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Let k be an algebraically closed field of characteristic $p > 0$, let R be a commutative ring and let $\mathbb{F}$ be an algebraically closed field of characteristic 0. We introduce the category $\overline{\mathcal{F}^{\Delta}_{Rpp_k}}$ of stable diagonal p-permutation functors over R. We prove that the category $\overline{\mathcal{F}^{\Delta}_{\mathbb{F}pp_k}}$ is semisimple and give a parametrization of its simple objects in terms of the simple diagonal p-permutation functors.

We also introduce the notion of a stable functorial equivalence over R between blocks of finite groups. We prove that if G is a finite group and if b is a block idempotent of kG with an abelian defect group D and Frobenius inertial quotient E, then there exists a stable functorial equivalence over $\mathbb{F}$ between the pairs (G, b) and $(D \rtimes E, 1)$.

1. Introduction

Various notions of equivalences between blocks of finite groups have been studied such as splendid Morita equivalence, splendid Rickard equivalence, p-permutation equivalence, isotypies and perfect isometries [Broué 1990; Boltje and Xu 2008; Boltje and Perepelitsky 2020]. These equivalences are related to prominent conjectures in modular representation theory such as Broué's abelian defect group conjecture [Linckelmann 2018, Conjecture 9.7.6], Puig's finiteness conjecture [Linckelmann 2018, Conjecture 6.4.2] and Donovan's conjecture [Linckelmann 2018, Conjecture 6.1.9].

In [Bouc and Yılmaz 2022] we introduced another equivalence of blocks, namely *functorial equivalence*, using the notion of diagonal p-permutation functors: Let k be an algebraically closed field of characteristic $p > 0$, let $\mathbb{F}$ be an algebraically closed field of characteristic 0 and let R be a commutative ring. We denote by Rpp_k^{Δ} the category whose objects are finite groups and for finite groups G and H whose morphisms from H to G are the Grothendieck group $RT^{\Delta}(G, H)$ of *diagonal* p-permutation (kG, kH)-bimodules. An R-linear functor from Rpp_k^{Δ} to ${}_R\mathsf{Mod}$ is called a *diagonal p-permutation functor*. To each pair (G, b) of a finite group G

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and a block idempotent b of kG, we associate a canonical diagonal p-permutation functor over R, denoted by $RT^{\Delta}_{G,b}$. If (H, c) is another such pair, we say that (G, b) and (H, c) are *functorially equivalent over* R if the functors $RT^{\Delta}_{G,b}$ and $RT^{\Delta}_{H,c}$ are isomorphic.

In [Bouc and Yılmaz 2022] we proved that the category of diagonal p-permutation functors over $\mathbb{F}$ is semisimple, parametrized simple functors and provided three equivalent descriptions of the decomposition of the functor $\mathbb{F}T^{\Delta}_{G,b}$ in terms of the simple functors [Bouc and Yılmaz 2022, Corollary 6.15 and Theorem 8.22]. We proved that the number of isomorphism classes of simple modules, the number of ordinary irreducible characters, and the defect groups are preserved under functorial equivalences over $\mathbb{F}$ [Bouc and Yılmaz 2022, Theorem 10.5]. Moreover we proved that for a given finite p-group D, there are only finitely many pairs (G, b), where G is a finite group and b is a block idempotent of kG, up to functorial equivalence over $\mathbb{F}$ [Bouc and Yılmaz 2022, Theorem 10.6] and we provided a sufficient condition for two blocks to be functorially equivalent over $\mathbb{F}$ in the situation of Broué's abelian defect group conjecture [Bouc and Yılmaz 2022, Theorem 11.1].

In this paper, we introduce the notion of *stable* diagonal p-permutation functors and *stable* functorial equivalences. We denote by $\overline{Rpp^{\Delta}_k}$ the quotient category of Rpp^{Δ}_k by the morphisms that factor through the trivial group. A *stable diagonal p-permutation functor over* R is an R-linear functor from $\overline{Rpp^{\Delta}_k}$ to ${}_R\mathsf{Mod}$, or equivalently, a diagonal p-permutation functor which vanishes at the trivial group. In particular, the simple diagonal p-permutation functors $S_{L,u,V}$ with $L \neq 1$ are (simple) stable diagonal p-permutation functors. Our first main result is the following.

