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**LORENTZ–SHIMOGAKI AND ARAZY–CWIKEL THEOREMS  
REVISITED**

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# LORENTZ–SHIMOGAKI AND ARAZY–CWIKEL THEOREMS REVISITED

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**By using the space  $L_0$  of finitely supported functions as a left endpoint on the interpolation scale of  $L_p$ -spaces, we present a new approach to the Lorentz–Shimogaki and Arazy–Cwikel theorems which covers the whole range of  $p, q \in (0, \infty]$ . In particular, we show that for  $0 \leq p < q < r < s \leq \infty$ ,**

$$\text{Int}(L_q, L_r) = \text{Int}(L_p, L_r) \cap \text{Int}(L_q, L_s)$$

**if the underlying space is  $(0, \alpha)$ ,  $\alpha \in (0, \infty]$  equipped with the Lebesgue measure. As a byproduct of our result, we solve a conjecture of Levitina, Sukochev and Zanin (2020).**

## 1. Introduction

Descriptions of interpolation spaces for couples of  $L_p$ -spaces for  $1 \leq p \leq \infty$  were extensively researched from the 1960s to the 80s, providing satisfying answers to most problems that were considered relevant at the time.

However, new questions arising from noncommutative analysis recently highlighted some gaps in our knowledge of this subject, especially for the case of  $p < 1$  of quasi-Banach spaces. In this paper, we revisit some important results of the literature [1; 21; 27], generalizing them and thus filling some of the holes that were revealed in the theory. In particular, we answer a question asked by Levitina, Sukochev and Zanin in [20] and already partially studied in [11] regarding the interpolation theory of sequence spaces (see Theorem 1.2). Besides this new result, this paper introduces a general approach that covers the range of all  $0 \leq p \leq \infty$  and is self-contained. It emphasizes the use of the space  $L_0$  of all finitely supported measurable functions. As far as the authors know this space rarely appears in interpolation theory (however, see [2; 16; 24] and [15]). We provide evidence that  $L_0$  is a suitable “left endpoint” on the interpolation scale of  $L_p$ -spaces, despite its possessing an atypical structure, that of a normed abelian group.

A function space  $E$  is an *interpolation space* for the couple  $(L_p, L_q)$  if any linear operator  $T$  bounded on  $L_p$  and  $L_q$  is also bounded on  $E$  (see Definition 2.4). This

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notion provides a way of transferring inequalities well known in  $L_p$ -spaces to more exotic ones. To both understand the range of applicability of this technique and be able to check whether it applies to a given function space  $E$ , we are interested in simple descriptions of interpolation spaces for the couple  $(L_p, L_q)$ .

This problem has a long history starting with the Calderón–Mityagin theorem (see [9; 22]) on the couple  $(L_1, L_\infty)$  and followed by Lorentz and Shimogaki’s [21] results on the couples  $(L_1, L_q)$  and  $(L_p, L_\infty)$  with  $1 \leq p, q \leq \infty$ . A remarkable result of Arazy and Cwikel then states that a space  $E$  is an interpolation space for the couple  $(L_p, L_q)$ ,  $1 < p < q < \infty$  if and only if it is an interpolation space for the couples  $(L_1, L_q)$  and  $(L_p, L_\infty)$ .

Describing interpolation spaces often comes down to understanding certain orders. In fact, at a very fundamental level, being an interpolation space can be understood as a monotonicity property. Indeed, given two compatible quasi-Banach spaces  $A, B$ , denote by  $\mathbb{C}(A, B)$  the set of operators  $A + B \rightarrow A + B$  that restrict to contractions on  $A$  and  $B$ . Consider the following order on  $A + B$ :

$$f \leq_{p,q} g \Leftrightarrow \exists T \in \mathbb{C}(A, B), \quad T(g) = f.$$

With this definition in mind,  $E$  is an interpolation space for the couple  $(A, B)$  if and only if

$$\forall f \in E, \forall g \in L_p + L_q, \quad g \leq_{p,q} f \Rightarrow g \in E.$$

In fact, the fundamental theorem of Calderón and Mityagin precisely describes the order  $\leq_{L_1, L_\infty}$  (from now on denoted by  $\ll_{\text{hd}}$ ). It states that for  $f, g \in L_1 + L_\infty$ ,

$$g \ll_{\text{hd}} f \Leftrightarrow g \leq_{L_1, L_\infty} f \Leftrightarrow \forall t > 0, \quad \int_0^t \mu(s, g) ds \leq \int_0^t \mu(s, f) ds,$$

where  $\mu(g) : t \rightarrow \mu(t, g)$  denotes the right-continuous decreasing rearrangement of  $g$ . We will call this order *head majorization*. Moreover, if  $f, g \in L_1$  and  $\|f\|_1 = \|g\|_1$ , then we write  $g \ll_{\text{hd}} f$ . Variants of this order allow to describe interpolation spaces for any couple  $(L_p, L_\infty)$ ,  $p \in (0, \infty)$  (see [21] for the Banach range and [8] for  $p < 1$ ).

