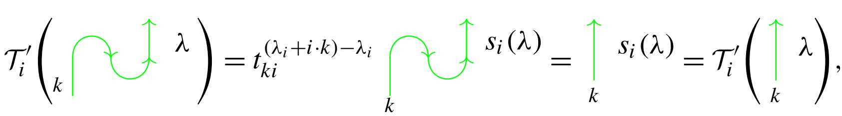


hold in $\mathsf{Com}(\mathcal{U}_Q)$.

Proof. When $\ell = i$ or $\ell = k$ with $i \cdot k = 0$, these relations follow from a straightforward computation, provided one is careful with the relevant parameters. For example, the first equality follows from the computation

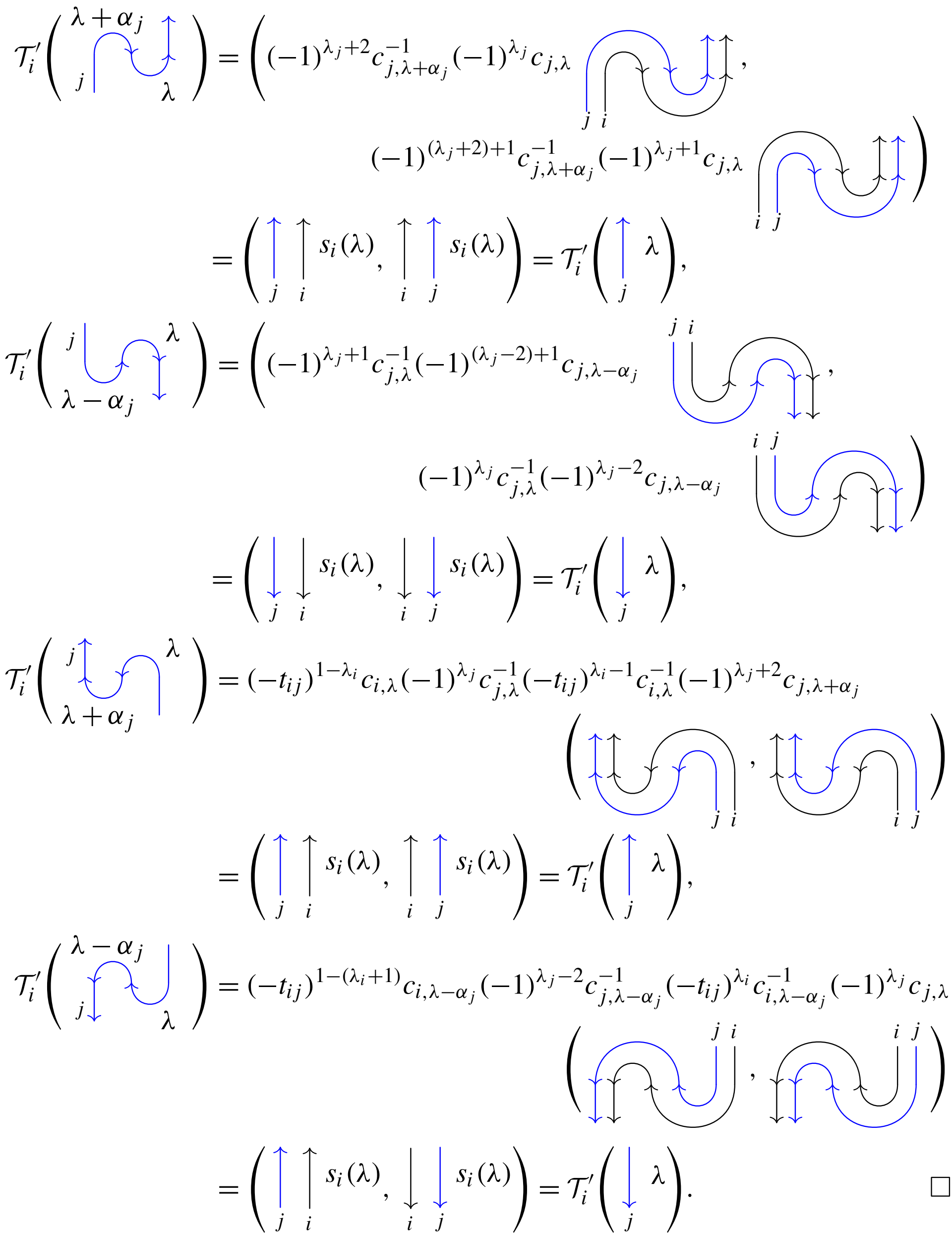


when $\ell = i$, and from



when $\ell = k$. We omit the other checks, as they are completely analogous.

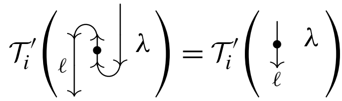
For $\ell = j$, the coefficients are more delicate. As $\mathcal{T}'_i(\mathcal{E}_j \mathbb{1}_\lambda)$ is a 2-term chain complex, we will use an ordered pair to describe its chain endomorphisms (with the convention that the first term in the pair corresponds to lower homological degree):



□

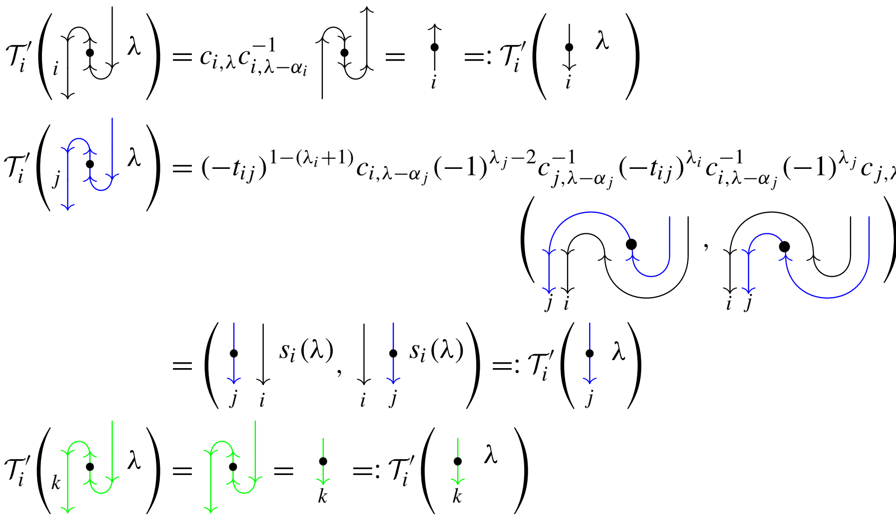
5B. ***Dot cyclicity.*** We verify the dot cyclicity relation given in Definition 3.3(2). Recall that, in our presentation given by Remark 3.4, the downward dot morphism is defined in terms of the upward dot morphism and *rightward* cap/cup morphisms. Dot cyclicity is then equivalent to the following.

Proposition 5.2. *For $\ell \in I$, the relation*



holds in $\mathsf{Com}(\mathcal{U}_Q)$.

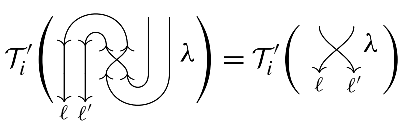
Proof. We compute the left-hand side, and verify the relations by comparing to the results of Section A.1, which give the value of $\mathcal{T}_i'$ on downward oriented dot 2-morphisms:



□

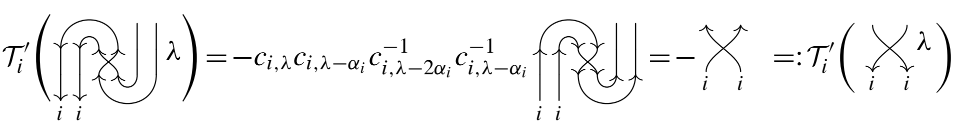
5C. *Crossing cyclicity.* We now verify the crossing cyclicity relations given in Definition 3.3(3). It suffices to prove cyclicity for the downward crossing, as the relations for the sideways crossings follow from this and the adjunction relations. We will use the value of the downward crossing from Section A.3, where (by definition) it is given in terms of the upward crossing and *rightward* cap/cup 2-morphisms.

Proposition 5.3. *For all $\ell, \ell' \in I$, the equation*

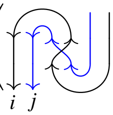


holds in $\mathsf{Com}(\mathcal{U}_Q)$.

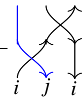
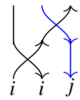
Proof. We compute the left-hand side, considering the three possibilities for each $\ell, \ell' \in I$ in relation to the fixed node $i \in I$. For both stands labeled i, we have

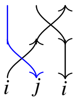
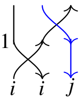
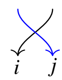


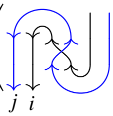
For strands labeled i and j, we have

$\mathcal{T}_i'\Big($  $\lambda\Big)$

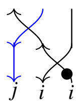
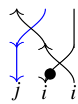
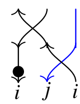
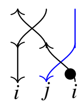
$= c^{-1}_{i,\lambda-\alpha_i-\alpha_j}(-t_{ij})^{1-(\lambda_i+1)}c_{i,\lambda-\alpha_j}(-1)^{\lambda_j-2}c^{-1}_{j,\lambda-\alpha_j}$

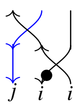
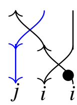
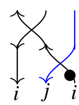
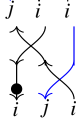
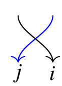
$c_{i,\lambda}(-t_{ij})^{\lambda_i-2}c^{-1}_{i,\lambda-\alpha_i-\alpha_j}(-1)^{\lambda_j+1}c_{j,\lambda-\alpha_i}\Big(-$  ,  $\Big)$

$=\Big(t_{ij}^{-1}t_{ji}^{-1}$  $, -t_{ij}^{-1}t_{ji}^{-1}$  $\Big) =: \mathcal{T}_i'\Big($  $\lambda\Big)$,

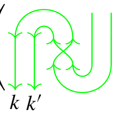
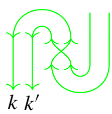
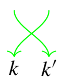
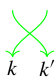
$\mathcal{T}_i'\Big($  $\lambda\Big)$

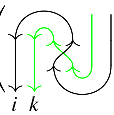
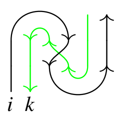
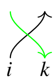
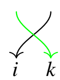
$= (-t_{ij})^{2-\lambda_i}c_{i,\lambda-\alpha_j}(-1)^{\lambda_j-1}c^{-1}_{j,\lambda-\alpha_i}c^{-1}_{i,\lambda}$

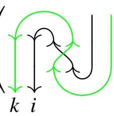
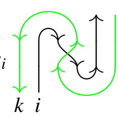
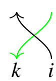
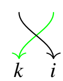
$t_{ij}c_{i,\lambda-\alpha_j}(-t_{ij})^{\lambda_i}c^{-1}_{i,\lambda-\alpha_j}(-1)^{\lambda_j}c_{j,\lambda}\Big($  $-$  ,  $-$  $\Big)$

$= t_{ij}^2t_{ji}\Big($  $-$  ,  $-$  $\Big) =: \mathcal{T}_i'\Big($  $\lambda\Big)$.

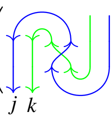
For crossings in which at least one strand is k-labeled and no strand is j-labeled, the relations are trivial to check. We compute

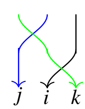
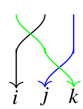
$\mathcal{T}_i'\Big($  $\lambda\Big) =$  $=$  $=: \mathcal{T}_i'\Big($  $\lambda\Big)$,

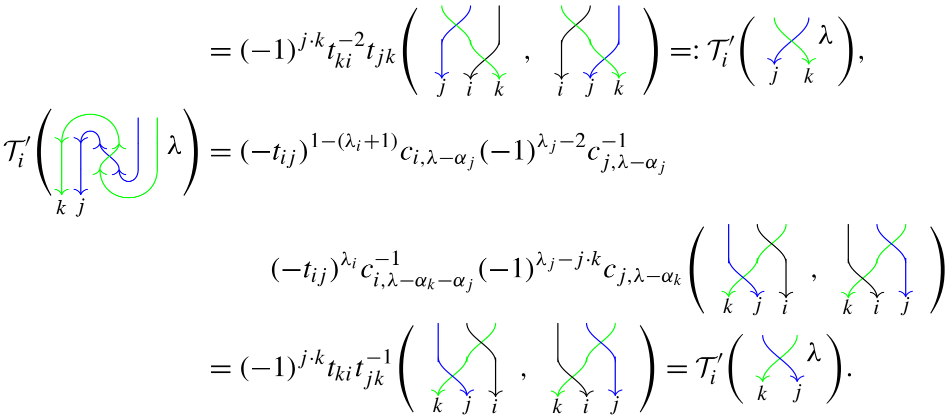
$\mathcal{T}_i'\Big($  $\lambda\Big) = c^{-1}_{i,\lambda-\alpha_i-\alpha_k}c_{i,\lambda}t_{ki}$  $= t_{ki}^2$  $=: \mathcal{T}_i'\Big($  $\lambda\Big)$,

$\mathcal{T}_i'\Big($  $\lambda\Big) = c_{i,\lambda-\alpha_k}c^{-1}_{i,\lambda-\alpha_i}$  $= t_{ki}^{-1}$  $=: \mathcal{T}_i'\Big($  $\lambda\Big)$.

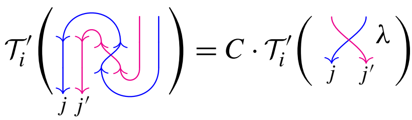
For strands labeled j and k, we compute

$\mathcal{T}_i'\Big($  $\lambda\Big) = (-t_{ij})^{1-(\lambda_i+1)}c_{i,\lambda-\alpha_j-\alpha_k}(-1)^{\lambda_j-2-j\cdot k}c^{-1}_{j,\lambda-\alpha_j-\alpha_k}$

$t_{ki}^{-1}(-t_{ij})^{\lambda_i}c^{-1}_{i,\lambda-\alpha_j}(-1)^{\lambda_j}c_{j,\lambda}\Big($  ,  $\Big)$



Finally, in the case of a crossing between strands labeled j and j', it's clear that



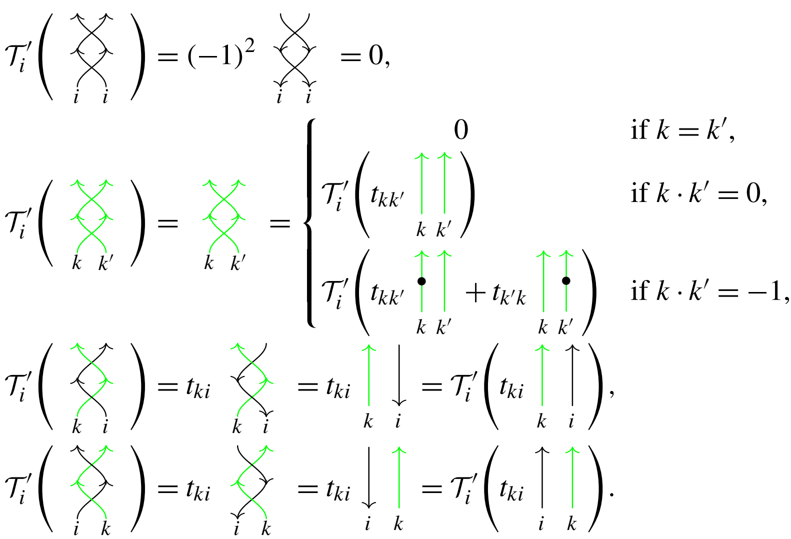
for some scalar C. A direct computation shows that

$$C = t_{ij}^{-1} t_{ij'} c_{i,\lambda-\alpha_{j'}} c^{-1}_{i,\lambda-\alpha_j} c_{j,\lambda} c^{-1}_{j,\lambda-\alpha_{j'}} c_{j',\lambda-\alpha_j} c^{-1}_{j',\lambda} (c_{j',\lambda-\alpha_j} c^{-1}_{j',\lambda} c_{j,\lambda} c^{-1}_{j,\lambda-\alpha_{j'}})^{-1} = 1. \qquad \square$$

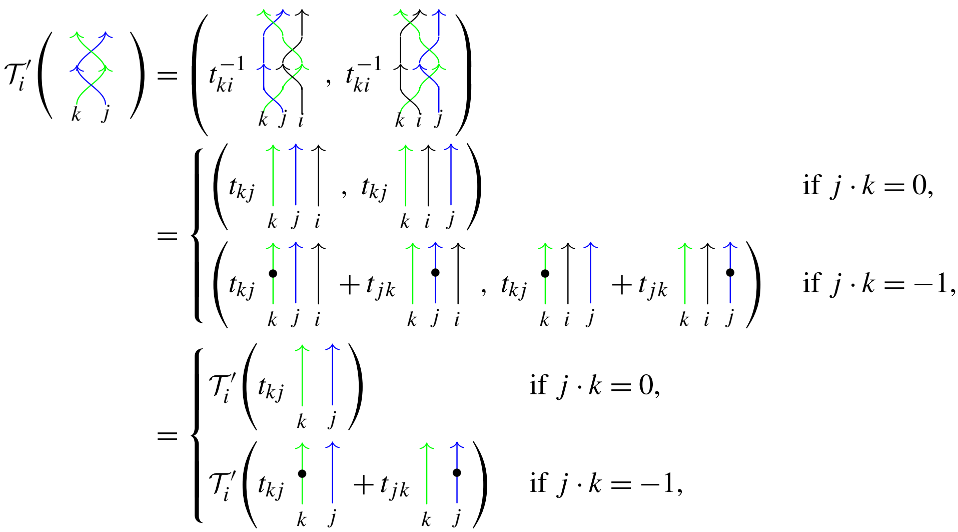
5D. *Quadratic KLR.*

Proposition 5.4. *$\mathcal{T}_i'$ preserves the quadratic KLR relation.*

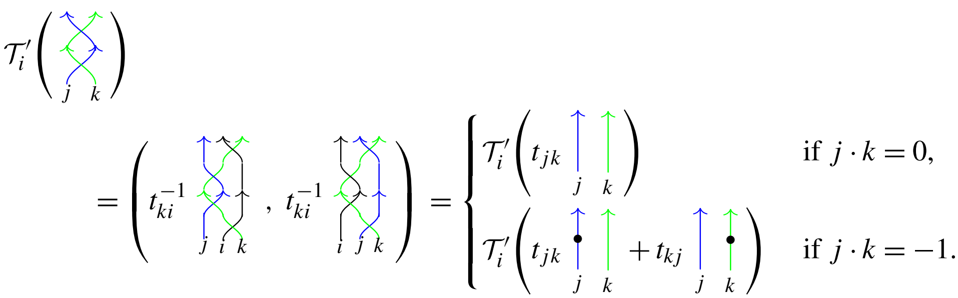
Proof. We verify (4) in Definition 3.3, first considering the cases that do not require homotopies. We compute



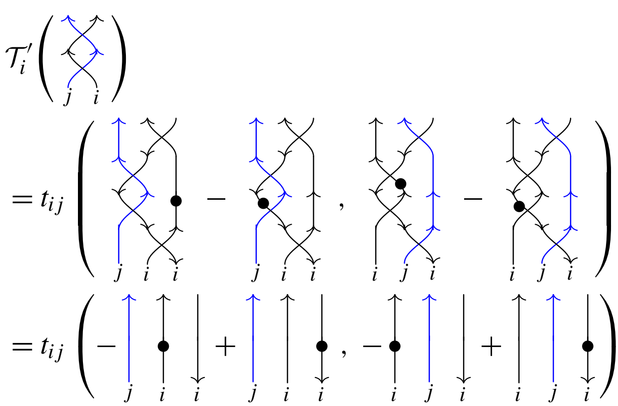
Our next four cases concern endomorphisms of chain complexes concentrated in two adjacent homological degrees; we denote endomorphisms of such complexes using ordered pairs. We compute

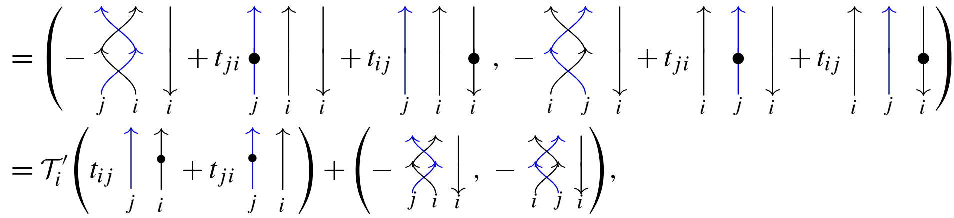


and similarly

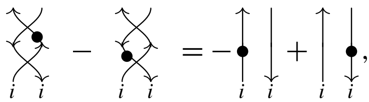


The remaining cases only hold up to chain homotopy. We compute

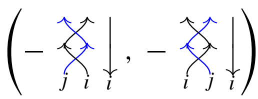




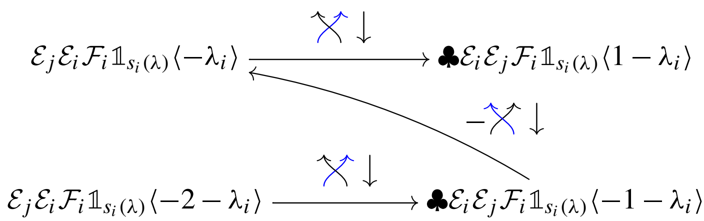
where in the second step we make use of the equality



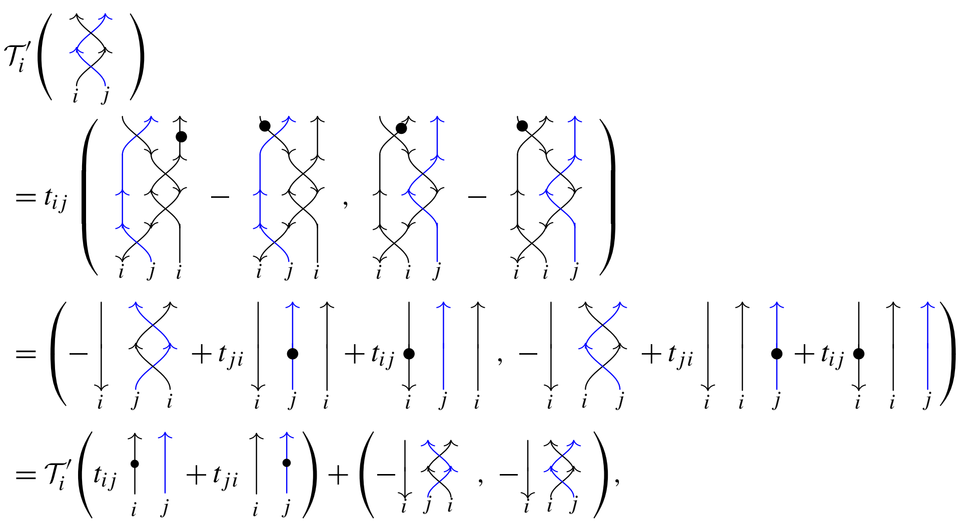
which holds in any weight. The result now follows since the chain endomorphism



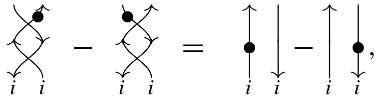
is null-homotopic with homotopy $h : \mathcal{T}_i'(\mathcal{E}_j\mathcal{E}_i\mathbb{1}_\lambda) \to \mathcal{T}_i'(\mathcal{E}_j\mathcal{E}_i\mathbb{1}_\lambda\langle 2\rangle)$ given by



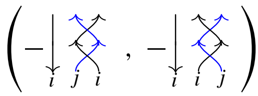
We similarly compute



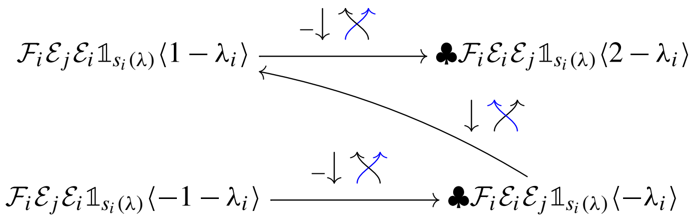
where in this case we use the equality



which holds in any weight. The relation is verified since the chain endomorphism



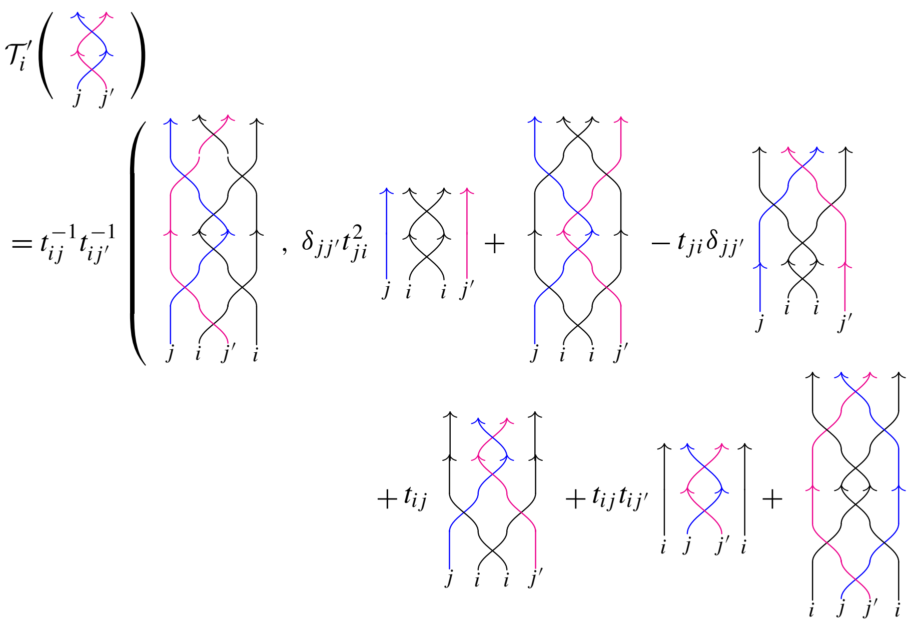
is null-homotopic, with homotopy given by

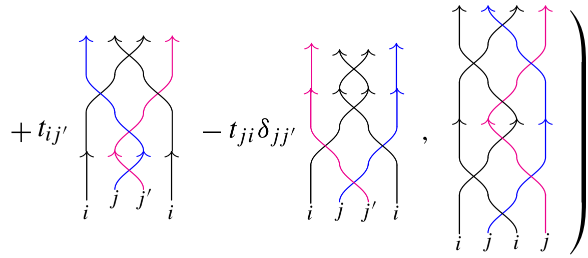


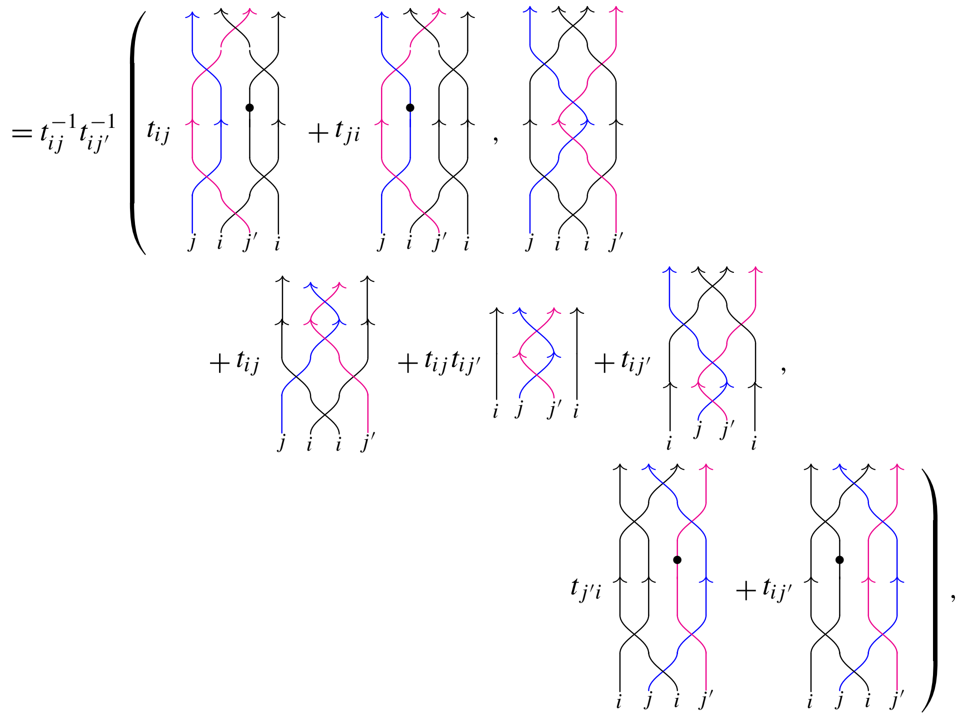
Finally, we compute the case in which strands are labeled j

$$\mathcal{T}_i'(\mathcal{E}_j\mathcal{E}_{j'}\mathbb{1}_\lambda) = \clubsuit\mathcal{E}_j\mathcal{E}_i\mathcal{E}_{j'}\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}$$
$$\to \mathcal{E}_j\mathcal{E}_i\mathcal{E}_i\mathcal{E}_{j'}\mathbb{1}_{s_i(\lambda)}\langle 1\rangle \oplus \mathcal{E}_i\mathcal{E}_j\mathcal{E}_{j'}\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle 1\rangle \to \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i\mathcal{E}_{j'}\mathbb{1}_{s_i(\lambda)}\langle 2\rangle,$$

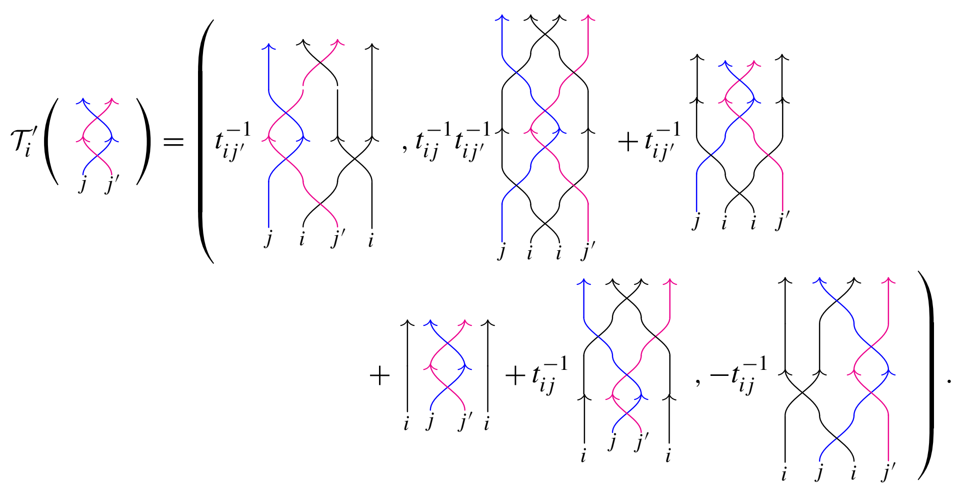
and we denote the relevant endomorphism as an ordered triple. We abuse notation for the component mapping between the terms in homological degree one: technically this should be given by a 2×2 matrix, but, in the interest of space, we add all terms in the relevant matrix, as the components are distinguished by their (co)domains.



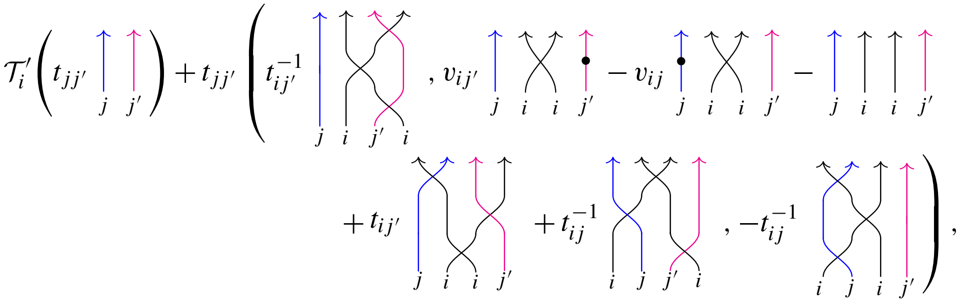




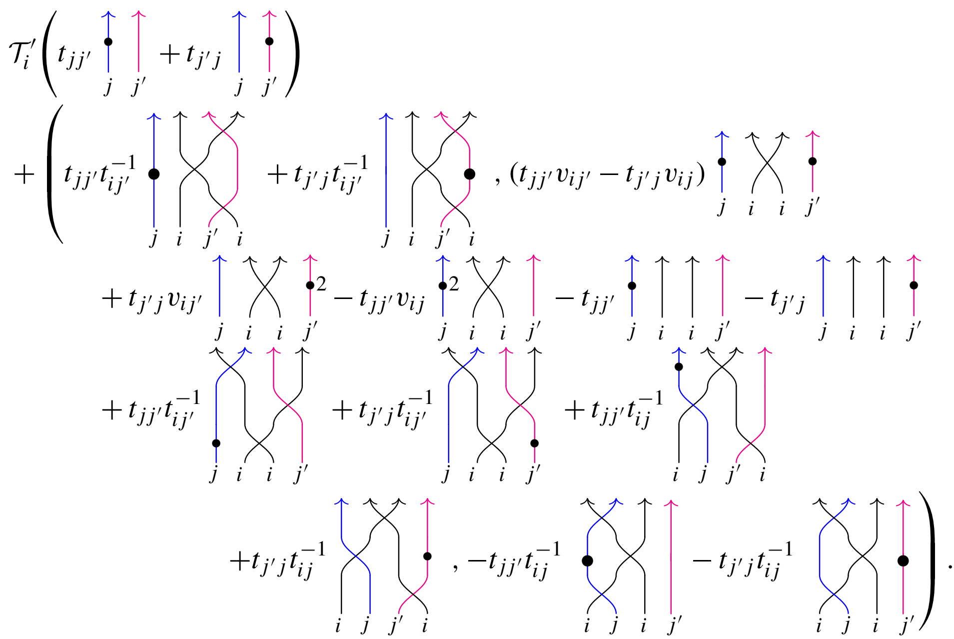
which vanishes if $j = j'$, as desired. If $j \neq j'$, we instead have



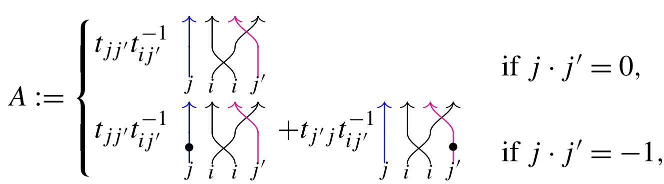
If $j \cdot j' = 0$ the right-hand side of the above simplifies to

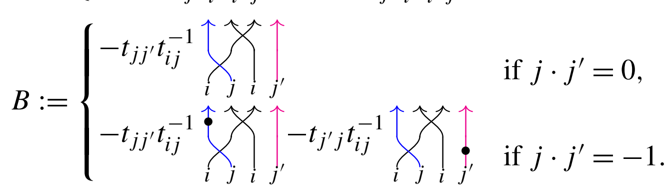


and if $j \cdot j' = -1$ it instead simplifies to

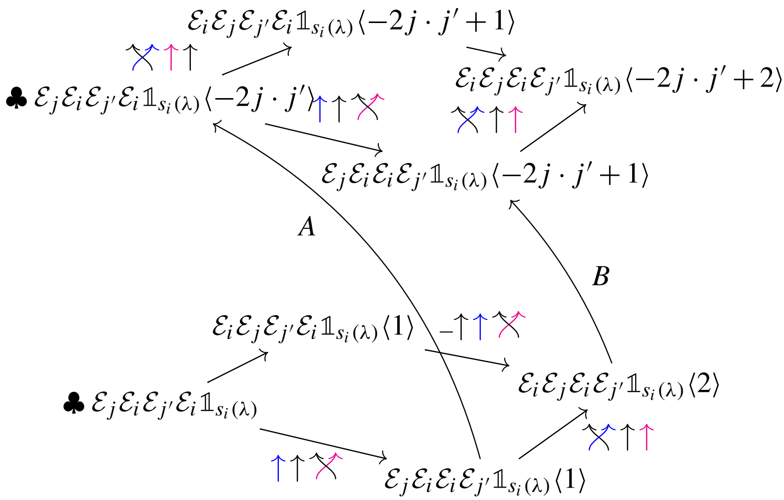


In both cases, the second summand (the "error term" preventing the relation from holding on the nose) is null-homotopic. The nonzero terms of both null-homotopies are defined as





and seen in the following diagram:

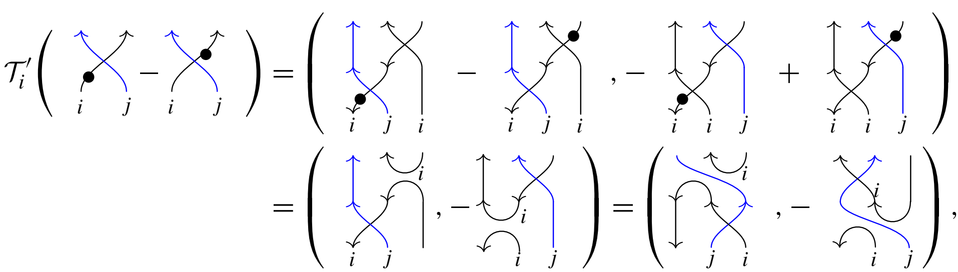


□

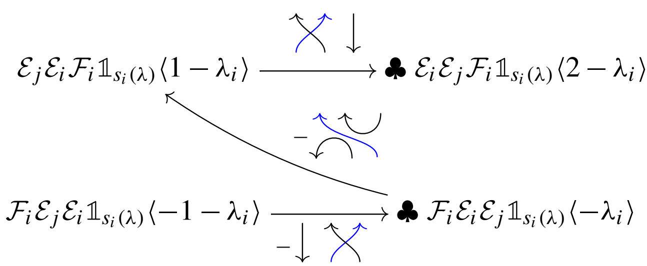
5E. *Dot slide.*

Proposition 5.5. *$\mathcal{T}_i'$ preserve the KLR dot sliding relation.*

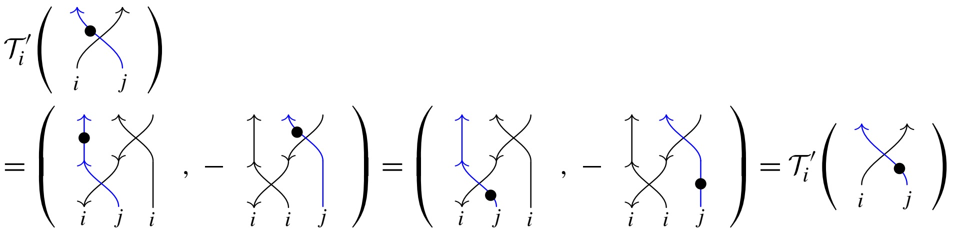
Proof. We verify (5) from Definition 3.3, only exhibiting the computations for crossings involving j- and j'-labeled strands (for $j \cdot i = -1 = j' \cdot i$), as all others are completely straightforward. For ij-crossings with dotted i-labeled strand, we compute



which is null-homotopic, as desired, via the homotopy

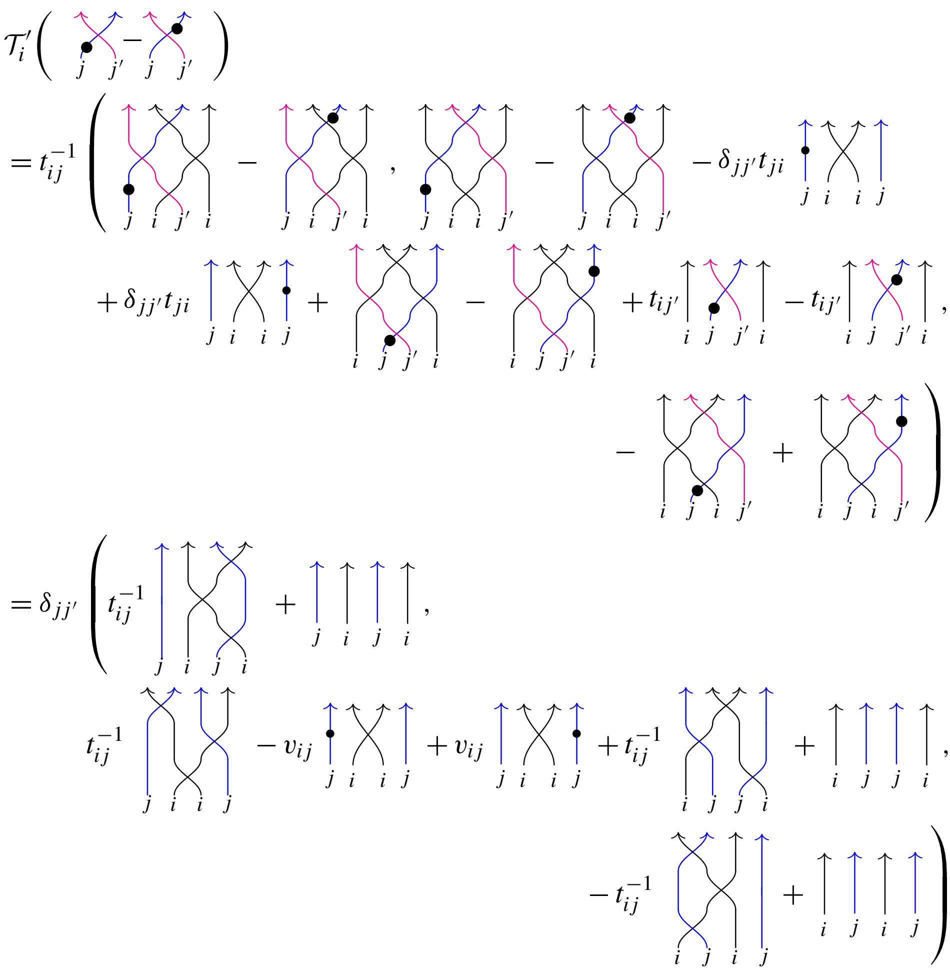


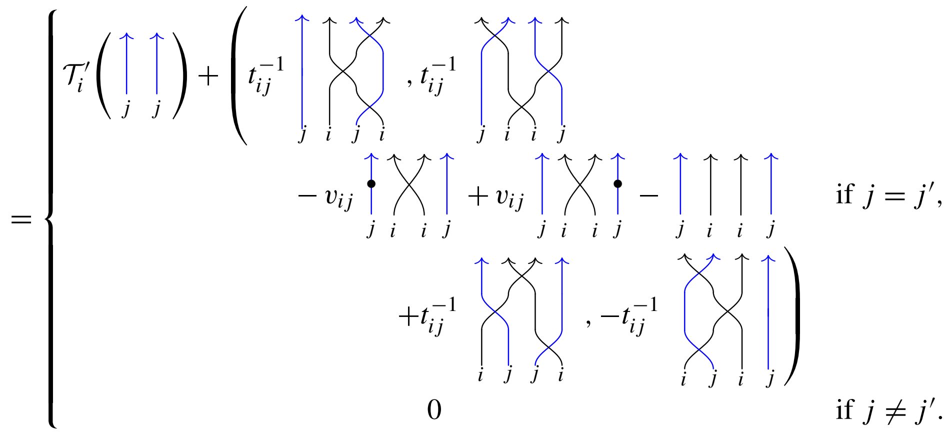
For the ij-crossing with dotted j-labeled strand, we have



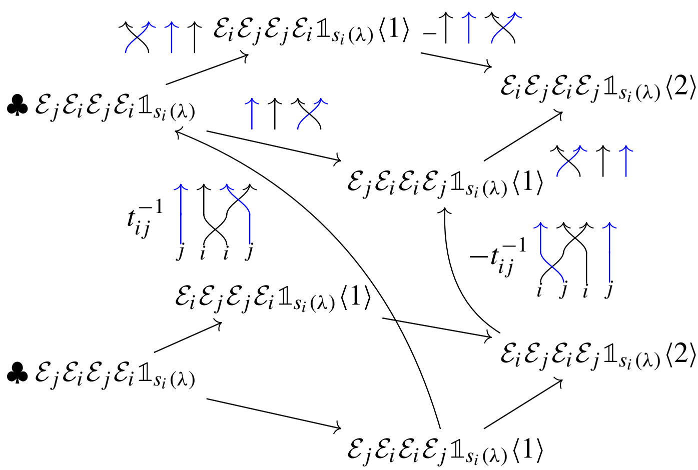
For dotted ji-crossings, neither case requires a chain homotopy, so we omit the computations, which are straightforward.

Finally, we consider dotted jj'-crossings. As in the proof of Proposition 5.4, our chain maps here map between complexes supported in three adjacent homological degrees, and we denote them as ordered triples. We have

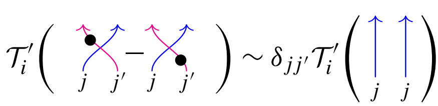




The relation thus holds on the nose unless $j = j'$, in which case the "error term" is null-homotopic, with homotopy given by



The verification that



is almost identical to the above case, so we omit the details. □

5F. *Cubic KLR.*

Proposition 5.6. *$\mathcal{T}_i'$ preserves the cubic KLR relation.*

Proof. We verify (6) in Definition 3.3, the "Reidemeister III"-like KLR relation. There are 27 cases to consider, depending on whether the label ℓ of each strand satisfies $i \cdot \ell = 2$, -1, or 0. To cover multiple cases at once, we will use the notation

$$\Delta_{abc} = \begin{cases} t_{ab} & \text{if } a = c \text{ and } a \cdot b = -1, \\ 0 & \text{else.} \end{cases}$$

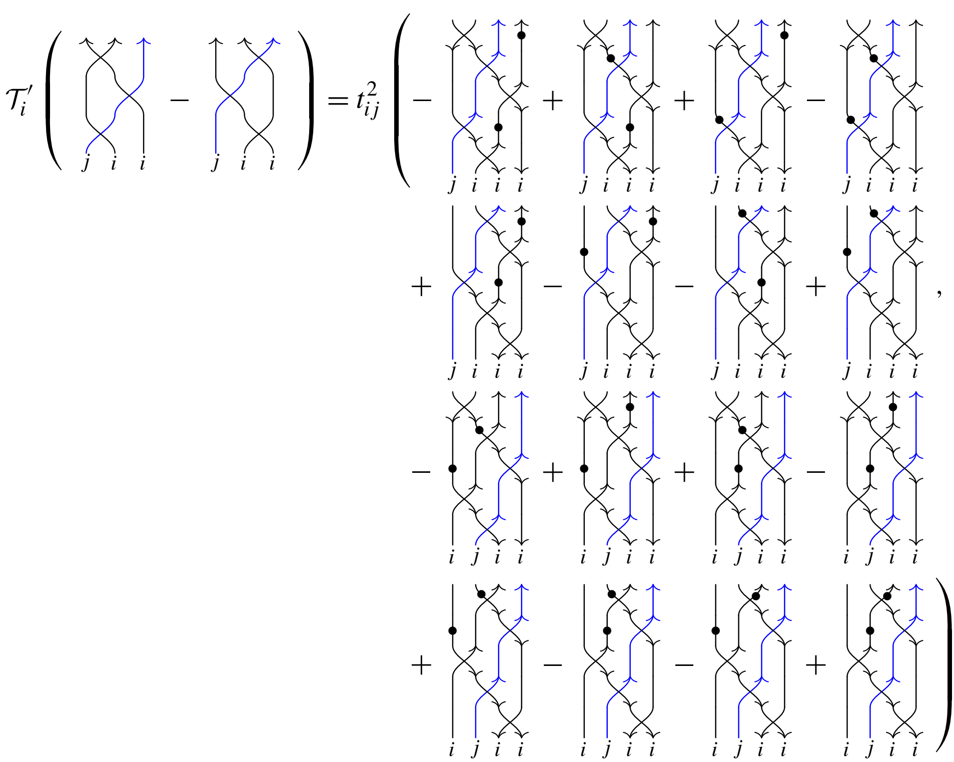
Note that $\Delta_{abc} = \Delta_{cba}$.

The relation holds on the nose (does not require a nonzero homotopy), except for the strand labelings in the list

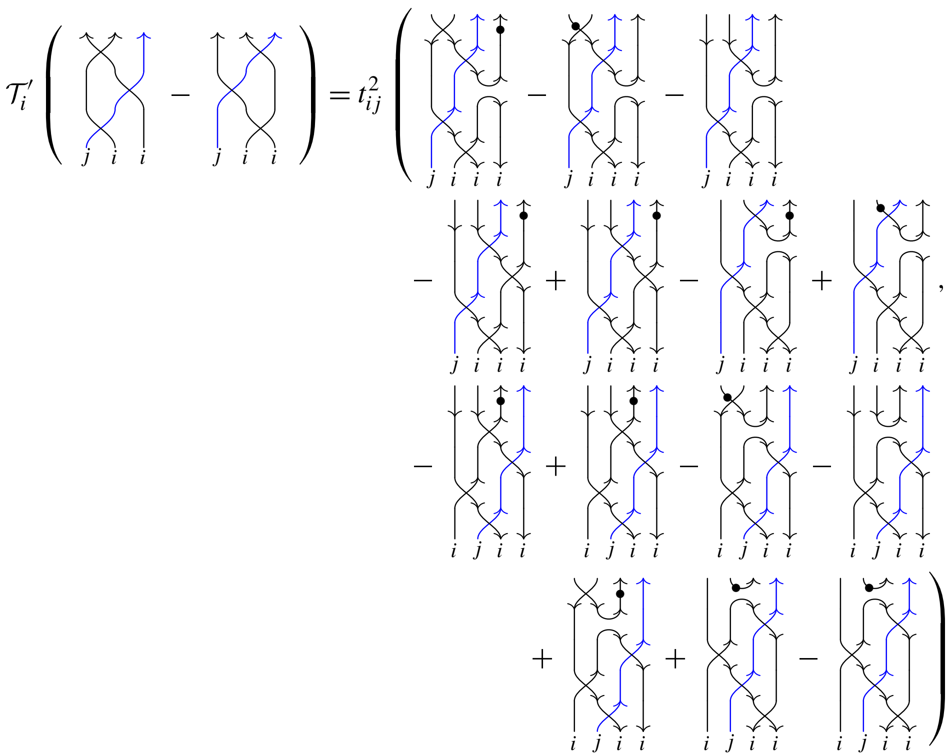
$$iji, \quad jkj', \quad jij', \quad jj'j,$$

where we continue with our conventions for strand labelings ($i \cdot j = -1 = i \cdot j'$ and $i \cdot k = 0 = i \cdot k'$). In the interest of space, we will explicitly exhibit three representative cases that do not require homotopies (to give the flavor of the computations required), exhibit the homotopy and verify the relation in the iji-labeled case, and exhibit the homotopy (but not include all the computations involved for the verification) in the remaining three cases.

In the jii-labeled case, the relation holds on the nose via the following computation, where, as above, we denote the chain map as an ordered pair:

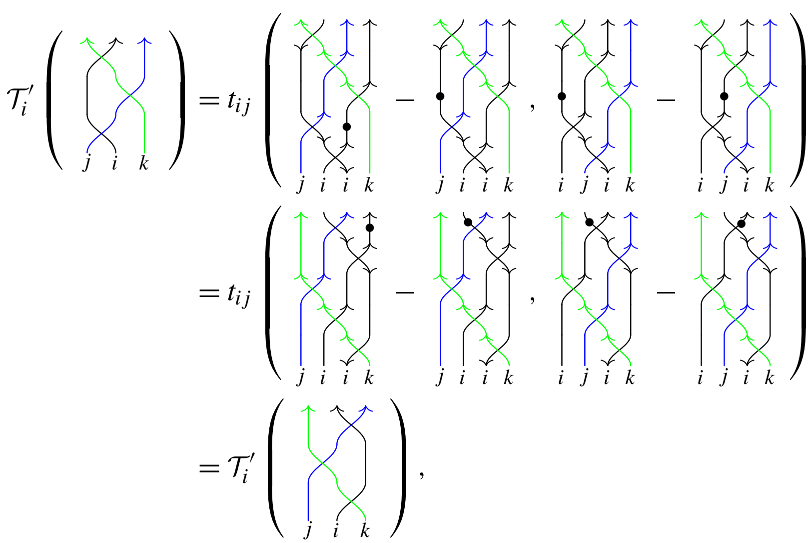


To simplify, we use the dot slide relation to move all dots to the top, and apply the cubic KLR relation to cancel terms, arriving at



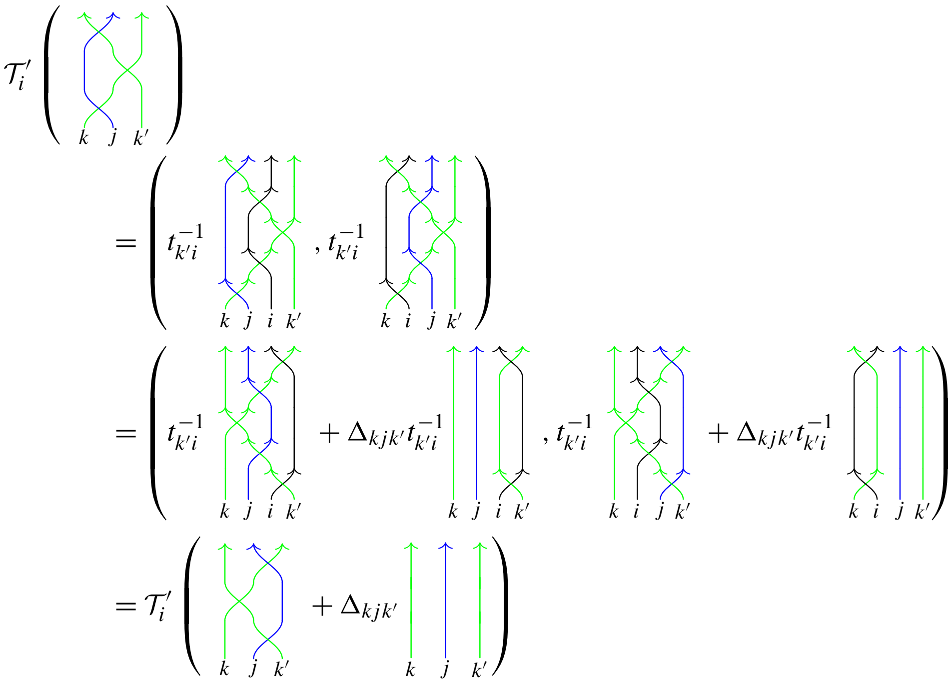
$$= 0.$$

For the jik-labeled case, we have



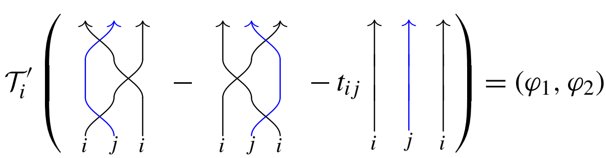
where in the middle step we use dot sliding and (3-2).

The kjk'-labeled case is given by

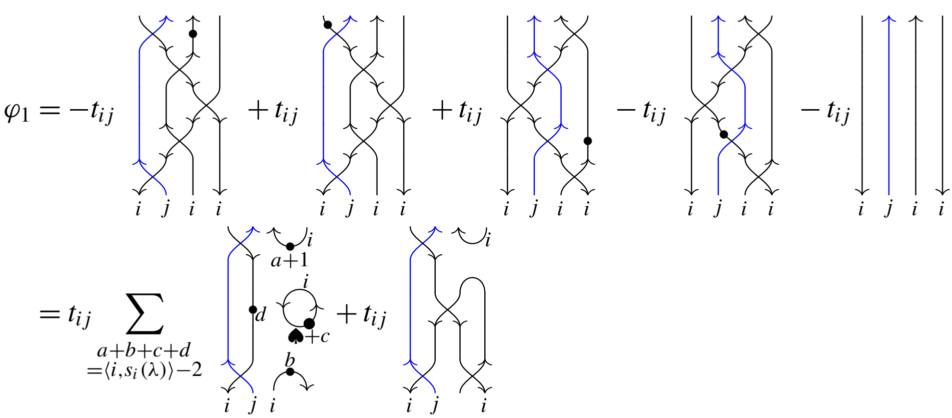


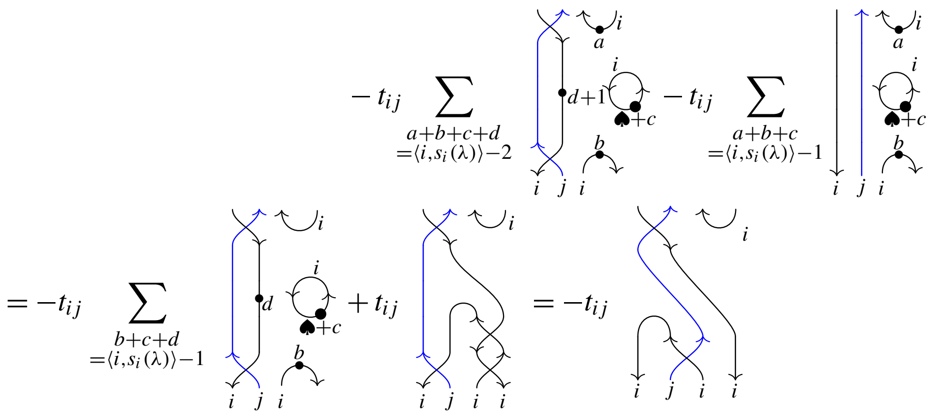
and all others that don't require a nonzero homotopy are similar to these cases.

We now consider the cases listed above that require chain homotopies. Considering the iji-labeled case, we compute that

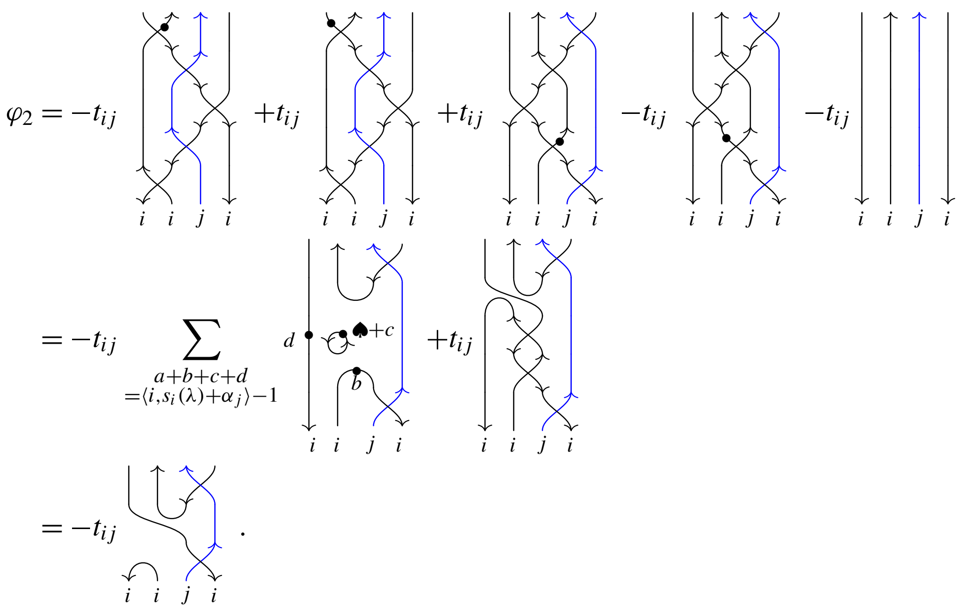


where

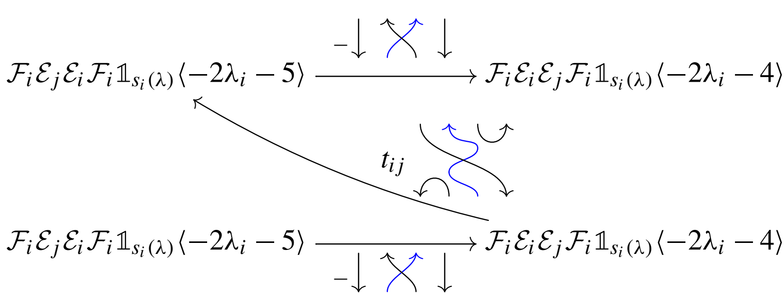




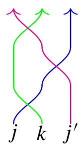
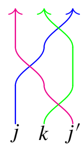
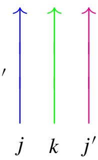
and



In both computations, we make extensive use of (3-2). It follows that this chain map is null-homotopic, with homotopy given by



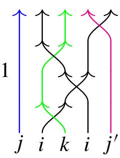
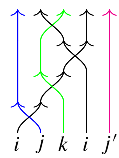
For the remaining cases, we provide the explicit homotopy between the relevant maps. We have

$$\mathcal{T}_i'\left(\right.$$  $$\left.\right) \sim \mathcal{T}_i'\left(\right.$$  $$+\Delta_{jkj'}$$  $$\left.\right)$$

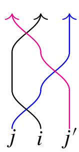
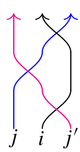
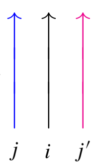
via the homotopy

$$\begin{array}{ccccc}
\clubsuit\mathcal{E}_{j'}\mathcal{E}_i\mathcal{E}_k\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)} & \longrightarrow & \begin{array}{c}\mathcal{E}_i\mathcal{E}_{j'}\mathcal{E}_k\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle 1\rangle \\ \oplus\mathcal{E}_{j'}\mathcal{E}_i\mathcal{E}_k\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle\end{array} & \longrightarrow & \mathcal{E}_i\mathcal{E}_{j'}\mathcal{E}_k\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle 2\rangle \\
 & \nwarrow^{(0\ h^1)} & & \nwarrow^{\binom{0}{h^2}} & \\
\clubsuit\mathcal{E}_j\mathcal{E}_i\mathcal{E}_k\mathcal{E}_{j'}\mathcal{E}_i\mathbb{1}_{s_i(\lambda)} & \longrightarrow & \begin{array}{c}\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k\mathcal{E}_{j'}\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle 1\rangle \\ \oplus\mathcal{E}_j\mathcal{E}_i\mathcal{E}_k\mathcal{E}_i\mathcal{E}_{j'}\mathbb{1}_{s_i(\lambda)}\langle 1\rangle\end{array} & \longrightarrow & \mathcal{E}_i\mathcal{E}_j\mathcal{E}_k\mathcal{E}_i\mathcal{E}_{j'}\mathbb{1}_{s_i(\lambda)}\langle 2\rangle
\end{array}$$

with

$$h^1 = \Delta_{jkj'}t_{ki}^{-1}t_{ij}^{-1}$$  $$,\quad h^2 = -\Delta_{jkj'}t_{ki}^{-1}t_{ij}^{-1}$$  .