Theorem 1.1. *The category $\overline{\mathcal{F}^{\Delta}_{\mathbb{F}pp_k}}$ of stable diagonal p-permutation functors over $\mathbb{F}$ is semisimple. The simple stable diagonal p-permutation functors are precisely the simple diagonal p-permutation functors $S_{L,u,V}$ with $L \neq 1$.*

Given a finite group G and a block idempotent b of kG, we define a stable diagonal p-permutation functor $\overline{RT^{\Delta}_{G,b}}$ similar to $RT^{\Delta}_{G,b}$; see Definition 4.1. Note that $\overline{RT^{\Delta}_{G,b}}$ is the zero functor if and only if b has defect 0. We say that two pairs (G, b) and (H, c) are *stably functorially equivalent over* R if the functors $\overline{RT^{\Delta}_{G,b}}$ and $\overline{RT^{\Delta}_{H,c}}$ are isomorphic. For a block algebra kGb, let $k(kGb)$ and $l(kGb)$ denote the number of irreducible ordinary characters and the number of irreducible Brauer characters of b, respectively.

Theorem 1.2. *Let b be a block idempotent of kG and let c be a block idempotent of kH.*

(i) *The pairs (G, b) and (H, c) are stably functorially equivalent over $\mathbb{F}$ if and only if the multiplicities of $S_{L,u,V}$ in $\mathbb{F}T^{\Delta}_{G,b}$ and $\mathbb{F}T^{\Delta}_{H,c}$ are the same for any simple diagonal p-permutation functor $S_{L,u,V}$ with $L \neq 1$. In this case, (G, b) and (H, c) are functorially equivalent over $\mathbb{F}$ if and only if $l(kGb) = l(kHc)$.*

(ii) *If the pairs (G, b) and (H, c) are stably functorially equivalent over $\mathbb{F}$, then b and c have isomorphic defect groups and one has*

$$k(kGb) - l(kGb) = k(kHc) - l(kHc).$$

We also consider the blocks with abelian defect groups and Frobenius inertial quotient.

Theorem 1.3. *Let G be a finite group, b a block idempotent of kG with a nontrivial abelian defect group D. Let $E = N_G(D, e_D)/C_G(D)$ denote the inertial quotient of b. Suppose that E acts freely on $D \setminus \{1\}$. Then*:

(i) *There exists a functorial equivalence over $\mathbb{F}$ between (G, b) and $(D \rtimes E, 1)$ if and only if $l(kGb) = l(k(D \rtimes E))$.*

(ii) *Suppose that E is abelian. Then there exists a functorial equivalence over $\mathbb{F}$ between (G, b) and $(D \rtimes E, 1)$ if and only if (G, b) and $(D \rtimes E, 1)$ are p-permutation equivalent.*

In Section 2 we recall diagonal p-permutation functors and functorial equivalences between blocks. In Section 3 we introduce the category of stable diagonal p-permutation functors and prove Theorem 1.1. In Section 4 we introduce the notion of stable functorial equivalences between blocks and prove Theorem 1.2. Finally, in Section 5 we prove Theorem 1.3.

2. Preliminaries

(a) Let (P, s) be a pair, where P is a p-group and s is a generator of a p'-group acting on P. We write $P\langle s\rangle := P \rtimes \langle s\rangle$ for the corresponding semidirect product. We say that two pairs (P, s) and (Q, t) are *isomorphic* and write $(P, s) \cong (Q, t)$, if there is a group isomorphism $f : P\langle s\rangle \to Q\langle t\rangle$ that sends s to a conjugate of t. We set $\mathrm{Aut}(P, s)$ to be the group of the automorphisms of the pair (P, s) and $\mathrm{Out}(P, s) = \mathrm{Aut}(P, s)/\mathrm{Inn}(P\langle s\rangle)$. Recall from [Bouc and Yılmaz 2020] that a pair (P, s) is called a D^Δ*-pair*, if $C_{\langle s\rangle}(P) = 1$.

(b) Let G and H be finite groups. We denote by $T(G)$ the Grothendieck group of p-permutation kG-modules and by $T^\Delta(G, H)$ the Grothendieck group of p-permutation (kG, kH)-bimodules whose indecomposable direct summands have twisted diagonal vertices. Let Rpp_k^Δ denote the following category:

- objects: finite groups.
- $\mathrm{Mor}_{Rpp_k^\Delta}(G, H) = R \otimes_{\mathbb{Z}} T^\Delta(H, G) = RT^\Delta(H, G)$.
- composition is induced from the tensor product of bimodules.
- $\mathrm{Id}_G = [kG]$.