Another phenomenon, this time specific to the study of interpolation theory of  $L_p$ -spaces, is that to guarantee that a space  $E$  is an interpolation space for the couple  $(L_p, L_q)$ ,  $p < q$  it is natural to impose two conditions: one which will impose that  $E$  is “on the right of  $L_p$ ” and one that will impose that  $E$  is “on the left of  $L_q$ ”. An example of such a result is the above-mentioned Arazy–Cwikel theorem but one can think also of convexity/concavity conditions or Boyd indices (see [18] for an overview and [8, Theorem 1.4]).

In this spirit, the natural counterpart of head majorization is *tail majorization* defined on  $L_0 + L_1$  by

$$g \ll_{\text{tl}} f \Leftrightarrow \forall t > 0, \int_t^\infty \mu(s, g) ds \leq \int_t^\infty \mu(s, f) ds.$$

We'll show later that this order is in fact equivalent to  $\leq_{L_0, L_1}$ . Moreover, if  $f, g \in L_1$  and  $\|f\|_1 = \|g\|_1$ , then we write  $g \prec_{\text{tl}} f$ . Remark that  $g \prec_{\text{tl}} f$  if and only if  $f \prec_{\text{hd}} g$ . Note that tail majorization coincides with the weak supermajorization of [13].

Let us now state our main theorem. Let  $\mathcal{X}$  be the linear space of all measurable functions. If not specified otherwise, the *underlying measure space* we are working on is  $(0, \infty)$  equipped with the Lebesgue measure  $m$ . We obtain:

**Theorem 1.1.** *Let  $E \subset \mathcal{X}$  be a quasi-Banach function space (a priori, not necessarily symmetric). Let  $p, q \in (0, \infty)$  such that  $p < q$ . Then:*

- (a)  *$E$  is an interpolation space for the couple  $(L_p, L_\infty)$  if and only if there exists  $c_{p,E} > 0$  such that for any  $f \in E$  and  $g \in L_p + L_\infty$ ,*

$$|g|^p \prec_{\text{hd}} |f|^p \Rightarrow g \in E \quad \text{and} \quad \|g\|_E \leq c_{p,E} \|f\|_E.$$

- (b)  *$E$  is an interpolation space for the couple  $(L_0, L_q)$  if and only if there exists  $c_{q,E} > 0$  such that for any  $f \in E$  and  $g \in L_0 + L_q$ ,*

$$|g|^q \prec_{\text{tl}} |f|^q \Rightarrow g \in E \quad \text{and} \quad \|g\|_E \leq c_{q,E} \|f\|_E.$$

- (c)  *$E$  is an interpolation space for the couples  $(L_0, L_q)$  and  $(L_p, L_\infty)$  if and only if it is an interpolation space for the couple  $(L_p, L_q)$ .*

This extends the results of Lorentz–Shimogaki and Arazy–Cwikel to the quasi-Banach setting and contributes to the two first questions asked by Arazy in [12, p. 232] in the particular case of  $L_p$ -spaces for  $0 < p < \infty$ . As mentioned before, our approach places  $L_0$  as a left endpoint on the interpolation scale of  $L_p$ -spaces, in sharp contrast to earlier results which focused mostly on Banach spaces and had  $L_1$  playing this part. An advantage of our approach is that it naturally encompasses every symmetric quasi-Banach space since they are all interpolation spaces for the couple  $(L_0, L_\infty)$  (see [2; 16]). On the contrary, there exist some symmetric Banach spaces which are *not* interpolation spaces for the couple  $(L_1, L_\infty)$  (see [26]). This led to some difficulties which were customarily circumvented with the help of various technical conditions such as the Fatou property (as appears, e.g., in [4]).