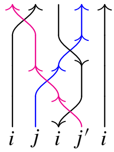
For the jij'-labeled case, we have

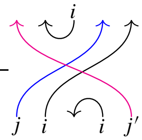
$$\mathcal{T}_i'\left(\right.$$  $$\left.\right) \sim \mathcal{T}_i'\left(\right.$$  $$+\Delta_{jij'}$$  $$\left.\right)$$

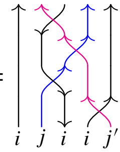
with chain homotopy given by (here we indicate the signs on the differential since they are not the usual ones, due to the homological shift on $\mathcal{T}_i'(\mathcal{E}_i\mathbb{1}_\lambda)$)

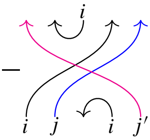
$$\begin{array}{ccc}
\mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_i\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle -1-\lambda_i\rangle & & \mathcal{E}_j\mathcal{E}_i\mathcal{F}_i\mathcal{E}_{j'}\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle -1-\lambda_i\rangle \\
\downarrow \binom{+}{-} & \nwarrow^{(h_1^1\ h_2^1)} & \downarrow \binom{+}{-} \\
\begin{array}{c}\mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_i\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle -\lambda_i\rangle \\ \oplus\mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_i\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle -\lambda_i\rangle\end{array} & & \begin{array}{c}\mathcal{E}_i\mathcal{E}_j\mathcal{F}_i\mathcal{E}_{j'}\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle -\lambda_i\rangle \\ \oplus\mathcal{E}_j\mathcal{E}_i\mathcal{F}_i\mathcal{E}_i\mathcal{E}_{j'}\mathbb{1}_{s_i(\lambda)}\langle -\lambda_i\rangle\end{array} \\
\downarrow (+\ +) & \nwarrow^{\binom{h_1^2}{h_2^2}} & \downarrow (+\ +) \\
\mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_i\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle 1-\lambda_i\rangle & & \mathcal{E}_i\mathcal{E}_j\mathcal{F}_i\mathcal{E}_i\mathcal{E}_{j'}\mathbb{1}_{s_i(\lambda)}\langle 1-\lambda_i\rangle
\end{array}$$

for

$h_1^1 = -$ 

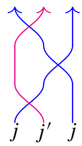
, $h_2^1 = -$ 

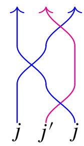
, $h_1^2 =$ 

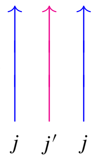
, $h_2^2 = -$ 

.

For the final case with $jj'j$-labeled strands, we have

$\mathcal{T}_i'\Big($ 

$\Big) \sim \mathcal{T}_i'\Big($ 

$+ \Delta_{jj'j}$ 

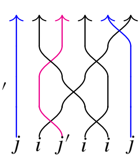
$\Big)$

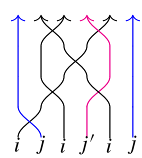
The relevant homotopy maps between chain complexes given as the triple composition of two term chain complexes, and is nonzero only when $j \neq j'$. We'll exhibit the homotopy assuming this, and that $j \cdot j' = 0$, as the homotopy is more involved when $j \cdot j' = -1$. The latter is only possible when the graph Γ corresponding to our Cartan datum has a length-three cycle (which in finite- or affine-type only occurs for $\widehat{\mathfrak{sl}_3}$).

We give the relevant homotopy, where, in the interest of space, we follow [26] in defining $\mathcal{E}_{\ell_1 \cdots \ell_k} := \mathcal{E}_{\ell_1} \cdots \mathcal{E}_{\ell_k}$; we also indicate the signs of the nonzero terms in the differentials, which are given up to sign by the relevant ji- (or $j'i$-) crossing:

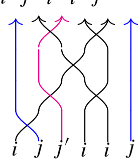
$$
\clubsuit \mathcal{E}_{jij'iji}
\underset{\begin{pmatrix}0 & 0 & h_3^1\end{pmatrix}}{\overset{\begin{pmatrix}+\\+\\+\end{pmatrix}}{\rightleftarrows}}
\begin{matrix}\mathcal{E}_{ijj'iji}\langle 1\rangle\\ \oplus\mathcal{E}_{jiij'ji}\langle 1\rangle\\ \oplus\mathcal{E}_{jij'iij}\langle 1\rangle\end{matrix}
\underset{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & h_{23}^2\\ 0 & h_{32}^2 & h_{33}^2\end{pmatrix}}{\overset{\begin{pmatrix}- & + & 0\\ - & 0 & +\\ 0 & - & +\end{pmatrix}}{\rightleftarrows}}
\begin{matrix}\mathcal{E}_{ijij'ji}\langle 2\rangle\\ \oplus\mathcal{E}_{ijj'iij}\langle 2\rangle\\ \oplus\mathcal{E}_{jiij'ij}\langle 2\rangle\end{matrix}
\underset{\begin{pmatrix}0\\0\\h_3^3\end{pmatrix}}{\overset{(+\ -\ +)}{\rightleftarrows}}
\clubsuit \mathcal{E}_{ijij'ij}\langle 3\rangle
$$

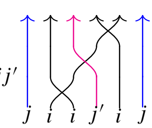
Herein, the maps in the homotopy are given by

$h_3^1 = t_{ij}^{-1} v_{ij} t_{ij'}^{-1} t_{jj'}$ 

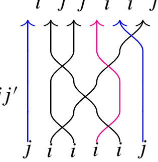
, $h_3^3 = -t_{ij}^{-1} v_{ij} t_{ij'}^{-1} t_{jj'}$ 

,

$h_{32}^2 = -t_{ij}^{-1} v_{ij} t_{ij'}^{-1} t_{jj'}$ 

, $h_{33}^2 = -v_{ij} t_{ij'}^{-1} t_{jj'}$ 

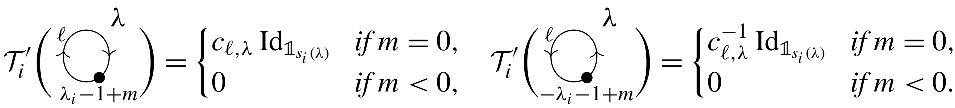
,

$h_{23}^2 = -t_{ij}^{-1} v_{ij} t_{ij'}^{-1} t_{jj'}$ 

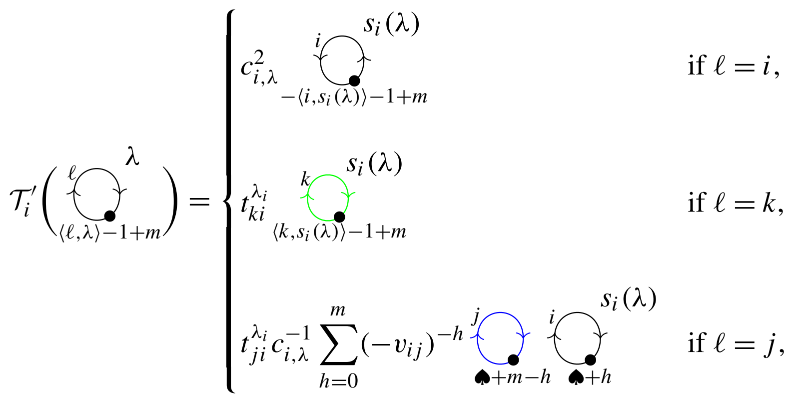
. □

5G. ***Bubble relations.*** We now verify that $\mathcal{T}_i'$ preserves (8) in Definition 3.3.

Proposition 5.7. *We have*

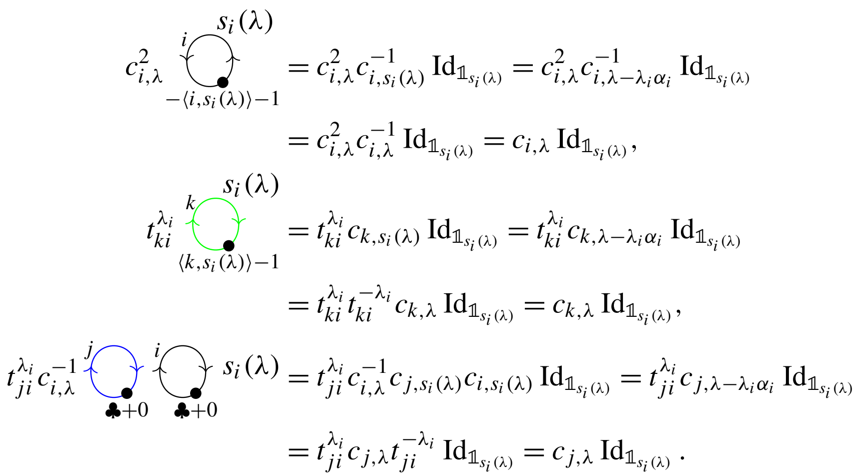


Proof. We'll give the proof only in the clockwise case, as the counterclockwise case is completely analogous. The computations in Section A.4 show that



which immediately gives the result in the $m<0$ case.

For $m=0$, we compute



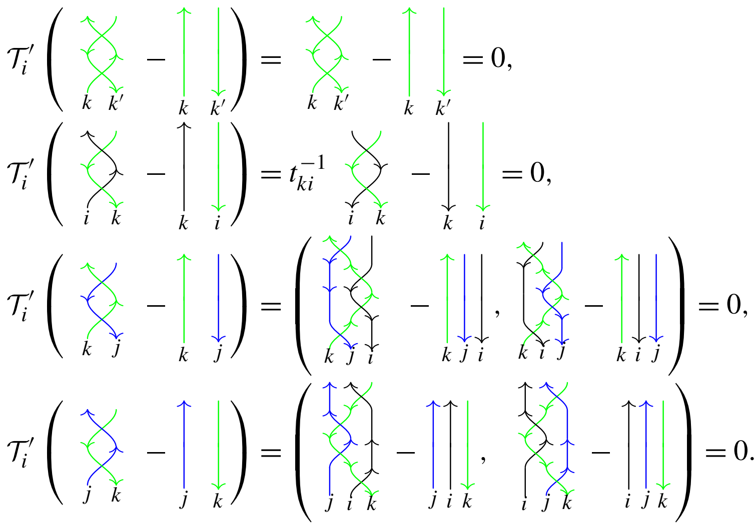
□

In Section A.4 we verify that the infinite Grassmannian relations from Section 3C2 are preserved by $\mathcal{T}_i'$.

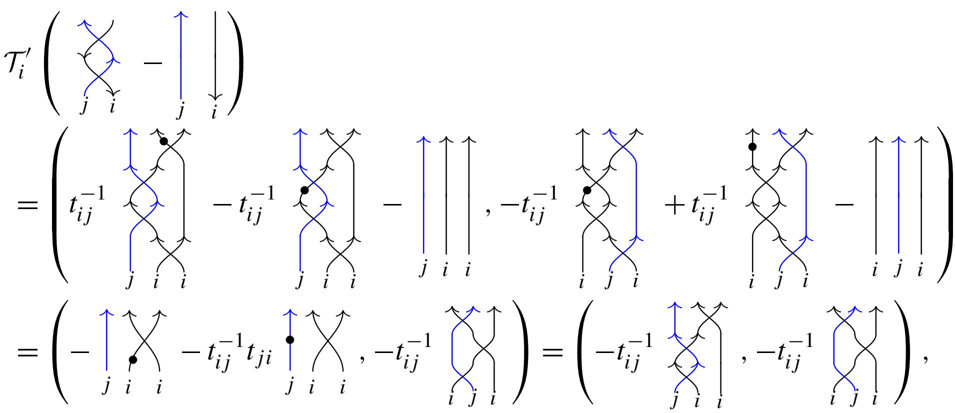
5H. ***Mixed EF relation.*** We now verify (7) in Definition 3.3.

Proposition 5.8. *$\mathcal{T}_i'$ preserves the mixed EF relations.*

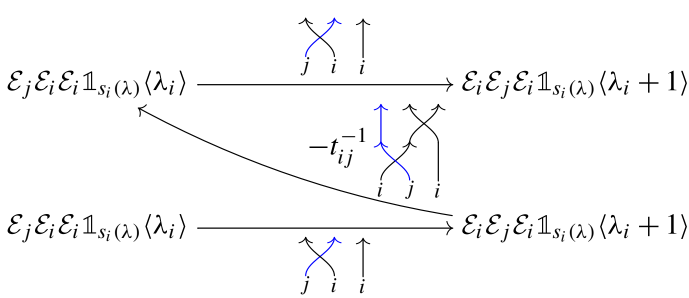
Proof. All cases involving k-labeled strands hold, and are trivial to verify. The following computations exhibit half the requisite checks, and the remaining follow almost identically. We make extensive use of Section A.2. We compute



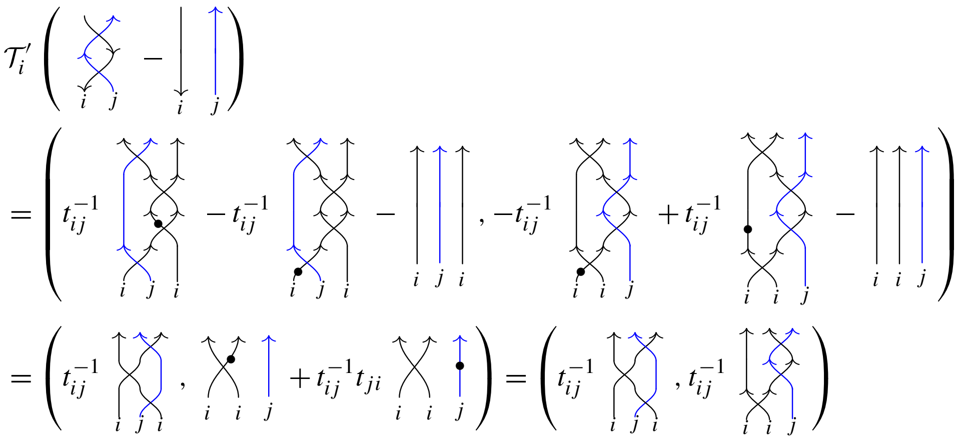
We now consider the cases requiring homotopies. We compute



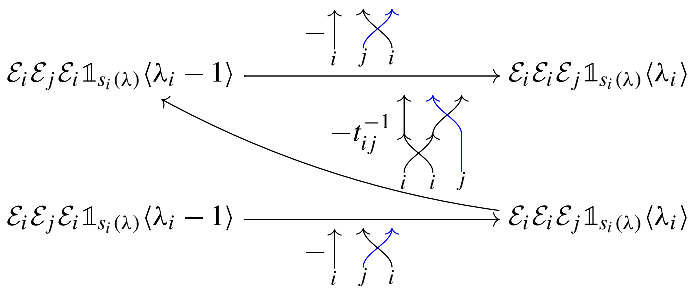
which is null-homotopic, with homotopy given by



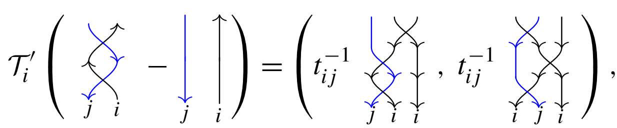
and



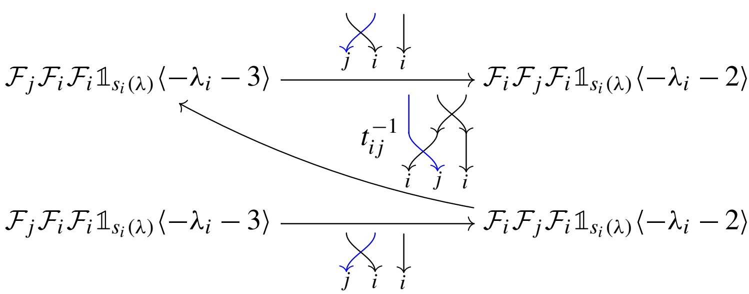
is null-homotopic with homotopy given by



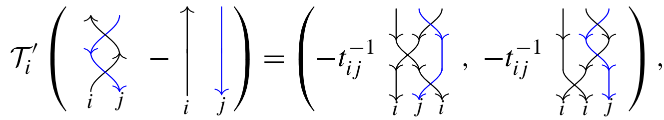
Similar computations show that



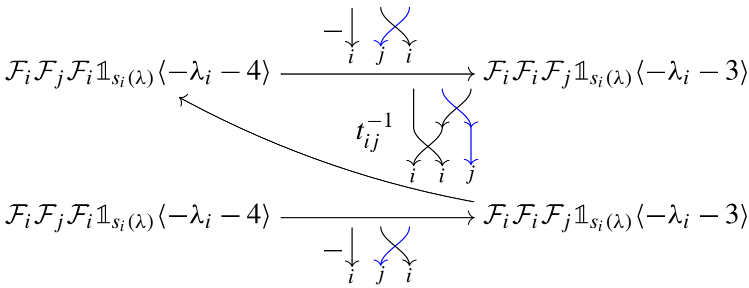
which is null-homotopic via the homotopy



and that



which is null-homotopic with homotopy given by

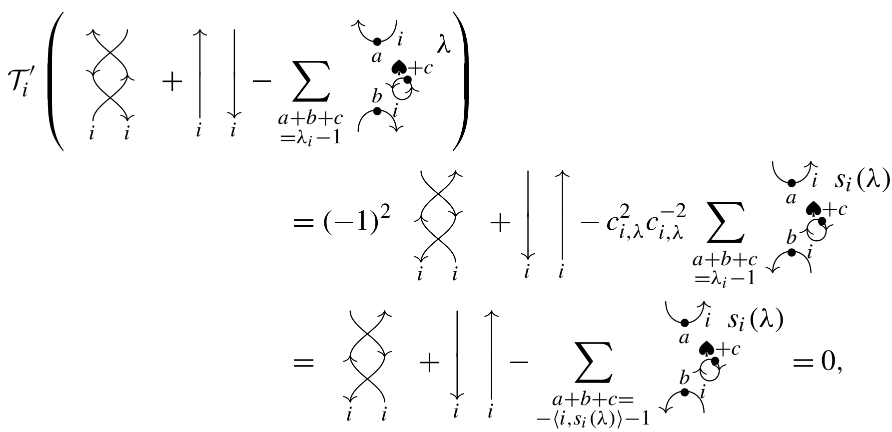


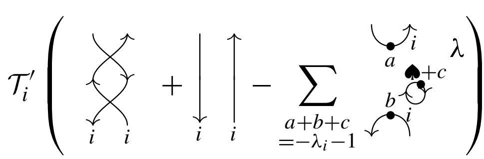
The final case, involving j- and j'-labeled strands (with $j \neq j'$), will be addressed in Proposition 5.10 below. ☐

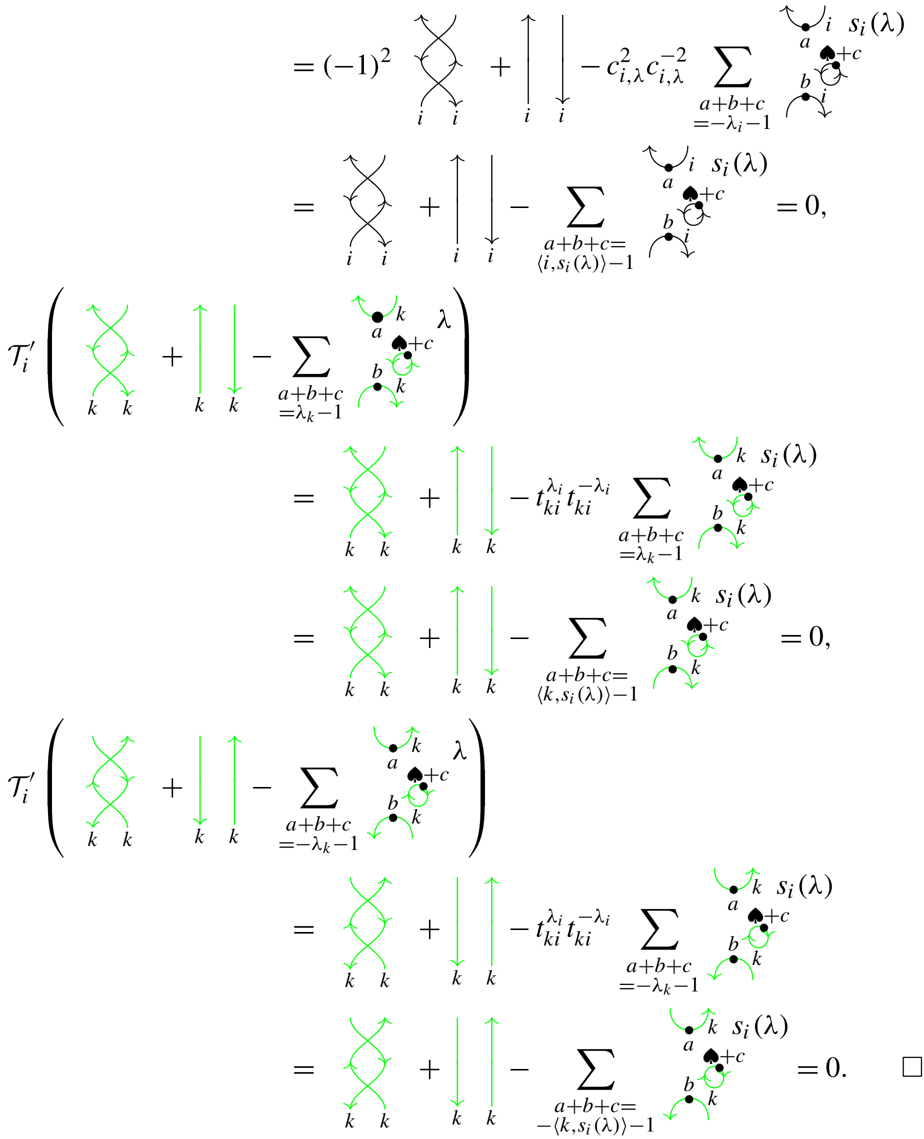
5I. *Extended $\mathfrak{sl}_2$ relations.* We now verify (9) in Definition 3.3.

Proposition 5.9. *$\mathcal{T}_i'$ preserves the extended $\mathfrak{sl}_2$ relations in the i- and k-labeled cases.*

Proof. In these cases, the relations hold on the nose, as we confirm:

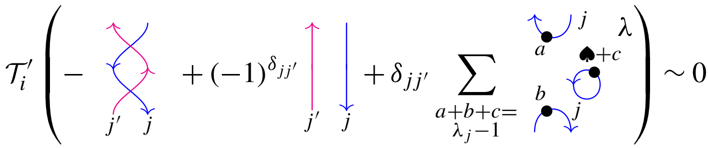






We conclude by considering the outstanding relations, i.e., the j-labeled extended $\mathfrak{sl}_2$ relations, and the jj'-labeled mixed EF relation. These are the most involved relations, in part because the homotopies involved are not necessarily unique. Indeed, if $\underline{\mathrm{Hom}}(X, Y)$ denotes the chain complex of all homogeneous maps (that are not necessarily degree zero, or chain maps) between complexes X and Y, then given any element $\alpha \in \underline{\mathrm{Hom}}^{-2}(X, Y)$, the element $d(\alpha) = d_Y\alpha - \alpha d_X$ can be added to any homotopy h without affecting $d_Y h + h d_X$. Our previous cases have not admitted such an α, but in the present case there exist (many) such α, given by any map $\mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle \to \mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1\rangle$.

Proposition 5.10. *The relation*

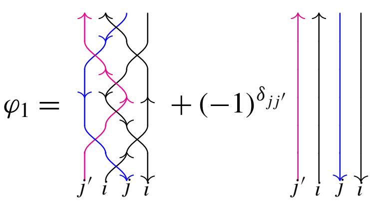
(5-1) 

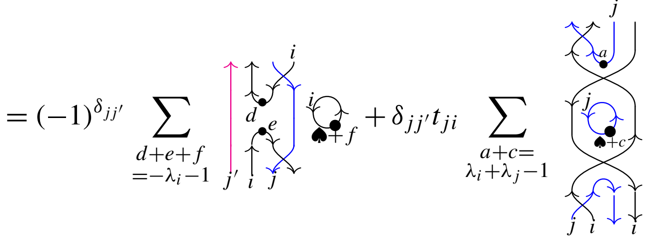
holds in $\mathsf{Com}(\mathcal{U}_Q)$.

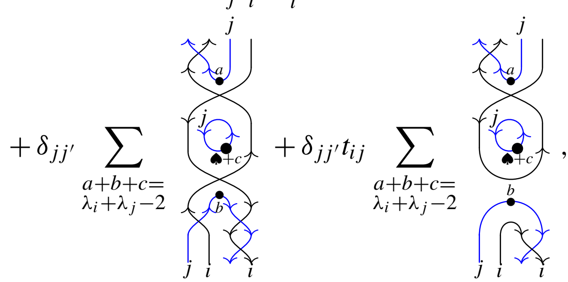
Proof. The left-hand side of (5-1) is given by

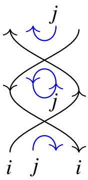
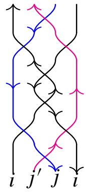
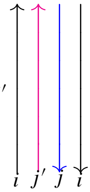
$$\begin{array}{ccc} \clubsuit\mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1\rangle & \xrightarrow{\varphi_1} & \clubsuit\mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1\rangle \\ \downarrow & & \downarrow \\ \begin{array}{c}\mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)} \\ \oplus\mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\end{array} & \xrightarrow{\left(\begin{smallmatrix}\varphi_2 & \varphi_4\\ \varphi_3 & \varphi_5\end{smallmatrix}\right)} & \begin{array}{c}\mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)} \\ \oplus\mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\end{array} \\ \downarrow & & \downarrow \\ \mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle & \xrightarrow{\varphi_6} & \mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle \end{array} \tag{5-2}$$

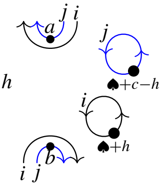
where the components of the chain map are given as follows (which can be verified by completely simplifying both sides of the equalities):

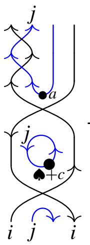
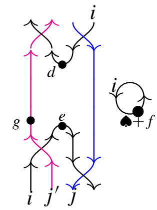


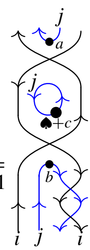
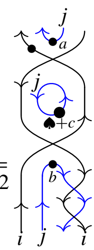


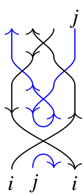
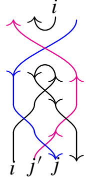
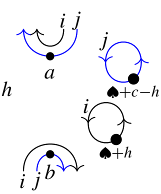


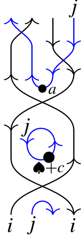
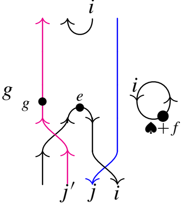
$$\varphi_2 = -t_{ji}^2\delta_{jj'}$$  $$+$$  $$+(-1)^{\delta_{jj'}}$$ 

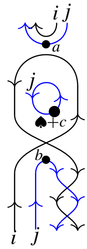
$$-\delta_{jj'}t_{ij}\sum_{\substack{a+b+c\\=\lambda_j-1}}\sum_{h=0}^{c}(-v_{ij})^{-\lambda_i-h}$$ 

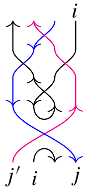
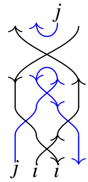
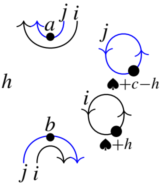
$$=\delta_{jj'}t_{ji}\sum_{\substack{a+c=\\ \lambda_i+\lambda_j-1}}$$  $$+t_{ij'}^{-1}(-1)^{\delta_{jj'}}\sum_{\substack{d+e+f+g\\=-\lambda_i-2}}(-v_{ij'})^g$$ 

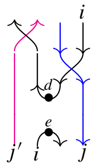
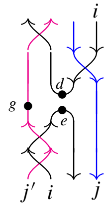
$$+\delta_{jj'}t_{ji}\sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-1}}$$  $$+\delta_{jj'}t_{ij}\sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$$  ,

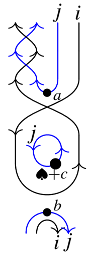
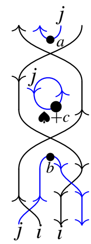
$$\varphi_3 = \delta_{jj'}t_{ji}$$  $$-t_{ij}$$  $$-\delta_{jj'}t_{ij}\sum_{\substack{a+b+c\\=\lambda_j-1}}\sum_{h=0}^{c}(-v_{ij})^{-\lambda_i-h}$$ 

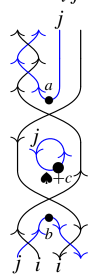
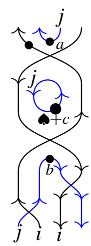
$$=\delta_{jj'}t_{ji}\sum_{\substack{a+c=\\ \lambda_i+\lambda_j-1}}$$  $$-t_{ij}t_{ij'}^{-1}(-1)^{\delta_{jj'}}\sum_{\substack{e+f+g\\=-\lambda_i-1}}(-v_{ij'})^g$$ 

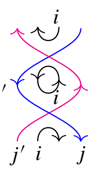
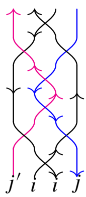
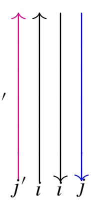
$$-\delta_{jj'}t_{ij}\sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$$  ,

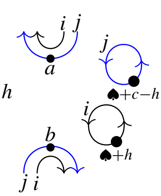
$$\varphi_4 = t_{ij'}$$  $$- \delta_{jj'} t_{ji}$$  $$+ \delta_{jj'} t_{ij} \sum_{\substack{a+b+c \\ =\lambda_j-1}} \sum_{h=0}^{c} (-v_{ij})^{-\lambda_i - h}$$ 

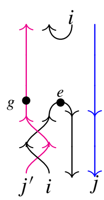
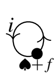
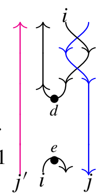
$$= (-1)^{\delta_{jj'}} \Bigg(\sum_{\substack{d+e+f= \\ -\lambda_i-1}}$$   $$- t_{ij'}^{-1} \sum_{\substack{d+e+f+g \\ =-\lambda_i-2}} (-v_{ij'})^{g}$$   $$\Bigg)$$

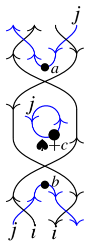
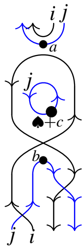
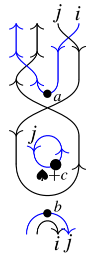
$$+ \delta_{jj'} t_{ij} \sum_{\substack{a+b+c= \\ \lambda_i+\lambda_j-2}}$$  $$- \delta_{jj'} t_{ji} \sum_{\substack{a+b+c= \\ \lambda_i+\lambda_j-1}}$$ 

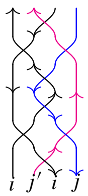
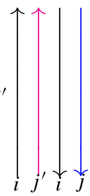
$$+ \delta_{jj'} \sum_{\substack{a+b+c= \\ \lambda_i+\lambda_j-2}}$$  $$- \delta_{jj'} t_{ij} \sum_{\substack{a+b+c= \\ \lambda_i+\lambda_j-2}}$$  ,

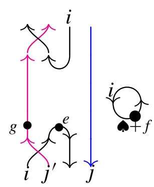
$$\varphi_5 = -t_{ij} t_{ij'}$$  $$+$$  $$+ (-1)^{\delta_{jj'}}$$ 

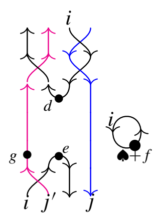
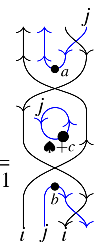
$$+ \delta_{jj'} t_{ij} \sum_{\substack{a+b+c \\ =\lambda_j-1}} \sum_{h=0}^{c} (-v_{ij})^{-\lambda_i - h}$$ 

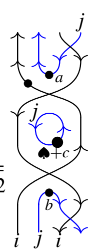
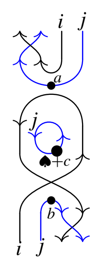
$$= t_{ij} t_{ij'}^{-1} (-1)^{\delta_{jj'}} \sum_{\substack{d+f+g= \\ -\lambda_i-1}} (-v_{ij'})^{g}$$   $$+ (-1)^{\delta_{jj'}} \sum_{\substack{d+e+f \\ =-\lambda_i-1}}$$  

$+\delta_{jj'} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$  $+\delta_{jj'} t_{ij} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$  $+\delta_{jj'} t_{ij} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$  ,

$\varphi_6 =$  $+(-1)^{\delta_{jj'}}$ 

$= t_{ij} t_{ij'}^{-1} (-1)^{\delta_{jj'}} \sum_{\substack{e+f+g\\ =-\lambda_i-1}} (-v_{ij'})^g$ 

$+ t_{ij'}^{-1} (-1)^{\delta_{jj'}} \sum_{\substack{d+e+f+g\\ =-\lambda_i-2}} (-v_{ij'})^g$  $+\delta_{jj'} t_{ji} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-1}}$ 

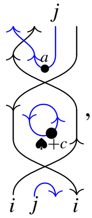
$+\delta_{jj'} t_{ij} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$  $+\delta_{jj'} t_{ij} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$  .

The chain map given in (5-2) is thus null-homotopic, with homotopy given by the diagram

$$
\begin{array}{ccccc}
\clubsuit \mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1\rangle & \longrightarrow & \begin{matrix} \mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)} \\ \oplus \mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)} \end{matrix} & \longrightarrow & \mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle \\
 & \nwarrow^{(h_1^1\ h_2^1)} & & \nwarrow^{\binom{h_1^2}{h_2^2}} & \\
\clubsuit \mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1\rangle & \longrightarrow & \begin{matrix} \mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)} \\ \oplus \mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)} \end{matrix} & \longrightarrow & \mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle
\end{array}
$$

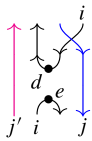
where

$$h_1^1 = \delta_{jj'} t_{ji} \sum_{\substack{a+c=\\ \lambda_i+\lambda_j-1}}$$

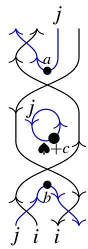


,

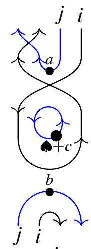
$$h_2^1 = (-1)^{\delta_{jj'}} \sum_{\substack{d+e+f\\ =-\lambda_i-1}}$$



$$+\, \delta_{jj'} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$$

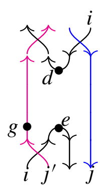


$$+\, \delta_{jj'} t_{ij} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$$

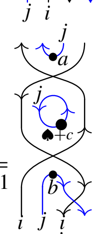


,

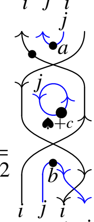
$$h_1^2 = -t_{ij'}^{-1} (-1)^{\delta_{jj'}} \sum_{\substack{d+e+f+g\\ =-\lambda_i-2}} (-v_{ij'})^g$$



$$-\, \delta_{jj'} t_{ji} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-1}}$$

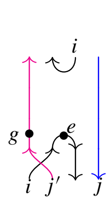


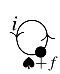
$$-\, \delta_{jj'} t_{ij} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$$



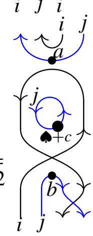
,

$$h_2^2 = t_{ij} t_{ij'}^{-1} (-1)^{\delta_{jj'}} \sum_{\substack{e+f+g\\ =-\lambda_i-1}} (-v_{ij'})^g$$





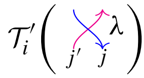
$$+\, \delta_{jj'} t_{ij} \sum_{\substack{a+b+c=\\ \lambda_i+\lambda_j-2}}$$



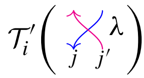
. □

It remains to verify the FE version of (5-1). We can proceed to compute as above, but in this case we can obtain the relation via a trick using the symmetry ω.

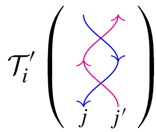
Indeed, note that, up to scalar factors, each map determining



is given by applying ω to the corresponding component in

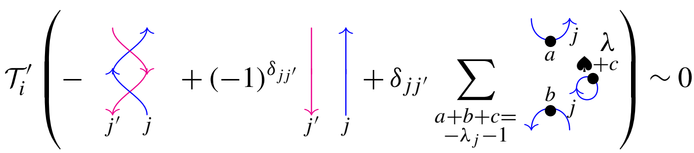


and exchanging the roles of j and j'. Upon taking the composition, the discrepancies between the relevant scalars cancel, and we find that the maps determining



are given by applying ω to the φ_i's. Similarly, the other terms in the relation are obtained by those in (5-1) via ω. It follows that we can "apply ω" to the proof of Proposition 5.10 (in weight $-\lambda$) to obtain the following.

Corollary 5.11. *The relation*

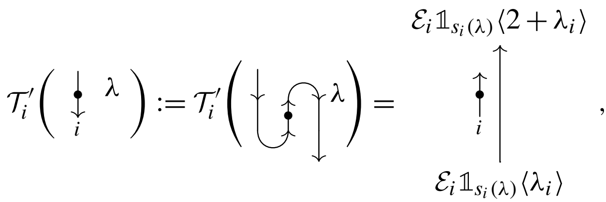


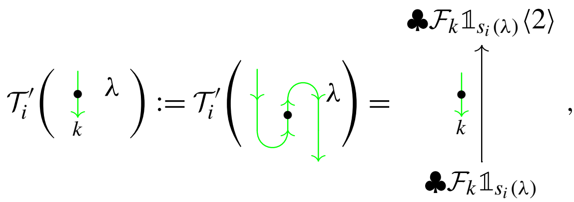
holds in $\mathsf{Com}(\mathcal{U}_Q)$.

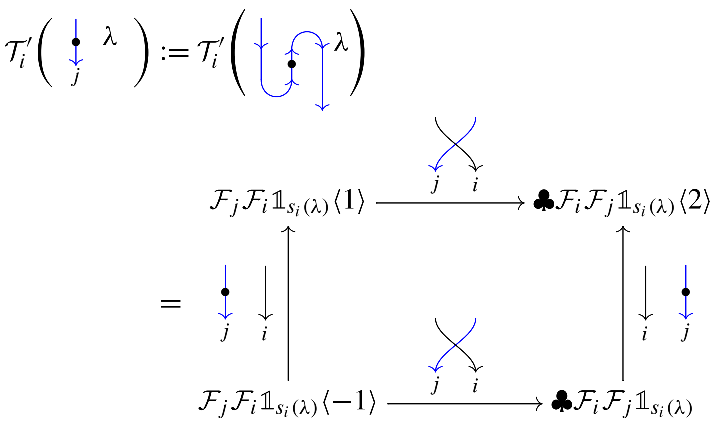
Appendix: Computation of $\mathcal{T}_{i,1}'$ for composite 2-morphisms

In light of Remark 3.4, we can compute the value of $\mathcal{T}_{i,1}'$ on downward dot and sideways and downward crossing 2-morphisms in terms of the presentation of these 2-morphisms in terms of upward dot and crossing 2-morphisms and cap/cup 2-morphisms. In Sections A.1, A.2, and A.3, we compute this value, and in Section A.4, we compute the value of $\mathcal{T}_{i,1}'$ on bubbles. Throughout, we employ our conventions that $i \cdot j = -1 = i \cdot j'$ and $i \cdot k = 0 = i \cdot k'$, but assume no other relation between j, j', k, and k'.

A.1. ***Value of $\mathcal{T}_{i,1}'$ for downward dot 2-morphisms.*** We compute $\mathcal{T}_{i,1}'$ on downward dot 2-morphisms using the right cyclicity relation. Each of the following is a direct consequence of the definitions in Sections 4B1 and 4B3:

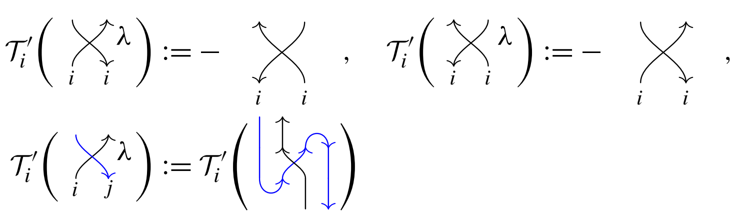


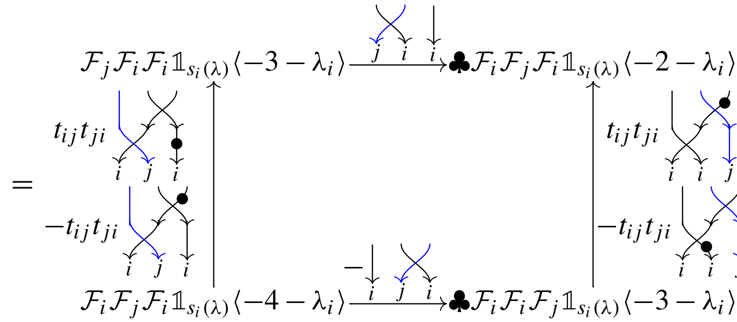


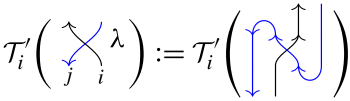


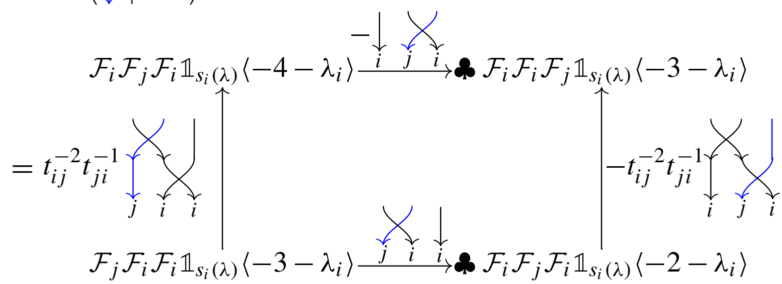
This agrees with the value in terms of left cyclicity, which is verified in Section 5B.

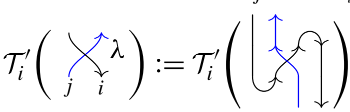
A.2. ***Value of $\mathcal{T}_{i,1}'$ on sideways crossing 2-morphisms.*** We explicitly compute the value on sideways crossings in terms of the images of upward crossings, caps, and cups. As above, each follows via a direct (but sometimes tedious) computation using the definitions in Sections 4B2 and 4B3. In the interest of space, we will omit displaying the domain and codomain of the image when they are 1-term complexes, as, save for the relevant shifts, they can be read from the diagram. We have

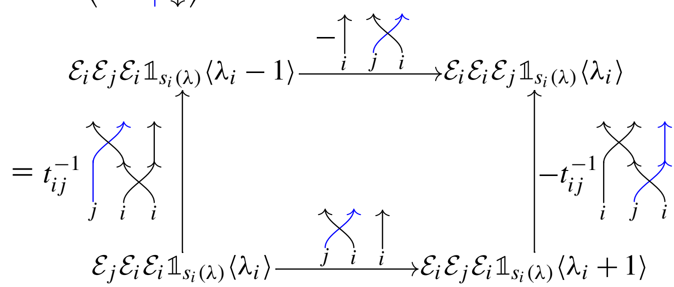


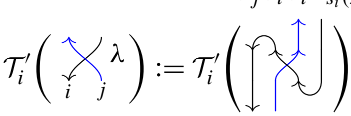

$\mathcal{F}_j\mathcal{F}_i\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -3-\lambda_i\rangle$
$\clubsuit\mathcal{F}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -2-\lambda_i\rangle$
$=$
$t_{ij}t_{ji}$
$-t_{ij}t_{ji}$
$t_{ij}t_{ji}$
$-t_{ij}t_{ji}$
$-$
$\mathcal{F}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -4-\lambda_i\rangle$
$\clubsuit\mathcal{F}_i\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle -3-\lambda_i\rangle$

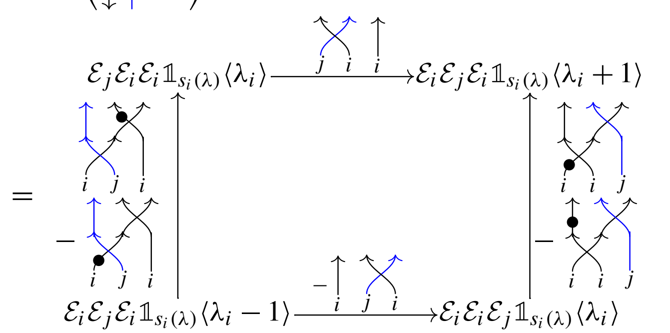

$\mathcal{T}_i'$
λ
$:= \mathcal{T}_i'$

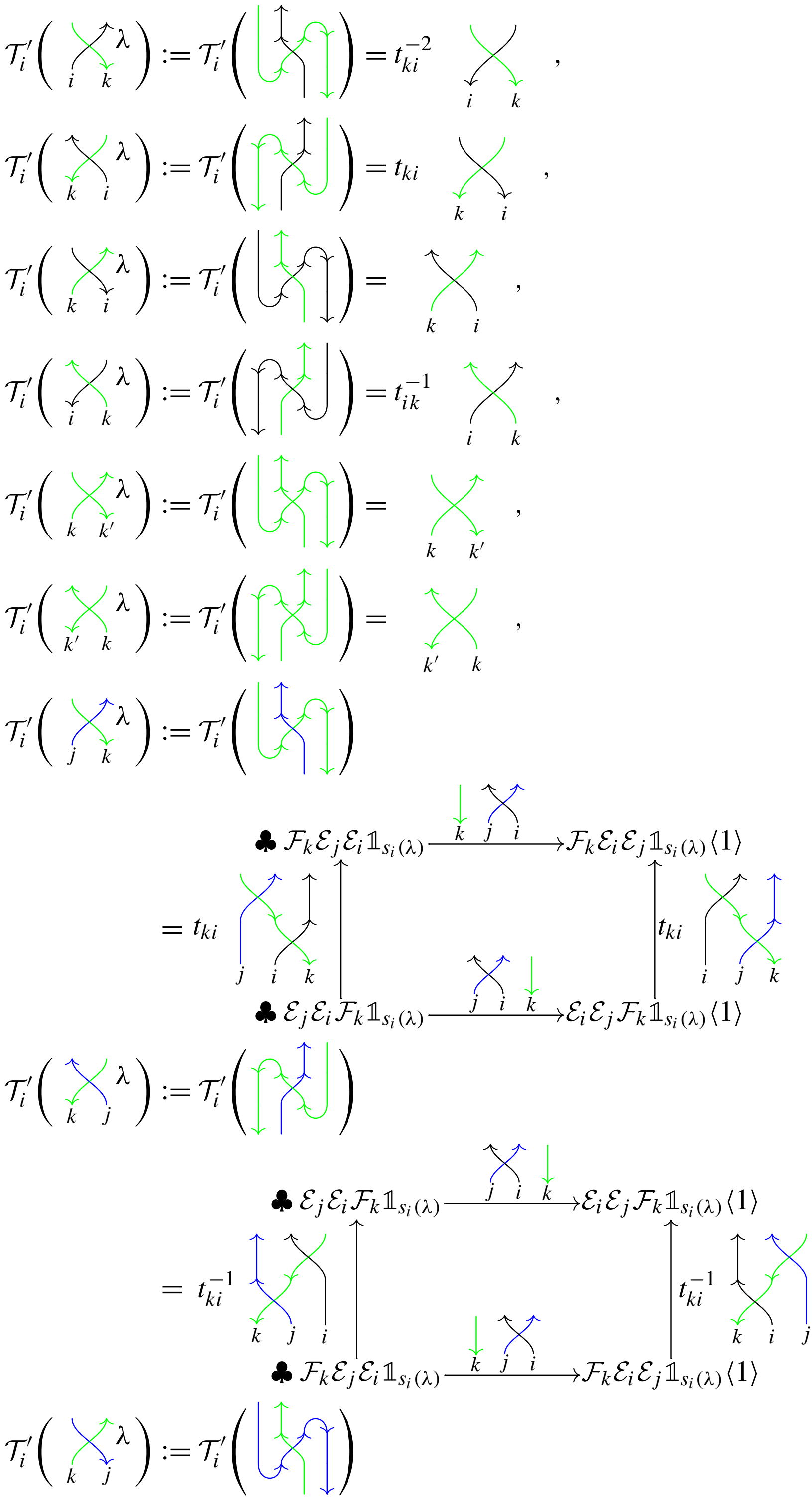

$\mathcal{F}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -4-\lambda_i\rangle$
$-$
$\clubsuit\ \mathcal{F}_i\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle -3-\lambda_i\rangle$
$= t_{ij}^{-2}t_{ji}^{-1}$
$-t_{ij}^{-2}t_{ji}^{-1}$
$\mathcal{F}_j\mathcal{F}_i\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -3-\lambda_i\rangle$
$\clubsuit\ \mathcal{F}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -2-\lambda_i\rangle$


$\mathcal{T}_i'$
λ
$:= \mathcal{T}_i'$

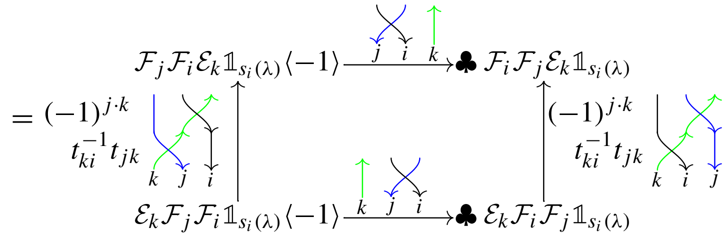

$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle \lambda_i-1\rangle$
$-$
$\mathcal{E}_i\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle \lambda_i\rangle$
$= t_{ij}^{-1}$
$-t_{ij}^{-1}$
$\mathcal{E}_j\mathcal{E}_i\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle \lambda_i\rangle$
$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle \lambda_i+1\rangle$

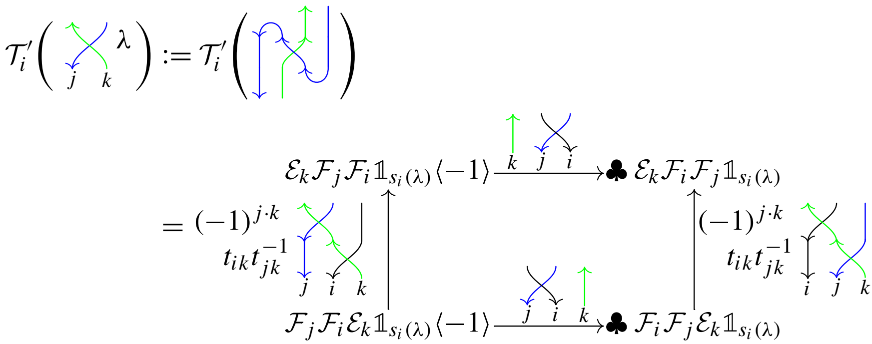

$\mathcal{T}_i'$
λ
$:= \mathcal{T}_i'$


$\mathcal{E}_j\mathcal{E}_i\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle \lambda_i\rangle$
$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle \lambda_i+1\rangle$
$=$
$-$
$-$
$-$
$\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle \lambda_i-1\rangle$
$\mathcal{E}_i\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle \lambda_i\rangle$

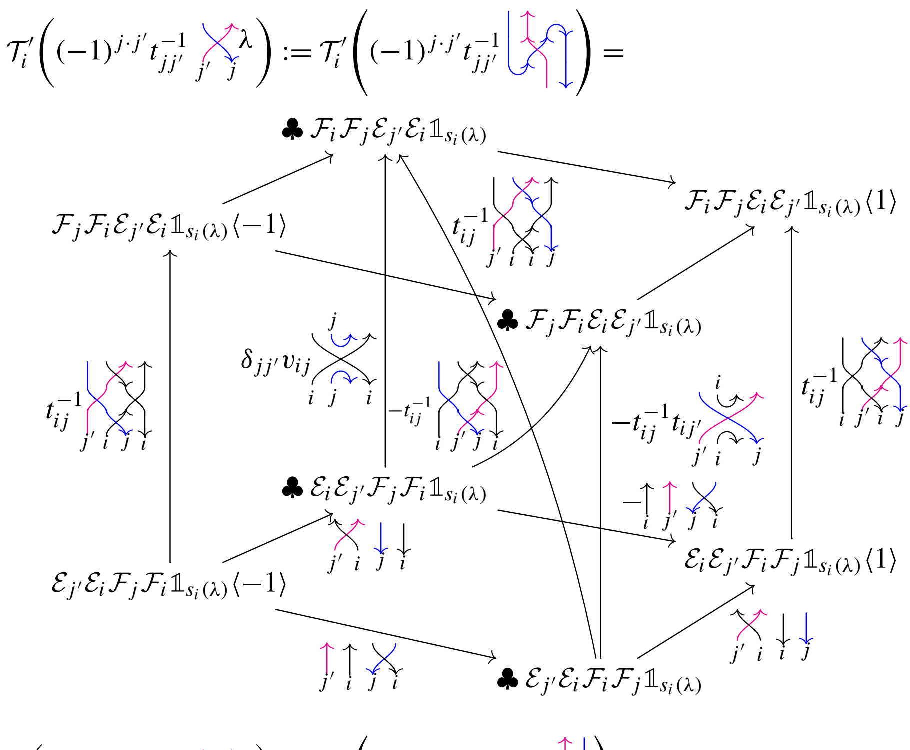


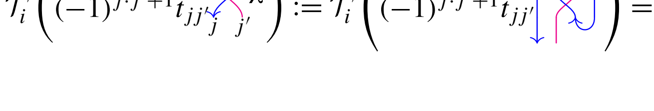
$\mathcal{T}_i'$:= $\mathcal{T}_i'$ = t_{ki}^{-2} ,
$\mathcal{T}_i'$:= $\mathcal{T}_i'$ = t_{ki} ,
$\mathcal{T}_i'$:= $\mathcal{T}_i'$ = ,
$\mathcal{T}_i'$:= $\mathcal{T}_i'$ = t_{ik}^{-1} ,
$\mathcal{T}_i'$:= $\mathcal{T}_i'$ = ,
$\mathcal{T}_i'$:= $\mathcal{T}_i'$ = ,
$\mathcal{T}_i'$:= $\mathcal{T}_i'$
$\clubsuit\ \mathcal{F}_k\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)} \to \mathcal{F}_k\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle$
$= t_{ki}$
t_{ki}
$\clubsuit\ \mathcal{E}_j\mathcal{E}_i\mathcal{F}_k\mathbb{1}_{s_i(\lambda)} \to \mathcal{E}_i\mathcal{E}_j\mathcal{F}_k\mathbb{1}_{s_i(\lambda)}\langle 1\rangle$
$\mathcal{T}_i'$:= $\mathcal{T}_i'$
$\clubsuit\ \mathcal{E}_j\mathcal{E}_i\mathcal{F}_k\mathbb{1}_{s_i(\lambda)} \to \mathcal{E}_i\mathcal{E}_j\mathcal{F}_k\mathbb{1}_{s_i(\lambda)}\langle 1\rangle$
$= t_{ki}^{-1}$
t_{ki}^{-1}
$\clubsuit\ \mathcal{F}_k\mathcal{E}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)} \to \mathcal{F}_k\mathcal{E}_i\mathcal{E}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle$
$\mathcal{T}_i'$:= $\mathcal{T}_i'$

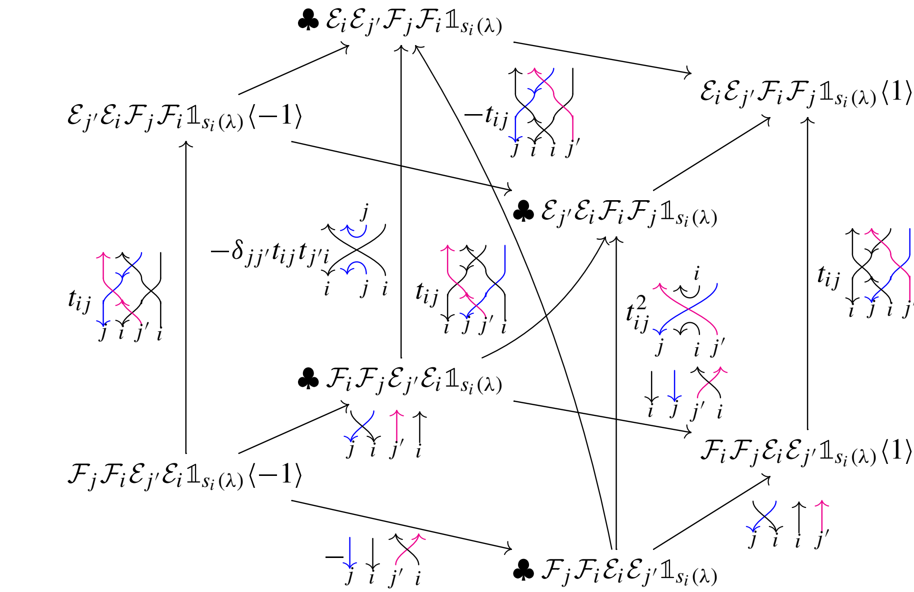

$\mathcal{F}_j\mathcal{F}_i\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}\langle -1\rangle$
$♣\ \mathcal{F}_i\mathcal{F}_j\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}$
$= (-1)^{j\cdot k}$
$t_{ki}^{-1}t_{jk}$
$(-1)^{j\cdot k}$
$t_{ki}^{-1}t_{jk}$
$\mathcal{E}_k\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1\rangle$
$♣\ \mathcal{E}_k\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}$