An R-linear functor from Rpp_k^Δ to ${}_R\mathsf{Mod}$ is called a *diagonal p-permutation functor* over R. Together with natural transformations, diagonal p-permutation functors form an abelian category $\mathcal{F}^\Delta_{Rpp_k}$.

(c) Recall from [Bouc and Yılmaz 2022] that the category $\mathcal{F}^\Delta_{\mathbb{F}pp_k}$ is semisimple. Moreover, the simple diagonal p-permutation functors, up to isomorphism, are parametrized by the isomorphism classes of triples (L, u, V), where (L, u) is a D^Δ-pair, and V is a simple $\mathbb{F}\mathrm{Out}(L, u)$-module (see [loc. cit., Sections 6 and 7] for more details on simple functors).

(d) Let G be a finite group and b a block idempotent of kG. Recall from [loc. cit.] that the block diagonal p-permutation functor $RT^\Delta_{G,b}$ is defined as

$$RT^\Delta_{G,b} : Rpp_k^\Delta \to {}_R\mathsf{Mod}, \quad H \mapsto RT^\Delta(H, G) \otimes_{kG} kGb.$$

See [loc. cit., Section 8] for the decomposition of $\mathbb{F}T^\Delta_{G,b}$ in terms of the simple functors $S_{L,u,V}$.

(e) Let b be a block idempotent of kG and let c be a block idempotent of kH. We say that the pairs (G, b) and (H, c) are *functorially equivalent* over R, if the corresponding diagonal p-permutation functors $RT^\Delta_{G,b}$ and $RT^\Delta_{H,c}$ are isomorphic in $\mathcal{F}^\Delta_{Rpp_k}$ [loc. cit., Definition 10.1]. By [loc. cit., Lemma 10.2] the pairs (G, b) and (H, c) are functorially equivalent over R if and only if there exists $\omega \in bRT^\Delta(G, H)c$ and $\sigma \in cRT^\Delta(H, G)b$ such that

$$\omega \cdot_H \sigma = [kGb] \quad \text{in} \quad bRT^\Delta(G, G)b \quad \text{and} \quad \sigma \cdot_G \omega = [kHc] \quad \text{in} \quad cRT^\Delta(H, H)c.$$

3. Stable diagonal p-permutation functors

In this section we introduce the category of stable diagonal p-permutation functors.

For a finite group G, let $P(G)$ denote the subgroup of $T(G)$ generated by the indecomposable projective kG-modules. Let also $\overline{T(G)}$ denote the quotient group $T(G)/P(G)$. For $X \in T(G)$, we denote by $\overline{X}$ the image of X in $\overline{T(G)}$. If H is another finite group, we define $P(G, H)$ and $\overline{T^\Delta(G, H)}$ similarly.

Lemma 3.1. *For finite groups G and H one has $P(G, H) = T^\Delta(G, 1) \circ T^\Delta(1, H)$.*

Proof. This follows from the fact that the projective indecomposable $k(G \times H)$-modules are of the form $P \otimes_k Q$, where P and Q are projective indecomposable kG and kH-modules, respectively. □

Definition 3.2. Let $\overline{Rpp_k^\Delta}$ denote the following category:

- objects: finite groups.
- $\mathrm{Mor}_{\overline{Rpp_k^\Delta}}(G, H) = R \otimes_{\mathbb{Z}} \overline{T^\Delta(H, G)} = \overline{RT^\Delta(H, G)}$.
- composition is induced from the tensor product of bimodules.
- $\mathrm{Id}_G = \overline{[kG]}$.

Definition 3.3. An R-linear functor $\overline{Rpp_k^\Delta} \to {}_R\mathsf{Mod}$ is called a *stable diagonal p-permutation functor* over R. Together with natural transformations, stable diagonal p-permutation functors form an abelian category $\overline{\mathcal{F}_{Rpp_k}^\Delta}$.

Remark 3.4. The functor

$$\Gamma : \overline{\mathcal{F}_{Rpp_k}^\Delta} \to \mathcal{F}_{Rpp_k}^\Delta$$

obtained by composition with the projection $Rpp_k^\Delta \to \overline{Rpp_k^\Delta}$ gives a description of $\overline{\mathcal{F}_{Rpp_k}^\Delta}$ as a full subcategory of $\mathcal{F}_{Rpp_k}^\Delta$. Moreover, Γ has a left adjoint Σ, constructed as follows: If F is a diagonal p-permutation functor over R and G is a finite group, set

$$\bar{F}(G) := F(G)/RT^\Delta(G, \mathbf{1})F(\mathbf{1}).$$

Then $\bar{F}$ is a diagonal p-permutation functor, equal to the quotient of F by the subfunctor generated by $F(\mathbf{1})$. Obviously, $\bar{F}$ vanishes at the trivial group, so it is a stable diagonal p-permutation functor. The functor $\Sigma : F \mapsto \bar{F}$ is a left adjoint to the above functor Γ. In particular, $\overline{\mathcal{F}_{Rpp_k}^\Delta}$ is a reflective subcategory of $\mathcal{F}_{Rpp_k}^\Delta$.