Compared to [8] where the first author investigates similar characterizations, the novelty of this theorem is statement (b) that deals with the space  $L_0$ . A deeper advantage of our new approach is that it no longer relies on Sparr's  $K$ -monotonicity result [27] for couples of  $L_p$ -spaces which was instrumental in [8].

Indeed, our strategy in this paper is different from the techniques used in [1; 2; 3; 4; 8; 9; 10; 11; 12; 16; 18; 21; 22; 27] and is based on partition lemmas,

which were originally developed in a deep paper due to Braverman and Mekler [7] devoted to the study of the symmetric Banach function spaces  $E$  such that the set  $\{f \in E : f \prec\prec_{\text{hd}} g\}$  coincides with the closure of the convex hull of its extreme points.

The approach of Braverman and Mekler was subsequently revised and redeveloped in [28] and precisely this revision constitutes the core of our approach in this paper.

We restate partition lemmas based on [28, Proposition 19] in Section 4. These lemmas allow us to restrict head and tail majorizations to very simple situations and reduce the problem to functions taking at most two values. Then, we deduce interpolation results from those structural lemmas.

Note that this scheme of proof is quite direct and in particular, does not involve at any point duality related arguments which apply only to Banach spaces [21] or more generally to  $L$ -convex quasi-Banach spaces [17; 25].

In Section 6, we pursue the same type of investigation, but in the setting of sequence spaces. The nondiffuse aspect of the underlying measure generates substantial technical difficulties. In particular, we require a new partition lemma which is not as efficient as those in Section 4 (compare Lemmas 6.2 and 4.6). This deficiency has been first pointed out to the authors by Cwikel. However, we are still able to resolve the conjecture of [20] (in the affirmative) by combining Lemma 6.1 with a Boyd-type argument which we borrow from Montgomery and Smith [23]. In particular, we substantially strengthen the results in [11]. Here is the precise statement:

**Theorem 1.2.** *Let  $E \subset \ell_\infty$  be a quasi-Banach sequence space and  $q \geq 1$ . The following conditions are equivalent:*

- (a) *There exists  $p < q$  such that  $E$  is an interpolation space for the couple  $(\ell^p, \ell^q)$ .*
- (b) *There exists  $c > 0$  such that for any  $u \in E$  and  $v \in \ell_\infty$ ,*

$$|v|^q \prec\prec_{\text{tl}} |u|^q \Rightarrow v \in E \quad \text{and} \quad \|v\|_E \leq c\|u\|_E.$$

- (c) *For any  $u \in E$  and  $v \in \ell_\infty$ ,*

$$|v|^q \prec\prec_{\text{tl}} |u|^q \Rightarrow v \in E.$$

In this section, we freely use results of Cwikel [10] and Cadilhac [8] to avoid repeating too many similar arguments.

Note that Theorem 1.2 was since proved independently in [5] where a deeper analysis of the interpolation scale of sequence spaces  $\ell_p$ ,  $0 \leq p \leq \infty$  is presented. In particular, it is shown in [5] that for any  $0 < p < q < \infty$ ,  $E$  is an interpolation space for  $(\ell_p, \ell_q)$  if and only if it is an interpolation space for  $(\ell_0, \ell_q)$  and  $(\ell_p, \ell_\infty)$ , thus providing a counterpart to our main theorem in the sequence setting.

## 2. Preliminaries

**Interpolation spaces.** The reader is referred to [6] for more details on interpolation theory and to [19] for an introduction to symmetric spaces. In the remainder of this section,  $p$  and  $q$  will denote two nonnegative reals such that  $p \leq q$ .

Let  $(\Omega, m)$  be any measure space (in particular the following definitions apply to  $\mathbb{N}$  equipped with the counting measure, i.e., sequence spaces). As previously mentioned,  $L_0(\Omega) \subset \mathcal{X}(\Omega)$  denotes the set of functions whose supports have finite measures, it is naturally equipped with the group norm

$$\|f\|_0 = m(\text{supp } f), \quad f \in L_0(\Omega).$$

The “norm” of a linear operator  $T : L_0(\Omega) \rightarrow L_0(\Omega)$ , is defined as

$$\|T\|_{L_0 \rightarrow L_0} = \sup_{f \in L_0} \frac{m(\text{supp}(Tf))}{m(\text{supp}(f))}.$$

**Definition 2.1.** A linear space  $E \subset \mathcal{X}(\Omega)$  becomes a quasi-Banach function space when equipped with a complete quasinorm  $\|\cdot\|_E$  such that:

- If  $f \in E$  and  $g \in \mathcal{X}(\Omega)$  are such that  $|g| \leq |f|$ , then  $g \in E$  and  $\|g\|_E \leq \|f\|_E$ .

