HANKEL OPERATORS ON $L^p(\mathbb{R}_+)$ AND THEIR $p$-COMPLETELY BOUNDED MULTIPLIERS

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We show that for any $1 < p < \infty$, the space $\text{Hank}_p(\mathbb{R}_+)$ is equal to the $w^*$-closure of the linear span of the operators $\theta_u : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ defined by $\theta_u f = f(u - \cdot)$, for $u > 0$. We deduce that $\text{Hank}_p(\mathbb{R}_+)$ is the dual space of $A_p(\mathbb{R}_+)$, a half-line analogue of the Figà-Talamanca–Herz algebra $A_p(\mathbb{R})$. Then we show that a function $m : \mathbb{R}_+^* \to \mathbb{C}$ is the symbol of a $p$-completely bounded multiplier $\text{Hank}_p(\mathbb{R}_+) \to \text{Hank}_p(\mathbb{R}_+)$ if and only if there exist $\alpha \in L^\infty(\mathbb{R}_+; L^p(\mathcal{S}))$ and $\beta \in L^\infty(\mathbb{R}_+; L^p'(\mathcal{S}))$ such that $m(s + t) = \langle \alpha(s), \beta(t) \rangle$ for a.e. $(s, t) \in \mathbb{R}_+^2$.

We also give analogues of these results in the (easier) discrete case.

1. Introduction

For any $u > 0$ and for any function $f : \mathbb{R}_+ \to \mathbb{C}$, let $\tau_u f : \mathbb{R}_+ \to \mathbb{C}$ be the shifted function defined by $\tau_u f = f(\cdot - u)$. Let $1 < p, p' < \infty$ be two conjugate indices. We say that a bounded operator $T : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ is Hankelian if $\langle T \tau_u f, g \rangle = \langle T f, \tau_u g \rangle$ for all $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$. Let $B(L^p(\mathbb{R}_+))$ denote the Banach space of all bounded operators on $L^p(\mathbb{R}_+)$. The main object of this paper is the subspace $\text{Hank}_p(\mathbb{R}_+) \subseteq B(L^p(\mathbb{R}_+))$ of all Hankel operators on $L^p(\mathbb{R}_+)$. The case $p = 2$ has received a lot of attention; see [Nikolski 2002; 2020; Peller 2003; Yafaev 2015; 2017a; 2017b]. The most important result in this case is that $\text{Hank}_2(\mathbb{R}_+)$ is isometrically isomorphic to the quotient space $L^\infty(\mathbb{R})/H^\infty(\mathbb{R})$, where $H^\infty(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$ is the classical Hardy space of essentially bounded functions whose Fourier transform has support in $\mathbb{R}_+$ (see [Nikolski 2020, Section IV.5.3] or [Peller 2003, Theorem I.8.1]). This result is the real line analogue of Nehari’s classical theorem describing Hankel operators on $\ell^2$ (see [Nikolski 2020, Theorem II.2.2.4], [Peller 2003, Theorem I.1.1] or [Power 1982, Theorem 1.3]). An equivalent formulation of the above result is that

$$\text{Hank}_2(\mathbb{R}_+) \simeq H^1(\mathbb{R})^*, \tag{1}$$

where $H^1(\mathbb{R}) \subseteq L^1(\mathbb{R})$ is the Hardy space of all integrable functions whose Fourier transform vanishes on $\mathbb{R}_-$.

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**Keywords:** $p$-complete boundedness, multipliers, Hankel operators.

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The first main result of this paper is that for any $1 < p < \infty$, the Banach space $\text{Hank}_p(\mathbb{R}_+)$ coincides with $\text{Span}^{w^*}\{\theta_u : u > 0\} \subset B(L^p(\mathbb{R}_+))$, where, for any $u > 0$, $\theta_u : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ is the Hankel operator defined by $\theta_u f = f(u - \cdot)$. As a consequence, we show that

\begin{equation}
\text{Hank}_p(\mathbb{R}_+) \simeq A_p(\mathbb{R}_+)^*,
\end{equation}

where $A_p(\mathbb{R}_+)$ is a half-line analogue of the Figà-Talamanca–Herz algebra $A_p(\mathbb{R})$ (see, e.g., [Derighetti 2011, Chapter 3]). We will see in Remark 4.2(a) that $A_2(\mathbb{R}_+) \simeq H^1(\mathbb{R})$. Thus, the duality result (2), established in Theorem 4.1, is an $L^p$-version of (1).

By a multiplier of $\text{Hank}_p(\mathbb{R}_+)$, we mean a $w^*$-continuous operator

$T : \text{Hank}_p(\mathbb{R}_+) \to \text{Hank}_p(\mathbb{R}_+)$

such that $T(\theta_u) = m(u)\theta_u$ for all $u > 0$, for some function $m : \mathbb{R}_+^* \to \mathbb{C}$. In this case, we set $T = T_m$ and it turns out that $m$ is necessarily bounded and continuous, see Lemma 4.5. The second main result of this paper is a characterization of $p$-completely bounded multipliers $T_m$. We refer to Section 2 for some background on $p$-complete boundedness, whose definition goes back to [Pisier 1990] (see also [Daws 2010; Le Merdy 1996; Pisier 2001]). We prove in Theorem 4.6 that $T_m : \text{Hank}_p(\mathbb{R}_+) \to \text{Hank}_p(\mathbb{R}_+)$ is a $p$-completely bounded multiplier if and only if there exist a measure space $(\Omega, \mu)$ and two essentially bounded measurable functions $\alpha : \mathbb{R}_+ \to L^p(\Omega)$ and $\beta : \mathbb{R}_+ \to L^{p'}(\Omega)$ such that $m(s + t) = \langle \alpha(s), \beta(t) \rangle$ for almost every $(s, t) \in \mathbb{R}_+^2$. This is a generalisation of [Arnold et al. 2022, Theorem 3.1]. Indeed, the result in [Arnold et al. 2022] provides a characterization of $S^1$-bounded multipliers on $H^1(\mathbb{R})$. Using (1), this yields a characterization of completely bounded multipliers on $\text{Hank}_2(\mathbb{R}_+)$, which is nothing but the case $p = 2$ of Theorem 4.6. See Remark 4.7 for more on this.

Let us briefly explain the plan of the paper. Section 2 contains some preliminary results. Section 3 is devoted to $\text{Hank}_p(\mathbb{N}) \subset B(\ell^p)$, the space of Hankel operators on $\ell^p$. We establish analogues of the aforementioned results in the discrete setting. Results for $\text{Hank}_p(\mathbb{N})$ are easier than those concerning $\text{Hank}_p(\mathbb{R}_+)$ and Section 3 can be considered as a warm up. The main results are stated and proved in Section 4.

2. Preliminaries

All our Banach spaces are complex ones. For any Banach spaces $X, Z$, we let $B(X, Z)$ denote the Banach space of all bounded operators from $X$ into $Z$ and we write $B(X)$ instead of $B(X, X)$ when $Z = X$. For any $x \in X$ and $x^* \in X^*$, the duality action $x^*(x)$ is denoted by $\langle x^*, x \rangle_{X^*, X}$, or simply by $\langle x^*, x \rangle$ if there is no risk of confusion.
We start with duality on tensor products. Let $X$, $Y$ be Banach spaces. Let $X \hat{\otimes} Y$ denote their projective tensor product [Diestel and Uhl 1977, Section VIII.1]. We will use the classical isometric identification

$$(X \hat{\otimes} Y)^* \simeq B(X, Y^*)$$

provided, e.g., by [Diestel and Uhl 1977, Corollary VIII.2.2]. More precisely, for any $\xi \in (X \hat{\otimes} Y)^*$, there exists a necessarily unique $R_{\xi} \in B(X, Y^*)$ such that $\xi(x \otimes y) = \langle R_{\xi}(x), y \rangle$ for all $x \in X$ and $y \in Y$. Moreover $\|R_{\xi}\| = \|\xi\|$ and the mapping $\xi \mapsto R_{\xi}$ is onto.

**Lemma 2.1.** Let $A \subset X$ and $B \subset Y$ such that $\text{Span}\{A\}$ is dense in $X$ and $\text{Span}\{B\}$ is dense in $Y$. Assume that $(R_i)_i$ is a bounded net of $B(X, Y^*)$. Then $R_i$ converges to some $R \in B(X, Y^*)$ in the $w^*$-topology if and only if $\langle R_i(x), y \rangle \to \langle R(x), y \rangle$ for all $x \in A$ and $y \in B$.

**Proof.** Assume the latter property. Since the algebraic tensor product $X \otimes Y$ is dense in $X \hat{\otimes} Y$, it implies that $\langle R_i, z \rangle \to \langle R, z \rangle$, for all $z$ belonging to a dense subspace of $X \hat{\otimes} Y$. Next, the boundedness of $(R_i)_i$ implies that $\langle R_i, z \rangle \to \langle R, z \rangle$, for all $z$ belonging to $X \hat{\otimes} Y$. The equivalence follows.

We will use the above duality principles in the case when $X = Y^*$ is an $L^p$-space $L^p(\Omega)$, for some index $1 < p < \infty$.

We now give a brief background on $p$-completely bounded maps, following [Pisier 1990] (see also [Daws 2010; Le Merdy 1996; Pisier 2001]). Let $1 < p < \infty$ and let $SQ_p$ denote the collection of quotients of subspaces of $L^p$-spaces, where we identify spaces which are isometrically isomorphic. Let $E$ be an $SQ_p$-space. Let $n \geq 1$ be an integer and let $[T_{ij}]_{1 \leq i,j \leq n} \in M_n \otimes B(E)$ be an $n \times n$ matrix with entries $T_{ij}$ in $B(E)$. We equip $M_n \otimes B(E)$ with the norm defined by

$$(4) \quad \|[T_{ij}]\| = \sup \left\{ \left( \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij}(x_j) \right)^p \right\}^{\frac{1}{p}} : x_1, \ldots, x_n \in E, \sum_{i=1}^{n} \|x_i\|^p \leq 1 \right\}. $$

If $S \subset B(E)$ is any subspace, then we let $M_n(S)$ denote $M_n \otimes S$ equipped with the induced norm.

Let $S_1$ and $S_2$ be subspaces of $B(E_1)$ and $B(E_2)$, respectively, for some $SQ_p$-spaces $E_1$ and $E_2$. Let $w: S_1 \to S_2$ be a linear map. For any $n \geq 1$, let $w_n: M_n(S_1) \to M_n(S_2)$ be defined by $w_n([T_{ij}]) = [w(T_{ij})]$, for any $[T_{ij}]_{1 \leq i,j \leq n} \in M_n(S_1)$. By definition, $w$ is called $p$-completely bounded if the maps $w_n$ are uniformly bounded. In this case, the $p\text{-cb}$ norm of $w$ is defined by $\|w\|_{p\text{-cb}} = \sup_n \|w_n\|$. We further say that $w$ is $p$-completely contractive if $\|w\|_{p\text{-cb}} \leq 1$ and that $w$ is a $p$-complete isometry if $w_n$ is an isometry for all $n \geq 1$. Note that the case $p = 2$ corresponds to the classical notion of completely bounded maps (see, e.g., [Paulsen 2002; Pisier 2001]).
We recall the following factorisation theorem of Pisier (see [Le Merdy 1996, Theorem 1.4; Pisier 1990, Theorem 2.1]), which extends Wittstock’s factorisation theorem [Paulsen 2002, Theorem 8.4].