Let G be a finite group. Recall that by [Bouc and Yılmaz 2022, Corollary 8.23(i)], the multiplicity of the simple diagonal p-permutation functor $S_{1,1,\mathbb{F}}$ in the representable functor $\mathbb{F}T^\Delta(-, G)$ is equal to the number $l(kG)$ of the isomorphism classes of simple kG-modules. Let $\mathcal{I}(-, G)$ denote the sum of simple subfunctors of $\mathbb{F}T^\Delta(-, G)$ isomorphic to $S_{1,1,\mathbb{F}}$. Let also $\mathbb{F}\mathrm{Proj}(-, G)$ denote the subfunctor of $\mathbb{F}T^\Delta(-, G)$ sending a finite group H to $\mathbb{F}\mathrm{Proj}(H, G)$.

Lemma 3.5. *The subfunctors $\mathcal{I}(-, G)$ and $\mathbb{F}\mathrm{Proj}(-, G)$ of the representable functor $\mathbb{F}T^\Delta(-, G)$ are equal.*

Proof. For finite groups G and H, the number of isomorphism classes of projective indecomposable $k(G \times H)$-modules, or equivalently, the number of isomorphism classes of simple $k(G \times H)$-modules is equal to the number of conjugacy classes of p'-elements of $G \times H$. Hence the $\mathbb{F}$-dimension of the evaluation $\mathbb{F}\mathrm{Proj}(H, G)$ is equal to

$$l(k(G \times H)) = l(kG)l(kH)$$

which is equal to the $\mathbb{F}$-dimension of $l(kG)S_{1,1,\mathbb{F}}(H)$ by [Bouc and Yılmaz 2022, Corollary 8.23(i)], and hence to the $\mathbb{F}$-dimension of $\mathcal{I}(H, G)$.

Note that $\mathbb{F}\mathrm{Proj}(-, G)$ is equal to the functor

$$\mathbb{F}T^\Delta(-, 1) \circ \mathbb{F}T^\Delta(1, G).$$

Moreover $S_{L,u,V}(1) = 0$ for $L \neq 1$, and hence $\mathbb{F}T^\Delta(1, G) = \mathcal{I}(1, G)$. Therefore,

$$\mathbb{F}\mathrm{Proj}(-, G) = \mathbb{F}T^\Delta(-, 1) \circ \mathbb{F}T^\Delta(1, G) = \mathbb{F}T^\Delta(-, 1) \circ \mathcal{I}(1, G) \subseteq \mathcal{I}(-, G).$$

Since the $\mathbb{F}$-dimensions of $\mathbb{F}\mathrm{Proj}(H, G)$ and $\mathcal{I}(H, G)$ are the same for any finite group H, it follows that $\mathbb{F}\mathrm{Proj}(-, G) = \mathcal{I}(-, G)$. □

Proof of Theorem 1.1. For a finite group G, the representable diagonal p-permutation functor $\mathbb{F}T^{\Delta}(-, G)$ decomposes as a direct sum of simple functors $S_{L,u,V}$, and hence we have

$$\mathbb{F}T^{\Delta}(-, G) \cong \mathcal{I}(-, G) \bigoplus_{\substack{(L,u,V)\\ L\neq 1}} S_{L,u,V}^{m_{L,u,V}},$$

for some nonnegative integers $m_{L,u,V}$, where (L, u, V) runs over a set of isomorphism classes of D^{Δ}-pairs (L, u) with $L \neq 1$, and simple $\mathbb{F}\mathrm{Out}(L, u)$-modules V. By Lemma 3.5, the representable stable diagonal p-permutation functor $\overline{\mathbb{F}T^{\Delta}(-, G)}$ is isomorphic to the direct sum

$$\bigoplus_{\substack{(L,u,V)\\ L\neq 1}} S_{L,u,V}^{m_{L,u,V}}$$

of simple diagonal p-permutation functors, and each of these simple functors is a simple stable diagonal p-permutation functor. Since the functor category $\overline{\mathcal{F}^{\Delta}_{\mathbb{F}pp_k}}$ is generated by the representable functors the result follows. □

4. Stable functorial equivalences

Let G and H be finite groups.