$\mathcal{T}_i'$
$:= \mathcal{T}_i'$
$\mathcal{E}_k\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1\rangle$
$♣\ \mathcal{E}_k\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}$
$= (-1)^{j\cdot k}$
$t_{ik}t_{jk}^{-1}$
$(-1)^{j\cdot k}$
$t_{ik}t_{jk}^{-1}$
$\mathcal{F}_j\mathcal{F}_i\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}\langle -1\rangle$
$♣\ \mathcal{F}_i\mathcal{F}_j\mathcal{E}_k\mathbb{1}_{s_i(\lambda)}$

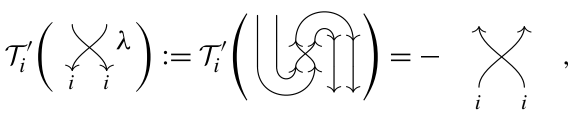
and

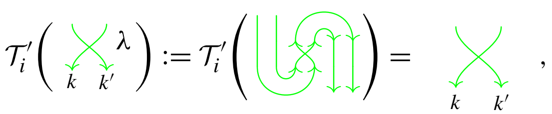

$\mathcal{T}_i'\left((-1)^{j\cdot j'}t_{jj'}^{-1}\right) := \mathcal{T}_i'\left((-1)^{j\cdot j'}t_{jj'}^{-1}\right) =$
$♣\ \mathcal{F}_i\mathcal{F}_j\mathcal{E}_{j'}\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}$
$\mathcal{F}_i\mathcal{F}_j\mathcal{E}_i\mathcal{E}_{j'}\mathbb{1}_{s_i(\lambda)}\langle 1\rangle$
$\mathcal{F}_j\mathcal{F}_i\mathcal{E}_{j'}\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle -1\rangle$
t_{ij}^{-1}
$♣\ \mathcal{F}_j\mathcal{F}_i\mathcal{E}_i\mathcal{E}_{j'}\mathbb{1}_{s_i(\lambda)}$
$\delta_{jj'}v_{ij}$
t_{ij}^{-1}
$-t_{ij}^{-1}$
$-t_{ij}^{-1}t_{ij'}$
t_{ij}^{-1}
$♣\ \mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}$
$\mathcal{E}_i\mathcal{E}_{j'}\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle 1\rangle$
$\mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle -1\rangle$
$♣\ \mathcal{E}_{j'}\mathcal{E}_i\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}$

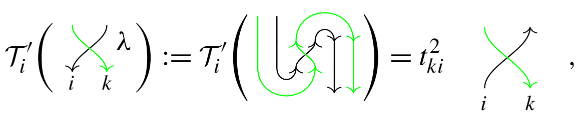

$\mathcal{T}_i'\left((-1)^{j\cdot j'+1}t_{jj'}\right) := \mathcal{T}_i'\left((-1)^{j\cdot j'+1}t_{jj'}\right) =$

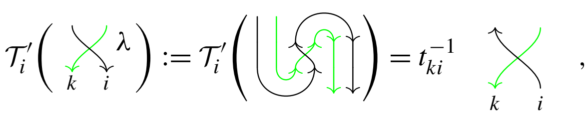


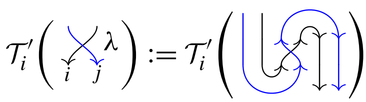
A.3. *Value of $\mathcal{T}'_{i,1}$ on downwards crossing 2-morphisms.* We have

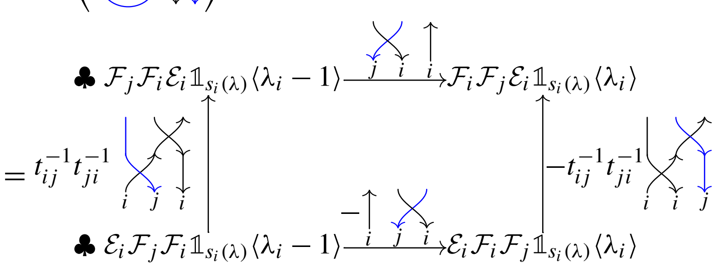


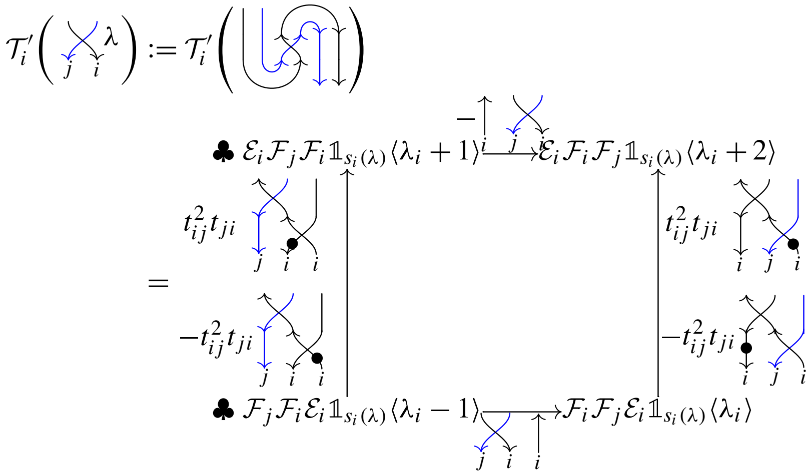




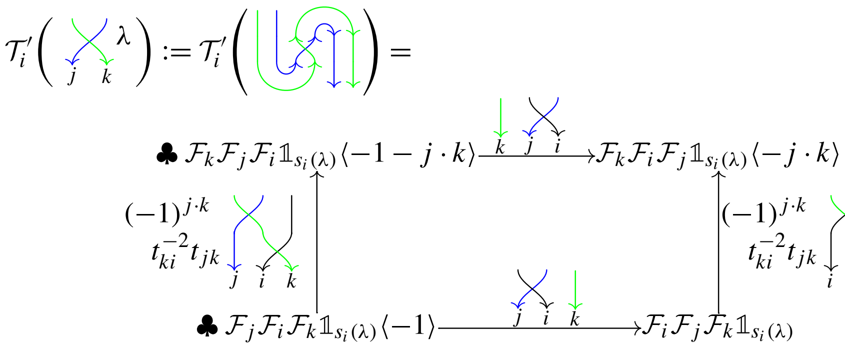




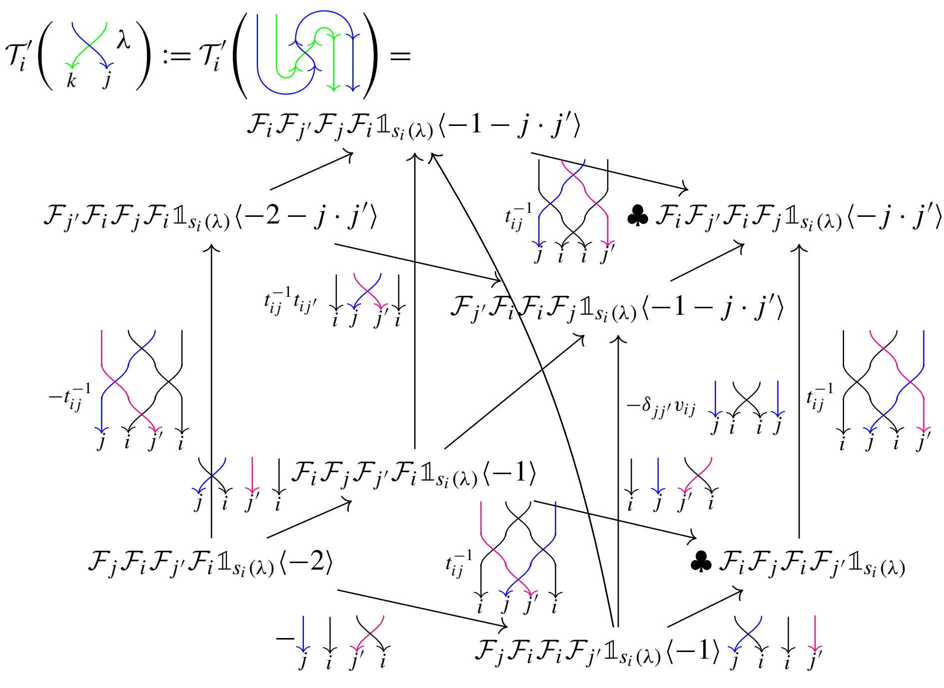




$\mathcal{T}_i'$ λ := $\mathcal{T}_i'$ =
♣ $\mathcal{E}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle\lambda_i+1\rangle$
$\mathcal{E}_i\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle\lambda_i+2\rangle$
$t_{ij}^2t_{ji}$
$-t_{ij}^2t_{ji}$
♣ $\mathcal{F}_j\mathcal{F}_i\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle\lambda_i-1\rangle$
$\mathcal{F}_i\mathcal{F}_j\mathcal{E}_i\mathbb{1}_{s_i(\lambda)}\langle\lambda_i\rangle$

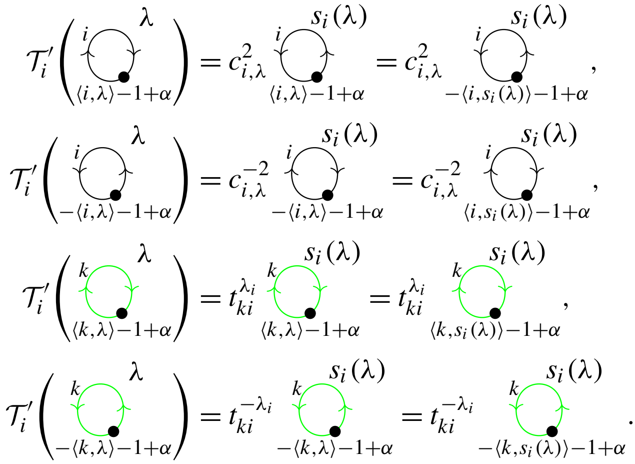


$\mathcal{T}_i'$ λ := $\mathcal{T}_i'$ =
♣ $\mathcal{F}_k\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle-1-j\cdot k\rangle$
$\mathcal{F}_k\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle-j\cdot k\rangle$
$(-1)^{j\cdot k}$ $t_{ki}^{-2}t_{jk}$
♣ $\mathcal{F}_j\mathcal{F}_i\mathcal{F}_k\mathbb{1}_{s_i(\lambda)}\langle-1\rangle$
$\mathcal{F}_i\mathcal{F}_j\mathcal{F}_k\mathbb{1}_{s_i(\lambda)}$

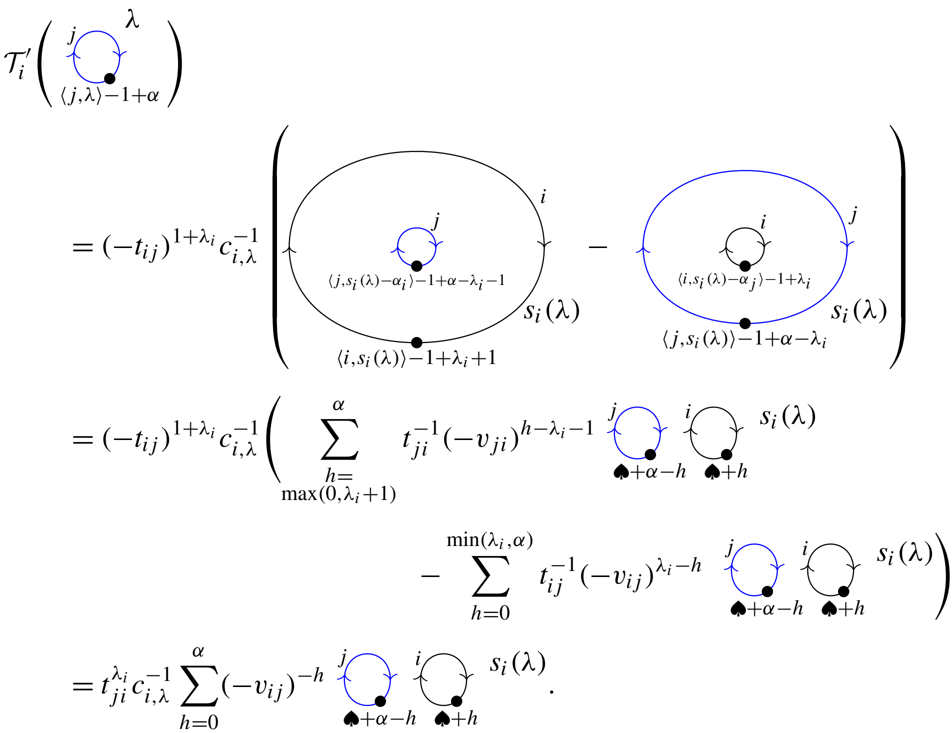


$\mathcal{T}_i'$ λ := $\mathcal{T}_i'$ =
$\mathcal{F}_i\mathcal{F}_{j'}\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle-1-j\cdot j'\rangle$
$\mathcal{F}_{j'}\mathcal{F}_i\mathcal{F}_j\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle-2-j\cdot j'\rangle$
t_{ij}^{-1}
♣ $\mathcal{F}_i\mathcal{F}_{j'}\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle-j\cdot j'\rangle$
$t_{ij}^{-1}t_{ij'}$
$\mathcal{F}_{j'}\mathcal{F}_i\mathcal{F}_i\mathcal{F}_j\mathbb{1}_{s_i(\lambda)}\langle-1-j\cdot j'\rangle$
$-t_{ij}^{-1}$
$-\delta_{jj'}v_{ij}$
t_{ij}^{-1}
$\mathcal{F}_i\mathcal{F}_j\mathcal{F}_{j'}\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle-1\rangle$
$\mathcal{F}_j\mathcal{F}_i\mathcal{F}_{j'}\mathcal{F}_i\mathbb{1}_{s_i(\lambda)}\langle-2\rangle$
t_{ij}^{-1}
♣ $\mathcal{F}_i\mathcal{F}_j\mathcal{F}_i\mathcal{F}_{j'}\mathbb{1}_{s_i(\lambda)}$
$\mathcal{F}_j\mathcal{F}_i\mathcal{F}_i\mathcal{F}_{j'}\mathbb{1}_{s_i(\lambda)}\langle-1\rangle$

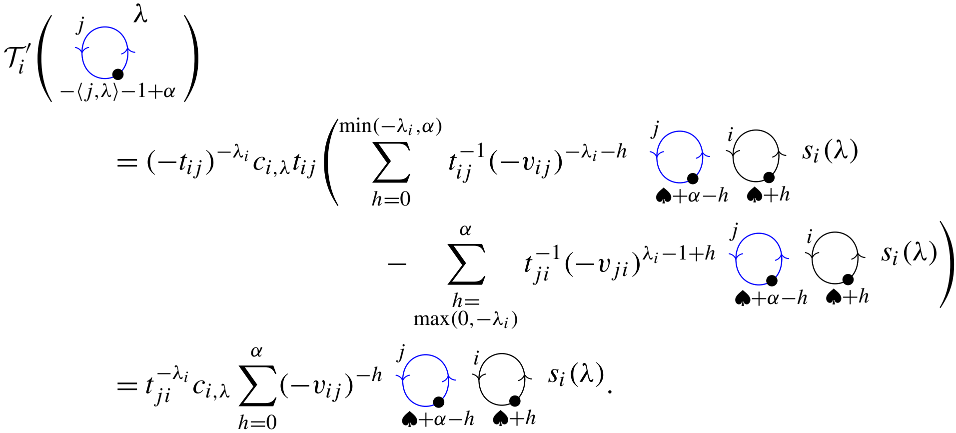
A.4. *Computation of $\mathcal{T}'_{i,1}$ on bubble 2-morphisms.* We compute the image of bubble 2-morphisms, and use them to explicitly verify that $\mathcal{T}'_{i,1}$ preserves the infinite Grassmannian relation:



For j-labeled bubbles, we use the bubble sliding relations from Section 3C3 (note that, in the first equation, the number of dots on the black circles equals zero for both summands):



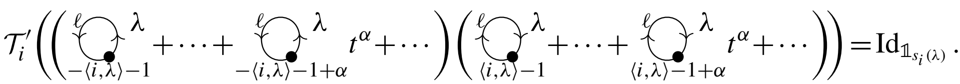
Similarly, the image of the counterclockwise bubble is given by



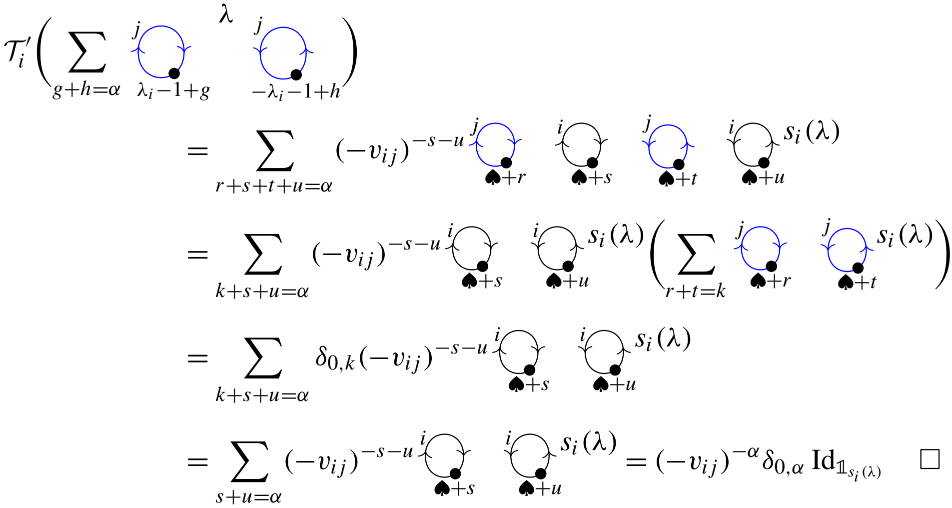
(In both cases, recall our convention that any sums with nonincreasing index are by definition zero.)

These computations for the images of bubbles under $\mathcal{T}_{i,1}'$ are only valid when the number of dots is positive; however, our next result shows that they also hold for bubbles with a negative number of dots (i.e., for fake bubbles; see Definition 3.3(8)).

Lemma 1. $\mathcal{T}_{i,1}'$ preserves the infinite Grassmannian relation:



Proof. The only nontrivial case is when the bubbles are j-labeled (for $i\cdot j=-1$), and here we compute the relation in degree α as follows:



Acknowledgments

This project began as a collaboration between Lauda and Mikhail Khovanov; the authors sincerely thank him for his generosity with ideas. We also thank Andrea Appel, Ben Elias, and Peter McNamara for valuable discussions. During the course of this work, Lauda was partially supported by the NSF grants DMS-1255334, DMS-1664240, DMS-1902092 and Army Research Office W911NF-20-1-0075, and Rose was partially supported by NSA grants H-98230-16-1-0003 and H-98230-17-1-0211, and Simons Collaboration Grant 523992.

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Received February 28, 2023. Revised October 19, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 328, No. 1, 2024
https://doi.org/10.2140/pjm.2024.328.77

FUSED HECKE ALGEBRA AND ONE-BOUNDARY ALGEBRAS

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This paper gives an algebraic presentation of the fused Hecke algebra which describes the centraliser of tensor products of the $U_q(\mathrm{gl}_N)$-representation labelled by a one-row partition of any size with vector representations. It is obtained through a detailed study of a new algebra that we call the symmetric one-boundary Hecke algebra. In particular, we prove that the symmetric one-boundary Hecke algebra is free over a ring of Laurent polynomials in three variables and we provide a basis indexed by a certain subset of signed permutations. We show how the symmetric one-boundary Hecke algebra admits the one-boundary Temperley–Lieb algebra as a quotient, and we also describe a basis of this latter algebra combinatorially in terms of signed permutations with avoiding patterns. The quotients corresponding to any value of N in gl_N (the Temperley–Lieb one corresponds to $N = 2$) are also introduced. Finally, we obtain the fused Hecke algebra, and in turn the centralisers for any value of N, by specialising and quotienting the symmetric one-boundary Hecke algebra. In particular, this generalises to the Hecke case the description of the so-called boundary seam algebra, which is then obtained (taking $N = 2$) as a quotient of the fused Hecke algebra.

1. Introduction

The usual Hecke algebra $H_n(q)$ appears in the quantum Schur–Weyl duality [14] describing the centralisers of tensor powers of the vector representation of the

MSC2020: primary 20C08; secondary 05E10, 20F36, 20G42.

Keywords: Hecke algebras, one-boundary, quantum groups, centraliser, fusion, signed permutations.

quantum group $U_q(\mathrm{gl}_N)$. If we denote L^N the vector representation of dimension N of $U_q(\mathrm{gl}_N)$, there is a surjective morphism

$$H_n \to \mathrm{End}_{U_q(\mathrm{gl}_N)}((L^N)^{\otimes n}).$$

The Hecke algebra H_n does not depend on N and plays its role for gl_N for any N. The dependence on N of the centraliser appears in the description of the kernel of the above map. Indeed, for a given N, the centraliser of $U_q(\mathrm{gl}_N)$ is isomorphic to the quotient of H_n by the q-antisymmetriser on $N+1$ points. This q-antisymmetriser is a minimal central idempotent of H_{N+1} (the quotient is trivial if $n \leq N$) and generates the kernel for any $n \geq N+1$. In particular, for $N = 2$, the resulting algebra is the well-known Temperley–Lieb algebra.

We would like to generalise the whole picture for tensor products of more general representations of $U_q(\mathrm{gl}_N)$. The fused Hecke algebra was introduced in [7] for this purpose. For $\vec{k} = (k_1, \dots, k_m)$ a vector of positive integers, we have a surjective morphism

$$H_{\vec{k}} \to \mathrm{End}_{U_q(\mathrm{gl}_N)}(L^N_{(k_1)} \otimes L^N_{(k_2)} \otimes \cdots \otimes L^N_{(k_m)}), \tag{1-1}$$

where $L^N_{(k)}$ is the k-th q-symmetrised power of L^N (in other words, the irreducible representation of $U_q(\mathrm{gl}_N)$ indexed by the one-row partition of size k) and $H_{\vec{k}}$ is called the fused Hecke algebra. Again, the algebra $H_{\vec{k}}$ does not depend on N and for large N is exactly the centraliser. The dependence of the centralisers on N appears in the kernel of the surjective map, of which an explicit description is conjectured in [7] and proved in some cases, including the ones we will study in this paper. For $N = 2$, the centralisers can be called the fused Temperley–Lieb algebras in our terminology. They appear in several recent works and are also known as seam algebras, valenced Temperley–Lieb algebra or Jones–Wenzl algebras [1; 10; 27].

There is no known presentation by generators and relations for the fused Hecke algebra $H_{\vec{k}}$ in general and in turn no known presentation for the $U_q(\mathrm{gl}_N)$-centralisers (see [10] for a study of this question for the Jones–Wenzl algebras, that is, for $U_q(\mathrm{gl}_2)$-centralisers),

This paper is concerned with the case where only the first representation in (1-1) is fused. Namely we fix $k > 0$ and $n \geq 0$, and we denote by $H_{k,n}$ the fused Hecke algebra corresponding to the following centraliser:

$$H_{k,n} \to \mathrm{End}_{U_q(\mathrm{gl}_N)}(L^N_{(k)} \otimes (L^N)^{\otimes n}). \tag{1-2}$$

This situation is commonly referred to as the one-boundary case. Such one-boundary centraliser algebras have been studied especially for $U_q(\mathrm{gl}_2)$, and also often with an infinite-dimensional module (like a Verma module) as the first factor; see [4; 13; 15] for recent works. For $N = 2$, the one-boundary case of the fused Temperley–Lieb algebra is referred to as the boundary seam algebra [16; 17; 19] and is a quotient of the one-boundary Temperley–Lieb algebra or blob algebra [8; 18; 20]. The presentation given in [19], even if not explicitly stated this way, can be seen as a description of the centraliser (1-2) in the gl_2 case.

The first main goal of this paper is to give an algebraic presentation of the fused Hecke algebra $H_{k,n}$ and of its quotients corresponding to the centralisers for any N. In particular, we obtain the boundary seam algebra ($N = 2$) explicitly as a quotient of the fused Hecke algebra $H_{k,n}$, and provide its generalisation for any $N > 2$.

The second main purpose of this paper is the introduction of a new algebra, which we denote $\mathcal{A}_n$ and call the symmetric one-boundary Hecke algebra. Roughly speaking, this 3-parameter algebra $\mathcal{A}_n$ allows to interpolate between all algebras $H_{k,n}$ for varying k. The word *symmetric* is meant to recall the fact that the representations allowed at the boundary are the q-symmetric powers. The algebra $\mathcal{A}_n$ admits as a quotient the 3-parameter one-boundary Temperley–Lieb algebra, which we denote here $\mathcal{C}_{n,2}$, and we also define naturally as quotients of $\mathcal{A}_n$ the generalisations $\mathcal{C}_{n,N}$ corresponding to gl_N for any $N > 2$.

We now describe more precisely, step by step, the algebras involved in the paper. It is well known, see, for example, [22], that the one-boundary centraliser in (1-2) is a quotient of the affine Hecke algebra. Moreover, since the partition (k) made of a line of k boxes has only two addable nodes, this quotient factors through a cyclotomic quotient of level 2. This is the starting point of our constructions.

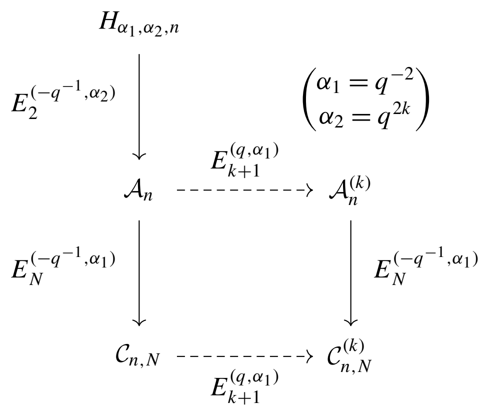
The starting point $H_{\alpha_1,\alpha_2,n}$. We start with the cyclotomic Hecke algebra of level 2 $H_{\alpha_1,\alpha_2,n}$ defined over the ring $\mathbb{C}[q^{\pm 1}, \alpha_1^{\pm 1}, \alpha_2^{\pm 2}]$ with three indeterminates. The indeterminates α_1 and α_2 correspond to the eigenvalues of the boundary, or type B, generator, while the eigenvalues of the other generators are q and $-q^{-1}$.

The algebra $H_{\alpha_1,\alpha_2,n}$ has a standard basis indexed by the signed permutations and we have a good understanding of the representation theory over the field of fractions $\mathbb{C}(q, \alpha_1, \alpha_2)$. Namely, the algebra is semisimple and the irreducible representations are indexed by bipartitions of n. Among these irreducible representations, four of them are of dimension 1 and they correspond to the following bipartitions:

$$(\square\cdots\square\,,\varnothing),\quad \left(\begin{matrix}\square\\ \vdots\\ \square\end{matrix}\,,\varnothing\right),\quad (\varnothing\,,\square\cdots\square),\quad \left(\varnothing\,,\begin{matrix}\square\\ \vdots\\ \square\end{matrix}\right),$$

$$(q,\alpha_1),\qquad (-q^{-1},\alpha_1),\qquad (q,\alpha_2),\qquad (-q^{-1},\alpha_2).$$

Each of these one-dimensional representations corresponds to a choice of eigenvalues for the generators, as indicated above. Moreover, explicit expressions for the minimal central idempotents corresponding to these representations are known (see Section 2B). These idempotents live in the algebra $H_{\alpha_1,\alpha_2,n}$ extended over the field of fractions $\mathbb{C}(q, \alpha_1, \alpha_2)$, and by simply removing the denominators in these explicit expressions, we obtain central quasiidempotents in the algebra $H_{\alpha_1,\alpha_2,n}$ well defined over the ring of polynomials. These quasiidempotents are denoted $E_n^{(x,y)}$, where (x, y) are the corresponding eigenvalues, and will be crucial to all our constructions. Indeed all algebras involved in the paper are obtained by quotienting by some of these quasiidempotents, as summarised in the picture below.



The full lines represent genuine quotients, while dashed lines represent quotients combined with a specialisation of the parameters α_1, α_2 as indicated in the diagram. We briefly detail each step of the diagram.

The algebra $\mathcal{A}_n$. The symmetric one-boundary Hecke algebra $\mathcal{A}_n$ is obtained from $H_{\alpha_1,\alpha_2,n}$ by quotienting out the quasiidempotent $E_2^{(-q^{-1},\alpha_2)}$. This quotient is the main object of study of Section 2. Quite naturally from its definition, the irreducible representations of the algebra $\mathcal{A}_n$ over the field of fractions are indexed by bipartitions with a one-row partition as the second component. Our first main result is that the algebra $\mathcal{A}_n$ is free over $\mathbb{C}[q^{\pm 1}, \alpha_1^{\pm 1}, \alpha_2^{\pm 2}]$ and we provide a basis in terms of signed permutations with avoiding patterns. We conclude this section with a technical fact, namely, that some of the remaining central quasiidempotents $E_n^{(x,y)}$ can be renormalised in $\mathcal{A}_n$ while still being well defined over $\mathbb{C}[q^{\pm 1}, \alpha_1^{\pm 1}, \alpha_2^{\pm 2}]$. This will be important for what follows in order for the subsequently defined quotients to behave well.

The algebras $\mathcal{C}_{n,N}$. In Section 3, for $N > 1$, the symmetric one-boundary N-centraliser algebras $\mathcal{C}_{n,N}$ are defined by further quotienting $\mathcal{A}_n$ by one of the remaining (and renormalised) quasiidempotent when $n \geq N$ (as well as the usual q-antisymmetriser on $N+1$ points). The relevant quasiidempotent is indicated in the diagram above and this definition leads easily to the description of the representation theory over the field of fractions: the irreducible representations are now indexed by bipartitions (λ, μ) where μ is a one-row partition and λ has strictly less than N rows.

The algebras $\mathcal{C}_{n,N}$ include the one-boundary Temperley–Lieb algebra, which is the case $N = 2$, and provide its natural generalisation for general N. The name comes from the fact that the algebra $\mathcal{C}_{n,N}$ somehow interpolates the $U_q(\mathrm{gl}_N)$-centralisers in (1-2) for fixed N and varying k. The case $N=2$ is examined in more details and again, we show that $\mathcal{C}_{n,2}$ is free over the ring $\mathbb{C}[q^{\pm 1}, \alpha_1^{\pm 1}, \alpha_2^{\pm 2}]$ with a basis also given in terms of signed permutations with certain avoiding patterns (this is where the renormalisation mentioned above is important). This description can be seen as a one-boundary generalisation of the description of the usual Temperley–Lieb algebra in terms of usual permutations with avoiding patterns.

The algebras $\mathcal{A}_n^{(k)}$. Section 4 is mainly devoted to the algebraic description of the fused Hecke algebra $H_{k,n}$. For this purpose, we define the algebra $\mathcal{A}_n^{(k)}$ as a specialisation of $\mathcal{A}_n$ followed by a quotient by another (renormalized) quasiidempotent for $n \geq k+1$. The specialisation replaces α_1 and α_2 by the eigenvalues of the boundary generator in the fused Hecke algebra. Their values are indicated in the diagram above together with the relevant quasiidempotent. The main result of the section is that the algebras $\mathcal{A}_n^{(k)}$ and $H_{k,n}$ are isomorphic. This leads to a presentation of the fused Hecke algebra in terms of generators and relations. Again, a basis of $\mathcal{A}_n^{(k)}$ in terms of signed permutations with avoiding patterns is provided.

The algebras $\mathcal{C}_{n,N}^{(k)}$. Lastly, in Section 5, the algebras $\mathcal{C}_{n,N}^{(k)}$ are defined by naturally completing the square of the picture above. Namely, they are defined either as specialisations and quotients of $\mathcal{C}_{n,N}$, or equivalently as quotients of $\mathcal{A}_n^{(k)}$. The algebras $\mathcal{C}_{n,N}^{(k)}$ are shown to be isomorphic to the $U_q(\mathrm{gl}_N)$-centraliser in (1-2) and this provides an algebraic description of the centraliser. We show that we have reobtained naturally with $\mathcal{C}_{n,2}^{(k)}$ the boundary seam algebra of [19] and thus the algebras $\mathcal{C}_{n,N}^{(k)}$ can be seen as the gl_N-generalisations. Following our results, a natural definition of the algebra $\mathcal{C}_{n,2}^{(k)}$ over $\mathbb{C}[q^{\pm 1}]$ is given (and here we differ from [19]) and it is shown to be free over $\mathbb{C}[q^{\pm 1}]$ with a basis given explicitly.

2. The symmetric one-boundary Hecke algebra

We will use the notations

$$[r]_x = \frac{x^r - x^{-r}}{x - x^{-1}} = x^{r-1} + x^{r-3} + \cdots + x^{1-r} \quad \text{and} \quad [r]_x! = [2]_x \ldots [r]_x.$$

We will be working with the ring $R = \mathbb{C}[q^{\pm 1}, \alpha_1^{\pm 1}, \alpha_2^{\pm 1}]$, where q, α_1, α_2 are indeterminates, and with its field of fractions $F = \mathbb{C}(q, \alpha_1, \alpha_2)$.

2A. ***The cyclotomic Hecke algebra of level 2.*** Let $n \geq 0$. We define the algebra $H_{\alpha_1,\alpha_2,n}$ as the algebra over R with generators g_i for $i = 0, 1, \ldots, n-1$ and defining relations

$$g_i\, g_{i+1}\, g_i = g_{i+1}\, g_i\, g_{i+1}, \quad 1 \le i \le n-2, \tag{2-1}$$

$$g_0\, g_1\, g_0\, g_1 = g_1\, g_0\, g_1\, g_0, \tag{2-2}$$

$$g_i\, g_j = g_j\, g_i, \quad |i-j| \ge 2, \tag{2-3}$$

$$(g_i - q)(g_i + q^{-1}) = 0, \quad 1 \le i \le n-1, \tag{2-4}$$

$$(g_0 - \alpha_1)(g_0 - \alpha_2) = 0. \tag{2-5}$$

By convention, $H_{\alpha_1,\alpha_2,0} = R$. The algebra is a quotient of the affine Hecke algebra of type A by the last relation. It is called a cyclotomic Hecke algebra of level 2 since this last relation is a quadratic characteristic relation for g_0. For what is recalled below about $H_{\alpha_1,\alpha_2,n}$, see, e.g., [11].

The algebra $H_{\alpha_1,\alpha_2,n}$ is free as an R-module and has a basis labelled by the elements of the Coxeter group of type B_n. We will abuse notations and denote B_n this Coxeter group. Its elements can be viewed as signed permutations, that is, those permutations ω on the set $\{-n, -n+1, \dots, -1, 1, 2, \dots, n\}$ such that we have $\omega(-i) = -\omega(i)$ for all $i \in \{1, 2, \dots, n\}$. We can represent the elements of B_n by words $b = b_1 b_2 \dots b_n$ where each of the numbers $1, 2, \dots, n$ appears once and is possibly barred (see, e.g., [25]). In this representation, b_i is the image of i by ω, where the bar notation is understood as a negative sign. The group B_n contains $n!\, 2^n$ elements.

We denote by s_i the transposition of i and $i+1$ in B_n (which thus also transposes $-i$ and $-(i+1)$), and by s_0 the transposition of -1 and 1. The group B_n is generated by s_i with $i = 0, \dots, n-1$. For an element $\omega \in B_n$, we write it as a reduced expression, that is, as a product $s_{i_1} \dots s_{i_k}$ with minimal k. We set $g_\omega = g_{i_1} \dots g_{i_k}$ in $H_{\alpha_1,\alpha_2,n}$. The element g_ω does not depend on the choice of the reduced expression, and the set $\{g_\omega\}_{\omega \in B_n}$ forms an R-basis of $H_{\alpha_1,\alpha_2,n}$. A standard choice of reduced forms leads to an explicit expression for the basis as the following product of sets:

$$\begin{Bmatrix} 1, \\ g_0 \end{Bmatrix} \cdot \begin{Bmatrix} 1, \\ g_1, \\ g_1\, g_0, \\ g_1\, g_0\, g_1 \end{Bmatrix} \cdot \begin{Bmatrix} 1, \\ g_2, \\ g_2\, g_1, \\ g_2\, g_1\, g_0, \\ g_2\, g_1\, g_0\, g_1, \\ g_2\, g_1\, g_0\, g_1\, g_2 \end{Bmatrix} \cdot \dots \cdot \begin{Bmatrix} 1, \\ g_{n-1}, \\ \vdots \\ g_{n-1} \cdots g_1\, g_0, \\ g_{n-1} \cdots g_1\, g_0\, g_1, \\ \vdots \\ g_{n-1} \cdots g_1\, g_0\, g_1 \cdots g_{n-1} \end{Bmatrix}. \tag{2-6}$$

Introducing the following notation for $0 \le m \le n$:

$$[n, m] = g_n \dots g_{m+1} g_m \quad \text{and} \quad [n, -m] = g_n \dots g_1\, g_0\, g_1 \dots g_m,$$

the basis elements can be written as

$$[n_1, m_1][n_2, m_2] \dots [n_k, m_k], \quad \text{with } 0 \le n_1 < n_2 < \dots < n_k \le n-1 \text{ and } |m_i| \le n_i.$$

The algebra $H_{\alpha_1,\alpha_2,n-1}$ is naturally a subalgebra of $H_{\alpha_1,\alpha_2,n}$, the one generated by $g_0, \dots, g_{n-2}$, where elements of B_{n-1} are naturally identified with elements of B_n leaving invariant the letter n.

Hecke algebra of type A. The algebra generated by $g_1, \dots, g_{n-1}$ with defining relations (2-1), (2-3), (2-4) is the usual Hecke algebra H_n of type A, associated to the symmetric group $\mathfrak{S}_n$ on n letters. It is naturally identified as the subalgebra of $H_{\alpha_1,\alpha_2,n}$ generated by $g_1, \dots, g_{n-1}$, and a basis of H_n is the subset $\{g_\omega\}_{\omega\in\mathfrak{S}_n}$, when the symmetric group is naturally considered as a subgroup of B_n.

The basis of H_n is made of those elements in (2-6) which do not contain g_0. The basis (2-6) is well adapted to the inclusion $B_{n-1} \subset B_n$. There is a different way to produce a basis of $H_{\alpha_1,\alpha_2,n}$ adapted to the inclusion $\mathfrak{S}_n \subset B_n$, which is the set of elements:

$$g_\omega \cdot g_0\, g_1 \dots g_{i_1} \dots g_0\, g_1 \dots g_{i_k}, \quad \omega \in \mathfrak{S}_n,\ n-1 \geq i_1 > \dots > i_k \geq 0. \tag{2-7}$$

2B. ***Central quasiidempotents in $H_{\alpha_1,\alpha_2,n}$.*** For $i = 0, 1, \dots, n-1$ and $\omega \in B_n$, we denote $\ell(\omega)$ the length of ω, which is the number of generators appearing in any reduced expression of ω. We denote $\ell_0(\omega)$ the number of times that the generator s_0 appears in a reduced expression for ω. This does not depend on the reduced expression since all braid relations in $H_{\alpha_1,\alpha_2,n}$ are homogeneous in g_0.

q-symmetriser and q-antisymmetriser in the Hecke algebra. First we discuss the standard quasiidempotents in the usual Hecke algebra H_n generated by $g_1, \dots, g_{n-1}$. Let $x \in \{q, -q^{-1}\}$ and set

$$\Lambda_n^x(g_1, \dots, g_{n-1}) = \sum_{\omega\in\mathfrak{S}_n} x^{\ell(\omega)} g_\omega. \tag{2-8}$$

By convention, $\Lambda_1^x = 1$. Using the basis in (2-6) without the elements containing g_0, we find the recursive formula

$$\Lambda_n^x(g_1, \dots, g_{n-1}) = \Lambda_{n-1}^x(g_1, \dots, g_{n-2})(1 + x g_{n-1} + \dots + x^{n-1} g_{n-1} \dots g_1).$$

It is well known and easy to check (see the proof of Proposition 2.1 below) that

$$\begin{aligned}\Lambda_n^x(g_1, \dots, g_{n-1})\, g_i &= g_i \Lambda_n^x(g_1, \dots, g_{n-1})\\ &= x\Lambda_n^x(g_1, \dots, g_{n-1}), \quad i = 1, \dots, n-1.\end{aligned}$$

It follows that these two elements are central in H_n and are quasiidempotents, namely,

$$(\Lambda_n^x(g_1, \dots, g_{n-1}))^2 = q^{\pm\frac{1}{2}(n(n-1))} [n]_q!\, \Lambda_n^x(g_1, \dots, g_{n-1}), \quad x = \pm q^{\pm 1}.$$

To find the coefficient, one needs to replace each g_i by x in the formula for the quasiidempotents. This is easily done using their recursive formula. We refer to the element with $x=q$ as the (unnormalised) q-symmetriser of H_n and to the element with $x=-q^{-1}$ as the (unnormalised) q-antisymmetriser of H_n.

The four quasiidempotents in $H_{\alpha_1,\alpha_2,n}$. Let $x\in\{q,-q^{-1}\}$ and $b\in\{1,2\}$. In what follows, it will be convenient to consider the indices of α_1,α_2 modulo 2, so that $\alpha_{b+1}=\alpha_1$ when $b=2$. We define

$$E_n^{(x,\alpha_b)} := \sum_{\omega\in B_n} z_\omega\, g_\omega,\qquad z_\omega := x^{\ell(\omega)-\ell_0(\omega)}(-\alpha_{b+1}^{-1})^{\ell_0(\omega)}. \tag{2-9}$$

By convention, $E_0^{(x,\alpha_b)}=1$. Using the standard basis in (2-6), a recursive formula for these elements is

$$E_n^{(x,\alpha_b)} = E_{n-1}^{(x,\alpha_b)}\biggl(1+\sum_{i=1}^{n-1} x^{n-i} g_{n-1}\dots g_i - x^{n-1}\alpha_{b+1}^{-1} g_{n-1}\dots g_1 g_0\Bigl(1+\sum_{i=1}^{n-1} x^i g_1\dots g_i\Bigr)\biggr). \tag{2-10}$$

Using the basis (2-7) adapted to the embedding $H_n\subset H_{\alpha_1,\alpha_2,n}$, we also have

$$E_n^{(x,\alpha_b)} = \Lambda_n^x(g_1,\dots,g_{n-1})\cdot(1-x^{n-1}\alpha_{b+1}^{-1} g_0 g_1\dots g_{n-1}) \dots (1-x\alpha_{b+1}^{-1} g_0 g_1)(1-\alpha_{b+1}^{-1} g_0). \tag{2-11}$$

Explicit examples for small n are

$$\begin{aligned} E_1^{(x,\alpha_b)} &= 1-\alpha_{b+1}\, g_0,\\ E_2^{(x,\alpha_b)} &= 1+xg_1-\alpha_{b+1}\, g_0 - x\alpha_{b+1}(g_1 g_0+g_0 g_1)\\ &\qquad - x^2\alpha_{b+1}\, g_1 g_0 g_1 + x\alpha_{b+1}^2\, g_0 g_1 g_0 + x^2\alpha_{b+1}^2\, g_0 g_1 g_0 g_1. \end{aligned}$$

We recall the important facts about the elements $E_n^{(x,\alpha_b)}$, implying in particular that they are central quasiidempotents of $H_{\alpha_1,\alpha_2,n}$.

Proposition 2.1. *Let $x=\pm q^{\pm 1}$ and $b\in\{1,2\}$. We have*

$$E_n^{(x,\alpha_b)} g_0 = g_0\, E_n^{(x,\alpha_b)} = \alpha_b\, E_n^{(x,\alpha_b)} \quad\textit{and}\quad E_n^{(x,\alpha_b)} g_i = g_i\, E_n^{(x,\alpha_b)} = x\, E_n^{(x,\alpha_b)} \tag{2-12}$$

for $i=1,\dots,n-1$, and

$$\begin{aligned} &(E_n^{(x,\alpha_b)})^2 = P_n(x,\alpha_b)\, E_n^{(x,\alpha_b)},\\ &P_n(x,\alpha_b) = q^{\pm\frac{1}{2}(n(n-1))}[n]_q!\prod_{i=0}^{n-1}\Bigl(1-q^{\pm 2i}\frac{\alpha_b}{\alpha_{b+1}}\Bigr). \end{aligned} \tag{2-13}$$

Proof. For any $\omega \in B_n$ and any $i \in \{0, 1, \dots, n-1\}$, we have $\ell(s_i\omega) = \ell(\omega) \pm 1$. If $\ell(s_i\omega) > \ell(\omega)$, then we have $g_{s_i w} = g_i\, g_w$. Therefore we can write

$$E_n^{(x,\alpha_b)} = \sum_{\substack{\omega\in B_n \\ \ell(s_i\omega)>\ell(\omega)}} (z_\omega\, g_\omega + z_{s_i\omega}\, g_i\, g_\omega). \tag{2-14}$$

Note that under the hypothesis that $\ell(s_i\omega) > \ell(\omega)$, we must have $z_{s_i\omega} = x z_\omega$ if $1 \le i \le n-1$ and $z_{s_i\omega} = -\alpha_{b+1}^{-1} z_\omega$ if $i = 0$. The defining relations (2-4) and (2-5) imply that

$$(g_0 - \alpha_b)(1 - \alpha_{b+1}^{-1} g_0) = 0 \quad \text{and} \quad (g_i - x)(g_i + x^{-1}) = 0 \tag{2-15}$$

for $i = 1, \dots, n-1$. Using the previous equations, we have, for example, if $1 \le i \le n-1$, that

$$\begin{aligned}(g_i - x)\, E_n^{(x,\alpha_b)} &= \sum_{\substack{\omega\in B_n \\ \ell(s_i\omega)>\ell(\omega)}} (g_i - x)(z_\omega\, g_\omega + x z_\omega g_i\, g_\omega) \\ &= \sum_{\substack{\omega\in B_n \\ \ell(s_i\omega)>\ell(\omega)}} x z_\omega (g_i - x)(x^{-1} + g_i)\, g_\omega = 0.\end{aligned} \tag{2-16}$$

The case $i = 0$ is similarly done. Moreover, similar arguments can be used when considering instead the product $E_n^{(x,\alpha_b)}\, g_i$ for $0 \le i \le n-1$. This proves (2-12).

Now the coefficient $P_n(x, \alpha_b)$ in (2-13) is found by replacing g_0 by α_b and the other g_i's by x in the formula for $E_n^{(x,\alpha_b)}$. The given formula for $P_n(x, \alpha_b)$ follows then easily from (2-11). □

Remark 2.2. Over the field of fractions F, or in a specialisation with $P_{n-1}(x, \alpha_b) \neq 0$, we have also the recursive formula

$$\begin{aligned}E_n^{(x,\alpha_b)} = E_{n-1}^{(x,\alpha_b)} &+ x\frac{E_{n-1}^{(x,\alpha_b)}\, g_{n-1}\, E_{n-1}^{(x,\alpha_b)}}{P_{n-2}(x, \alpha_b)} \\ &- x^{2(n-1)} \alpha_{b+1}^{-1} \frac{E_{n-1}^{(x,\alpha_b)}\, g_{n-1} \cdots g_1\, g_0\, g_1 \cdots g_{n-1}\, E_{n-1}^{(x,\alpha_b)}}{P_{n-1}(x, \alpha_b)}.\end{aligned} \tag{2-17}$$

2C. ***The symmetric one-boundary Hecke algebra $\mathcal{A}_n$.*** We define below the main object of this section that we call the symmetric one-boundary Hecke algebra.

Definition 2.3. Let $n \ge 0$. We define the symmetric one-boundary Hecke algebra $\mathcal{A}_n$ as the algebra over R which is the quotient of $H_{\alpha_1,\alpha_2,n}$ by the relation

$$E_2^{(-q^{-1},\alpha_2)} = 0. \tag{2-18}$$

It is understood that $\mathcal{A}_n = H_{\alpha_1,\alpha_2,n}$ if $n = 0, 1$.

Using the explicit expression of $E_2^{(-q^{-1},\alpha_2)}$, this is equivalent to imposing the following relation:

$$g_0\,g_1\,g_0\,g_1 = -q^2\alpha_1^2 + q\alpha_1^2 g_1 + q^2\alpha_1\,g_0 - q\alpha_1(g_1\,g_0 + g_0\,g_1) + \alpha_1 g_1\,g_0\,g_1 + q g_0\,g_1\,g_0. \tag{2-19}$$

Semisimple representation theory. Here we extend the algebras $H_{\alpha_1,\alpha_2,n}$ and $\mathcal{A}_n$ over the field of fractions F, and denote them $FH_{\alpha_1,\alpha_2,n}$ and $F\mathcal{A}_n$ to avoid any confusion. The representation theory of $FH_{\alpha_1,\alpha_2,n}$ is well known [3; 12; 21], and can be described in terms of bipartitions and Young tableaux.

A partition λ of n, denoted $\lambda \vdash n$, is a decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$ such that $\lambda_1 + \dots + \lambda_k = n$. We also say that λ is a partition *of size* n and denote $|\lambda| = n$. We identify partitions with their Young diagrams: the Young diagram of λ is a left-justified array of rows of boxes such that the j-th row (we count from top to bottom) contains λ_j boxes. The number of nonempty rows is the length $\ell(\lambda)$ of λ. By convention, the empty set $\varnothing$ is the only partition of $n = 0$.

A standard tableau of shape λ is a bijective filling of the boxes of λ by numbers $1, \dots, n$ such that the entries strictly increase along any row and down any column of the diagram. We denote by d_λ the number of standard tableaux of shape λ. From the representation theory of the symmetric group, or from the Robinson–Schensted correspondence, we have

$$\sum_{\lambda \vdash n} d_\lambda^2 = n!. \tag{2-20}$$

A bipartition of size n is a pair of partitions (λ, μ) such that $|\lambda| + |\mu| = n$. We denote $\mathrm{Par}_2(n)$ the set of bipartitions of n. A standard tableau of shape (λ, μ) is a bijective filling of the boxes of λ and μ by the numbers $1, \dots, n$ such that the entries strictly increase along any row and down any column of the two diagrams. The number of standard tableaux of shape (λ, μ) is easily seen to be

$$d_{\lambda,\mu} = \binom{n}{|\lambda|} d_\lambda\, d_\mu. \tag{2-21}$$

The set of irreducible representations of $FH_{\alpha_1,\alpha_2,n}$ is indexed by the bipartitions of size n, and we will denote $V_{(\lambda,\mu)}$ the irreducible representations indexed by $(\lambda, \mu) \in \mathrm{Par}_2(n)$ so that

$$\mathrm{Irr}(FH_{\alpha_1,\alpha_2,n}) = \{V_{(\lambda,\mu)} \mid (\lambda, \mu) \in \mathrm{Par}_2(n)\}.$$

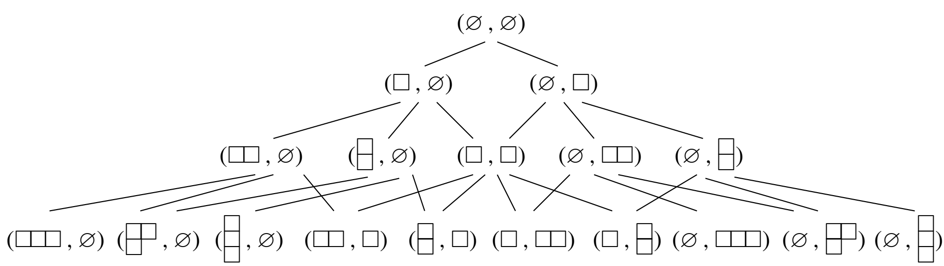
There are four one-dimensional representations of $FH_{\alpha_1,\alpha_2,n}$ for $n \geq 2$ and the parametrisation is made such that they correspond to the following bipartitions of n, with the given corresponding values, respectively, of g_0 and of g_i, $i = 1, \dots, n-1$:

$$
\begin{array}{llll}
(\square\cdots\square\,,\varnothing), & \left(\begin{array}{c}\square\\ \vdots\\ \square\end{array},\varnothing\right), & (\varnothing\,,\square\cdots\square), & \left(\varnothing\,,\begin{array}{c}\square\\ \vdots\\ \square\end{array}\right),\\
g_0 \mapsto \alpha_1, & g_0 \mapsto \alpha_1, & g_0 \mapsto \alpha_2, & g_0 \mapsto \alpha_2,\\
g_i \mapsto q, & g_i \mapsto -q^{-1}, & g_i \mapsto q, & g_i \mapsto -q^{-1}.
\end{array}
\tag{2-22}
$$

Moreover, the branching rules expressing the restriction from $H_{\alpha_1,\alpha_2,n}$ to $H_{\alpha_1,\alpha_2,n-1}$ are given by inclusion of bipartitions (or more precisely, of their Young diagrams), as shown in the beginning of the Bratteli graph below. We refer to an appendix in [7] for a discussion of Bratteli diagrams and of quotients of semisimple algebras by central idempotents.

The parametrisation of the irreducible representations is uniquely fixed by these requirements, and the dimension of the irreducible representation $V_{(\lambda,\mu)}$ is the number of standard tableaux of shape (λ, μ):

$$\dim V_{(\lambda,\mu)} = d_{(\lambda,\mu)} = \binom{n}{|\lambda|} d_\lambda\, d_\mu,$$



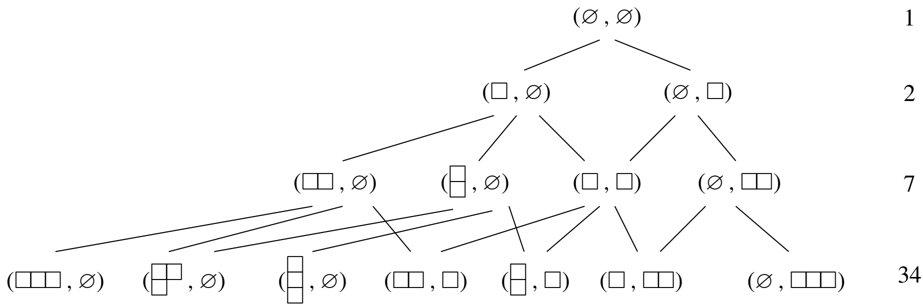
Thanks to its properties recalled in Proposition 2.1, the element $E_m^{(x,\alpha_b)}$ in $FH_{\alpha_1,\alpha_2,m}$ is a nonzero element proportional to the minimal central idempotent corresponding to the one-dimensional representation associated to (x, α_b). It means that it is nonzero in this one-dimensional representation of $FH_{\alpha_1,\alpha_2,m}$ and acts as 0 in any other irreducible representation. Now if $n \geq m$, it follows that $E_m^{(x,\alpha_b)}$ seen as an element of $H_{\alpha_1,\alpha_2,n}$ is nonzero in an irreducible representation if and only if this irreducible representation contains in its restriction to $H_{\alpha_1,\alpha_2,m}$ the given one-dimensional representation. Therefore the quotient by $E_m^{(x,\alpha_b)} = 0$ removes exactly these irreducible representations.

In the particular case of $F\mathcal{A}_n$, which is the quotient of $H_{\alpha_1,\alpha_2,n}$ by the relation $E_2^{(-q^{-1},\alpha_2)} = 0$, we recall the indexing of one-dimensional representations set up in (2-22), and we find that the disappearing representations are those $V_{(\lambda,\mu)}$ with μ having at least two nonempty rows. We summarise this discussion in the following proposition.

Proposition 2.4. *The algebra $F\mathcal{A}_n$ is semisimple with the following set of irreducible representations*:

$$\mathrm{Irr}(F\mathcal{A}_n) = \{V_{(\lambda,\mu)} \mid (\lambda, \mu) \in \mathrm{Par}_2(n) \text{ and } \ell(\mu) < 2\}.$$

The Bratteli diagram for the algebras $\mathcal{A}_n$ is obtained from the Bratteli diagram for the algebras $H_{\alpha_1,\alpha_2,n}$ above, where all bipartitions with more than one row in the second component are removed:



The dimension of $F\mathcal{A}_n$ can be easily calculated, by summing the squares of the dimensions of the irreducible representations:

$$\dim(F\mathcal{A}_n) = \sum_{i=0}^{n} \sum_{\lambda \vdash n-i} (\dim V_{(\lambda,(i))})^2 = \sum_{i=0}^{n} \binom{n}{i}^2 \sum_{\lambda \vdash n-i} d_\lambda^2 = \sum_{i=0}^{n} \binom{n}{i}^2 (n-i)!,$$

where we first split the sum according to the size of the second partition μ, which must be a single line of i boxes, and then we use successively (2-21) and (2-20). The dimensions for $n = 0, 1, 2, 3$ are written in the diagram above.

Remark 2.5. The above description of the representations of $F\mathcal{A}_n$ is also valid for many specialisations of the parameters α_1, α_2, q in $\mathcal{A}_n$, namely, those specialisations such that the algebra $H_{\alpha_1,\alpha_2,n}$ is semisimple. This happens if q^2 is not a root of unity of order $e \leq n$ and $\alpha_1 \neq \alpha_2 q^{\pm 2i}$ for $i = 1, \dots, n-1$, see [2].

An R-basis of $\mathcal{A}_n$. A word $b = b_1 b_2 \dots b_n$ of B_n is said to be $\bar{1}\bar{2}$-avoiding if all barred numbers in b appear in decreasing order, see [25]. Put differently, b avoids the pattern $\bar{1}\bar{2}$ if there are no two indices $1 \leq i < j \leq n$ such that $b_i = \bar{m}_1$ and $b_j = \bar{m}_2$ with $0 < m_1 < m_2$. For instance, $35\bar{6}1\bar{4}\bar{2}$ is $\bar{1}\bar{2}$-avoiding in B_6 while $35\bar{4}1\bar{6}\bar{2}$ is not because of the subsequence $\bar{4}\bar{6}$.

We will denote by $B_n(\bar{1}\bar{2})$ the subset of all signed permutations in B_n which are $\bar{1}\bar{2}$-avoiding. A word $b = b_1 b_2 \dots b_n$ corresponding to a permutation in $B_n(\bar{1}\bar{2})$ can be written as follows: choose i numbers in $\{1, 2, \dots, n\}$ that will be barred, choose i positions among n to place these barred numbers in decreasing order in b, and then permute the remaining $n-i$ numbers in the remaining $n-i$ positions in b.

It follows that

$$|B_n(\bar{1}\bar{2})| = \sum_{i=0}^{n} (n-i)! \binom{n}{i}^2. \tag{2-23}$$

We are now ready to give a basis of $\mathcal{A}_n$ over R. Thus, this also gives a basis for any specialisation of q, α_1, α_2 to nonzero complex numbers.

Theorem 2.6. *The algebra $\mathcal{A}_n$ is free over R with basis given by the set of elements g_ω corresponding to $\bar{1}\bar{2}$-avoiding signed permutations*, *i.e.*, *the basis is*

$$\{g_\omega \mid \omega \in B_n(\bar{1}\bar{2})\}. \tag{2-24}$$

Proof. We first prove that the set (2-24) is a spanning set

$$\mathcal{A}_n = \operatorname{span}_R\{g_\omega \mid \omega \in B_n(\bar{1}\bar{2})\}. \tag{2-25}$$

Since $\mathcal{A}_n$ is a quotient of $H_{\alpha_1,\alpha_2,n}$, it is clearly linearly generated by the set of elements g_ω with $\omega \in B_n$. To show that this spanning set can be reduced to (2-25), we proceed by induction on the length of elements in B_n.

Any signed permutation $\omega \in B_n$ of length $\ell(\omega) < 4$ is such that at most one number in $\{1, 2, \dots, n\}$ is mapped to a barred number. Therefore, all signed permutations ω with $\ell(\omega) < 4$ are $\bar{1}\bar{2}$-avoiding and the associated elements g_ω belong to the spanning set (2-25).

Suppose now that all elements g_ω with $\ell(\omega) \le m$, for some fixed integer $m \ge 4$, belong to the span of the set (2-24). Consider an element g_ω with $\ell(\omega) = m+1$ such that $\omega \notin B_n(\bar{1}\bar{2})$. It follows by [28, Lemma 2.1] that ω must contain $s_0 s_1 s_0 s_1$ in some reduced expression. This implies in turn that there is a reduced expression for g_ω that contains $g_0 g_1 g_0 g_1$. Relation (2-19) can hence be used to express g_ω as a linear combination of terms of length less than $m+1$. By induction hypothesis, we therefore conclude that g_ω can be written in terms of the set (2-24).

