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HANKEL OPERATORS ON $L^p(\mathbb{R}_+)$ AND THEIR p-COMPLETELY BOUNDED MULTIPLIERS

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We show that for any $1 < p < \infty$, the space $\mathrm{Hank}_p(\mathbb{R}_+) \subseteq B(L^p(\mathbb{R}_+))$ of all Hankel operators on $L^p(\mathbb{R}_+)$ is equal to the w^*-closure of the linear span of the operators $\theta_u : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ defined by $\theta_u f = f(u-\cdot)$, for $u > 0$. We deduce that $\mathrm{Hank}_p(\mathbb{R}_+)$ is the dual space of $A_p(\mathbb{R}_+)$, a half-line analogue of the Figà-Talamanca–Herz algebra $A_p(\mathbb{R})$. Then we show that a function $m : \mathbb{R}_+^* \to \mathbb{C}$ is the symbol of a p-completely bounded multiplier $\mathrm{Hank}_p(\mathbb{R}_+) \to \mathrm{Hank}_p(\mathbb{R}_+)$ if and only if there exist $\alpha \in L^\infty(\mathbb{R}_+; L^p(\Omega))$ and $\beta \in L^\infty(\mathbb{R}_+; L^{p'}(\Omega))$ such that $m(s+t) = \langle \alpha(s), \beta(t)\rangle$ for a.e. $(s,t) \in \mathbb{R}_+^{*2}$. We also give analogues of these results in the (easier) discrete case.

1. Introduction

For any $u > 0$ and for any function $f : \mathbb{R}_+ \to \mathbb{C}$, let $\tau_u f : \mathbb{R}_+ \to \mathbb{C}$ be the shifted function defined by $\tau_u f = f(\cdot - u)$. Let $1 < p, p' < \infty$ be two conjugate indices. We say that a bounded operator $T : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ is Hankelian if $\langle T\tau_u f, g\rangle = \langle Tf, \tau_u g\rangle$ for all $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$. Let $B(L^p(\mathbb{R}_+))$ denote the Banach space of all bounded operators on $L^p(\mathbb{R}_+)$. The main object of this paper is the subspace $\mathrm{Hank}_p(\mathbb{R}_+) \subseteq B(L^p(\mathbb{R}_+))$ of all Hankel operators on $L^p(\mathbb{R}_+)$.

The case $p = 2$ has received a lot of attention; see [Nikolski 2002; 2020; Peller 2003; Yafaev 2015; 2017a; 2017b]. The most important result in this case is that $\mathrm{Hank}_2(\mathbb{R}_+)$ is isometrically isomorphic to the quotient space $L^\infty(\mathbb{R})/H^\infty(\mathbb{R})$, where $H^\infty(\mathbb{R}) \subset L^\infty(\mathbb{R})$ is the classical Hardy space of essentially bounded functions whose Fourier transform has support in $\mathbb{R}_+$ (see [Nikolski 2020, Section IV.5.3] or [Peller 2003, Theorem I.8.1]). This result is the real line analogue of Nehari's classical theorem describing Hankel operators on ℓ^2 (see [Nikolski 2020, Theorem II.2.2.4], [Peller 2003, Theorem I.1.1] or [Power 1982, Theorem 1.3]). An equivalent formulation of the above result is that

$$\mathrm{Hank}_2(\mathbb{R}_+) \simeq H^1(\mathbb{R})^*, \tag{1}$$

where $H^1(\mathbb{R}) \subseteq L^1(\mathbb{R})$ is the Hardy space of all integrable functions whose Fourier transform vanishes on $\mathbb{R}_-$.

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The first main result of this paper is that for any $1 < p < \infty$, the Banach space $\text{Hank}_p(\mathbb{R}_+)$ coincides with $\overline{\text{Span}}^{w^*}\{\theta_u : u > 0\} \subset B(L^p(\mathbb{R}_+))$, where, for any $u > 0$, $\theta_u : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ is the Hankel operator defined by $\theta_u f = f(u-\cdot)$. As a consequence, we show that

$$\text{Hank}_p(\mathbb{R}_+) \simeq A_p(\mathbb{R}_+)^*, \tag{2}$$

where $A_p(\mathbb{R}_+)$ is a half-line analogue of the Figà-Talamanca–Herz algebra $A_p(\mathbb{R})$ (see, e.g., [Derighetti 2011, Chapter 3]). We will see in Remark 4.2(a) that $A_2(\mathbb{R}_+) \simeq H^1(\mathbb{R})$. Thus, the duality result (2), established in Theorem 4.1, is an L^p-version of (1).

By a multiplier of $\text{Hank}_p(\mathbb{R}_+)$, we mean a w^*-continuous operator

$$T : \text{Hank}_p(\mathbb{R}_+) \to \text{Hank}_p(\mathbb{R}_+)$$

such that $T(\theta_u) = m(u)\theta_u$ for all $u > 0$, for some function $m : \mathbb{R}_+^* \to \mathbb{C}$. In this case, we set $T = T_m$ and it turns out that m is necessarily bounded and continuous, see Lemma 4.5. The second main result of this paper is a characterization of p-completely bounded multipliers T_m. We refer to Section 2 for some background on p-complete boundedness, whose definition goes back to [Pisier 1990] (see also [Daws 2010; Le Merdy 1996; Pisier 2001]). We prove in Theorem 4.6 that $T_m : \text{Hank}_p(\mathbb{R}_+) \to \text{Hank}_p(\mathbb{R}_+)$ is a p-completely bounded multiplier if and only if there exist a measure space (Ω, μ) and two essentially bounded measurable functions $\alpha : \mathbb{R}_+ \to L^p(\Omega)$ and $\beta : \mathbb{R}_+ \to L^{p'}(\Omega)$ such that $m(s+t) = \langle \alpha(s), \beta(t) \rangle$ for almost every $(s, t) \in \mathbb{R}_+^{*2}$. This is a generalisation of [Arnold et al. 2022, Theorem 3.1]. Indeed, the result in [Arnold et al. 2022] provides a characterization of S^1-bounded multipliers on $H^1(\mathbb{R})$. Using (1), this yields a characterization of completely bounded multipliers on $\text{Hank}_2(\mathbb{R}_+)$, which is nothing but the case $p = 2$ of Theorem 4.6. See Remark 4.7 for more on this.

Let us briefly explain the plan of the paper. Section 2 contains some preliminary results. Section 3 is devoted to $\text{Hank}_p(\mathbb{N}) \subset B(\ell^p)$, the space of Hankel operators on ℓ^p. We establish analogues of the aforementioned results in the discrete setting. Results for $\text{Hank}_p(\mathbb{N})$ are easier than those concerning $\text{Hank}_p(\mathbb{R}_+)$ and Section 3 can be considered as a warm up. The main results are stated and proved in Section 4.

2. Preliminaries

All our Banach spaces are complex ones. For any Banach spaces X, Z, we let $B(X, Z)$ denote the Banach space of all bounded operators from X into Z and we write $B(X)$ instead of $B(X, X)$ when $Z = X$. For any $x \in X$ and $x^* \in X^*$, the duality action $x^*(x)$ is denoted by $\langle x^*, x \rangle_{X^*,X}$, or simply by $\langle x^*, x \rangle$ if there is no risk of confusion.

We start with duality on tensor products. Let X, Y be Banach spaces. Let $X \widehat{\otimes} Y$ denote their projective tensor product [Diestel and Uhl 1977, Section VIII.1]. We will use the classical isometric identification

$$(X \widehat{\otimes} Y)^* \simeq B(X, Y^*) \tag{3}$$

provided, e.g., by [Diestel and Uhl 1977, Corollary VIII.2.2]. More precisely, for any $\xi \in (X \widehat{\otimes} Y)^*$, there exists a necessarily unique $R_\xi \in B(X, Y^*)$ such that $\xi(x \otimes y) = \langle R_\xi(x), y\rangle$ for all $x \in X$ and $y \in Y$. Moreover $\|R_\xi\| = \|\xi\|$ and the mapping $\xi \mapsto R_\xi$ is onto.

Lemma 2.1. *Let $A \subset X$ and $B \subset Y$ such that* $\operatorname{Span}\{A\}$ *is dense in X and* $\operatorname{Span}\{B\}$ *is dense in Y. Assume that $(R_\iota)_\iota$ is a bounded net of $B(X, Y^*)$. Then R_ι converges to some $R \in B(X, Y^*)$ in the w^*-topology if and only if $\langle R_\iota(x), y\rangle \to \langle R(x), y\rangle$ for all $x \in A$ and $y \in B$.*

Proof. Assume the latter property. Since the algebraic tensor product $X \otimes Y$ is dense in $X \widehat{\otimes} Y$, it implies that $\langle R_\iota, z\rangle \to \langle R, z\rangle$, for all z belonging to a dense subspace of $X \widehat{\otimes} Y$. Next, the boundedness of $(R_\iota)_\iota$ implies that $\langle R_\iota, z\rangle \to \langle R, z\rangle$, for all z belonging to $X \widehat{\otimes} Y$. The equivalence follows. □

We will use the above duality principles in the case when $X = Y^*$ is an L^p-space $L^p(\Omega)$, for some index $1 < p < \infty$.

We now give a brief background on p-completely bounded maps, following [Pisier 1990] (see also [Daws 2010; Le Merdy 1996; Pisier 2001]). Let $1 < p < \infty$ and let SQ_p denote the collection of quotients of subspaces of L^p-spaces, where we identify spaces which are isometrically isomorphic. Let E be an SQ_p-space. Let $n \geq 1$ be an integer and let $[T_{ij}]_{1\leq i,j\leq n} \in M_n \otimes B(E)$ be an $n \times n$ matrix with entries T_{ij} in $B(E)$. We equip $M_n \otimes B(E)$ with the norm defined by

$$\|[T_{ij}]\| = \sup\Big\{\Big(\sum_{i=1}^n \Big\|\sum_{j=1}^n T_{ij}(x_j)\Big\|^p\Big)^{\frac{1}{p}} : x_1, \ldots, x_n \in E,\ \sum_{i=1}^n \|x_i\|^p \leq 1\Big\}. \tag{4}$$

If $S \subset B(E)$ is any subspace, then we let $M_n(S)$ denote $M_n \otimes S$ equipped with the induced norm.

Let S_1 and S_2 be subspaces of $B(E_1)$ and $B(E_2)$, respectively, for some SQ_p-spaces E_1 and E_2. Let $w : S_1 \to S_2$ be a linear map. For any $n \geq 1$, let $w_n : M_n(S_1) \to M_n(S_2)$ be defined by $w_n([T_{ij}]) = [w(T_{ij})]$, for any $[T_{ij}]_{1\leq i,j\leq n} \in M_n(S_1)$. By definition, w is called p-completely bounded if the maps w_n are uniformly bounded. In this case, the *p-cb* norm of w is defined by $\|w\|_{p-cb} = \sup_n \|w_n\|$. We further say that w is p-completely contractive if $\|w\|_{p-cb} \leq 1$ and that w is a p-complete isometry if w_n is an isometry for all $n \geq 1$. Note that the case $p = 2$ corresponds to the classical notion of completely bounded maps (see, e.g., [Paulsen 2002; Pisier 2001]).

We recall the following factorisation theorem of Pisier (see [Le Merdy 1996, Theorem 1.4; Pisier 1990, Theorem 2.1]), which extends Wittstock's factorisation theorem [Paulsen 2002, Theorem 8.4].