Definition 4.1. Let b a block idempotent of kG. The stable diagonal p-permutation functor $\overline{RT^{\Delta}_{G,b}}$ is defined as

$$\overline{RT^{\Delta}_{G,b}} : \overline{Rpp^{\Delta}_k} \to {}_R\mathsf{Mod}, \quad H \mapsto \overline{RT^{\Delta}(H, G) \otimes_{kG} kGb}.$$

See Section 2(d) for the definition of $RT^{\Delta}_{G,b}$ and note that $\overline{RT^{\Delta}_{G,b}} = \Sigma(RT^{\Delta}_{G,b})$.

Definition 4.2. Let b be a block idempotent of kG and let c be a block idempotent of kH. We say that the pairs (G, b) and (H, c) are *stably functorially equivalent over* R, if their corresponding stable diagonal p-permutation functors $\overline{RT^{\Delta}_{G,b}}$ and $\overline{RT^{\Delta}_{H,c}}$ are isomorphic in $\overline{\mathcal{F}^{\Delta}_{Rpp_k}}$.

Lemma 4.3. *Let b be a block idempotent of kG and let c be a block idempotent of kH.*

(a) *The pairs (G, b) and (H, c) are stably functorially equivalent over R if and only if there exists $\omega \in bRT^{\Delta}(G, H)c$ and $\sigma \in cRT^{\Delta}(H, G)b$ such that*

$$\omega \cdot_H \sigma = [kGb] + [P] \text{ in } bRT^{\Delta}(G, G)b \text{ and } \sigma \cdot_G \omega = [kHc] + [Q] \text{ in } cRT^{\Delta}(H, H)c$$

for some $P \in R\mathrm{Proj}(kGb, kGb)$ and $Q \in R\mathrm{Proj}(kHc, kHc)$.

(b) *If the pairs (G, b) and (H, c) are functorially equivalent over R, then they are also stably functorially equivalent over R.*

Proof. By the Yoneda lemma, the pairs (G, b) and (H, c) are stably functorially equivalent over R if and only if there exists $\bar{\omega} \in \overline{bRT^{\Delta}(G, H)c}$ and $\bar{\sigma} \in \overline{cRT^{\Delta}(H, G)b}$ such that

$$\bar{\omega} \cdot_H \bar{\sigma} = \overline{[kGb]} \quad \text{in} \quad \overline{bRT^{\Delta}(G, G)b} \quad \text{and} \quad \bar{\sigma} \cdot_G \bar{\omega} = \overline{[kHc]} \quad \text{in} \quad \overline{cRT^{\Delta}(H, H)c}.$$

Hence (a) follows and (b) is clear. □

Proof of Theorem 1.2. (i) The first statement follows from Theorem 1.1 and the second statement follows since by [Bouc and Yılmaz 2022, Corollary 8.23(i)] the multiplicity of the simple functor $S_{1,1,\mathbb{F}}$ in $\mathbb{F}T^{\Delta}_{G,b}$ is equal to $l(kGb)$.

(ii) The first statement follows from the proof of [loc. cit., Theorem 10.5(iii)] and the second statement follows from the proof of [loc. cit., Theorem 10.5(ii)]. □

5. Blocks with Frobenius inertial quotient

(a) Recall the assumptions of Theorem 1.3: Let G be a finite group, b a block idempotent of kG with a nontrivial abelian defect group D. Let (D, e_D) be a maximal b-Brauer pair and let $E = N_G(D, e_D)/C_G(D)$ denote the inertial quotient of b. Suppose that E acts freely on $D \backslash \{1\}$. This condition is equivalent to requiring $D \rtimes E$ be a Frobenius group. Let $\mathcal{F}_b$ be the fusion system of b with respect to (D, e_D). Then $\mathcal{F}_b$ is equal to the fusion system $\mathcal{F}_{D \rtimes E}(D)$ on D determined by $D \rtimes E$.

Under these assumptions, we know from [Linckelmann 2018, Theorem 10.5.1] that there is a stable equivalence of Morita type between (G, b) and $(D \rtimes E, 1)$, induced by a bimodule with an endopermutation source. Since D is abelian, by [Rickard 1996, Theorem 7.2] this bimodule admits an endosplit p-permutation resolution, which yields a stable p-permutation equivalence between (G, b) and $(D \rtimes E, 1)$; see, for instance, [Linckelmann 2018, Remark 9.11.7]. In particular (G, b) and $(D \rtimes E, 1)$ are stably functorially equivalent.