**Definition 2.2.** A quasi-Banach function space  $E \subset \mathcal{X}(\Omega)$  is called symmetric if

- $f \in E$  and  $g \in \mathcal{X}(\Omega)$  are such that  $\mu(f) = \mu(g)$ , then  $g \in E$  and  $\|g\|_E = \|f\|_E$ .

**Definition 2.3** (bounded operator on a couple of quasi-Banach function spaces). Let  $X$  and  $Y$  be quasi-Banach function spaces. We say that a linear operator  $T$  is bounded on  $(X, Y)$  if  $T$  is defined from  $X + Y$  to  $X + Y$  and restricts to a bounded operator from  $X$  to  $X$  and from  $Y$  to  $Y$ . Set

$$\|T\|_{(X,Y) \rightarrow (X,Y)} = \max(\|T\|_{X \rightarrow X}, \|T\|_{Y \rightarrow Y}).$$

Let us recall the precise abstract definition of an interpolation space (see [6; 19]).

**Definition 2.4** (interpolation space between function spaces). Let  $X, Y$  and  $Z$  be either quasi-Banach function spaces on  $\Omega$  or  $L_0(\Omega)$ . We say that  $Z$  is an interpolation space for the couple  $(X, Y)$  if  $X \cap Y \subset Z \subset X + Y$  and any bounded operator on  $(X, Y)$  restricts to a bounded operator on  $Z$ . Denote by  $\text{Int}(X, Y)$  the set of interpolation spaces for the couple  $(X, Y)$ .

For quasi-Banach spaces, the above definition is equivalent to a seemingly stronger quantitative property.

**Proposition 2.5.** *Let  $X, Y, Z$  be quasi-Banach function spaces. If  $Z$  is an interpolation space for the couple  $(X, Y)$ , then there exists a constant  $c(X, Y, Z) > 0$  such that for any bounded operator  $T$  on  $(X, Y)$ ,*

$$\|T\|_{Z \rightarrow Z} \leq c(X, Y, Z) \cdot \|T\|_{(X,Y) \rightarrow (X,Y)}.$$

The best possible value of  $c(X, Y, Z)$  is called *interpolation constant of  $Z$  with respect to the couple  $(X, Y)$* .

*Proof.* In [19, Lemma I.4.3], the assertion is proved for Banach spaces. The argument for quasi-Banach spaces is identical (because it relies on the closed graph theorem, which holds for  $F$ -spaces, and hence for quasi-Banach spaces).  $\square$

***K-functional and E-functional.*** In the remainder of the subsection,  $X, Y$  and  $Z$  will denote function spaces which are either quasi-Banach, or  $L_0$ .

**Definition 2.6.** Let  $f \in X + Y$  and  $t > 0$ . Define

$$K_t(f, X, Y) := \inf_{g+h=f} \|g\|_X + t\|h\|_Y \quad \text{and} \quad E_t(f, X, Y) := \inf_{\|g\|_X \leq t} \|f - g\|_Y.$$

These two notions are closely related to one another (see [24]) and the  $K$ -functional in particular plays a major role in the study of general interpolation spaces. Note that

$$K_t(f, L_1, L_\infty) = \int_0^t \mu(s, f) ds \quad \text{and} \quad E_t(f, L_0, L_1) = \int_t^\infty \mu(s, f) ds.$$

Thus the head and tail majorizations we consider can be in fact expressed in terms of  $K$  and  $E$  functionals. We say that  $Z$  is *K-monotone* with respect to the couple  $(X, Y)$  if  $X \cap Y \subset Z \subset X + Y$  and for any  $f \in Z, g \in X + Y$ ,

$$\forall t > 0, \quad K_t(g, X, Y) \leq K_t(f, X, Y) \Rightarrow g \in Z.$$

Similarly,  $Z$  is *E-monotone* with respect to the couple  $(X, Y)$  if  $X \cap Y \subset Z \subset X + Y$  and for any  $f \in Z, g \in X + Y$ ,

$$\forall t > 0, \quad E_t(g, X, Y) \leq E_t(f, X, Y) \Rightarrow g \in Z.$$

**Remark 2.7.** It is clear from the definitions that if  $Z$  is either  $E$ -monotone or  $K$ -monotone for the couple  $(X, Y)$  then  $Z$  is an interpolation space for  $(X, Y)$ .