Theorem 2.2. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be measure spaces and let $1 < p < \infty$. Let $S \subseteq B(L^p(\Omega_1))$ be a unital subalgebra. Let $w : S \to B(L^p(\Omega_2))$ be a linear map and let $C \geq 0$ be a constant. The following assertions are equivalent.*

(i) *The map w is p-completely bounded and $\|w\|_{p-cb} \leq C$.*

(ii) *There exist an SQ_p-space E, a unital, nondegenerate p-completely contractive homomorphism $\pi : S \to B(E)$ as well as operators $V : L^p(\Omega_2) \to E$ and $W : E \to L^p(\Omega_2)$ such that $\|V\|\|W\| \leq C$ and for any $x \in S$, $w(x) = W\pi(x)V$.*

Remark 2.3. Let $1 < p < \infty$ and let p' be its conjugate index. Let E be an SQ_p-space. Then by assumption, there exist a measure space (Ω, μ) and two closed subspaces $E_2 \subseteq E_1 \subseteq L^p(\Omega)$ such that $E = E_1/E_2$. Then $E_1^\perp \subseteq E_2^\perp \subseteq L^{p'}(\Omega)$ and we have an isometric identification

$$E^* \simeq \frac{E_2^\perp}{E_1^\perp}, \tag{5}$$

by the classical duality between subspaces and quotients of Banach spaces. More explicitly, let $f \in E_1$ and let $g \in E_2^\perp$. Let $\dot{f} \in E$ denote the class of f modulo E_2 and let $\dot{g} \in E^*$ denote the element associated to the class of g modulo $E_1^\perp$ through the identification (5). Then we have

$$\langle \dot{g}, \dot{f} \rangle_{E^*, E} = \langle g, f \rangle_{L^{p'}, L^p}. \tag{6}$$

We now turn to Bochner spaces. Let (Σ, ν) be a measure space and let X be a Banach space. For any $1 \leq p \leq \infty$, we let $L^p(\Sigma; X)$ denote the space of all measurable functions $\phi : \Sigma \to X$ (defined up to almost everywhere zero functions) such that the norm function $t \mapsto \|\phi(t)\|$ belongs to $L^p(\Sigma)$. This is a Banach space for the norm $\|\phi\|_p$, defined as the $L^p(\Sigma)$-norm of $\|\phi(\,\cdot\,)\|$ (see, e.g., [Diestel and Uhl 1977, Chapters I and II]).

Assume that p is finite and note that in this case, $L^p(\Sigma) \otimes X$ is dense in $L^p(\Sigma; X)$. Let p' be the conjugate index of p. For all $\phi \in L^p(\Sigma; X)$ and $\psi \in L^{p'}(\Sigma; X^*)$, the function $t \mapsto \langle \psi(t), \phi(t) \rangle_{X^*, X}$ belongs to $L^1(\Sigma)$ and the resulting duality paring $\langle \psi, \phi \rangle := \int_\Omega \langle \psi(t), \phi(t) \rangle \, d\nu(t)$ extends to an isometric embedding $L^{p'}(\Sigma; X^*) \hookrightarrow L^p(\Sigma; X)^*$. Furthermore, this embedding is onto if X is reflexive, that is,

$$L^{p'}(\Sigma; X^*) \simeq L^p(\Sigma; X)^* \quad \text{if } X \text{ is reflexive.} \tag{7}$$

We refer to [Diestel and Uhl 1977, Corollary III.2.13 and Section IV.1] for these results and complements.

Let (Σ, ν) and (Ω, μ) be two measure spaces. Then we have an isometric identification

$$L^p(\Sigma; L^p(\Omega)) \simeq L^p(\Sigma \times \Omega),$$

from which it follows that for any $T \in B(L^p(\Sigma))$, the tensor extension

$$T \otimes I_{L^p(\Omega)} : L^p(\Sigma) \otimes L^p(\Omega) \to L^p(\Sigma) \otimes L^p(\Omega)$$

extends to a bounded operator $T \overline{\otimes} I_{L^p(\Omega)}$ on $L^p(\Sigma \times \Omega)$, whose norm is equal to the norm of T. The following is elementary.

Lemma 2.4. *The mapping $\pi : B(L^p(\Sigma)) \to B(L^p(\Sigma \times \Omega))$ defined by $\pi(T) = T \overline{\otimes} I_{L^p(\Omega)}$ is a p-complete isometry.*

Proof. Let $n \geq 1$ and let $J_n = \{1, \dots, n\}$. It follows from (4) that $M_n(B(L^p(\Sigma))) = B(\ell_n^p(L^p(\Sigma)))$ and hence $M_n(B(L^p(\Sigma))) = B(L^p(J_n \times \Sigma))$ isometrically. Likewise, we have $M_n(B(L^p(\Sigma \times \Omega))) = B(L^p(J_n \times \Sigma \times \Omega))$ isometrically. Through these identifications,

$$[T_{ij} \overline{\otimes} I_{L^p(\Omega)}] = [T_{ij}] \overline{\otimes} I_{L^p(\Omega)},$$

for all $[T_{ij}]_{1 \leq i,j \leq n}$ in $M_n(B(L^p(\Sigma)))$. The result follows at once. □

We finally state an important result concerning Schur products on $B(\ell_I^p)$-spaces. Let I be an index set, let $1 < p < \infty$ and let ℓ_I^p denote the discrete L^p-space over I. Let $(e_t)_{t \in I}$ be its canonical basis. To any $T \in B(\ell_I^p)$, we associate a matrix of complex numbers, $[a_{st}]_{s,t \in I}$, defined by $a_{st} = \langle T(e_t), e_s \rangle$, for all $s, t \in I$. Following [Pisier 2001, Chapter 5], we say that a bounded family $\{\varphi(s,t)\}_{(s,t) \in I^2}$ of complex numbers is a bounded Schur multiplier on $B(\ell_I^p)$ if for all $T \in B(\ell_I^p)$, with matrix $[a_{st}]_{s,t \in I}$, the matrix $[\varphi(s,t)a_{st}]_{s,t \in I}$ represents an element of $B(\ell_I^p)$. In this case, the mapping $[a_{st}] \to [\varphi(s,t)a_{st}]$ is a bounded operator from $B(\ell_I^p)$ into itself. We note that $\{\varphi(s,t)\}_{(s,t) \in I^2}$ is a bounded Schur multiplier with norm $\leq C$ if and only if for all $n \geq 1$, all $[a_{ij}]_{1 \leq i,j \leq n}$ in M_n and all $t_1, \dots, t_n, s_1, \dots, s_n$ in I, we have

$$\|[\varphi(s_i, t_j)a_{ij}]\|_{B(\ell_n^p)} \leq C \|[a_{ij}]\|_{B(\ell_n^p)}. \tag{8}$$

In the sequel, we apply the above definitions to the case when $I = \mathbb{R}_+^*$.

Theorem 2.5. *Let $\varphi : \mathbb{R}_+^{*2} \to \mathbb{C}$ be a continuous bounded function. Let $1 < p, p' < \infty$ be conjugate indices and let $C \geq 0$ be a constant. The following assertions are equivalent.*

(i) *The family $\{\varphi(s,t)\}_{(s,t) \in \mathbb{R}_+^{*2}}$ is a bounded Schur multiplier on $B(\ell_{\mathbb{R}_+^*}^p)$, with norm $\leq C$.*

(ii) *There exist a measure space (Ω, μ) and two functions $\alpha \in L^\infty(\mathbb{R}_+; L^p(\Omega))$ and $\beta \in L^\infty(\mathbb{R}_+; L^{p'}(\Omega))$ such that $\|\alpha\|_\infty \|\beta\|_\infty \leq C$ and $\varphi(s,t) = \langle \alpha(s), \beta(t) \rangle_{L^p, L^{p'}}$ for almost every $(s,t) \in \mathbb{R}_+^{*2}$.*

Proof. According to [Coine 2018, Section 4.1], (ii) is equivalent to the fact that as an element of $L^\infty(\mathbb{R}_+^2)$,

(ii') φ is a bounded Schur multiplier on $B(L^p(\mathbb{R}_+))$.

It further follows from [Herz 1974, Lemmas 1 and 2] that since φ is continuous, (ii') is equivalent to (i). The result follows. □

3. Hankel operators on ℓ^p and their multipliers

In this section we work on the sequence spaces $\ell^p = \ell^p_{\mathbb{N}}$, where $\mathbb{N} = \{0, 1, \dots\}$. For any $1 < p < \infty$, we let $(e_n)_{\geq 0}$ denote the classical basis of ℓ^p. For any $T \in B(\ell^p)$, the associated matrix $[t_{ij}]_{i,j\geq 0}$ is given by $t_{ij} = \langle T(e_j), e_i\rangle$, for all $i, j \geq 0$.

Let $\mathrm{Hank}_p(\mathbb{N}) \subseteq B(\ell^p)$ be the subspace of all $T \in B(\ell^p)$ whose matrix is Hankelian, i.e., has the form $[c_{i+j}]_{i,j\geq 0}$ for some sequence $(c_k)_{k\geq 0}$ of complex numbers.

Let p' be the conjugate index of p and regard $\ell^p \otimes \ell^{p'} \subset B(\ell^p)$ in the usual way. We set

$$\gamma_k = \sum_{i+j=k} e_i \otimes e_j$$

for any $k \geq 0$. Then each γ_k belongs to $\mathrm{Hank}_p(\mathbb{N})$, and $\|\gamma_k\| = 1$. Indeed, the matrix of γ_k is $[c_{i+j}]_{i,j\geq 0}$ with $c_k = 1$ and $c_l = 0$ for all $l \neq k$.

Lemma 3.1. *For any $1 < p < \infty$, the space $\mathrm{Hank}_p(\mathbb{N})$ is the w^*-closure of the linear span of $\{\gamma_k : k \geq 0\}$.*

Proof. It is plain that $\mathrm{Hank}_p(\mathbb{N})$ is a w^*-closed subspace of $B(\ell^p)$, hence one inclusion is straightforward.

To check the other one, consider $T \in \mathrm{Hank}_p(\mathbb{N})$. By the definition of this space, there is a sequence $(c_k)_{k\geq 0}$ of $\mathbb{C}$ such that

$$\langle T(e_j), e_i\rangle = c_{i+j}, \quad \text{for all } i, j \geq 0.$$

For any $n \geq 1$, let K_n be the Fejér kernel defined by

$$K_n(t) = \sum_{k=-n}^{n} \Big(1 - \frac{|k|}{n}\Big) e^{int}, \quad t \in \mathbb{R}.$$

Then let $T_n \in B(\ell^p)$ be the finite rank operator whose matrix is $[\widehat{K_n}(i+j)c_{i+j}]_{i,j\geq 0}$. Note that

$$T_n = \sum_{k=0}^{n} \Big(1 - \frac{|k|}{n}\Big) c_k\, \gamma_k \in \mathrm{Span}\{\gamma_k : k \geq 0\}.$$

We show that $\|T_n\| \leq \|T\|$. To see this, let $\alpha = (\alpha_j)_{j\geq 0} \in \ell^p$ and $(\beta_m)_{m\geq 0} \in \ell^{p'}$. We have that

$$\begin{aligned}\langle T_n(\alpha), \beta\rangle &= \sum_{m,j\geq 0} \widehat{K_n}(m+j) c_{m+j} \alpha_j \beta_m \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t) \sum_{m,j\geq 0} c_{m+j}\alpha_j\beta_m e^{-i(m+j)t}\, dt.\end{aligned}$$

Since $K_n \geq 0$, we deduce

$$|\langle T_n(\alpha), \beta\rangle| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t) \Big|\sum_{m,j\geq 0} c_{m+j}\alpha_j\beta_m e^{-i(m+j)t}\Big|\, dt.$$

Now for all $t \in [-\pi, \pi]$, we have

$$\begin{aligned}\Big|\sum_{m,j\geq 0} c_{m+j}\alpha_j\beta_m e^{-i(m+j)t}\Big| &= \Big|\sum_{m,j\geq 0} c_{m+j}(e^{-ijt}\alpha_j)(e^{-imt}\beta_m)\Big| \\ &= \Big|\big\langle T((e^{-ijt}\alpha_j)_{j\geq 0}), (e^{-imt}\beta_m)_{m\geq 0}\big\rangle\Big| \\ &\leq \|T\| \Big(\sum_{j\geq 0} |e^{-ijt}\alpha_j|^p\Big)^{\frac{1}{p}} \Big(\sum_{m\geq 0} |e^{-imt}\beta_m|^{p'}\Big)^{\frac{1}{p'}} \\ &\leq \|T\|\|\alpha\|_p\|\beta\|_{p'},\end{aligned}$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1,$$

we therefore obtain that $|\langle T_n(\alpha), \beta\rangle| \leq \|T\|\|\alpha\|_p\|\beta\|_{p'}$. This proves that $\|T_n\| \leq \|T\|$, as requested.