**Theorem 2.2.** Let \((\Omega_1, \mu_1)\) and \((\Omega_2, \mu_2)\) be measure spaces and let \(1 < p < \infty\). Let \(S \subseteq B(L^p(\Omega_1))\) be a unital subalgebra. Let \(w : S \to B(L^p(\Omega_2))\) be a linear map and let \(C \geq 0\) be a constant. The following assertions are equivalent.

1. The map \(w\) is \(p\)-completely bounded and \(\|w\|_{p-cb} \leq C\).
2. There exist an \(S Q_p\)-space \(E\), a unital, nondegenerate \(p\)-completely contractive homomorphism \(\pi : S \to B(E)\) as well as operators \(V : L^p(\Omega_2) \to E\) and \(W : E \to L^p(\Omega_2)\) such that \(\|V\|\|W\| \leq C\) and for any \(x \in S\), \(w(x) = W\pi(x)V\).

**Remark 2.3.** Let \(1 < p < \infty\) and let \(p'\) be its conjugate index. Let \(E\) be an \(S Q_p\)-space. Then by assumption, there exist a measure space \((\Omega, \mu)\) and two closed subspaces \(E_2 \subseteq E_1 \subseteq L^p(\Omega)\) such that \(E = E_1/E_2\). Then \(E_1^\perp \subseteq E_2^\perp \subset L^{p'}(\Omega)\) and we have an isometric identification

\[
E^* \simeq \frac{E_2^\perp}{E_1^\perp},
\]

by the classical duality between subspaces and quotients of Banach spaces. More explicitly, let \(f \in E_1\) and let \(g \in E_2^\perp\). Let \(\hat{f} \in E\) denote the class of \(f\) modulo \(E_2\) and let \(\hat{g} \in E^*\) denote the element associated to the class of \(g\) modulo \(E_1^\perp\) through the identification (5). Then we have

\[
\langle \hat{g}, \hat{f} \rangle_{E^*, E} = \langle g, f \rangle_{L^{p'}, L^p}.
\]

We now turn to Bochner spaces. Let \((\Sigma, \nu)\) be a measure space and let \(X\) be a Banach space. For any \(1 \leq p \leq \infty\), we let \(L^p(\Sigma; X)\) denote the space of all measurable functions \(\phi : \Sigma \to X\) (defined up to almost everywhere zero functions) such that the norm function \(t \mapsto \|\phi(t)\|\) belongs to \(L^p(\Sigma)\). This is a Banach space for the norm \(\|\phi\|_p\), defined as the \(L^p(\Sigma)\)-norm of \(\|\phi(\cdot)\|\) (see, e.g., [Diestel and Uhl 1977, Chapters I and II]).

Assume that \(p\) is finite and note that in this case, \(L^p(\Sigma) \otimes X\) is dense in \(L^p(\Sigma; X)\). Let \(p'\) be the conjugate index of \(p\). For all \(\phi \in L^p(\Sigma; X)\) and \(\psi \in L^{p'}(\Sigma; X^*)\), the function \(t \mapsto \langle \psi(t), \phi(t) \rangle_{X^*, X}\) belongs to \(L^1(\Sigma)\) and the resulting duality paring \(\langle \psi, \phi \rangle := \int_\Omega \langle \psi(t), \phi(t) \rangle d\nu(t)\) extends to an isometric embedding \(L^{p'}(\Sigma; X^*) \hookrightarrow L^p(\Sigma; X)^*\). Furthermore, this embedding is onto if \(X\) is reflexive, that is,

\[
L^{p'}(\Sigma; X^*) \simeq L^p(\Sigma; X)^* \quad \text{if} \ X \text{ is reflexive.}
\]

We refer to [Diestel and Uhl 1977, Corollary III.2.13 and Section IV.1] for these results and complements.
Let \((\Sigma, \nu)\) and \((\Omega, \mu)\) be two measure spaces. Then we have an isometric identification
\[
L^p(\Sigma; L^p(\Omega)) \simeq L^p(\Sigma \times \Omega),
\]
from which it follows that for any \(T \in B(L^p(\Sigma))\), the tensor extension
\[
T \otimes I_{L^p(\Omega)} : L^p(\Sigma) \otimes L^p(\Omega) \to L^p(\Sigma) \otimes L^p(\Omega)
\]
extends to a bounded operator \(T \otimes I_{L^p(\Omega)}\) on \(L^p(\Sigma \times \Omega)\), whose norm is equal to the norm of \(T\). The following is elementary.

**Lemma 2.4.** The mapping \(\pi : B(L^p(\Sigma)) \to B(L^p(\Sigma \times \Omega))\) defined by \(\pi(T) = T \otimes I_{L^p(\Omega)}\) is a \(p\)-complete isometry.

**Proof.** Let \(n \geq 1\) and let \(J_n = \{1, \ldots, n\}\). It follows from (4) that \(M_n(B(L^p(\Sigma))) = B(\ell_n^p(L^p(\Sigma)))\) and hence \(M_n(B(L^p(\Sigma))) = B(L^p(J_n \times \Sigma))\) isometrically. Likewise, we have \(M_n(B(L^p(\Sigma \times \Omega))) = B(L^p(J_n \times \Sigma \times \Omega))\) isometrically. Through these identifications,
\[
[T_{ij} \otimes I_{L^p(\Omega)}] = [T_{ij}] \otimes I_{L^p(\Omega)},
\]
for all \([T_{ij}]_{1 \leq i, j \leq n}\) in \(M_n(B(L^p(\Sigma)))\). The result follows at once. \(\square\)

We finally state an important result concerning Schur products on \(B(\ell^p_I)\)-spaces.

Let \(I\) be an index set, let \(1 < p < \infty\) and let \(\ell^p_I\) denote the discrete \(L^p\)-space over \(I\). Let \((e_t)_{t \in I}\) be its canonical basis. To any \(T \in B(\ell^p_I)\), we associate a matrix of complex numbers, \([a_{st}]_{s, t \in I}\), defined by \(a_{st} = \langle T(e_t), e_s\rangle\), for all \(s, t \in I\). Following [Pisier 2001, Chapter 5], we say that a bounded family \(\{\varphi(s, t)\}_{s, t \in I}\) of complex numbers is a bounded Schur multiplier on \(B(\ell^p_I)\) if for all \(T \in B(\ell^p_I)\), with matrix \([a_{st}]_{s, t \in I}\), the matrix \([\varphi(s, t)a_{st}]_{s, t \in I}\) represents an element of \(B(\ell^p_I)\). In this case, the mapping \([a_{st}] \to [\varphi(s, t)a_{st}]\) is a bounded operator from \(B(\ell^p_I)\) into itself. We note that \(\{\varphi(s, t)\}_{s, t \in I}\) is a bounded Schur multiplier with norm \(\leq C\) if and only if for all \(n \geq 1\), all \([a_{ij}]_{1 \leq i, j \leq n}\) in \(M_n\) and all \(t_1, \ldots, t_n, s_1, \ldots, s_n\) in \(I\), we have
\[
\|[\varphi(s_i, t_j)a_{ij}]\|_{B(\ell^p_I)} \leq C \|[a_{ij}]\|_{B(\ell^p_I)}.
\]

In the sequel, we apply the above definitions to the case when \(I = \mathbb{R}^*_+\).

**Theorem 2.5.** Let \(\varphi : \mathbb{R}^*_+^{2} \to \mathbb{C}\) be a continuous bounded function. Let \(1 < p, p' < \infty\) be conjugate indices and let \(C \geq 0\) be a constant. The following assertions are equivalent.

(i) The family \(\{\varphi(s, t)\}_{(s, t) \in \mathbb{R}^*_+^{2}}\) is a bounded Schur multiplier on \(B(\ell^p_{\mathbb{R}^*_+})\), with norm \(\leq C\).

(ii) There exist a measure space \((\Omega, \mu)\) and two functions \(\alpha \in L^{\infty}(\mathbb{R}^*_+; L^p(\Omega))\) and \(\beta \in L^{\infty}(\mathbb{R}^*_+; L^{p'}(\Omega))\) such that \(\|\alpha\|_{\infty}\|\beta\|_{\infty} \leq C\) and \(\varphi(s, t) = (\alpha(s), \beta(t))_{L^p, L^{p'}}\) for almost every \((s, t) \in \mathbb{R}^*_+^{2}\).
Proof. According to [Coine 2018, Section 4.1], (ii) is equivalent to the fact that as an element of $L^\infty(\mathbb{R}_+^2)$,

(ii’) $\varphi$ is a bounded Schur multiplier on $B(L^p(\mathbb{R}_+))$.

It further follows from [Herz 1974, Lemmas 1 and 2] that since $\varphi$ is continuous, (ii’) is equivalent to (i). The result follows. $\square$

3. Hankel operators on $\ell^p$ and their multipliers

In this section we work on the sequence spaces $\ell^p = \ell^p_N$, where $N = \{0, 1, \ldots\}$. For any $1 < p < \infty$, we let $(e_n)_{n \geq 0}$ denote the classical basis of $\ell^p$. For any $T \in B(\ell^p)$, the associated matrix $[t_{ij}]_{i, j \geq 0}$ is given by $t_{ij} = \langle T(e_j), e_i \rangle$, for all $i, j \geq 0$.

Let $\text{Hank}_p(N) \subseteq B(\ell^p)$ be the subspace of all $T \in B(\ell^p)$ whose matrix is Hankelian, i.e., has the form $[c_{i+j}]_{i, j \geq 0}$ for some sequence $(c_k)_{k \geq 0}$ of complex numbers.

Let $p'$ be the conjugate index of $p$ and regard $\ell^p \otimes \ell^{p'} \subset B(\ell^p)$ in the usual way. We set

$$\gamma_k = \sum_{i+j=k} e_i \otimes e_j$$

for any $k \geq 0$. Then each $\gamma_k$ belongs to $\text{Hank}_p(N)$, and $\|\gamma_k\| = 1$. Indeed, the matrix of $\gamma_k$ is $[c_{i+j}]_{i, j \geq 0}$ with $c_k = 1$ and $c_l = 0$ for all $l \neq k$.

Lemma 3.1. For any $1 < p < \infty$, the space $\text{Hank}_p(N)$ is the $w^*$-closure of the linear span of $\{\gamma_k : k \geq 0\}$.

Proof. It is plain that $\text{Hank}_p(N)$ is a $w^*$-closed subspace of $B(\ell^p)$, hence one inclusion is straightforward.