**Symmetry of interpolation spaces.** In this subsection, we show that a quasi-Banach interpolation space for a couple of symmetric spaces can always be renormed into a symmetric space. Note that similar results can be found in the literature, see, for example, [19, Theorem 2.1].

As usual, we will use the term *measure preserving* for a measurable map  $\omega$  between measure spaces  $(\Omega_1, \mathcal{A}_1, m_1)$  and  $(\Omega_2, \mathcal{A}_2, m_2)$  verifying,

$$\forall A \in \mathcal{A}_1, \quad \omega(A) \in \mathcal{A}_2 \quad \text{and} \quad m_2(\omega(A)) = m_1(A).$$

**Lemma 2.8.** Assume that  $\Omega$  is  $(0, 1), (0, \infty)$  or  $\mathbb{N}$ . Let  $0 \leq f, g \in L_0(\Omega) + L_\infty(\Omega)$  and let  $\varepsilon > 0$ . Assume that  $\mu(f) = \mu(g)$ . There exists a measure preserving map  $\omega : \text{supp}(g) \rightarrow \text{supp}(f)$  such that  $(1 + \varepsilon)(f \circ \omega) \geq g$ .

*Proof. Case 1.* Suppose first that  $\mu(\infty, f) = \mu(\infty, g) = 0$ .

Define, for any  $n \in \mathbb{Z}$ ,

$$F_n = \{t : (1 + \varepsilon)^n < f(t) \leq (1 + \varepsilon)^{n+1}\}, \quad G_n = \{t : (1 + \varepsilon)^n < g \leq (1 + \varepsilon)^{n+1}\}.$$

By assumption,  $m(F_n) = m(G_n)$  for every  $n \in \mathbb{Z}$ . Let  $\omega_n : G_n \rightarrow F_n$  be an arbitrary measure preserving bijection.

Define the measure preserving map  $\omega : \text{supp}(g) \rightarrow \text{supp}(f)$  by concatenating  $\omega_n : G_n \rightarrow F_n$ ,  $n \in \mathbb{Z}$ . For every  $t \in G_n$ , we have

$$f(\omega(t)) \geq (1 + \varepsilon)^n, \quad g \leq (1 + \varepsilon)^{n+1}.$$

Thus,

$$(1 + \varepsilon) f(\omega(t)) \geq g, \quad t \in \text{supp}(g).$$

This completes the proof of Case 1.

**Case 2.** Let  $\delta$  such that  $(1 + \delta)^2 = (1 + \varepsilon)$ . Let  $a = \mu(\infty, f) = \mu(\infty, g) > 0$ . Define, for any  $n \geq 1$ ,

$$F_n = \{t : a(1 + \delta)^n < f(t) \leq a(1 + \delta)^{n+1}\}, \quad G_n = \{t : a(1 + \delta)^n < g \leq a(1 + \delta)^{n+1}\}$$

and

$$F_0 = \{t : (1 + \delta)^{-1}a \leq f(t) \leq (1 + \delta)a\}, \quad G_0 = \{t : 0 < g \leq (1 + \delta)a\}.$$

By assumption, for any  $n \geq 1$ ,  $m(G_n) = m(F_n)$  and  $m(G_0) = m(F_0) = \infty$ . For any  $n \geq 0$ , choose a measure preserving bijection  $\omega_n$  from  $G_n$  to  $F_n$ .

Define the measure preserving map  $\omega : \text{supp}(g) \rightarrow \text{supp}(f)$  by concatenating the  $\omega_n$ 's. For any  $n \geq 0$  and any  $t \in G_n$ ,

$$f(\omega(t)) \geq a(1 + \delta)^{n-1}, \quad g \leq a(1 + \delta)^{n+1}.$$

Thus,

$$(1 + \delta)^2 f(\omega(t)) = (1 + \varepsilon) f(\omega(t)) \geq g, \quad t \in \text{supp}(g). \quad \square$$

**Lemma 2.9.** *Assume that  $\Omega$  is  $(0, 1)$ ,  $(0, \infty)$  or  $\mathbb{N}$ . Let  $E, A, B \subset (L_0 + L_\infty)(\Omega)$  be quasi-Banach function spaces. Assume that  $A$  and  $B$  are symmetric and that  $E$  is an interpolation space for the couple  $(A, B)$ . Then  $E$  admits an equivalent symmetric quasinorm.*