For all $i, j \geq 0$,

$$\langle T_n(e_j), e_i\rangle = \widehat{K_n}(i+j)\, \langle Te_j, e_i\rangle \to \langle Te_j, e_i\rangle,$$

when $n \to \infty$. Hence $T_n \to T$ in the w^*-topology, by Lemma 2.1. Consequently, T belongs to the w^*-closure of $\text{Span}\{\gamma_k : k \geq 0\}$. □

Remark 3.2. (a) Nehari's celebrated theorem (see, e.g., [Nikolski 2020, Theorem II.2.2.4], [Peller 2003, Theorem I.1.1] or [Power 1982, Theorem 1.3]) asserts that

$$\text{Hank}_2(\mathbb{N}) \simeq \frac{L^\infty(\mathbb{T})}{H^\infty(\mathbb{T})}. \tag{9}$$

Here $\mathbb{T}$ stands for the unit circle of $\mathbb{C}$ and $H^\infty(\mathbb{T}) \subset L^\infty(\mathbb{T})$ is the Hardy space of functions whose negative Fourier coefficients vanish. The isometric isomorphism $J : L^\infty(\mathbb{T})/H^\infty(\mathbb{T}) \to \text{Hank}_2(\mathbb{N})$ providing (9) is defined as follows. Given any $F \in L^\infty(\mathbb{T})$, let $\dot{F}$ denote its class modulo $H^\infty(\mathbb{T})$. Then $J(\dot{F})$ is the operator whose matrix is equal to $[\widehat{F}(-i-j-1)]_{i,j\geq 0}$.

(b) We remark that $\mathrm{Hank}_p(\mathbb{N}) \subseteq \mathrm{Hank}_2(\mathbb{N})$. To see this, note that if $T \in \mathrm{Hank}_p(\mathbb{N})$, then because of the symmetry in its matrix representation due to being a Hankelian matrix, T has the same matrix representation as T^*, and therefore T extends to a bounded operator on $\ell^{p'}$. By interpolation, T extends to a bounded operator on ℓ^2, which is represented by the same matrix as T. Hence, T belongs to $\mathrm{Hank}_2(\mathbb{N})$.

However for $1 < p \neq 2 < \infty$, there is no description of $\mathrm{Hank}_p(\mathbb{N})$ similar to Nehari's theorem.

(c) The definition of $\mathrm{Hank}_p(\mathbb{N})$ readily extends to the case $p = 1$ isometrically:

$$\mathrm{Hank}_1(\mathbb{N}) \simeq \ell^1.$$

Indeed, let $J_1 : \ell^1 \to \mathrm{Hank}_1(\mathbb{N})$ be defined by

$$J_1(c) = \sum_{k=0}^{\infty} c_k \gamma_k, \quad c = (c_k)_{k \geq 0} \in \ell^1.$$

Next, let $J_2 : \mathrm{Hank}_1(\mathbb{N}) \to \ell^1$ be defined by $J_2(T) = T(e_0)$. Then J_1, J_2 are contractions and it is easy to check that they are inverse to each other. Hence J_1 is an isometric isomorphism.

We say that a sequence $m = (m_k)_{k \geq 0}$ in $\mathbb{C}$ is the symbol of a multiplier on $\mathrm{Hank}_p(\mathbb{N})$ if there is a w^*-continuous operator $T_m : \mathrm{Hank}_p(\mathbb{N}) \to \mathrm{Hank}_p(\mathbb{N})$ such that

$$T_m(\gamma_k) = m_k \gamma_k, \quad k \geq 0.$$

Such an operator is uniquely defined. In this case, $m \in \ell^\infty$ and $\|m\|_\infty \leq \|T_m\|$.

The following is a simple extension of [Pisier 2001, Theorems 6.1 and 6.2].

Theorem 3.3. *Let $1 < p < \infty$, let $C \geq 0$ be a constant and let $m = (m_k)_{k \geq 0}$ be a sequence in $\mathbb{C}$. The following assertions are equivalent.*

(i) *m is the symbol of a p-completely bounded multiplier on $\mathrm{Hank}_p(\mathbb{N})$, and*

$$\|T_m : \mathrm{Hank}_p(\mathbb{N}) \to \mathrm{Hank}_p(\mathbb{N})\|_{p-cb} \leq C.$$

(ii) *There exist a measure space (Ω, μ), and bounded sequences $(\alpha_i)_{i \geq 0}$ in $L^p(\Omega)$ and $(\beta_j)_{j \geq 0}$ in $L^{p'}(\Omega)$ such that $m_{i+j} = \langle \alpha_i, \beta_j \rangle$, for every $i, j \geq 0$, and*

$$\sup_{i \geq 0} \|\alpha_i\|_p \sup_{j \geq 0} \|\beta_j\|_{p'} \leq C.$$

Proof. By homogeneity, we may assume that $C = 1$ throughout this proof.

Assume (i). Let $\kappa : \ell^p_{\mathbb{Z}} \to \ell^p_{\mathbb{Z}}$ be defined by $\kappa((a_k)_{k \in \mathbb{Z}}) = (a_{-k})_{k \in \mathbb{Z}}$, let $J : \ell^p_{\mathbb{N}} \to \ell^p_{\mathbb{Z}}$ be the canonical embedding and let $Q : \ell^p_{\mathbb{Z}} \to \ell^p_{\mathbb{N}}$ be the canonical projection. Define $q : B(\ell^p_{\mathbb{Z}}) \to B(\ell^p_{\mathbb{N}})$ by $q(T) = Q \kappa T J$. According to the easy implication (ii) $\Longrightarrow$ (i) of Theorem 2.2, the mapping q is p-completely contractive. We note that if $[t_{i,j}]_{(i,j) \in \mathbb{Z}^2}$ is the matrix of some $T \in B(\ell^p_{\mathbb{Z}})$, then the matrix of $q(T)$ is equal to $[t_{-i,j}]_{(i,j) \in \mathbb{N}^2}$.

Let $\mathcal{M}_p(\mathbb{Z}) \subseteq B(\ell^p_{\mathbb{Z}})$ be the space of all bounded Fourier multipliers on $\ell^p_{\mathbb{Z}}$; this is a unital subalgebra. Let $T \in \mathcal{M}_p(\mathbb{Z})$ and let $\phi \in L^\infty(\mathbb{T})$ denote its symbol. Then the matrix of T is equal to $[\widehat{\phi}(i-j)]_{(i,j)\in\mathbb{Z}^2}$, hence the matrix of $q(T)$ is equal to $[\widehat{\phi}(-i-j)]_{(i,j)\in\mathbb{N}^2}$. Hence, $q(T)$ is Hankelian. We can therefore consider the restriction map

$$q_{|\mathcal{M}_p(\mathbb{Z})} : \mathcal{M}_p(\mathbb{Z}) \to \mathrm{Hank}_p(\mathbb{N}).$$

Let $\mathtt{s} : \ell^p_{\mathbb{Z}} \to \ell^p_{\mathbb{Z}}$ be the shift operator defined by $\mathtt{s}(e_j) = e_{j+1}$, for all $j \in \mathbb{Z}$. We observe (left to the reader) that

$$q(\mathtt{s}^{-k}) = \gamma_k, \quad k \in \mathbb{N}. \tag{10}$$

We assume that $T_m : \mathrm{Hank}_p(\mathbb{N}) \to \mathrm{Hank}_p(\mathbb{N})$ is p-completely contractive. Consider $w : \mathcal{M}_p(\mathbb{Z}) \to \mathrm{Hank}_p(\mathbb{N}) \subseteq B(\ell^p)$ defined by $w := T_m \circ q_{|\mathcal{M}_p(\mathbb{Z})}$. Then w is p-completely contractive. Applying Theorem 2.2 to w, we obtain an SQ_p-space E, a contractive homomorphism $\pi : \mathcal{M}_p(\mathbb{Z}) \to B(E)$ and contractive maps $V : \ell^p_{\mathbb{N}} \to E$ and $W : E \to \ell^p_{\mathbb{N}}$ such that

$$w(T) = W\pi(T)V, \quad T \in \mathcal{M}_p(\mathbb{Z}). \tag{11}$$

Let $i, j \geq 0$. By (10), we have

$$w(\mathtt{s}^{-(i+j)}) = T_m(q(\mathtt{s}^{-(i+j)})) = T_m(\gamma_{i+j}) = m_{i+j}\gamma_{i+j},$$

hence $\langle w(\mathtt{s}^{-(i+j)})e_i, e_j\rangle = m_{i+j}$. Consequently, from (11), we obtain that

$$m_{i+j} = \langle \pi(\mathtt{s}^{-(i+j)})V(e_i), W^*(e_j)\rangle_{E,E^*}.$$

The mapping π is multiplicative, hence this implies that

$$m_{i+j} = \langle \pi(\mathtt{s}^{-i})V(e_i), \pi(\mathtt{s}^{-j})^* W^*(e_j)\rangle_{E,E^*}.$$

Set $x_i := \pi(\mathtt{s}^{-i})V(e_i) \in E$ and $y_j := \pi(\mathtt{s}^{-j})^* W^*(e_j) \in E^*$. Then, for all $i, j \geq 0$ we have $\|x_i\| \leq 1$, $\|y_j\| \leq 1$ and $m_{i+j} = \langle x_i, y_j\rangle_{E,E^*}$.

Let us now apply Remark 2.3. As in the latter, consider a measure space (Ω, μ) and closed subspaces $E_2 \subset E_1 \subset L^p(\Omega)$ such that $E = E_1/E_2$. Recall (5). For any $i \geq 0$, pick $\alpha_i \in E_1$ such that $\|\alpha_i\|_p = \|x_i\|$ and $\dot{\alpha_i} = x_i$. Likewise, for any $j \geq 0$, pick $\beta_j \in E_2^\perp$ such that $\|\beta_j\|_{p'} = \|y_j\|$ and $\dot{\beta_j} = y_j$. Then for all $i, j \geq 0$, we both have $\|\alpha_i\|_p \leq 1$, $\|\beta_j\|_{p'} \leq 1$ and $m_{i+j} = \langle \alpha_i, \beta_j\rangle_{L^p, L^{p'}}$. This proves (ii).