Now, since the set (2-24) is a spanning set of $\mathcal{A}_n$ over R, it is also a spanning set of $F\mathcal{A}_n$ over F. Moreover, the dimension of $F\mathcal{A}_n$ over F was calculated after Proposition 2.4 and it coincides with the cardinality of the spanning set. Therefore, the set (2-24) is a basis of $F\mathcal{A}_n$ and in particular is linearly independent over F. Thus it is also linearly independent over R (since $R \subset F$). We conclude that the set (2-24) is an R-basis of $\mathcal{A}_n$. □

Remark 2.7. The elements of B_n which avoid the pattern $\bar{1}\bar{2}$ are in fact the same as the elements which do not contain $s_0 s_1 s_0 s_1$ in any reduced expression, see [28, Lemma 2.1].

2D. ***Quasiidempotents in $\mathcal{A}_n$.*** Let $n \geq 2$. In $\mathcal{A}_n$, the element $E_n^{(-q^{-1},\alpha_2)}$, which is proportional to $E_2^{(-q^{-1},\alpha_2)}$ is equal to 0. Thus among the four central quasiidempotents $E_n^{(x,\alpha_b)}$ of $H_{\alpha_1,\alpha_2,n}$, only three remain nonzero in $\mathcal{A}_n$:

$$E_n^{(-q^{-1},\alpha_1)}, \quad E_n^{(q,\alpha_2)}, \quad E_n^{(q,\alpha_1)}.$$

In $H_{\alpha_1,\alpha_2,n}$, no common factor appears in the coefficients of these elements. This will now be different in $\mathcal{A}_n$, where a nontrivial common factor in the ring R may sometimes be factored out. Thus we can define renormalised elements defined over R by removing this common factor, and it will be important for later use to do so. In fact, only the two first elements in the list above factorise generically over the ring R, and this is the content of the following statement.

Proposition 2.8. *Let $n \geq 2$. In $\mathcal{A}_n$ we have*

$$E_n^{(-q^{-1},\alpha_1)} = \alpha_2^{-1} \prod_{i=0}^{n-2} \left(1 - \frac{\alpha_1}{\alpha_2} q^{-2i}\right) \tilde{E}_n^{(-q^{-1},\alpha_1)}, \tag{2-26}$$

$$E_n^{(q,\alpha_2)} = q^{\frac{1}{2}(n(n-1))} [n]_q!\, \tilde{E}_n^{(q,\alpha_2)}, \tag{2-27}$$

where the renormalised elements are given by

$$\begin{aligned}\tilde{E}_n^{(-q^{-1},\alpha_1)} &= \Lambda_n(g_1, \dots, g_{n-1}) \\ &\cdot \left(\alpha_2 + \alpha_1 q^{-(n-2)} [n-1]_q - \sum_{i=0}^{n-1} (-1)^i q^{-i} g_0 \dots g_i\right),\end{aligned} \tag{2-28}$$

$$\begin{aligned}\tilde{E}_n^{(q,\alpha_2)} = \tilde{E}_{n-1}^{(q,\alpha_2)} \big((1-q^2)(1 + q g_{n-1} + \dots + q^{n-2} g_{n-1} \dots g_2) \\ + q^{n-1} g_{n-1} \dots g_1 (1 - \alpha_1^{-1} g_0)\big),\end{aligned} \tag{2-29}$$

with the convention that $\tilde{E}_1^{(q,\alpha_2)} = E_1^{(q,\alpha_2)} = (1 - \alpha_1^{-1} g_0)$.

Proof. **Step 1.** First we prove (2-26). Denote $x = -q^{-1}$ and

$$X_n^0 = (1 - x^{n-1} \alpha_2^{-1} g_0\, g_1 \dots g_{n-1})(1 - x^{n-2} \alpha_2^{-1} g_0\, g_1 \dots g_{n-2}) \dots (1 - \alpha_2^{-1} g_0).$$

Recall from (2-11) that we have

$$E_n^{(x,\alpha_1)} = \Lambda_n \cdot X_n^0, \quad \Lambda_n = \Lambda_n^x(g_1, \dots, g_{n-1}).$$

We use induction on n. For $n = 2$, the formula is easy to check by direct calculation. We write the definition of $E_2^{(x,\alpha_1)}$ and use the defining relation of $\mathcal{A}_n$ to replace $g_0\, g_1\, g_0\, g_1$. Now, let $n \geq 3$ and write the above formula as

$$E_n^{(x,\alpha_1)} = \Lambda_n \cdot (1 - x^{n-1} \alpha_2^{-1} g_0\, g_1 \dots g_{n-1})\, X_{n-1}^0. \tag{2-30}$$

Note that

$$\Lambda_n = A\cdot\Lambda^x_{n-1}(g_1,\dots,g_{n-2}) = B\cdot\Lambda^x_{n-1}(g_2,\dots,g_{n-1}) \quad \text{for some } A, B\in H_{\alpha_1,\alpha_2,n}.$$

Besides, we have

$$\Lambda^x_{n-1}(g_2,\dots,g_{n-1})\, g_0\, g_1\dots g_{n-1} = g_0\, g_1\dots g_{n-1}\Lambda^x_{n-1}(g_1,\dots,g_{n-2})$$

using the braid relations. Therefore, in (2-30), we can bring $\Lambda^x_{n-1}(g_1,\dots,g_{n-2})$ in front of X^0_{n-1} and thus use the induction hypothesis. At this point, we have

$$E_n^{(x,\alpha_1)} = \gamma_{n-1}\Lambda_n\cdot(1 - x^{n-1}\alpha_2^{-1}\, g_0\, g_1\dots g_{n-1}) \cdot\Big(\alpha_2 + \alpha_1 q^{-(n-3)}[n-2]_q - \sum_{i=0}^{n-2} x^i g_0\dots g_i\Big), \tag{2-31}$$

with the coefficient γ_{n-1} given by the induction hypothesis. Now we are going to use the defining relation of $\mathcal{A}_n$ in the following form:

$$(1-q^{-1}g_1)\, g_0\, g_1\, g_0 = (1-q^{-1}g_1)(q\alpha_1^2 - q\alpha_1 g_0 + \alpha_1 g_0\, g_1)$$

and the fact that $\Lambda_n = C\cdot(1-q^{-1}g_1)$ for some $C\in H_{\alpha_1,\alpha_2,n}$ to make the following calculation, recalling that $\Lambda_n\, g_k = x\Lambda_n$ for $k = 1,\dots,n-1$:

$$\begin{aligned}
&\Lambda_n\cdot g_0\, g_1\dots g_{n-1}\sum_{i=0}^{n-2} x^i g_0\, g_1\dots g_i\\
&\quad= \Lambda_n\cdot(q\alpha_1^2 - q\alpha_1 g_0 + \alpha_1 g_0\, g_1)\, g_2\dots g_{n-1}\sum_{i=0}^{n-2} x^i g_1\dots g_i\\
&\quad= \Lambda_n\cdot(x^{n-2}q\alpha_1^2 - x^{n-2}q\alpha_1 g_0 + \alpha_1 g_0\, g_1\dots g_{n-1})\sum_{i=0}^{n-2} x^i g_1\dots g_i\\
&\quad= \Lambda_n\cdot\Big(q^{-(n-3)}x^{n-2}\alpha_1^2[n-1]_q\\
&\qquad\qquad + q^{-(n-2)}\alpha_1[n-1]_q\, g_0\, g_1\dots g_{n-1} - x^{n-2}q\alpha_1\sum_{i=0}^{n-2} x^i g_0\, g_1\dots g_i\Big).
\end{aligned}$$

We have used the braid relations to move $g_1\dots g_i$ through $g_0\, g_1\dots g_{n-1}$ (getting $g_2\dots g_{i+1}$), and that $\sum_{i=0}^{n-2} x^i g_1\dots g_i = q^{-(n-2)}[n-1]_q$ when all g's are replaced by $x = -q^{-1}$.

It remains only to use this formula in (2-31) and to collect the various terms. Omitting $\gamma_{n-1}\Lambda_n$, one finds directly that the coefficient in front of $g_0\, g_1\dots g_i$ when $i < n-1$ is $-x^i(1-q^{-2(n-2)}\alpha_1/\alpha_2)$. Then easy manipulations give that the coefficients in front of 1 and in front of $g_0\, g_1\dots g_{n-1}$ are respectively,

$$\Big(1-q^{-2(n-2)}\frac{\alpha_1}{\alpha_2}\Big)(\alpha_2+\alpha_1 q^{-(n-2)}[n-1]_q) \quad\text{and}\quad -\Big(1-q^{-2(n-2)}\frac{\alpha_1}{\alpha_2}\Big)x^{n-1}.$$

This concludes the verification of (2-26).

Step 2. Now we prove (2-27) with similar methods, using induction on n. Once again, the case $n=2$ is directly verified using the explicit expression for $E_2^{(q,\alpha_2)}$ and the defining relation of $\mathcal{A}_n$ to replace $g_0\, g_1\, g_0\, g_1$. For $n \geq 3$, the recursive formula (2-10) allows us to write

$$E_n^{(q,\alpha_2)} = E_{n-1}^{(q,\alpha_2)}\Bigg(1+\sum_{i=1}^{n-1} q^{n-i} g_{n-1}\dots g_i - q^{n-1}\alpha_1^{-1}\, g_{n-1} \dots g_1\, g_0\Bigg(1+\sum_{i=1}^{n-1} q^i g_1 \dots g_i\Bigg)\Bigg). \tag{2-32}$$

Then, we use in (2-32) the defining relation of $\mathcal{A}_n$, which can be rewritten as

$$(-\alpha_1^{-1})(1-\alpha_1^{-1}\, g_0)\, g_1\, g_0\, g_1 = q(1-\alpha_1^{-1}\, g_0)(-q+g_1-\alpha_1^{-1}\, g_1\, g_0),$$

together with the fact that $E_{n-1}^{(q,\alpha_2)} = C\cdot(1-\alpha_1^{-1}\, g_0)$ for some $C \in H_{\alpha_1,\alpha_2,n}$ to get

$$E_n^{(q,\alpha_2)} = E_{n-1}^{(q,\alpha_2)}\Bigg(1+\sum_{i=2}^{n-1} q^{n-i} g_{n-1}\dots g_i + q^{n-1} g_{n-1}\dots g_1(1-\alpha_1^{-1}\, g_0) + q^{n+1} g_{n-1}\dots g_2(-q+g_1-\alpha_1^{-1}\, g_1\, g_0) \cdot\Bigg(1+\sum_{i=2}^{n-1} q^{i-1} g_2\dots g_i\Bigg)\Bigg). \tag{2-33}$$

Recalling that $E_{n-1}^{(q,\alpha_2)} g_k = q E_{n-1}^{(q,\alpha_2)}$ for $k=1,\dots,n-2$, we can use the braid relations and the Hecke relation to obtain

$$E_{n-1}^{(q,\alpha_2)} g_{n-1}\dots g_2\Bigg(1+\sum_{i=2}^{n-1} q^{i-1} g_2\dots g_i\Bigg) = E_{n-1}^{(q,\alpha_2)} q^{n-2}\Bigg(1+\sum_{i=2}^{n-1} q^{n-i} g_{n-1}\dots g_i\Bigg),$$

$$E_{n-1}^{(q,\alpha_2)} g_{n-1}\dots g_2\, g_1(1-\alpha_1^{-1}\, g_0)\Bigg(1+\sum_{i=2}^{n-1} q^{i-1} g_2\dots g_i\Bigg) = E_{n-1}^{(q,\alpha_2)} q^{n-2}[n-1]_q\, g_{n-1}\dots g_2\, g_1(1-\alpha_1^{-1}\, g_0).$$

Replacing these results in (2-33) and combining terms together, it is found that

$$E_n^{(q,\alpha_2)} = E_{n-1}^{(q,\alpha_2)} q^{n-1}[n]_q\Bigg((1-q^2)\Bigg(1+\sum_{i=2}^{n-1} q^{n-i} g_{n-1}\dots g_i\Bigg) + q^{n-1} g_{n-1}\dots g_1(1-\alpha_1^{-1}\, g_0)\Bigg).$$

The proof is completed by using the induction hypothesis on $E_{n-1}^{(q,\alpha_2)}$. □

3. The one-boundary Temperley–Lieb algebra and its gl_N-generalisations

Let $N \geq 2$. In this section we define the symmetric one-boundary N-centraliser algebras $\mathcal{C}_{n,N}$ as quotients of the algebra $\mathcal{A}_n$. The meaning of this definition will be clear from the point of view of representation theory, and will result in a natural description of the semisimple representation theory of $\mathcal{C}_{n,N}$. Besides, our motivation and the origin of the terminology comes from the use we will make of the algebras $\mathcal{C}_{n,N}$ to describe $U_q(\mathrm{gl}_N)$-centralisers in Section 5.

We will then study in details the case $N = 2$, showing that we recover the generic 3-parameter one-boundary Temperley–Lieb algebra, for which we will describe a basis using the signed permutations from the preceding section.

3A. *Definition.* We have defined in (2-8) the q-antisymmetriser $\Lambda_{N+1}^{-q^{-1}}(g_1, \dots, g_N)$ of the usual Hecke algebra generated by $g_1, \dots, g_N$. From Section 2D we have

$$\begin{aligned}(3\text{-}1)\quad & \tilde{E}_n^{(-q^{-1},\alpha_1)} \\ &= \Lambda_n(g_1, \dots, g_{n-1}) \cdot \left(\alpha_2 + \alpha_1 q^{-(n-2)}[n-1]_q - \sum_{i=0}^{n-1}(-1)^i q^{-i} g_0 \dots g_i\right),\end{aligned}$$

which is a renormalisation in $\mathcal{A}_n$ of the quasiidempotent $\tilde{E}_N^{(-q^{-1},\alpha_1)}$. We propose:

Definition 3.1. We define the symmetric one-boundary N-centraliser algebra $\mathcal{C}_{n,N}$ to be the quotient of the algebra $\mathcal{A}_n$ by the relations

$$\tilde{E}_N^{(-q^{-1},\alpha_1)} = 0, \tag{3-2}$$

$$\Lambda_{N+1}^{-q^{-1}}(g_1, \dots, g_N) = 0. \tag{3-3}$$

It is understood that $\mathcal{C}_{n,N} = \mathcal{A}_n$ when $n < N$.

Semisimple representation theory. Here we extend the algebra $\mathcal{C}_{n,N}$ over the field of fractions F and denote it $F\mathcal{C}_{n,N}$. The description below is also valid for specialisations of the parameters satisfying the semisimplicity conditions for $H_{\alpha_1,\alpha_2,n}$ in Remark 2.5.

Proposition 3.2. *The algebra $F\mathcal{C}_{n,N}$ is semisimple with the following set of irreducible representations*

$$\mathrm{Irr}(F\mathcal{C}_{n,N}) = \{V_{(\lambda,\mu)} \mid (\lambda, \mu) \in \mathrm{Par}_2(n) \text{ with } \ell(\lambda) \leq N-1 \text{ and } \ell(\mu) \leq 1\}.$$

Proof. In the algebra $\mathcal{C}_{n,N}$, we have

$$E_N^{(-q^{-1},\alpha_1)} = 0 \quad \text{and} \quad E_2^{(-q^{-1},\alpha_2)} = 0.$$

Reproducing the same reasoning as in the preceding section before Proposition 2.4, we find that cancelling these two elements kills the irreducible representation $V_{(\lambda,\mu)}$

of $FH_{\alpha_1,\alpha_2,n}$ if and only if either λ has strictly more than $N-1$ rows or μ has strictly more than one row.

It remains to argue that, in the remaining irreducible representations the last relation (3-3) cancelling the q-antisymmetriser in N generators $g_1,\dots,g_N$ is satisfied. This can be checked rather directly, using the explicit description of the irreducible representations of $H_{\alpha_1,\alpha_2,n}$ [3; 12; 21] and using the same sort of methods than those used in [7] for the q-symmetriser.

Otherwise, note that the claim is equivalent to the fact that the one-dimensional representation of FH_{N+1} given by $g_1,\dots,g_N \mapsto -q^{-1}$ (that is, indexed by a one-column partition) does not appear when we restrict to FH_{N+1} the irreducible representations of $FH_{\alpha_1,\alpha_2,N+1}$ indexed by bipartitions (λ,μ) with $\ell(\lambda)\le N-1$ and $\ell(\mu)\le 1$. These restrictions are expressed in terms of Littlewood–Richardson coefficients (see, e.g., [5; 24]). This implies in particular the easy claim above since, by what is called the Pieri rule, the Littlewood–Richardson coefficient $d^{\nu}_{\lambda,\mu}$ is 0 with λ and μ as above and ν the one-column partition of length $N+1$. □

The Bratteli diagram for the algebras $F\mathcal{C}_{n,N}$ is thus obtained from the Bratteli diagram for the algebras $F\mathcal{A}_n$ given before, removing all bipartitions with N rows or more in the first component. The dimension of $F\mathcal{C}_{n,N}$ is then calculated as

$$\dim(F\mathcal{C}_{n,N}) = \sum_{i=0}^{n}\binom{n}{i}^2 \sum_{\substack{\lambda\vdash i\\ \ell(\lambda)<N}} d_\lambda^2.$$

For $N=2$, the algebra $F\mathcal{C}_{n,2}$ will be studied in details in the next subsection.

For $N=3$, the sum of the squares of d_λ's in the formula above is the Catalan number $\frac{1}{i+1}\binom{2i}{i}$. Moreover, the series of dimensions start with $1, 2, 7, 33, 183$ and is the series labelled A086618 in [26]. We note that, similarly to the situation $N=2$ discussed in Remark 3.6, the dimension of $F\mathcal{C}_{n,3}$ is the number of signed permutations of $\{-n,\dots,-1,1,\dots,n\}$ which are 4321-avoiding [9].

In general, we may ask whether the algebra $\mathcal{C}_{n,N}$ is free over the ring R and we may look for a basis indexed by a natural subset of signed permutations, for example, those avoiding certain patterns. This is what we are going to do in details for $N=2$ in the next subsection.

Remark 3.3. The representation theory shows that over F the relation (3-3) is actually implied by the others. This is not true over R but this is also true in any specialisation such that the algebra $H_{\alpha_1,\alpha_2,n}$ is semisimple (Remark 2.5).

3B. ***The case N = 2 (the Temperley–Lieb situation).*** The definition of $\mathcal{C}_{n,N}$ can be written slightly differently and more explicitly when $N=2$.

Proposition 3.4. *The algebra $\mathcal{C}_{n,2}$ is the quotient of the algebra $H_{\alpha_1,\alpha_2,n}$ by the relations*:

$$g_1 g_0 g_1 = q(\alpha_1+\alpha_2)(q-g_1) - q^2 g_0 + q(g_0 g_1 + g_1 g_0), \tag{3-4}$$

$$g_i g_{i+1} g_i = q^3 - q^2(g_i+g_{i+1}) + q(g_i g_{i+1} + g_{i+1} g_i), \quad i=1,\dots,n-2. \tag{3-5}$$

Proof. Using Proposition 2.8, the first relation is obtained easily by writing explicitly $\tilde{E}_2^{(-q^{-1},\alpha_1)} = 0$. The second relation for $i=1$ is $\Lambda_3^{-q^{-1}}(g_1, g_2) = 0$. It implies, by suitable conjugation, the second relations for any $i \geq 1$. The last statement is that (3-4) implies the defining relation of $\mathcal{A}_n$:

$$g_0 g_1 g_0 g_1 = -q^2\alpha_1^2 + q\alpha_1^2 g_1 + q^2\alpha_1 g_0 - q\alpha_1(g_1 g_0 + g_0 g_1) + \alpha_1 g_1 g_0 g_1 + q g_0 g_1 g_0.$$

This is easy to see since multiplying (3-4) on the left or on the right by g_0 gives

$$g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0 = q\alpha_1\alpha_2(q-g_1) + q g_0 g_1 g_0, \tag{3-6}$$

which, combined with (3-4), produces the desired relation. □

A Temperley–Lieb presentation. We make the slight change of generators as

$$e_0 := \alpha_2 - g_0, \qquad e_i := q - g_i, \quad i = 1, 2, \dots, n-1. \tag{3-7}$$

Then it is an easy exercise to check that the algebra $\mathcal{C}_{n,2}$ can be equivalently presented as generated by e_i for $i = 0, 1, \dots, n-1$ with the defining relations

$$e_i^2 = (q+q^{-1})\, e_i, \qquad 1 \leq i \leq n-1, \tag{3-8}$$

$$e_0^2 = (\alpha_2 - \alpha_1)\, e_0, \tag{3-9}$$

$$e_i e_j = e_j e_i, \qquad |i-j| \geq 2, \tag{3-10}$$

$$e_i e_{i\pm1} e_i = e_i, \qquad 1 \leq i,\ i \pm 1 \leq n-1, \tag{3-11}$$

$$e_1 e_0 e_1 = (q^{-1}\alpha_2 - q\alpha_1)\, e_1. \tag{3-12}$$

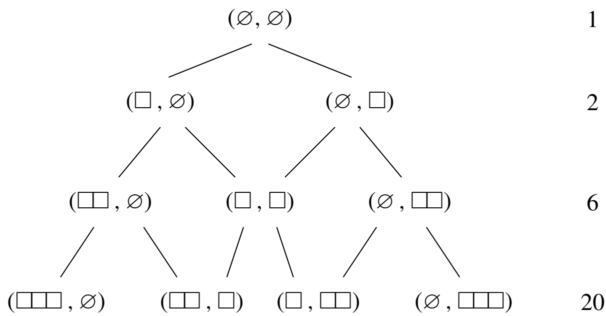
We recover a three-parameter version of the one-boundary Temperley–Lieb algebra, or blob algebra, see for example [8; 16; 18; 19; 20].

Semisimple representation theory. We take $N = 2$ in the Proposition 3.2.

Proposition 3.5. *The algebra $F\mathcal{C}_{n,2}$ is semisimple with the following set of irreducible representations*:

$$\mathrm{Irr}(F\mathcal{C}_{n,2}) = \{V_{(\lambda,\mu)} \mid (\lambda, \mu) \in \mathrm{Par}_2(n) \text{ and } \ell(\lambda), \ell(\mu) \leq 1\}.$$

The Bratteli diagram for the algebras $F\mathcal{C}_{n,2}$ is obtained from the Bratteli diagram for the algebras $H_{\alpha_1,\alpha_2,n}$, where all bipartitions with more than one row in any component are removed. One finds the Pascal triangle, whose beginning is given here:



The dimension of $F\mathcal{C}_{n,2}$ is then easily calculated:

$$\dim(F\mathcal{C}_{n,2}) = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}. \tag{3-13}$$

An R-basis of $\mathcal{C}_{n,2}$. Now we produce an R-basis of $\mathcal{C}_{n,2}$ in terms of signed permutations, and also explicitly in terms of the generators.

We consider the signed permutations with the following avoiding patterns: $(\pm 1, -2)$ and $(\pm 3, 2, \pm 1)$. This means that in the word $b_1 b_2 \dots b_n$ giving the signed permutations ($b_i \in \{\pm 1, \dots, \pm n\}$ is the image of i), we never have:

- For $i < j$: $b_j < 0$ and $|b_i| < |b_j|$ (in words, a negative number b_j is never preceded by a smaller number when ignoring signs).
- For $i < j < k$: $b_j > 0$ and $|b_i| > b_j > |b_k|$ (in words, a positive number is never in the middle of a decreasing sequence of length 3, when ignoring signs).

We denote by $B_n(\bar{1}\bar{2}, 1\bar{2}, 321, \bar{3}21, 32\bar{1}, \bar{3}2\bar{1})$ the set of signed permutations with these avoiding patterns. These elements are called fully commutative top elements in [28]. For example, $3\bar{2}45\bar{1}$ is in this set for $n = 5$, while $3\bar{2}\bar{5}41$ is not for three reasons: the subsequences $3\bar{5}$, $\bar{2}\bar{5}$ and $\bar{5}41$.

In terms of the standard basis elements (2-6), it is proved in [28, Corollary 5.6] that the set g_ω with $\omega \in B_n(\bar{1}\bar{2}, 1\bar{2}, 321, \bar{3}21, 32\bar{1}, \bar{3}2\bar{1})$ corresponds to all elements of the form

$$[n_1, m_1][n_2, m_2] \dots [n_r, m_r], \tag{3-14}$$

$$\text{with } \begin{cases} 0 \le n_1 < n_2 < \cdots < n_r \le n-1 \text{ and } m_i \le n_i, \\ 0 = m_1 = \cdots = m_s < m_{s+1} < \cdots < m_r. \end{cases}$$

The cardinality of the set $B_n(\bar{1}\bar{2}, 1\bar{2}, 321, \bar{3}21, 32\bar{1}, \bar{3}2\bar{1})$ has been calculated in [28, Proposition 5.9] or [19, Appendix B], and it is found that

$$|B_n(\bar{1}\bar{2}, 1\bar{2}, 321, \bar{3}21, 32\bar{1}, \bar{3}2\bar{1})| = \binom{2n}{n}. \tag{3-15}$$

Remark 3.6. An alternative description of $B_n(\bar{1}\bar{2}, 1\bar{2}, 321, \bar{3}21, 32\bar{1}, \bar{3}2\bar{1})$ is as follows. A signed permutation is in particular a permutation of $\{-n, \dots, -1, 1, \dots, n\}$. For these permutations on $2n$ elements, there is the usual meaning of being 321-avoiding (no strictly decreasing subsequence of length 3 in the sequence of images $b_{-n} \dots b_{-1} b_1 \dots b_n$). We leave to the reader to check that the set $B_n(\bar{1}\bar{2}, 1\bar{2}, 321, \bar{3}21, 32\bar{1}, \bar{3}2\bar{1})$ coincides with the set of signed permutations which are 321-avoiding as permutations of $\{-n, \dots, -1, 1, \dots, n\}$. See [9] for a proof that this latter set is indeed of cardinal $\binom{2n}{n}$.

Theorem 3.7. *The algebra $\mathcal{C}_{n,2}$ is free over R with basis consisting of elements in (3-14), that is,*

$$\{g_\omega \mid \omega \in B_n(\bar{1}\bar{2}, 1\bar{2}, 321, \bar{3}21, 32\bar{1}, \bar{3}2\bar{1})\}. \tag{3-16}$$

One can replace each generator g_i by e_i in the expressions in (3-14) and this clearly also gives a basis of $\mathcal{C}_{n,2}$. This basis can be found in [19, Appendix B].

Proof. We proceed similarly to the proof of Theorem 2.6. All signed permutations $\omega \in B_n$ with length $\ell(\omega) < 3$ avoid the patterns in (3-16). It is shown in [28] (see Theorem 4.1 and Corollary 5.6) that an element ω which does not avoid the patterns of (3-16) contains in a reduced expression $s_1 s_0 s_1$ or $s_i s_{i+1} s_i$ for $i \geq 1$. This means that the defining relations (3-4) and (3-5) can be used to express g_ω in terms of elements of smaller length. Using induction on the length, we can therefore conclude that (3-16) is a spanning set for $\mathcal{C}_{n,2}$ over R. The rest follows by comparing the cardinality (3-15) with the dimension (3-13) of $F\mathcal{C}_{n,2}$. □

3C. ***Quasiidempotents in the one-boundary Temperley–Lieb algebra $\mathcal{C}_{n,2}$.*** Let $n \geq 2$. Among the four central quasiidempotents of $H_{\alpha_1,\alpha_2,n}$, the two $E_n^{(-q^{-1},\alpha_i)}$ are equal to 0 in $\mathcal{C}_{n,2}$. The one with α_2 was cancelled to define $\mathcal{A}_n$, and the one with α_1 was cancelled to define $\mathcal{C}_{n,2}$. So only the following two remain:

$$E_n^{(q,\alpha_1)}, \qquad E_n^{(q,\alpha_2)}.$$

It turns out that the expression of these two remaining quasiidempotents in $\mathcal{C}_{n,2}$ simplifies compared to their original definition in $H_{\alpha_1,\alpha_2,n}$ and a global factor appears, as shown in the following result. Note that the following result was already true in $\mathcal{A}_n$ only for $E_n^{(q,\alpha_2)}$ (Proposition 2.8). So the novelty here is that it becomes also true for $E_n^{(q,\alpha_1)}$ in $\mathcal{C}_{n,2}$. Below, we use again the notation modulo 2 for the indices of α_b.

Proposition 3.8. *Let $n \geq 2$ and $b \in \{1, 2\}$. In $\mathcal{C}_{n,2}$ we have*

$$E_n^{(q,\alpha_b)} = q^{\frac{1}{2}(n(n-1))} [n]_q!\, \tilde{E}_n^{(q,\alpha_b)}, \tag{3-17}$$

where the renormalised element is given recursively by

$$\tilde{E}_n^{(q,\alpha_b)} = \tilde{E}_{n-1}^{(q,\alpha_b)}\big((1-q^2)(1+qg_{n-1}+\cdots+q^{n-2}g_{n-1}\dots g_2) + q^{n-1}g_{n-1}\dots g_1(1-\alpha_{b+1}^{-1}g_0)\big), \tag{3-18}$$

with the convention that $\tilde{E}_1^{(q,\alpha_b)} = E_1^{(q,\alpha_b)} = (1-\alpha_{b+1}^{-1}g_0)$.

Proof. The case $n=2$ is easily verified by a direct calculation in $\mathcal{C}_{2,2}$. Now suppose that the formula (3-17) is true for some $n \geq 2$. To show that it is true for $n+1$, one can use the recurrence relation (2-10) for $E_{n+1}^{(q,\alpha_b)}$ in $H_{\alpha_1,\alpha_2,n+1}$ and then replace $g_1\, g_0\, g_1$ using the defining relation (3-4) of $\mathcal{C}_{n+1,2}$. With the help of the property $E_n^{(q,\alpha_b)} g_0 = \alpha_b\, E_n^{(q,\alpha_b)}$, the result simplifies to

$$E_{n+1}^{(q,\alpha_b)} = E_n^{(q,\alpha_b)}\big(1+qg_n+\cdots +q^{n-1}g_n\dots g_2 - q^{n+3}g_n\dots g_2(1+qg_2+\cdots+q^{n-1}g_2\dots g_n) + q^n g_n\dots g_1(1+q^2+q^3g_2+\cdots+q^{n+1}g_2\dots g_n)(1-\alpha_{b+1}^{-1}g_0)\big).$$

The remaining terms can be simplified using the Hecke relation (2-4), the braid relations (2-1) and (2-3), and the property $E_n^{(q,\alpha_b)} g_i = q\, E_n^{(q,\alpha_b)}$ for $i=1,\dots,n-1$ to arrive at

$$E_{n+1}^{(q,\alpha_b)} = E_n^{(q,\alpha_b)} q^n [n+1]_q \big((1-q^2)(1+qg_n+\cdots+q^{n-1}g_n\dots g_2) + q^n g_n\dots g_1(1-\alpha_{b+1}^{-1}g_0)\big). \tag{3-19}$$

The proof is completed by using the induction hypothesis. □

4. The fused Hecke algebra

Let $k \geq 1$. We briefly recall the definition of the fused Hecke algebra $H_{k,n}$ in the particular case where only the k first strands are fused, and refer to [6; 7; 23] for more details. The fused Hecke algebra $H_{k,n}$ is defined for a nonzero complex number q satisfying

$$q^{2i} \neq 1, \quad i = 1,\dots,k, \tag{4-1}$$

since denominators of the form $q^{2i}-1$, with $i=1,\dots,k$, appear in its definition. Equivalently, we will consider $H_{k,n}$ to be defined over the ring generated by $\mathbb{C}[q^{\pm 1}]$ and $(q^{2i}-1)^{-1}$ for $i=1,\dots,k$:

$$\mathbb{C}^{(k)}[q^{\pm1}] = \mathbb{C}[q^{\pm1}, (q^2-1)^{-1}\dots(q^{2k}-1)^{-1}]. \tag{4-2}$$

Remark 4.1. The fused Hecke algebra can also be defined for $q=\pm1$ (and is called the algebra of fused permutations in [7]), but we will not consider this possibility here, since this would require to replace from the beginning the algebra $H_{\alpha_1,\alpha_2,n}$ of Section 2 by a different algebra.

4A. *Definition of $H_{k,n}$*. Let $n \geq 0$. We denote H_{k+n} the Hecke algebra associated to the symmetric group $\mathfrak{S}_{k+n}$, with generators $\sigma_i,\ i = 1, \dots, k+n-1$. Its definition was given at the end of Section 2A, in terms of generators g_i's, but we change the names of the generators to avoid any confusion. We define it over $\mathbb{C}^{(k)}[q^{\pm 1}]$.

The standard basis elements of H_{k+n} are denoted σ_ω, where $\omega \in \mathfrak{S}_{k+n}$. Recall that $\sigma_\omega = \sigma_{i_1} \dots \sigma_{i_l}$ if $\omega = s_{i_1} \dots s_{i_l}$ is a reduced expression in terms of the adjacent transpositions $s_i = (i, i+1)$.

The (normalised) q-symmetriser of the algebra H_k is

$$P_k = \frac{\sum_{\omega \in \mathfrak{S}_k} q^{\ell(\omega)} \sigma_\omega}{\sum_{\omega \in \mathfrak{S}_k} q^{2\ell(\omega)}} = \frac{q^{-k(k-1)/2}}{[k]_q!} \sum_{\omega \in \mathfrak{S}_k} q^{\ell(\omega)} \sigma_\omega, \tag{4-3}$$

and is well defined over the ring $\mathbb{C}^{(k)}[q^{\pm 1}]$, see (4-2). We see the q-symmetriser P_k as an element of H_{k+n} through the natural embedding of $\mathfrak{S}_k$ in $\mathfrak{S}_{k+n}$, where $\mathfrak{S}_k$ acts on the first k letters.

The q-symmetriser is a primitive central idempotent of H_k such that we can write $P_k\, \sigma_i = \sigma_i\, P_k = q\, P_k$ for $i = 1, \dots, k-1$ and $P_k P_i = P_i P_k = P_k$ for all $i \leq k$. It satisfies the recursive formula

$$q^{k-1}[k]_q P_k = P_{k-1}(1 + q\sigma_{k-1} + q^2 \sigma_{k-1}\, \sigma_{k-2} + \dots + q^{k-1} \sigma_{k-1}\, \sigma_{k-2} \dots \sigma_1). \tag{4-4}$$

Definition 4.2. The fused Hecke algebra $H_{k,n}$ is the algebra of the form $P_k H_{k+n} P_k$.

A basis of the algebra $H_{k,n}$ is indexed by the double cosets $\mathfrak{S}_k \backslash \mathfrak{S}_{k+n} / \mathfrak{S}_k$ of the subgroup $\mathfrak{S}_k$ in $\mathfrak{S}_{k+n}$. In each of these cosets, there is a unique representative of minimal length (see [11]), and we will identify the set of double cosets $\mathfrak{S}_k \backslash \mathfrak{S}_{k+n} / \mathfrak{S}_k$ with the set of minimal-length representatives. The standard basis of $H_{k,n}$ is

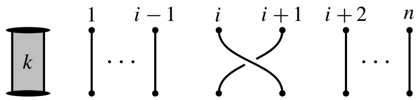
$$\{P_k\, \sigma_\omega\, P_k \mid \omega \in \mathfrak{S}_k \backslash \mathfrak{S}_{k+n} / \mathfrak{S}_k\}.$$

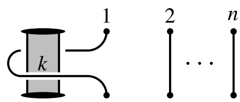
The dimension of the algebra $H_{k,n}$ is the number of double cosets in $\mathfrak{S}_k \backslash \mathfrak{S}_{k+n} / \mathfrak{S}_k$, or of what were called fused permutations in [7]. It is the number of ways to connect a row of $n+1$ dots to another such row, with the requirement that k edges start and k edges arrive at the first dot of each row, while the usual rule of a single edge at each dot applies for the other dots. It will be convenient to draw and to refer to the first dot as an ellipse. So from now on, we have two lines each consisting of one ellipse followed by n dots.

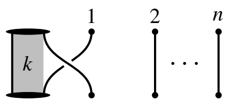
To count such fused permutations, one has first to choose how many edges from, say, the top ellipse will go to a dot. This is choosing $i \in \{1, 2, \dots, \min\{k, n\}\}$. Then one needs to chose i bottom dots among n where to put these i edges and i top dots among n which will be connected to the bottom ellipse. Finally, one can choose an arbitrary permutation diagram between the remaining two lines of $n - i$ dots which are not connected to the ellipses. It follows from this discussion that

$$\dim(H_{k,n}) = \sum_{i=0}^{\min\{k,n\}} (n-i)! \binom{n}{i}^2. \tag{4-5}$$

For any word x in the generators σ_i of H_{k+n} and their inverses σ_i^{-1}, the diagrammatic representation of the element $P_k\, x\, P_k$ of $H_{k,n}$ is obtained by drawing the usual braid-like picture for x between two rows of $k+n$ dots, and then fusing in one large dot (or ellipse) the k first top dots and similarly for the k first bottom dots. For example, we define the following elements of $H_{k,n}$:

(4-6) $S_i :=$ 

(4-7) $S_0 :=$ 

(4-8) $T :=$ 

where $i = 1, 2, \dots, n-1$, which algebraically correspond to

$$S_i = P_k\, \sigma_{k+i}\, P_k = P_k\, \sigma_{k+i} = \sigma_{k+i}\, P_k, \tag{4-9}$$

$$S_0 = P_k\, \sigma_k\, \sigma_{k-1} \dots \sigma_2\, \sigma_1^2 \sigma_2 \dots \sigma_{k-1}\, \sigma_k\, P_k, \tag{4-10}$$

$$T = P_k\, \sigma_k\, P_k. \tag{4-11}$$

We have used that σ_{k+i} commutes with P_k when $i \geq 1$.

Eigenvalues of the generator S_0. We will need to know the characteristic equation satisfied by S_0 in order to relate the algebra $H_{k,n}$ to a cyclotomic quotient $H_{\alpha_1,\alpha_2,n}$ for the correct values of α_1, α_2.

Proposition 4.3. *In $H_{k,n}$ we have*

$$(S_0 - q^{2k} P_k)(S_0 - q^{-2} P_k) = 0. \tag{4-12}$$

Proof. First, we show the following relation between S_0 and T:

$$S_0 = (q - q^{-1})\, q^{k-1} [k]_q T + P_k. \tag{4-13}$$

To check this, we start with the defining formula for S_0 and use the quadratic relation for σ_1. We find

$$S_0 = (q - q^{-1})\, P_k\, \sigma_k\, \sigma_{k-1} \dots \sigma_2\, \sigma_1 \sigma_2 \dots \sigma_{k-1}\, \sigma_k\, P_k \\ + P_k\, \sigma_k\, \sigma_{k-1} \dots \sigma_2^2 \dots \sigma_{k-1}\, \sigma_k\, P_k.$$

Using the braid relations and the property of P_k, the first term becomes

$$\begin{aligned}(q-q^{-1})\,P_k\,\sigma_1\,\sigma_2\ldots\sigma_{k-1}\,\sigma_k\,\sigma_{k-1}\ldots\sigma_2\,\sigma_1\,P_k &= (q-q^{-1})\,q^{2(k-1)}\,P_k\,\sigma_k\,P_k\\ &= (q-q^{-1})\,q^{2(k-1)}\,T.\end{aligned}$$

We proceed similarly with the remaining term, which ultimately gives (4-13).

The proof of the proposition is concluded by calculating the eigenvalues of T. We have

$$(T-q\,P_k)(T+q^{-k}[k]_q^{-1}\,P_k)=0. \tag{4-14}$$

To check this equality, we first use the recurrence relation for P_k to write

$$\sigma_k\,P_k\,\sigma_k = q^{-k+1}[k]_q^{-1}\,P_{k-1}\,\sigma_k(1+q\sigma_{k-1}+q^2\sigma_{k-1}\,\sigma_{k-2}+\cdots+q^{k-1}\,\sigma_{k-1}\ldots\sigma_1)\,\sigma_k.$$

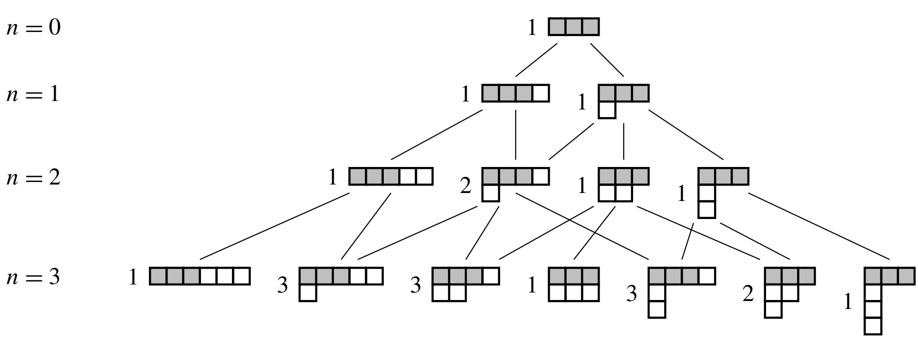
Then we proceed as follows, using $P_k\,P_{k-1}=P_k$, the braid relations and $\sigma_i\,P_k=q\,P_k$ if $i<k$:

$$\begin{aligned}T^2=P_k\,\sigma_k\,P_k\,\sigma_k\,P_k &= q^{-k+1}[k]_q^{-1}\,P_k(1+(q+q^3+q^5+\cdots+q^{2k-1}-q^{-1})\,\sigma_k)\,P_k\\ &= q^{-k+1}[k]_q^{-1}\,P_k+(q-q^{-k}[k]_q^{-1})\,T.\end{aligned}$$

This concluded the verification. □

The semisimple representation theory of $H_{k,n}$. Here we work over $\mathbb{C}(q)$ (or we assume that q^2 is not a root of unity of order $\le k+n$).

The algebra $\mathbb{C}(q)H_{k,n}$ is semisimple and its irreducible representations were described in [7]. They are indexed by partitions $\lambda \vdash k+n$ such that $\lambda_1 \ge k$ (that is, the first line of λ contains at least k boxes). For example, for $n=0$, there is a single irreducible representation, indexed by a line of k boxes. The branching rules are given by inclusion of partitions. For example, when $k=3$, the beginning of the Bratteli diagram is as follows:



We have shaded the three fixed boxes in the first row of each partition. Next to each partition is the dimension of the corresponding irreducible representation.

We emphasise that the dimension is not the number of standard tableaux strictly speaking, but is the number of standard fillings of the nonshaded boxes by $1, \dots, n$.

It is easy to see that, when $2 \le n \le k$, there are three one-dimensional representations for $H_{k,n}$. They are given by the following partitions:

$$\lambda = (k+n) : S_0 \mapsto q^{2k},\ S_i \mapsto q, \tag{4-15}$$

$$\lambda = (k, 1^n) : S_0 \mapsto q^{-2},\ S_i \mapsto -q^{-1}, \tag{4-16}$$

$$\lambda = (k, n) : S_0 \mapsto q^{-2}, S_i \mapsto q, \tag{4-17}$$

where we give the associated values of the elements S_0 and S_i $(i \ge 1)$. These are easily obtained from the description in [7]. The case of $\lambda = (k+n)$ is immediate and the level $n = 1$ together with the given branching rules give all the other values of S_0. Calculating the eigenvalue of S_i is also immediate from the description in [7] since $S_i = P_k\, \sigma_{k+i}$ with σ_{k+i} commuting with P_k.

When $n > k$, there is only two remaining one-dimensional representations, the one corresponding to $\lambda = (k, n)$ being removed from the list (it would not make sense for $n > k$).

Remark 4.4. Let $n \le k$. Anticipating a little bit, the algebra $H_{k,n}$ is going to be obtained (below) as a specialisation of the parameters α_1, α_2 of the algebra $\mathcal{A}_n$. This specialisation is semisimple over $\mathbb{C}(q)$, and thus we have a bijection between the irreducible representations of $\mathbb{C}(q)\, H_{k,n}$ and the representations described in Section 2:

$$\mathrm{Irr}(F\mathcal{A}_n) = \{V_{(\lambda,\mu)} \mid (\lambda, \mu) \in \mathrm{Par}_2(n) \text{ with } \ell(\mu) \le 1\}.$$

The bijection maps (λ, μ) to the partition made of a first line of $k + |\mu|$ boxes, and with the diagram of λ for the remaining lines. This is well defined since n, and in turn $|\lambda|$, is less or equal to k.

If $n > k$, the specialisation of $\mathcal{A}_n$ is not semisimple anymore and the algebra $H_{k,n}$ will be obtained as a quotient of this nonsemisimple specialisation of $\mathcal{A}_n$. Therefore we cannot immediately identify the irreducible representations of $H_{k,n}$ with a subset of the irreducible representations of $\mathcal{A}_n$ as soon as $n > k$.

4B. *An algebraic description of $H_{k,n}$.* In this section, we use the algebra $\mathcal{A}_n$ from Section 2 to obtain an algebraic presentation of the fused Hecke algebra $H_{k,n}$. We are going to work with the following specialisation of the parameters α_1, α_2:

$$\alpha_1 = q^{-2}, \quad \alpha_2 = q^{2k}. \tag{4-18}$$

This is motivated by Proposition 4.3 since these two values are the eigenvalues of the element S_0. For these specific values, we have the following factorisation result for one of the quasiidempotents of $\mathcal{A}_n$, as a corollary of our results in Section 2D.

Corollary 4.5. *In $\mathcal{A}_{k+1}$, when (α_1, α_2) are specialised as in (4-18), the element $E_{k+1}^{(q,\alpha_1)}$ factorises as*

$$E_{k+1}^{(q,\alpha_1)} = [k+1]_q \, E'^{(q,\alpha_1)}_{k+1},$$

where $E'^{(q,\alpha_1)}_{k+1}$ is in $\mathcal{A}_{k+1}$ with coefficients in $\mathbb{C}[q^{\pm 1}]$.

Proof. It is clear that $E_{k+1}^{(q,\alpha_1)} - E_{k+1}^{(q,\alpha_2)}$ is divisible by $(\alpha_1 - \alpha_2)$. With the given specialisation, this means that it is divisible by $q^{2(k+1)} - 1$ and thus by $[k+1]_q$. Now Proposition 2.8 shows that $E_{k+1}^{(q,\alpha_2)}$ is also divisible by $[k+1]_q$ (actually, already over R). Thus we get the desired statement. □

This allows us to define, in the specialisation of $\mathcal{A}_{k+1}$, the following element with coefficients in $\mathbb{C}[q^{\pm 1}]$:

$$E'^{(q,\alpha_1)}_{k+1} := \frac{1}{[k+1]_q} E_{k+1}^{(q,\alpha_1)}.$$

Definition 4.6. The algebra $\mathcal{A}_n^{(k)}$ is defined over $\mathbb{C}^{(k)}[q^{\pm 1}]$ as the specialisation of $\mathcal{A}_n$ corresponding to (4-18) with the additional defining relation if $n > k$:

$$E'^{(q,\alpha_1)}_{k+1} = 0. \tag{4-19}$$

Note that it is not clear at once if over $\mathbb{C}(q)$ the algebra $\mathcal{A}_n^{(k)}$ is semisimple for all $n \geq 0$ since the specialisation (4-18) falls into the nonsemisimple regime (see Remarks 2.5 and 4.4). However, note that the specialisation is semisimple if $n \leq k$. Moreover, for $n > k$, we have taken a quotient of the nonsemisimple specialisation, so it may well be that we obtain again something semisimple. Actually, the algebra $\mathcal{A}_n^{(k)}$ over $\mathbb{C}(q)$ will turn out to be semisimple as a consequence of its isomorphism with the fused Hecke algebra $H_{k,n}$ which is known to be semisimple [7].

Remark 4.7. The additional relation when $n \geq k+1$ is the analogue in the Hecke case of the additional relation that is needed to pass from the one-boundary Temperley–Lieb algebra to the boundary seam algebra. We refer to [19] and [17] where this additional relation was interpreted in terms of a quasiidempotent of the one-boundary Temperley–Lieb algebra (see also Section 5C).

A spanning set of $\mathcal{A}_n^{(k)}$. First, we find a spanning set for $\mathcal{A}_n^{(k)}$. Consider the signed permutations with the following avoiding patterns: $\bar{1}\bar{2}$ and $\overline{k+1}\,\bar{k}\ldots\bar{1}$. We denote by $B_n(\bar{1}\bar{2}, \overline{k+1}\,\bar{k}\ldots\bar{1})$ the subset of these signed permutations.

The first condition defining $B_n(\bar{1}\bar{2}, \overline{k+1}\,\bar{k}\ldots\bar{1})$ is the same as the one giving the basis of $\mathcal{A}_n$ in Theorem 2.6. The second one adds the condition that in the word $b_1 b_2 \ldots b_n$ giving the signed permutations, we never have a strictly decreasing sequence of length $k+1$ of barred numbers. Both together mean that all barred numbers appear in decreasing orders, and that at most k barred numbers are present. So to count the number of elements in $B_n(\bar{1}\bar{2}, \overline{k+1}\,\bar{k}\ldots\bar{1})$, we reason as follows on the word $b_1 b_2 \dots b_n$ giving the signed permutation. We choose i numbers in $\{1, \dots, n\}$ that will be barred, with the condition that $i \le k$; then choose i positions among n to place these barred numbers in decreasing order; and finally we choose a permutation of the remaining $n-i$ numbers to be placed (not barred) in the remaining $n-i$ positions. This gives

$$|B_n(\bar{1}\bar{2}, \overline{k+1}\,\bar{k} \dots \bar{1})| = \sum_{i=0}^{\min(k,n)} \binom{n}{i}^2 (n-i)!. \tag{4-20}$$

Note that the above discussion shows the alternative description

$$B_n(\bar{1}\bar{2}, \overline{k+1}\,\bar{k} \dots \bar{1}) = B_n(\bar{1}\bar{2}) \cap \{\omega \in B_n \mid \ell_0(\omega) < k+1\}. \tag{4-21}$$

Let us emphasise that if $n \le k$, then the set $B_n(\bar{1}\bar{2}, \overline{k+1}\,\bar{k} \dots \bar{1})$ is the same as $B_n(\bar{1}\bar{2})$ since there are no signed permutations in B_n with more than k barred numbers.

Proposition 4.8. *The following set is a spanning set of $\mathcal{A}_n^{(k)}$:*

$$\{g_\omega \mid \omega \in B_n(\bar{1}\bar{2}, \overline{k+1}\,\bar{k} \dots \bar{1})\}. \tag{4-22}$$

Proof. Recall the basis $\{g_\omega \mid \omega \in B_n(\bar{1}\bar{2})\}$ of $\mathcal{A}_n$ from Theorem 2.6. There is a unique basis element containing n occurrences of g_0 and all the others contain strictly less. In other words, there is a unique element $\omega \in B_n(\bar{1}\bar{2})$ with maximal ℓ_0, which is $\ell_0(w) = n$. This element corresponds to the sequence $\bar{n}, \overline{n-1}, \dots, \bar{1}$, and the corresponding basis element is

$$g_0 \cdot g_1 g_0 \cdot \dots \cdot g_{n-1} \dots g_1 g_0. \tag{4-23}$$

Lemma 4.9. *The element (4-23) appears with coefficient $(-1)^n \alpha_2^{-n} q^{(n-1)n} [n]_q!$ when $E_n^{(q,\alpha_1)}$ is expanded in the basis $\{g_\omega \mid \omega \in B_n(\bar{1}\bar{2})\}$ of $\mathcal{A}_n$.*

Proof. The proof is by induction on n. The statement is immediate when $n = 1$. Then we use the recurrence formula for the quasiidempotent:

$$E_{n+1}^{(q,\alpha_1)} = E_n^{(q,\alpha_1)} \left(1 + \sum_{i=1}^{n} q^{n-i+1} g_n \dots g_i - \alpha_2^{-1} q^n g_n \dots g_1 g_0 \left(1 + \sum_{i=1}^{n} q^i g_1 \dots g_i\right)\right).$$

Using the induction hypothesis and keeping the only terms which can contribute to the relevant coefficient, we have to study

$$-\gamma_n \alpha_2^{-1} q^n g_0 \cdot g_1 g_0 \cdot \dots \cdot g_{n-1} \dots g_1 g_0 \cdot g_n \dots g_1 g_0 \left(1 + \sum_{i=1}^{n} q^i g_1 \dots g_i\right),$$

where γ_n is the coefficient given by the induction hypothesis. For dealing with the terms in the sum, note that they produce the appearance of $g_0 g_1 g_0 g_1$ (the leftmost g_0 is the g_0 just to the left of g_n). So we can use the defining relation (2-19) of $\mathcal{A}_n$ and keep only the term not reducing the occurrences of g_0. This amounts to simply replacing $g_0\,g_1\,g_0\,g_1$ by $qg_0\,g_1\,g_0$. Thus all g_1's in the sum are replaced by q. Similarly, if an element g_i hits the expression on the left of the parenthesis, it will move through $g_n \dots g_1\,g_0,\ g_{n-1} \dots g_1\,g_0, \dots,$ becoming successively $g_{i-1},\ g_{i-2}$ and so on until it becomes g_1 and hits again a $g_0\,g_1\,g_0$. The same reasoning as before allows to replace it by q. So all generators $g_1, \dots, g_n$ appearing in the parenthesis are replaced by q and we are left with $(1+q^2+\cdots+q^{2n}) = q^n[n+1]_q$. □

The lemma implies that in the renormalised quasiidempotent $E'^{(q,\alpha_1)}_{k+1}$, the coefficient in front of the element (4-23) with $n = k+1$ is

$$(-1)^{k+1}\alpha_2^{-(k+1)}q^{k(k+1)}[k]_q!,$$

since we have divided $E^{(q,\alpha_1)}_{k+1}$ by $[k+1]_q$. This coefficient is invertible in the ring $\mathbb{C}^{(k)}[q^{\pm 1}]$.

Now we will show that every element g_ω with $\ell_0(\omega) \ge k+1$ can be rewritten in $\mathcal{A}_n^{(k)}$ in terms of elements $g_{\omega'}$ with $\ell_0(\omega') < k+1$. Indeed, if the number of occurrences of g_0 is at least $k+1$ then write g_ω in the standard form (2-6) and pick $k+1$ consecutive occurrences of g_0. Then using the braid relations we have

$$g_j \cdot g_i \dots g_1\,g_0 = g_i \dots g_1\,g_0 \cdot g_{j+1} \quad \text{for all } 0 < j < i.$$

This allows to find a subexpression of g_ω:

$$g_{i_1} \dots g_1\,g_0 \cdot g_{i_2} \dots g_1\,g_0 \cdot \dots \cdot g_{i_{k+1}} \dots g_1\,g_0, \quad i_1 < i_2 \dots < i_{k+1}.$$

Moving some g_i's to the left using commutation relations, we see at once that the element (4-23) with $n = k+1$ appears. Thanks to the lemma and its consequence stated just after, we can use the relation $E'^{(q,\alpha_1)}_{k+1} = 0$ of $\mathcal{A}_n^{(k)}$ to rewrite this element in terms of $g_{\omega'}$ with $\ell_0(\omega') < k+1$. Thus we have strictly reduced the number of occurrences of g_0, and by induction we conclude that $\mathcal{A}_n^{(k)}$ is spanned by elements $g_{\omega'}$ with $\ell_0(\omega') < k+1$.

Now if an element ω with $\ell_0(\omega) < k+1$ has the pattern $\bar{1}\bar{2}$ then as shown in the proof of Theorem 2.6 we can use the defining relation of $\mathcal{A}_n$ allowing to rewrite $g_0\,g_1\,g_0\,g_1$ to write g_ω in terms of $g_{\omega'}$ with $\omega' \in B_n(\bar{1}\bar{2})$. In doing so, note that the 0-length ℓ_0 never increases since the relation we use rewrites $g_0\,g_1\,g_0\,g_1$ in terms of elements with 2 occurrences of g_0 or less. □

Isomorphism theorem. We are now ready to state the main result of this section.

Theorem 4.10. *For any n, an algebra isomorphism from $\mathcal{A}_n^{(k)}$ to $H_{k,n}$ is given by*

(4-24) $$\phi : \mathcal{A}_n^{(k)} \to H_{k,n}, \qquad 1 \mapsto P_k, \qquad g_i \mapsto S_i, \quad i = 0, 1, \dots, n-1.$$

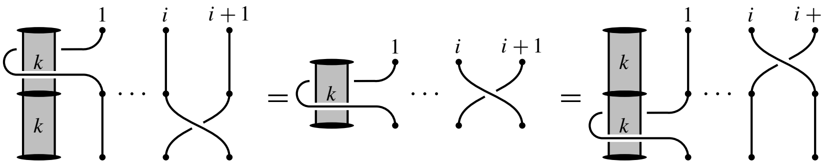
Corollary 4.11. *Over $\mathbb{C}^{(k)}[q^{\pm 1}]$, the following set is a basis of $\mathcal{A}_n^{(k)}$ and its image by ϕ is a basis of $H_{k,n}$:*

$$\{g_\omega \mid \omega \in B_n(\bar{1}\bar{2}, \overline{k+1}\,\bar{k} \dots \bar{1})\}.$$

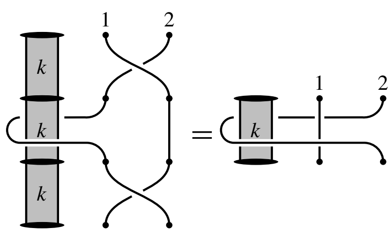
Proof. This follows from the fact that this set is a spanning set (Proposition 4.8) with cardinality given by (4-20), which is equal to the dimension of $H_{k,n}$, and from the isomorphism of $\mathcal{A}_n^{(k)}$ with $H_{k,n}$. □

4C. *Proof of Theorem 4.10.*

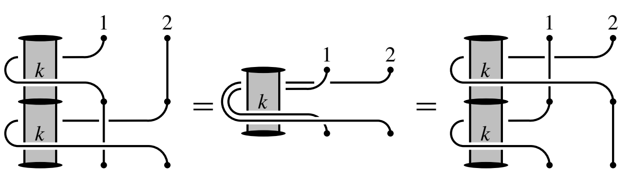
The morphism property of ϕ. By definition, the projector P_k acts as the unit element in $H_{k,n}$. We must verify that the elements $S_i \in H_{k,n}$ for $i = 0, 1, \dots, n$ satisfy the same defining relations (2-1)–(2-5), (2-19) and (4-19) as the elements $g_i \in \mathcal{A}_n^{(k)}$. Relations (2-1), (2-3) for $i, j \neq 0$ and (2-4) directly follow from the definition of the fused Hecke algebra. It is also readily apparent that (2-3) for $i = 0$ is verified, since S_0 and S_j with $j \geq 2$ are elements that each act on different strands. In diagrams, we have (ignoring unaffected strands in the illustrations for simplicity):

(4-25) 

Consider now relation (2-2). We start by computing the following (again ignoring unaffected strands):

(4-26) $S_1 S_0 S_1 =$ 

Now, in order to show that (2-2) holds for the images, one must show that S_0 commutes with $S_1 S_0 S_1$. This is seen using the isotopy of diagrams and the fact that an ellipse can be replaced by a sum of braids acting on the k fused strands. This means that the strand labelled 2, which encircles both the strand 1 and the k fused strands, can be moved up or down around any middle ellipse and around all encircled strands, and hence the commutativity. In diagrams, we have

(4-27) 

The quadratic relation (2-5) for g_0 is preserved thanks to Proposition 4.3.

Then to show that relation (2-19) is preserved, we start with a lemma.