*Proof.* Let  $f \in E$  and  $g \in L_0 + L_\infty$ . Assume that  $\mu(g) \leq \mu(f)$ . By Lemma 2.8, there exists a map  $\omega : \text{supp}(g) \rightarrow \text{supp}(f)$  such that for any  $t \in \text{supp}(g)$ ,

$$2|f \circ \omega(t)| \geq |g(t)|.$$

Define, for any  $h \in \mathcal{X}(\Omega)$ ,

$$T(h) := \begin{cases} \frac{g}{f \circ \omega} h \circ \omega & \text{on } \text{supp}(g), \\ 0, & \text{elsewhere.} \end{cases}$$

Since  $\omega$  is measure preserving,  $T$  is bounded on  $A$  and  $B$  of norm less than 2. Let  $c_E$  be the interpolation constant of  $E$  for the couple  $(A, B)$  (as in Proposition 2.5). We know that  $Tf = g \in E$  and

$$(2-1) \quad \|g\| \leq 2c_E \|f\|.$$

Define, for any  $f \in E$ ,

$$\|f\|_{E'} = \inf_{\mu(g) \geq \mu(f)} \|g\|_E.$$

By (2-1),  $\|f\|_{E'} \leq \|f\|_E \leq 2c_E \|f\|_{E'}$  and  $(E, \|\cdot\|_{E'})$  is a symmetric space.  $\square$

**Remark 2.10.** It is not difficult to see that if the underlying measure space  $\Omega$  contains both a continuous part and atoms, then Lemma 2.9 is no longer true for  $A = L_p(\Omega)$ ,  $B = L_q(\Omega)$  and  $p < 1$ . However, one can observe that if  $A$  and  $B$  are *fully symmetric* (i.e., interpolation spaces between  $L_1(\Omega)$  and  $L_\infty(\Omega)$ ), Lemma 2.9 remains valid for any  $\Omega$ . This is reminiscent of the conditions required in [27, Section 4].

### 3. Interpolation for the couple $(L_0, L_q)$

In this section,  $\Omega = (0, \infty)$  (for brevity, we omit  $\Omega$  in the notations). We investigate some basic properties of the interpolation couple  $(L_0, L_q)$ . First, we provide a statement analogous to Proposition 2.5 and applicable to  $L_0$ .

Since the closed graph theorem does not apply to  $L_0$  (it is not an  $F$ -space), our proof uses concrete constructions that rely on the structure of the underlying measure space.

For any  $f \in \mathcal{X}$ , denote by  $M_f$  the multiplication operator  $g \mapsto f \cdot g$ .

**Theorem 3.1.** *Let  $E$  be a quasi-Banach function space and  $q \in (0, \infty]$ . Assume that  $E$  is an interpolation space for the couple  $(L_0, L_q)$ . Then, there exists a constant  $c$  such that for any contraction  $T$  on  $(L_0, L_q)$ ,  $\|T\|_{E \rightarrow E} \leq c$ .*

*Proof.* Let  $(A_n)_{n \geq 1}$  be a partition of  $(0, \infty)$  such that  $m(A_n) = \infty$  for every  $n \geq 1$ .

Let  $\gamma_n : A_n \rightarrow A_n^c$  be a measure preserving bijective transform. Set

$$(U_n x)(t) = \begin{cases} x(\gamma_n(t)), & t \in A_n, \\ 0, & t \in A_n^c, \end{cases} \quad (V_n x)(t) = \begin{cases} x(\gamma_n^{-1}(t)), & t \in A_n^c, \\ 0, & t \in A_n. \end{cases}$$

Obviously,  $U_n$  and  $V_n$  are bounded operators on the couple  $(L_0, L_q)$ . By assumption,  $U_n, V_n : E \rightarrow E$  are bounded mappings.

Note that

$$V_n U_n = M_{\chi_{A_n^c}}, \quad n \geq 1.$$

Let us argue by contradiction. For any  $n \geq 1$ , choose an operator  $T_n$  which is a contraction on  $(L_0, L_q)$  and such that

$$(3-1) \quad \|T_n\|_{E \rightarrow E} \geq 4^n \cdot \max\{\|U_n\|_{E \rightarrow E}, \|V_n\|_{E \rightarrow E}, 1\}^2.$$