To check the other one, consider $T \in \text{Hank}_p(N)$. By the definition of this space, there is a sequence $(c_k)_{k \geq 0}$ of $\mathbb{C}$ such that

$$\langle T(e_j), e_i \rangle = c_{i+j}, \quad \text{for all } i, j \geq 0.$$ 

For any $n \geq 1$, let $K_n$ be the Fejér kernel defined by

$$K_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right)e^{int}, \quad t \in \mathbb{R}.$$ 

Then let $T_n \in B(\ell^p)$ be the finite rank operator whose matrix is $[\widehat{K}_n(i+j)c_{i+j}]_{i, j \geq 0}$. Note that

$$T_n = \sum_{k=0}^{n} \left(1 - \frac{|k|}{n}\right)c_k \gamma_k \in \text{Span}\{\gamma_k : k \geq 0\}.$$
We show that $\|T_n\| \leq \|T\|$. To see this, let $\alpha = (\alpha_j)_{j \geq 0} \in \ell^p$ and $(\beta_m)_{m \geq 0} \in \ell^{p'}$. We have that

$$\langle T_n(\alpha), \beta \rangle = \sum_{m, j \geq 0} K_n(m + j) c_{m+j} \alpha_j \beta_m$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t) \sum_{m, j \geq 0} c_{m+j} \alpha_j \beta_m e^{-i(m+j)t} \, dt.$$ 

Since $K_n \geq 0$, we deduce

$$|\langle T_n(\alpha), \beta \rangle| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t) \left| \sum_{m, j \geq 0} c_{m+j} \alpha_j \beta_m e^{-i(m+j)t} \right| \, dt.$$ 

Now for all $t \in [-\pi, \pi]$, we have

$$\left| \sum_{m, j \geq 0} c_{m+j} \alpha_j \beta_m e^{-i(m+j)t} \right| = \left| \sum_{m, j \geq 0} c_{m+j} (e^{-ijt} \alpha_j)(e^{-imt} \beta_m) \right|$$

$$= \left| \langle T((e^{-ijt} \alpha_j)_{j \geq 0}), (e^{-imt} \beta_m)_{m \geq 0} \rangle \right|$$

$$\leq \|T\| \left( \sum_{j \geq 0} |e^{-ijt} \alpha_j|^p \right)^{\frac{1}{p}} \left( \sum_{m \geq 0} |e^{-imt} \beta_m|^p \right)^{\frac{1}{p'}}$$

$$\leq \|T\| \|\alpha\|_p \|\beta\|_{p'},$$

Since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t) \, dt = 1,$$

we therefore obtain that $|\langle T_n(\alpha), \beta \rangle| \leq \|T\| \|\alpha\|_p \|\beta\|_{p'}$. This proves that $\|T_n\| \leq \|T\|$, as requested.

For all $i, j \geq 0$,

$$\langle T_n(e_j), e_i \rangle = \langle \widehat{K}_n(i + j), T e_j, e_i \rangle \rightarrow \langle T e_j, e_i \rangle,$$

when $n \to \infty$. Hence $T_n \rightarrow T$ in the $w^*$-topology, by Lemma 2.1. Consequently, $T$ belongs to the $w^*$-closure of Span$\{\gamma_k : k \geq 0\}$. □

**Remark 3.2.** (a) Nehari’s celebrated theorem (see, e.g., [Nikolski 2020, Theorem II.2.2.4], [Peller 2003, Theorem I.1.1] or [Power 1982, Theorem 1.3]) asserts that

$$\text{Hank}_2(\mathbb{N}) \simeq \frac{L^\infty(\mathbb{T})}{H^\infty(\mathbb{T})}.$$ 

Here $\mathbb{T}$ stands for the unit circle of $\mathbb{C}$ and $H^\infty(\mathbb{T}) \subset L^\infty(\mathbb{T})$ is the Hardy space of functions whose negative Fourier coefficients vanish. The isometric isomorphism $J : L^\infty(\mathbb{T})/H^\infty(\mathbb{T}) \rightarrow \text{Hank}_2(\mathbb{N})$ providing (9) is defined as follows. Given any $F \in L^\infty(\mathbb{T})$, let $\hat{F}$ denote its class modulo $H^\infty(\mathbb{T})$. Then $J(\hat{F})$ is the operator whose matrix is equal to $[\hat{F}(-i - j - 1)]_{i,j\geq 0}$.
(b) We remark that $\text{Hank}_p(\mathbb{N}) \subseteq \text{Hank}_2(\mathbb{N})$. To see this, note that if $T \in \text{Hank}_p(\mathbb{N})$, then because of the symmetry in its matrix representation due to being a Hankelian matrix, $T$ has the same matrix representation as $T^*$, and therefore $T$ extends to a bounded operator on $\ell^p$. By interpolation, $T$ extends to a bounded operator on $\ell^2$, which is represented by the same matrix as $T$. Hence, $T$ belongs to $\text{Hank}_2(\mathbb{N})$.

However for $1 < p \neq 2 < \infty$, there is no description of $\text{Hank}_p(\mathbb{N})$ similar to Nehari’s theorem.

(c) The definition of $\text{Hank}_p(\mathbb{N})$ readily extends to the case $p = 1$ isometrically:

$$\text{Hank}_1(\mathbb{N}) \simeq \ell^1.$$ 

Indeed, let $J_1 : \ell^1 \to \text{Hank}_1(\mathbb{N})$ be defined by

$$J_1(c) = \sum_{k=0}^{\infty} c_k \gamma_k, \quad c = (c_k)_{k \geq 0} \in \ell^1.$$ 

Next, let $J_2 : \text{Hank}_1(\mathbb{N}) \to \ell^1$ be defined by $J_2(T) = T(e_0)$. Then $J_1, J_2$ are contractions and it is easy to check that they are inverse to each other. Hence $J_1$ is an isometric isomorphism.

We say that a sequence $m = (m_k)_{k \geq 0}$ in $\mathbb{C}$ is the symbol of a multiplier on $\text{Hank}_p(\mathbb{N})$ if there is a $w^*$-continuous operator $T_m : \text{Hank}_p(\mathbb{N}) \to \text{Hank}_p(\mathbb{N})$ such that

$$T_m(\gamma_k) = m_k \gamma_k, \quad k \geq 0.$$ 

Such an operator is uniquely defined. In this case, $m \in \ell^\infty$ and $\|m\|_\infty \leq \|T_m\|$.

The following is a simple extension of [Pisier 2001, Theorems 6.1 and 6.2].

**Theorem 3.3.** Let $1 < p < \infty$, let $C \geq 0$ be a constant and let $m = (m_k)_{k \geq 0}$ be a sequence in $\mathbb{C}$. The following assertions are equivalent.

(i) $m$ is the symbol of a $p$-completely bounded multiplier on $\text{Hank}_p(\mathbb{N})$, and

$$\|T_m : \text{Hank}_p(\mathbb{N}) \to \text{Hank}_p(\mathbb{N})\|_{p-\text{cb}} \leq C.$$ 

(ii) There exist a measure space $(\Omega, \mu)$, and bounded sequences $(\alpha_i)_{i \geq 0}$ in $L^p(\Omega)$ and $(\beta_j)_{j \geq 0}$ in $L^{p'}(\Omega)$ such that $m_{i+j} = \langle \alpha_i, \beta_j \rangle$, for every $i, j \geq 0$, and

$$\sup_{i \geq 0} \|\alpha_i\|_p \sup_{j \geq 0} \|\beta_j\|_{p'} \leq C.$$ 

**Proof.** By homogeneity, we may assume that $C = 1$ throughout this proof.

Assume (i). Let $\kappa : \ell^p_Z \to \ell^p_Z$ be defined by $\kappa((a_k)_{k \in \mathbb{Z}}) = (a_{-k})_{k \in \mathbb{Z}}$, let $J : \ell^p_N \to \ell^p_Z$ be the canonical embedding and let $Q : \ell^p_Z \to \ell^p_N$ be the canonical projection. Define $q : B(\ell^p_Z) \to B(\ell^p_N)$ by $q(T) = Q \kappa T J$. According to the easy implication (ii) $\Rightarrow$ (i) of Theorem 2.2, the mapping $q$ is $p$-completely contractive. We note that if $[t_{i,j}]_{(i,j) \in \mathbb{Z}^2}$ is the matrix of some $T \in B(\ell^p_Z)$, then the matrix of $q(T)$ is equal to $[t_{-i,j}]_{(i,j) \in \mathbb{Z}^2}$. 

Let \( \mathcal{M}_p(\mathbb{Z}) \subseteq B(\ell^p_\mathbb{Z}) \) be the space of all bounded Fourier multipliers on \( \ell^p_\mathbb{Z} \); this is a unital subalgebra. Let \( T \in \mathcal{M}_p(\mathbb{Z}) \) and let \( \phi \in L^\infty(\mathbb{T}) \) denote its symbol. Then the matrix of \( T \) is equal to \( [\hat{\phi}(i-j)]_{(i,j)\in\mathbb{Z}^2} \), hence the matrix of \( q(T) \) is equal to \( [\hat{\phi}(-i-j)]_{(i,j)\in\mathbb{N}^2} \). Hence, \( q(T) \) is Hankelian. We can therefore consider the restriction map

\[
q|_{\mathcal{M}_p(\mathbb{Z})} : \mathcal{M}_p(\mathbb{Z}) \to \text{Hank}_p(\mathbb{N}).
\]

Let \( s : \ell^p_\mathbb{Z} \to \ell^p_\mathbb{Z} \) be the shift operator defined by \( s(e_j) = e_{j+1} \), for all \( j \in \mathbb{Z} \). We observe (left to the reader) that

\[
q(s^{-k}) = \gamma_k, \quad k \in \mathbb{N}.
\]

We assume that \( T_m : \text{Hank}_p(\mathbb{N}) \to \text{Hank}_p(\mathbb{N}) \) is \( p \)-completely contractive. Consider \( w : \mathcal{M}_p(\mathbb{Z}) \to \text{Hank}_p(\mathbb{N}) \subseteq B(\ell^p) \) defined by \( w := T_m \circ q|_{\mathcal{M}_p(\mathbb{Z})} \). Then \( w \) is \( p \)-completely contractive. Applying Theorem 2.2 to \( w \), we obtain an \( SQ_p \)-space \( E \), a contractive homomorphism \( \pi : \mathcal{M}_p(\mathbb{Z}) \to B(E) \) and contractive maps \( V : \ell^p_\mathbb{N} \to E \) and \( W : E \to \ell^p_\mathbb{N} \) such that

\[
w(T) = W\pi(T)V, \quad T \in \mathcal{M}_p(\mathbb{Z}).
\]

Let \( i, j \geq 0 \). By (10), we have

\[w(s^{-(i+j)}) = T_m(q(s^{-(i+j)}))) = T_m(\gamma_{i+j}) = m_{i+j},\]

hence \( \langle w(s^{-(i+j)})e_i, e_j \rangle = m_{i+j} \). Consequently, from (11), we obtain that

\[m_{i+j} = \langle \pi(s^{-(i+j)})V(e_i), W^*(e_j) \rangle_{E,E^*}.
\]

The mapping \( \pi \) is multiplicative, hence this implies that

\[m_{i+j} = \langle \pi(s^{-i})V(e_i), \pi(s^{-j})*W^*(e_j) \rangle_{E,E^*}.
\]

Set \( x_i := \pi(s^{-i})V(e_i) \in E \) and \( y_j := \pi(s^{-j})*W^*(e_j) \in E^* \). Then, for all \( i, j \geq 0 \) we have \( \|x_i\| \leq 1, \|y_j\| \leq 1 \) and \( m_{i+j} = \langle x_i, y_j \rangle_{E,E^*} \).