Conversely, assume (ii). By [Pisier 2001, Corollary 8.2], the family $\{m_{i+j}\}_{(i,j)\in\mathbb{N}^2}$ induces a p-completely contractive Schur multiplier on $B(\ell^p)$. It is clear that the restriction of this Schur multiplier maps $\mathrm{Hank}_p(\mathbb{N})$ into itself. More precisely, it maps γ_k to $m_k\gamma_k$ for all $k \geq 0$. Hence m is the symbol of a p-completely contractive multiplier on $\mathrm{Hank}_p(\mathbb{N})$. □

4. Hankel operators on $L^p(\mathbb{R}_+)$

Throughout we let $1 < p < \infty$ and we let p' denote its conjugate index. For any $u > 0$, we set $\tau_u f := f(\,\cdot - u)$, for all $f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$. Let

$$\mathrm{Hank}_p(\mathbb{R}_+) \subseteq B(L^p(\mathbb{R}_+))$$

be the space of Hankelian operators on $L^p(\mathbb{R}_+)$, consisting of all bounded operators $T : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ such that

$$\langle T\tau_u f, g\rangle = \langle Tf, \tau_u g\rangle,$$

for all $f \in L^p(\mathbb{R}_+)$, $g \in L^{p'}(\mathbb{R}_+)$ and $u > 0$.

For any $u > 0$, let $\theta_u : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ be defined by $\theta_u f = f(u - \cdot\,)$. Note that θ_u is a Hankelian operator on $L^p(\mathbb{R}_+)$. Indeed, for all $f \in L^p(\mathbb{R}_+)$, $g \in L^{p'}(\mathbb{R}_+)$ and $v > 0$, we have

$$\langle \theta_u \tau_v f, g\rangle = \int_v^u f(u-s)g(s-v)\,ds = \langle \theta_u f, \tau_v g\rangle$$

if $v < u$, and $\langle \theta_u \tau_v f, g\rangle = \langle \theta_u f, \tau_v g\rangle = 0$ if $v \geq u$. The operators θ_u are the continuous counterparts of the operators γ_k from Section 3. From this point of view, part (1) of Theorem 4.1 below is an analogue of Lemma 3.1. However its proof is more delicate.

We introduce a new space $A_p(\mathbb{R}_+) \subseteq C_0(\mathbb{R}_+)$ by

$$A_p(\mathbb{R}_+) := \Big\{ F = \sum_{n=1}^{\infty} f_n * g_n : f_n \in L^p(\mathbb{R}_+),\ g_n \in L^{p'}(\mathbb{R}_+) \text{ and } \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_{p'} < \infty \Big\},$$

and we equip it with the norm

$$\|F\|_{A_p} = \inf \Big\{ \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_{p'} \Big\}, \tag{12}$$

where the infimum runs over all possible representations of F as above. The space $A_p(\mathbb{R}_+)$ is a half-line analogue of the classical Figà-Talamanca–Herz algebra $A_p(\mathbb{R})$; see, e.g., [Derighetti 2011]. The classical arguments showing that the latter is a Banach space show as well that (12) is a norm on $A_p(\mathbb{R}_+)$ and that $A_p(\mathbb{R}_+)$ is a Banach space.

It follows from the above definitions that there exists a (necessarily unique) contractive map

$$Q_p : L^p(\mathbb{R}_+) \mathbin{\widehat{\otimes}} L^{p'}(\mathbb{R}_+) \to A_p(\mathbb{R}_+)$$

such that $Q_p(f \otimes g) = f * g$, for all $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$. Moreover Q_p is a quotient map. Hence the adjoint

$$Q_p^* : A_p(\mathbb{R}_+)^* \to B(L^p(\mathbb{R}_+))$$

of Q_p is an isometry. This yields an isometric identification $A_p(\mathbb{R}_+)^* \simeq \ker(Q_p)^\perp$ $(= \mathrm{ran}(Q_p^*))$.

We observe that

$$\ker(Q_p)^\perp = \overline{\mathrm{Span}}^{w^*}\{\theta_u : u > 0\}. \tag{13}$$

To prove this, we note that

$$\langle \theta_u, f \otimes g\rangle = \langle \theta_u(f), g\rangle = (f * g)(u), \tag{14}$$

for all $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R}_+)$ and $u > 0$. Hence,

$$\Big\langle \theta_u, \sum_{n=1}^{\infty} f_n \otimes g_n \Big\rangle = \Big(\sum_{n=1}^{\infty} f_n * g_n \Big)(u)$$

for all sequences $(f_n)_n$ in $L^p(\mathbb{R}_+)$ and $(g_n)_n$ in $L^{p'}(\mathbb{R}_+)$ such that

$$\sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_{p'} < \infty,$$

and all $u > 0$. This implies that $\mathrm{Span}\{\theta_u : u > 0\}_\perp = \ker(Q_p)$, and (13) follows.

Theorem 4.1. (1) *The space* $\mathrm{Hank}_p(\mathbb{R}_+)$ *is equal to the* w^**-closure of the linear span of* $\{\theta_u : u > 0\}$.

(2) *We have an isometric identification*

$$\mathrm{Hank}_p(\mathbb{R}_+) \simeq A_p(\mathbb{R}_+)^*.$$

Proof. Part (2) follows from part (1) and the discussion preceding the statement of Theorem 4.1. For any $f \in L^p(\mathbb{R}_+)$, $g \in L^{p'}(\mathbb{R}_+)$ and $u > 0$, the functionals $T \mapsto \langle T\tau_u f, g\rangle$ and $T \mapsto \langle Tf, \tau_u g\rangle$ are w^*-continuous on $B(L^p(\mathbb{R}_+))$. Consequently, $\mathrm{Hank}_p(\mathbb{R}_+)$ is w^*-closed. Hence $\mathrm{Hank}_p(\mathbb{R}_+)$ contains the w^*-closure of $\mathrm{Span}\{\theta_u : u > 0\}$. To prove the reverse inclusion, it suffices to show, by (13), that

$$\mathrm{Hank}_p(\mathbb{R}_+) \subset \ker(Q_p)^\perp.$$

We will use a double approximation process. First, let k, l in $C_c(\mathbb{R})$, the space of continuous functions with compact support. To any $T \in B(L^p(\mathbb{R}_+))$, we associate $T_{k,l} \in B(L^p(\mathbb{R}_+))$ defined by

$$\langle T_{k,l}(f), g\rangle = \int_{\mathbb{R}} \langle T(\tau_u k \cdot f), \tau_{-u} l \cdot g\rangle\, du, \qquad f \in L^p(\mathbb{R}_+),\ g \in L^{p'}(\mathbb{R}_+).$$

We note that

$$\begin{aligned}\int_{\mathbb{R}} \big|\langle T(\tau_u k \cdot f), \tau_{-u} l \cdot g\rangle\big|\, du &\leq \|T\|_p \Big(\int_{\mathbb{R}} \|\tau_u k f\|_p^p\, du\Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}} \|\tau_{-u} l g\|_{p'}^{p'}\, du\Big)^{\frac{1}{p'}} \\ &= \|T\|_p \|f\|_p \|g\|_{p'} \|k\|_p \|l\|_{p'}.\end{aligned}$$

Thus, $T_{k,l}$ is well-defined and $\|T_{k,l}\| \le \|T\|\|k\|_p\|l\|_{p'}$. We are going to show that

$$T \in \mathrm{Hank}_p(\mathbb{R}_+) \implies T_{k,l} \in \ker(Q_p)^\perp. \tag{15}$$

Let $\alpha \in C_c(\mathbb{R}_+)^+$ such that $\|\alpha\|_1 = 1$. Let $R_\alpha \in B(L^p(\mathbb{R}_+))$ be defined by

$$R_\alpha(f) = \alpha * f, \quad f \in L^p(\mathbb{R}_+).$$

We show that $(TR_\alpha)_{k,l}$ belongs to $\ker(Q_p)^\perp$ if $T \in \mathrm{Hank}_p(\mathbb{R}_+)$, and we use these auxiliary operators to establish (15).

We fix some $T \in \mathrm{Hank}_p(\mathbb{R}_+)$. Let $z \in \ker(Q_p)$. Since $C_c(\mathbb{R}_+)$ is both dense in $L^p(\mathbb{R}_+)$ and $L^{p'}(\mathbb{R}_+)$, it follows, e.g., from [Derighetti 2011, Chapter 3, Proposition 6] that there exist sequences $(f_n)_{n\ge1}$ and $(g_n)_{n\ge1}$ in $C_c(\mathbb{R}_+)$ such that $\sum_{n=1}^\infty \|f_n\|_p\|g_n\|_{p'} < \infty$ and $z = \sum_{n=1}^\infty f_n \otimes g_n$. Since $z \in \ker(Q_p)$, we have $\sum_{n=1}^\infty f_n * g_n = 0$, pointwise.

We write $R_\alpha f = \int_{\mathbb{R}_+} f(s)\tau_s\alpha\, ds$ as a Bochner integral, for all $f \in C_c(\mathbb{R}_+)$. A simple application of Fubini's theorem leads to

$$k * l \cdot f_n * g_n = \int_{\mathbb{R}}\int_{\mathbb{R}_+} (\tau_u k \cdot f_n)(s)\tau_s(\tau_{-u} l \cdot g_n)\, ds du,$$

for all $n \ge 1$. We deduce that

$$\begin{aligned}
\sum_{n=1}^\infty \langle (TR_\alpha)_{k,l}(f_n), g_n\rangle &= \sum_{n=1}^\infty \int_{\mathbb{R}} \langle TR_\alpha(\tau_u k \cdot f_n), \tau_{-u} l \cdot g_n\rangle\, du \\
&= \sum_{n=1}^\infty \int_{\mathbb{R}} \langle T((\tau_u k \cdot f_n) * \alpha), \tau_{-u} l \cdot g_n\rangle\, du \\
&= \sum_{n=1}^\infty \int_{\mathbb{R}}\int_{\mathbb{R}_+} (\tau_u k \cdot f_n)(s)\langle T(\tau_s\alpha), \tau_{-u} l \cdot g_n\rangle\, ds du \\
&= \sum_{n=1}^\infty \int_{\mathbb{R}}\int_{\mathbb{R}_+} \langle T(\alpha), (\tau_u k \cdot f_n)(s)\tau_s(\tau_{-u} l \cdot g_n)\rangle\, ds du \\
&= \sum_{n=1}^\infty \langle T(\alpha), k * l \cdot f_n * g_n\rangle \\
&= \Big\langle T(\alpha), k * l \cdot \sum_{n=1}^\infty f_n * g_n \Big\rangle = 0.
\end{aligned}$$

This shows that $(TR_\alpha)_{k,l}$ belongs to $\ker(Q_p)^\perp$.