Hereafter we give an alternative proof of the existence of this stable functorial equivalence, which relies only on Theorem 1.1. Together with Theorem 1.2, this now implies Part (i) of Theorem 1.3. Part (ii) of Theorem 1.3 follows from Part (i) and [Linckelmann 2018, Theorem 10.5.10].

(b) Let $S_{L,u,V}$ be a simple diagonal p-permutation functor such that L is nontrivial and isomorphic to a subgroup of D. Recall that by [Bouc and Yılmaz 2022, Theorem 8.22] the multiplicity of $S_{L,u,V}$ in $\mathbb{F}T^{\Delta}_{G,b}$ is equal to the $\mathbb{F}$-dimension of

$$\bigoplus_{(P,e_P)\in[\mathcal{F}_b]} \ \bigoplus_{\pi\in[N_G(P,e_P)\backslash\mathcal{P}_{(P,e_P)}(L,u)/\mathrm{Aut}(L,u)]} \mathbb{F}\mathrm{Proj}(ke_P C_G(P), u) \otimes_{\mathrm{Aut}(L,u)_{\overline{(P,e_P,\pi)}}} V,$$

where $[\mathcal{F}_b]$ denotes a set of isomorphism classes of objects in $\mathcal{F}_b$, $\mathcal{P}_{(P,e_P)}$ is the set of group isomorphisms $\pi : L \to P$ with $\pi i_u \pi^{-1} \in \mathrm{Aut}_{\mathcal{F}_b}(P, e_P)$, and $\mathrm{Aut}(L, u)_{\overline{(P,e_P,\pi)}}$ is the stabilizer in $\mathrm{Aut}(L, u)$ of the G-orbit of (P, e_P, π). Since b

is a block with Frobenius inertial quotient, the block $kC_G(P)e_P$ is nilpotent for every nontrivial subgroup P of D; see, for instance, [Linckelmann 2018, Theorem 10.5.2]. Therefore, we have $l(ke_P C_G(P)) = 1$, and hence the multiplicity formula reduces to

$$\bigoplus_{(P,e_P)\in[\mathcal{F}_b]} \bigoplus_{\pi\in[N_G(P,e_P)\backslash\mathcal{P}_{(P,e_P)}(L,u)/\mathrm{Aut}(L,u)]} V^{\mathrm{Out}(L,u)_{\overline{(P,e_P,\pi)}}}.$$

Let $\mathcal{Q}_{D\rtimes E,p}$ denote the set of pairs (P,s) of p-subgroups P of $D\rtimes E$ and p'-elements s of $N_{D\rtimes E}(P)$. Let also $[\mathcal{Q}_{D\rtimes E,p}]$ denote a set of representatives of $D\rtimes E$-orbits on $\mathcal{Q}_{D\rtimes E,p}$ under the conjugation map. Recall from [Bouc and Yılmaz 2022, Corollary 7.4] that the multiplicity of $S_{L,u,V}$ in $\mathbb{F}T^{\Delta}_{D\rtimes E}$ is equal to the $\mathbb{F}$-dimension of

$$\bigoplus_{\substack{(P,s)\in[\mathcal{Q}_{D\rtimes E,p}]\\ (\tilde{P},\tilde{s})\cong(L,u)}} V^{N_{D\rtimes E}(P,s)},$$

where for a pair $(P,s)\in\mathcal{Q}_{D\rtimes E,p}$ with $(\tilde{P},\tilde{s})\cong(L,u)$, we fix an isomorphism $\phi_{P,s}: L\to P$ with $\phi_{P,s}({}^u l) = {}^s\phi_{P,s}(l)$ for all $l\in L$ and we view V as an $\mathbb{F}N_{D\rtimes E}(P,s)$-module via the group homomorphism

$$N_G(P,s)\to \mathrm{Out}(L,u) \tag{1}$$

that sends $g\in N_G(P,s)$ to the image of $\phi_{P,s}^{-1}\circ i_g\circ\phi_{P,s}$ in $\mathrm{Out}(L,u)$.

(c) Let $\mathcal{P}_b(G,L,u)$ denote the set of triples (P,e,π) where $(P,e)\in\mathcal{F}_b$ and $\pi\in\mathcal{P}_{(P,e_P)}(L,u)$. Let also $\mathcal{Q}_{D\rtimes E,p}(L,u)$ denote the set of pairs (P,s) in $\mathcal{Q}_{D\rtimes E,p}$ with the property that $(\tilde{P},\tilde{s})\cong(L,u)$.