Let us now apply Remark 2.3. As in the latter, consider a measure space \( (\Omega, \mu) \) and closed subspaces \( E_2 \subset E_1 \subset L^p(\Omega) \) such that \( E = E_1/E_2 \). Recall (5). For any \( i \geq 0 \), pick \( \alpha_i \in E_1 \) such that \( \|\alpha_i\|_p = \|x_i\| \) and \( \alpha_i = x_i \). Likewise, for any \( j \geq 0 \), pick \( \beta_j \in E_2^\perp \) such that \( \|\beta_j\|_{p'} = \|y_j\| \) and \( \beta_j = y_j \). Then for all \( i, j \geq 0 \), we both have \( \|\alpha_i\|_p \leq 1, \|\beta_j\|_{p'} \leq 1 \) and \( m_{i+j} = \langle \alpha_i, \beta_j \rangle_{L^p,L^{p'}} \). This proves (ii).

Conversely, assume (ii). By [Piéron 2001, Corollary 8.2], the family \( \{m_{i+j}\}_{(i,j)\in\mathbb{N}^2} \) induces a \( p \)-completely contractive Schur multiplier on \( B(\ell^p) \). It is clear that the restriction of this Schur multiplier maps \( \text{Hank}_p(\mathbb{N}) \) into itself. More precisely, it maps \( \gamma_k \) to \( m_k\gamma_k \) for all \( k \geq 0 \). Hence \( m \) is the symbol of a \( p \)-completely contractive multiplier on \( \text{Hank}_p(\mathbb{N}) \).

\[\square\]
4. Hankel operators on $L^p(\mathbb{R}_+)$

Throughout we let $1 < p < \infty$ and we let $p'$ denote its conjugate index. For any $u > 0$, we set $\tau_u f := f(\cdot - u)$, for all $f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$. Let

$$\text{Hank}_p(\mathbb{R}_+) \subseteq B(L^p(\mathbb{R}_+))$$

be the space of Hankelian operators on $L^p(\mathbb{R}_+)$, consisting of all bounded operators $T : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ such that

$$\langle T \tau_u f, g \rangle = \langle T f, \tau_u g \rangle,$$

for all $f \in L^p(\mathbb{R}_+)$, $g \in L^{p'}(\mathbb{R}_+)$ and $u > 0$.

For any $u > 0$, let $\theta_u : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ be defined by $\theta_u f = f(u - \cdot)$. Note that $\theta_u$ is a Hankelian operator on $L^p(\mathbb{R}_+)$. Indeed, for all $f \in L^p(\mathbb{R}_+)$, $g \in L^{p'}(\mathbb{R}_+)$ and $v > 0$, we have

$$\langle \theta_u \tau_v f, g \rangle = \int_v^u f(u - s)g(s - v) \, ds = \langle \theta_u f, \tau_v g \rangle$$

if $v < u$, and $\langle \theta_u \tau_v f, g \rangle = \langle \theta_u f, \tau_v g \rangle = 0$ if $v \geq u$. The operators $\theta_u$ are the continuous counterparts of the operators $\gamma_k$ from Section 3. From this point of view, part (1) of Theorem 4.1 below is an analogue of Lemma 3.1. However its proof is more delicate.

We introduce a new space $A_p(\mathbb{R}_+) \subseteq C_0(\mathbb{R}_+)$ by

$$A_p(\mathbb{R}_+):= \left\{ F = \sum_{n=1}^\infty f_n * g_n : f_n \in L^p(\mathbb{R}_+), \ g_n \in L^{p'}(\mathbb{R}_+) \text{ and } \sum_{n=1}^\infty \| f_n \|_p \| g_n \|_{p'} < \infty \right\},$$

and we equip it with the norm

$$\| F \|_{A_p} = \inf \left\{ \sum_{n=1}^\infty \| f_n \|_p \| g_n \|_{p'} \right\},$$

(12)

where the infimum runs over all possible representations of $F$ as above. The space $A_p(\mathbb{R}_+)$ is a half-line analogue of the classical Figà-Talamanca–Herz algebra $A_p(\mathbb{R})$; see, e.g., [Derighetti 2011]. The classical arguments showing that the latter is a Banach space show as well that (12) is a norm on $A_p(\mathbb{R}_+)$ and that $A_p(\mathbb{R}_+)$ is a Banach space.

It follows from the above definitions that there exists a (necessarily unique) contractive map

$$Q_p : L^p(\mathbb{R}_+) \hat{\otimes} L^{p'}(\mathbb{R}_+) \to A_p(\mathbb{R}_+)$$

such that $Q_p(f \otimes g) = f * g$, for all $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$. Moreover $Q_p$ is a quotient map. Hence the adjoint

$$Q_p^* : A_p(\mathbb{R}_+)^* \to B(L^p(\mathbb{R}_+))$$
of $Q_p$ is an isometry. This yields an isometric identification $A_p(\mathbb{R}_+)^* \simeq \ker(Q_p)^\perp (= \operatorname{ran}(Q_p^*))$.

We observe that

\begin{equation}
\ker(Q_p)^\perp = \overline{\operatorname{span}}_{w^*} \{ \theta_u : u > 0 \}.
\end{equation}

To prove this, we note that

\begin{equation}
\langle \theta_u, f \otimes g \rangle = \langle \theta_u(f), g \rangle = (f * g)(u),
\end{equation}

for all $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R}_+)$ and $u > 0$. Hence,

$$\left\{ \theta_u, \sum_{n=1}^{\infty} f_n \otimes g_n \right\} = \left( \sum_{n=1}^{\infty} f_n * g_n \right)(u)$$

for all sequences $(f_n)_n$ in $L^p(\mathbb{R}_+)$ and $(g_n)_n$ in $L^{p'}(\mathbb{R}_+)$ such that

$$\sum_{n=1}^{\infty} \| f_n \|_p \| g_n \|_{p'} < \infty,$$

and all $u > 0$. This implies that $\overline{\operatorname{span}}_{w^*} \{ \theta_u : u > 0 \} = \ker(Q_p)$, and (13) follows.

**Theorem 4.1.** (1) The space $\operatorname{Hank}_p(\mathbb{R}_+)$ is equal to the $w^*$-closure of the linear span of $\{ \theta_u : u > 0 \}$.

(2) We have an isometric identification

$$\operatorname{Hank}_p(\mathbb{R}_+) \simeq A_p(\mathbb{R}_+)^*.$$

**Proof.** Part (2) follows from part (1) and the discussion preceding the statement of Theorem 4.1. For any $f \in L^p(\mathbb{R}_+)$, $g \in L^{p'}(\mathbb{R}_+)$ and $u > 0$, the functionals $T \mapsto \langle T \tau_u f, g \rangle$ and $T \mapsto \langle T f, \tau_u g \rangle$ are $w^*$-continuous on $B(L^p(\mathbb{R}_+))$. Consequently, $\operatorname{Hank}_p(\mathbb{R}_+)$ is $w^*$-closed. Hence $\operatorname{Hank}_p(\mathbb{R}_+)$ contains the $w^*$-closure of $\overline{\operatorname{span}}_{w^*} \{ \theta_u : u > 0 \}$. To prove the reverse inclusion, it suffices to show, by (13), that

$$\operatorname{Hank}_p(\mathbb{R}_+) \subset \ker(Q_p)^\perp.$$

We will use a double approximation process. First, let $k, l$ in $C_c(\mathbb{R})$, the space of continuous functions with compact support. To any $T \in B(L^p(\mathbb{R}_+))$, we associate $T_{k,l} \in B(L^p(\mathbb{R}_+))$ defined by

$$\langle T_{k,l}(f), g \rangle = \int_{\mathbb{R}} \langle T(\tau_u k \cdot f), \tau_u -l \cdot g \rangle \, du, \quad f \in L^p(\mathbb{R}_+), \ g \in L^{p'}(\mathbb{R}_+).$$

We note that

$$\int_{\mathbb{R}} \left| \langle T(\tau_u k \cdot f), \tau_u -l \cdot g \rangle \right| \, du \leq \| T \|_p \left( \int_{\mathbb{R}} \| \tau_u k f \|_p \, du \right)^{\frac{1}{p'}} \left( \int_{\mathbb{R}} \| \tau_u f \|_{p'} \, du \right)^{\frac{1}{p'}} = \| T \|_p \| f \|_p \| g \|_{p'} \| k \|_p \| l \|_{p'}.$$
Thus, $T_{k,l}$ is well-defined and $\|T_{k,l}\| \leq \|T\|\|k\|_p\|l\|_p$. We are going to show that

$$T \in \text{Hank}_p(\mathbb{R}_+) \Rightarrow T_{k,l} \in \text{ker}(Q_p) \perp. \tag{15}$$

Let $\alpha \in C_c(\mathbb{R}_+) \perp$ such that $\|\alpha\|_1 = 1$. Let $R_\alpha \in B(L^p(\mathbb{R}_+))$ be defined by

$$R_\alpha(f) = \alpha * f, \quad f \in L^p(\mathbb{R}_+).$$

We show that $(TR_\alpha)_{k,l}$ belongs to $\text{ker}(Q_p) \perp$ if $T \in \text{Hank}_p(\mathbb{R}_+)$, and we use these auxiliary operators to establish (15).

We fix some $T \in \text{Hank}_p(\mathbb{R}_+)$. Let $z \in \text{ker}(Q_p)$. Since $C_c(\mathbb{R}_+)$ is both dense in $L^p(\mathbb{R}_+)$ and $L^{p'}(\mathbb{R}_+)$, it follows, e.g., from [Derighetti 2011, Chapter 3, Proposition 6] that there exist sequences $(f_n)_{n \geq 1}$ and $(g_n)_{n \geq 1}$ in $C_c(\mathbb{R}_+)$ such that $\sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_{p'} < \infty$ and $z = \sum_{n=1}^{\infty} f_n \otimes g_n$. Since $z \in \text{ker}(Q_p)$, we have $\sum_{n=1}^{\infty} f_n \otimes g_n = 0$, pointwise.

We write $R_\alpha f = \int_{\mathbb{R}_+} f(s) \tau_s \alpha \, ds$ as a Bochner integral, for all $f \in C_c(\mathbb{R}_+)$. A simple application of Fubini’s theorem leads to

$$k * l \cdot f_n \otimes g_n = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (\tau_u k \cdot f_n)(s) \tau_s (\tau_{-u} l \cdot g_n) \, ds \, du,$$

for all $n \geq 1$. We deduce that

$$\sum_{n=1}^{\infty} \langle (TR_\alpha)_{k,l}(f_n), g_n \rangle = \sum_{n=1}^{\infty} \int_{\mathbb{R}_+} \langle TR_\alpha(\tau_u k \cdot f_n), \tau_{-u} l \cdot g_n \rangle \, du$$

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}_+} \langle T((\tau_u k \cdot f_n) \ast \alpha), \tau_{-u} l \cdot g_n \rangle \, du$$

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} (\tau_u k \cdot f_n)(s) \langle T(\tau_s \alpha), \tau_{-u} l \cdot g_n \rangle \, ds \, du$$

$$= \sum_{n=1}^{\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \langle T(\alpha), (\tau_u k \cdot f_n)(s) \tau_s (\tau_{-u} l \cdot g_n) \rangle \, ds \, du$$

$$= \sum_{n=1}^{\infty} \langle T(\alpha), k * l \cdot \sum_{n=1}^{\infty} f_n \otimes g_n \rangle$$

$$= \langle T(\alpha), k * l \cdot \sum_{n=1}^{\infty} f_n \otimes g_n \rangle = 0.$$

This shows that $(TR_\alpha)_{k,l}$ belongs to $\text{ker}(Q_p) \perp$.