For z, f_n, g_n as above, write

$$\sum_{n=1}^\infty \langle T_{k,l}(f_n), g_n\rangle = \sum_{n=1}^\infty \langle T_{k,l}(f_n), g_n\rangle - \sum_{n=1}^\infty \langle (TR_\alpha)_{k,l}(f_n), g_n\rangle.$$

Then we have

$$
\begin{aligned}
\Big|\sum_{n=1}^{\infty}\langle T_{k,l}(f_n), g_n\rangle\Big| \\
&\le \sum_{n=1}^{\infty}\int_{\mathbb{R}}\big|\langle T(\tau_u k\cdot f_n-(\tau_u k\cdot f_n)*\alpha), \tau_{-u}l\cdot g_n\rangle\big|\,du \\
&\le \sum_{n=1}^{\infty}\|T\|\Big(\int_{\mathbb{R}}\|\tau_u k\cdot f_n-(\tau_u k\cdot f_n)*\alpha\|_p^p\,du\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}}\|\tau_{-u}l\cdot g_n\|_{p'}^{p'}\,du\Big)^{\frac{1}{p'}} \\
&\le \|T\|\|l\|_{p'}\sum_{n=1}^{\infty}\|g_n\|_{p'}\Big(\int_{\mathbb{R}}\|\tau_u k\cdot f_n-\Big(\tau_u k\cdot f_n\Big)*\alpha\|_p^p\,du\Big)^{\frac{1}{p}}.
\end{aligned}
$$

Recall that by assumption, $\alpha\ge 0$ and $\int_{\mathbb{R}_+}\alpha(s)ds=1$. Then we deduce from above that

$$
\begin{aligned}
\Big|\sum_{n=1}^{\infty}\langle T_{k,l}(f_n), g_n\rangle\Big| \\
&\le \|T\|\|l\|_{p'}\sum_{n=1}^{\infty}\|g_n\|_{p'}\Big(\int_{\mathbb{R}}\Big\|\int_{\mathbb{R}_+}\alpha(s)(\tau_u k\cdot f_n-\tau_s(\tau_u k\cdot f_n))ds\Big\|_p^p\,du\Big)^{\frac{1}{p}} \\
&\le \|T\|\|l\|_{p'}\sum_{n=1}^{\infty}\|g_n\|_{p'}\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\alpha(s)\big\|\tau_u k\cdot f_n-\tau_s(\tau_u k\cdot f_n)\big\|_p^p\,ds du\Big)^{\frac{1}{p}}.
\end{aligned}
$$

The integral in the right-hand side satisfies

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\alpha(s)\big\|\tau_u k\cdot f_n-\tau_s(\tau_u k\cdot f_n)\big\|_p^p\,ds du\Big)^{\frac{1}{p}} \\
&\le \Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\alpha(s)\big\|\tau_u k\cdot f_n-\tau_{s+u}k\cdot f_n\big\|_p^p ds du\Big)^{\frac{1}{p}} \\
&\qquad+\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\alpha(s)\big\|\tau_{s+u}k\cdot f_n-\tau_s(\tau_u k\cdot f_n)\big\|_p^p\,ds du\Big)^{\frac{1}{p}} \\
&\le \Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\alpha(s)\big\|\tau_u((k-\tau_s k)\cdot f_n)\big\|_p^p\,ds du\Big)^{\frac{1}{p}} \\
&\qquad+\Big(\int_{\mathbb{R}}\int_{\mathbb{R}_+}\alpha(s)\big\|\tau_{s+u}k\cdot(f_n-\tau_s f_n)\big\|_p^p ds du\Big)^{\frac{1}{p}} \\
&\le \sup_{s\in\operatorname{supp}(\alpha)}\Big(\int_{\mathbb{R}}\big\|\tau_u(k-\tau_s k)\cdot f_n\big\|_p^p du\Big)^{\frac{1}{p}}+\sup_{s\in\operatorname{supp}(\alpha)}\Big(\int_{\mathbb{R}}\big\|\tau_{s+u}k\cdot(f_n-\tau_s f_n)\big\|_p^p du\Big)^{\frac{1}{p}} \\
&= \sup_{s\in\operatorname{supp}(\alpha)}\|k-\tau_s k\|_p\|f_n\|_p+\sup_{s\in\operatorname{supp}(\alpha)}\|k\|_p\|f_n-\tau_s f_n\|_p.
\end{aligned}
$$

Hence we obtain that

$$
\begin{aligned}
&\Big|\sum_{n=1}^{\infty}\langle T_{k,l}(f_n), g_n\rangle\Big| \\
&\le \|T\|\|l\|_{p'}\sum_{n=1}^{\infty}\|g_n\|_{p'}\big(\sup_{s\in\operatorname{supp}(\alpha)}\|k-\tau_s k\|_p\|f_n\|_p+\sup_{s\in\operatorname{supp}(\alpha)}\|k\|_p\|f_n-\tau_s f_n\|_p\big).
\end{aligned}
$$

Given $\epsilon > 0$, choose M such that

$$\sum_{n=M+1}^{\infty} \|f_n\|_p \|g_n\|_{p'} < \epsilon.$$

We may find $s_0 > 0$ such that for all $s \in (0, s_0)$ and all $1 \le n \le M$, we have that

$$\|k - \tau_s k\|_p \le \frac{\epsilon \|k\|_p}{\sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_{p'}} \quad \text{and} \quad \|f_n - \tau_s f_n\|_p \le \frac{\epsilon}{M \|g_n\|_{p'}}.$$

We may now choose α so that $\operatorname{supp}(\alpha) \subseteq (0, t_0)$. Then we obtain from above that

$$\begin{aligned}
&\Big|\sum_{n=1}^{\infty} \langle T_{k,l}(f_n), g_n\rangle\Big| \\
&\quad \le \|T\| \|l\|_{p'} \Big(\epsilon \|k\|_p + \sum_{n=1}^{M} \|g_n\|_{p'} \cdot \sup_{s \in \operatorname{supp}(\alpha)} \|k\|_p \|f_n - \tau_s f_n\|_p \\
&\qquad\qquad + \sum_{n=M+1}^{\infty} \|g_n\|_{p'} \cdot \sup_{s \in \operatorname{supp}(\alpha)} \|k\|_p \|f_n - \tau_s f_n\|_p \Big) \\
&\quad \le \|T\| \|l\|_{p'} \Big(2\epsilon \|k\|_p + \sum_{n=M+1}^{\infty} 2 \|k\|_p \|g_n\|_{p'} \|f_n\|_p \Big) \\
&\quad \le 4\epsilon \|T\| \|l\|_{p'} \|k\|_p.
\end{aligned}$$

Since ϵ was arbitrary, this shows that $\sum_{n=1}^{\infty} \langle T_{k,l}(f_n), g_n\rangle = 0$. Since $z = \sum_{n=1}^{\infty} f_n \otimes g_n$ was an arbitrary element of $\ker(Q_p)$, we obtain (15).

Next, we construct a sequence $(T_{k_n,l_n})_n$ which tends to T in the w^*-topology of $B(L^p(\mathbb{R}_+))$. In the sequel, we assume that k, l in $C_c(\mathbb{R})$ are such that

$$\|k\|_p = 1, \quad \|l\|_{p'} = 1 \quad \text{and} \quad \int_{\mathbb{R}} k(-s) l(s)\, ds = 1. \tag{16}$$

Consider any $f, g \in C_c(\mathbb{R}_+)$. We have

$$\begin{aligned}
&\big|\langle T(f), g\rangle - \langle T_{k,l}(f), g\rangle\big| \\
&\quad = \Big| \int_{\mathbb{R}} \langle T(k(-s) f), l(s) g\rangle - \langle T(\tau_s k \cdot f), \tau_{-s} l \cdot g\rangle\, ds \Big| \\
&\quad \le \int_{\mathbb{R}} \big|\langle T((k(-s) - \tau_s k) f), l(s) g\rangle\big|\, ds + \int_{\mathbb{R}} \big|\langle T(\tau_s k \cdot f), (l(s) - \tau_{-s} l) g\rangle\big|\, ds \\
&\quad \le \|T\| \Big(\int_{\mathbb{R}} \big\|(k(-s) - \tau_s k) f\big\|_p^p ds \Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}} \|l(s) g\|_{p'}^{p'}\, ds \Big)^{\frac{1}{p'}} \\
&\qquad\qquad + \|T\| \Big(\int_{\mathbb{R}} \|\tau_s k \cdot f\|_p^p ds \Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}} \|(l(s) - \tau_{-s} l) g\|_{p'}^{p'}\, ds \Big)^{\frac{1}{p'}} \\
&\quad \le \|T\| \|g\|_{p'} \Big(\int_{\mathbb{R}_+} |f(t)|^p \|\tau_t \check{k} - \check{k}\|_p^p\, dt \Big)^{\frac{1}{p}} \\
&\qquad\qquad + \|T\| \|f\|_p \Big(\int_{\mathbb{R}_+} |g(t)|^{p'} \|\tau_{-t} l - l\|_{p'}^{p'}\, dt \Big)^{\frac{1}{p'}}.
\end{aligned}$$

Here $\check{k}$ denotes the function $s \mapsto k(-s)$.

Now for $n \geq 1$, set

$$k_n := \frac{\chi_{[-n,n]}}{(2n)^{\frac{1}{p}}} \quad \text{and} \quad l_n := \frac{\chi_{[-n,n]}}{(2n)^{\frac{1}{p'}}},$$

where $\chi_{[-n,n]}$ is the indicator function of the interval $[-n,n]$. Then $\|k_n\|_p = \|l_n\|_{p'} = 1$ and $\int_{\mathbb{R}} k_n(-s)l_n(s)\,ds = 1$ as in (16). Let $K = \mathrm{supp}(f) \cup \mathrm{supp}(g)$ and let $r = \sup(K)$. Note that $\check{k_n} = k_n$ and that we have

$$\sup_{t\in K} \|\tau_t k_n - k_n\|_p \leq \left(\frac{r}{n}\right)^{\frac{1}{p}} \quad \text{and} \quad \sup_{t\in K} \|\tau_{-t} l_n - l_n\|_{p'} \leq \left(\frac{r}{n}\right)^{\frac{1}{p'}}.$$

Therefore,

$$\left|\langle T(f), g\rangle - \langle T_{k_n,l_n}(f), g\rangle\right| \leq \frac{2r}{n}\|T\|\|f\|_p\|g\|_{p'},$$

hence $\langle T_{k_n,l_n}(f), g\rangle \xrightarrow[n\to\infty]{} \langle T(f), g\rangle$. Since $\|T_{k_n,l_n}\| \leq \|T\|$ for all $n \geq 1$, this implies, by Lemma 2.1, that $T_{k_n,l_n} \to T$ in the w^*-topology of $B(L^p(\mathbb{R}_+))$. Consequently, $T \in \ker(Q_p)^\perp$ as expected. □

Remark 4.2. (a) For any $1 \leq p \leq \infty$, let $H^p(\mathbb{R}) \subset L^p(\mathbb{R})$ be the subspace of all $f \in L^p(\mathbb{R})$ whose Fourier transform has support in $\mathbb{R}_+$. Recall the factorisation property

$$H^1(\mathbb{R}) = H^2(\mathbb{R}) \times H^2(\mathbb{R}).$$

More precisely, the product $h_1h_2 \in H^1(\mathbb{R})$ and $\|h_1h_2\|_1 \leq \|h_1\|_2\|h_2\|_2$ for all $h_1, h_2 \in H^2(\mathbb{R})$ and conversely, for all $h \in H^1(\mathbb{R})$, there exist $h_1, h_2 \in H^2(\mathbb{R})$ such that $h = h_1h_2$ and $\|h\|_1 = \|h_1\|_2\|h_2\|_2$.