If $(P,e,\pi)\in\mathcal{P}_b(G,L,u)$, then $\pi i_u\pi^{-1}\in\mathrm{Aut}_{\mathcal{F}_b}(P,e_P)$ by definition and since $\mathcal{F}_b$ is equal to $\mathcal{F}_{D\rtimes E}(D)$, it follows that there exists a p'-element s of $N_{D\rtimes E}(P)$ with $\pi i_u\pi^{-1} = i_s$. This implies by [Bouc and Yılmaz 2022, Lemma 3.3] that $(\tilde{P},\tilde{s})\cong(L,u)$ and therefore we have a map

$$\Psi:\mathcal{P}_b(G,L,u)\to\mathcal{Q}_{D\rtimes E,p}(L,u),\quad (P,e,\pi)\mapsto(P,s).$$

Lemma 5.1. *The map* Ψ *induces a bijection*

$$\overline{\Psi}:[G\backslash\mathcal{P}_b(G,L,u)/\mathrm{Aut}(L,u)]\to[\mathcal{Q}_{D\rtimes E,p}(L,u)].$$

Proof. First we show that the map $\overline{\Psi}$ is well-defined. Let (P,e,π) and (Q,f,ρ) be two elements in $\mathcal{P}_b(G,L,u)$ that lie in the same $G\times\mathrm{Aut}(L,u)$-orbit. We need to show that $\overline{\Psi}(P,e,\pi)=\overline{\Psi}(Q,f,\rho)$. Write $\Psi(P,e,\pi)=(P,s)$ and $\Psi(Q,f,\rho)=(Q,t)$. Let $g\in G$ and $\varphi\in\mathrm{Aut}(L,u)$ such that

$$g\cdot(P,e,\pi)\cdot\varphi=(Q,f,\rho).$$

Then (P,e) and (Q,f) lie in the same isomorphism class in $[\mathcal{F}_b]$ and hence P and Q are $D\rtimes E$-conjugate since $\mathcal{F}_b=\mathcal{F}_{D\rtimes E}(D)$. Thus, there exists $h\in D\rtimes E$

with $i_g = i_h : P \to Q$. Hence $\rho = i_g\pi\varphi = i_h\pi\varphi : L \to Q$. Since $\varphi \in \mathrm{Aut}(L,u)$, one has $\varphi \circ i_u = i_u \circ \varphi$. Therefore,

$$i_t = \rho i_u \rho^{-1} = i_h\pi\varphi i_u\varphi^{-1}\pi^{-1}i_{h^{-1}} = i_h\pi i_u\pi^{-1}i_{h^{-1}} = i_h i_s i_{h^{-1}} = i_{hsh^{-1}}.$$

This shows that $(Q,t) = h\cdot(P,s)$ and hence the map $\overline{\Psi}$ is well-defined.

Now we show that $\overline{\Psi}$ is surjective. Let $(P,s) \in \mathcal{Q}_{D\rtimes E,p}(L,u)$. Since $(\tilde{P},\tilde{s}) \cong (L,u)$, again by [Bouc and Yılmaz 2022, Lemma 3.3], there exists $\pi : L \to P$ such that $\pi i_u = i_s\pi$, i.e., $\pi i_u\pi^{-1} = i_s : P \to P$. Since $\mathcal{F}_{D\rtimes E}(D) = \mathcal{F}_b$, it follows that there exists $g \in N_G(P,e)$ with $i_s = i_g$, and hence $(P,e,\pi) \in \mathcal{P}_b(G,L,u)$ with $\overline{\Psi}(P,e,\pi) = (P,s)$. Thus, $\overline{\Psi}$ is surjective.

Finally, we show that $\overline{\Psi}$ is injective. Let $(P,e,\pi), (Q,f,\rho) \in \mathcal{P}_b(G,L,u)$ be elements with $\overline{\Psi}(P,e,\pi) = \overline{\Psi}(Q,f,\rho)$. Write $(P,s) = \Psi(P,e,\pi)$ and $(Q,t) = \Psi(Q,f,\rho)$. Then there exists $h \in D\rtimes E$ such that