Lemma 4.12. *For k any positive integer, the following relations hold*:

$$TS_1TS_1 - qTS_1T = q^{1-k}[k]_q^{-1}(S_1T - q^{-1}S_1TS_1), \tag{4-28}$$

$$S_1TS_1T - qTS_1T = q^{1-k}[k]_q^{-1}(TS_1 - q^{-1}S_1TS_1). \tag{4-29}$$

Proof. Similarly as has been done before in this section, one can use the algebraic expressions of T and S_1, the properties of the projector P_k as well as the braid relations to show that

$$\begin{aligned} TS_1T &= P_k\,\sigma_k\,\sigma_{k+1}\,P_k\,\sigma_k\,P_k \\ &= q^{1-k}[k]_q^{-1}(S_1TS_1 + q^{k-1}[k-1]_q\,P_k\,\sigma_k\,\sigma_{k-1}\,\sigma_{k+1}\,\sigma_k\,P_k). \end{aligned} \tag{4-30}$$

Multiplying the previous equation by S_1 on the right, and then using the Hecke relation and braid relations, it is found that

$$\begin{aligned} &TS_1TS_1 \\ &= q^{1-k}[k]_q^{-1}\big((q-q^{-1})\,S_1TS_1 + S_1T + q^k[k-1]_q\,P_k\,\sigma_k\,\sigma_{k-1}\,\sigma_{k+1}\,\sigma_k\,P_k\big). \end{aligned} \tag{4-31}$$

So, combining the two previous results, relation (4-28) is found. To obtain (4-29), multiply by S_1 on the left instead. □

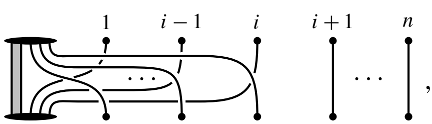
One can now use (4-13) to write equation (4-28) in terms of S_0, which gives

$$\begin{aligned} &S_0\,S_1\,S_0\,S_1 \\ &= -q^{-2}P_k + q^{-3}S_1 + S_0 - q^{-1}(S_0S_1 + S_1S_0) + q^{-2}S_1\,S_0\,S_1 + q\,S_0\,S_1\,S_0. \end{aligned} \tag{4-32}$$

The previous relation corresponds to (2-19) with parameters α_1 and α_2 as in (4-18).

Finally, when $n > k$, we must show that the additional relation (4-19) is satisfied in $H_{k,n}$. If $E'^{(q,\alpha_1)}_{k+1}$ were nonzero, this would imply that this element would also be nonzero in the algebra $H_{k,n}$ extended over $\mathbb{C}(q)$. This in turn would imply the existence of a one-dimensional representation $S_0 \mapsto q^{-2}$ and $S_i \mapsto q$, $i \geq 1$. We already discussed the nonexistence of such a one-dimensional representation around equations (4-15)–(4-17).

Surjectivity of ϕ. For $i = 1, 2, \ldots, \min\{k, n\}$, we define the element $U_i \in H_{k,n}$ which consists of the diagram where the i last strands of the top ellipse go out to the i first bottom circles (without crossing each other), and similarly for the strands of the bottom ellipse. It is illustrated as

(4-33) $U_i :=$ 

, $i = 1, 2, \ldots, \min\{k, n\}$.

Here it is understood that there are $k-i$ straight strands in the gray zone. If we denote

$$\sigma_{k,i} := \sigma_k\, \sigma_{k+1} \dots \sigma_{k+i-1}, \tag{4-34}$$

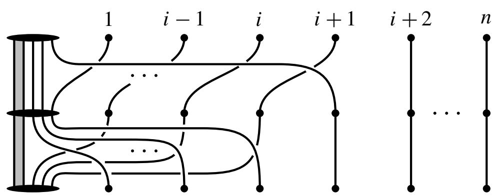
then the algebraic expression of the element U_i is given by

$$U_i = P_k\, \sigma_{k,i}\, \sigma_{k-1,i} \dots \sigma_{k-i+1,i}\, P_k. \tag{4-35}$$

The elements P_k, U_i for $i = 1, 2, \dots, \min\{k, n\}$ and S_i for $i = 1, 2, \dots, n-1$ generate the fused Hecke algebra $H_{k,n}$. Indeed, a basis for $H_{k,n}$ consists of the set of fused braid diagrams corresponding to fused permutations, which were described when counting the dimension (4-5) of $H_{k,n}$. A generic fused braid diagram can be obtained by multiplying the element U_i on the left and on the right by elements S_j for $j = 1, \dots, n-1$.

We are now ready to prove that the map ϕ is surjective. It suffices to show that the generators of $H_{k,n}$ belong to the image of ϕ. We already know that P_k and S_i for $i = 1, \dots, n-1$ belong to the image by definition of the map ϕ. We will show that it is also the case for the elements U_i by induction on i.

For $i = 1$, we have $U_1 = T$, which belongs to the image because of (4-13). Suppose now that U_i belongs to the image for some integer $1 \le i < \min\{k, n\}$. Then, the following element of $H_{k,n}$ also belongs to the image of ϕ:

(4-36) $\qquad T\, S_1 S_2 \dots S_i\, U_i =$ 

Using the algebraic expressions (4-11), (4-9) and (4-35) as well as the properties of the projector P_k, we can write

$$T\, S_1 S_2 \dots S_i\, U_i = P_k\, \sigma_{k,i+1}\, P_k\, \sigma_{k,i}\, \sigma_{k-1,i} \dots \sigma_{k-i+1,i}\, P_k. \tag{4-37}$$

Using now the property (4-4) for the middle projector, we get

$$T\, S_1 S_2 \dots S_i\, U_i = q^{1-k}[k]_q^{-1} \sum_{m=1}^{k} q^{k-m}\, P_k\, \sigma_{k,i+1} (\sigma_{k-1}\, \sigma_{k-2} \dots \sigma_m)\, \sigma_{k,i}\, \sigma_{k-1,i} \dots \sigma_{k-i+1,i}\, P_k,$$

where the interior of the parenthesis in the case $m = k$ is understood to be 1. Separating the previous sum in two at $m = k - i$, and absorbing braid generators in the right-most projector P_k for the terms with $m < k-i$, we find

$$\begin{aligned}(4\text{-}38)\quad & T S_1 S_2 \dots S_i U_i \\ &= [k]_q^{-1}[k-i]_q P_k\, \sigma_{k,i+1}(\sigma_{k-1}\,\sigma_{k-2}\dots\sigma_{k-i})\,\sigma_{k,i}\,\sigma_{k-1,i}\dots\sigma_{k-i+1,i}\,P_k \\ &\quad + q[k]_q^{-1}\sum_{m=k-i+1}^{k} q^{-m} P_k\,\sigma_{k,i+1}(\sigma_{k-1}\,\sigma_{k-2}\dots\sigma_m) \\ &\qquad\qquad\qquad\qquad \cdot\sigma_{k,i}\,\sigma_{k-1,i}\dots\sigma_{k-i+1,i}\,P_k.\end{aligned}$$

All terms in the sum of the previous equation do not act on the $k-i$ first fused strands. Therefore, these terms correspond to diagrams where only the i last fused strands go out of the top and bottom ellipses, and they can be obtained from a multiplication of the diagram U_i with diagrams S_j. Hence, by the induction hypothesis, they belong to the image of ϕ. Since we have supposed that $i < k$ and since $q^{2m}-1$ is invertible for $m = 1, \dots, k$, the first term in (4-38) is, up to an invertible factor:

$$\begin{aligned}(4\text{-}39)\quad & P_k\,\sigma_{k,i+1}\,\sigma_{k-1}\,\sigma_{k,i}\,\sigma_{k-2}\sigma_{k-1,i}\dots\sigma_{k-i}\sigma_{k-i+1,i}\,P_k \\ &\qquad = P_k\,\sigma_{k,i+1}\,\sigma_{k-1,i+1}\,\sigma_{k-2,i+1}\dots\sigma_{k-i,i+1}\,P_k = U_{i+1}.\end{aligned}$$

Therefore, the element U_{i+1} belongs to the image of ϕ. By induction, we conclude that all the elements U_i for $i = 1, 2, \dots \min\{k, n\}$ belong to the image.

Injectivity of ϕ. At this point, we have shown that $\phi : \mathcal{A}_n^{(k)} \to H_{k,n}$ is a surjective morphism. By comparing the cardinality (4-20) of the spanning set (4-22) for $\mathcal{A}_n^{(k)}$ with the dimension (4-5) of $H_{k,n}$, we deduce that $\dim(\mathcal{A}_n^{(k)}) \le \dim(H_{k,n})$. Hence both algebras have the same dimension and ϕ is an isomorphism.

4D. ***Towards a definition of*** $\mathcal{A}_n^{(k)}$ ***over*** $\mathbb{C}[q^{\pm 1}]$**.** The fused Hecke algebra $H_{k,n}$ is not directly defined over $\mathbb{C}[q^{\pm1}]$. Nevertheless, the presentation by generators and relations of Definition 4.6 could be taken as it is over $\mathbb{C}[q^{\pm1}]$. However, note that the resulting algebra would then not be free over $\mathbb{C}[q^{\pm1}]$ with the correct dimension, that is, Corollary 4.11 would not be true over $\mathbb{C}[q^{\pm1}]$ since we used that $[k]_q!$ is invertible in $\mathbb{C}^{(k)}[q^{\pm1}]$ to prove Proposition 4.8.

So we believe that Definition 4.6 is not the correct one to take over $\mathbb{C}[q^{\pm1}]$. The key to this problem is the following conjectural result.

Conjecture 4.13. *In* $\mathcal{A}_{k+1}$, *when* (α_1, α_2) *are specialised to* (q^{-2}, q^{2k}), *the element* $E_{k+1}^{(q,\alpha_1)}$ *factorises as*

$$(4\text{-}40)\qquad E_{k+1}^{(q,\alpha_1)} = [k+1]_q!\,\tilde{E}_{k+1}^{(q,\alpha_1)},$$

where $\tilde{E}_{k+1}^{(q,\alpha_1)}$ *is in* $\mathcal{A}_{k+1}$ *with coefficients in* $\mathbb{C}[q^{\pm1}]$.

To support this conjecture, we have checked it explicitly for small values of k. Note moreover that it generalises Corollary 4.5 which already identified the factor $[k+1]_q$ (but not the full q-factorial). Finally, we are able to prove this statement in the quotient $\mathcal{C}_{n,2}$ of $\mathcal{A}_n$ (see Section 5C).

Just for this subsection, we are going to assume that this conjecture is true, thereby allowing to define, when α_1, α_2 are specialised as before, an element $\tilde{E}_{k+1}^{(q,\alpha_1)}$ in $\mathcal{A}_n$ with coefficients in $\mathbb{C}[q^{\pm 1}]$. Now the correct definition over $\mathbb{C}[q^{\pm 1}]$ of the algebra $\mathcal{A}_n^{(k)}$ that we promote is:

Definition 4.14. The algebra $\mathcal{A}_n^{(k)}$ is the specialisation over $\mathbb{C}[q^{\pm 1}]$ of $\mathcal{A}_n$ corresponding to $(\alpha_1, \alpha_2) = (q^{-2}, q^{2k})$ with the additional defining relation if $n > k$:

$$\tilde{E}_{k+1}^{(q,\alpha_1)} = 0. \tag{4-41}$$

Now, we can prove the analogue of Corollary 4.11.

Proposition 4.15. *If Conjecture 4.13 holds, the algebra $\mathcal{A}_n^{(k)}$ is free over $\mathbb{C}[q^{\pm 1}]$ with basis*

$$\{g_\omega \mid \omega \in B_n(\bar{1}\bar{2}, \overline{k+1}\,\bar{k} \dots \bar{1})\}.$$

Proof. First the above set is now a spanning set over $\mathbb{C}[q^{\pm 1}]$. Indeed, following the proof of Proposition 4.8, we see that we have now removed all factors in front of the element in $\mathcal{A}_{k+1}$ that we need to rewrite using $\tilde{E}_{k+1}^{(q,\alpha_1)} = 0$. So the same proof works now over $\mathbb{C}[q^{\pm 1}]$. The freeness follows immediately from the already proved isomorphism with $H_{k,n}$ over the field of fractions $\mathbb{C}(q)$. □

5. Centralisers of $U_q(\mathfrak{gl}_N)$ and the boundary seam algebra ($N = 2$)

Let $N > 1$ and let $k > 0$. In this final section, we combine the preceding sections to complete the following square by defining the algebras $\mathcal{C}_{n,N}^{(k)}$:

$$\begin{array}{ccc} \mathcal{A}_n & \rightsquigarrow & \mathcal{A}_n^{(k)} \\ \downarrow & & \downarrow \\ \mathcal{C}_{n,N} & \rightsquigarrow & \mathcal{C}_{n,N}^{(k)} \end{array}$$

and we show their connections with the centralisers of $U_q(\mathrm{gl}_N)$ as discussed in the introduction. We then study in details the case $N = 2$ to show that we have finally recovered the so-called boundary seam algebra from [19].

As in the preceding section, we are going to work, unless otherwise specified over the ring $\mathbb{C}^{(k)}[q^{\pm 1}]$.

5A. *Definition of $\mathcal{C}_{n,N}^{(k)}$.* The following definition has two equivalent forms, due to the two paths in the square above to reach $\mathcal{C}_{n,N}^{(k)}$. Recall that the specialisation and the relation (5-1) were by definition how to go from $\mathcal{A}_n$ to $\mathcal{A}_n^{(k)}$, while the relations (5-2)–(5-3) were by definition how to go from $\mathcal{A}_n$ to $\mathcal{C}_{n,N}$.

Definition 5.1. Over the ring $\mathbb{C}^{(k)}[q^{\pm 1}]$ we have:

• The algebra $\mathcal{C}^{(k)}_{n,N}$ is the specialisation of $\mathcal{C}_{n,N}$ for $\alpha_1 = q^{-2}$ and $\alpha_2 = q^{2k}$, with the additional defining relation if $n > k$:

$$E'^{(q,\alpha_1)}_{k+1} = 0. \tag{5-1}$$

• Equivalently, the algebra $\mathcal{C}^{(k)}_{n,N}$ is the quotient of $\mathcal{A}^{(k)}_n$ by the relations

$$\tilde{E}^{(-q^{-1},\alpha_1)}_N = 0, \tag{5-2}$$

$$\Lambda^{-q^{-1}}_{N+1}(g_1, \dots, g_N) = 0. \tag{5-3}$$

Remark 5.2. Exactly as discussed in Section 4D for the algebra $\mathcal{A}^{(k)}_n$, we emphasise that a good definition of $\mathcal{C}^{(k)}_{n,N}$ over $\mathbb{C}[q^{\pm 1}]$ would require to prove that the idempotent $E^{(q,\alpha_1)}_{k+1}$ factorises as

$$E^{(q,\alpha_1)}_{k+1} = [k+1]_q!\, \tilde{E}^{(q,\alpha_1)}_{k+1}.$$

Then we would define $\mathcal{C}^{(k)}_{n,N}$ by replacing (5-1) by $\tilde{E}^{(q,\alpha_1)}_{k+1} = 0$.

5B. ***Isomorphism with the centralisers.*** We start by relating $\mathcal{C}^{(k)}_{n,N}$ to the fused Hecke algebra $H_{k,n}$.

Proposition 5.3. *The algebra $\mathcal{C}^{(k)}_{n,N}$ is isomorphic to the quotient of $H_{k,n}$ by the relations*

$$P_k\, \Lambda_{N+1}(\sigma_k, \dots, \sigma_{k+N-1})\, P_k = 0, \tag{5-4}$$

$$P_k\, \Lambda_{N+1}(\sigma_{k+1}, \dots, \sigma_{k+N})\, P_k = 0. \tag{5-5}$$

Proof. From the isomorphism of $\mathcal{A}^{(k)}_n$ with $H_{k,n}$, it remains only to prove that the quasiidempotent $\tilde{E}^{(-q^{-1},\alpha_1)}_N$ and the antisymmetriser $\Lambda^{-q^{-1}}_{N+1}(g_1, \dots, g_N)$ of $\mathcal{A}^{(k)}_n$ are mapped to the correct elements in $H_{k,n}$.

First, it is directly seen that

$$\begin{aligned} \phi(\Lambda_{N+1}(g_1, \dots, g_N)) &= \Lambda_{N+1}(S_1, \dots, S_N) \\ &= P_k\, \Lambda_{N+1}(\sigma_{k+1}, \dots, \sigma_{k+N})\, P_k. \end{aligned} \tag{5-6}$$

Then, using an explicit basis for $\mathfrak{S}_{N+1}$, it is seen that

$$\begin{aligned} &\Lambda_{N+1}(\sigma_k, \dots, \sigma_{k+N-1}) \\ &\quad = \Lambda_N(\sigma_{k+1}, \dots, \sigma_{k+N-1})(1 - q^{-1}\sigma_k + \dots + (-q^{-1})^N \sigma_k\, \sigma_{k+1} \dots \sigma_{k+N-1}). \end{aligned} \tag{5-7}$$

Therefore, by definition of the elements S_i and T and by the properties of P_k we can write

$$\begin{aligned} &P_k\, \Lambda_{N+1}(\sigma_k, \dots, \sigma_{k+N-1})\, P_k \\ &\quad = \Lambda_N(S_1, \dots, S_{N-1})(P_k - q^{-1} T + \dots + (-q^{-1})^N T\, S_1 \dots S_{N-1}). \end{aligned} \tag{5-8}$$

Define the element t of $\mathcal{A}_n^{(k)}$ by

(5-9) $$g_0 = (q^{-1}\alpha_2 - q\alpha_1)\,t + q^2\alpha_1,$$

where we recall that $\alpha_1 = q^{-2}$ and $\alpha_2 = q^{2k}$. Since $q^{2k} - 1$ is invertible, this indeed defines the element t. Then we can rewrite the formula (2-28) obtained for $\tilde{E}_N^{(-q^{-1},\alpha_1)}$ as

(5-10) $$\tilde{E}_N^{(-q^{-1},\alpha_1)} = \Lambda_N(g_1,\dots,g_{N-1})\cdot(\alpha_2 - q^2\alpha_1)\Big(1 - q^{-1}\sum_{i=0}^{N-1}(-q^{-1})^i\, t g_1\dots g_i\Big).$$

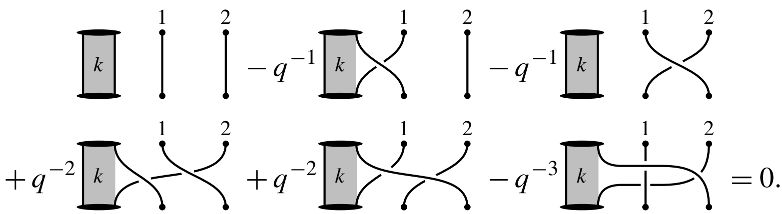
Note that, because of (4-13) (taking into account the specialisation of α_1 and α_2), the image of t under the map ϕ is T. Therefore, by comparing (5-8) with (5-10), it is seen that

(5-11) $$\phi(\tilde{E}_N^{(-q^{-1},\alpha_1)}) = (q^{2k}-1)\,P_k\,\Lambda_{N+1}(\sigma_k,\dots,\sigma_{k+N-1})\,P_k.$$

We conclude using again that $q^{2k}-1$ is invertible. □

Example. Take $N = 2$ and let us write the first relation (5-4) in diagrams like:

(5-12)



In words, we plug in the usual q-antisymmetriser (here on 3 strands) using the last strand coming out of the ellipse as a first strand. In terms of the generators $U_1 := T$ and S_1 of $H_{k,2}$, the previous relation is

(5-13) $$P_k - q^{-1}T - q^{-1}S_1 + q^{-2}S_1T + q^{-2}TS_1 - q^{-3}S_1TS_1 = 0.$$

A similar description works for any $N \geq 2$.

The second relation (5-5) is just the usual q-antisymmetriser on $N+1$ strands, which is plugged in using the $N+1$ first dots (and not using at all the strands coming out of the ellipse).

Isomorphism with the centraliser. In this paragraph only, we will work over the field of fractions $\mathbb{C}(q)$. Using the notations of the introduction, consider the centraliser

$$\mathcal{Z}_{k,n,N} = \mathrm{End}_{U_q(\mathrm{gl}_N)}(L^N_{(k)} \otimes (L^N)^{\otimes n}).$$

Combining what we have obtained so far with the results from [7] on these centralisers, we get the following description of $\mathcal{Z}_{k,n,N}$.

Corollary 5.4. *For all k, n, N as before, we have that $\mathcal{Z}_{k,n,N}$ is isomorphic to $\mathcal{C}_{n,N}^{(k)}$.*

Proof. From [7, Section 5], we have that the fused Hecke algebra $H_{k,n}$ surjects onto the centraliser $\mathcal{Z}_{k,n,N}$. Moreover, it is also clear from this construction that both relations (5-4) and (5-5) are satisfied in the image (since the expressions between the projectors are already 0 in the usual Schur–Weyl duality with $U_q(\mathrm{gl}_N)$). Moreover, it was proved in [7, Section 9] that, for q^2 not a root of unity or over $\mathbb{C}(q)$, the first relation (5-4) is enough to generate the kernel, and this proves that the quotient of $H_{k,n}$ by (5-4) and (5-5) is isomorphic to $\mathcal{Z}_{k,n,N}$. With Proposition 5.3, this concludes the proof. □

Remark 5.5. The proof shows that, over $\mathbb{C}(q)$ or for q^2 not a root of unity, the second relation (5-5) is implied by the first. This was already noticed at the level of $\mathcal{C}_{n,N}$, where it was shown, using the semisimple representation theory in Section 3 that relation (5-2) implies (5-3).

Remark 5.6. The representation theory of $\mathcal{C}_{n,N}^{(k)}$ over $\mathbb{C}(q)$ or when q^2 is not a root of unity is described as follows. Starting with the algebra $\mathcal{A}_n^{(k)}$, which is the fused Hecke algebra, for which the irreducible representations were indexed by partitions $\lambda \vdash k+n$ with $\lambda_1 \geq k$, we simply remove all those which have strictly more than N lines. This is in agreement with the known decomposition of the tensor product of $U_q(\mathrm{gl}_N)$-representations. We will give more details for $N=2$ in Section 5C below.

5C. *The boundary seam algebra (N = 2).* For $N=2$, using the methods and the terminology of [7], the centraliser $Z_{k,n,2}$ could be called the fused Temperley–Lieb algebra, since it can be described by multiplying the usual Temperley–Lieb algebra by a suitable projector on the left and on the right. In our case here, where only the first representation is fused, the fused Temperley–Lieb algebra was introduced in [19] and called the boundary seam algebra (see also [16; 17]). We will show how it is recovered as the algebra $\mathcal{C}_{n,2}^{(k)}$.

First, recall that the algebra $\mathcal{C}_{n,2}$ was identified in Section 3 as the one-boundary Temperley–Lieb algebra, using the following change of generators:

$$e_0 := \alpha_2 - g_0, \qquad e_i := q - g_i, \quad i = 1, 2, \dots, n-1. \tag{5-14}$$

The presentation of $\mathcal{C}_{n,2}$ in terms of these generators was given explicitly in equations (3-8)–(3-12). Here we complete the presentation of $\mathcal{C}_{n,2}^{(k)}$ in terms of the same generators.

Proposition 5.7. *The algebra $\mathcal{C}_{n,2}^{(k)}$ is the specialisation of the one-boundary Temperley–Lieb algebra $\mathcal{C}_{n,2}$ corresponding to $\alpha_1 = q^{-2}$ and $\alpha_2 = q^{2k}$, and the additional relation, if $n \geq k+1$:*

$$u_1 u_2 \dots u_{k+1} = 0, \tag{5-15}$$

where, *for* $m = 0, \dots, k$,

$$u_{m+1} := \sum_{r=0}^{m-1} (-q)^r \left(1 - q^{2(m-r)} \frac{\alpha_1}{\alpha_2}\right) e_m\, e_{m-1} \dots e_{m+1-r} + (-q)^m \alpha_2^{-1} e_m\, e_{m-1} \dots e_0.$$

Proof. According to Definition 5.1, it remains to describe, if $n \geq k+1$, the following relation of $\mathcal{C}_{n,2}^{(k)}$ in terms of the generators $e_0, e_1, \dots, e_{n-1}$:

$$\frac{1}{[k+1]_q} E_{k+1}^{(q,\alpha_1)} = 0. \tag{5-16}$$

We have done most of the work in Proposition 3.8 which gives that, for any $1 \leq m < n$, we have in $\mathcal{C}_{n,2}$:

$$E_{m+1}^{(q,\alpha_1)} = q^{\frac{1}{2}(m(m+1))} [m+1]_q!\, \tilde{E}_{m+1}^{(q,\alpha_1)}, \tag{5-17}$$

where $\tilde{E}_{m+1}^{(q,\alpha_1)}$ is defined recursively by $\tilde{E}_1^{(q,\alpha_1)} = (1 - \alpha_2^{-1} g_0)$ and

$$\tilde{E}_{m+1}^{(q,\alpha_1)} = \tilde{E}_m^{(q,\alpha_1)}\big((1-q^2)(1 + q g_m + \dots + q^{m-1} g_m \dots g_2) + q^m g_m \dots g_1 (1 - \alpha_2^{-1} g_0)\big). \tag{5-18}$$

We rewrite this recursive definition using (5-14) together with the properties $\tilde{E}_m^{(q,\alpha_1)} e_0 = (\alpha_2 - \alpha_1) \tilde{E}_m^{(q,\alpha_1)}$ and $\tilde{E}_m^{(q,\alpha_1)} e_i = 0$ for $1 \leq i \leq m-1$. As an intermediate step, it is found that, for $1 \leq i \leq m+1$,

$$\tilde{E}_m^{(q,\alpha_1)} q^{m+1-i} (q - e_m)(q - e_{m-1}) \dots (q - e_i) = \tilde{E}_m^{(q,\alpha_1)} \sum_{r=0}^{m+1-i} (-1)^r q^{2(m+1-i)-r} e_m\, e_{m-1} \dots e_{m+1-r}. \tag{5-19}$$

Using (5-19) in (5-18), and rearranging sums, the result $\tilde{E}_{m+1}^{(q,\alpha_1)} = \tilde{E}_m^{(q,\alpha_1)} u_{m+1}$ is achieved with u_{m+1} as in the proposition. Now up to some unnecessary invertible power of q, the relation reads

$$[k]_q!\, \tilde{E}_{k+1}^{(q,\alpha_1)} = [k]_q!\, u_1 u_2 \dots u_{k+1} = 0.$$

The claim follows from the invertibility of $[k]_q!$ in the ring $\mathbb{C}^{(k)}[q^{\pm 1}]$. □

The elements u_m and $\tilde{E}_m^{(q,\alpha_1)}$ appear in [17] (up to global factors of α_2 and $-q^{-1}$) as generalised Wenzl–Jones factors and generalised Wenzl–Jones projectors respectively for the one-boundary Temperley–Lieb algebra. Now, using the preceding proposition, it can be directly verified that the following mappings give an antiisomorphism from $\mathcal{C}_{n,2}^{(k)}$ to the boundary seam algebra (with the notations of [17])

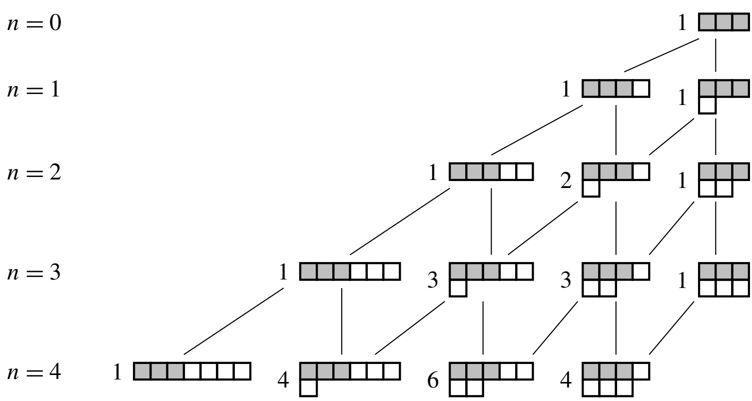
$$e_0 \mapsto q^{k-1}(q - q^{-1})\, e_n, \qquad e_i \mapsto e_{n-i}, \quad 1 \leq i \leq n-1. \tag{5-20}$$

Let us also mention that a recursive formula, similar to (2-17) in $H_{\alpha_1,\alpha_2,n}$, holds for $\tilde{E}_m^{(q,\alpha_1)}$ when q is not a root of unity (or over $\mathbb{C}(q)$) in a specialisation such that $\prod_{r=0}^{m-2}\big(1-q^{2r}\frac{\alpha_1}{\alpha_2}\big) \neq 0$:

$$\tilde{E}_m^{(q,\alpha_1)} = \left(1-q^{2(m-1)}\frac{\alpha_1}{\alpha_2}\right)\tilde{E}_{m-1}^{(q,\alpha_1)} - q\,\frac{\tilde{E}_{m-1}^{(q,\alpha_1)}e_{m-1}\,\tilde{E}_{m-1}^{(q,\alpha_1)}}{\prod_{r=0}^{m-3}\big(1-q^{2r}\frac{\alpha_1}{\alpha_2}\big)}. \tag{5-21}$$

Semisimple representation theory. Here we work over $\mathbb{C}(q)$ or we assume that q^2 is not a root of unity. The representation theory of the boundary seam algebra $\mathcal{C}_{n,2}^{(k)}$ is easily obtained from the one of the fused Hecke algebra $\mathcal{A}_n^{(k)}$. Recall from Section 4 that the irreducible representations of $\mathcal{A}_n^{(k)}$ were indexed by partitions $\lambda \vdash k+n$ with $\lambda_1 \geq k$. The quotient giving the boundary seam algebra $\mathcal{C}_{n,2}^{(k)}$ consists simply in removing all those which have strictly more than two lines.

The beginning of the Bratteli diagram, for example, for $k=3$, of the algebras $\mathcal{C}_{n,2}^{(k)}$ is as follows:



We can see the special behaviour starting at the value $n=k+1=4$. The irreducible representations, at level n, are indexed by a positive integer h, which is the number of boxes added in the second row, satisfying $0 \leq 2h \leq k+n$. It is easy to see recursively from the branching graph that the dimension of the corresponding irreducible representation is

$$\binom{n}{h} - \binom{n}{h-k-1},$$

with the understanding that $\binom{n}{h-k-1} = 0$ if $h \leq k$. Note that comparing with [19], our h is their $\frac{1}{2}(n+k-d)$. The dimension of the algebra is

$$\dim C_{n,2}^{(k)} = \binom{2n}{n} - \binom{2n}{n-k-1}. \tag{5-22}$$

Definition over $\mathbb{C}[q^{\pm 1}]$. We have originally defined $\mathcal{C}_{n,2}^{(k)}$ over $\mathbb{C}^{(k)}[q^{\pm 1}]$. The presentation put forward in Proposition 5.7 can be used without problem over $\mathbb{C}[q^{\pm 1}]$. In our notation, this means to define the algebra $\mathcal{C}_{n,2}^{(k)}$ over $\mathbb{C}[q^{\pm 1}]$ as follows.

Definition 5.8. Over $\mathbb{C}[q^{\pm 1}]$, the algebra $\mathcal{C}_{n,2}^{(k)}$ is the specialisation of $\mathcal{C}_{n,2}$ for $\alpha_1 = q^{-2}$ and $\alpha_2 = q^{2k}$, with the additional defining relation if $n > k$:

$$\tilde{E}_{k+1}^{(q,\alpha_1)} = 0, \tag{5-23}$$

where the renormalised quasiidempotent $\tilde{E}_{k+1}^{(q,\alpha_1)}$ was obtained in Proposition 3.8 and recalled in (5-18).

With this definition, we can prove that we get an algebra which is free over $\mathbb{C}[q^{\pm 1}]$ with dimension equal to (5-22). In fact, we may check that the following set of elements is a $\mathbb{C}[q^{\pm 1}]$-basis:

$$[n_1, m_1][n_2, m_2] \dots [n_r, m_r], \qquad \text{with } \begin{cases} 0 \le n_1 < n_2 < \cdots < n_r \le n-1 \text{ and } m_i \le n_i, \\ 0 = m_1 = \cdots = m_s < m_{s+1} < \cdots < m_r, \quad s < k+1. \end{cases}$$

Without the condition $s < k+1$ in the second line, we already know that this set is a spanning set for $\mathcal{C}_{n,2}$, see (3-14). The relation $\tilde{E}_{k+1}^{(q,\alpha_1)} = 0$ further allows to rewrite any element $[n_1, 0][n_2, 0] \dots [n_{k+1}, 0]$ in terms of elements with fewer g_0 (smaller s). This works over $\mathbb{C}[q^{\pm 1}]$ since the element we need to rewrite appears with an invertible coefficient in $\tilde{E}_{k+1}^{(q,\alpha_1)}$. We refer to the proof of Proposition 4.8 for more details. The above set is of the correct cardinality [19, Appendix B], that is, equation (5-22), and thus is a basis over $\mathbb{C}[q^{\pm 1}]$.

Remark 5.9. If we specialise q to a complex number such that $q^{2i} \neq 1$ for $i = 1, \dots, k$, of course Definition 5.8 recovers Definition 5.1. But now Definition 5.8 allows to consider the cases where $q^{2i} = 1$ for some $i = 1, \dots, k$. We note that we differ here from [19] where the defining relations, when $q^{2i} \neq 1$, were modified according to the value of q and the dimension of the algebra resultingly depended on q.

Acknowledgements

The authors thank Yvan Saint-Aubin for many interesting discussions. L. Poulain d'Andecy is supported by Agence Nationale de la Recherche Projet AHA ANR-18-CE40-0001 and the international research project AAPT of the CNRS. M. Zaimi is grateful to the LMR for its hospitality and is supported by the international research project AAPT of the CNRS. Zaimi also holds an Alexander-Graham-Bell scholarship from the Natural Sciences and Engineering Research Council of Canada (NSERC).

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Received August 7, 2023. Revised December 15, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 328, No. 1, 2024
https://doi.org/10.2140/pjm.2024.328.119

ON THE GENERIC BEHAVIOR OF THE SPECTRAL NORM

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Our main result is that for any closed symplectic manifold, the spectral norm of the iterates of a Hamiltonian diffeomorphism is locally uniformly bounded away from zero C^∞-generically.

1. Introduction

We show that for a Hamiltonian diffeomorphism φ of a closed symplectic manifold M, the spectral norm over $\mathbb{Q}$ of the iterates φ^k is locally uniformly bounded away from zero C^∞-generically in φ, without any additional assumptions on M.

The question of the behavior of the sequence $\gamma(\varphi^k)$ of spectral norms goes back to the work of Polterovich [2002]. Recently, there has been renewed interest in the problem whether and when this sequence is bounded away from zero. There are several reasons for this question, amounting roughly speaking to the fact that one can obtain pretty strong results on the symplectic dynamics of φ when the sequence is *not* bounded away from zero:

$$\underline{\gamma}(\varphi) := \liminf_{k\to\infty} \gamma(\varphi^k) = 0. \tag{1-1}$$

Among these are, for instance, Lagrangian Poincaré recurrence [Ginzburg and Gürel 2018; Joksimović and Seyfaddini 2023], and the variant of the strong closing lemma from [Cineli and Seyfaddini 2022]. Simultaneously, fairly explicit criteria for this sequence to be bounded away from zero have been established, based on the crossing energy theorem from [Ginzburg and Gürel 2014; 2018]; see, e.g., [Cineli et al. 2022] and Theorem 3.1. Let us now provide some more context for the question.

First, note that the condition (1-1) can be interpreted as that φ is γ-rigid or, in other words, a γ-approximate identity.

This notion is a particular case of a much more general concept. Namely, consider a class of diffeomorphisms φ or even homeomorphisms of a manifold M, which we assume here to be closed. For instance, this can be the class of all diffeomorphisms or

Çineli is supported by ERC Starting Grant 851701 via a postdoctoral fellowship. Ginzburg is supported by NFS grant DMS-2304206 and Simons Foundation Travel Support MP-TSM-00002529. Gürel is supported by NSF grant DMS-2304207 and Simons Foundation Travel Support 855299.
MSC2020: 37J11, 37J46, 53D40.
Keywords: spectral norm, periodic points, Hamiltonian diffeomorphisms, Floer homology.

of Hamiltonian diffeomorphisms when M is symplectic, etc. Assume that this class is equipped with some norm $\|\cdot\|$, e.g., the C^0- or C^1-norm or the γ- or Hofer-norm in the Hamiltonian case. A map φ is said to be $\|\cdot\|$-*rigid* if $\varphi^{k_i} \to \mathrm{id}$ with respect to $\|\cdot\|$, i.e., $\|\varphi^{k_i}\| \to 0$, for some sequence $k_i \to \infty$. The term "rigid" is somewhat overused in dynamics and also frequently confused with structural stability, and in [Ginzburg and Gürel 2019a] we proposed to call such a map φ a $\|\cdot\|$-*approximate identity*, or a $\|\cdot\|$-*a.i.* for the sake of brevity. We refer the reader to, e.g., [Bramham 2015; Ginzburg and Gürel 2019a; Cineli and Seyfaddini 2022] for a further discussion of approximate identities, aka rigid maps, in different contexts. Here we only mention that C^r-a.i. is obviously C^s-a.i. for any $s \leq r$ and, when M is aspherical or $M = \mathbb{CP}^n$, a C^0-a.i. is also a γ-a.i.; see [Buhovsky et al. 2021; Shelukhin 2022b].

Zeroing in on γ-a.i.'s we note that there are rather few examples of such maps. The most dynamically interesting examples are Hamiltonian pseudorotations. This class of maps has been extensively studied in a variety of settings by dynamical systems methods and more recently from the perspective of symplectic topology and Floer theory; see, for example, [Anosov and Katok 1970; Avila et al. 2020; Bramham 2015; Fayad and Katok 2004; Ginzburg and Gürel 2018; Joksimović and Seyfaddini 2023; Le Roux and Seyfaddini 2022].

While the official definitions of Hamiltonian pseudorotations vary, these are, roughly speaking, Hamiltonian diffeomorphisms with a finite and minimal possible number of periodic points (in the sense of Arnold's conjecture); see [Ginzburg and Gürel 2018; Shelukhin 2020; 2021]. For instance, when $M = \mathbb{CP}^n$ this number is $n+1$. Most likely, for many symplectic manifolds this condition can be relaxed. Namely, in all examples of Hamiltonian diffeomorphisms φ with finitely many periodic points, all periodic points are fixed points and their number is minimal possible. Thus φ is a pseudorotation. For a certain class of manifolds M, including $\mathbb{CP}^n$, this has been established rigorously under a minor nondegeneracy assumption; see [Shelukhin 2022a] and also [Çineli et al. 2022]. In all examples to date of Hamiltonian diffeomorphisms φ with finitely many periodic points, φ is nondegenerate.

In general, the relation between pseudorotations and γ-a.i.'s is not obvious. All known Hamiltonian pseudorotations are γ-a.i.'s and for $M = \mathbb{CP}^n$ this is proved in [Ginzburg and Gürel 2018] by using the results from [Ginzburg and Gürel 2009a]. The converse is not true: for instance any element φ of a Hamiltonian torus action is a γ-a.i., although φ need not have isolated fixed points. (It is conceivable that for a strongly nondegenerate γ-a.i., the periodic points are necessarily the fixed points: in the obvious notation, $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$. However, a map φ with the latter property need not be a γ-a.i. For instance, $\gamma(\varphi^k)$ can grow linearly for such a map; see Remark 4.10.)

Most closed symplectic manifolds (M, ω) admit no pseudorotations, that is, every Hamiltonian diffeomorphism of M has infinitely many periodic points. This statement (for a specific manifold M) is usually referred to as the Conley conjecture. To date, the Conley conjecture has been shown to hold unless there exists $A \in \pi_2(M)$ such that $\langle[\omega], A\rangle > 0$ and $\langle c_1(TM), A\rangle > 0$; see [Çineli 2018; Ginzburg and Gürel 2015; 2019b]. In particular, the Conley conjecture holds when M is symplectically aspherical or negative monotone. For a broad class of closed symplectic manifolds, φ has infinitely many periodic points C^∞-generically; see [Ginzburg and Gürel 2009b; Sugimoto 2021] and Section 4B.

Although the classes of Hamiltonian pseudorotations and γ-a.i.'s are certainly different, there is a clear parallel between these two classes and their existence conditions on M.

Conjecture. Let M be closed symplectic manifold.

(i) The manifold M admits no γ-a.i.'s unless there exists $A \in \pi_2(M)$ such that $\langle[\omega], A\rangle > 0$ and $\langle c_1(TM), A\rangle > 0$.

(ii) A Hamiltonian diffeomorphism $\varphi : \mathbb{CP}^n \to \mathbb{CP}^n$ is a γ-a.i. if and only if all iterates φ^k are Morse–Bott nondegenerate and $\dim \mathrm{H}_*(\mathrm{Fix}(\varphi^k); \mathbb{F}) = n+1$ for all $k \in \mathbb{N}$ and any ground field $\mathbb{F}$.

This conjecture is supported by some evidence. For instance, M does not admit periodic Hamiltonian diffeomorphisms φ (i.e., $\varphi^N = \mathrm{id}$ for some $N > 1$) when M satisfies the conditions of (i); see [Atallah and Shelukhin 2023; Polterovich 2002]. In addition, $\mathrm{Fix}(\varphi^k)$ is Morse–Bott nondegenerate whenever φ is periodic. This is a consequence of the equivariant Darboux lemma; see, e.g., [Guillemin and Sternberg 1984, Theorem 22.2]. Aspherical or negative monotone symplectic manifolds do not admit C^1-a.i.'s; see [Polterovich 2002] and [Sugimoto 2023]. Further results along these lines can be found in [Atallah and Shelukhin 2023]. In [Cineli et al. 2022] both assertions are proved in dimension two for strongly nondegenerate Hamiltonian diffeomorphisms; see Corollary 3.4. In the setting of (i) the sequence of the spectral norms $\gamma(\varphi^p)$ over $\mathbb{Z}/p\mathbb{Z}$, where p ranges through all primes, is separated away from zero [Shelukhin 2023]. As we have already mentioned the "if" part of (ii) is established in [Ginzburg and Gürel 2018] without any nondegeneracy assumption when $|\mathrm{Per}(\varphi)| = n+1$. With this in mind, part (ii) of the conjecture asserts, in particular, that every pseudorotation of $\mathbb{CP}^n$ is strongly nondegenerate.

Remark 1.1. While part (ii) of the conjecture might extend to some other ambient symplectic manifolds M, some restriction on M is necessary. For instance, the torus $\mathbb{T}^{2n}$ equipped with an irrational symplectic structure admits a Hamiltonian diffeomorphism φ such that the conditions of (ii) are satisfied but $\gamma(\varphi^k) \to \infty$; see [Zehnder 1987] and also [Cineli 2023] for further constructions of this type with complicated dynamics.

In a similar vein, the main result of this paper can be thought of as the γ-a.i. analogue of the aforementioned theorem on the C^∞-generic Conley conjecture, although at this moment the proof of the latter requires some additional conditions on the underlying manifold; see Section 4B.

Remark 1.2. Overall, rather little is known about the behavior of the γ-norm under iterations. For a certain class of manifolds, including $\mathbb{CP}^n$, the spectral norm is a priori bounded from above [Entov and Polterovich 2003; Kislev and Shelukhin 2021]. However, such manifolds appear to be rare; see Remark 4.10. Also, the sequence $\gamma(\varphi^k)$ is bounded from above when $\operatorname{supp}\varphi$ is displaceable in M, but not much beyond these facts and the results of this paper is known about the behavior of this sequence. For instance, when M is a surface of positive genus, it is not known if $\gamma(\varphi^k)$ necessarily grows linearly or can be bounded from above when φ is strongly nondegenerate or, at the opposite extreme, autonomous and $\operatorname{supp}\varphi$ is not displaceable.

Remark 1.3. It is worth keeping in mind that in contrast with some other dynamics concepts, in most if not all settings a.i.'s are sensitive to reparametrization. To be more specific, let an a.i. φ be the time-one map of the flow of a vector field X and let ψ be the time-one map of fX for some function $f > 0$. Then, in general, ψ need not be an a.i. For instance, assume that X is a solid rotation vector field on $M = S^2$ and f is not constant. Then one can show that ψ is not a C^0-a.i., and hence not a C^r-a.i. for any $r \geq 0$. Apparently, the same is true for the γ-norm, but this fact is yet to be proved rigorously; cf. item (ii) of the Conjecture.

2. Preliminaries and notation

We very briefly set our notation and conventions which are quite standard and spelled out in more detail in, for example, [Cineli and Seyfaddini 2022]. The reader may find it convenient to jump to Section 3 and consult this section only as needed.

Throughout the paper, all manifolds, functions and maps are assumed to be C^∞-smooth unless specifically stated otherwise.

Let (M^{2n}, ω) be a closed symplectic manifold. A *Hamiltonian diffeomorphism* $\varphi = \varphi_H = \varphi_H^1$ is the time-one map of the time-dependent flow $\varphi^t = \varphi_H^t$ of a 1-periodic in time Hamiltonian $H : S^1 \times M \to \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. We set $H_t = H(t, \cdot)$. The Hamiltonian vector field X_H of H is defined by $i_{X_H}\omega = -dH$. We say that φ is *nondegenerate* if all fixed points of φ are nondegenerate, and *strongly nondegenerate* if all periodic points of φ are nondegenerate. We will denote by $\operatorname{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms of (M, ω).

Recall that the *spectral norm*, also known as the *γ-norm*, of φ is defined as

$$\gamma(\varphi) = \inf_H \{\mathrm{c}(H) + \mathrm{c}(H^{\mathrm{inv}}) \mid \varphi = \varphi_H\},$$

where $H^{\text{inv}}(x) = -H_t(\varphi_H^t(x))$ is the Hamiltonian generating the flow $(\varphi_H^t)^{-1}$ and $\mathrm{c} = \mathrm{c}_{[M]}$ is the spectral invariant associated with the fundamental class $[M] \in \mathrm{H}_{2n}(M)$. (Here we can take as H^{inv} any Hamiltonian generating this flow with the same time/space average as H.) The infimum is taken over all 1-periodic in time Hamiltonians H generating φ, i.e., $\varphi = \varphi_H$. The *Hofer norm* of φ is defined as

$$\|\varphi\|_H = \inf_H \int_{S^1} (\max_M H_t - \min_M H_t)\, dt,$$

where the infimum is again taken over all 1-periodic in time Hamiltonians H generating φ. Then

$$\gamma(\varphi) \leq \|\varphi\|_H.$$

We refer the reader to, e.g., [Oh 2005a; 2005b; Schwarz 2000; Viterbo 1992] and also, e.g., [Cineli and Seyfaddini 2022; Entov and Polterovich 2003; Ginzburg and Gürel 2009a; Kislev and Shelukhin 2021; Polterovich 2001; Usher 2008; 2011], for the original treatment and a detailed discussion of spectral invariants and these norms.

Here we are interested in the behavior of $\gamma(\varphi^k)$, $k \in \mathbb{N}$, and in particular in the question when this sequence is bounded away from zero. As in the introduction, set

$$\underline{\gamma}(\varphi) = \liminf_{k\to\infty} \gamma(\varphi^k) \in [0,\ \infty].$$

These definitions implicitly depend on the construction of the filtered Floer homology $\mathrm{HF}^a(H)$ for the action window $(-\infty,\ a)$. In this paper we do not in general assume that the class $[\omega]$ is rational or that φ is nondegenerate. Hence, we feel, a word is due on the specifics of the definitions.

Assume first that H is nondegenerate. Then we utilize Pardon's VFC package [2016], to define the filtered Floer homology $\mathrm{HF}^a(H)$ over $\mathbb{Q}$ and spectral invariants; see, for example, [Cineli and Seyfaddini 2022; Usher 2008]. To be more specific, $\mathrm{HF}^a(H)$ is the homology of the subcomplex $\mathrm{CF}^a(H)$ of the Floer complex $\mathrm{CF}(H)$ generated by Floer chains with action below a. Virtually any choice of the *Novikov field* can be used here. We take the standard Novikov field

$$\Lambda = \Big\{ \sum_{A\in\Gamma} b_A A \mid b_A \in \mathbb{Q} \text{ and } \#\{b_A \neq 0,\ \omega(A) > c\} < \infty \text{ for all } c \in \mathbb{R} \Big\},$$

where $\Gamma = \pi_2(M)/(\ker[\omega] \cap \ker c_1(TM))$. Alternatively, we could have used the universal Novikov field. Then, for any $\alpha \in \mathrm{H}_*(M) \otimes \Lambda$, the spectral invariant $\mathrm{c}_\alpha(H)$ is defined as

$$\mathrm{c}_\alpha(H) = \inf\{a \in \mathbb{R} \mid \alpha \in \operatorname{im} \iota_a\}, \tag{2-1}$$

where

$$\iota_a : \mathrm{HF}^a(H) \to \mathrm{HF}(H) \cong \mathrm{H}_*(M) \otimes \Lambda \tag{2-2}$$

is the natural inclusion-induced map and the identification on the right is the PSS-isomorphism. We note that all spectral invariants necessarily belong to the action spectrum $\mathcal{S}(H)$ of H when H is nondegenerate [Usher 2008].

When H is not necessarily nondegenerate, we set

$$\mathrm{c}_\alpha(H) := \inf_{\tilde{H}\geq H} \mathrm{c}_\alpha(\tilde{H}) = \sup_{\tilde{H}\leq H} \mathrm{c}_\alpha(\tilde{H}) = \lim_{\tilde{H}\to H} \mathrm{c}_\alpha(\tilde{H}),$$

where $\tilde{H}$ is nondegenerate and the convergence $\tilde{H}\to H$ is taken to be C^0. The second and third equalities and the existence of the limit follow from that c_α is monotone and $\mathrm{c}_\alpha(\tilde{H}+k) = \mathrm{c}_\alpha(\tilde{H})+k$ for any constant function k. Alternatively, we could have set

$$\mathrm{HF}^a(H) = \varinjlim_{\tilde{H}\geq H} \mathrm{HF}^a(\tilde{H}),$$

and then used (2-1) and (2-2) to get the same result.

Defined in this way, spectral invariants c_α can be easily shown to have all the standard properties: $\mathrm{c}_\alpha(H)$ is monotone and Lipschitz continuous in H with Lipschitz constant one; $\mathrm{c}_\alpha(H+k) = \mathrm{c}_\alpha(H)+k$ for any constant function k; etc. (We refer the reader to, e.g., [Cineli and Seyfaddini 2022] for more details.) The exception is that $\mathrm{c}_\alpha(H)$ has been proven to be spectral, i.e., an element of $\mathcal{S}(H)$, only when $[\omega]$ is rational or H is nondegenerate; see [Entov and Polterovich 2003; Oh 2005a; Usher 2008].

3. Main results

The key to bounding $\underline{\gamma}$ from below is the following fact connecting the behavior of $\gamma(\varphi^k)$ with the dynamics of φ and, in particular, its hyperbolic points.

Theorem 3.1. *Let $\varphi: M\to M$ be a Hamiltonian diffeomorphism of a closed symplectic manifold M with more than* $\dim \mathrm{H}_*(M)$ *hyperbolic periodic points. Then* $\underline{\gamma}(\varphi)>0$. *Also, $\underline{\gamma}$ is locally uniformly bounded away from zero near φ, i.e., there exists* $\delta>0$, *possibly depending on φ, and a sufficiently C^∞-small neighborhood $\mathcal{U}$ of φ such that*

$$\underline{\gamma}(\psi) > \delta \quad \textit{for all } \psi\in\mathcal{U}.$$

Without the "Also" part, this theorem was originally proved in [Cineli et al. 2022]. We give a complete proof in Section 4. Let us emphasize that in Theorem 3.1 we impose no nondegeneracy requirements on φ, and also that the property of φ to have more than $\dim \mathrm{H}_*(M)$ hyperbolic periodic points, or more than any fixed number of hyperbolic periodic points, is open in C^1-topology.

Example 3.2. Assume that M is a closed surface and $\mathrm{h_{top}}(\varphi)>0$. Then φ has infinitely many hyperbolic periodic points [Katok 1980]. Hence, $\underline{\gamma}(\varphi)>0$, and

$\underline{\gamma}(\psi) > \delta$ for some $\delta > 0$ and all ψ which are C^∞-close to φ. Also note in connection with Theorem 3.3 and Corollary 3.4 below that $\mathrm{h_{top}} > 0$ is a C^∞-generic condition in dimension two [Le Calvez and Sambarino 2022].

The requirement of the theorem that the number of hyperbolic points is greater than $\dim \mathrm{H}_*(M)$ can be further relaxed by looking only at the odd/even-degree homology of M, depending on whether $n = \dim M/2$ is odd or even; see Remark 4.2.

The main result of the paper is the following theorem relying on Theorem 3.1 and the strong closing lemma from [Cineli and Seyfaddini 2022].

Theorem 3.3. *Let M be a closed symplectic manifold. The function $\underline{\gamma}$ is locally uniformly bounded away from zero on a C^∞-open and dense set of Hamiltonian diffeomorphisms $\varphi : M \to M$, i.e., for every φ in this set there exists $\delta > 0$, possibly depending on φ but not on ψ, such that*

$$\underline{\gamma}(\psi) > \delta,$$

whenever ψ is sufficiently C^∞-close to φ.

We note that we do not assert here that in general the set of Hamiltonian diffeomorphisms φ with $\underline{\gamma}(\varphi) > 0$ is itself C^∞-open, but rather that this set contains a set which is C^∞-open and dense. Nor do we impose any restrictions on the (symplectic) topology of M or require any of the iterates φ^k to be nondegenerate. The proof of Theorem 3.3 given in Section 4A is based on a variant of the Birkhoff–Lewis–Moser theorem. The key new ingredient of the proof is the strong closing lemma from [Cineli and Seyfaddini 2022]. It is also worth pointing out that if we replaced the statement that the set is C^∞-dense by that it is C^1-dense, the theorem would turn into an easy consequence of already known facts; see Remark 4.5.

In several situations, Theorem 3.3 can be made slightly more precise. For instance, we have the following result, also originally proved in [Cineli et al. 2022] without the "Also" part.

Corollary 3.4. *Assume that M is a surface and φ is strongly nondegenerate. Then $\underline{\gamma}(\varphi) > 0$ when M has positive genus. When M is the two-sphere, $\underline{\gamma}(\varphi) = 0$ if and only if φ is a pseudorotation. Also, $\underline{\gamma}$ is locally uniformly bounded from 0 on the set of all strongly nondegenerate Hamiltonian diffeomorphisms φ when M has positive genus and on the set of such φ with at least three fixed points when $M = S^2$.*

Proof. When M has positive genus, a Conley conjecture type argument guarantees that φ has infinitely many hyperbolic periodic points; see [Franks and Handel 2003; Ginzburg and Gürel 2015; Salamon and Zehnder 1992] or [Le Calvez and Sambarino 2022]. Thus, in this case, the statement follows directly from Theorem 3.1.

Concentrating on $M = S^2$, first note that for all, not necessarily nondegenerate, pseudorotations of $\mathbb{CP}^n$, the sequence $\gamma(\varphi^k)$ contains a subsequence converging to zero, and hence $\underline{\gamma}(\varphi)=0$; see [Ginzburg and Gürel 2018]. In the opposite direction, when $M=S^2$, the existence of one positive hyperbolic periodic point is enough to ensure that $\underline{\gamma}(\varphi)>0$ and, moreover, $\underline{\gamma}$ is locally uniformly bounded away from zero; see Remark 4.2. Hence, more generally, without any nondegeneracy assumption, if $\underline{\gamma}(\varphi)=0$, then all periodic points of φ are elliptic. For strongly nondegenerate Hamiltonian diffeomorphisms φ, this forces φ to be a pseudorotation. □

Since the Hofer norm is bounded from below by the spectral norm, we have the following.

Corollary 3.5. *In all results from this section, we can replace the spectral norm by the Hofer norm.*

We refer the reader to Section 4 for further refinements of Theorems 3.1 and 3.3.

Remark 3.6. Throughout the paper all homology groups are taken over $\mathbb{Q}$. This choice of the background coefficient field is necessitated by the use of Floer theory for an arbitrary closed symplectic manifold M. When M is weakly monotone, $\mathbb{Q}$ can be replaced by any coefficient field.

4. Proofs and refinements

In Section 4A, we prove Theorems 3.1 and 3.3. In Section 4B, we refine the latter result under certain additional assumptions on M and further comment on the class of γ-a.i.'s.

4A. *Proofs of Theorems 3.1 and 3.3.*

Proof of Theorem 3.1. By the conditions of the theorem, for some $N\in\mathbb{N}$, the Hamiltonian diffeomorphism φ has more than $\dim \mathrm{H}_*(M)$ hyperbolic N-periodic points. We denote the set of these points by $\mathcal{K}$. Thus $|\mathcal{K}|>\dim \mathrm{H}_*(M)$ and clearly $\mathcal{K}$ is a locally maximal hyperbolic set. Furthermore, every point in $\mathcal{K}$ is also ℓN-periodic for all $\ell\in\mathbb{N}$. For $\epsilon>0$, denote by $b_\epsilon(\varphi)$ the number of bars in the barcode of φ of length greater than ϵ including infinite bars; see, for example, [Cineli et al. 2021]. Then, we claim that, for a sufficiently small $\epsilon>0$ and any $\ell\in\mathbb{N}$,

$$b_\epsilon(\varphi^{\ell N})\geq \dim \mathrm{H}_*(M)+\left\lceil\frac{|\mathcal{K}|-\dim \mathrm{H}_*(M)}{2}\right\rceil>\dim \mathrm{H}_*(M). \tag{4-1}$$

In particular, $\varphi^{\ell N}$ has at least one finite bar of length greater than $\epsilon>0$.

This inequality is essentially a consequence of [Cineli et al. 2021, Proposition 3.8 and 6.2]. To prove (4-1), first note that the number of infinite bars in the barcode of any Hamiltonian diffeomorphism is equal to $\dim \mathrm{H}_*(M)$. Secondly, it follows from [Cineli et al. 2021, Proposition 6.2] and the proof of [Cineli et al. 2021, Proposition 3.8] that every periodic point in $\mathcal{K}$ appears as an "end point" of a bar of length greater than $\epsilon > 0$. Combining these two facts, we conclude that $\varphi^{\ell N}$ has at least $\lceil(|\mathcal{K}| - \dim \mathrm{H}_*(M))/2\rceil$ finite bars of length greater than $\epsilon > 0$, and (4-1) follows.

Furthermore, since the crossing energy lower bound in [Cineli et al. 2021, Theorem 6.1] is stable under C^∞-small perturbations of the Hamiltonian, for every positive $\eta < \epsilon$ the same is true for any Hamiltonian diffeomorphism Ψ which is C^∞-close to φ^N. Namely,

$$b_\eta(\Psi^\ell) > \dim \mathrm{H}_*(M),$$

and hence the barcode of Ψ^ℓ has a finite bar of length greater than η.

Also, recall that as is proved in [Kislev and Shelukhin 2021, Theorem A], for any φ,

$$\beta_{\max}(\varphi) \le \gamma(\varphi),$$

where the left-hand side is the *boundary depth*, i.e., the longest finite bar in the barcode of φ. Thus, for a sufficiently small $\eta > 0$,

$$\eta < \beta_{\max}(\Psi^\ell) \le \gamma(\Psi^\ell). \tag{4-2}$$

Next, set $\delta = \eta/2$ and arguing by contradiction, assume that there exist ψ sufficiently C^∞-close to φ and a sequence $k_i \to \infty$ such that

$$\gamma(\psi^{k_i}) < \delta.$$

Since the sequence k_i is infinite and there are only finitely many residues modulo N, there exists a pair $k_i < k_j$ such that

$$k_j - k_i = \ell N$$

for some $\ell \in \mathbb{N}$.

Clearly, $\Psi = \psi^N$ is C^∞-close to φ^N when ψ is sufficiently C^∞-close to φ, and hence (4-2) holds. Then by the triangle inequality for γ, we have

$$\eta < \gamma(\Psi^\ell) \le \gamma(\psi^{k_j}) + \gamma(\psi^{-k_i}) < 2\delta = \eta.$$

This contradiction concludes the proof of the theorem. □

Remark 4.1. It might be worth a second to examine how the invariants of φ involved in the proof depend on the isotopy φ_H^t in $\mathrm{Ham}(M, \omega)$ generated by H and its lift to the universal covering of the group. Namely, $\gamma(\varphi)$ is a priori independent of the isotopy only on the universal covering. On $\mathrm{Ham}(M, \omega)$ it is defined by passing to the infimum over often infinitely many elements. However, the boundary depth $\beta_{\max}$ is well defined on $\mathrm{Ham}(M, \omega)$. In the proof we bound $\beta_{\max}(\varphi)$ from below (see, e.g., [Usher 2011]) and that bounds $\gamma(\varphi)$ from below regardless of the lift [Kislev and Shelukhin 2021].