It is immediate that

$$\begin{aligned} T_n &= M_{\chi_{A_n}} T_n M_{\chi_{A_n}} + M_{\chi_{A_n^c}} T_n M_{\chi_{A_n}} + M_{\chi_{A_n}} T_n M_{\chi_{A_n^c}} + M_{\chi_{A_n^c}} T_n M_{\chi_{A_n^c}} \\ &= T_{1,n} + V_n T_{2,n} + T_{3,n} U_n + V_n T_{4,n} U_n, \end{aligned}$$

where

$$T_{1,n} = M_{\chi_{A_n}} T_n M_{\chi_{A_n}}, \quad T_{2,n} = U_n T_n M_{\chi_{A_n}}, \quad T_{3,n} = M_{\chi_{A_n}} T_n V_n, \quad T_{4,n} = U_n T_n V_n.$$

By quasitriangle inequality, we have

$$\|T_n\|_{E \rightarrow E} \leq C_E^2 \cdot \left( \sum_{k=1}^4 \|T_{k,n}\|_{E \rightarrow E} \right) \cdot \max\{\|U_n\|_{E \rightarrow E}, \|V_n\|_{E \rightarrow E}, 1\}^2.$$

Let  $k_n \in \{1, 2, 3, 4\}$  be such that

$$\|T_{k_n,n}\|_{E \rightarrow E} = \max_{1 \leq k \leq 4} \|T_{k,n}\|_{E \rightarrow E}.$$

We, therefore, have

$$(3-2) \quad \|T_n\|_{E \rightarrow E} \leq 4C_E^2 \|T_{k_n,n}\|_{E \rightarrow E} \cdot \max\{\|U_n\|_{E \rightarrow E}, \|V_n\|_{E \rightarrow E}, 1\}^2.$$

Set  $S_n = T_{k_n,n}$ . Note that  $\|S_n\|_{L_0 \rightarrow L_0} \leq 1$  and  $\|S_n\|_{L_q \rightarrow L_q} \leq 1$ . A combination of (3-1) and (3-2) yields

$$\|S_n\|_{E \rightarrow E} \geq 4^{n-1} C_E^{-2}, \quad n \geq 1.$$

Note that  $S_n = M_{\chi_{A_n}} S_n M_{\chi_{A_n}}$ . Set

$$S = \sum_{n \geq 1} S_n.$$

Since the  $S_n$ 's are in direct sum, we have

$$\|S\|_{L_0 \rightarrow L_0} = \sup_{n \geq 1} \|S_n\|_{L_0 \rightarrow L_0} \leq 1 \quad \text{and} \quad \|S\|_{L_q \rightarrow L_q} = \sup_{n \geq 1} \|S_n\|_{L_q \rightarrow L_q} \leq 1.$$

Moreover,  $E$  is an interpolation space for the couple  $(L_0, L_q)$ , it follows that  $S : E \rightarrow E$  is bounded.

For any  $n \geq 1$ , choose  $f_n \in E$  such that  $\|f_n\|_E \leq 1$  and  $\|S_n f_n\|_E \geq 4^{n-2} C_E^{-2}$ . Recall that  $S_n = M_{\chi_{A_n}} S_n$ . Hence, we may assume without loss of generality that  $f_n$  is supported on  $A_n$ . Thus,  $S(f_n) = S_n(f_n)$  and

$$\|S(f_n)\|_E = \|S_n(f_n)\|_E \geq 4^{n-2} C_E^{-2}.$$

This contradicts the boundedness of  $S$ . □

**Remark 3.2.** Theorem 3.1 above remains true for other underlying measure spaces:

- *For sequence spaces.* Indeed, in the proof of Theorem 3.1, we only use properties of the underlying measure space in the first sentence, namely when we consider a partition of  $(0, \infty)$  into countably many sets, each of them isomorphic to  $(0, \infty)$ . Since a partition satisfying the same property exists for  $\mathbb{Z}_+$ , Theorem 3.1 remains true for interpolation spaces between  $\ell_0$  and  $\ell_q$ .
- *For  $(0, 1)$ .* The same general idea applies in this case but some modifications have to be made because the maps  $\gamma_n$  introduced in the proof cannot be assumed to be measure-preserving. The details are left to the reader.

**Lemma 3.3.** *Let  $E, Y \subset L_0 + L_\infty$  be quasi-Banach function spaces. Assume that  $Y$  is symmetric and that  $E$  is an interpolation space for the couple  $(L_0, Y)$ . Then  $E$  admits an equivalent symmetric quasinorm.*

*Proof.* The argument follows that in Lemma 2.9 mutatis mutandi. □

The following assertion is a special case of Theorem 1.1 and an important ingredient in the proof of the latter theorem.