$$h\cdot(P,s) = (Q,t).$$

Again, there exists $g \in G$ such that $i_g = i_h : P \to Q$. Define

$$\varphi := \pi^{-1}\circ i_g^{-1}\circ\rho : L \to L.$$

One has

$$\begin{aligned}\varphi\circ i_u &= \pi^{-1}\circ i_g^{-1}\circ\rho\circ i_u = \pi^{-1}\circ i_g^{-1}\circ i_t\circ\rho = \pi^{-1}\circ i_g^{-1}\circ i_g\circ i_s\circ i_g^{-1}\circ\rho\\ &= \pi^{-1}\circ i_s\circ i_g^{-1}\circ\rho = i_u\circ\pi^{-1}\circ i_g^{-1}\circ\rho = i_u\circ\varphi\end{aligned}$$

which shows that $\varphi \in \mathrm{Aut}(L,u)$. Moreover, one has

$$g\cdot(P,e,\pi)\cdot\varphi = (Q,f,\rho)$$

and so the map $\overline{\Psi}$ is injective. □

Lemma 5.2. *Let* $(P,e,\pi) \in [G\backslash\mathcal{P}_b(G,L,u)/\mathrm{Aut}(L,u)]$ *and* $(P,s) = \overline{\Psi}(P,e,\pi) \in [\mathcal{Q}_{D\rtimes E,p}]$. *Then the image of* $N_{D\rtimes E}(P,s)$ *in* $\mathrm{Out}(L,u)$ *is equal to* $\mathrm{Out}(L,u)_{\overline{(P,e,\pi)}}$.

Proof. We have $\pi i_u\pi^{-1} = i_s$ and hence the image of $N_{D\rtimes E}(P,s)$ is given by

$$N_{D\rtimes E}(P,s) \to \mathrm{Out}(L,u), \quad h \mapsto \overline{\pi^{-1}\circ i_h\circ\pi}$$

Note that since ${}^hs = s$, we have $i_h i_s = i_s i_h$, i.e., $i_h\pi i_u\pi^{-1} = \pi i_u\pi^{-1}i_h$. Therefore the image is

$$\begin{aligned}&\{\overline{\pi^{-1}i_h\pi} \mid h\in D\rtimes E, i_h : P\to P, {}^hs = s\}\\ &\qquad = \{\overline{\pi^{-1}i_g\pi} \mid g\in N_G(P,e), i_g\pi i_u\pi^{-1} = \pi i_u\pi^{-1}i_g\}\\ &\qquad = \{\overline{\pi^{-1}i_g\pi} \in \mathrm{Out}(L,u) \mid \pi^{-1}i_g\pi = i_g, g\in N_G(P,e)\}\\ &\qquad = \mathrm{Out}(L,u)_{\overline{(P,e,\pi)}}\end{aligned}$$

as was to be shown. □

Proof of Theorem 1.3. We need to show that for any $L \neq 1$, the multiplicities of a simple diagonal p-permutation functor $S_{L,u,V}$ in $\mathbb{F}T^{\Delta}_{G,b}$ and in $\mathbb{F}T^{\Delta}_{D \rtimes E}$ are equal. But this follows from Lemmas 5.1 and 5.2. Now Part (i) follows from Theorem 1.2(i), and Part (ii) follows from [Linckelmann 2018, Theorem 10.5.10]. □

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References

[Boltje and Perepelitsky 2020] R. Boltje and P. Perepelitsky, "p-permutation equivalences between blocks of group algebras", 2020. arXiv 2007.09253

[Boltje and Xu 2008] R. Boltje and B. Xu, "On p-permutation equivalences: between Rickard equivalences and isotypies", *Trans. Amer. Math. Soc.* **360**:10 (2008), 5067–5087. MR Zbl

[Bouc and Yılmaz 2020] S. Bouc and D. Yılmaz, "Diagonal p-permutation functors", *J. Algebra* **556** (2020), 1036–1056. MR Zbl

[Bouc and Yılmaz 2022] S. Bouc and D. Yılmaz, "Diagonal p-permutation functors, semisimplicity, and functorial equivalence of blocks", *Adv. Math.* **411** (2022), art. id. 108799. MR Zbl

[Broué 1990] M. Broué, "Isométries parfaites, types de blocs, catégories dérivées", pp. 61–92 in *Représentations linéaires des groupes finis*, Astérisque **181-182**, 1990. MR Zbl

[Linckelmann 2018] M. Linckelmann, *The block theory of finite group algebras*, *II*, Cambridge University Press, 2018. MR Zbl

[Rickard 1996] J. Rickard, "Splendid equivalences: derived categories and permutation modules", *Proc. London Math. Soc.* (3) **72**:2 (1996), 331–358. MR Zbl

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