Recall that by definition,

$$A_2(\mathbb{R}_+) = \left\{\sum f_n * g_n : f_n, g_n \in L^2(\mathbb{R}_+), \sum \|f_n\|_2\|g_n\|_2 < \infty\right\}.$$

It therefore follows from the above factorisation property and the identification of $L^2(\mathbb{R}_+)$ with $H^2(\mathbb{R})$ via the Fourier transform that

$$A_2(\mathbb{R}_+) = \{\hat{h} : h \in H^1(\mathbb{R})\},$$

with $\|\hat{h}\|_{A_2(\mathbb{R}_+)} = \|h\|_{H^1(\mathbb{R})}$. Therefore, we have an isometric identification

$$A_2(\mathbb{R}_+) \cong H^1(\mathbb{R}).$$

Since $H^1(\mathbb{R})^\perp = H^\infty(\mathbb{R})$, we have

$$H_1(\mathbb{R})^* \cong \frac{L^\infty(\mathbb{R})}{H^\infty(\mathbb{R})}.$$

Applying Theorem 4.1(2), we recover the well-known fact (see [Nikolski 2020, Section IV.5.3] or [Peller 2003, Theorem I.8.1]) that

$$\mathrm{Hank}_2(\mathbb{R}_+) \cong \frac{L^\infty(\mathbb{R})}{H^\infty(\mathbb{R})}.$$

(b) Notice that $\mathrm{Hank}_p(\mathbb{R}_+) \subseteq \mathrm{Hank}_2(\mathbb{R}_+)$. Indeed, suppose that $T \in \mathrm{Hank}_p(\mathbb{R}_+)$ and note that the adjoint mapping $T^* \in B(L^{p'}(\mathbb{R}_+))$ coincides with T on $L^p(\mathbb{R}_+) \cap L^{p'}(\mathbb{R}_+)$. To see this, take $f, g \in L^p(\mathbb{R}_+) \cap L^{p'}(\mathbb{R}_+)$ and observe that $f \otimes g - g \otimes f$ belongs to $\ker(Q_p)$. This implies that $\langle T(f), g\rangle = \langle T(g), f\rangle$. Therefore, T coincides with T^* on $L^p(\mathbb{R}_+) \cap L^{p'}(\mathbb{R}_+)$. It then follows by interpolation that T extends to a bounded operator on $L^2(\mathbb{R}_+)$, say $\widetilde{T}$. Since T and $\widetilde{T}$ coincide on $L^p(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ and T is Hankelian, it follows from the definition of Hankel operators that $\widetilde{T}$ is also a Hankel operator and hence belongs to $\mathrm{Hank}_2(\mathbb{R}_+)$.

(c) The definition of $\mathrm{Hank}_p(\mathbb{R}_+)$ extends to the case $p = 1$. In analogy with Remark 3.2(c), we have an isometric identification

$$\mathrm{Hank}_1(\mathbb{R}_+) \simeq M(\mathbb{R}_+^*),$$

where $M(\mathbb{R}_+^*)$ denotes the space of all bounded Borel measures on $\mathbb{R}_+^*$. To establish this, we first note that for all $f \in L^1(\mathbb{R}_+)$, the function $u \mapsto \theta_u(f)$ is bounded and continuous from $\mathbb{R}_+^*$ into $L^1(\mathbb{R}_+)$. Hence for all $\nu \in M(\mathbb{R}_+^*)$, we may define $H_\nu \in B(L^1(\mathbb{R}_+))$ by

$$H_\nu(f) = \int_{\mathbb{R}_+^*} \theta_u(f)\, d\nu(u), \qquad f \in L^1(\mathbb{R}_+). \tag{17}$$

It is clear that H_ν is Hankelian. It follows from (14) that

$$\langle H_\nu(f), g\rangle = \int_{\mathbb{R}_+^*} (f * g)(u)\, d\nu(u), \qquad f \in L^1(\mathbb{R}_+),\ g \in L^\infty(\mathbb{R}_+).$$

We note that the mapping $\nu \mapsto H_\nu$ is a one-to-one contraction from $M(\mathbb{R}_+^*)$ into $\mathrm{Hank}_1(\mathbb{R}_+)$. We shall now prove that this mapping is an onto isometry.

We use the isometric identification $M(\mathbb{R}_+^*) \simeq C_0(\mathbb{R}_+^*)^*$ provided by the Riesz theorem and we regard $L^1(\mathbb{R}_+) \subseteq M(\mathbb{R}_+^*)$ in the obvious way. Let $T \in \mathrm{Hank}_1(\mathbb{R}_+)$. We observe that for all $h, f \in L^1(\mathbb{R}_+)$ and all $g \in C_0(\mathbb{R}_+^*)$, we have

$$\langle T(h * f), g\rangle = \langle T(h), f * g\rangle \tag{18}$$

Indeed, write $h * f = \int_0^\infty f(s)\tau_s h\, ds$. This implies that $T(h*f) = \int_0^\infty f(s) T(\tau_s h)\, ds$, hence

$$\langle T(h * f), g\rangle = \int_0^\infty f(s)\langle T\tau_s h, g\rangle\, ds = \int_0^\infty f(s)\langle Th, \tau_s g\rangle\, ds = \langle T(h), f * g\rangle.$$

Let $(h_n)_{n\geq 1}$ be a norm one approximate unit of $L^1(\mathbb{R}_+)$. Then $(T(h_n))_{n\geq 1}$ is a bounded sequence of $L^1(\mathbb{R}_+)$. Hence it admits a cluster point $\nu \in M(\mathbb{R}_+^*)$

in the w^*-topology of $M(\mathbb{R}_+^*)$. Thus, for all $g \in C_0(\mathbb{R}_+^*)$, the complex number $\int_{\mathbb{R}_+^*} g(u)\, d\nu(u)$ is a cluster point of the sequence $(\langle T(h_n), g\rangle)_{n\geq 1}$. Furthermore, we have $\|\nu\| \leq \|T\|$. Let $f \in L^1(\mathbb{R}_+)$ and let $g \in C_0(\mathbb{R}_+^*)$. Since $h_n * f \to f$ in $L^1(\mathbb{R}_+)$, we have that $\langle T(h_n * f), g\rangle \to \langle T(f), g\rangle$. By (18), we may write $\langle T(h_n * f), g\rangle = \langle T(h_n), f * g\rangle$. We deduce that

$$\langle T(f), g\rangle = \int_{\mathbb{R}_+^*} (f * g)(u)\, d\nu(u).$$

This implies that $T = H_\nu$, see (17), which concludes the proof.

Definition 4.3. We say that a function $m : \mathbb{R}_+^* \to \mathbb{C}$ is the symbol of a multiplier on $\mathrm{Hank}_p(\mathbb{R}_+)$ if there exist a w^*-continuous operator $T_m : \mathrm{Hank}_p(\mathbb{R}_+) \to \mathrm{Hank}_p(\mathbb{R}_+)$ such that for every $u > 0$, $T_m(\theta_u) = m(u)\theta_u$. (Note that such an operator T_m is necessarily unique.)

Remark 4.4. Suppose that $T_m : \mathrm{Hank}_p(\mathbb{R}_+) \to \mathrm{Hank}_p(\mathbb{R}_+)$ is a multiplier as defined above. Using Theorem 4.1(2), let $S_m : A_p(\mathbb{R}_+) \to A_p(\mathbb{R}_+)$ be the operator such that $S_m^* = T_m$. For $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$, we have, by (14),

$$\begin{aligned}
[S_m(f * g)](u) &= \langle \theta_u, S_m(f * g)\rangle \\
&= \langle T_m(\theta_u), f * g\rangle \\
&= m(u)\langle \theta_u, f * g\rangle \\
&= m(u)(f * g)(u).
\end{aligned}$$

We deduce that $S_m(F) = m \cdot F$, for every $F \in A_p(\mathbb{R}_+)$.

Conversely, if $m : \mathbb{R}_+^* \to \mathbb{C}$ is such that $S_m : A_p(\mathbb{R}_+) \to A_p(\mathbb{R}_+)$ given by $S_m(F) = m \cdot F$ is well-defined and bounded, then S_m^* is a multiplier on $\mathrm{Hank}_p(\mathbb{R}_+)$.

Lemma 4.5. *If $m : \mathbb{R}_+^* \to \mathbb{C}$ is the symbol of a multiplier on* $\mathrm{Hank}_p(\mathbb{R}_+)$, *then m is continuous and bounded.*

Proof. For all $u > 0$, we have $m(u)\theta_u = T_m(\theta_u)$, hence $|m(u)| \leq \|T_m\|$. Thus, m is bounded. For any $a > 0$, let $\chi_{(0,a)}$ be the indicator function of the interval $(0, a)$. Then $m \cdot \chi_{(0,a)} * \chi_{(0,a)}$ belongs to $A_p(\mathbb{R}_+)$, hence to $C_b(\mathbb{R}_+^*)$, by Remark 4.4. Since $\chi_{(0,a)} * \chi_{(0,a)} > 0$ on $(0, 2a)$, it follows that m is continuous on $(0, 2a)$. Thus, m is continuous on $\mathbb{R}_+^*$. □

Theorem 4.6. *Let $1 < p < \infty$, let $C \geq 0$ be a constant and let $m : \mathbb{R}_+^* \to \mathbb{C}$ be a function. The following assertions are equivalent.*

(i) *m is the symbol of a p-completely bounded multiplier on* $\mathrm{Hank}_p(\mathbb{R}_+)$, *and*

$$\|T_m : \mathrm{Hank}_p(\mathbb{R}_+) \to \mathrm{Hank}_p(\mathbb{R}_+)\|_{p-cb} \leq C.$$

(ii) *m is continuous and there exist a measure space (Ω, μ) and two functions $\alpha \in L^\infty(\mathbb{R}_+; L^p(\Omega))$ and $\beta \in L^\infty(\mathbb{R}_+; L^{p'}(\Omega))$ such that $\|\alpha\|_\infty \|\beta\|_\infty \leq C$ and $m(s+t) = \langle \alpha(s), \beta(t)\rangle$, for almost every $(s,t) \in \mathbb{R}_+^{*2}$.*

Proof. By homogeneity, we may assume that $C = 1$ throughout this proof.

Assume (i). The continuity of m follows from Lemma 4.5. Let $T_m : \mathrm{Hank}_p(\mathbb{R}_+) \to \mathrm{Hank}_p(\mathbb{R}_+)$ be the p-completely contractive multiplier associated with m. Let $\kappa : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be defined by $(\kappa f)(t) = f(-t)$, for all $f \in L^p(\mathbb{R})$. Let $J : L^p(\mathbb{R}_+) \to L^p(\mathbb{R})$ be the canonical embedding and let $Q : L^p(\mathbb{R}) \to L^p(\mathbb{R}_+)$ be the canonical projection defined by $Qf = f_{|\mathbb{R}_+}$. Let $q : B(L^p(\mathbb{R})) \to B(L^p(\mathbb{R}_+))$ be given by $q(T) = Q\kappa T J$, for all $T \in B(L^p(\mathbb{R}))$. Applying the easy implication (ii) $\Longrightarrow$ (i) of Theorem 2.2 we obtain that q is p-completely contractive.

Let $\mathcal{M}_p(\mathbb{R}) \subseteq B(L^p(\mathbb{R}))$ denote the subalgebra of bounded Fourier multipliers. Let us show that if $T \in \mathcal{M}_p(\mathbb{R})$, then $q(T) \in \mathrm{Hank}_p(\mathbb{R}_+)$. For any $s \in \mathbb{R}$, recall $\tau_s \in B(L^p(\mathbb{R}))$ given by $\tau_s(f) = f(\,\cdot - s)$. Note that $\tau_s \in \mathcal{M}_p(\mathbb{R})$ and that $\mathcal{M}_p(\mathbb{R}) = \overline{\mathrm{Span}}^{w^*}\{\tau_s : s \in \mathbb{R}\}$. For all $f \in L^p(\mathbb{R}_+)$, we have

$$q(\tau_s)f = Q\tau(f(\,\cdot - s)) = Q(f(-(\,\cdot + s))) = \{t \in \mathbb{R}_+ \mapsto f(-t-s)\}.$$

Hence, if $s \geq 0$, then $q(\tau_s) = 0$ and if $s < 0$, then $q(\tau_s) = \theta_{-s}$. It is plain that q is w^*-continuous. Since $\mathrm{Hank}_p(\mathbb{R}_+)$ is w^*-closed, we deduce that q maps $\mathcal{M}_p(\mathbb{R})$ into $\mathrm{Hank}_p(\mathbb{R}_+)$.