Remark 4.2. When $n = \dim M/2$ is odd, it is sufficient to require in Theorem 3.1 that the number of hyperbolic periodic points is greater than $b = \dim \mathrm{H}_{\mathrm{odd}}(M)$. For instance, this is the case when M is a surface. Indeed, in the proof of the theorem, by taking N even and sufficiently large, we can guarantee that the number of positive hyperbolic N-periodic points is greater than b. Such points necessarily have even Conley–Zehnder index, and hence contribute to the odd-degree homology of M under the isomorphism $\mathrm{HF}_*(\varphi^N) \cong \mathrm{H}_{*+n}(M)$. Likewise, when n is even, it suffices to require the number of hyperbolic periodic points to be greater than $\dim \mathrm{H}_{\mathrm{even}}(M)$.

Proof of Theorem 3.3. To prove the theorem, it suffices to show that every C^∞-open set $\mathcal{U}$ in the group of Hamiltonian diffeomorphisms contains an open subset $\mathcal{W}$ such that $\underline{\gamma}(\varphi) > \delta$ for all $\varphi \in \mathcal{W}$ and some $\delta = \delta(\mathcal{W}) > 0$ independent of φ. Indeed, then fixing $\mathcal{W}$ for every $\mathcal{U}$ we can take the union of sets $\mathcal{W}$ for all $\mathcal{U}$ as the desired open and dense subset.

Let $q = \dim \mathrm{H}_*(M)$. For any $\mathcal{U}$, there are two alternatives:

(i) There exists $\varphi \in \mathcal{U}$ with more than q periodic points.

(ii) Every $\varphi \in \mathcal{U}$ has at most q periodic points.

Let us first focus on case (i). Pick $\varphi \in \mathcal{U}$ with more than q periodic points and fix $q+1$ of them. Denote these points by $x_0, \dots, x_q$, and note that arbitrarily C^∞-close to φ there exists a Hamiltonian diffeomorphism $\varphi' \in \mathcal{U}$ such that $x_0, \dots, x_q$ are nondegenerate periodic points of φ'. This is essentially a linear algebra fact and to construct φ', it suffices to perturb φ near these points, changing $D\varphi$ slightly. (Note that φ' may have many other periodic points, nondegenerate or not. We can ensure in addition that φ' is strongly nondegenerate, but we do not need this fact.) We replace φ by φ', keeping the notation φ.

If all periodic points $x_0, \dots, x_q$ are hyperbolic, we can take as $\mathcal{W}$ any C^∞-small neighborhood of φ by Theorem 3.1.

If one of the points $x_0, \dots, x_q$ is not hyperbolic, we argue by perturbing φ again. Namely, recall that by the Birkhoff–Lewis–Moser theorem (see [Moser 1977]), whenever φ has a nonhyperbolic, nondegenerate periodic point x, there exists an arbitrarily C^∞-small perturbation $\varphi' \in \mathcal{U}$ of φ with infinitely many periodic points near x. Moreover, φ' can be chosen so that infinitely many of these periodic points are hyperbolic; see [Arnaud 1992, Proposition 8.2]. (This follows from the proof of the Birkhoff–Lewis–Moser theorem.) Thus, again by Theorem 3.1, we can take a sufficiently C^∞-small neighborhood of φ' as $\mathcal{W}$.

To deal with case (ii), we need the following quantitative variant of the strong closing lemma:

Lemma 4.3 [Cineli and Seyfaddini 2022]. *Let ψ be a Hamiltonian diffeomorphism of a closed symplectic manifold M. Assume that there is a closed ball $V \subset M$*

containing no periodic points of ψ, *that is*, $V \cap \mathrm{Per}(\psi) = \varnothing$. *Let* $G \geq 0$ *be a Hamiltonian supported in* V *and such that*

$$\mathrm{c}(G) > \underline{\gamma}(\psi).$$

Then the composition $\psi\varphi_G$ *has a periodic orbit passing through* V.

Pick a nondegenerate Hamiltonian diffeomorphism $\varphi \in \mathcal{U}$, where $\mathcal{U}$ is as in case (ii). Such a map exists since $\mathcal{U}$ is C^∞-open and the set of nondegenerate Hamiltonian diffeomorphisms is C^∞-dense (and open). We will show that there exists $\delta > 0$ such that $\underline{\gamma}(\psi) > \delta$ for all $\psi \in \mathcal{U}$ which are C^∞-close to φ. Hence, in this case, we can take a small C^∞-neighborhood of φ as $\mathcal{W}$.

Lemma 4.4. *Let* (M, ω) *be a closed symplectic manifold. Suppose that there exists a* C^∞*-open* $\mathcal{U} \subset \mathrm{Ham}(M, \omega)$ *such that all* $\varphi \in \mathcal{U}$ *have at most* $q = \dim H_*(M)$ *periodic points. Then the function* $\underline{\gamma} : \mathcal{U} \to [0, \infty)$ *is locally uniformly bounded away from zero at every nondegenerate* $\varphi \in \mathcal{U}$.

Note that the proof of Theorem 3.3 will be completed once we prove Lemma 4.4. To prove the lemma, arguing by contradiction, fix a nondegenerate $\varphi \in \mathcal{U}$ and assume that there exists a sequence $\psi_i \to \varphi$ in $\mathcal{U}$ such that

$$\underline{\gamma}(\psi_i) \to 0.$$

Here and below convergence of maps is always understood in the C^∞-sense.

We claim that when i is large enough, all periodic points of ψ_i are close to periodic points of φ, and hence there exists a closed ball $V \subset M$ containing no periodic points of any of these maps. Indeed, since φ is nondegenerate and

$$|\mathrm{Fix}(\varphi)| \leq |\mathrm{Per}(\varphi)| \leq q = \dim \mathrm{H}_*(M),$$

by the Arnold conjecture (see [Fukaya and Ono 1999; Liu and Tian 1998] and also [Pardon 2016]),

$$\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi) \quad \text{and} \quad |\mathrm{Per}(\varphi)| = |\mathrm{Fix}(\varphi)| = q.$$

Furthermore, when i is large enough, $\psi_i \in \mathcal{U}$ is also nondegenerate since $\psi_i \to \varphi$. Therefore, again by the Arnold conjecture,

$$\mathrm{Per}(\psi_i) = \mathrm{Fix}(\psi_i) \quad \text{and} \quad |\mathrm{Per}(\psi_i)| = |\mathrm{Fix}(\psi_i)| = q.$$

It follows that $\mathrm{Per}(\psi_i)$ converges to $\mathrm{Per}(\varphi)$.

Next, take $G \geq 0$ as in Lemma 4.3, which is supported in V and small enough so that $\varphi\varphi_G \in \mathcal{U}$. Hence, $\psi_i\varphi_G \in \mathcal{U}$ when i is large; for $\psi_i \to \varphi$ and thus $\psi_i\varphi_G \to \varphi\varphi_G$. On the other hand, due to the assumption that $\underline{\gamma}(\psi_i) \to 0$, we have

$$\mathrm{c}(G) > \underline{\gamma}(\psi_i),$$

when again i is sufficiently large. By the strong closing lemma, the composition $\psi_i\varphi_G$ has a periodic orbit passing through V. On the other hand, the fixed points of ψ_i (or equivalently the periodic points) are among the fixed points of $\psi_i\varphi$ because $\operatorname{supp} G \subset V$. It follows that

$$|\mathrm{Per}(\psi_i\varphi_G)| \geq q+1,$$

when i is large enough, which is impossible since $\psi_i\varphi_G \in \mathcal{U}$. This contradiction completes the proof of Lemma 4.4 and hence of Theorem 3.3. □

Remark 4.5. If in Theorem 3.3 we were to find a C^1-dense (and open) set of Hamiltonian diffeomorphisms rather than C^∞-dense, the argument would be considerably simpler. Namely, in this case it would be enough to first construct a map φ with just one hyperbolic periodic point. Once this is done, we could apply the results from [Hayashi 1997; Xia 1996] to create nontrivial transverse homoclinic intersections, and hence a horseshoe (see [Katok and Hasselblatt 1995]) by a C^1-small perturbation. As a consequence, the perturbed map ψ would have infinitely many hyperbolic periodic points. For any $m \in \mathbb{N}$, having at least m such points is a C^1-open property. Now we can take any $m > \dim \mathrm{H}_*(M)$.

4B. ***Sugimoto manifolds and further remarks.*** As is shown in [Sugimoto 2021], a strongly nondegenerate Hamiltonian diffeomorphism φ of a closed symplectic manifold M^{2n} has either a nonhyperbolic periodic point or infinitely many hyperbolic periodic points when M meets one of the following requirements:

(i) n is odd.

(ii) $\mathrm{H}_{\mathrm{odd}}(M) \neq 0$.

(iii) the minimal Chern number of M is greater than 1.

Below we refer to a closed symplectic manifold meeting at least one of these requirements as a *Sugimoto manifold*. For this class of manifolds Theorem 3.3 has a more direct proof and can be slightly refined. We do this in two steps.

Denote by $\mathcal{V}_m$, $m \in \mathbb{N}$, the set of Hamiltonian diffeomorphisms with at least m hyperbolic points. Note that we do not require the elements of $\mathcal{V}_m$ to be strongly nondegenerate.

Proposition 4.6. *Let M be a Sugimoto manifold. Then for any $m \in \mathbb{N}$ the set $\mathcal{V}_m$ is C^1-open and C^∞-dense in the space of all Hamiltonian diffeomorphisms.*

Proof. The statement that $\mathcal{V}_m$ is C^1-open is obvious. (It is essential here that m is finite.) To show that it is C^∞-dense we argue as in [Sugimoto 2021] and the proof of Theorem 3.3. Let φ be a Hamiltonian diffeomorphism. To prove the proposition, we need to find $\psi \in \mathcal{V}_m$ arbitrarily C^∞-close to φ. Since the set of strongly nondegenerate Hamiltonian diffeomorphisms is C^∞-dense, we can assume that φ is in this class. As shown in [Sugimoto 2021], φ has infinitely many hyperbolic periodic points or a (nondegenerate) nonhyperbolic point. In the former case, $\varphi \in \mathcal{V}_m$ for all $m \in \mathbb{N}$. In the latter case, by [Arnaud 1992, Proposition 8.2], for any $m \in \mathbb{N}$ there exists $\psi \in \mathcal{V}_m$ arbitrarily close to φ. □

As an immediate consequence, we obtain a slightly more precise variant of the main result from [Sugimoto 2021]:

Corollary 4.7. *Assume that M is a Sugimoto manifold. Then C^∞-generically a Hamiltonian diffeomorphism φ of M has infinitely many hyperbolic periodic points.*

The key difference with [Sugimoto 2021] is that the periodic points of φ here are specified to be hyperbolic. The residual set in this corollary is, of course,

$$\mathcal{V} := \bigcap_{m \in \mathbb{N}} \mathcal{V}_m.$$

We note that this set is not C^1- and even C^∞-open. However, one can require in addition φ to be strongly nondegenerate. Indeed, the set of such maps is residual and its intersection with $\mathcal{V}$ is still a residual set.

Closer to the immediate subject of the paper we have the following refinement of Theorem 3.3 and Corollary 3.4:

Corollary 4.8. *Assume that M is a Sugimoto manifold. Then $\underline{\gamma}$ is locally uniformly bounded away from zero on a C^1-open and C^∞-dense set of Hamiltonian diffeomorphisms of M.*

Here we can take any $\mathcal{V}_m$ with $m > \dim \mathrm{H}_*(M)$ as a C^1-open and C^∞-dense set, where $\underline{\gamma}$ is locally uniformly bounded away from zero. Note also that in this corollary we can again replace the spectral norm by the Hofer norm.

Remark 4.9. In contrast with Theorem 3.3, C^∞-generic existence of infinitely many periodic points is not known to hold without some additional assumptions on M. The class of Sugimoto manifolds is the broadest to date for which such existence has been proved [Sugimoto 2021]. (See also [Ginzburg and Gürel 2009b] for the original result and a different approach.)

Remark 4.10. Continuing the discussion from the introduction and Remark 1.2, we give some "textbook" examples where $\gamma(\varphi^k)$ grows linearly, and hence $\underline{\gamma}(\varphi) = \infty$, and at the same time all periodic points of φ are fixed points: $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$. Namely, let $H : M \to \mathbb{R}$ be a nonconstant autonomous Hamiltonian such that H has only finitely many critical values and all nonconstant periodic orbits of the flow of H are noncontractible. Set $\varphi = \varphi_H$. Then, as is easy to see, $\gamma(\varphi^k)$ grows linearly and the only periodic points of φ are the critical points of H. For instance, we can take $H = \sin 2\pi\theta$, where θ is the first angular coordinate θ on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Alternatively, let $(\mathbb{T}^4, \omega)$ be a Zehnder's torus, that is, a torus equipped with a sufficiently irrational translation invariant symplectic structure ω (see [Zehnder 1987]), and again let $\theta : \mathbb{T}^4 \to \mathbb{R}/\mathbb{Z}$ be a fixed angular coordinate. Then the flow of H given by the same formula has no periodic orbits at all, contractible or not, other than the critical points of H: the 3-dimensional tori $\theta = \frac{1}{2}$ and $\theta = \frac{3}{2}$. In both cases, $\gamma(\varphi^k) = 2k$. More surprisingly, there exists a Hamiltonian diffeomorphism $\varphi : S^2 \times S^2 \to S^2 \times S^2$ such that $\gamma(\varphi^k)$ grows linearly; see [Shelukhin 2022a, Remark 8] and [Polterovich and Rosen 2014, Theorem 6.2.6], although the argument is quite indirect.

In all these examples, $\dim \mathrm{H}_*(\mathrm{Fix}(\varphi)) = \dim \mathrm{H}_*(M)$ over any field, in addition to the condition that $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$. Loosely following [Atallah and Shelukhin 2023], we call such a map φ a *generalized pseudorotation*. Generalized pseudorotations from the above examples have simple dynamics. However, this is not necessarily so in general. For instance, in dimension six and higher Morse–Bott nondegenerate, generalized pseudorotations φ with positive topological entropy have been recently constructed in [Cineli 2023]. Such a generalized pseudorotation can be neither a C^0-a.i. since $\mathrm{h}_{\mathrm{top}}(\varphi) > 0$ (see [Avila et al. 2020]) nor a γ-a.i. In fact, $\gamma(\varphi^k)$ also grows linearly since M is aspherical and $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$ has finitely many connected components.

Acknowledgements

Parts of this work were carried out while Ginzburg and Gürel were visiting the IMJ-PRG, Paris, France, in May 2023 and also during the Summer 2023 events Symplectic Dynamics Workshop at INdAM, Rome, Italy, and From Smooth to C^0 Symplectic Geometry: Topological Aspects and Dynamical Implications Conference at CIRM, Luminy, France. The authors would like to thank these institutes for warm hospitality and support.

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Received October 10, 2023. Revised February 2, 2024.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 328, No. 1, 2024
https://doi.org/10.2140/pjm.2024.328.137

PEGO THEOREM ON COMPACT GROUPS

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The Pego theorem characterizes the precompact subsets of the square-integrable functions on $\mathbb{R}^n$ via the Fourier transform. We prove the analogue of the Pego theorem on (not necessarily abelian) compact groups.

1. Introduction

Characterizing precompact subsets is one of the classical topics in function space theory. It is well known that the Arzelà–Ascoli theorem characterizes a precompact subset of the space of continuous functions over a compact Hausdorff space. The celebrated Riesz–Kolmogorov theorem provides a characterization of precompact subsets of $L^p(\mathbb{R}^n)$. We refer to [8] for a historical account of it. Weil [14, page 52] extended it to the Lebesgue spaces over locally compact groups. See [7] for its extension to the Banach function spaces over locally compact groups.

In 1985, Pego [13] used the Riesz–Kolmogorov theorem to find a characterization of precompact subsets of $L^2(\mathbb{R}^n)$ via certain decay of the Fourier transform.

Theorem 1.1. [13, Theorems 2 and 3] *Let K be a bounded subset of $L^2(\mathbb{R}^n)$. Then, the following are equivalent*:

(i) *K is precompact.*

(ii) $\int_{|x|>r} |f(x)|^2\,dx \to 0$ *and* $\int_{|\xi|>r} |\hat{f}(\xi)|^2\,d\xi \to 0$ *as* $r \to \infty$, *both uniformly for f in K.*

(iii) $\int_{\mathbb{R}^n} |f(x+y) - f(x)|^2\,dx \to 0$ *as* $y \to 0$, *and* $\int_{\mathbb{R}^n} |\hat{f}(\xi+\omega) - \hat{f}(\xi)|^2\,d\xi \to 0$ *as* $\omega \to 0$, *both uniformly for f in K.*

An application of this theorem to information theory has been provided in [13].

Pego-type theorems have also been studied via the short-time Fourier and wavelet transforms [2], the Laplace transform [11] and the Laguerre and Hankel transforms [10]. The Pego theorem has been extended to the locally compact abelian groups with some technical assumptions [5]. Using the Pontryagin duality and the Arzelà–Ascoli theorem, the authors in [6] showed that the technical assumptions

MSC2020: primary 43A30, 43A77; secondary 22C05.
Keywords: compact group, Fourier transform, compactness.

are redundant. For the L^1-space analogue of the Pego theorem over locally compact abelian groups, see [12].

In Section 2, we present preliminaries on compact groups. In Section 3, using Weil's compactness theorem, we extend Theorem 1.1 to (not necessarily abelian) compact groups; see Theorem 3.4.

2. Fourier analysis on compact groups

Let G be a compact Hausdorff group. Let m_G denote the normalized positive Haar measure on G. Let $L^p(G)$ denote the p-th Lebesgue space w.r.t. the measure m_G. The norm on the space $L^p(G)$ is denoted by $\|\cdot\|_p$.

We denote by $\widehat{G}$ the space consisting of all irreducible unitary representations of G up to the unitary equivalence. The set $\widehat{G}$ is known as the unitary dual of G and is equipped with the discrete topology. Note that the representation space $\mathcal{H}_\pi$ of $\pi \in \widehat{G}$ is a complex Hilbert space and finite-dimensional. Denote by d_π the dimension of $\mathcal{H}_\pi$.

Let $\Lambda \subset \widehat{G}$. Assume that $\{(X_\pi, \|\cdot\|_\pi) : \pi \in \Lambda\}$ is a family of Banach spaces. For $1 \le p < \infty$, we denote by $\ell^p\text{-}\bigoplus_{\pi\in\wedge} X_\pi$ the Banach space

$$\Big\{(x_\pi) \in \prod_{\pi\in\wedge} X_\pi : \sum_{\pi\in\wedge} d_\pi \|x_\pi\|_\pi^p < \infty\Big\}$$

endowed with the norm $\|(x_\pi)\|_{\ell^p\text{-}\bigoplus_{\pi\in\wedge} X_\pi} := \big(\sum_{\pi\in\wedge} d_\pi \|x_\pi\|_\pi^p\big)^{1/p}$. Denote by $\ell^\infty\text{-}\bigoplus_{\pi\in\wedge} X_\pi$ the Banach space

$$\big\{(x_\pi) \in \prod_{\pi\in\wedge} X_\pi : \sup_{\pi\in\wedge} \|x_\pi\|_\pi < \infty\big\}$$

endowed with the norm $\|(x_\pi)\|_{\ell^\infty\text{-}\bigoplus_{\pi\in\wedge} X_\pi} := \sup_{\pi\in\wedge} \|x_\pi\|_\pi$. Similarly, denote by $c_0\text{-}\bigoplus_{\pi\in\wedge} X_\pi$ the space consisting of (x_π) from $\ell^\infty\text{-}\bigoplus_{\pi\in\wedge} X_\pi$ such that $x_\pi \to 0$ as $\pi \to \infty$, i.e., for any given $\epsilon > 0$ there exists a finite set $\Lambda_\epsilon \subset \Lambda$ such that $\|x_\pi\|_\pi < \epsilon$ for all $\pi \in \Lambda \setminus \Lambda_\epsilon$. Note that $c_0\text{-}\bigoplus_{\pi\in\wedge} X_\pi$ is a closed subspace of $\ell^\infty\text{-}\bigoplus_{\pi\in\wedge} X_\pi$.

For $1 \le p < \infty$, let $\mathcal{B}_p(\mathcal{H}_\pi)$ denote the space of all bounded linear operators T on $\mathcal{H}_\pi$ such that $\|T\|_{\mathcal{B}_p(\mathcal{H}_\pi)} := (\mathrm{tr}(|T|^p))^{1/p} < \infty$. The space $\mathcal{B}_2(\mathcal{H}_\pi)$ is called the space of the Hilbert–Schmidt operators on the Hilbert space $\mathcal{H}_\pi$. The space $\mathcal{B}_2(\mathcal{H}_\pi)$ is a Hilbert space endowed with the inner product

$$\langle T, S\rangle_{\mathcal{B}_2(\mathcal{H}_\pi)} := \mathrm{tr}(TS^*).$$

Let $\mathcal{B}(\mathcal{H}_\pi)$ denote the space consisting of all bounded linear operators on $\mathcal{H}_\pi$ endowed with the operator norm.

Let $f \in L^1(G)$. The Fourier transform of f is defined by

$$\hat{f}(\pi) = \int_G f(t)\pi(t)^* \, dm_G(t), \quad \pi \in \widehat{G}.$$

The Fourier transform operator $f \mapsto \hat{f}$ from $L^1(G)$ into $\ell^\infty\text{-}\bigoplus_{\pi\in\widehat{G}} \mathcal{B}(\mathcal{H}_\pi)$ is injective and bounded. By the Riemann–Lebesgue lemma, we know that $\hat{f} \in c_0\text{-}\bigoplus_{\pi\in\widehat{G}} \mathcal{B}(\mathcal{H}_\pi)$. The convolution of $f, g \in L^1(G)$ is given by

$$f * g(x) = \int_G f(xy^{-1})g(y)\, dm_G(y).$$

Then, $\widehat{f*g}(\pi) = \hat{g}(\pi)\hat{f}(\pi)$, $\pi \in \widehat{G}$. For $y \in G$, the right translation R_y of $f \in L^p(G)$ is given by $R_y(f)(x) = f(xy)$, $x \in G$. Then, $\widehat{R_y f}(\pi) = \pi(y)\hat{f}(\pi)$, $\pi \in \widehat{G}$.

For more information on compact groups, we refer to [4; 9].

Throughout the paper, G will denote a (not necessarily abelian) compact Hausdorff group. The identity of G is denoted by e. We will denote by I_{d_π} the $d_\pi \times d_\pi$ identity matrix.

3. Pego theorem on compact groups

We discuss the characterization of precompact subsets of square-integrable functions on G in terms of the Fourier transform. We need the following definitions.

Let $K \subset L^p(G)$. Define $\widehat{K} := \{\hat{f} : f \in L^p(G)\}$. K is said to be *uniformly $L^p(G)$-equicontinuous* if for any given $\epsilon > 0$ there exists an open neighborhood O of e such that

$$\|R_y f - f\|_p < \epsilon, \quad f \in K \text{ and } y \in O.$$

Let $F \subset \ell^p\text{-}\bigoplus_{\pi\in\widehat{G}} \mathcal{B}_p(\mathcal{H}_\pi)$. F is said to have *uniform $\ell^p\text{-}\bigoplus_{\pi\in\widehat{G}} \mathcal{B}_p(\mathcal{H}_\pi)$-decay* if for any given $\epsilon > 0$ there exists a finite set $A \subset \widehat{G}$ such that

$$\|\phi\|_{\ell^p\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A} \mathcal{B}_p(\mathcal{H}_\pi)} < \epsilon, \quad \phi \in F.$$

Let us begin with some important lemmas.

Lemma 3.1. *Let $K \subset L^p(G)$, where $p \in [1, 2]$. If K is uniformly $L^p(G)$-equicontinuous then $\widehat{K}$ has uniform $\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}} \mathcal{B}_{p'}(\mathcal{H}_\pi)$-decay.*

Proof. Let $(e_U)_{U\in\Lambda}$ be a Dirac net on G; see [1, page 24]. By the Riemann–Lebesgue lemma [9, Theorem 28.40], $\widehat{e_U} \in c_0\text{-}\bigoplus_{\pi\in\widehat{G}} \mathcal{B}(\mathcal{H}_\pi)$. Then, there exists a finite set $A \subset \widehat{G}$ such that

$$\|\widehat{e_U}(\pi)\|_{\mathcal{B}(\mathcal{H}_\pi)} \le \tfrac{1}{2}, \quad \pi \in \widehat{G} \setminus A.$$

Let $f \in K$. We denote by $\widehat{e_U}\hat{f}$ the pointwise product of $\widehat{e_U}$ and $\hat{f}$. Now,

$$\begin{aligned}
&\|\hat{f}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)} \\
&\quad\le \|\hat{f}-\widehat{e_U}\hat{f}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)} + \|\widehat{e_U}\hat{f}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)} \\
&\quad\le \|\widehat{f-f*e_U}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)} + \|\hat{f}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)} \sup_{\pi\in\widehat{G}\setminus A}\|\widehat{e_U}(\pi)\|_{\mathcal{B}(\mathcal{H}_\pi)} \\
&\quad\le \|\widehat{f-f*e_U}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)} + \tfrac{1}{2}\|\hat{f}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)}.
\end{aligned}$$

Then, applying the Hausdorff–Young inequality [9, Theorem 31.22], we get

$$\begin{aligned}
\|\hat{f}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)} &\le 2\,\|\widehat{f-f*e_U}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}}\mathcal{B}_{p'}(\mathcal{H}_\pi)} \\
&\le 2\,\|f-f*e_U\|_p \\
&= 2\Big(\int_G |f(x)-f*e_U(x)|^p\,dm_G(x)\Big)^{1/p} \\
&= 2\Big(\int_G \Big|\int_G (f(x)-f(xy^{-1}))e_U(y)\,dm_G(y)\Big|^p\,dm_G(x)\Big)^{1/p}.
\end{aligned}$$

Therefore, using the Minkowski integral inequality, we obtain

$$\begin{aligned}
\|\hat{f}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)} &\le 2\int_G\Big(\int_G |f(x)-f(xy^{-1})|^p\,dm_G(x)\Big)^{1/p} e_U(y)\,dm_G(y) \\
&\le 2\sup_{y\in U}\Big(\int_G |f(x)-f(xy^{-1})|^p\,dm_G(x)\Big)^{1/p}.
\end{aligned}$$

Let $\epsilon > 0$. Since K is uniformly $L^p(G)$-equicontinuous, there exists an open neighborhood O of e such that

$$\|R_y f - f\|_p < \frac{\epsilon}{2}, \quad f \in K \text{ and } y \in O.$$

By [1, Lemma 1.6.5, page 24], we get that there exists $U \in \Lambda$ such that

$$\|R_y f - f\|_p < \frac{\epsilon}{2}, \quad f \in K \text{ and } y \in U.$$

Hence,

$$\|\hat{f}\|_{\ell^{p'}\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_{p'}(\mathcal{H}_\pi)} < \epsilon, \quad f \in K. \qquad \square$$

Lemma 3.2. *Let K be a subset of $L^{p'}(G)$, where $p \in [1,2]$. If $\widehat{K}$ has uniform $\ell^p\text{-}\bigoplus_{\pi\in\widehat{G}}\mathcal{B}_p(\mathcal{H}_\pi)$-decay then K is uniformly $L^{p'}(G)$-equicontinuous.*

Proof. Let $\epsilon > 0$. Since $\widehat{K}$ has uniform $\ell^p\text{-}\bigoplus_{\pi\in\widehat{G}}\mathcal{B}_p(\mathcal{H}_\pi)$-decay, there exists a finite set $A \subset \widehat{G}$ such that

$$\|\hat{f}\|_{\ell^p\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_p(\mathcal{H}_\pi)} < \frac{\epsilon}{4}, \quad f \in K.$$

Let $f \in K$ and $y \in G$. Then, applying [9, Corollary 31.25], we obtain

$$\begin{aligned}
\|R_y f - f\|_{p'} &\le \|\widehat{R_y f - f}\|_{\ell^p\text{-}\bigoplus_{\pi\in\widehat{G}}\mathcal{B}_p(\mathcal{H}_\pi)} \\
&= \Big(\sum_{\pi\in\widehat{G}} d_\pi \|\widehat{R_y f}(\pi) - \hat{f}(\pi)\|^p_{\mathcal{B}_p(\mathcal{H}_\pi)}\Big)^{1/p} \\
&\le \Big(\sum_{\pi\in A} d_\pi \|\pi(y)\hat{f}(\pi) - \hat{f}(\pi)\|^p_{\mathcal{B}_p(\mathcal{H}_\pi)}\Big)^{1/p} \\
&\qquad + \Big(\sum_{\pi\in\widehat{G}\setminus A} d_\pi \|\pi(y)\hat{f}(\pi) - \hat{f}(\pi)\|^p_{\mathcal{B}_p(\mathcal{H}_\pi)}\Big)^{1/p} \\
&\le \sup_{\pi\in A} \|\pi(y) - I_{d_\pi}\|_{\mathcal{B}(\mathcal{H}_\pi)} \Big(\sum_{\pi\in A} d_\pi \|\hat{f}(\pi)\|^p_{\mathcal{B}_p(\mathcal{H}_\pi)}\Big)^{1/p} \\
&\qquad + \sup_{\pi\in\widehat{G}\setminus A} \|\pi(y) - I_{d_\pi}\|_{\mathcal{B}(\mathcal{H}_\pi)} \Big(\sum_{\pi\in\widehat{G}\setminus A} d_\pi \|\hat{f}(\pi)\|^p_{\mathcal{B}_p(\mathcal{H}_\pi)}\Big)^{1/p} \\
&\le M \sup_{\pi\in A} \|\pi(y) - I_{d_\pi}\|_{\mathcal{B}(\mathcal{H}_\pi)} + \frac{\epsilon}{2},
\end{aligned}$$

where M is a positive number such that $\big(\sum_{\pi\in A} d_\pi \|\hat{f}(\pi)\|^p_{\mathcal{B}_p(\mathcal{H}_\pi)}\big)^{1/p} \le M$.

Let $\pi \in A$. Using continuity of π, we obtain that there exists a neighborhood O_π of e such that

$$\|\pi(y) - I_{d_\pi}\|_{\mathcal{B}(\mathcal{H}_\pi)} < \frac{\epsilon}{2M}, \quad y \in O_\pi.$$

Assume that $O = \bigcap_{\pi\in A} O_\pi$. Then,

$$\|\pi(y) - I_{d_\pi}\|_{\mathcal{B}(\mathcal{H}_\pi)} < \frac{\epsilon}{2M}, \quad \pi \in A \text{ and } y \in O.$$

Hence,

$$\|R_y f - f\|_{p'} < \epsilon, \quad f \in K \text{ and } y \in O.$$

□

The following corollary is a generalization of [13, Theorem 1] studied on $\mathbb{R}^n$, and [5, Theorem 1] and [3, Lemma 2.5] studied on locally compact abelian groups. This is also an improvement of the corresponding result on compact abelian groups in the sense that we do not assume boundedness of the subset of $L^2(G)$.

Corollary 3.3. *Let $K \subset L^2(G)$. Then, K is uniformly $L^2(G)$-equicontinuous if and only if $\widehat{K}$ has uniform ℓ^2-$\bigoplus_{\pi\in\widehat{G}}\mathcal{B}_2(\mathcal{H}_\pi)$-decay.*

Proof. This is a direct consequence of Lemmas 3.1 and 3.2. □

Now, we present our main result, that is, the Pego theorem over compact groups. It is a consequence of the Weil theorem and above corollary.

Theorem 3.4. *Let K be a bounded subset of $L^2(G)$. Then, the following are equivalent:*

(i) *K is precompact.*

(ii) *K is uniformly $L^2(G)$-equicontinuous.*

(iii) *$\widehat{K}$ has uniform ℓ^2-$\bigoplus_{\pi\in\widehat{G}}\mathcal{B}_2(\mathcal{H}_\pi)$-decay.*

Proof. For any given $\epsilon > 0$ we have that

$$\sup_{f\in K}\|f\chi_{G\setminus G}\|_2 = 0 < \epsilon.$$

Therefore, (i) and (ii) are equivalent by the Weil theorem [14, page 52] (or see [7, Theorems 3.1 and 3.3]). Further, (ii) and (iii) are equivalent by Corollary 3.3. □

The following gives an example of a set $K \subset L^2(G)$ which is not precompact but K is uniformly $L^2(G)$-equicontinuous and $\widehat{K}$ has uniform ℓ^2-$\bigoplus_{\pi\in\widehat{G}}\mathcal{B}_2(\mathcal{H}_\pi)$-decay.

Example 3.5. *Consider the set $K = \{n\chi_G : n \in \mathbb{N}\} \subset L^2(G)$ as given in* [7, Example 4.2]. *Since K consists of only constant functions, it is clear that K is uniformly $L^2(G)$-equicontinuous. By Corollary 3.3, $\widehat{K}$ has uniform ℓ^2-$\bigoplus_{\pi\in\widehat{G}}\mathcal{B}_2(\mathcal{H}_\pi)$-decay. Since K is not bounded, K is not precompact.*

Now, with the help of our main result Theorem 3.4, we show that certain subsets of $L^2(G)$ are precompact.

Example 3.6. (i) *Let $r \in \mathbb{R}$. Consider the set $K = \left\{\frac{r}{n}\chi_G : n \in \mathbb{N}\right\} \subset L^2(G)$. Since $\left\{\frac{r}{n} : n \in \mathbb{N}\right\}$ is bounded and K consists of only constant functions, it follows that K is bounded and uniformly $L^2(G)$-equicontinuous. Therefore, by Theorem 3.4, K is precompact.*

(ii) *Let A be a finite subset of $\widehat{G}$. Assume that K is a bounded subset of the linear span of the set consisting of matrix entries* [4, page 139] *of elements in A. Note that the matrix entries are bounded functions. For $f \in K$, using the Schur orthogonality relations* [4, Theorem 5.8] *we obtain that*

$$\|\hat{f}\|_{\ell^2\text{-}\bigoplus_{\pi\in\widehat{G}\setminus A}\mathcal{B}_2(\mathcal{H}_\pi)} = 0.$$

Thus, $\widehat{K}$ has uniform ℓ^2-$\bigoplus_{\pi\in\widehat{G}}\mathcal{B}_2(\mathcal{H}_\pi)$-decay. Hence, by Theorem 3.4, K is precompact. In particular, the convex hull of the set consisting of matrix entries of elements in A is precompact.

Acknowledgements

The author is supported by the NBHM postdoctoral fellowship with reference number 0204/3/2021/R&D-II/7356 from the Department of Atomic Energy (DAE), Government of India. The author is grateful to Sundaram Thangavelu for his useful comments.

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Received February 21, 2023. Revised December 14, 2023.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 328, No. 1, 2024
https://doi.org/10.2140/pjm.2024.328.145

MAXIMAL DEGREE OF A MAP OF SURFACES

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Given closed possibly nonorientable surfaces M, N, we prove that if a map $f : M \to N$ has geometric degree $d > 0$, then $\chi(M) \le d \cdot \chi(N)$. We give all necessary comments on the definition and properties of geometric degree, which can be defined for any map. Our proof is based on the factorization theorem of Edmonds, a simple natural proof of which is also presented.

1. Introduction

Through this paper, we set M, N to be closed connected surfaces, possibly nonorientable.

Given a map $f : M \to N$, we define its *geometric degree* $\operatorname{Deg} f$ as a minimal cardinality of the preimage of a regular value among all smooth maps in the homotopy class of f. Note that if M and N are orientable, then this number coincides with the usual notion of degree; we discuss features of the definition and prove some of its properties in Section 2.2.

Our goal is to prove the following fact:

Theorem 1. *Let $f : M \to N$ be a map of geometric degree $d > 0$. Then $\chi(M) \le d \cdot \chi(N)$.*

This fact is well known; apparently it was first proved by Kneser [1930] in the case of orientable surfaces. It looks very similar to the assertions "*all maps $S^2 \to N$ are nullhomotopic*" and "*if $\chi(M) > \chi(N)$, then $d = 0$*". These statements can be easily proved using universal covering or the intersection form in cohomology. However, similar elementary approaches to Theorem 1 are not known to the author.

The possible ways to prove Theorem 1 are rather a bit more technical. For orientable surfaces, one can use the Milnor–Wood inequality [1971, Theorem 1.1] or the Gromov norm [1999, §5.35]. In this paper, we present the most elementary proof including the factorization theorem of Edmonds. First, let us recall the notation.

Suppose we have a 2-submanifold with boundary $K \subset M$, such that for every component $K_i \subset K$ its boundary ∂K_i is connected. Collapsing each K_i to a single

MSC2020: primary 57K20; secondary 55M25.
Keywords: surfaces, geometric degree, branched covering.

point (see, e.g., [Whitehead 1978, p. 23]) we obtain a smooth manifold Q. The factorization map $p : M \to Q$ is called *a pinch map*.

A smooth map of closed surfaces $q : Q \to N$ is called *a branched covering* if it is a local diffeomorphism outside of a finite subset $P \subset M$ and near every $x_0 \in P$ one can write q as $z \mapsto z^k$, $k > 1$, in some local complex coordinates. The number k is called *the index of the branch point* x_0.

Theorem 2. *Every map $f : M \to N$ is either homotopic to the composition of a pinch map $p : M \to Q$ with a branched covering $q : Q \to N$ for some closed surface Q, or homotopic to a map whose image is a graph embedded into N.*

This theorem is due to Edmonds [1979]. He proved the theorem for surfaces with boundary as well, considering maps f whose restriction to the boundary is a $(\operatorname{Deg} f)$-sheeted covering. The proof of Edmonds was corrected and improved by Skora [1987].

We present another proof of Theorem 2 which is simpler and more natural in some ways. Namely, it does not use induction and constructs the factorization in one step. The idea is to take a triangulation of N and consider a map h homotopic to f which is transversal to $\operatorname{sk}^1(N)$ and has a minimal number of edges of $h^{-1}(\operatorname{sk}^1(N))$, and then to deform h over each triangle. This approach was inspired by Lurie's proof of the Dehn–Nielsen theorem [2009, Lecture 38]; see also [Farb and Margalit 2012, §8.3.1].

Unlike [Edmonds 1979] and [Skora 1987], in our proof we do not control the degree of a branched covering. Also, our approach does not deal with surfaces with boundary, the corresponding generalization is possible but it would require some additional work. Finally, we do not use the theory of absolute degree — see Section 2.2 for some remarks on this subject — so we tried to make our reasoning completely self-contained.

2. Preliminaries

2.1. ***Conventions and notation on surfaces and transversality.*** We use the term *surface* for a 2-manifold, and *closed* with respect to a manifold means that it is compact without boundary. All manifolds will be assumed to be infinitely smooth, as well as maps. The maps we construct sometimes will be not smooth and should be smoothed if needed, but this inaccuracy will not cause difficulties; see, e.g., [Hirsch 1976, Chapter 8].

Every open subset of a surface $U \subset M$ that we take is supposed to be "sufficiently nice" — namely, it should be an interior of a compact 2-submanifold with boundary. We refer to this boundary as ∂U (the notation $\partial \overline{U}$ here would be formally correct, but it is more cumbersome).

When we *cut* a closed surface M *along a closed curve* $C \subset M$, we assume to obtain as a result a compact surface with boundary M', and each point of C will double in M'.

Given surfaces M, N, we say that a map $f : M \to N$ is *transversal* to a stratified subset $\mathcal{S} \subset N$, if f is transversal to its every stratum. Namely, every vertex $y \in \mathcal{S}$ must be a regular value of f and for every edge $C \subset \mathcal{S}$ and any $x \in f^{-1}(C)$ there must be a vector $v \in T_x M$ such that $df(v) \notin T_{f(x)}C$. By the implicit function theorem, $f^{-1}(\mathcal{S}) \subset M$ is a stratified subset. If $\mathcal{S}$ is closed, then the set of maps transversal to $\mathcal{S}$ is a dense and open set in $C^\infty(M, N)$. See, for example, [Goresky and MacPherson 1988, Part 1, §1] for details.

We say that a loop $\varphi : S^1 \to S^1$ has *index* $i \geq 0$, if $[\varphi] = \pm i \in \pi_1(S^1)$.

2.2. *Geometric degree.* Recall, we define a *degree* of a map $f : M \to N$ as a minimal $d \in \mathbb{Z}_{\geq 0}$ such that there is a smooth map $h : M \to N$ homotopic to f and there is a regular value $y \in N$ such that $|h^{-1}(y)| = d$. It is known as *geometric degree*, but in later sections, we will simply call it a *degree* and denote it as $\operatorname{Deg} f$.

The degree theory began with the work of Hopf [1928; 1930] and was developed by Olum [1953] and Epstein [1966]. The most important properties of geometric degree in dimension 2 were proved by Kneser [1928; 1930].

Here we state and sketch the proofs of a few properties of degree. We consider not famous properties, but only those that will be used in Section 4 in order to prove Theorem 1. For a more detailed review, see, e.g., [Brown and Schirmer 2001] or [Sklyarenko 2008] in addition to [Epstein 1966] and [Olum 1953]. As usual, we suppose M, N to be closed connected surfaces, but one can similarly formulate and prove corresponding statements for any pair of closed manifolds of the same dimension.

2.2.1. *Degree of a branched covering.* Suppose a map $f : M \to N$ is *orientation-true*. This means that it takes orientation-preserving/reversing loops in M to orientation-preserving/reversing loops in N, respectively. Clearly, this is equivalent to the equality $f^*(w_1(N)) = w_1(M)$ for Stiefel–Whitney classes, or to the fact $f^*(\mathbb{Z}_N) \simeq \mathbb{Z}_M$, where $\mathbb{Z}_M$ denotes the orientation local system of M with fiber $\mathbb{Z}$. Then f induces a homomorphism $H^2(N; \mathbb{Z}_N) \to H^2(M; \mathbb{Z}_M)$. Here both groups are isomorphic to $\mathbb{Z}$, so the homomorphism $\mathbb{Z} \to \mathbb{Z}$ is a multiplication by an integer called the *cohomological degree* of f, denoted by $\deg f$.

Note that for non-orientation-true maps one can similarly define a cohomological degree as a residue (mod 2), but not as an integer because of $H^2(M; f^*(\mathbb{Z}_N)) \simeq \mathbb{Z}_2$. For more details on local systems, see, for example, [Spanier 1993] or [Whitehead 1978, Chapter VI].

Proposition 3. *If the map* $f : M \to N$ *is orientation-true*, *then* $\operatorname{Deg} f \geq \deg f$.

Proof. Homotope f so that a regular value $y \in N$ has $\operatorname{Deg} f$ preimages. Choose a local orientation of M and N. That allows us to define *the sign* for every preimage $x_i \in f^{-1}(y)$, so that the sum equals $\deg f$. (Formally, here we use local cohomology; see, e.g., [Hirsch 1976, Chapter 5]). □

In fact, the opposite inequation also holds, so we have $\operatorname{Deg} f = \deg f$ for an orientation-true f. The idea of a proof is to cancel a pair of preimages of y which have different signs (see, e.g., [Epstein 1966, p. 380] for dimension > 2), but for surfaces, this strategy is quite a bit more complicated (see, e.g., [Melikhov 2004, Lemma 2]). Another strategy is to apply Theorem 2 (our proof uses Proposition 3, but not the equality $\deg f = \operatorname{Deg} f$ for an orientation-true f).

Corollary 4. *For a k-sheeted branched covering $f : M \to N$ we have* $\operatorname{Deg} f = k$.

Proof. Clearly, f is orientation-true and $\deg f = k$, so by Proposition 3 we have $\operatorname{Deg} f \geq k$. On the other hand, the regular values of f have k preimages, so $\operatorname{Deg} f \leq k$. □

Corollary 5. *If $p : M \to Q$ is an orientation-true pinch map and $q : Q \to N$ is a k-sheeted branched covering, then we have* $\operatorname{Deg}(q \circ p) = k$.

Proof. The reasoning is the same: use the functoriality of cohomology and Proposition 3. □

2.2.2. *Degree of the composition of maps.*

Proposition 6. *Take any map $f : M \to N$. Let $g : N' \to N$ be a k-sheeted covering such that N' is connected and f lifts to N', i.e., there is $f' : M \to N'$ such that $f = g \circ f'$. Then* $\operatorname{Deg} f = k \cdot \operatorname{Deg} f'$.

Proof. Homotope f so that the regular value $y \in N$ has $\operatorname{Deg} f$ preimages. This homotopy can be lifted to a homotopy of f'. So we obtain that every $y' \in g^{-1}(y)$ is a regular value of f', and this immediately implies that $\operatorname{Deg} f \geq k \cdot \operatorname{Deg} f'$.

The opposite inequality is true since for any finite subset $P \subset N'$ we can homotope f' so that each point of P becomes a regular value with $\operatorname{Deg} f'$ preimages. □

In fact, the last argument shows that $\operatorname{Deg}(f_2 \circ f_1) \leq \operatorname{Deg} f_1 \cdot \operatorname{Deg} f_2$ for any maps f_1, f_2. The opposite inequality in Proposition 6 is nontrivial (and does not hold if g is a branched covering) because of the following pathology.

Remark 7. If $f_1 : S^2 \to \mathbb{RP}^2$ is the universal covering, then $\operatorname{Deg} f_1 = 2$ by Corollary 4. If $f_2 : \mathbb{RP}^2 \to S^2$ is a map collapsing a projective line $l \subset \mathbb{RP}^2$ to a point, then $\operatorname{Deg} f_2 = 1$ since it is nonnullhomotopic (the homomorphism $f_2^* H^2(S^2; \mathbb{Z}_2) \to H^2(\mathbb{RP}^2; \mathbb{Z}_2)$ is nonzero).

However, the composition $f_2 \circ f_1$ is nullhomotopic: it takes both hemispheres of the domain S^2 surjectively to the range S^2, but with different orientations. So $\operatorname{Deg}(f_2 \circ f_1) = 0$.

Remark 8. If $f_2 : \mathbb{RP}^2 \to S^2$ is as above, we have $\operatorname{Deg} f_2 = 1$. If $f_3 : S^2 \to S^2$ is any map of degree 7 as an element of $\pi_2(S^2)$, then $\operatorname{Deg} f_3 = 7$ according to Corollary 4. However, $\operatorname{Deg}(f_3 \circ f_2) = 1$, and $f_3 \circ f_2 \sim f_2$.

Indeed, as one can see, there are exactly two homotopy classes of maps $\mathbb{RP}^2 \to S^2$, since the obstruction for such maps to be homotopic lies in $H^2(\mathbb{RP}^2; \pi_2(S^2)) \simeq \mathbb{Z}_2$ (see, e.g., [Whitehead 1978, Chapter VI, §6]). One also can directly construct a homotopy of $f_3 \circ f_2$ to f_2. So we have another example when $\operatorname{Deg}(f_3 \circ f_2) \neq \operatorname{Deg} f_3 \cdot \operatorname{Deg} f_2$. In particular, we note that here f_2 is homotopic to a pinch map, and as f_3 we can take a branched covering.

3. The factorization theorem

3.1. *A map with a minimal graph.*

Proof of Theorem 2. Take some triangulation of N and denote its 1-skeleton by $\mathcal{T} \subset N$. Consider maps $h : M \to N$ homotopic to f which are transversal to $\mathcal{T}$. Then $h^{-1}(\mathcal{T})$ is an embedded graph in M, call it Γ, possibly with *isolated circles* whose images do not cover the vertices of $\mathcal{T}$.

We take h such that Γ has a minimal number of edges $E(\Gamma)$. Each isolated circle is counted as one edge. Then we observe that Γ has the following three properties which we will prove in Section 3.2.

Claim 9. For every edge of Γ the images of its endpoints do not coincide.

For every component $A \subset M \setminus \Gamma$ its image $h(A)$ is contained in a certain triangle $B \subset N$ and $h(\partial A) \subset \partial B$. By Claim 9, for every component $\alpha \subset \partial A$ we may assume that either α is an isolated circle, or α is mapped to ∂B monotonously with index $i_\alpha \neq 0$. In the last case, we call α *essential*.

Claim 10. Either Γ has no isolated circles, or Γ is a union of such circles and has no vertices.

In the case when Γ is a union of isolated circles, the image $h(M)$ is contained in $N \setminus \operatorname{sk}^0(\mathcal{T})$. Note that $N \setminus \operatorname{sk}^0(\mathcal{T})$ can be deformation retracted onto the dual graph of $\mathcal{T}$. This proves the theorem in that case.

Further we will assume that Γ has no isolated circles. Take a triangle $B \subset N$ and a component $A \subset h^{-1}(B)$. Orient ∂B ant ∂A so that $h|_{\partial A}$ preserves the orientation.

Claim 11. The boundary ∂A consists of one component with index 1, or A is orientable and all the components of ∂A have the same orientation with respect to A.

If A is nonorientable, then $h|_A$ is homotopic to a pinch map so that the homotopy is stationary in a neighborhood of ∂A. Indeed, in that case by Claim 11 the boundary ∂A is a circle with index 1. Since h takes ∂A to ∂B monotonously, the restriction of h to a small neighborhood $W \supset \partial A$ is injective. Then $h|_A$ is homotopic

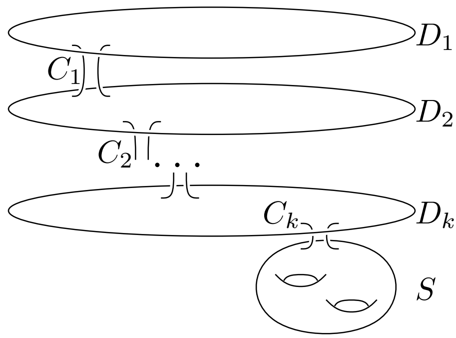

Figure 1. Presentation of A as the connected sum.

to the composition of the collapse of $A \setminus W$ to a point and a homeomorphism $A/(A \setminus W) \to B$ because of contractibility of B.

Otherwise, orient A. We can present A as a connected sum of a certain number of disks $D_1, \dots, D_k$, and a closed surface S (which is possibly a sphere). We may assume that they are joined by cylinders $C_1, \dots, C_k$ in that order; see Figure 1.

Homotope h so that each C_j is mapped to a single point as well as S. Then on every D_j we can homotope h to a branched covering with one critical point of index $i_{\partial D_j}$ or to a diffeomorphism if $i_{\partial D_j} = 1$. This is possible because the restriction $h|_{\partial D_j} : \partial D_j \to \partial B$ is a $i_{\partial D_j}$-sheeted covering. The homotopy is assumed to be stationary in a neighborhood of ∂A and the resulting h is assumed to preserve the orientation. Finally, homotope h near $C_1, \dots, C_{k-1}$ to a branched covering with two branched points of index 2 (see, for instance, [Gabai and Kazez 1987, Figure 2.1]). The subsurface $C_k \cup S \subset A$ remains pinched.

Repeat this for all components $A \subset M \setminus \Gamma$, and the proof of Theorem 2 is complete. □

3.2. *Properties of the minimal graph.*

Proof of Claim 9. Recall that h takes vertices of Γ to vertices of $\mathcal{T}$ and takes interiors of edges of Γ to interiors of edges of $\mathcal{T}$. By transversality of h to the vertices of $\mathcal{T}$, the half-edges of any vertex $v \in \Gamma$ are in bijective correspondence with the half-edges of $h(v) \in \mathcal{T}$. Therefore, since $\mathcal{T}$ has no loops, Γ has no loops either.

Take an edge $e \subset \Gamma$ with endpoints v and w. Suppose $h(v) = h(w)$. Let $e' \subset \mathcal{T}$ be a (closed) edge that contains $h(e)$. Take a small tubular neighborhood $U \supset e'$ and a tubular neighborhood $V \supset e$ such that $h(V) \subset U$ and $h(\partial V) \subset \partial U$.

Since $h(v) = h(w)$, the image $h(V)$ does not cover the whole of e'. Then we can homotope h on V and squeeze $h(V)$ outside U, so that $h(V) \subset \partial U$. When we

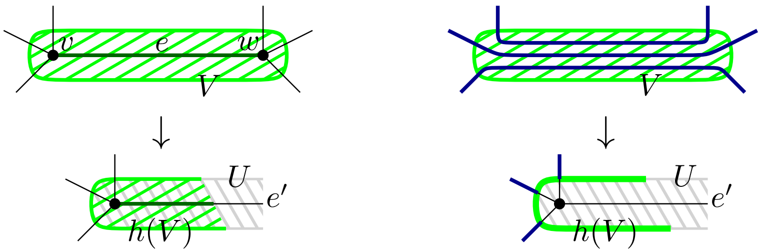

Figure 2. Collapsing of an edge with the same images of the endpoints.

redefine Γ as the preimage of $\mathcal{T}$ under new h, we will see that the edge e disappears and other edges incident on v and w are modified as in Figure 2.

Thus, $E(\Gamma)$ decreased at least by 1, which contradicts the minimality. $\square$

Proof of Claim 10. Suppose Γ has isolated circles as well as vertices. Then find a component $A \subset M \setminus \Gamma$ whose boundary includes at least one isolated circle α_0 and at least one essential component α_1 (to do this, one can consider the dual graph of Γ, mark blue the edges dual to isolated circles, mark red the edges dual to essential curves, and then find a vertex incident to edges with different colors).

Take a triangle $B \subset N$ such that $h(A) \subset B$. Take points $x_0 \in \alpha_0$ and $x_1 \in \alpha_1$ such that $h(x_0) = h(x_1) = y \in \partial B$. Take a non-self-intersecting path γ in A from x_0 to x_1. Then $h \circ \gamma$ is a loop inside the triangle B with basepoint $y \in \partial B$.

Similarly to the proof of Claim 9, we can homotope h to compress $h(\gamma)$ to y and then to squeeze $h(\gamma)$ outside B, so that the homotopy is stationary outside a small neighborhood of γ.

As a result, when we redefine Γ as the preimage of $\mathcal{T}$ under new h, the edges α_0 and α_1 will join together in one curve; see Figure 3. Thus $E(\Gamma)$ decreased by 1, which contradicts the minimality. $\square$

Proof of Claim 11. Suppose the hypothesis of the claim is violated. Then one can find two points $x_1, x_2 \in \partial A$ such that $h(x_1) = h(x_2) = y \in N$ and a non-self-intersecting curve γ in A from x_1 to x_2 which admits a coorientation agreed with the orientation of ∂A at x_1 and x_2. (Indeed, if A is nonorientable, we can take any

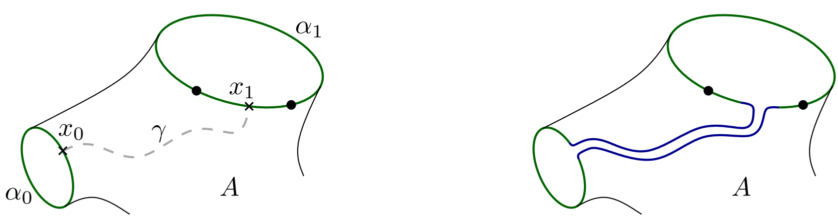

Figure 3. Join of an isolated circle with an essential curve.

x_1, x_2 with the same image and then choose γ properly. And if A is orientable, we take x_1 and x_2 on the components of ∂A with different orientations with respect to A and take any γ.)

Then we can homotope h in a small neighborhood of γ similarly to the proof of Claim 10. Suppose x_1 belongs to the oriented edge $v_1 w_1$ of Γ, and x_2 belongs to the oriented edge $v_2 w_2$. Then after the homotopy Γ modifies as in Figure 3 so that these edges will be replaced by the edges $v_1 v_2$ and $w_1 w_2$.

Note that the homotopy will not change $E(\Gamma)$. But also note that $h(v_1) = h(v_2)$ and $h(w_1) = h(w_2)$, which contradicts Claim 9. □

Remark 12. Clearly, if f is the composition of a pinch map with a branched covering, then $\operatorname{Deg} f \neq 0$. In the case $\operatorname{Deg} f = 0$ one can strengthen Theorem 2 as follows: *f is homotopic to the composition of a retraction of M to a graph $\Gamma' \subset M$ with a projection $\Gamma' \to N$*. Note that as Γ' we can take the dual graph of Γ above, but it may be not isomorphic to the dual graph of $\mathcal{T}$.

4. Estimation of the degree

Note that in Theorem 2 the pinched subsurface of M may be assumed to be connected (or empty). Also, in order to prove Theorem 1, we note the following remark.

Proposition 13. *The resulting map in our proof of Theorem 2 cannot both pinch a nonorientable subsurface of M and have a branch point.*

Note that this assertion refers to the decomposition obtained just in our proof of Theorem 2. Of course, one can compose a pinch of a crosscap with a branched covering (e.g., as in Remark 8), but this is not our case.

Proof. Suppose that the obtained pinch map $p : M \to Q$ collapses a Möbius band $L \subset M$ (and, possibly, some other subsurface). Note that we can "move" L across M. Namely, we can replace L by a point and for any component $A \subset M \setminus \Gamma$ glue a Möbius band instead of any point inside A defining p on it as a collapse to a single point.

For such an A we take a component of $M \setminus \Gamma$ with a disconnected boundary or a component whose boundary has index > 1. Otherwise, if there is no such component, the resulting map will have no branched points, as we can see from the final part of the proof of Theorem 2, and our proposition holds.

Then, after the "moving" of L into A, the statement of Claim 11 does not hold for the obtained map. But the moving of L does not change Γ, which contradicts the minimality of $E(\Gamma)$. □

Proposition 14. *For the factorization $M \xrightarrow{p} Q \xrightarrow{q} N$ from our proof of Theorem 2, if q is a d-fold branched covering, then* $\operatorname{Deg}(q \circ p) = d$.

This assertion is nontrivial in view of Remark 8.

Proof. Note that a pinch map p cannot be contractible since the homomorphism $p^* : H^2(Q; \mathbb{Z}_2) \to H^2(M; \mathbb{Z}_2)$ is nonzero. Therefore $\operatorname{Deg} p = 1$.

Suppose p is not orientation-true. Then by Proposition 13 we have that q is a covering (without branch points). Then $\operatorname{Deg}(q \circ p) = d$ by Proposition 6.

Suppose p is orientation-true. Then $\operatorname{Deg}(q \circ p) = d$ by Corollary 5. □

Proof of Theorem 1. We are given a map $f : M \to N$ of geometric degree $d > 0$. Applying Theorem 2, we obtain a factorization $M \xrightarrow{p} Q \xrightarrow{q} N$. Denote the set of critical values of q by $B \subset N$ (it is finite, possibly empty). By Proposition 14 the number of sheets of q equals d. Then

$$\chi(Q \setminus q^{-1}(B)) = d \cdot \chi(N \setminus B),$$

and therefore $\chi(Q) \leq d \cdot \chi(N)$ (this is similar to the reasoning in the Riemann–Hurwitz formula). To complete our proof, note that $\chi(M) \leq \chi(Q)$. □

Appendix: Stable maps and apparent contours

Now let us show one unexpected application of the factorization theorem.

A map of surfaces $f : M \to N$ is called *generic* if its critical points Σ_f are folds and cusps. A generic map is called *stable* if the set of its critical values $f(\Sigma_f)$ has only transversal self-crossings, called *nodes*. For more details, see, for example [Arnold et al. 2012, Part 1] or [Yamamoto 2017] (see also [Ryabichev 2020]). Denote the number of nodes by $n(f)$.

Note that stable maps form a dense open subset in $C^\infty(M, N)$. Pinches and branched coverings, which we have considered above, are not stable, but one can deform them to stable maps by an arbitrary small homotopy.