For $z$, $f_n$, $g_n$ as above, write

$$\sum_{n=1}^{\infty} \langle T_{k,l}(f_n), g_n \rangle = \sum_{n=1}^{\infty} \langle T_{k,l}(f_n), g_n \rangle - \sum_{n=1}^{\infty} \langle (TR_\alpha)_{k,l}(f_n), g_n \rangle.$$
Then we have
\[ \left| \sum_{n=1}^{\infty} \langle T_{k,l}(f_n), g_n \rangle \right| \]
\[ \leq \sum_{n=1}^{\infty} \int_{\mathbb{R}} \left| (T(\tau_u k \cdot f_n - (\tau_u k \cdot f_n) \ast \alpha), \tau_{-u} l \cdot g_n) \right| du \]
\[ \leq \sum_{n=1}^{\infty} \| T \| \left( \int_{\mathbb{R}} \| \tau_u k \cdot f_n - (\tau_u k \cdot f_n) \ast \alpha \|_{p}^p du \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}} \| \tau_{-u} l \cdot g_n \|_{p'}^p du \right)^{\frac{1}{p'}} \]
\[ \leq \| T \| \| I \|_{p'} \sum_{n=1}^{\infty} \| g_n \|_{p'} \left( \int_{\mathbb{R}} \| \tau_u k \cdot f_n - (\tau_u k \cdot f_n) \ast \alpha \|_{p}^p du \right)^{\frac{1}{p}}. \]

Recall that by assumption, \( \alpha \geq 0 \) and \( \int_{\mathbb{R}^+} \alpha(s) ds = 1 \). Then we deduce from above that
\[ \left| \sum_{n=1}^{\infty} \langle T_{k,l}(f_n), g_n \rangle \right| \]
\[ \leq \| T \| \| I \|_{p'} \sum_{n=1}^{\infty} \| g_n \|_{p'} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^+} \alpha(s)(\tau_u k \cdot f_n - \tau_s(\tau_u k \cdot f_n)) ds \right)^{\frac{1}{p}} du \right)^{\frac{1}{p}} \]
\[ \leq \| T \| \| I \|_{p'} \sum_{n=1}^{\infty} \| g_n \|_{p'} \left( \int_{\mathbb{R}} \int_{\mathbb{R}^+} \alpha(s) \| \tau_u k \cdot f_n - \tau_s(\tau_u k \cdot f_n) \|_{p}^p ds du \right)^{\frac{1}{p}}. \]

The integral in the right-hand side satisfies
\[ \left( \int_{\mathbb{R}} \int_{\mathbb{R}^+} \alpha(s) \| \tau_u k \cdot f_n - \tau_s(\tau_u k \cdot f_n) \|_{p}^p ds du \right)^{\frac{1}{p}} \]
\[ \leq \left( \int_{\mathbb{R}} \int_{\mathbb{R}^+} \alpha(s) \| \tau_u k \cdot f_n - \tau_{s+u} k \cdot f_n \|_{p}^p ds du \right)^{\frac{1}{p}} \]
\[ + \left( \int_{\mathbb{R}} \int_{\mathbb{R}^+} \alpha(s) \| \tau_{s+u} k \cdot f_n - \tau_s(\tau_u k \cdot f_n) \|_{p}^p ds du \right)^{\frac{1}{p}} \]
\[ \leq \left( \int_{\mathbb{R}} \int_{\mathbb{R}^+} \alpha(s) \| \tau_u ((k - \tau_s k) \cdot f_n) \|_{p}^p ds du \right)^{\frac{1}{p}} \]
\[ + \left( \int_{\mathbb{R}} \int_{\mathbb{R}^+} \alpha(s) \| \tau_{s+u} k \cdot (f_n - \tau_s f_n) \|_{p}^p ds du \right)^{\frac{1}{p}} \]
\[ \leq \sup_{s \in \text{supp}(\alpha)} \left( \int_{\mathbb{R}} \| \tau_u (k - \tau_s k) \cdot f_n \|_{p}^p du \right)^{\frac{1}{p}} + \sup_{s \in \text{supp}(\alpha)} \left( \int_{\mathbb{R}} \| \tau_{s+u} k \cdot (f_n - \tau_s f_n) \|_{p}^p du \right)^{\frac{1}{p}} \]
\[ = \sup_{s \in \text{supp}(\alpha)} \| k - \tau_s k \|_{p} \| f_n \|_{p} + \sup_{s \in \text{supp}(\alpha)} \| k \|_{p} \| f_n - \tau_s f_n \|_{p}. \]

Hence we obtain that
\[ \left| \sum_{n=1}^{\infty} \langle T_{k,l}(f_n), g_n \rangle \right| \]
\[ \leq \| T \| \| I \|_{p'} \sum_{n=1}^{\infty} \| g_n \|_{p'} \left( \sup_{s \in \text{supp}(\alpha)} \| k - \tau_s k \|_{p} \| f_n \|_{p} + \sup_{s \in \text{supp}(\alpha)} \| k \|_{p} \| f_n - \tau_s f_n \|_{p} \right). \]
Given $\epsilon > 0$, choose $M$ such that
\[
\sum_{n=M+1}^\infty \|f_n\|_p \|g_n\|_{p'} < \epsilon.
\]
We may find $s_0 > 0$ such that for all $s \in (0, s_0)$ and all $1 \leq n \leq M$, we have that
\[
\|k - \tau_s k\|_p \leq \frac{\epsilon \|k\|_p}{\sum_{n=1}^M \|f_n\|_p \|g_n\|_{p'}} \quad \text{and} \quad \|f_n - \tau_s f_n\|_p \leq \frac{\epsilon}{M \|g_n\|_{p'}}.
\]
We may now choose $\alpha$ so that $\text{supp}(\alpha) \subseteq (0, t_0)$. Then we obtain from above that
\[
\left| \sum_{n=1}^\infty \langle T_{k,l}(f_n), g_n \rangle \right|
\leq \|T\| \|I\|_{p'} \left( \epsilon \|k\|_p + \sum_{n=1}^M \|g_n\|_{p'} \cdot \sup_{s \in \text{supp}(\alpha)} \|k\|_p \|f_n - \tau_s f_n\|_p 
+ \sum_{n=M+1}^\infty \|g_n\|_{p'} \cdot \sup_{s \in \text{supp}(\alpha)} \|k\|_p \|f_n - \tau_s f_n\|_p \right)
\leq \|T\| \|I\|_{p'} \left( 2\epsilon \|k\|_p + \sum_{n=M+1}^\infty 2\|k\|_p \|g_n\|_{p'} \|f_n\|_p \right)
\leq 4\epsilon \|T\| \|I\|_{p'} \|k\|_p.
\]
Since $\epsilon$ was arbitrary, this shows that $\sum_{n=1}^\infty \langle T_{k,l}(f_n), g_n \rangle = 0$. Since $z = \sum_{n=1}^\infty f_n \otimes g_n$ was an arbitrary element of $\text{ker}(Q_p)$, we obtain (15).

Next, we construct a sequence $(T_{k_n,l_n})_n$ which tends to $T$ in the $w^*$-topology of $B(L^p(\mathbb{R}_+))$. In the sequel, we assume that $k, l \in C_c(\mathbb{R})$ are such that
\[
(16) \quad \|k\|_p = 1, \quad \|l\|_{p'} = 1 \quad \text{and} \quad \int_\mathbb{R} k(-s)l(s) \, ds = 1.
\]
Consider any $f, g \in C_c(\mathbb{R}_+)$. We have
\[
\left| \langle T(f), g \rangle - \langle T_{k,l}(f), g \rangle \right|
= \left| \int_\mathbb{R} \langle T(k(-s)f), l(s)g \rangle - \langle T(\tau_s k \cdot f), \tau_{-s} l \cdot g \rangle \, ds \right|
\leq \int_\mathbb{R} \left| \langle T((k(-s) - \tau_s k)f), l(s)g \rangle \right| \, ds + \int_\mathbb{R} \left| \langle T(\tau_s k \cdot f), (l(s) - \tau_{-s} l)g \rangle \right| \, ds
\leq \|T\| \left( \int_\mathbb{R} \|k(-s) - \tau_s k\|_p \|f\|_p \, ds \right)^{\frac{1}{p'}} \left( \int_\mathbb{R} \|l(s)g\|_{p'} \, ds \right)^{\frac{1}{p'}}
+ \|T\| \left( \int_\mathbb{R} \|\tau_s k \cdot f\|_p \, ds \right)^{\frac{1}{p'}} \left( \int_\mathbb{R} \|\tau_{-s} l\|_{p'} \, ds \right)^{\frac{1}{p'}}
\leq \|T\| \|g\|_{p'} \left( \int_{\mathbb{R}_+} |f(t)|^p \|\tau_{i \hat{k}} \tilde{k} - \hat{k}\|_p \, dt \right)^{\frac{1}{p'}}
+ \|T\| \|f\|_p \left( \int_{\mathbb{R}_+} |g(t)|^p \|\tau_{-l} l - l\|_{p'} \, dt \right)^{\frac{1}{p'}}.
\]
Here $\hat{k}$ denotes the function $s \mapsto k(-s)$.  


Now for \( n \geq 1 \), set
\[
k_n := \frac{\chi([-n,n])}{(2n)^\frac{1}{p}} \quad \text{and} \quad l_n := \frac{\chi([-n,n])}{(2n)^{\frac{1}{p'}}},
\]
where \( \chi([-n,n]) \) is the indicator function of the interval \([-n,n]\). Then \( \|k_n\|_p = \|l_n\|_{p'} = 1 \) and \( \int_{\mathbb{R}} k_n(-s)l_n(s)\,ds = 1 \) as in (16). Let \( K = \text{supp}(f) \cup \text{supp}(g) \) and let \( r = \sup(K) \). Note that \( k_n = k_n \) and that we have
\[
\sup_{t \in K} \|\tau_t k_n - k_n\|_p \leq \left( \frac{r}{n} \right)^{\frac{1}{p}} \quad \text{and} \quad \sup_{t \in K} \|\tau_t l_n - l_n\|_{p'} \leq \left( \frac{r}{n} \right)^{\frac{1}{p'}}.
\]
Therefore,
\[
\left| \langle T(f), g \rangle - \langle T_{k_n,l_n}(f), g \rangle \right| \leq \frac{2r}{n} \|T\| \|f\|_p \|g\|_{p'},
\]
hence \( \langle T_{k_n,l_n}(f), g \rangle \xrightarrow{n \to \infty} \langle T(f), g \rangle \). Since \( \|T_{k_n,l_n}\| \leq \|T\| \) for all \( n \geq 1 \), this implies, by Lemma 2.1, that \( T_{k_n,l_n} \to T \) in the \( w^* \)-topology of \( B(L^p(\mathbb{R}^+)) \). Consequently, \( T \in \ker(Q_p)^\perp \) as expected. \( \square \)

**Remark 4.2.** (a) For any \( 1 \leq p \leq \infty \), let \( H^p(\mathbb{R}) \subset L^p(\mathbb{R}) \) be the subspace of all \( f \in L^p(\mathbb{R}) \) whose Fourier transform has support in \( \mathbb{R}_+ \). Recall the factorisation property
\[
H^1(\mathbb{R}) = H^2(\mathbb{R}) \times H^2(\mathbb{R}).
\]

More precisely, the product \( h_1h_2 \in H^1(\mathbb{R}) \) and \( \|h_1h_2\|_1 \leq \|h_1\|_2\|h_2\|_2 \) for all \( h_1, h_2 \in H^2(\mathbb{R}) \) and conversely, for all \( h \in H^1(\mathbb{R}) \), there exist \( h_1, h_2 \in H^2(\mathbb{R}) \) such that \( h = h_1h_2 \) and \( \|h\|_1 = \|h_1\|_2\|h_2\|_2 \).