Consider the mapping

$$q_0 := q_{|\mathcal{M}_p(\mathbb{R})} : \mathcal{M}_p(\mathbb{R}) \to \mathrm{Hank}_p(\mathbb{R}_+)$$

and set

$$\Gamma := T_m \circ q_0 : \mathcal{M}_p(\mathbb{R}) \to B(L^p(\mathbb{R}_+)).$$

It follows from above that

$$\Gamma(\tau_{-s}) = m(s)\theta_s, \quad s > 0. \tag{19}$$

Since q is p-completely contractive, Γ is also p-completely contractive. Applying Theorem 2.2 to Γ, we obtain the existence of an SQ_p-space E, a unital p-completely contractive, nondegenerate homomorphism $\pi : \mathcal{M}_p(\mathbb{R}) \to B(E)$ as well as operators $V : L^p(\mathbb{R}_+) \to E$ and $W : E \to L^p(\mathbb{R}_+)$ such that $\|V\|\|W\| \leq 1$ and for every $x \in \mathcal{M}_p(\mathbb{R})$, $\Gamma(x) = W\pi(x)V$.

Let $c : L^1(\mathbb{R}) \to \mathcal{M}_p(\mathbb{R})$ be defined by $[c(g)](f) = g * f$, for all $g \in L^1(\mathbb{R})$ and $f \in L^p(\mathbb{R})$. Let $\lambda : L^1(\mathbb{R}) \to B(E)$ be given by $\lambda = \pi \circ c$. Then λ is a contractive, nondegenerate homomorphism. By [de Pagter and Ricker 2008, Remark 2.5], there exists $\sigma : \mathbb{R} \to B(E)$, a bounded strongly continuous representation such that for all $g \in L^1(\mathbb{R})$, $\lambda(g) = \int_{\mathbb{R}} g(t)\sigma(t)\,dt$ (defined in the strong sense). Let us show that

$$\Gamma(\tau_{-s}) = W\sigma(-s)V, \quad s > 0. \tag{20}$$

Let $\eta \in L^1(\mathbb{R})_+$ be such that $\int_{\mathbb{R}} \eta(t)\,dt = 1$. For any $r > 0$, let $\eta_r(t) = r\eta(rt)$. Since $\sigma : \mathbb{R} \to B(E)$ is strongly continuous, the function $t \mapsto \langle \sigma(t)x, x^*\rangle$ is continuous and we have

$$\int_{\mathbb{R}} \eta_r(-s-t)\langle \sigma(t)x, x^*\rangle\,dt \xrightarrow[r\to\infty]{} \langle \sigma(-s)x, x^*\rangle, \tag{21}$$

for all $x \in E$ and $x^* \in E^*$. Since the left-hand side in (21) is equal to

$$\langle \pi(c(\eta_r(-s-\cdot)))x, x^*\rangle,$$

we obtain, by Lemma 2.1, that $\pi(c(\eta_r(-s-\cdot))) \to \sigma(-s)$ in the w^*-topology of $B(E)$. This implies that $W\pi(c(\eta_r(-s-\cdot)))V \to W\sigma(-s)V$ in the w^*-topology of $B(L^p(\mathbb{R}_+))$. We next show that $W\pi(c(\eta_r(-s-\cdot)))V \to \Gamma(\tau_{-s})$ in the w^*-topology of $B(L^p(\mathbb{R}_+))$, which will complete the proof of (20). Since

$$W\pi(c(\eta_r(-s-\cdot)))V = \Gamma(c(\eta_r(-s-\cdot)))$$

and Γ is w^*-continuous, it suffices to show that $c(\eta_r(-s-\cdot)) \to \tau_{-s}$ in the w^*-topology of $B(L^p(\mathbb{R}))$. To see this, let $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$. We have that

$$\begin{aligned}\langle c(\eta_r(-s-\cdot))f, g\rangle &= \langle \eta_r(-s-\cdot) * f, g\rangle \\ &= \langle \delta_{-s} * \eta_r * f, g\rangle \\ &\to \langle \delta_{-s} * f, g\rangle = \langle \tau_{-s} f, g\rangle.\end{aligned}$$

By Lemma 2.1 again, this proves that $c(\eta_r(-s-\cdot)) \to \tau_{-s}$ in the w^*-topology, as expected.

Given any $\epsilon > 0$, let $m_\epsilon : \mathbb{R}_+^* \to \mathbb{C}$ be defined by

$$m_\epsilon(t) = m(t+\epsilon), \qquad t > 0.$$

Let $f \in L^p(\mathbb{R}_+)$ be given by $f = \epsilon^{-\frac{1}{p}}\chi_{(0,\epsilon)}$ and let $g \in L^{p'}(\mathbb{R}_+)$ be given by $g = \epsilon^{-\frac{1}{p'}}\chi_{(0,\epsilon)}$. For any $s, t > 0$, set

$$\alpha_\epsilon(s) := \sigma\big(-s - \tfrac{\epsilon}{2}\big)V(\tau_s f) \quad \text{and} \quad \beta_\epsilon(t) := \sigma\big(-t - \tfrac{\epsilon}{2}\big)^* W^*(\tau_t g).$$

Since σ is strongly continuous, α_ϵ and β_ϵ are continuous. By (19) and (20), we have that

$$\begin{aligned}\langle \alpha_\epsilon(s), \beta_\epsilon(t)\rangle_{E,E^*} &= \big\langle \sigma\big(-s-\tfrac{\epsilon}{2}\big)V(\tau_s f), \sigma\big(-t-\tfrac{\epsilon}{2}\big)^* W^*(\tau_t g)\big\rangle \\ &= \langle W\sigma(-s-t-\epsilon)V(\tau_s f), \tau_t g\rangle \\ &= \langle (\Gamma(\tau_{-s-t-\epsilon}))(\tau_s f), \tau_t g\rangle \\ &= m(s+t+\epsilon)\langle \theta_{s+t+\epsilon}(\tau_s f), \tau_t g\rangle \\ &= m_\epsilon(s+t)\langle \epsilon^{-1/p}\chi_{(t,t+\epsilon)}, \epsilon^{-1/p'}\chi_{(t,t+\epsilon)}\rangle \\ &= m_\epsilon(s+t),\end{aligned}$$

for all $s, t > 0$. Moreover, $\|\alpha_\epsilon(s)\| \leq \|V\|$ and $\|\beta_\epsilon(t)\| \leq \|W\|$ for all $t, s > 0$. Since α_ϵ and β_ϵ are continuous, this implies that $\alpha_\epsilon \in L^\infty(\mathbb{R}_+; E)$, $\beta_\epsilon \in L^\infty(\mathbb{R}_+; E^*)$ and $\|\alpha_\epsilon\|_\infty \|\beta_\epsilon\|_\infty \leq \|V\|\|W\| \leq 1$.

We now show that the SQ_p-space E can be replaced by an L^p-space in the above factorisation property of m_ϵ. Following Remark 2.3, assume that $E = E_1/E_2$, with $E_2 \subseteq E_1 \subseteq L^p(\Omega)$, and for all $f \in E_1$, let $\dot{f} \in E$ denote the class of f. Recall (5) and for all $g \in E_2^\perp$, let $\dot{g} \in E^*$ denote the class of g. Since E is a quotient of E_1, we have an isometric embedding $E^* \subseteq E_1^*$. More precisely,

$$E^* = \frac{E_2^\perp}{E_1^\perp} \hookrightarrow \frac{L^{p'}(\Omega)}{E_1^\perp} = E_1^*.$$

This induces an isometric embedding

$$L^1(\mathbb{R}_+; E^*) \subseteq L^1(\mathbb{R}_+; E_1^*).$$

Since E^* and E_1^* are reflexive, we may apply the identifications

$$L^1(\mathbb{R}_+; E^*)^* \simeq L^\infty(\mathbb{R}_+; E) \quad \text{and} \quad L^1(\mathbb{R}_+; E_1^*)^* \simeq L^\infty(\mathbb{R}_+; E_1)$$

provided by (7). By the Hahn–Banach theorem, we deduce the existence of $\widetilde{\alpha_\epsilon} \in L^\infty(\mathbb{R}_+; E_1)$ such that $\|\widetilde{\alpha_\epsilon}\|_\infty = \|\alpha_\epsilon\|_\infty$ and the functional $L^1(\mathbb{R}_+; E_1^*) \to \mathbb{C}$ induced by $\widetilde{\alpha_\epsilon}$ extends the functional $L^1(\mathbb{R}_+; E^*) \to \mathbb{C}$ induced by α_ϵ. It is easy to check that the latter means that $\dot{\widetilde{\alpha_\epsilon}}(s) = \alpha_\epsilon(s)$ almost everywhere on $\mathbb{R}_+$. Likewise, there exist $\widetilde{\beta_\epsilon} \in L^\infty(\mathbb{R}_+; E_2^\perp)$ such that $\|\widetilde{\beta_\epsilon}\|_\infty = \|\beta_\epsilon\|_\infty$ and $\dot{\widetilde{\beta_\epsilon}}(t) = \beta_\epsilon(t)$ almost everywhere on $\mathbb{R}_+$. Regard $\widetilde{\alpha_\epsilon}$ as an element of $L^\infty(\mathbb{R}_+, L^p(\Omega))$ and $\widetilde{\beta_\epsilon}$ as an element of $L^\infty(\mathbb{R}_+, L^{p'}(\Omega))$. By (6), we then have

$$\langle \alpha_\epsilon(s), \beta_\epsilon(t) \rangle_{E,E^*} = \langle \widetilde{\alpha_\epsilon}(s), \widetilde{\beta_\epsilon}(t) \rangle_{L^p, L^{p'}},$$

for almost every $(s, t) \in \mathbb{R}_+^{*2}$.

We therefore obtain that $m_\epsilon : \mathbb{R}_+^* \to \mathbb{C}$ satisfies condition (ii) of the theorem (with $C = 1$).

Define $\varphi : \mathbb{R}_+^{*2} \to \mathbb{C}$ by $\varphi(s, t) = m(s + t)$. Likewise, for any $\epsilon > 0$, define $\varphi_\epsilon : \mathbb{R}_+^{*2} \to \mathbb{C}$ by $\varphi(s, t) = m_\epsilon(s + t)$. Since m is continuous, the functions φ and φ_ϵ are continuous. It follows from above that for all $\epsilon > 0$, φ_ϵ satisfies condition (ii) in Theorem 2.5, with $C = 1$. The latter theorem therefore implies that the family $\{\varphi_\epsilon(s, t)\}_{(s,t)\in\mathbb{R}_+^{*2}}$ is a bounded Schur multiplier on $B(\ell^p_{\mathbb{R}_+^*})$, with norm less than one. Thus for all $[a_{ij}]_{1\leq i,j\leq n}$ in M_n and for all $t_1, \ldots, t_n, s_1, \ldots, s_n$ in $\mathbb{R}_+^*$, we have $\|[\varphi_\epsilon(s_i, t_j)a_{ij}]\|_{B(\ell_n^p)} \leq \|[a_{ij}]\|_{B(\ell_n^p)}$. Since m is continuous, $\varphi_\epsilon \to \varphi$ pointwise when $\epsilon \to 0$. We deduce that φ satisfies (8) with $C = 1$ for all $[a_{ij}]_{1\leq i,j\leq n}$ in M_n and all $t_1, \ldots, t_n, s_1, \ldots, s_n$ in $\mathbb{R}_+^*$. Consequently, the family $\{\varphi(s, t)\}_{(s,t)\in\mathbb{R}_+^{*2}}$ is a bounded Schur multiplier on $B(\ell^p_{\mathbb{R}_+^*})$, with norm less than one. Applying the implication (i) $\Longrightarrow$ (ii) in Theorem 2.5, we deduce the assertion (ii) of Theorem 4.6.