Theorem 15. *Every map of closed surfaces $f : M \to N$ is homotopic to a stable map f' such that $n(f') = 0$, i.e., the set of critical values $f'(\Sigma_{f'})$ is a collection of nonintersecting non-self-intersecting curves* (*possibly with cusps*) *in N.*

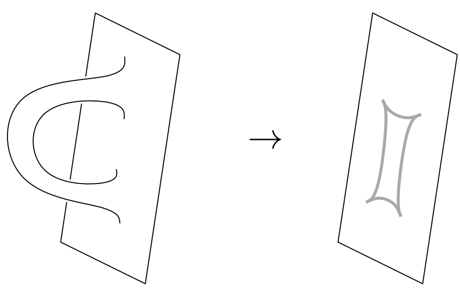

Figure 4. Collapsing of a handle adds a fold curve with 4 cusps.

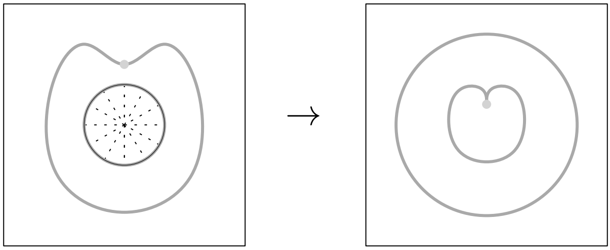

Figure 5. Collapsing of a Möbius band via adding two fold curves with 1 cusp.

This fact connects the study of apparent contours of T. Yamamoto with the theory of branched coverings; see, e.g., [Gabai and Kazez 1987]. Note that if $N = \mathbb{R}^2$, then Theorem 15 can be proved without the factorization theorem, and in the case $N = S^2$, Theorem 15 was proved in [Yamamoto 2010, Theorem 1.4].

Proof. Applying Theorem 2 to the map f, we obtain a factorization $M \xrightarrow{p} Q \xrightarrow{q} N$.

The branched covering q can be homotopied so that each branch point of index i will turn into a fold circle with $i+2$ cusps. See, for example, [Arnold et al. 2012, Part 1, §1.8] for more details.

A collapsing of an orientable handle can be turned into a fold curve with 4 cusps, just as a projection of a plane with a handle in $\mathbb{R}^3$ to the plane; see Figure 4 (see also the discussion in [Yamamoto 2017, §3.2]).

Finally, a collapsing of a Möbius band is homotopic into a map with a fold curve along the Möbius band and with a fold curve with one cusp around it; see Figure 5 (on the left, the opposite points on the inner circle should be identified, we show them connected by dotted lines; see also [Yamamoto 2010, Figures 11 and 12]).

The described homotopies are local, the images of these curves are small and we may assume that they do not cross each other. □

Acknowledgements

I wish to express my gratitude to Ian Agol, who drew my attention to Edmonds' paper [1979] answering my mathoverflow question, and also to Allan Edmonds, who took part in the discussion and provided a link to Skora's article [1987]. I am very thankful to participants of the Moscow seminar "Geometric walks", in particular to Alex Gorin, for a fruitful discussion of the approaches to Theorem 1. I want to thank Sashachka Pilipyuk for some critical remarks on this text. I am also grateful to the anonymous referee, whose comments helped make the text much clearer. Finally, I am thankful to the Independent University of Moscow and the Young Russian Mathematics award for financial support.

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Received August 16, 2023. Revised February 11, 2024.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 328, No. 1, 2024
https://doi.org/10.2140/pjm.2024.328.157

A VIRO–ZVONILOV-TYPE INEQUALITY FOR Q-FLEXIBLE CURVES OF ODD DEGREE

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We define an analogue of the Arnold surface for odd degree flexible curves, and we use it to double branch cover Q-flexible embeddings, where Q-flexible is a condition to be added to the classical notion of a flexible curve. This allows us to obtain a Viro–Zvonilov-type inequality: an upper bound on the number of nonempty ovals of a curve of odd degree. We investigate our method for flexible curves in a quadric to derive a similar bound in two cases. We also digress around a possible definition of nonorientable flexible curves, for which our method still works and a similar inequality holds.

Let $F \subset \mathbb{CP}^2$ be a flexible curve of odd degree m. We denote as $\ell^{\pm}$ and ℓ^0 the number of ovals of the curve $\mathbb{R}F \subset \mathbb{RP}^2$ that bound from the outside a component of $\mathbb{RP}^2 \smallsetminus \mathbb{R}F$ which has positive, negative or zero Euler characteristic, respectively. In particular, ℓ^+ is the number of empty ovals, and $\ell^0 + \ell^-$ is that of nonempty ones. O. Viro and V. Zvonilov [1992] proved the following upper bound for the number of nonempty ovals:

$$\ell^0 + \ell^- \leqslant \frac{(m-3)^2}{4} + \frac{m^2 - h(m)^2}{4h(m)^2},$$

with $h(m)$ denoting the biggest prime power that divides m. Their proof relied on taking a branched cover of $\mathbb{CP}^2$, ramified over the surface F. Usually, it is a good choice to take doubly sheeted branched covers, but this is not possible in this setting where m is odd. Odd degree curves are a different story compared to even degree ones, one reason being the nonexistence of the Arnold surface in $\mathbb{S}^4$ ($\mathbb{R}F$ is not null-homologous in $H_1(\mathbb{RP}^2; \mathbb{Z}/2)$, and neither is F in $H_2(\mathbb{CP}^2; \mathbb{Z}/2)$). In the present paper, we give a definition of an analogue of the Arnold surface in $\overline{\mathbb{CP}}^2$ for odd degree curves. This means that, under a certain condition of being Q-flexible (up to taking another conic Q with empty real part and pseudoholomorphic, this is always satisfied by pseudoholomorphic curves), we are allowed to take the double branched cover of $\overline{\mathbb{CP}}^2$ ramified over a perturbation of this Arnold surface. This

MSC2020: primary 14P25, 57M12; secondary 14H45, 57S25.
Keywords: algebraic curves, flexible curves, Hilbert's 16th problem, double branched covers, nonorientable surfaces.

condition is also always satisfied by algebraic curves. We will show the following result, by methods analogous to Viro and Zvonilov.

Theorem 3.11. *Let F be a Q-flexible curve of odd degree m. Then*

$$\ell^0 + \ell^- \leqslant \frac{(m-1)^2}{4}.$$

If equality holds, then the curve is type I.

It is worth mentioning that this is not quite Zvonilov's bound $(m-1)(m-3)/4$ [1979], which works for any flexible curve that intersects a real line generically (this condition being the degree-one analogue of our Q-flexibility), and in particular for any pseudoholomorphic curve. However, it appears that Q-flexibility and this condition by Zvonilov are independent for general flexible curves.

In Section 1, we discuss some constructions in $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ seen as 2-fold branched covers of the standard 4-sphere. In Section 2, we construct the Arnold surface for odd degree curves, and we describe the behavior of the real part of the curve under this construction. In Section 3, we prove the inequality. In Section 4, we review our method for curves in a quadric to produce a result which, to our knowledge, is new even for algebraic curves. In Section 5, we compare our inequality to Viro and Zvonilov's, and we investigate the possible notion of nonorientable flexible curves, for which our method still applies to derive a similar bound.

1. Preliminaries

Throughout this paper, all surfaces will be assumed to be connected, and all embeddings are smooth.

The complex conjugation $\mathrm{conj} : \mathbb{CP}^2 \to \mathbb{CP}^2$ is defined in homogeneous coordinates by $\mathrm{conj}([z_0 : z_1 : z_2]) = [\overline{z_0} : \overline{z_1} : \overline{z_2}]$, and $\mathbb{RP}^2 \subset \mathbb{CP}^2$ is the fix-point set $\mathrm{Fix}(\mathrm{conj})$. Here, Q will always denote a generic real conic with empty real part (for instance, the Fermat conic given by the equation $z_0^2 + z_1^2 + z_2^2 = 0$). In particular, it is a smoothly embedded 2-sphere $Q \subset \mathbb{CP}^2$ which represents the homology class $[Q] = 2[\mathbb{CP}^1]$ in $H_2(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}$, the choice of a generator being the homology class of any complex line.

Flexible and Q-flexible curves. A *real plane algebraic curve* is a real *nonsingular* homogenous polynomial $X \in \mathbb{R}[x_0 : x_1 : x_2]$. By the *real part* of the curve, we mean the set

$$\mathbb{R}X = \{[x_0 : x_1 : x_2] \in \mathbb{RP}^2 \mid X(x_0, x_1, x_2) = 0\},$$

and by the *complexification* of the curve, we mean

$$\mathbb{C}X = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid X(z_0, z_1, z_2) = 0\}.$$

Evidently, $\mathbb{C}X$ is invariant under complex conjugation. If $m = \deg(X) \geqslant 1$, then we see that $[\mathbb{C}X] = m[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$, and that $\mathbb{C}X$ is a surface of genus $g = (m-1)(m-2)/2$. Also, the tangent space of the complex curve is related to that of the real curve in the following sense:

$$\text{for all x} \in \mathbb{R}\text{X}, \quad T_x\mathbb{C}X = T_x\mathbb{R}X \oplus \boldsymbol{i} \cdot T_x\mathbb{R}X.$$

We define a *flexible* curve, in the sense of Viro [1984], as follows:

Definition 1.1. Let F be a closed oriented surface embedded in $\mathbb{CP}^2$. The surface F is called a degree $m \geqslant 1$ *flexible curve* if

(i) $\operatorname{conj}(F) = F$;

(ii) $\chi(F) = -m^2 + 3m$;

(iii) $[F] = m[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$;

(iv) for all $x \in \mathbb{R}F$, $T_xF = T_x\mathbb{R}F \oplus \boldsymbol{i} \cdot T_x\mathbb{R}F$, where $\mathbb{R}F = F \cap \mathbb{RP}^2$.

The following classical results are known for flexible curves (see [Rokhlin 1978]):

(i) If b_0 denotes the 0-th Betti number, then $b_0(\mathbb{R}F) \leqslant g+1 = (m-1)(m-2)/2+1$. Curves with $b_0(\mathbb{R}F) = g+1$ are called *M-curves*. On the other hand, curves with $b_0(\mathbb{R}F) = 0$ if m is even and $b_0(\mathbb{R}F) = 1$ if $m \geqslant 3$ is odd are called *minimal* curves (note that there are no minimal curves in degree one).

(ii) If m is even, then each component of $\mathbb{R}F$ is contractible in $\mathbb{RP}^2$, and if m is odd, all but one components of $\mathbb{R}F$ are. Contractible components of $\mathbb{R}F$ are called *ovals*. An odd degree curve can never have $\mathbb{R}F = \varnothing$ (it always has the noncontractible component).

(iii) A flexible curve F is said to be a *type I* (*resp. type II*) curve if $F \setminus \mathbb{R}F$ has two connected components (*resp.* is connected). An M-curve is always type I, and a minimal curve is always type II.

What makes flexible curves so different from algebraic curves is the lack of rigidity, mainly seen with the Bézout theorem, which, in particular, implies that a degree m algebraic curve generically intersects Q transversely in exactly $2m$ points.

Definition 1.2. A flexible curve F of degree $m \geqslant 1$ is called *Q-flexible* if $F \pitchfork Q$ consists of $2m$ points, necessarily swapped pairwise by complex conjugation.

Two double branched covers. There is a well-known diffeomorphism between $\mathbb{CP}^2/\text{conj}$ and $\mathbb{S}^4$ (see [Kuiper 1974]). We denote the associated (cyclic) 2-fold branched cover as $p : (\mathbb{CP}^2, \mathbb{RP}^2) \to (\mathbb{S}^4, \mathscr{R})$, with $\mathscr{R} = p(\mathbb{RP}^2)$ an embedded $\mathbb{RP}^2$ in $\mathbb{S}^4$. We also let $\mathscr{Q} = p(Q)$, the image of the preferred conic under that branched cover. From the fact that Q does not intersect the branch locus $\mathbb{RP}^2$, we see that the restriction $p : Q \to \mathscr{Q}$ is an unbranched 2-fold cover, with the conjugation being an orientation-preserving involution generating the group of deck transformations. As such, we see that $\mathscr{Q}$ is also an embedded $\mathbb{RP}^2$ in $\mathbb{S}^4$.

Given a closed embedded surface F^2 in a (closed oriented) 4-manifold X^4, we denote as $e(X, F)$ the normal Euler number of the embedding $F \subset X$; that is, the Euler class of the normal bundle νF. It is also equal to the self-intersection number $F \cdot F$, which is defined by counting signed intersection points between F and a small perturbation F' of F in the normal direction. If F is oriented, then it corresponds to the intersection form of X evaluated on $[F] \in H_2(X; \mathbb{Z})$.

Proposition 1.3. *We have the following normal Euler numbers*:

$$e(\mathbb{CP}^2, \mathbb{RP}^2) = -1, \quad e(\mathbb{CP}^2, Q) = +4, \quad e(\mathbb{S}^4, \mathscr{R}) = -2 \quad \textit{and} \quad e(\mathbb{S}^4, \mathscr{Q}) = +2.$$

Proof. The conic Q is oriented and $[Q] = 2[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$, so we obtain $e(\mathbb{CP}^2, Q) = +4$. Next, because $\mathbb{RP}^2$ is Lagrangian in $\mathbb{CP}^2$, we have that the normal bundle $\nu\mathbb{RP}^2$ and the tangent bundle $T\mathbb{RP}^2$ are anti-isomorphic, and thus, for the Euler class, $e(\nu\mathbb{RP}^2) = -e(T\mathbb{RP}^2) = -\chi(\mathbb{RP}^2) = -1$. Finally, the computations of $e(\mathbb{S}^4, \mathscr{R})$ and $e(\mathbb{S}^4, \mathscr{Q})$ come from the next lemma. □

Lemma 1.4. *Given a* 2*-fold branched cover* $f : (Y^4, \tilde{B}^2) \to (X^4, B^2)$, *and given* F *an embedded closed surface in* X, *we denote as* $\tilde{F}$ *the lift* $p^{-1}(F)$.

(i) *If* $F \pitchfork B$, *possibly with* $F \cap B = \varnothing$, *then* $e(Y, \tilde{F}) = 2e(X, F)$.

(ii) *If* $F \subset B$, *then* $e(Y, \tilde{F}) = \frac{1}{2}e(X, F)$.

Proof. One has to inspect what happens in each case individually. In the first, note that the lift of a perturbation is a perturbation of the lift, and one can ensure that the self-intersection points occur away from the ramification locus B. As such, each of these points lifts to two intersections, and the orientations agree because f is orientation-preserving.

The second case can be deduced from the first. Let $\tilde{F}'$ be a small transverse perturbation of $\tilde{F}$. Letting $\tau : Y \to Y$ denote the involution that spans $\mathrm{Aut}(f)$, and letting $F' = f(\tilde{F}')$, we see that F' is a perturbation of F and $\tilde{F}' \cup \tau(\tilde{F}')$ is the lift of F'. By the first case, we obtain $e(\tau(\tilde{F}')) = e(\tilde{F}') = 2e(F') = 2e(F)$. Moreover, we have $2e(\tilde{F}) = e(\tilde{F}' \cup \tau(\tilde{F}')) = e(\tilde{F}') + e(\tau(\tilde{F}')) = 4e(F)$. □

We now wish to consider the 2-fold branched cover of $\mathbb{S}^4$, ramified over $\mathscr{Q}$ this time. It is possible to make some computations to find an orientation-reversing involution of $\mathbb{S}^4$ which swaps $\mathscr{R}$ and $\mathscr{Q}$. Alternatively, taking any orientation-reversing free involution of $\mathbb{S}^4$, this maps $\mathscr{R}$ to a projective plane with normal Euler number $+2$, and this is always isotopic to $\mathscr{Q}$ in $\mathbb{S}^4$. Tracking this isotopy produces the involution needed. As such, we see that the smooth 4-manifold obtained as the

$$\begin{array}{ccccc}
\mathbb{RP}^2 & \subset & \mathbb{CP}^2 & \supset & Q \\
\| & & \big\downarrow p & & \big\downarrow\!\!\downarrow \\
\mathscr{R} & \subset & \mathbb{S}^4 & \supset & \mathscr{Q} \\
\big\uparrow\!\!\uparrow & & \big\uparrow \tilde{p} & & \| \\
\overline{Q} & \subset & \overline{\mathbb{CP}}^2 & \supset & \overline{\mathbb{RP}}^2
\end{array}$$

Figure 1. The two branched coverings of interest and their associated branch loci. The arrows marked $\twoheadrightarrow$ denote an unbranched 2-fold cover from a 2-sphere to a real projective plane.

double branched cover of $\mathbb{S}^4$ ramified along $\mathscr{Q}$ is diffeomorphic[1] to $\overline{\mathbb{CP}}^2$. We let $\tilde{p} : \overline{\mathbb{CP}}^2 \to \mathbb{S}^4$ denote that double branched cover.

Define $\overline{Q} = \tilde{p}^{-1}(\mathscr{R})$. We see that $\overline{\mathbb{RP}}^2$ and $\overline{Q}$ are respectively embeddings of $\mathbb{RP}^2$ and $\mathbb{S}^2$ in $\overline{\mathbb{CP}}^2$. Using Lemma 1.4 again, we can compute the normal Euler numbers.

Proposition 1.5. *We have* $e(\overline{\mathbb{CP}}^2, \overline{\mathbb{RP}}^2) = +1$ *and* $e(\overline{\mathbb{CP}}^2, \overline{Q}) = -4$.

In Figure 1 we depict a summary of the different maps in play.

2. The Arnold surface of an odd degree flexible curve

For a flexible curve $F \subset \mathbb{CP}^2$, let $A^+(F) = F/\mathrm{conj} = p(F)$. It is an embedded surface in $\mathbb{S}^4$ with boundary $\partial A^+(F) \subset \mathscr{R}$ identified with $\mathbb{R}F$, and it is orientable if and only if F is type I.

If the curve has *even* degree, then $\mathbb{R}F$ is null-homologous, and thus exactly one component of $\mathbb{RP}^2 \smallsetminus \mathbb{R}F$ is nonorientable (it is a punctured Möbius band). Let $\mathbb{RP}^2_\pm$ be the closure of the two possible subsets of $\mathbb{RP}^2 \smallsetminus \mathbb{R}F$ that have $\partial\mathbb{RP}^2_\pm = \partial F$. We choose $\mathbb{RP}^2_-$ to be the one containing the punctured Möbius band (i.e., $\mathbb{RP}^2_+$ is orientable, and $\mathbb{RP}^2_-$ has exactly one nonorientable component). In the case where F is an *algebraic* curve of even degree, the polynomial P defining it can be chosen in such a way that

$$\mathbb{RP}^2_\pm = \{[x_0 : x_1 : x_2] \in \mathbb{RP}^2 \mid \pm P(x_0, x_1, x_2) \geqslant 0\}.$$

In Figure 2, we depict such an example for an algebraic curve.

Definition 2.1. Given a flexible curve F of even degree, we let

$$\mathcal{A}(F) = A^+(F) \cup p(\mathbb{RP}^2_+) \subset \mathbb{S}^4,$$

and we call it the *Arnold surface* of F.

[1]In fact, it is sufficient to obtain that the double branch cover of $\mathbb{S}^4$ ramified over $\mathscr{Q}$ is a homology $\overline{\mathbb{CP}}^2$, as will be the case for curves on quadrics in a later section.

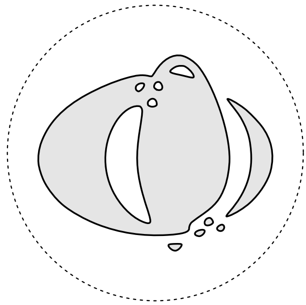


Figure 2. The set $\mathbb{RP}^2_+$, shaded, for Gudkov's M-sextic.

In odd degrees, it is not possible to define a surface in this way; we need to go to $\overline{\mathbb{CP}}^2$ first. Take F to be a flexible curve of odd degree. We denote as $J \subset \mathbb{RP}^2$ the noncontractible component of $\mathbb{R}F$ and as o an oval of $\mathbb{R}F$. Let $J^+ = p(J)$, $o^+ = p(o)$, and set $\bar{J} = \tilde{p}^{-1}(J^+) \subset \overline{Q}$ and $\bar{o} = \tilde{p}^{-1}(o^+) \subset \overline{Q}$. Observe that $\tilde{p}$ restricts to an unbranched 2-fold covering $\tilde{p} : \bar{J} \to J^+$ and $\tilde{p} : \bar{o} \to o^+$.

Proposition 2.2. *We have $\bar{J} \cong \mathbb{S}^1$ and $\bar{o} \cong \mathbb{S}^1 \sqcup \mathbb{S}^1$, and the unbranched coverings $\tilde{p} : \bar{J} \to J^+$ and $\tilde{p} : \bar{o} \to o^+$ are respectively the nontrivial and the trivial 2-fold coverings of the circle.*

Proof. Isotope J in $\mathbb{RP}^2$ to be $J = \mathbb{R}X$ with $X \in \mathbb{R}[x_0 : x_1 : x_2]$ a degree-one nonsingular homogeneous polynomial. Note that we do not need to look at what happens *outside* of $\mathbb{RP}^2$ for the claim. In particular, $\mathbb{C}X \pitchfork Q$ is two points. Letting $G^+ = p(\mathbb{C}X)$ and $\overline{G} = \tilde{p}^{-1}(G^+)$, we obtain

$$J^+ = \partial G^+ \quad \text{and} \quad \bar{J} = \partial \overline{G}.$$

Moreover, $G^+ \pitchfork \mathscr{Q}$ is one point, for the two points in $\mathbb{C}X \pitchfork Q$ are swapped pairwise by conjugation. In particular, the covering $\tilde{p} : \overline{G} \to G^+$ is a 2-fold branched cover of the disc G^+ (for in degree one, $\mathbb{C}X$ is a sphere and $\mathbb{R}X$ is type I), with one branch point in its interior. This is unique, and it is known to induce the nontrivial cover on the boundary, so the first claim follows (see Figure 3).

For the other claim, an oval o bounds a disc D embedded in $\mathbb{RP}^2$, and is thus disjoint from Q. Therefore, the disc $D/\mathrm{conj} \subset \mathscr{R}$ bounded by o^+ lifts in $\overline{\mathbb{CP}}^2$ to two disjoint discs in $\overline{Q}$. This means that $p : \bar{o} \to o^+$ is the trivial covering, and $\bar{o}$ is two circles. □

We let $\overline{\mathbb{R}}F$ be the set

$$\bar{J} \cup \bigcup_{o \text{ oval}} \bar{o} \subset \overline{Q}.$$

The previous statement implies that every oval of $\mathbb{R}F$ gets doubled in $\overline{\mathbb{R}}F$, whereas the noncontractible component J does not.

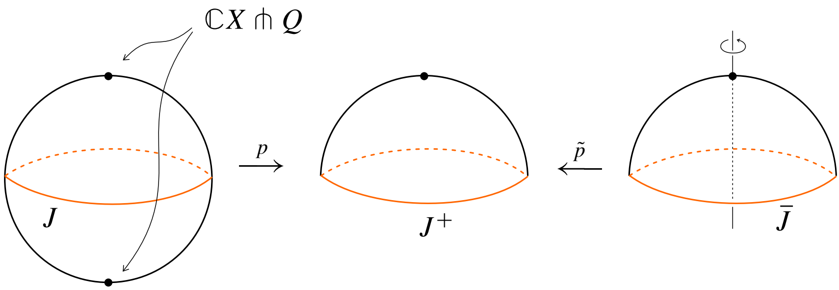


Figure 3. The restrictions $p : \mathbb{C}X \to G^+$ and $\tilde{p} : \overline{G} \to G^+$, the second one being a branched covering. On the right, the rotation by 180° generates the group of deck transformations.

Proposition 2.3. *Let o_1 and o_2 be ovals of $\mathbb{R}F$.*

(i) *The set $\overline{Q} \setminus \bar{J}$ is two open discs, each containing one of the two components of $\bar{o}_1$.*

(ii) *If $o_1 \subset o_2$ (where inclusion means that o_1 is contained in the orientable component of $\mathbb{RP}^2 \setminus o_2$), then $\overline{o_1} \subset \overline{o_2}$, in the following sense: $\overline{Q} \setminus \overline{o_2}$ has three components, one being a cylinder containing $\bar{J}$, and the other two being discs each containing a component of $\overline{o_1}$.*

Proof. This comes from the observation that the covering $\tilde{p} : \overline{Q} \to \mathscr{Q}$ is the quotient of the 2-sphere $\overline{Q}$ by a fixed-point free involution (that is, the antipodal map), as well as the fact that $p : (\mathbb{RP}^2, \mathbb{R}F) \to (\mathscr{R}, p(\mathbb{R}F))$ is a diffeomorphism of the pair. □

This means that the real scheme $\mathbb{R}F$ can be seen *doubled* in $\overline{\mathbb{R}}F$, as Figure 4 depicts. Now, define $\bar{A}^+(F) = \tilde{p}^{-1}(A^+(F)) = \tilde{p}^{-1}(F/\mathrm{conj})$. For the analogue of $\mathbb{RP}^2_+$, there are two subsets $\overline{Q}_\pm$ of $\overline{Q} \setminus \overline{\mathbb{R}}F$ that have $\partial \overline{Q}_\pm = \overline{\mathbb{R}}F$, and those are diffeomorphic, exchanged by "symmetry" of $\overline{Q}$ along $\bar{J}$. To be more precise, we

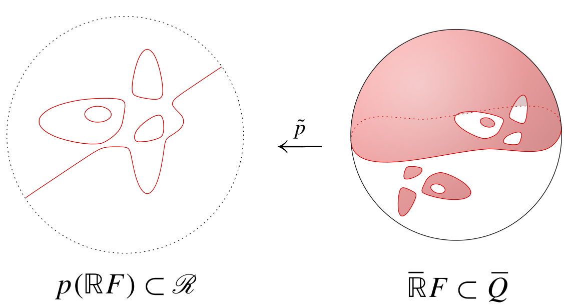


Figure 4. The set $\overline{Q}_+$, shaded, for an algebraic curve of degree 7 with real scheme $\langle J \sqcup 2 \sqcup 1\langle 1\rangle\rangle$ (in Viro notation), obtained as a perturbation of three ellipses and a line.

denote as $\overline{Q}_\pm$ the closure of these two sets, with a choice involved in labeling one $\overline{Q}_+$ and the other $\overline{Q}_-$.

Definition 2.4. The *Arnold surface* of a flexible curve F of odd degree is the surface $\mathcal{A}(F) = \overline{A}^+(F) \cup \overline{Q}_+ \subset \overline{\mathbb{CP}}^2$.

3. Proving the inequality

The idea is that we would like to take the 2-fold branched cover of $\overline{\mathbb{CP}}^2$ ramified along the Arnold surface. This is not yet possible in this odd degree setting, for the surface $\mathcal{A}(F)$ is not null-homologous in $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}/2)$ (we will see that it has an odd self-intersection number). In fact, this limitation is what led Viro and Zvonilov to consider $h(m)$-sheeted branched covers, where $h(m)$ denotes the highest prime power that divides m. However, in our favorable setting, we can perturb the Arnold surface, with the important feature that it preserves the structure of the curve $\overline{\mathbb{R}}F$ inside $\overline{Q}$. One last remark is that we could *not* apply the same construction to a Q-flexible curve $F \subset \mathbb{CP}^2$ directly, because the conic Q has an *even* homology class.

Branching over the Arnold surface. We are first interested in computing the normal Euler number of $\mathcal{A}(F) \subset \overline{\mathbb{CP}}^2$. Recall that if $F \subset X$ is a closed surface in a closed oriented 4-manifold, then the Euler class $e(\nu F) \in H^2(F; \mathbb{Z}_w)$ corresponds to the self-intersection of F (here, $\mathbb{Z}_w$ means coefficients twisted by $w_1(\nu F)$, and $w_1(X) = 0$ implies $w_1(\nu F) = w_1(F)$, from which twisted Poincaré duality readily gives $H^2(F; \mathbb{Z}_w) \cong \mathbb{Z}$).

In the case where $\partial F \neq \varnothing$ however, one needs to choose a fixed nonvanishing section θ of $\nu F|_{\partial F}$, and consider a relative Euler class (see [Sharafutdinov 1973]):

$$e_\theta(X, F) = e_\theta(\nu F) \in H^2(F, \partial F; \mathbb{Z}_w) \cong \mathbb{Z}.$$

This Euler class corresponds to the integer obstruction to extend this section θ to the whole νF. However, if one needs to glue two surfaces F_1 and F_2 along their common boundary $\partial F_1 = \partial F_2$ and compute the Euler number of $F_1 \cup_\partial F_2$ in terms of relative Euler numbers of F_1 and F_2, there are two things to be careful about:

(1) The bundle $\Lambda = (\nu F_1 \cap \nu F_2)|_{\partial F_i}$ over ∂F_i needs to be rank one.

(2) This bundle Λ needs to have a nonvanishing section θ.

If both conditions are satisfied, the section θ gives rise to the *same* section of $\nu F_1|_{\partial F_1}$ and $\nu F_2|_{\partial F_2}$. This can be used to define relative Euler numbers $e_\theta(X, F_i)$. Since, in the closed case, the number $e(X, F)$ does not depend on the choice of the (possibly vanishing) global section of νF, we obtain the relation

$$e(X, F_1 \cup_\partial F_2) = e_\theta(X, F_1) + e_\theta(X, F_2).$$

For instance, if $F \subset \mathbb{CP}^2$ is a flexible curve of even degree $m = 2k$ with nonempty real part $\mathbb{R}F$, one sees that $\Lambda = (\nu\mathbb{RP}^2_+ \cap \nu F)|_{\mathbb{R}F}$ is the trivial line bundle over $\mathbb{R}F$. If θ denotes a section of the normal bundle $\mathbb{R}F$ in $\mathbb{RP}^2$, then $\boldsymbol{i}\theta$ is a section of Λ, and letting $\mathscr{R}_+ = \mathbb{RP}^2_+/\text{conj}$ and $A^+(F) = F/\text{conj}$, it also induces a section $\hat{\theta}$ of $(\nu\mathscr{R}_+ \cap \nu A^+(F))|_{\partial A^+(F)}$. By a careful examination, one can use Lemma 1.4 to compute

$$e_{\hat{\theta}}(\mathbb{S}^4, A^+(F)) = \tfrac{1}{2} e_{\boldsymbol{i}\theta}(\mathbb{CP}^2, F) = \tfrac{1}{2} F \cdot F = 2k^2,$$

because F is closed, and

$$e_{\hat{\theta}}(\mathbb{S}^4, \mathscr{R}_+) = 2e_{\boldsymbol{i}\theta}(\mathbb{CP}^2, \mathbb{RP}^2_+) = -2\chi(\mathbb{RP}^2_+),$$

because $\mathbb{RP}^2$ is Lagrangian. This means that the Arnold surface $\mathcal{A}(F) \subset \mathbb{S}^4$ has normal Euler number

$$e(\mathbb{S}^4, \mathcal{A}(F)) = 2k^2 - 2\chi(\mathbb{RP}^2_+).$$

If $F \subset \mathbb{CP}^2$ is now a flexible curve of *odd* degree, the normal bundle of $\mathbb{R}F$ in $\mathbb{RP}^2$ is a nontrivial line bundle over $\mathbb{R}F$ (to be more precise, exactly one connected component of this bundle is the nonorientable line bundle over the circle: the component associated to the pseudoline $J \subset \mathbb{R}F$). As such, there is no nonvanishing section θ of Λ, and it does not give rise to a section $\boldsymbol{i}\theta$ of $\nu F|_{\mathbb{R}F}$. However, the subbundle $\boldsymbol{i}\Lambda \subset \nu F|_{\mathbb{R}F}$ can be seen as a field of lines of $\nu F|_{\mathbb{R}F}$ (instead of a section being a vector field).

In general, let $\Lambda \subset \nu F|_{\partial F}$ be a line subbundle. As done in [Guillou and Marin 1980, §3], one can still consider the integer obstruction

$$\tilde{e}_\Lambda(X, F) \in H^2(F, \partial F; \mathbb{Z}_w)$$

to extend this field of line to the whole νF. In the case where Λ *does* have a section θ, we have $\tilde{e}_\Lambda(X, F) = 2e_\theta(X, F)$.

Back to where F is a flexible curve of odd degree, and letting $A^+(F) = F/\text{conj}$, we see that $\boldsymbol{i}\Lambda$ induces a line subbundle $\hat{\Lambda}$ of $\nu A^+(F)|_{\partial A^+(F)}$. From an application of Lemma 1.4,

$$\tilde{e}_{\hat{\Lambda}}(\mathbb{S}^4, A^+(F)) = \tfrac{1}{2}\tilde{e}_{\boldsymbol{i}\Lambda}(\mathbb{CP}^2, F) = \tfrac{1}{2} \cdot 2e(\mathbb{CP}^2, F) = m^2,$$

because F is closed. This means that, in the above sense, we have $e(\mathbb{S}^4, A^+(F)) = m^2/2$, although this is a noninteger value.

To ease out the exposition, we will allow ourselves to write half-integer Euler numbers and to use Lemma 1.4 with half-integers. It will be understood that we use the obstruction $\tilde{e}$ when needed. We will also omit the choice of the field of lines in the subscript, as all surfaces will ultimately become closed at the end of computations.

Proposition 3.1. *We have* $e(\overline{\mathbb{CP}}^2, \mathcal{A}(F)) = m^2 - 2$.

Proof. Recall that we defined $\bar{A}^+(F) = \tilde{p}^{-1}(A^+(F))$, and $\mathcal{A}(F) = \bar{A}^+(F) \cup \bar{Q}_+$. Using Lemma 1.4 twice, we compute that $e(\overline{\mathbb{CP}}^2, \bar{A}^+(F)) = m^2$. Now, we simply make use of the fact that $e(\overline{\mathbb{CP}}^2, \bar{Q}_+) = -2$. Indeed, $\bar{Q} = \bar{Q}_+ \cup \bar{Q}_-$, thus

$$-4 = e(\overline{\mathbb{CP}}^2, \bar{Q}) = e(\overline{\mathbb{CP}}^2, \bar{Q}_+) + e(\overline{\mathbb{CP}}^2, \bar{Q}_-),$$

and because $\bar{Q}_+$ and $\bar{Q}_-$ are swapped by the (orientation-preserving) involution of $\overline{\mathbb{CP}}^2$ spanning $\mathrm{Aut}(\tilde{p})$, we obtain

$$e(\overline{\mathbb{CP}}^2, \bar{Q}_+) = e(\overline{\mathbb{CP}}^2, \bar{Q}_-),$$

from which we derive $e(\overline{\mathbb{CP}}^2, \bar{Q}_\pm) = -2$. Alternatively, this can be obtained from the following lemma. □

Lemma 3.2. *Let X be a submanifold of $\bar{Q}$. Then $e(\overline{\mathbb{CP}}^2, X) = -2\chi(X)$.*

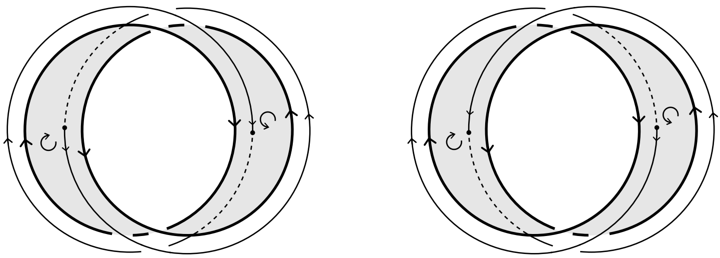
Proof. The submanifold $\mathbb{RP}^2 \subset \mathbb{CP}^2$ being Lagrangian, and the covering $p : \mathbb{CP}^2 \to \mathbb{S}^4$ being branched exactly on $p(\mathbb{RP}^2)$, we observe that $\nu\mathscr{R} \cong -T\mathscr{R}$ in $\mathbb{S}^4$. However, the covering $\tilde{p} : \overline{\mathbb{CP}}^2 \to \mathbb{S}^4$ is *unbranched* in a regular neighborhood of $\mathscr{R}$, whence $\nu\bar{Q} \cong -2T\bar{Q}$. In particular, for the Euler classes, we have $e(\overline{\mathbb{CP}}^2, X) = e(\nu X) = -2e(TX) = -2\chi(X)$. □

Because $\mathcal{A}(F)$ has an odd self-intersection, we see that it cannot be null-homologous in $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}/2)$. In fact, because this group has rank one, being $\mathbb{Z}/2$-null-homologous is equivalent to having an even self-intersection. There is another surface, however, which is not null-homologous and transverse to $\mathcal{A}(F)$: the surface $\overline{\mathbb{RP}}^2$. If F is a Q-flexible curve of odd degree m, the transverse intersection $F \pitchfork Q$ is $2m$ points. This implies that $\mathcal{A}(F)$ intersects $\overline{\mathbb{RP}}^2$ transversely in m points. The surface $\mathcal{A}(F) \cup \overline{\mathbb{RP}}^2$ is therefore immersed with m transverse crossings only. We will describe how to resolve those double points to obtain an embedded surface.

Firstly, in a closed oriented 4-manifold X, let $\Sigma \subset X$ be the image of a closed surface through an immersion, with only one transverse self-intersection point $x \in \Sigma$. Take $B \subset X$ to be a small 4-ball around x, which meets Σ in two disks intersecting transversely at their common center x. The boundary of those discs is a Hopf link $\partial B \cap \Sigma \subset \partial B \cong \mathbb{S}^3$, which bounds a Hopf band $H \subset B$. We call the surface Σ' defined by a choice of a gluing of a Hopf band H to $\Sigma \setminus B$ a *smoothing of the singularity* of the immersed surface $\Sigma \subset X$.

Lemma 3.3. *The resulting surface Σ' is an embedded surface in X with $\chi(\Sigma') = \chi(\Sigma) - 1$, $e(X, \Sigma') = e(X, \Sigma) \pm 2$, and we have freedom in the choice.*

Proof. Regarding the claim about the normal Euler numbers, we use similar arguments as in [Yamada 1995, §5]. Note that if B is a small 4-ball around the double point $x \in \Sigma$, then the Hopf link $\partial B \cap \Sigma$ comes with two possible choices of



$e(\Sigma') = e(\Sigma) \pm 2$ $\qquad$ $e(\Sigma') = e(\Sigma) \mp 2$

Figure 5. The two possible smoothings of a singularity of an immersion, given by both choices of orientation of the Hopf link.

orientations. Each determines a unique (up to isotopy fixing the boundary) oriented Hopf band H inducing that orientation. A transverse push-off $s(\Sigma)$ of Σ can be assumed to be parallel to Σ near x, and the intersection $s(\Sigma) \cap \Sigma \cap H$ is two points with the same sign. Finally, we see that those signs are opposite to one another in both choices of orientations of $\partial B \cap \Sigma$ (see Figure 5).

The fact that $\chi(\Sigma') = \chi(\Sigma) - 1$ is simply a matter of using the formula $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ twice (here, all the sets involved are cellular subspaces). Indeed, if H denotes the Hopf band that is glued to $\Sigma \setminus B$, then

$$\chi(\Sigma') = \chi(\Sigma \setminus B \cup_\partial H) = \chi(\Sigma \setminus B) + \chi(H) - \chi(\mathbb{S}^1 \sqcup \mathbb{S}^1) = \chi(\Sigma \setminus B),$$

and

$$\chi(\Sigma) = \chi(\Sigma \setminus B \cup_\partial B \cap \Sigma) = \chi(\Sigma \setminus B) + \chi(B \cap \Sigma) - \chi(\mathbb{S}^1 \sqcup \mathbb{S}^1) = \chi(\Sigma \setminus B) + 1,$$

by noting that $B \cap \Sigma$ is topologically a wedge of two discs. □

Consider $F \subset \mathbb{CP}^2$ a Q-flexible curve of odd degree m. The Arnold surface $\mathcal{A}(F)$ needs not be orientable, and as said before, there is no 2-fold branched cover of $(\overline{\mathbb{CP}}^2, \mathcal{A}(F))$. Recall that $\mathcal{A}(F) \pitchfork \overline{\mathbb{RP}}^2$ is m points, and as such, $\mathcal{A}(F) \cup \overline{\mathbb{RP}}^2$ is an immersed surface with m double points. Applying the previous smoothing of the singularities at each of those m points, this yields a surface $\mathcal{X}(F) \subset \overline{\mathbb{CP}}^2$, with

$$\chi(\mathcal{X}(F)) = \chi(\mathcal{A}(F) \cup \overline{\mathbb{RP}}^2) - m;$$
$$e(\overline{\mathbb{CP}}^2, \mathcal{X}(F)) = e(\overline{\mathbb{CP}}^2, \mathcal{A}(F) \cup \overline{\mathbb{RP}}^2) + 2r, \quad r \in \{-m, \dots, m\}.$$

Here, r is not free to take *all* the possible values in $\{-m, \dots, m\}$. However, the extremal values $\pm m$ are always realizable. Define $\mathcal{X}(F)$ to be the one where we pick up a $+2$ every time (that is, $r = +m$). Two applications of the topological Riemann–Hurwitz formula give $\chi(\mathcal{A}(F)) = \chi(F) - m + 1$. Therefore, we have

$$\chi(\mathcal{X}(F)) = -m^2 + 2.$$

Next, we compute

$$e(\overline{\mathbb{CP}}^2, \mathcal{X}(F)) = e(\overline{\mathbb{CP}}^2, \mathcal{A}(F) \cup \overline{\mathbb{RP}}^2) + 2m = m^2 + 2m - 1.$$

Take Y^4 to be the 2-fold cover of $\overline{\mathbb{CP}}^2$ branched over $\mathcal{X}(F)$. This has been made possible because the surface $\mathcal{X}(F)$ has zero homology mod 2: $[\mathcal{X}(F)] = 0 \in H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}/2)$ (see [Gompf and Stipsicz 1999, §6.3] or [Nagami 2000, Corollary 2.10]). Indeed,

$$H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}/2) = \{0, [\overline{\mathbb{RP}}^2]\},$$

and $\mathcal{A}(F)$ intersects $\overline{\mathbb{RP}}^2$ in an odd number m of points. Therefore, we deduce that $[\mathcal{A}(F)] = [\overline{\mathbb{RP}}^2]$ in $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}/2)$. Now, adding $\overline{\mathbb{RP}}^2$ and smoothing the singularities means that $\mathcal{X}(F) \cap \overline{\mathbb{RP}}^2 = \varnothing$, and as such $[\mathcal{X}(F)] = 0$ in $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}/2)$. We denote as $\Theta : Y^4 \to (\overline{\mathbb{CP}}^2, \mathcal{X}(F))$ the 2-fold branched cover. The previous computations of $\chi(\mathcal{X}(F))$ and $e(\overline{\mathbb{CP}}^2, \mathcal{X}(F))$ will allow us to obtain homological information about the 4-manifold Y.

Proposition 3.4. *The homology groups $H_1(Y; \mathbb{Z})$ and $H_3(Y; \mathbb{Z})$ are torsion. In particular, for Betti numbers, we have $b_1(Y) = b_3(Y) = 0$.*

Proof. In order to show that $H_1(Y; \mathbb{Z})$ is torsion, it is sufficient to know that $H_1(Y; \mathbb{Z}/2) = 0$, for any free part $\mathbb{Z}^p < H_1(Y; \mathbb{Z})$ would give p copies of $\mathbb{Z}/2$ in $H_1(Y; \mathbb{Z}/2)$. We use a generalization of the Gysin sequence, as stated in [Lee and Weintraub 1995, Theorem 1]:

$$H_1(\overline{\mathbb{CP}}^2, \mathcal{X}(F); \mathbb{Z}/2) \to H_1(Y, *; \mathbb{Z}/2) \to H_1(\overline{\mathbb{CP}}^2, \mathcal{X}(F); \mathbb{Z}/2).$$

Here, $H_1(Y, *; \mathbb{Z}/2) \cong \tilde{H}_1(Y; \mathbb{Z}/2)$ the reduced homology group, and we have $H_1(\overline{\mathbb{CP}}^2, \mathcal{X}(F); \mathbb{Z}/2) = 0$, by looking at the homology long exact sequence of the pair $(\overline{\mathbb{CP}}^2, \mathcal{X}(F))$. This provides $H_1(Y; \mathbb{Z}/2) = 0$, as claimed. For $b_3(Y) = 0$, this is a consequence of $b_1(Y) = 0$ and Poincaré duality. □

An educated guess is that Y *may* be simply connected, just like the usual branched cover of $\mathbb{CP}^2$ branched over an *algebraic* curve $\{P(x_0, x_1, x_2) = 0\}$, given as the algebraic surface $\{P(x_0, x_1, x_2) = w^2\} \subset \mathbb{CP}(1, 1, 1, m/2)$, is simply connected (see [Wilson 1978]). However, we have enough information to compute all the homological invariants of Y that will be useful. We recall the Hirzebruch formula for the signature of 2-fold branched covers.

Theorem 3.5 [Hirzebruch 1969, Section 3]. *Let $f : (Y, B) \to (X, A)$ be a cyclic 2-fold branched cover, with X and Y both closed oriented 4-manifolds, A a closed surface and f orientation-preserving. Then, we have*

$$\sigma(Y) = 2\sigma(X) - \tfrac{1}{2}e(X, A).$$

Proposition 3.6. *We have*

$$\chi(Y) = m^2+4, \quad b_2(Y) = m^2+2, \quad \sigma(Y) = \frac{-m^2-2m-3}{2},$$
$$b_2^+(Y) = \frac{(m-1)^2}{4} \quad \text{and} \quad b_2^-(Y) = \frac{3m^2+2m+7}{4},$$

where $b_2^+(Y)$ and $b_2^-(Y)$ respectively denote the maximal ranks of the subspaces of $H_2(Y;\mathbb{Z})$ on which the intersection form Q_Y is positive and negative definite.

Proof. First, the topological Riemann–Hurwitz formula again yields

$$\chi(Y) = 2\chi(\overline{\mathbb{CP}}^2) - \chi(\mathcal{X}(F)) = m^2+4.$$

Next, we use the Theorem 3.5 with the branched cover Θ to obtain

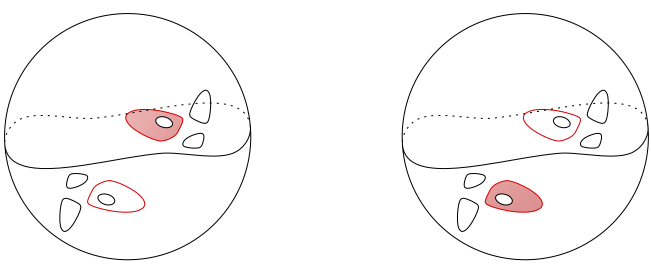
$$\sigma(Y) = 2\sigma(\overline{\mathbb{CP}}^2) - \tfrac{1}{2}e(\overline{\mathbb{CP}}^2, \mathcal{X}(F)) = -2 - \frac{m^2+2m-1}{2} = \frac{-m^2-2m-3}{2}.$$

Now, because of Proposition 3.4, we see that $\chi(Y) = 2 + b_2(Y)$. This provides

$$b_2^+(Y) + b_2^-(Y) = b_2(Y) \quad \text{and} \quad b_2^+(Y) - b_2^-(Y) = \sigma(Y),$$

which we can easily solve for $b_2^\pm(Y)$. □

Proving the inequality. We will now mostly mimic the proof of Viro and Zvonilov [1992]. Note that the construction of $\mathcal{X}(F)$ from $\mathcal{A}(F) \cup \overline{\mathbb{RP}}^2$ happens away from a neighborhood $\overline{Q}$. In particular, we still see $\overline{\mathbb{R}}F$ embedded inside $\mathcal{X}(F)$. Given an oval $o \subset \mathbb{R}F$, recall that $\mathbb{RP}^2 \setminus o$ has two connected components, one of which is a punctured disc (the other being a punctured Möbius band). Letting $C(o) \subset \mathscr{R}$ be the image of that component under $p : \mathbb{CP}^2 \to \mathbb{S}^4$, we see that $\tilde{p}^{-1}(C(o)) \subset \overline{Q}$ is diffeomorphic to two disjoint copies of $C(o)$. We denote as $C_\pm(o)$ each of these copies, with the property that $C_\pm(o) \subset \overline{Q}_\pm$ (see Figure 6).



$C_-(o) \subset \overline{Q}_-$ $C_+(o) \subset \overline{Q}_+$

Figure 6. Using the same scheme $\langle J \sqcup 2 \sqcup 1\langle 1\rangle\rangle$ as in the example of Figure 4, we take o to be the only nonempty oval. In the shaded regions, we depict $C_\pm(o)$, where part of the boundary $\partial C_\pm(o)$ is $\bar{o}$.

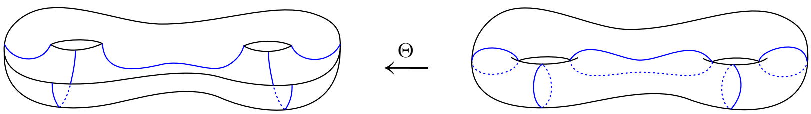


Figure 7. The "pseudo" branched cover $\tilde{C}(o) \to C_-(o)$.

We see that $C_+(o)$ is totally included in the ramification locus of the branched cover $\Theta : Y \to \overline{\mathbb{CP}}^2$, and that $C_-(o)$ intersects this ramification locus only at its boundary $\partial C_-(o) \supset \bar{o}$. We let $\tilde{C}(o) = \Theta^{-1}(C_-(o))$. The restriction $\Theta : \tilde{C}(o) \to C_-(o)$ is not a branched cover, but it is close enough: it maps the boundaries $\partial\tilde{C}(o) \to \partial C_-(o)$ diffeomorphically, and is two-to-one on the interior. Because $C_-(o)$ is planar (that is, a sphere with holes), we have that $\tilde{C}(o)$ is obtained as gluing two spheres with holes along their boundary components. Additionally, $\mathrm{Aut}(\Theta)$ is a $\mathbb{Z}/2$ spanned by $\tau : Y \to Y$ an orientation-preserving involution. This involution τ swaps those two planar surfaces in Y that glue to $\tilde{C}(o)$ and fixes their common boundary. As such, we have shown the next result.

Proposition 3.7. *For any oval $o \subset \mathbb{R}F$, $\tilde{C}(o)$ is an oriented surface in Y of genus b the number of ovals directly contained in o. The restriction $\Theta : \tilde{C}(o) \to C_-(o)$, shown in Figure 7, is the result of the quotient of the surface Σ_b by reflection along a plane of symmetry that cuts it into two planar surfaces.* □

The same construction works for J: there are two path-connected subsets $D_\pm(J)$ of $\overline{Q}_\pm$ that have $\overline{J}$ as a part of their boundary. Letting $\tilde{D}(J) = \Theta^{-1}(D_-(J))$, we have that $\tilde{D}(J)$ is a surface of genus e the number of exterior ovals in $\mathbb{R}F$ (those not included in any other), and the restriction $\Theta : \tilde{D}(J) \to D_-(J)$ is again the quotient of Σ_e by reflection along a plane in the middle.

Given an oval $o \subset \mathbb{R}F$, we denote as $\chi(o) = \chi(C_-(o))$ the Euler characteristic of the connected subset of $\mathbb{RP}^2 \smallsetminus \mathbb{R}F$ bounded by o from outside. Similarly, we let $\chi(J) = \chi(D_-(J))$. One remarks that $\chi(o) \leqslant 1$, with equality if and only if o is empty, and that $\chi(J) = 1 - e$ with e the number of exterior ovals.

Proposition 3.8. *Let $o, o' \subset \mathbb{R}F$ be ovals, and denote again by J the noncontractible component of $\mathbb{R}F$.*

(1) *We have $Q_Y(\tilde{C}(o), \tilde{C}(o)) = -4\chi(o)$ and $Q_Y(\tilde{D}(J), \tilde{D}(J)) = -4\chi(J) = 4(e-1)$.*

(2) *We have $Q_Y(\tilde{C}(o), \tilde{D}(J)) = 0$. If $o \neq o'$, then $Q_Y(\tilde{C}(o), \tilde{C}(o')) = 0$.*

Proof. For the first claim, observe $e(\overline{\mathbb{CP}}^2, C_-(o)) = -2\chi(o)$ and $e(\overline{\mathbb{CP}}^2, D_-(J)) = -2\chi(J)$, by using Lemma 3.2. Next, from Lemma 1.4, we can see that

$$e(Y, \tilde{C}(o)) = 2e(\overline{\mathbb{CP}}^2, C_-(o)) \quad \text{and} \quad e(Y, \tilde{D}(J)) = 2e(\overline{\mathbb{CP}}^2, D_-(J)).$$

To derive $Q_Y(\tilde{C}(o), \tilde{C}(o))$ and $Q_Y(\tilde{D}(J), \tilde{D}(J))$, we remark that $\tilde{C}(o)$ and $\tilde{D}(J)$ are orientable surfaces, so the self-intersection and the evaluation of the intersection form agree.

For the second claim, distinct ovals o and o' cannot satisfy $C_-(o) \cap C_-(o') \neq \varnothing$, even if one is included inside the other (but it is possible that $C_-(o) \cap C_+(o') \neq \varnothing$). The same goes for $C_-(o) \cap D_-(J) = \varnothing$. As such, the surfaces $\tilde{C}(o)$, $\tilde{C}(o')$ and $\tilde{D}(J)$ are nonintersecting in Y. □

The homology classes of the surfaces $\tilde{C}(o_i)$, $i \in [\![1, \ell]\!]$, and $\tilde{D}(J)$ were respectively denoted as β_i and β_0 by Viro and Zvonilov (where ℓ denotes the number of ovals in $\mathbb{R}F$). They showed the following result.

Lemma 3.9 [Viro and Zvonilov 1992, Lemma 1.3]. *Let $h = p^r$ be a prime power. Let $\nu : Y \to X$ be an h-sheeted cyclic covering between two n-manifolds, branched over a codimension-two subset $A \subset X$. Let $B \subset X$ be a membrane, let b be the class in $H_k(X, A)$ determined by B, and let β be the class in $H_k(Y)$ determined by $\nu^{-1}(B)$, oriented coherently with B. Let $\tau : Y \to Y$ be a generator of* Aut(ν), *and let $\varrho = 1 - \tau \in (\mathbb{Z}/p)[\text{Aut}(\nu)]$. Recall the Smith long exact sequence in homology (with coefficients in $\mathbb{Z}/p$):*

$$\cdots \to H^{\varrho}_{k+1}(Y) \xrightarrow{\partial} H_k(X, A) \oplus H_k(A) \xrightarrow{\alpha_k} H_k(Y) \xrightarrow{\varrho_*} H^{\varrho}_k(Y) \to \cdots .$$

Then, the restriction $\tilde{\alpha}_k : H_k(X, A) \to H_k(Y)$ maps b to β, and

(1) *α_{n-1} is monic if $H^{\varrho}_n(Y) = 0$;*

(2) *$\tilde{\alpha}_{n-2}$ is monic if X is connected and $H_{n-1}(Y) = 0$;*

(3) *if $\lfloor (n+1)/2 \rfloor \leqslant k < n-2$, then α_k is monic if X and A are connected and if $H_i(Y) = 0$ for all $k+1 \leqslant i \leqslant n-1$.*

We can now prove an analogue to their Corollary 1.5.C.

Corollary 3.10. *The set $\{\tilde{C}(o_i) \mid 1 \leqslant i \leqslant \ell\} \cup \{\tilde{D}(J)\}$ has rank at least ℓ (where ℓ is the number of ovals $o_1, \dots, o_\ell$ of the curve). If the family has rank $\ell + 1$, then the curve is type I.*

Proof. We can apply Lemma 3.9 in our setting, where $\nu = \Theta : Y \to (\overline{\mathbb{CP}}^2, \mathcal{X}(F))$ and $h = 2$. We then see that

$$\tilde{\alpha}_2 : H_2(\overline{\mathbb{CP}}^2, \mathcal{X}(F); \mathbb{Z}/2) \to H_2(Y; \mathbb{Z}/2)$$

is injective, because $\overline{\mathbb{CP}}^2$ is connected and $H_3(Y;\mathbb{Z}/2)=0$ (Proposition 3.4). Noting that $\tilde{\alpha}_2(C_-(o)) = \tilde{C}(o)$ and $\tilde{\alpha}_2(D_-(J)) = \tilde{D}(J)$, the claim follows from the very same arguments as in [Viro and Zvonilov 1992, §2.4]. □

Recall that $\ell^\pm$ and ℓ^0 denote the number of ovals of the curve that bound from the outside a component of $\mathbb{RP}^2 \smallsetminus \mathbb{R}F$ with positive/negative or zero Euler characteristic, respectively. The previous results finally wraps up to yield our main theorem.

Theorem 3.11. *Let F be a Q-flexible curve of odd degree m. Then*

$$\ell^0 + \ell^- \leqslant \frac{(m-1)^2}{4}.$$

If equality holds, then the curve is type I.

Proof. Take the maximal subset $\mathcal{P}$ of $\{\tilde{C}(o_i) \mid 1 \leqslant i \leqslant \ell\} \cup \{\tilde{D}(J)\}$ that spans a subspace of $H_2^+(Y)$, and let $r = \operatorname{rank}(\mathcal{P})$. Then, we obtain $r \leqslant b_2^+(Y)$. Moreover, because of $Q_Y(\tilde{C}(o), \tilde{C}(o)) = -4\chi(o)$ and similarly for $\tilde{D}(J)$, observe that $\mathcal{P}$ has exactly $\ell^0 + \ell^- + 1$ elements (assuming that there is at least one oval to have $\tilde{D}(J) \in \mathcal{P}$; if there are none, the theorem is vacuous). Therefore, because of Corollary 3.10, we deduce $r \geqslant \#\mathcal{P} - 1 = \ell^0 + \ell^-$. This produces

$$\ell^0 + \ell^- \leqslant b_2^+(Y),$$

which is the claimed inequality. The extremal case also follows from an almost word-for-word proof as in [Viro and Zvonilov 1992]. □

4. Curves on a quadric

We investigate our method for flexible curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$, with either of its antiholomorphic involutions $c_1(x, y) = (\bar{x}, \bar{y})$ or $c_2(x, y) = (\bar{y}, \bar{x})$. This is motivated by recent work from Zvonilov [2022], which generalizes [Viro and Zvonilov 1992] to flexible curves on almost-complex 4-manifolds. For a survey of results regarding curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$, we refer the reader to [Matsuoka 1991] or [Gilmer 1991]. We will also need the following result.

Theorem 4.1 [Letizia 1984, §3]. *There are diffeomorphisms $\mathbb{CP}^1 \times \mathbb{CP}^1/c_1 \cong \mathbb{S}^4$ and $\mathbb{CP}^1 \times \mathbb{CP}^1/c_2 \cong \overline{\mathbb{CP}}^2$.*

More precisely, the differential structure on $\mathbb{CP}^1 \times \mathbb{CP}^1 \smallsetminus \operatorname{Fix}(c_i)/c_i$ extends to the standard one on $\mathbb{S}^4$ or $\overline{\mathbb{CP}}^2$, respectively.

Note that in the present work, we do not make any assumption regarding $\gcd(a, b)$ with $[F] = (a, b) \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1)$, contrary to [Zvonilov 2022] where there is no result if $\gcd(a, b) = 1$.

Curves on a hyperboloid. Consider the space $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ with its involution $c_1 : ([x_0 : x_1], [y_0 : y_1]) \mapsto ([\bar{x_0} : \bar{x_1}], [\bar{y_0} : \bar{y_1}])$. We call (X, c_1) the *hyperboloid*. Let $\mathfrak{R} = \mathrm{Fix}(c_1) = \mathbb{RP}^1 \times \mathbb{RP}^1$. We consider $\mathfrak{Q}$ to be a generic real algebraic curve of bidegree (2, 2) and with empty real part $\mathbb{R}\mathfrak{Q} = \varnothing \subset \mathfrak{R}$. We will prove the following result.

Theorem 4.5. *Let F be a $\mathfrak{Q}$-flexible curve in the hyperboloid with bidegree (a, b) where both a and b are odd. Let $\ell^{\pm}$ and ℓ^0 denote the number of ovals of the curve that bound from the outside a subset with positive, negative or zero Euler characteristic, respectively. Then*

$$\ell^- + \ell^0 \leqslant \frac{ab+1}{2}.$$

Note that $H_2(X; \mathbb{Z})$ is a $\mathbb{Z} \oplus \mathbb{Z}$ spanned by the homology classes of algebraic curves of bidegree $(1, 0)$ and $(0, 1)$. We have a notion of a ($\mathfrak{Q}$-)flexible curve in this setting too.