Recall that by definition,
\[
A_2(\mathbb{R}_+) = \left\{ \sum f_n * g_n : f_n, g_n \in L^2(\mathbb{R}_+), \sum \|f_n\|_2 \|g_n\|_2 < \infty \right\}.
\]
It therefore follows from the above factorisation property and the identification of \( L^2(\mathbb{R}_+) \) with \( H^2(\mathbb{R}) \) via the Fourier transform that
\[
A_2(\mathbb{R}_+) = \left\{ \hat{h} : h \in H^1(\mathbb{R}) \right\},
\]
with \( \|\hat{h}\|_{A_2(\mathbb{R}_+)} = \|h\|_{H^1(\mathbb{R})} \). Therefore, we have an isometric identification
\[
A_2(\mathbb{R}_+) \cong H^1(\mathbb{R}).
\]
Since \( H^1(\mathbb{R})^\perp = H^\infty(\mathbb{R}) \), we have
\[
H^1(\mathbb{R})^* \cong \frac{L^\infty(\mathbb{R})}{H^\infty(\mathbb{R})}.
\]
Applying Theorem 4.1(2), we recover the well-known fact (see [Nikolski 2020, Section IV.5.3] or [Peller 2003, Theorem I.8.1]) that
\[ \text{Hank}_2(\mathbb{R}_+) \cong \frac{L^\infty(\mathbb{R})}{H^\infty(\mathbb{R})}. \]

(b) Notice that Hank\(p\)(\(\mathbb{R}_+\)) \(\subseteq\) Hank\(2\)(\(\mathbb{R}_+\)). Indeed, suppose that \(T \in \text{Hank}_p(\mathbb{R}_+)\) and note that the adjoint mapping \(T^* \in B(L^p(\mathbb{R}_+))\) coincides with \(T\) on \(L^p(\mathbb{R}_+)\cap L^p(\mathbb{R}_+)\). To see this, take \(f, g \in L^p(\mathbb{R}_+)\cap L^p(\mathbb{R}_+)\) and observe that \(f \otimes g - g \otimes f\) belongs to \(\ker(Q_p)\). This implies that \(\langle T(f), g \rangle = \langle T(g), f \rangle\). Therefore, \(T\) coincides with \(T^*\) on \(L^p(\mathbb{R}_+)\cap L^p(\mathbb{R}_+)\). It then follows by interpolation that \(T\) extends to a bounded operator on \(L^2(\mathbb{R}_+)\), say \(\widetilde{T}\). Since \(T\) and \(\widetilde{T}\) coincide on \(L^p(\mathbb{R}_+)\cap L^2(\mathbb{R}_+)\) and \(T\) is Hankelian, it follows from the definition of Hankel operators that \(\widetilde{T}\) is also a Hankel operator and hence belongs to Hank\(2\)(\(\mathbb{R}_+\)).

(c) The definition of Hank\(p\)(\(\mathbb{R}_+\)) extends to the case \(p = 1\). In analogy with Remark 3.2(c), we have an isometric identification
\[ \text{Hank}_1(\mathbb{R}_+) \simeq M(\mathbb{R}_+^*), \]
where \(M(\mathbb{R}_+^*)\) denotes the space of all bounded Borel measures on \(\mathbb{R}_+^*\). To establish this, we first note that for all \(f \in L^1(\mathbb{R}_+)\), the function \(u \mapsto \theta_u(f)\) is bounded and continuous from \(\mathbb{R}_+^*\) into \(L^1(\mathbb{R}_+)\). Hence for all \(v \in M(\mathbb{R}_+^*)\), we may define \(H_v \in B(L^1(\mathbb{R}_+))\) by
\[ H_v(f) = \int_{\mathbb{R}_+^*} \theta_v(f) \, d\nu(u), \quad f \in L^1(\mathbb{R}_+). \]

It is clear that \(H_v\) is Hankelian. It follows from (14) that
\[ \langle H_v(f), g \rangle = \int_{\mathbb{R}_+^*} (f * g)(u) \, d\nu(u), \quad f \in L^1(\mathbb{R}_+), \ g \in L^\infty(\mathbb{R}_+). \]

We note that the mapping \(v \mapsto H_v\) is a one-to-one contraction from \(M(\mathbb{R}_+^*)\) into Hank\(1\)(\(\mathbb{R}_+\)). We shall now prove that this mapping is an onto isometry.

We use the isometric identification \(M(\mathbb{R}_+^*) \simeq C_0(\mathbb{R}_+^*)^*\) provided by the Riesz theorem and we regard \(L^1(\mathbb{R}_+) \subseteq M(\mathbb{R}_+^*)\) in the obvious way. Let \(T \in \text{Hank}_1(\mathbb{R}_+)\). We observe that for all \(h, f \in L^1(\mathbb{R}_+)\) and all \(g \in C_0(\mathbb{R}_+^*)\), we have
\[ \langle T(h * f), g \rangle = \langle T(h), f * g \rangle \]
Indeed, write \(h * f = \int_0^\infty f(s) \tau_s h \, ds\). This implies that \(T(h * f) = \int_0^\infty f(s) T(\tau_s h) \, ds\), hence
\[ \langle T(h * f), g \rangle = \int_0^\infty f(s) \langle T \tau_s h, g \rangle \, ds = \int_0^\infty f(s) \langle T h, \tau_s g \rangle \, ds = \langle T(h), f * g \rangle. \]
Let \((h_n)_{n \geq 1}\) be a norm one approximate unit of \(L^1(\mathbb{R}_+)\). Then \((T(h_n))_{n \geq 1}\) is a bounded sequence of \(L^1(\mathbb{R}_+)\). Hence it admits a cluster point \(v \in M(\mathbb{R}_+^*)\).
in the $w^*$-topology of $M(\mathbb{R}_+^*)$. Thus, for all $g \in C_0(\mathbb{R}_+^*)$, the complex number 
$\int_{\mathbb{R}_+^*} g(u) \, dv(u)$ is a cluster point of the sequence $(\langle T(h_n), g \rangle)_{n \geq 1}$. Furthermore, we have $\|v\| \leq \|T\|$. Let $f \in L^1(\mathbb{R}_+)$ and let $g \in C_0(\mathbb{R}_+^*)$. Since $h_n \ast f \to f$ in $L^1(\mathbb{R}_+)$, we have that $(T(h_n \ast f), g) \to (T(f), g)$. By (18), we may write $\langle T(h_n \ast f), g \rangle = \langle T(h_n), f \ast g \rangle$. We deduce that 
\[ \langle T(f), g \rangle = \int_{\mathbb{R}_+^*} (f \ast g)(u) \, dv(u). \]
This implies that $T = H_v$, see (17), which concludes the proof.

**Definition 4.3.** We say that a function $m : \mathbb{R}_+^* \to \mathbb{C}$ is the symbol of a multiplier on 
$\text{Hank}_p(\mathbb{R}_+)$ if there exist a $w^*$-continuous operator $T_m : \text{Hank}_p(\mathbb{R}_+) \to \text{Hank}_p(\mathbb{R}_+)$ 
such that for every $u > 0$, $T_m(\theta_u) = m(u)\theta_u$. (Note that such an operator $T_m$ is 
necessarily unique.)

**Remark 4.4.** Suppose that $T_m : \text{Hank}_p(\mathbb{R}_+) \to \text{Hank}_p(\mathbb{R}_+)$ is a multiplier as defined 
above. Using Theorem 4.1(2), let $S_m : A_p(\mathbb{R}_+) \to A_p(\mathbb{R}_+)$ be the operator such 
that $S_m^* = T_m$. For $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$, we have, by (14),
\[ [S_m(f \ast g)](u) = \langle \theta_u, S_m(f \ast g) \rangle \]
\[ = \langle T_m(\theta_u), f \ast g \rangle \]
\[ = m(u) \langle \theta_u, f \ast g \rangle \]
\[ = m(u) (f \ast g)(u). \]
We deduce that $S_m(F) = m \cdot F$, for every $F \in A_p(\mathbb{R}_+)$. Conversely, if $m : \mathbb{R}_+^* \to \mathbb{C}$ is such that $S_m : A_p(\mathbb{R}_+) \to A_p(\mathbb{R}_+)$ given by 
$S_m(F) = m \cdot F$ is well-defined and bounded, then $S_m^*$ is a multiplier on $\text{Hank}_p(\mathbb{R}_+)$. 

**Lemma 4.5.** If $m : \mathbb{R}_+^* \to \mathbb{C}$ is the symbol of a multiplier on $\text{Hank}_p(\mathbb{R}_+)$, then $m$ is 
continuous and bounded.

**Proof.** For all $u > 0$, we have $m(u)\theta_u = T_m(\theta_u)$, hence $|m(u)| \leq \|T_m\|$. Thus, $m$ is 
bounded. For any $a > 0$, let $\chi_{(0,a)}$ be the indicator function of the interval $(0, a)$. Then 
$m \cdot \chi_{(0,a)} \ast \chi_{(0,a)}$ belongs to $A_p(\mathbb{R}_+)$, hence to $C_b(\mathbb{R}_+^*)$, by Remark 4.4. Since 
$\chi_{(0,a)} \ast \chi_{(0,a)} > 0$ on $(0, 2a)$, it follows that $m$ is continuous on $(0, 2a)$. Thus, $m$ is 
continuous on $\mathbb{R}_+^*$. \[ \square \]

**Theorem 4.6.** Let $1 < p < \infty$, let $C \geq 0$ be a constant and let $m : \mathbb{R}_+^* \to \mathbb{C}$ be a 
function. The following assertions are equivalent.

(i) $m$ is the symbol of a $p$-completely bounded multiplier on $\text{Hank}_p(\mathbb{R}_+)$, and 
\[ \|T_m : \text{Hank}_p(\mathbb{R}_+) \to \text{Hank}_p(\mathbb{R}_+)\|_{p\text{-cb}} \leq C. \]
(ii) \( m \) is continuous and there exist a measure space \( (Ω, μ) \) and two functions \( α ∈ L^∞(ℝ_+; L^p(Ω)) \) and \( β ∈ L^∞(ℝ_+; L^p(Ω)) \) such that \( ||α||_∞ ||β||_∞ ≤ C \) and \( m(s + t) = ⟨α(s), β(t)⟩ \), for almost every \( (s, t) ∈ ℝ^2_+ \).

Proof. By homogeneity, we may assume that \( C = 1 \) throughout this proof.