Conversely, assume (ii). Following Lemma 2.4, let

$$\pi : B(L^p(\mathbb{R}_+)) \to B(L^p(\mathbb{R}_+ \times \Omega))$$

be the p-completely isometric homomorphism defined by $\pi(T) = T \overline{\otimes} I_{L^p(\Omega)}$. This map is w^*-continuous. Indeed, let $(T_\iota)_\iota$ be a bounded net of $B(L^p(\mathbb{R}_+))$ converging to some $T \in B(L^p(\mathbb{R}_+))$ in the w^*-topology. For any $f \in L^p(\mathbb{R}_+)$, $g \in L^{p'}(\mathbb{R}_+)$, $\varphi \in L^p(\Omega)$ and $\psi \in L^{p'}(\Omega)$, we have

$$\big\langle \pi(T_\iota), (f \otimes \varphi) \otimes (g \otimes \psi) \big\rangle = \langle T_\iota f, g \rangle_{L^p(\mathbb{R}_+), L^{p'}(\mathbb{R}_+)} \langle \varphi, \psi \rangle_{L^p(\Omega), L^{p'}(\Omega)},$$

where the duality pairing in the left-hand side refers to the identification

$$\big(L^p(\mathbb{R}_+ \times \Omega) \widehat{\otimes} L^{p'}(\mathbb{R}_+ \times \Omega)\big)^* \simeq B(L^p(\mathbb{R}_+ \times \Omega)).$$

Since $\langle T_\iota f, g \rangle \to \langle T f, g \rangle$, we deduce that

$$\langle \pi(T_\iota), (f \otimes \varphi) \otimes (g \otimes \psi) \rangle \to \langle \pi(T), (f \otimes \varphi) \otimes (g \otimes \psi) \rangle.$$

Since $L^p(\mathbb{R}_+) \otimes L^p(\Omega)$ and $L^{p'}(\mathbb{R}_+) \otimes L^{p'}(\Omega)$ are dense in $L^p(\mathbb{R}_+ \times \Omega)$ and $L^{p'}(\mathbb{R}_+ \times \Omega)$, respectively, we deduce that $\pi(T_\iota) \to \pi(T)$ in the w^*-topology, by Lemma 2.1. This proves that π is w^*-continuous.

Let $V : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+; L^p(\Omega)) \simeq L^p(\mathbb{R}_+ \times \Omega)$ be defined by

$$V(f) = f\alpha, \quad f \in L^p(\mathbb{R}_+).$$

This is a well-defined contraction. Likewise we define a contraction

$$W : L^p(\mathbb{R}_+ \times \Omega) \to L^p(\mathbb{R}_+)$$

by setting

$$W^*(g) = g\beta, \quad g \in L^{p'}(\mathbb{R}_+).$$

It follows from above and from the implication (ii) $\Longrightarrow$ (i) of Theorem 2.2 that the mapping

$$w : B(L^p(\mathbb{R}_+)) \to B(L^p(\mathbb{R}_+)), \quad w(T) = W\pi(T)V$$

is a w^*-continuous p-complete contraction.

We claim that for all $u > 0$, we have

$$w(\theta_u) = m(u)\theta_u. \tag{22}$$

To prove this, consider $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$. For all $u > 0$, we have

$$\langle w(\theta_u) f, g \rangle = \langle \pi(\theta_u) V(f), W^*(g) \rangle = \langle \pi(\theta_u)(f\alpha), (g\beta) \rangle.$$

By the definitions of π and θ_u, we have $\pi(\theta_u)(f\alpha) = (f\alpha)(u - \cdot\,)$. Consequently,

$$\langle w(\theta_u) f, g \rangle = \int_0^u f(u-t) g(t) \langle \alpha(u-t), \beta(t) \rangle \, dt, \quad u > 0.$$

Let $h \in L^1(\mathbb{R}_+)$ be an auxiliary function. Then using Fubini's theorem and setting $s = u - t$ in due place, we obtain that

$$\int_0^\infty \langle w(\theta_u) f, g\rangle h(u)\, du = \int_0^\infty \int_t^\infty h(u) f(u-t) g(t) \langle \alpha(u-t), \beta(t)\rangle\, du dt$$
$$= \int_0^\infty \int_0^\infty h(s+t) f(s) g(t) \langle \alpha(s), \beta(t)\rangle\, ds dt.$$

Applying the a.e. equality $m(s+t) = \langle \alpha(s), \beta(t)\rangle$ and reversing this computation, we deduce that

$$\int_0^\infty \langle w(\theta_u) f, g\rangle h(u)\, du = \int_0^\infty m(u)(f * g)(u) h(u)\, du.$$

Since h is arbitrary, this implies that $\langle w(\theta_u) f, g\rangle = m(u)(f * g)(u)$ for a.e. $u > 0$. Equivalently, $\langle w(\theta_u) f, g\rangle = m(u)\langle \theta_u f, g\rangle$ for a.e. $u > 0$. It is plain that $u \mapsto \theta_u$ is w^*-continuous on $B(L^p(\mathbb{R}_+))$. Since w is w^*-continuous, the function $u \mapsto \langle w(\theta_u) f, g\rangle$ is continuous as well. Since m is assumed continuous, we deduce that $\langle w(\theta_u) f, g\rangle = m(u)\langle \theta_u f, g\rangle$ for all $u > 0$. This yields (22), for all $u > 0$.

By part (1) of Theorem 4.1 and the w^*-continuity of w, the identity (22) implies that $\mathrm{Hank}_p(\mathbb{R}_+)$ is an invariant subspace of w. Further the restriction of w to $\mathrm{Hank}_p(\mathbb{R}_+)$ is the multiplier associated to m. The assertion (i) follows. □

Remark 4.7. We proved in [Arnold et al. 2022, Theorem 3.1] that a continuous function $m : \mathbb{R}_+^* \to \mathbb{C}$ is the symbol of an S^1-bounded Fourier multiplier on $H^1(\mathbb{R})$, with S^1-bounded norm $\leq C$, if and only if there exist a Hilbert space $\mathcal{H}$ and two functions $\alpha, \beta \in L^\infty(\mathbb{R}_+; \mathcal{H})$ such that $\|\alpha\|_\infty \|\beta\|_\infty \leq C$ and $m(s+t) = \langle \alpha(t), \beta(s)\rangle_{\mathcal{H}}$ for almost every $(s, t) \in \mathbb{R}_+^{*2}$. It turns out that using (1), a mapping $S : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ is an S^1-bounded Fourier multiplier with S^1-bounded norm $\leq C$ if and only if $S^* : \mathrm{Hank}_2(\mathbb{R}_+) \to \mathrm{Hank}_2(\mathbb{R}_+)$ is a completely bounded multiplier with completely bounded norm $\leq C$. See [Arnold et al. 2022, Remark 3.4] for more on this. Thus the statement in [Arnold et al. 2022, Theorem 3.1] is equivalent to the case $p = 2$ of Theorem 4.6. In this regard, Theorem 4.6 can be regarded as a p-analogue of [Arnold et al. 2022, Theorem 3.1].

Remark 4.8. Let $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$. For any $s, t > 0$, we may write

$$(f * g)(s+t) = \int_{\mathbb{R}} f(s+r) g(t-r)\, dr.$$

Equivalently,

$$(f * g)(s+t) = \langle \tau_{-s} f, \tau_t \check{g}\rangle_{L^p(\mathbb{R}_+), L^{p'}(\mathbb{R}_+)}.$$

According to the implication (ii) $\Longrightarrow$ (i) of Theorem 4.6 and Remark 4.4, $f * g$ is therefore a pointwise multiplier of $A_p(\mathbb{R}_+)$, with norm less than or equal to $\|f\|_p \|g\|_{p'}$. We deduce that every $F \in A_p(\mathbb{R}_+)$ is a pointwise multiplier of $A_p(\mathbb{R}_+)$, with norm less than or equal to $\|F\|_{A_p}$. This means that $A_p(\mathbb{R}_+)$ is a Banach algebra for the pointwise product.

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References

[Arnold et al. 2022] L. Arnold, C. L. Merdy, and S. Zadeh, "S^1-bounded Fourier multipliers on $H^1(\mathbb{R})$ and functional calculus for semigroups", 2022. arXiv 2203.16829

[Coine 2018] C. Coine, "Schur multipliers on $\mathcal{B}(L^p, L^q)$", *J. Operator Theory* **79**:2 (2018), 301–326. MR Zbl

[Daws 2010] M. Daws, "p-operator spaces and Figà-Talamanca–Herz algebras", *J. Operator Theory* **63**:1 (2010), 47–83. MR Zbl

[Derighetti 2011] A. Derighetti, *Convolution operators on groups*, Lecture Notes of the Unione Matematica Italiana **11**, Springer, 2011. MR Zbl

[Diestel and Uhl 1977] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys **15**, American Mathematical Society, Providence, RI, 1977. MR Zbl

[Herz 1974] C. Herz, "Une généralisation de la notion de transformée de Fourier-Stieltjes", *Ann. Inst. Fourier* (*Grenoble*) **24**:3 (1974), xiii, 145–157. MR Zbl

[Le Merdy 1996] C. Le Merdy, "Factorization of p-completely bounded multilinear maps", *Pacific J. Math.* **172**:1 (1996), 187–213. MR Zbl

[Nikolski 2002] N. K. Nikolski, *Operators*, *functions*, *and systems*: *an easy reading*, *I*: *Hardy*, *Hankel*, *and Toeplitz*, vol. 1, Mathematical Surveys and Monographs **92**, American Mathematical Society, Providence, RI, 2002. MR Zbl

[Nikolski 2020] N. K. Nikolski, *Toeplitz matrices and operators*, Cambridge Studies in Advanced Mathematics **182**, Cambridge University Press, 2020. MR Zbl

[de Pagter and Ricker 2008] B. de Pagter and W. J. Ricker, "R-bounded representations of $L^1(G)$", *Positivity* **12**:1 (2008), 151–166. MR

[Paulsen 2002] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, 2002. MR Zbl

[Peller 2003] V. V. Peller, *Hankel operators and their applications*, Springer, 2003. MR Zbl

[Pisier 1990] G. Pisier, "Completely bounded maps between sets of Banach space operators", *Indiana Univ. Math. J.* **39**:1 (1990), 249–277. MR Zbl

[Pisier 2001] G. Pisier, *Similarity problems and completely bounded maps*, Lecture Notes in Mathematics **1618**, Springer, 2001. MR Zbl

[Power 1982] S. C. Power, *Hankel operators on Hilbert space*, Research Notes in Mathematics **64**, Pitman, Boston, 1982. MR Zbl

[Yafaev 2015] D. R. Yafaev, "On finite rank Hankel operators", *J. Funct. Anal.* **268**:7 (2015), 1808–1839. MR Zbl

[Yafaev 2017a] D. R. Yafaev, "Hankel and Toeplitz operators: continuous and discrete representations", *Opuscula Math.* **37**:1 (2017), 189–218. MR Zbl

[Yafaev 2017b] D. R. Yafaev, "Quasi-diagonalization of Hankel operators", *J. Anal. Math.* **133** (2017), 133–182. MR Zbl

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