Definition 4.2. Let $F \subset X$ be a closed, connected and oriented surface. We call F a *bidegree (a, b) flexible curve* if the following conditions hold:

(1) $\mathrm{conj}(F) = F$.

(2) $[F] = (a, b)$ in $H_2(X; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

(3) $\chi(F) = 2 - 2(a-1)(b-1)$.

(4) If $\mathbb{R}F = F \cap \mathfrak{R}$, then for all $x \in \mathbb{R}F$, $T_x F = T_x \mathbb{R}F \oplus \boldsymbol{i} \cdot T_x \mathbb{R}F$.

If, additionally, $F \pitchfork \mathfrak{Q}$ is $2(a+b)$ points, then F is said to be *$\mathfrak{Q}$-flexible*.

Note that if both a and b are odd, then $\mathbb{R}F$ is some number of ovals (null-homologous curves in $\mathfrak{R}$), and some nonzero number of parallel copies of a curve with homology class (α, β) in $H_1(\mathfrak{R}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, where $0 \leqslant \alpha \leqslant a$ and $0 \leqslant \beta \leqslant b$ are both odd and coprime, and $\pi_1(\mathfrak{R}) = H_1(\mathfrak{R}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is spanned by the real parts of bidegree $(1, 0)$ and $(0, 1)$ algebraic curves. In the case of an oval o, the complement $\mathfrak{R} \setminus o$ has two connected components, one of which is a disk and is called the *interior* of that oval, and we say that o bounds it *from the outside*.

We observe that $\mathfrak{R}$ is a null-homologous torus, and $\mathfrak{Q}$ is a torus with homology class $(2, 2)$, both in $H_2(X; \mathbb{Z})$. Therefore

$$e(X, \mathfrak{R}) = 0 \quad \text{and} \quad e(X, \mathfrak{Q}) = 8.$$

Denoting as $p : X \to X/c_1 \cong \mathbb{S}^4$ the 2-fold branched cover, we set $\mathscr{R} = p(\mathfrak{R})$ and $\mathscr{Q} = p(\mathfrak{Q})$. Observe that $\mathscr{R}$ is a torus and $\mathscr{Q}$ is a Klein bottle. Finally, letting $\tilde{p} : \overline{X} \to \mathbb{S}^4$ be the 2-fold branched cover of $\mathbb{S}^4$ ramified along $\mathscr{Q}$ (which exists because $[\mathscr{Q}] = 0 \in H_2(\mathbb{S}^4; \mathbb{Z}/2) \cong 0$), we set $\overline{\mathfrak{R}} = \tilde{p}^{-1}(\mathscr{Q})$ and $\overline{\mathfrak{Q}} = \tilde{p}^{-1}(\mathscr{R})$.

$$\begin{array}{ccccc}
\mathfrak{R} & \subset & X & \supset & \mathfrak{Q} \\
\| & & \downarrow{\scriptstyle p} & & \twoheaddownarrow \\
\mathscr{R} & \subset & \mathbb{S}^4 & \supset & \mathscr{Q} \\
\twoheaduparrow & & \uparrow{\scriptstyle \tilde{p}} & & \| \\
\overline{\mathfrak{Q}} & \subset & \overline{X} & \supset & \overline{\mathfrak{R}}
\end{array}$$

Figure 8. The branched covers in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ with its hyperboloid structure, with the same notation conventions as in Figure 1.

Consecutive applications of Lemma 1.4 yield

$$e(\mathbb{S}^4, \mathscr{R}) = 0, \quad e(\mathbb{S}^4, \mathscr{Q}) = 4, \quad e(\overline{X}, \overline{\mathfrak{R}}) = 2 \quad \text{and} \quad e(\overline{X}, \overline{\mathfrak{Q}}) = 0.$$

The situation is depicted in Figure 8.

The topological Riemann–Hurwitz formula gives $\chi(\overline{X}) = 4$, and Theorem 3.5 provides $\sigma(\overline{X}) = -2$. A similar reasoning as in Proposition 3.4 ensures that $H_1(\overline{X}; \mathbb{Z}/2) = 0$, and thus that $H_1(\overline{X}; \mathbb{Z})$ is torsion. In particular,

$$b_1(\overline{X}) = b_3(\overline{X}) = 0 \quad \text{and} \quad b_2(\overline{X}) = -\sigma(\overline{X}) = 2.$$

This suggests that $\overline{X}$ may be diffeomorphic to $\overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2$, but this will not be needed.

Consider a $\mathfrak{Q}$-flexible curve $F \subset X$ of bidegree (a, b), where a and b are both odd. In particular,

$$\chi(F) = -2ab + 2a + 2b \text{ and } e(X, F) = 2ab.$$

Letting $A^+(F) = p(F)$ and $\bar{A}^+(F) = \tilde{p}^{-1}(A^+(F))$, one checks that

$$\chi(\bar{A}^+(F)) = -2ab + a + b \quad \text{and} \quad e(\bar{A}^+(F)) = 2ab.$$

In order to understand $\overline{\mathbb{R}}F = \tilde{p}^{-1}(\partial A^+(F)) \subset \overline{\mathfrak{Q}}$, it is necessary to describe the unbranched 2-fold covering $\tilde{p} : \overline{\mathfrak{Q}} \to \mathscr{R}$, which is a nontrivial 2-fold cover of the torus (nontriviality can be deduced by the same argument as in the proof of the next proposition). There are only three such coverings, each given by the subgroups $2\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z} \oplus 2\mathbb{Z}$ and $G = \{(x, y) \in \mathbb{Z}^2 \mid x + y \equiv 0 \mod 2\}$.

Proposition 4.3. *The covering $\tilde{p} : \overline{\mathfrak{Q}} \to \mathscr{R}$ corresponds to the subgroup G.*

Proof. Assume it corresponds to the subgroup $2\mathbb{Z} \oplus \mathbb{Z}$ (the argument is the same with the other). Let γ be a curve with homology class $(0, 1)$ in $\mathscr{R}$. Its preimage is therefore two parallel copies of it. The situation is depicted in Figure 9.

Now, let C be a generic bidegree $(0, 1)$ algebraic curve, so that $\partial A^+(C) = \gamma$. Then $A^+(C) \pitchfork \mathscr{Q}$ is one point, so that the map $\tilde{p} : \bar{A}^+(C) \to A^+(C)$ is a branched

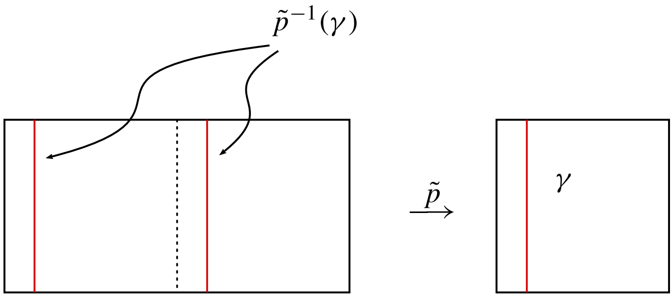


Figure 9. The unbranched 2-fold covering of the torus corresponding to the subgroup $2\mathbb{Z} \oplus \mathbb{Z}$, and its effect on the curve with homology class $(0, 1)$.

covering that restricts to an unbranched covering of the boundary. An application of the Riemann–Hurwitz formula gives $\chi(\bar{A}^+(C)) = 1$, and $\bar{A}^+(C)$ has at most two boundary components. Therefore, there is no other choice but the same situation as depicted in Figure 3. That is, $\bar{A}^+(C)$ is a disk, and $\tilde{p}^{-1}(\gamma) = \partial \bar{A}^+(C)$ is connected. This is excluded, by assumption. The same argument with a bidegree $(1, 0)$ algebraic curve works for the subgroup $\mathbb{Z} \oplus 2\mathbb{Z}$. □

The covering corresponding to the subgroup G is depicted in Figure 10. If a curve $\gamma \subset \mathscr{R}$ has homology class (α, β) with both α and β odd (and coprime), then its preimage is two parallel copies $\tilde{p}^{-1}(\gamma) \subset \overline{\mathfrak{Q}}$.

Recalling that $\mathbb{R}F$ is some ovals and some number of parallel copies of an (α, β) curve with α and β coprime and odd, we have the following immediate facts:

(1) Each copy of the (α, β) curve is doubled (indeed, the homotopy class of that curve belongs to the subgroup G).

(2) Each oval is doubled.

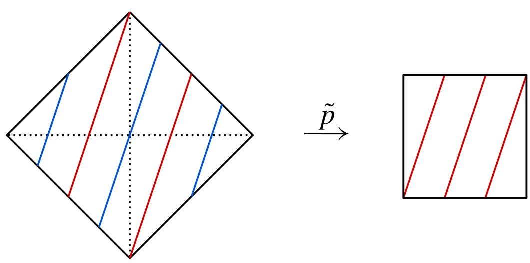


Figure 10. The unbranched covering $\tilde{p} : \overline{\mathfrak{Q}} \to \mathscr{R}$ corresponding to the subgroup $G = \{(x, y) \in \mathbb{Z}^2 \mid x + y \equiv 0 \mod 2\} \subset \mathbb{Z}^2$. On the left, the preimage of the $(3, 1)$ curve is two parallel copies of a $(1, -2)$ curve. It is understood that the two tori are represented by the two squares, whose opposite sides are identified.

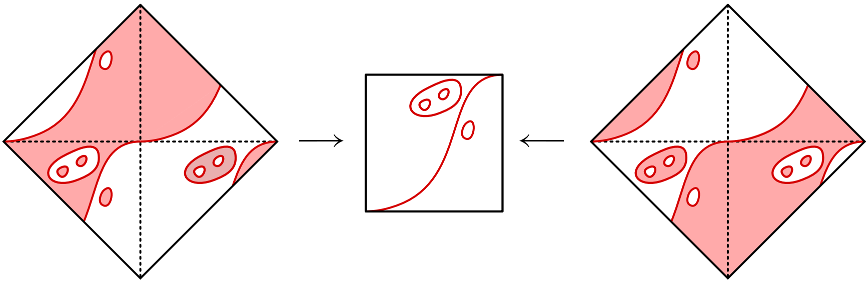


Figure 11. In the middle, a curve with real scheme $\langle(1,1), 1\sqcup 1\langle 2\rangle\rangle$. On the left and on the right, the two possible choices $\overline{\mathfrak{Q}}_\pm$.

(3) The preimage respects mutual position of ovals, as in Proposition 2.3 (that is, an oval inside another lifts to two copies inside the other two copies).

Hence, we see that $\overline{\mathfrak{Q}} \setminus \overline{\mathbb{R}}F$ has two diffeomorphic subsets $\overline{\mathfrak{Q}}_\pm$ with the property $\partial\overline{\mathfrak{Q}}_\pm = \overline{\mathbb{R}}F$ (we provide an example in Figure 11). We therefore have

$$\chi(\overline{\mathfrak{Q}}_+) = \chi(\overline{\mathfrak{Q}}_-) \quad \text{and} \quad \chi(\overline{\mathfrak{Q}}) = \chi(\overline{\mathfrak{Q}}_+) + \chi(\overline{\mathfrak{Q}}_-),$$

so that $\chi(\overline{\mathfrak{Q}}_\pm) = 0$. The same argument gives $e(\overline{X}, \overline{\mathfrak{Q}}_\pm) = 0$.

Therefore, we can define the *Arnold surface* of the curve as $\mathcal{A}(F) = \overline{A}^+(F) \cup \overline{\mathfrak{Q}}_+$. Note that $\mathcal{A}(F) \pitchfork \overline{\mathfrak{R}}$ is $a+b$ points (coming from the $2(a+b)$ points in $F \pitchfork \mathfrak{Q}$). We consider the immersed surface $\mathcal{A}(F) \cup \overline{\mathfrak{R}}$, and we let $\mathcal{X}(F)$ be the smoothing of its singularities, as provided by Lemma 3.3, where we choose the smoothing that satisfies

$$e(\overline{X}, \mathcal{X}(F)) = e(\overline{X}, \mathcal{A}(F) \cup \overline{\mathfrak{R}}) + 2(a+b).$$

Proposition 4.4. *Let $F \subset X$ be a $\mathfrak{Q}$-flexible curve of bidegree (a, b) with both a and b odd. The surface $\mathcal{X}(F)$ has zero homology in $H_2(\overline{X}; \mathbb{Z}/2)$ and satisfies*

$$\chi(\mathcal{X}(F)) = -2ab - a - b \quad \textit{and} \quad e(\overline{X}, \mathcal{X}(F)) = 2ab + 2a + 2b + 2.$$

Proof. Computing $\chi(\mathcal{X}(F))$ and $e(\overline{X}, \mathcal{X}(F))$ is straightforward. To prove that $[\mathcal{X}(F)] = 0 \in H_2(\overline{X}; \mathbb{Z}/2)$, it suffices to show that $\mathcal{A}(F)$ and $\overline{\mathfrak{R}}$ are homologous mod 2. Note that by the previous computations, $b_2(\overline{X}) = -\sigma(\overline{X}) = 2$, so that $\overline{X}$ is a negative definite smooth 4-manifold. By virtue of Donaldson's theorem, this means that the intersection form of $\overline{X}$ is, up to a change of basis, that of $\overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2}$. We consider a basis of $H_2(\overline{X}, \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ that diagonalizes this intersection form, and we will show that $\mathcal{A}(F)$ and $\overline{\mathfrak{R}}$ both realize the homology class $(1, 1)$ in $H_2(\overline{X}; \mathbb{Z}/2)$.

Because $e(\overline{X}, \mathcal{A}(F))$ and $e(\overline{X}, \overline{\mathfrak{R}})$ are both even, this rules out the two classes $(1,0)$ and $(0,1)$. As such, it suffices to show that $\mathcal{A}(F)$ and $\overline{\mathfrak{R}}$ are both not null-homologous in $H_2(\overline{X}; \mathbb{Z}/2)$ (if $\mathcal{A}(F)$ was null-homologous, we could directly take the 2-fold covering of $\overline{X}$ ramified along $\mathcal{A}(F)$, without adding $\overline{\mathfrak{R}}$).

By Theorem 5.5, we have the congruences

$$e(\overline{X}, \mathcal{A}(F)) + 2\chi(\mathcal{A}(F)) \equiv q([\mathcal{A}(F)]) \mod 4,$$
$$e(\overline{X}, \overline{\mathfrak{R}}) + 2\chi(\overline{\mathfrak{R}}) \equiv q([\overline{\mathfrak{R}}]) \mod 4,$$

for some quadratic function $q : H_2(\overline{X}; \mathbb{Z}/2) \to \mathbb{Z}/4$. From the previous computations, this yields

$$2ab \equiv q([\mathcal{A}(F)]) \mod 4 \qquad \text{and} \qquad 2 \equiv q([\overline{\mathfrak{R}}]) \mod 4.$$

Since a and b are both odd, this means that $2ab \not\equiv 0 \mod 4$. As such, we cannot have $q([\mathcal{A}(F)]) = 0$ and $q([\overline{\mathfrak{R}}]) = 0$, thus implying $[\mathcal{A}(F)] \neq 0$ and $[\overline{\mathfrak{R}}] \neq 0$. □

Let Y denote the 2-fold covering of $\overline{X}$ ramified along $\mathcal{X}(F)$. By Proposition 3.4, $H_1(Y)$ is torsion, and computations of $\chi(Y) = 2ab + a + b + 8$ and $\sigma(Y) = -ab - a - b - 5$ yield

$$b_2^+(Y) = \frac{ab+1}{2}.$$

This implies the following result.

Theorem 4.5. *Let F be a $\mathfrak{Q}$-flexible curve in the hyperboloid with bidegree (a, b) where both a and b are odd. Let $\ell^{\pm}$ and ℓ^0 denote the number of ovals of the curve that bound from the outside a subset with positive, negative or zero Euler characteristic, respectively. Then*

$$\ell^- + \ell^0 \leqslant \frac{ab+1}{2}.$$

Proof. We denote as $\Theta : Y \to \overline{X}$ the double branched cover of $(\overline{X}, \mathcal{X}(F))$. For any oval $o \subset \mathbb{R}F$, $\mathfrak{R} \smallsetminus \mathbb{R}F$ has exactly two path-connected components that have o as a part of their boundary. One is a punctured disc, and the other is a punctured torus. We denote as $C(o) \subset \mathscr{R}$ the image under $p : X \to \mathbb{S}^4$ of the punctured disc component, and as $C_{\pm}(o) \subset \overline{\mathfrak{Q}}_{\pm}$ the preimages under $\tilde{p} : \overline{X} \to \mathbb{S}^4$ of $C(o)$. We set $\tilde{C}(o) = \Theta^{-1}(C_-(o))$. For an analogue of $\tilde{D}(J)$, there is a subtlety. Indeed, in $\mathbb{R}F$, there may be several parallel copies of an (α, β)-curve in $H_1(\mathfrak{R}; \mathbb{Z})$, where α and β are both odd and coprime. Each of these curves will lift in $\overline{\mathfrak{Q}}$ to two copies. If $\mathbb{R}F$ contains ovals, then it is possible to choose one connected component D_- of $\overline{\mathfrak{Q}} \smallsetminus \overline{\mathbb{R}F}$ that has one of those curves as a boundary component, and at least one oval as another boundary component, and which is included in $\overline{\mathfrak{Q}}_-$. Define $\tilde{D} = \Theta^{-1}(D_-)$. By computations analogous to the $\mathbb{CP}^2$ case, we have the following:

(1) If $o \subset \mathbb{R}F$ is an oval, then $e(Y, \tilde{C}(o)) = -4\chi(o)$, where $\chi(o) = \chi(C_-(o))$. In particular, $e(Y, \tilde{C}(o)) \leqslant 0$ if and only if o is a nonempty oval.

(2) $Q_Y(\tilde{D}, \tilde{D}) \leqslant 0$.

(3) If $o \neq o'$ are two distinct ovals, then $Q_Y(\tilde{C}(o), \tilde{C}(o')) = 0$ and $Q_Y(\tilde{C}(o), \tilde{D}) = 0$.

We can now apply Lemma 3.9 to the family composed of the collection of the $\tilde{C}(o)$ and of $\tilde{D}$. □

Curves on an ellipsoid. We now consider the other antiholomorphic involution $c_2 : ([x_0 : x_1], [y_0 : y_1]) \mapsto ([\bar{y_0} : \bar{y_1}], [\bar{x_0} : \bar{x_1}])$ on $X = \mathbb{CP}^1 \times \mathbb{CP}^1$. This time, we have

$$\mathfrak{R} = \mathrm{Fix}(c_2) = \{(x, \bar{x}) \mid x \in \mathbb{CP}^1\} \cong \mathbb{S}^2,$$

and $X/c_2 \cong \overline{\mathbb{CP}}^2$. Algebraic curves in (X, c_2) necessarily have a bidegree of the form (m, m) for some $m \geqslant 1$. Consider a purely imaginary bidegree $(2, 2)$ algebraic curve $\mathfrak{Q}$, and define flexible curves and $\mathfrak{Q}$-flexible curves as before. Note that we still keep the same basis for $H_2(X; \mathbb{Z})$ as in the case of the hyperboloid.

Theorem 4.6. *Let F be a bidegree (m, m) $\mathfrak{Q}$-flexible curve on the ellipsoid, with m odd. Let $\ell^{\pm}$ and ℓ^0 denote the number of connected components of $\mathfrak{R} \setminus \mathbb{R}F$ with positive, negative or zero Euler characteristic. Then*

$$\ell^0 + \ell^- \leqslant \frac{m^2+1}{2}.$$

We have

$$e(X, \mathfrak{R}) = -2 \quad \text{and} \quad e(X, \mathfrak{Q}) = 8,$$

because $[\mathfrak{R}] = (\pm 1, \mp 1) \in H_2(X; \mathbb{Z})$ (depending on a choice of orientation) and $[\mathfrak{Q}] = (2, 2)$. Denoting the branched cover as $p : X \to \overline{\mathbb{CP}}^2$, we see that, letting $\mathscr{R} = p(\mathfrak{R})$ and $\mathscr{Q} = p(\mathfrak{Q})$,

$$e(\overline{\mathbb{CP}}^2, \mathscr{R}) = -4 \quad \text{and} \quad e(\overline{\mathbb{CP}}^2, \mathscr{Q}) = 4.$$

In particular, $\mathscr{Q}$ is a null-homologous Klein bottle in $H_2(\overline{\mathbb{CP}}^2; \mathbb{Z}/2)$, because it has even normal Euler number. This means that there is a well-defined 2-fold branched cover $\tilde{p} : \overline{X} \to \overline{\mathbb{CP}}^2$ ramified along $\mathscr{Q}$. We let $\overline{\mathfrak{R}} = \tilde{p}^{-1}(\mathscr{Q})$ and $\overline{\mathfrak{Q}} = \tilde{p}^{-1}(\mathscr{R})$, so that

$$e(\overline{X}, \overline{\mathfrak{R}}) = 2 \quad \text{and} \quad e(\overline{X}, \overline{\mathfrak{Q}}) = -8.$$

A direct computation provides

$$\chi(\overline{X}) = 6 \quad \text{and} \quad \sigma(\overline{X}) = -4,$$

with $H_1(\overline{X})$ torsion. This is evidence to think that $\overline{X} \cong 4\overline{\mathbb{CP}}^2$. What will be useful is knowing that $\overline{X}$ is negative definite, and so has intersection form $-I_4$ by Donaldson's theorem.

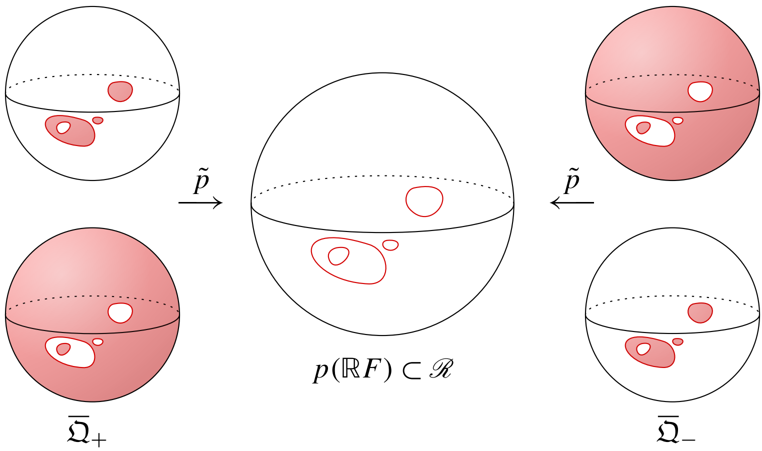


Figure 12. The two possible subsets $\overline{\mathfrak{Q}}_\pm$, shaded. It is understood that the two spheres in the first row are Q_1, and the two in the second are Q_2.

This time, the restriction $\tilde{p} : \overline{\mathfrak{Q}} \to \mathscr{R}$ is a two-fold covering of the 2-sphere, and is necessarily trivial. We set $\overline{\mathfrak{Q}} = Q_1 \sqcup Q_2$. Let $\tau : \overline{X} \to \overline{X}$ be the involution of $\overline{X}$ spanning $\mathrm{Aut}(\tilde{p})$. Denote as R_1 and R_2 the two subsets of $\mathscr{R} \setminus p(\mathbb{R}F)$ with $\partial R_i = p(\mathbb{R}F)$, and define

$$\overline{\mathfrak{Q}}_+ = Q_1 \cap \tilde{p}^{-1}(R_1) \sqcup Q_2 \cap \tau(\tilde{p}^{-1}(R_1)),$$
$$\overline{\mathfrak{Q}}_- = Q_2 \cap \tilde{p}^{-1}(R_1) \sqcup Q_1 \cap \tau(\tilde{p}^{-1}(R_1)).$$

We refer to Figure 12 for a representation. Of course, this definition depends on the choices of the labeling Q_i of the two components of $\overline{\mathfrak{Q}}$, as well as the choice of the labeling of the R_i. But ultimately, the inequality we obtain will not depend on these choices.

This allows for a definition of $\mathcal{A}(F)$ such that $e(\overline{\mathfrak{Q}}_+) = \frac{1}{2}e(\overline{\mathfrak{Q}})$ and $\chi(\overline{\mathfrak{Q}}_+) = \frac{1}{2}\chi(\overline{\mathfrak{Q}})$. We obtain

$$\chi(\mathcal{A}(F)) = -2m^2 + 2m + 2 \quad \text{and} \quad e(\mathcal{A}(F)) = 2m^2 - 4.$$

Another key difference from the cases of $\mathbb{CP}^2$ and of the hyperboloid is that the second homology $H_2(\overline{X}; \mathbb{Z}/2)$ now has rank four (the intersection form of $\overline{X}$ is $-I_4$). To show that $\mathcal{A}(F)$ and $\overline{\mathfrak{R}}$ are homologous mod 2 and not null-homologous, we need to eliminate more cases. We consider a basis of $H_2(\overline{X}; \mathbb{Z})$ that diagonalizes the intersection form of $\overline{X}$. It also descends to a basis of $H_2(\overline{X}; \mathbb{Z}/2)$. If $(a, b, c, d) \in H_2(\overline{X}; \mathbb{Z}/2)$ denotes the homology class of $\mathcal{A}(F)$ or $\overline{\mathfrak{R}}$, with $a, b, c, d \in \{0, 1\}$, then the fact that $e(\overline{X}, \mathcal{A}(F))$ and $e(\overline{X}, \overline{\mathfrak{R}})$ are even implies that $a+b+c+d \equiv 0 \mod 2$. There are 8 remaining cases: $(0, 0, 0, 0)$, $(1, 1, 1, 1)$, and the six cases of the type

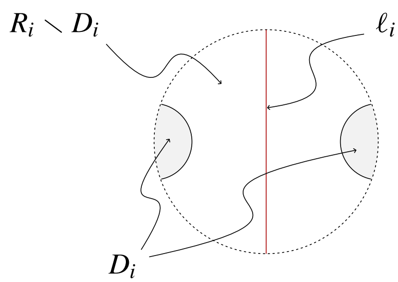


Figure 13. The core of a Möbius strip can be seen as a real line in the associated real projective plane.

(1, 0, 1, 0) with two nonzero coefficients. Theorem 5.5 rules out the zero homology class, as well as the (1, 1, 1, 1) one. Let $\mathcal{X}(F)$ be $\mathcal{A}(F)\cup\overline{\mathfrak{R}}$ with all $2m$ singularities removed accordingly to Lemma 3.3. This gives

$$\chi(\mathcal{X}(F)) = -2m^2 - 2m + 2 \quad \text{and} \quad e(\overline{X}, \mathcal{X}(F)) = 2m^2 + 4m - 2.$$

One last application of Theorem 5.5 provides $q([\mathcal{X}(F)]) \equiv 0 \mod 4$. In particular, if, without loss of generality, we have $[\overline{\mathfrak{R}}] = (1, 1, 0, 0)$, then this means that there are only two choices:

$$[\mathcal{A}(F)] = (1, 1, 0, 0) \text{ or } (0, 0, 1, 1).$$

That is, either $\mathcal{X}(F)$ is null-homologous, in which case $[\overline{\mathfrak{R}}] = [\mathcal{A}(F)]$, or it is a characteristic surface if $[\overline{\mathfrak{R}}] \neq [\mathcal{A}(F)]$. Assuming that $\mathcal{X}(F)$ is characteristic, the Guillou–Marin congruence (Theorem 5.10) applies and gives

$$\beta(\overline{X}, \mathcal{X}(F)) \equiv -(m+1)^2 \equiv 0 \text{ or } 4 \mod 8,$$

by inspection of the squares of odd integers mod 8. Because the surface $\mathcal{X}(F)$ has high genus, this method will a priori not yield any contradiction.

Proposition 4.7. *The surface $\mathcal{A}(F)$ is homologous to $\overline{\mathfrak{R}}$ mod 2.*

Proof. We start by describing the generators of the homology $H_2(\overline{X}; \mathbb{Z}/2)$. Consider a complex line $\mathbb{CP}^1 \subset \overline{\mathbb{CP}}^2$ such that $\mathbb{CP}^1 \cap \mathcal{Q} = \varnothing$. This means that $\mathbb{CP}^1$ lifts to two spheres S_1 and S_2 in $\overline{X}$, each with $e(\overline{X}, S_i) = -1$. Moreover, because they are disjoint, we have $Q_{\overline{X},\mathbb{Z}/2}(S_1, S_2) \equiv 0 \mod 2$, i.e., they are linearly independent. There are two more generators that come from the following construction.

$\mathcal{Q}$ is a Klein bottle, which can be seen as the desingularization of two real projective planes R_1 and R_2, with $R_1 \pitchfork R_2 = \{*\}$ and $e(\overline{\mathbb{CP}}^2, R_i) = 1$. By Lemma 3.3, we see that this is possible from the computation $e(\overline{\mathbb{CP}}^2, \mathcal{Q}) = 4$. Let $x \in R_1 \pitchfork R_2$ be the transverse intersection, and let $D_i \subset R_i$ be a small disc centered at x. $R_i \setminus D_i$ is a Möbius strip, whose core ℓ_i can be seen as a real projective line in R_i (see Figure 13).

This real line separates a complex line L_i into two components $L_i^{\pm}$ with $\partial L_i^{\pm} = \ell_i$. Let $\Sigma_i = \tilde{p}^{-1}(L_i^+)$. From $L_i^+ \pitchfork \mathscr{Q} = \ell_i = \partial L_i^+$, we see that Σ_i is a sphere in $\overline{X}$ with $e(\overline{X}, \Sigma_i) = -1$. Moreover, $Q_{\mathbb{Z}/2}(\Sigma_1, \Sigma_2) \equiv 0 \mod 2$ because $\Sigma_1 \cap \Sigma_2 = \varnothing$. Finally, because $L_i^{\pm} \cap \mathbb{CP}^1 \subset \overline{\mathbb{CP}}^2 \setminus \mathscr{Q}$, we have $\Sigma_i \pitchfork S_J$ is an even number of points, and thus $Q_{\mathbb{Z}/2}(\Sigma_i, S_j) \equiv 0 \mod 2$. As such, $(S_1, S_2, \Sigma_1, \Sigma_2)$ is a basis for the homology $H_2(\overline{X}; \mathbb{Z}/2)$. From [Nagami 2000, Lemma 3.4], the surface $\tilde{p}^{-1}(\mathbb{CP}^1) \cup \overline{\mathfrak{R}}$ is mod 2 characteristic in $\overline{X}$, and as such, we have $[\overline{\mathfrak{R}}]_{\mathbb{Z}/2} = [S_1] + [S_2]$. In order to show that $\mathcal{A}(F)$ and $\overline{\mathfrak{R}}$ are homologous mod 2, it suffices to prove that $Q_{\overline{X},\mathbb{Z}/2}(\mathcal{A}(F), \Sigma_i) \equiv 0 \mod 2$ for $i = 1, 2$. Equivalently, we need to show that $\mathcal{A}(F) \pitchfork \Sigma_i$ is an *even* number of points for $i = 1, 2$. Intersection points in $\mathcal{A}(F) \pitchfork \Sigma_i$ come in two types:

(1) intersections between $\tilde{p}^{-1}(F^+)$ and Σ_i; this number equals that of intersection points between F^+ and $\mathbb{CP}^1$, which is itself even because $e(\overline{\mathbb{CP}}^2, F^+) = 2m^2$;

(2) intersections between $\overline{\mathfrak{Q}}_+$ and Σ_i, itself also equal to $\#\mathscr{R} \pitchfork \mathbb{CP}^1$, which is even because $e(\overline{\mathbb{CP}}^2, \mathscr{R}) = -4$. □

To prove Theorem 4.6, we apply the same method as before. Given a connected component $U \subset \mathfrak{R} \setminus \mathbb{R}F$ (or equivalently, $U \subset \mathscr{R} \setminus p(\mathbb{R}F)$), the lift $\tilde{p}^{-1}(U)$ is two disjoint copies of U. We let $C_{\pm}(U)$ denote those copies, with the condition that $C_{\pm}(U) \subset \overline{\mathfrak{Q}}_{\pm}$. As before, let $\tilde{C}(U) = \Theta^{-1}(C_-(U))$, with $\Theta : Y \to \overline{X}$ the double branched cover of $\overline{X}$ ramified over $\mathcal{X}(F)$. We have

$$e(Y, \tilde{C}(U)) = -4\chi(U).$$

If Q_Y is the intersection form of Y, and if U and V are two distinct components of $\mathfrak{R} \setminus \mathbb{R}F$, then there are two possibilities:

(1) $U \cap V = \varnothing$, in which case $C_-(U) \cap C_-(V) = \varnothing$, and thus $Q_Y(\tilde{C}(U), \tilde{C}(V)) = 0$.

(2) $U \cap V$ is a component of $\mathbb{R}F$, in which case $C_-(U) \subset Q_i$ and $C_-(V) \subset Q_j$, with $\{i, j\} = \{1, 2\}$. In particular, we still have $C_-(U) \cap C_-(V) = \varnothing$.

Finally, one applies the same arguments as before to the family $\{\tilde{C}(U)\}_{\chi(U) \leqslant 0}$ to obtain the claimed bound.

5. Further comments

Other ways to resolve the singularities. In order to take the 2-fold branched cover, we added $\overline{\mathbb{RP}}^2$ to $\mathcal{A}(F)$. This led us to resolve the m singularities that arose. As suggested by Zvonilov in a personal communication, one could be tempted to use blow-ups and see what effect this has. But in order to ensure that the new surface $\mathcal{X}(F)$ is still connected, we cannot blow-up all m singularities. Doing this procedure to $m - 1$ of those, and gluing a Hopf band for the last as we did previously, leads to the very same bound. That is, the 4-manifold Y which is the double branched cover of $(m\overline{\mathbb{CP}}^2, \mathcal{X}(F))$ still has $b_2^+(Y) = (m-1)^2/4$.

Comparisons of our inequality. Given a prime number p and an integer $m \in \mathbb{N}^\star$, we denote as $\nu_p(m) = \max\{n \in \mathbb{N} \mid p^n \text{ divides } m\}$ the p-adic valuation of m. Define the function $h : \mathbb{N}^\star \to \mathbb{N}$ by

$$h(m) = \max_{p \text{ prime}} p^{\nu_p(m)}.$$

That is, $h(m)$ is the largest prime power that divides m. Viro and Zvonilov's inequality, which holds for flexible curves, is

$$\ell^0 + \ell^- \leqslant \frac{(m-3)^2}{4} + \frac{m^2 - h(m)^2}{4h(m)^2}.$$

We denote as $VZ(m)$ and $S(m)$ the bounds obtained by Viro and Zvonilov and ours, respectively. That is,

$$VZ(m) = \frac{(m-3)^2}{4} + \frac{m^2 - h(m)^2}{4h(m)^2} \quad \text{and} \quad S(m) = \frac{(m-1)^2}{4}.$$

For infinitely many degrees m, one has $S(m) < VZ(m)$. But in infinitely many others (e.g., when m is a prime power), the converse holds. However, both are far from sharp estimates that can be obtained from considerations for *algebraic* curves that come from Bézout theorem computations. That is, there are degrees m for which $VZ(m)$ and $S(m)$ are both not realized as upper bounds for $\ell^0 + \ell^-$. For instance, Zvonilov [1979] has the sharper estimate, valid for pseudoholomorphic curves,

$$\ell^0 + \ell^- \leqslant \frac{(m-1)(m-3)}{4}.$$

If one starts with Viro and Zvonilov's inequality in the case where $m+2$ is a prime power and the curve is Q-flexible of degree m, then one can perturb its union with the conic Q into a nonsingular degree $m+2$ flexible curve (which will have the same real set, for $\mathbb{R}Q = \varnothing$), and obtain

$$\ell^0 + \ell^- \leqslant \frac{(m-1)^2}{4} + \frac{(m+2)^2 - h(m+2)^2}{4h(m+2)^2} = \frac{(m-1)^2}{4}.$$

That is, one can derive our Theorem 3.11 from Viro and Zvonilov's when $m+2$ is a prime power. On a side note, if the famous twin prime conjecture happens to be true, this means that there are infinitely many degrees m for which $VZ(m) < S(m)$ and the bound $S(m)$ is a corollary of their bound.

By some easy number-theoretic considerations, one can show that there are infinitely many odd degrees m such that neither of m and $m+2$ are prime powers, and for which $S(m) < VZ(m)$. Indeed, the difference of the upper bounds in both inequalities is

$$VZ(m) - S(m) = \frac{1}{4}\Big(\Big[\frac{m}{h(m)}\Big]^2 - 4m + 7\Big).$$

With $m_p = 1287 \times 429^{12p+1}$, one has $5 \mid m_p+2$ and $7 \mid p+2$, and $h(m_p) \in o(m_p^{19/40})$. In particular, the difference diverges to $+\infty$ on the degrees m_p.

The same conclusion can be derived when comparing the inequalities of Theorems 4.5 and 4.6 with Zvonilov's work [2022]. It turns out that, for a curve of bidegree (a, b) with a and b coprime, Zvonilov has no possibility to take a cyclic covering, and there is no inequality in those cases.

Nonorientable flexible curves. There is a new object that could be interesting to study: nonorientable flexible curve. The motivation comes from the observation that in the operation of taking double branched covers, orientability of the ramification locus is disregarded. This is not the case for other cyclic branched covers (and the methods from [Viro and Zvonilov 1992] cannot apply to nonorientable surfaces). We propose the following nonorientable analogue of Definition 1.1.

Definition 5.1. Let $F \subset \mathbb{CP}^2$ be a closed, connected and nonorientable surface. We call F a *nonorientable degree m and genus h flexible curve* if the following conditions hold:

(i) $\chi(F) = 2 - h$.

(ii) $\mathrm{conj}(F) = F$.

(iii) $e(\mathbb{CP}^2, F) = m^2$.

(iv) For any $x \in \mathbb{R}F = F \cap \mathbb{RP}^2$, we have $T_x F = T_x \mathbb{R}F \oplus \boldsymbol{i} \cdot T_x \mathbb{R}F$.

What plays the role of asking that the integral homology class of F is m times a generator $[\mathbb{CP}^1]$ in $H_2(\mathbb{CP}^2; \mathbb{Z})$ is the condition $e(\mathbb{CP}^2, F) = m^2$. In the traditional orientable case, we also had the condition that $\chi(F) = -m^2 + 3m$. This was a requirement of extremality in the genus bound proved by Kronheimer and Mrowka.

Theorem 5.2 (Thom conjecture, [Kronheimer and Mrowka 1994]). *Let $F \subset \mathbb{CP}^2$ be a smoothly embedded oriented and connected surface with $[F] = m\,[\mathbb{CP}^1]$. Then $\chi(F) \leqslant -m^2 + 3m$.*

One could ask whether the implication

$$e(\mathbb{CP}^2, F) = m^2 \implies \chi(F) \leq -m^2 + 3m$$

holds for closed, connected, nonorientable surfaces F smoothly embedded in $\mathbb{CP}^2$. In fact, self-intersection numbers of nonorientable surfaces need not be squares. Given any $m \in \mathbb{Z}$, set $\Sigma(m)$ to be the collection of all smoothly embedded, connected and nonorientable surfaces $F \subset \mathbb{CP}^2$ with $e(\mathbb{CP}^2, F) = m$. We can define the following *nonorientable genus function* of $\mathbb{CP}^2$:

$$\tilde{g} : \mathbb{Z} \to \mathbb{Z}_{\leqslant 1}, \qquad m \mapsto \max_{F \in \Sigma(m)} \chi(F).$$

We will later prove the following result.

Theorem 5.3. *Here, $k \in \mathbb{N}^\star$ denotes a nonnegative integer.*

(1) *We have $\tilde{g}(0) = 0$.*

(2) *Let $\ell \in \{0, 1\}$ have the same parity as k. Then, on negative integers, we have*

$$\tilde{g}(-k) = 2 - \frac{k+\ell}{2}.$$

(3) *On even positive integers, we have*

$$\tilde{g}(4k) = 4 - 2k \quad (\textit{for } k \geqslant 2) \qquad \textit{and} \qquad \tilde{g}(4k+2) = 3 - 2k.$$

We also have the special values $\tilde{g}(2) = 1$ and $\tilde{g}(4) = 0$.

(4) *On odd positive integers, we have the bounds*

$$\tilde{g}(4k+1) \geqslant 2 - 2k \quad \textit{and} \quad \tilde{g}(4k+3) \geqslant 1 - 2k.$$

We also have the special values $\tilde{g}(1) = 0$, $\tilde{g}(3) = 1$, $\tilde{g}(5) = 0$, $\tilde{g}(7) = -1$ and $\tilde{g}(9) = -2$.

In the previous theorem, one can now look at the values of $\tilde{g}(m^2)$. We obtain

$$\begin{cases} \tilde{g}(m^2) = \frac{8-m^2}{2} & \text{if } m \text{ is even,} \\ \tilde{g}(m^2) \geqslant \frac{5-m^2}{2} & \text{if } m \text{ is odd.} \end{cases}$$

In particular, we see that the nonorientable analogue $\tilde{g}(m^2) \leqslant -m^2 + 3m$ of Theorem 5.2 has the quadratic term off by 50%. Nonorientable flexible curves still share some properties with traditional flexible curves. More precisely, we show the following.

Proposition 5.4. *Let $F \subset \mathbb{CP}^2$ be a nonorientable flexible curve of degree m. Then*

(1) *$\chi(F)$ is an even integer;*

(2) *$\mathbb{R}F$ realizes the nontrivial homology class in $H_1(\mathbb{RP}^2; \mathbb{Z})$ if and only if m is odd, and it has exactly one pseudoline in this case;*

(3) *F satisfies the Harnack bound: $b_0(\mathbb{R}F) \leqslant 3 - \chi(F)$.*

Proof. The first claim is a consequence of the following result, which is a generalization of the well-known Whitney congruence.

Theorem 5.5 [Yamada 1995, Theorems 1.2 and 1.4]. *Let X be a closed, connected, oriented 4-manifold.*

(1) *If $H_1(X; \mathbb{Z}) = 0$, define $q : H_2(X; \mathbb{Z}/2) \to \mathbb{Z}/4$ by setting, for $\xi \in H_2(X; \mathbb{Z}/2)$, $q(\xi) = Q_X(\tilde{\xi}, \tilde{\xi}) \mod 4$, where $\tilde{\xi} \in H_2(X; \mathbb{Z})$ is any integral lift of ξ. Then, for any embedded, closed, connected (not necessarily orientable) surface $F \subset X$,*

$$e(X, F) + 2\chi(F) \equiv q([F]) \mod 4.$$

(2) *Without the assumption that* $H_1(X;\mathbb{Z})=0$, *the map*

$$q : H_2(X;\mathbb{Z}/2) \to \mathbb{Z}/4$$

defined by $q([F]) = e(X, F) + 2\chi(F)$ *is a well-defined* $\mathbb{Z}/4$*-quadratic map.*

Indeed, if $e(\mathbb{CP}^2, F) = m^2$, then one inspects two cases, depending on the parity of m. If m is even, then $[F] = 0 \in H_2(\mathbb{CP}^2;\mathbb{Z}/2)$, and if m is odd, then $[F]$ is the generator of $H_2(\mathbb{CP}^2;\mathbb{Z}/2)$. Both cases yield $\chi(F) \equiv 0 \mod 2$.

For the other claims, the classical proofs for flexible curves, found, for instance, in Viro's lecture notes, work word for word. □

We call a nonorientable flexible curve of degree m *Q-flexible* if, as before, the intersection $F \pitchfork Q$ is $2m$ points. Then, we have the following result.

Theorem 5.6. *Let* $F \subset \mathbb{CP}^2$ *be a nonorientable Q-flexible of odd degree* m. *Then*

$$\ell^0 + \ell^- \leqslant -\frac{\chi(F)}{2} - \frac{m^2-1}{4} + m.$$

Proof. The only difference with traditional flexible curves is that one needs to do all the computations in terms of $\chi(F)$. Indeed, starting at the level of $\mathbb{S}^4$, the surface F/conj needs not be orientable anymore. One checks that, for the Arnold surface, we have

$$\chi(\mathcal{A}(F)) = \chi(F) - m + 1 \quad \text{and} \quad e(\overline{\mathbb{CP}}^2, \mathcal{A}(F)) = m^2 - 2,$$

and for the smoothing $\mathcal{X}(F)$, we obtain

$$\chi(\mathcal{X}(F)) = \chi(F) - 3m + 2 \quad \text{and} \quad e(\overline{\mathbb{CP}}^2, \mathcal{X}(F)) = m^2 + 2m - 1.$$

This gives, denoting as Y the double branched cover of $(\overline{\mathbb{CP}}^2, \mathcal{X}(F))$,

$$b_2^+(Y) = -\frac{\chi(F)}{2} - \frac{m^2-1}{4} + m.$$

Note that $\chi(F) \leqslant 0$ is necessarily even, as seen in Proposition 5.4. □

In regards to Theorem 5.3, we conjecture the following.

Conjecture 5.7. *The lower bounds for* $\tilde{g}$ *over nonnegative odd integers are equalities.*

If this holds, then one may add to the definition of a nonorientable flexible curve F of degree m that they must satisfy the extremal bound $\chi(F) = \tilde{g}(m)$. In this case, Theorem 5.6 becomes

$$\ell^0 + \ell^- \leqslant m - 1,$$

whereas the Harnack bound gives

$$b_0(\mathbb{R}F) \leqslant \frac{m^2+1}{2}.$$

This is to be compared to

$$b_0(\mathbb{R}F) \leqslant \frac{m^2-3m+4}{2} \sim \frac{m^2}{2} \quad \text{and} \quad \ell^0+\ell^- \leqslant \frac{m^2-2m+1}{4} \sim \frac{m^2}{4}$$

for traditional Q-flexible curves.

Proof of Theorem 5.3. The two steps of the proof are to

(1) obtain upper bounds for $\chi(F)$ given $e(\mathbb{CP}^2, F) = m$, and

(2) construct a surface realizing that upper bound.

To this end, we will use the following.

Theorem 5.8 [Levine et al. 2015, Theorem 10.1]. *Let X be a closed, connected, oriented, positive definite 4-manifold with $H_1(X;\mathbb{Z}) = 0$, and let $F \subset X$ be a closed, connected, nonorientable surface with nonorientable genus $h(F) = 2-\chi(F)$. Denote as $\ell(F)$ the minimal self-intersection of an integral lift of $[F] \in H_2(X;\mathbb{Z}/2)$. Then*

$$e(X, F) \geq \ell(F) - 2h(F).$$

This allows us to obtain the upper bounds

$$\tilde{g}(-k) \leqslant 2 - \frac{k+\ell}{2}$$

for $k \in \mathbb{N}^\star$ and $\ell \in \{0, 1\}$ having the same parity as k. Indeed, if $F \subset \mathbb{CP}^2$ has $e(\mathbb{CP}^2, F) = -k$, then $\ell(F) = \ell$, because $[F] \neq 0 \in H_2(\mathbb{CP}^2;\mathbb{Z}/2)$ if and only if $-k$ is odd, in which case a complex line is an integral lift of F with minimal self-intersection.

Another method (which worked for the orientable Thom conjecture in degree 4, for instance) is to make use of homological information of the double branched cover of $\mathbb{CP}^2$ ramified along F. More precisely, we have the following.

Proposition 5.9. *Let $F \subset \mathbb{CP}^2$ be a closed, connected surface such that $[F] = 0 \in H_2(\mathbb{CP}^2;\mathbb{Z}/2)$ (or equivalently, such that $e(\mathbb{CP}^2, F)$ is even). Then*

$$\chi(F) \leqslant 4 - \frac{e(\mathbb{CP}^2, F)}{2}.$$

Proof. Let Y denote the double branched cover of $\mathbb{CP}^2$ ramified over F. We compute

$$\chi(Y) = 6 - \chi(F) \quad \text{and} \quad \sigma(Y) = 2 - \frac{e(\mathbb{CP}^2, F)}{2}.$$

Moreover, by reasoning analogous to the proof of Proposition 3.4, we have $b_1(Y) = b_3(Y) = 0$, so that $b_2(Y) = 4 - \chi(F)$. If one considers any orientable surface $\Sigma \subset \mathbb{CP}^2$ which is not null-homologous in $H_2(\mathbb{CP}^2;\mathbb{Z})$, and which is transverse to F, we see that $e(\mathbb{CP}^2, \Sigma) > 0$, and Σ lifts in Y to a surface $\tilde{\Sigma}$ with $e(Y, \tilde{\Sigma}) = 2e(\mathbb{CP}^2, \Sigma) > 0$. This implies that

$$b_2^+(Y) = \frac{b_2(Y)+\sigma(Y)}{2} \geqslant 1,$$

yielding the inequality that was claimed. □

Note that unless $e(\mathbb{CP}^2, F) \geqslant 8$, the previous result only gives trivialities, since $\chi(F) \leqslant 1$ for a nonorientable surface. This is enough to obtain the upper bounds

$$\tilde{g}(4k) \leqslant 4 - 2k \quad \text{and} \quad \tilde{g}(4k+2) \leqslant 3 - 2k$$

for $k \in \mathbb{N}^\star$. Note that if $k = 1$, then the bound $\tilde{g}(4) \leqslant 2$ is vacuous.

What remains to do is

(1) compute the special values of $\tilde{g}$ at 0, 1, 2, 3, 4, 5, 7 and 9;
(2) construct surfaces $F \subset \mathbb{CP}^2$ that realize the upper bounds for $\tilde{g}(-k)$, $\tilde{g}(4k)$ and $\tilde{g}(4k+2)$;
(3) construct surfaces $F \subset \mathbb{CP}^2$ to derive lower bounds for $\tilde{g}(4k+1)$ and $\tilde{g}(4k+3)$.

To obtain upper bounds for $\tilde{g}$ on odd integers $\leqslant 9$, we will need the following.

Theorem 5.10 [Guillou and Marin 1980]. *Let $F \subset X$ be a mod 2 characteristic surface in a closed, connected, oriented 4-manifold with $H_1(X; \mathbb{Z}) = 0$. Then*

$$\sigma(X) - e(X, F) \equiv 2\beta(X, F) \mod 16,$$

where $\beta(X, F)$ is the Brown invariant of the embedding.

We shall recall what $\beta(X, F)$ is. The Guillou–Marin form

$$\varphi : H_1(F; \mathbb{Z}/2) \to \mathbb{Z}/4$$

is defined as follows. Because $H_1(X; \mathbb{Z}) = 0$, any immersed circle $\mathscr{C} \looparrowright F$ bounds an immersed orientable surface $\mathscr{D} \looparrowright X$. Isotope $\mathscr{D}$ (relatively to its boundary) such that it is transverse to F. The normal bundle $\nu\mathscr{D}$ of $\mathscr{D}$ in X is trivial, and as such, so is $\nu\mathscr{D}|_{\mathscr{C}}$. Considering the normal bundle $\nu\mathscr{C}$ of $\mathscr{C}$ in F as a subbundle $\nu\mathscr{C} < \nu\mathscr{D}|_{\mathscr{C}}$, count the number $n(\mathscr{D})$ of right-handed half-twists with respect to the fixed trivialization $\nu\mathscr{D}|_{\mathscr{C}} \cong \mathscr{C} \times \mathbb{R}^2$. Define

$$(*) \qquad \varphi(\mathscr{C}) = n(\mathscr{D}) + 2\mathscr{D} \cdot F + 2e(F, \mathscr{C}) \mod 4,$$

where $\mathscr{D} \cdot F$ is the number of transverse intersection points $F \pitchfork \mathscr{D}$ taken mod 2. Then this definition does not depend on any of the choices made, and $\varphi(\mathscr{C})$ depends only on the homology class $[\mathscr{C}] \in H_1(F; \mathbb{Z}/2)$. This defines a quadratic map

$$\varphi : H_1(F; \mathbb{Z}/2) \to \mathbb{Z}/4,$$

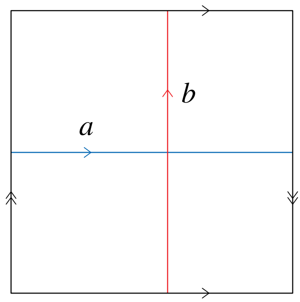


Figure 14. The standard basis for the first homology of the Klein bottle.

to which one can regard its *Brown invariant*

$$\beta(X, F) \overset{\text{def.}}{=} \left(\frac{1}{\sqrt{2}}\right)^{b_1(F;\mathbb{Z}/2)} \sum_{x \in H_1(F;\mathbb{Z}/2)} \sqrt{-1}^{\varphi(x)}. \qquad (**)$$

For instance, we can compute the possible values of $\beta(\mathbb{CP}^2, K)$ where $K \subset \mathbb{CP}^2$ is a Klein bottle. We refer to Figure 14 for a choice of generators a and b of $H_1(K; \mathbb{Z}) = \{0, a, b, a+b\}$.

From $\varphi(a+b) = \varphi(a) + \varphi(b) + 2a \cdot b = \varphi(a) + \varphi(b) + 2$, it suffices to compute $\varphi(a)$ and $\varphi(b)$. One checks that $\varphi(a) \in \{1, 3\}$ and $\varphi(b) \in \{0, 2\}$, by computing each term in $(*)$. Therefore, it suffices to inspect each of the four cases individually, and plug the values in $(**)$ to obtain

$$\beta(\mathbb{CP}^2, K) \in \{0, 2, 6\}.$$

Proposition 5.11. *We have* $\tilde{g}(1) \leqslant 0$, $\tilde{g}(3) \leqslant 1$, $\tilde{g}(5) \leqslant 0$, $\tilde{g}(7) \leqslant -1$ *and* $\tilde{g}(9) \leqslant -2$.

Proof. Note that Theorem 5.5 gives

$$e(\mathbb{CP}^2, F) \in \{1, 5, 9\} \implies \chi(F) \text{ is even},$$
$$e(\mathbb{CP}^2, F) \in \{3, 7\} \implies \chi(F) \text{ is odd}.$$

In particular, from $\chi(F) \leqslant 1$ always, we see that the only nontrivial bounds are for $\tilde{g}(7)$ and $\tilde{g}(9)$.

(1) Assume that $F \subset \mathbb{CP}^2$ has $e(\mathbb{CP}^2, F) = 7$ and $\chi(F) = 1$ (that is, F is diffeomorphic to a projective plane). Then, the Guillou–Marin congruence (Theorem 5.10) gives

$$1 - 7 \equiv 2\beta(\mathbb{CP}^2, F) \mod 16.$$

A simple calculation as before for the possible values of $\beta(\mathbb{CP}^2, K)$ gives that, in the case of the projective plane, $\beta(\mathbb{CP}^2, \mathbb{RP}^2) = 1$, a contradiction.

(2) If F now has $e(\mathbb{CP}^2, F) = 9$ and $\chi(F) = 0$ (that is, F is a Klein bottle), we can use the Guillou–Marin congruence again to derive

$$\beta(\mathbb{CP}^2, F) \equiv 4 \mod 8.$$

The previous calculations gave $\beta(\mathbb{CP}^2, K) \in \{0, 2, 6\}$, which is a contradiction. □

Now, the only things that remain to do are to construct surfaces that realize the upper bounds obtained thus far, and compute upper bounds for $\tilde{g}$ in the special cases not covered yet. For constructions of surfaces, we will make use of *local surfaces*. Recall the so-called Whitney–Massey theorem.

Theorem 5.12 [Massey 1969]. *Let $F \subset \mathbb{S}^4$ be a closed, connected, nonorientable surface. Then*

$$e(\mathbb{S}^4, F) \in \{2\chi(F) - 4, 2\chi(F), \dots, 4 - 2\chi(F)\}.$$

All tuples (e, χ) satisfying this condition are realizable by a closed, connected, nonorientable surface.

A surface F embedded in the 4-ball $\mathbb{B}^4$ that realizes an admissible tuple (e, χ) will be called a *local surface*. Note that the previous theorem ensures that those always exist for any admissible tuple.

Proposition 5.13. *All upper bounds obtained for $\tilde{g}$ so far are sharp. We have*

$$\tilde{g}(4k+1) \geqslant 2 - 2k \quad \textit{and} \quad \tilde{g}(4k+3) \geqslant 1 - 2k.$$

Proof. Fix $k \in \mathbb{N}^\star$ a nonnegative integer.

(1) If $k = 2p$ is even, consider a local surface $F \subset \mathbb{B}^4 \subset \mathbb{CP}^2$ of genus p and self-intersection $-2p$. Then $\chi(F) = 2 - p = 2 - k/2$ and $e(\mathbb{CP}^2, F) = -k$. If now $k = 2p+1$ is odd, choose $F \subset \mathbb{B}^4$ a local surface of genus $p+1$ and self-intersection $-2(p+1)$. Embed the 4-ball $\mathbb{B}^4$ in $\mathbb{CP}^2$ away from a fixed complex line $L \subset \mathbb{CP}^2$, and consider the surface $F\#L$ obtained by connecting F and L with a small tube. Then, by noting that L is a 2-sphere with self-intersection $+1$, we have

$$\chi(F\#L) = \chi(F) = 2 - (p+1) = 2 - \frac{k+1}{2}$$

and

$$e(\mathbb{CP}^2, F\#L) = -2(p+1) + 1 = -k.$$

In both cases, this implies that

$$\tilde{g}(-k) \geqslant 2 - \frac{k+\ell}{2}.$$

(2) Assume that $k \geqslant 2$. Let $F \subset \mathbb{B}^4$ be a local surface of genus $2(k-1)$ and self-intersection $4(k-1)$. Embed the 4-ball away from the conic Q, and consider the surface $F\#Q$ (again, by connecting them with a small tube). We compute, using that Q is a 2-sphere with self-intersection $+4$,

$$\chi(F\#Q) = 4-2k \quad \text{and} \quad e(\mathbb{CP}^2, F\#Q) = 4k,$$

yielding the lower bound $\tilde{g}(4k) \geqslant 4-2k$. Note that in the case $k=1$, this works, but gives an orientable surface (the 2-sphere Q).

(3) Let $F \subset \mathbb{B}^4$ be a local surface of genus $2k-1$ and self-intersection $4k-2$. Tubing with the conic Q gives

$$\chi(F\#Q) = 3-2k \quad \text{and} \quad e(\mathbb{CP}^2, F\#Q) = 4k+2,$$

which provides us with $\tilde{g}(4k+2) \geqslant 3-2k$.

(4) One can do the same to derive the bounds

$$\tilde{g}(4k+1) \geqslant 2-2k \quad \text{and} \quad \tilde{g}(4k+3) \geqslant 1-2k.$$

Indeed, taking F to be a local surface of genus $2k$ (resp. $2k+1$) and self-intersection $4k$ (resp. $4k+2$), this can be embedded away from a complex line L. Looking at the surface $F\#L$ gives the lower bounds. The special cases for $\tilde{g}(5), \dots, \tilde{g}(9)$ are covered by this construction. For $\tilde{g}(1)$, it suffices to consider a local Klein bottle with self-intersection 0 and tubing it with a complex line, and for $\tilde{g}(3)$, the conic Q can be tubed to $\mathbb{RP}^2 = \mathrm{Fix}(\mathrm{conj})$. □

To conclude the proof of Theorem 5.3, there only remains to compute three special values that have not been covered yet:

(1) $\tilde{g}(0) = 0$, because the Euler characteristic must be even, and a local Klein bottle with zero self-intersection gives a lower bound.
(2) $\tilde{g}(2) = 1$, since a local projective plane with self-intersection $+2$ works.
(3) $\tilde{g}(4) = 0$, as the Euler characteristic must be even, and a local Klein bottle with self-intersection $+4$ suffices.

Acknowledgements

The author is very grateful to T. Fiedler for all the insight he gave and for all the discussions. The author would also like to thank M. Golla for his very nice explanations, and D. Moussard for all the precious time she gave. The author is also deeply thankful to the anonymous referee for helpful comments and suggestions. This work received support from the University Research School EUR-MINT (state support managed by the National Research Agency for Future Investments program bearing the reference ANR-18-EURE-0023).

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Received May 15, 2023. Revised February 15, 2024.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 328 No. 1 January 2024