Assume (i). The continuity of \( m \) follows from Lemma 4.5. Let \( T_m : \text{Hank}_p(ℝ_+) → \text{Hank}_p(ℝ_+) \) be the \( p \)-completely contractive multiplier associated with \( m \). Let \( \kappa : L^p(ℝ) → L^p(ℝ) \) be defined by \( (κf)(t) = f(−t) \), for all \( f ∈ L^p(ℝ) \). Let \( J : L^p(ℝ_+) → L^p(ℝ) \) be the canonical projection defined by \( Qf = f|_{ℝ^+} \). Let \( q : B(L^p(ℝ)) → B(L^p(ℝ_+)) \) be given by \( q(T) = Qκ T J \), for all \( T ∈ B(L^p(ℝ)) \). Applying the easy implication (ii) \( ⇔ \) (i) of Theorem 2.2 we obtain that \( q \) is \( p \)-completely contractive.

Let \( \mathcal{M}_p(ℝ) \subseteq B(L^p(ℝ)) \) denote the subalgebra of bounded Fourier multipliers. Let us show that if \( T ∈ \mathcal{M}_p(ℝ) \), then \( q(T) ∈ \text{Hank}_p(ℝ_+) \). For any \( s ∈ ℝ \), recall \( τ_s ∈ B(L^p(ℝ)) \) given by \( τ_s(f) = f(−s) \). Note that \( τ_s ∈ \mathcal{M}_p(ℝ) \) and that \( \mathcal{M}_p(ℝ) = \text{Span} w^∗ \{τ_s : s ∈ ℝ \} \). For all \( f ∈ L^p(ℝ_+) \), we have

\[
q(τ_s)f = Qτ(f(−s)) = Q(f(−(−s+\bar{s}))) = \{ t ∈ ℝ_+ ↦ f(−t − s) \}.
\]

Hence, if \( s ≥ 0 \), then \( q(τ_s) = 0 \) and if \( s < 0 \), then \( q(τ_s) = τ_{−s} \). It is plain that \( q \) is \( w^∗ \)-continuous. Since \( \text{Hank}_p(ℝ_+) \) is \( w^∗ \)-closed, we deduce that \( q \) maps \( \mathcal{M}_p(ℝ) \) into \( \text{Hank}_p(ℝ_+) \).

Consider the mapping

\[
q_0 := q|_{\mathcal{M}_p(ℝ)} : \mathcal{M}_p(ℝ) → \text{Hank}_p(ℝ_+)
\]

and set

\[
Γ := T_m ◦ q_0 : \mathcal{M}_p(ℝ) → B(L^p(ℝ_+)).
\]

It follows from above that

\[
Γ(τ_{−s}) = m(s)τ_s, \quad s > 0.
\]

Since \( q \) is \( p \)-completely contractive, \( Γ \) is also \( p \)-completely contractive. Applying Theorem 2.2 to \( Γ \), we obtain the existence of an \( SQ_p \)-space \( E \), a unital \( p \)-completely contractive, nondegenerate homomorphism \( π : \mathcal{M}_p(ℝ) → B(E) \) as well as operators \( V : L^p(ℝ_+) → E \) and \( W : E → L^p(ℝ_+) \) such that \( ||V|| ||W|| ≤ 1 \) and for every \( x ∈ \mathcal{M}_p(ℝ) \), \( Γ(x) = Wπ(x)V \).

Let \( c : L^1(ℝ) → \mathcal{M}_p(ℝ) \) be defined by \( [c(g)](f) = g ∗ f \), for all \( g ∈ L^1(ℝ) \) and \( f ∈ L^p(ℝ) \). Let \( λ : L^1(ℝ) → B(E) \) be given by \( λ = π ◦ c \). Then \( λ \) is a contractive, nondegenerate homomorphism. By [de Pagter and Ricker 2008, Remark 2.5], there exists \( σ : ℝ → B(E) \), a bounded strongly continuous representation such that for all \( g ∈ L^1(ℝ) \), \( λ(g) = \int ℝ g(t)σ(t)dt \) (defined in the strong sense). Let us show that

\[
Γ(τ_{−s}) = Wσ(−s)V, \quad s > 0.
\]
Let \( \eta \in L^1(\mathbb{R}_+) \) be such that \( \int_{\mathbb{R}} \eta(t) \, dt = 1 \). For any \( r > 0 \), let \( \eta_r(t) = r \eta(rt) \).

Since \( \sigma : \mathbb{R} \to B(E) \) is strongly continuous, the function \( t \mapsto \langle \sigma(t)x, x^* \rangle \) is continuous and we have

\[
(21) \quad \int_{\mathbb{R}} \eta_r(-(s-t)) \langle \sigma(t)x, x^* \rangle \, dt \xrightarrow{t \to \infty} \langle \sigma(-s)x, x^* \rangle,
\]

for all \( x \in E \) and \( x^* \in E^* \). Since the left-hand side in (21) is equal to

\[
\langle \pi(c(\eta_r(-(s-\cdot))))x, x^* \rangle,
\]

we obtain, by Lemma 2.1, that \( \pi(c(\eta_r(-(s-\cdot)))) \to \sigma(-s) \) in the \( w^* \)-topology of \( B(E) \). This implies that \( W \pi(c(\eta_r(-(s-\cdot))))V \to W \sigma(-s)V \) in the \( w^* \)-topology of \( B(L^p(\mathbb{R}_+)) \). We next show that \( W \pi(c(\eta_r(-(s-\cdot))))V \to \Gamma(t_s) \) in the \( w^* \)-topology of \( B(L^p(\mathbb{R}_+)) \), which will complete the proof of (20). Since

\[
W \pi(c(\eta_r(-(s-\cdot))))V = \Gamma(c(\eta_r(-(s-\cdot))))
\]

and \( \Gamma \) is \( w^* \)-continuous, it suffices to show that \( c(\eta_r(-(s-\cdot))) \to \tau_{-s} \) in the \( w^* \)-topology of \( B(L^p(\mathbb{R})) \). To see this, let \( f \in L^p(\mathbb{R}) \) and \( g \in L^{p'}(\mathbb{R}) \). We have that

\[
\langle c(\eta_r(-(s-\cdot)))f, g \rangle = \langle \eta_r(-(s-\cdot)) \ast f, g \rangle = \langle \delta_{-s} \ast \eta_r \ast f, g \rangle \to \langle \delta_{-s} \ast f, g \rangle = \langle \tau_{-s}f, g \rangle.
\]

By Lemma 2.1 again, this proves that \( c(\eta_r(-(s-\cdot))) \to \tau_{-s} \) in the \( w^* \)-topology, as expected.

Given any \( \epsilon > 0 \), let \( m_\epsilon : \mathbb{R}_+^* \to \mathbb{C} \) be defined by

\[
m_\epsilon(t) = m(t + \epsilon), \quad t > 0.
\]

Let \( f \in L^p(\mathbb{R}_+) \) be given by \( f = \epsilon^{-1/p} \chi_{(0, \epsilon)} \) and let \( g \in L^{p'}(\mathbb{R}_+) \) be given by \( g = \epsilon^{-1/p'} \chi_{(0, \epsilon)} \). For any \( s, t > 0 \), set

\[
\alpha_\epsilon(s) := \sigma(-s - \frac{\epsilon}{2})V(t_s f) \quad \text{and} \quad \beta_\epsilon(t) := \sigma(-t - \frac{\epsilon}{2})W^*(\tau_t g).
\]

Since \( \sigma \) is strongly continuous, \( \alpha_\epsilon \) and \( \beta_\epsilon \) are continuous. By (19) and (20), we have that

\[
\langle \alpha_\epsilon(s), \beta_\epsilon(t) \rangle_{E, E^*} = \langle \sigma(-s - \frac{\epsilon}{2})V(t_s f), \sigma(-t - \frac{\epsilon}{2})W^*(\tau_t g) \rangle
\]

\[
= \langle W \sigma(-s - t - \epsilon)V(t_s f), \tau_t g \rangle
\]

\[
= \langle (\Gamma(t_{-s-t-\epsilon})(t_s f), \tau_t g \rangle
\]

\[
= m(s + t + \epsilon)\langle \theta_{s+t+\epsilon}(t_s f), \tau_t g \rangle
\]

\[
= m_\epsilon(s + t)\langle \epsilon^{-1/p} \chi_{(t,t+\epsilon)}, \epsilon^{-1/p'} \chi_{(t,t+\epsilon)} \rangle
\]

\[
= m_\epsilon(s + t),
\]
for all $s$, $t > 0$. Moreover, $\|\alpha_\epsilon(s)\| \leq \|V\|$ and $\|\beta_\epsilon(t)\| \leq \|W\|$ for all $t$, $s > 0$. Since $\alpha_\epsilon$ and $\beta_\epsilon$ are continuous, this implies that $\alpha_\epsilon \in L^\infty(\mathbb{R}_+; E)$, $\beta_\epsilon \in L^\infty(\mathbb{R}_+; E^*)$ and $\|\alpha_\epsilon\|_\infty \|\beta_\epsilon\|_\infty \leq \|V\| \|W\| \leq 1$.

We now show that the $SQ_p$-space $E$ can be replaced by an $L^p$-space in the above factorisation property of $m_\epsilon$. Following Remark 2.3, assume that $E = E_1/E_2$, with $E_2 \subseteq E_1 \subseteq L^p(\Omega)$, and for all $f \in E_1$, let $\hat{f} \in E$ denote the class of $f$. Recall (5) and for all $g \in E_2^\perp$, let $\hat{g} \in E^*$ denote the class of $g$. Since $E$ is a quotient of $E_1$, we have an isometric embedding $E^* \subseteq E_1^*$. More precisely,

$$E^* = \frac{E_2^\perp}{E_1^\perp} \hookrightarrow \frac{L^p(\Omega)}{E_1^\perp} = E_1^*.$$ 

This induces an isometric embedding

$$L^1(\mathbb{R}_+; E^*) \subseteq L^1(\mathbb{R}_+; E_1^*).$$

Since $E^*$ and $E_1^*$ are reflexive, we may apply the identifications

$$L^1(\mathbb{R}_+; E^*)^* \simeq L^\infty(\mathbb{R}_+; E) \quad \text{and} \quad L^1(\mathbb{R}_+; E_1^*)^* \simeq L^\infty(\mathbb{R}_+; E_1)$$

provided by (7). By the Hahn–Banach theorem, we deduce the existence of $\tilde{\alpha}_\epsilon \in L^\infty(\mathbb{R}_+; E_1)$ such that $\|\tilde{\alpha}_\epsilon\|_\infty = \|\alpha_\epsilon\|_\infty$ and the functional $L^1(\mathbb{R}_+; E_1^*) \to \mathbb{C}$ induced by $\tilde{\alpha}_\epsilon$ extends the functional $L^1(\mathbb{R}_+; E^*) \to \mathbb{C}$ induced by $\alpha_\epsilon$. It is easy to check that the latter means that $\tilde{\alpha}_\epsilon(s) = \alpha_\epsilon(s)$ almost everywhere on $\mathbb{R}_+$. Likewise, there exist $\tilde{\beta}_\epsilon \in L^\infty(\mathbb{R}_+; E_2^\perp)$ such that $\|\tilde{\beta}_\epsilon\|_\infty = \|\beta_\epsilon\|_\infty$ and $\tilde{\beta}_\epsilon(t) = \beta_\epsilon(t)$ almost everywhere on $\mathbb{R}_+$. Regard $\tilde{\alpha}_\epsilon$ as an element of $L^\infty(\mathbb{R}_+, L^p(\Omega))$ and $\tilde{\beta}_\epsilon$ as an element of $L^\infty(\mathbb{R}_+, L^p(\Omega'))$. By (6), we then have

$$\langle \alpha_\epsilon(s), \beta_\epsilon(t) \rangle_{E,E^*} = \langle \tilde{\alpha}_\epsilon(s), \tilde{\beta}_\epsilon(t) \rangle_{L^p, L^p'}$$

for almost every $(s, t) \in \mathbb{R}^2_+$.

We therefore obtain that $m_\epsilon : \mathbb{R}_+^2 \to \mathbb{C}$ satisfies condition (ii) of the theorem (with $C = 1$).

Define $\varphi : \mathbb{R}_+^2 \to \mathbb{C}$ by $\varphi(s, t) = m(s + t)$. Likewise, for any $\epsilon > 0$, define $\varphi_\epsilon : \mathbb{R}_+^2 \to \mathbb{C}$ by $\varphi_\epsilon(s, t) = m_\epsilon(s + t)$. Since $m$ is continuous, the functions $\varphi$ and $\varphi_\epsilon$ are continuous. It follows from above that for all $\epsilon > 0$, $\varphi_\epsilon$ satisfies condition (ii) in Theorem 2.5, with $C = 1$. The latter theorem therefore implies that the family $\{\varphi_\epsilon(s, t)\}_{(s, t) \in \mathbb{R}^2_+}$ is a bounded Schur multiplier on $B(\ell^p_\mathbb{R}_+)$, with norm less than one. Thus for all $[a_{ij}]_{1 \leq i, j \leq n}$ in $M_n$ and for all $t_1, \ldots, t_n, s_1, \ldots, s_n$ in $\mathbb{R}_+^*$, we have $\|[\varphi_\epsilon(s, t) a_{ij}]\|_{B(\ell^p_\mathbb{R}_+)} \leq \|[a_{ij}]\|_{B(\ell^p_\mathbb{R}_+)}$. Since $m$ is continuous, $\varphi_\epsilon \to \varphi$ pointwise when $\epsilon \to 0$. We deduce that $\varphi$ satisfies (8) with $C = 1$ for all $[a_{ij}]_{1 \leq i, j \leq n}$ in $M_n$ and all $t_1, \ldots, t_n, s_1, \ldots, s_n$ in $\mathbb{R}_+^*$. Consequently, the family $\{\varphi(s, t)\}_{(s, t) \in \mathbb{R}^2_+}$ is a bounded Schur multiplier on $B(\ell^p_\mathbb{R}_+)$, with norm less than one. Applying the implication $(i) \implies (ii)$ in Theorem 2.5, we deduce the assertion (ii) of Theorem 4.6.
Conversely, assume (ii). Following Lemma 2.4, let
\[ \pi : B(L^p(\mathbb{R}_+)) \to B(L^p(\mathbb{R}_+ \times \Omega)) \]
be the \( p \)-completely isometric homomorphism defined by \( \pi(T) = T \otimes I_{L^p(\Omega)} \). This map is \( w^* \)-continuous. Indeed, let \( (T_i) \) be a bounded net of \( B(L^p(\mathbb{R}_+)) \) converging to some \( T \in B(L^p(\mathbb{R}_+)) \) in the \( w^* \)-topology. For any \( f, g \in L^p(\mathbb{R}_+), \varphi \in L^p(\Omega) \) and \( \psi \in L^{p'}(\Omega) \), we have
\[
\langle \pi(T_i), (f \otimes \varphi) \otimes (g \otimes \psi) \rangle = \langle T_i f, g \rangle_{L^p(\mathbb{R}_+), L^{p'}(\mathbb{R}_+)} \langle \varphi, \psi \rangle_{L^p(\Omega), L^{p'}(\Omega)},
\]
where the duality pairing in the left-hand side refers to the identification
\[
(L^p(\mathbb{R}_+ \times \Omega) \otimes L^{p'}(\mathbb{R}_+ \times \Omega))^* \simeq B(L^p(\mathbb{R}_+ \times \Omega)).
\]
Since \( \langle T_i f, g \rangle \to \langle Tf, g \rangle \), we deduce that
\[
\langle \pi(T_i), (f \otimes \varphi) \otimes (g \otimes \psi) \rangle \to \langle \pi(T), (f \otimes \varphi) \otimes (g \otimes \psi) \rangle.
\]
Since \( L^p(\mathbb{R}_+) \otimes L^p(\Omega) \) and \( L^{p'}(\mathbb{R}_+) \otimes L^{p'}(\Omega) \) are dense in \( L^p(\mathbb{R}_+ \times \Omega) \) and \( L^{p'}(\mathbb{R}_+ \times \Omega) \), respectively, we deduce that \( \pi(T_i) \to \pi(T) \) in the \( w^* \)-topology, by Lemma 2.1. This proves that \( \pi \) is \( w^* \)-continuous.

Let \( V : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+; L^p(\Omega)) \simeq L^p(\mathbb{R}_+ \times \Omega) \) be defined by
\[ V(f) = f \alpha, \quad f \in L^p(\mathbb{R}_+). \]
This is a well-defined contraction. Likewise we define a contraction
\[ W : L^p(\mathbb{R}_+ \times \Omega) \to L^p(\mathbb{R}_+) \]
by setting
\[ W^*(g) = g \beta, \quad g \in L^{p'}(\mathbb{R}_+). \]
It follows from above and from the implication (ii) \( \Rightarrow \) (i) of Theorem 2.2 that the mapping
\[ w : B(L^p(\mathbb{R}_+)) \to B(L^p(\mathbb{R}_+)), \quad w(T) = W \pi(T) V \]
is a \( w^* \)-continuous \( p \)-complete contraction.

We claim that for all \( u > 0 \), we have
\[
w(\theta_u) = m(u) \theta_u.
\]
To prove this, consider \( f \in L^p(\mathbb{R}_+) \) and \( g \in L^{p'}(\mathbb{R}_+) \). For all \( u > 0 \), we have
\[
\langle w(\theta_u)f, g \rangle = \langle \pi(\theta_u)V(f), W^*(g) \rangle = \langle \pi(\theta_u)(f \alpha), (g \beta) \rangle.
\]
By the definitions of \( \pi \) and \( \theta_u \), we have \( \pi(\theta_u)(f \alpha) = (f \alpha)(u \cdot \cdot) \). Consequently,
\[
\langle w(\theta_u)f, g \rangle = \int_0^u f(u - t) g(t) \alpha(u - t), \beta(t) \ dt, \quad u > 0.
\]

HANKEL OPERATORS ON \( L^p(\mathbb{R}_+) \)
Let $h \in L^1(\mathbb{R}_+)$ be an auxiliary function. Then using Fubini’s theorem and setting $s = u - t$ in due place, we obtain that
\[
\int_0^\infty \langle w(\theta_u) f, g \rangle h(u) \, du = \int_0^\infty \int_t^\infty h(u) f(u-t) g(t) \langle \alpha(u-t), \beta(t) \rangle \, dudt = \int_0^\infty \int_0^s h(s+t) f(s) g(t) \langle \alpha(s), \beta(t) \rangle \, dsdt.
\]
Applying the a.e. equality $m(s + t) = \langle \alpha(s), \beta(t) \rangle$ and reversing this computation, we deduce that
\[
\int_0^\infty \langle w(\theta_u) f, g \rangle h(u) \, du = \int_0^\infty m(u)(f * g)(u)h(u) \, du.
\]
Since $h$ is arbitrary, this implies that $\langle w(\theta_u) f, g \rangle = m(u)(f * g)(u)$ for a.e. $u > 0$. Equivalently, $\langle w(\theta_u) f, g \rangle = m(u)\langle \theta_u f, g \rangle$ for a.e. $u > 0$. It is plain that $u \mapsto \theta_u$ is $w^*$-continuous on $B(L^p(\mathbb{R}_+))$. Since $w$ is $w^*$-continuous, the function $u \mapsto \langle w(\theta_u) f, g \rangle$ is continuous as well. Since $m$ is assumed continuous, we deduce that $\langle w(\theta_u) f, g \rangle = m(u)\langle \theta_u f, g \rangle$ for all $u > 0$. This yields (22), for all $u > 0$.

By part (1) of Theorem 4.1 and the $w^*$-continuity of $w$, the identity (22) implies that $\text{Hank}_p(\mathbb{R}_+)$ is an invariant subspace of $w$. Further the restriction of $w$ to $\text{Hank}_p(\mathbb{R}_+)$ is the multiplier associated to $m$. The assertion (i) follows. 

**Remark 4.7.** We proved in [Arnold et al. 2022, Theorem 3.1] that a continuous function $m : \mathbb{R}_+^* \to \mathbb{C}$ is the symbol of an $S^1$-bounded Fourier multiplier on $H^1(\mathbb{R})$, with $S^1$-bounded norm $\leq C$, if and only if there exist a Hilbert space $\mathcal{H}$ and two functions $\alpha, \beta \in L^\infty(\mathbb{R}_+; \mathcal{H})$ such that $\|\alpha\|_{\infty}\|\beta\|_{\infty} \leq C$ and $m(s+t) = \langle \alpha(t), \beta(s) \rangle_{\mathcal{H}}$ for almost every $(s, t) \in \mathbb{R}_+^2$. It turns out that using (1), a mapping $S : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ is an $S^1$-bounded Fourier multiplier with $S^1$-bounded norm $\leq C$ if and only if $S^* : \text{Hank}_2(\mathbb{R}_+) \to \text{Hank}_2(\mathbb{R}_+)$ is a completely bounded multiplier with completely bounded norm $\leq C$. See [Arnold et al. 2022, Remark 3.4] for more on this. Thus the statement in [Arnold et al. 2022, Theorem 3.1] is equivalent to the case $p = 2$ of Theorem 4.6. In this regard, Theorem 4.6 can be regarded as a $p$-analogue of [Arnold et al. 2022, Theorem 3.1].

**Remark 4.8.** Let $f \in L^p(\mathbb{R}_+)$ and $g \in L^{p'}(\mathbb{R}_+)$. For any $s, t > 0$, we may write
\[
(f * g)(s + t) = \int_\mathbb{R} f(s + r) g(t - r) \, dr.
\]
Equivalently,
\[
(f * g)(s + t) = \langle \tau_{-s} f, \tau_t \gamma \rangle_{L^p(\mathbb{R}_+), L^{p'}(\mathbb{R}_+)}.\]
According to the implication (ii) $\implies$ (i) of Theorem 4.6 and Remark 4.4, $f * g$ is therefore a pointwise multiplier of $A_p(\mathbb{R}_+)$, with norm less than or equal to $\|f\|_p \|g\|_{p'}$. We deduce that every $F \in A_p(\mathbb{R}_+)$ is a pointwise multiplier of $A_p(\mathbb{R}_+)$, with norm less than or equal to $\|F\|_{A_p}$. This means that $A_p(\mathbb{R}_+)$ is a Banach algebra for the pointwise product.
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