**Corollary 3.4.** *Let  $X$  be a quasi-Banach function space and  $q \in (0, \infty)$ . Assume that  $L_0 \cap L_q \subset X \subset L_0 + L_q$  and that for any  $f \in E$  and  $g \in L_0 + L_q$ ,*

$$|g|^q \prec\prec_{\text{id}} |f|^q \Rightarrow g \in E.$$

*Then  $E$  is an interpolation space for the couple  $(L_0, L_q)$ .*

*Proof.* It is clear that the condition on  $X$  is equivalent to  $E$ -monotonicity with respect to the couple  $(L_0, L_q)$  so by Remark 2.7,  $E$  is an interpolation space for the couple  $(L_0, L_q)$ . □

Corollary 3.4 applies in particular to  $L_p$ -spaces,  $p \leq q$ . We decided to add a more precise statement and to provide a direct proof of the latter.

**Corollary 3.5.** *Let  $p, q \in (0, \infty)$  such that  $p < q$ . Then,  $L_p$  is an interpolation space for the couple  $(L_0, L_q)$ . More precisely, if  $T$  is a contraction on  $(L_0, L_q)$ , then  $T$  is a contraction on  $L_p$ .*

*Proof.* Let us first consider characteristic functions. Let  $E$  be a set with finite measure. Since  $T$  is a contraction on  $L_0$ , the measure of the support of  $T(\chi_E)$  is less than  $m(E)$ . So by Hölder's inequality, setting  $r = (p^{-1} - q^{-1})^{-1}$ , we have

$$\|T(\chi_E)\|_p \leq \|T(\chi_E)\|_q \cdot m(E)^{1/r} \leq \|\chi_E\|_q \cdot m(E)^{1/r} = \|\chi_E\|_p.$$

First, consider the case  $p \leq 1$ . Let  $f \in L_p$  be a step function, i.e.,

$$f = \sum_{i \in \mathbb{N}} a_i \chi_{E_i},$$

where  $a_i \in \mathbb{C}$  and the sets  $E_i$  are disjoint sets with finite measure. By the  $p$ -triangular inequality we write

$$\|Tf\|_p^p \leq \sum_{i \in \mathbb{N}} |a_i|^p \|T(\chi_{E_i})\|_p^p \leq \sum_{i \in \mathbb{N}} |a_i|^p \|\chi_{E_i}\|_p^p = \|f\|_p^p.$$

Since  $T : L_p \rightarrow L_p$  is bounded by Corollary 3.4 and since step functions are dense in  $L_p$ , it follows that  $T : L_p \rightarrow L_p$  is a contraction (for  $p \leq 1$ ).

Now consider the case  $p > 1$ . Since  $p < q$ , it follows that  $q > 1$ . By the preceding paragraph,  $T : L_1 \rightarrow L_1$  is a contraction. By complex interpolation,  $T : L_p \rightarrow L_p$  is also contraction. □

#### 4. Construction of contractions on $(L_0, L_q)$ and $(L_p, L_\infty)$

Let  $p, q \in (0, \infty)$ . In this section, we are interested in the following question. Given functions  $f$  and  $g$  in  $L_0 + L_q$  (resp.  $L_p + L_\infty$ ), does there exist a bicontraction  $T$  on  $(L_0, L_q)$  (resp.  $(L_p, L_\infty)$ ) such that  $T(f) = g$ ? We show that such an operator exists provided that  $|g|^q \prec\prec_{tl} |f|^q$  (resp.  $|g|^p \prec\prec_{hd} |f|^p$ ). This directly implies a necessary condition for a symmetric space to be an interpolation space for the couple  $(L_0, L_q)$  (resp.  $(L_p, L_\infty)$ ) which will be exploited in the next section.

Our method of proof is very direct. We construct the bicontraction  $T$  as direct sums of very simple operators. This is made possible by two partition lemmas that enable us to understand the orders  $\prec\prec_{tl}$  and  $\prec\prec_{hd}$  as direct sums of simple situations.

**Partition lemmas.** We state our first lemma without proof since it essentially repeats that of Proposition 19 in [28].

**Lemma 4.1.** *Let  $f, g \in L_1$  be positive decreasing step functions. Assume that  $g \prec_{hd} f$ . There exists a sequence of intervals  $\{I_k, J_k\}_{k \geq 0}$  of  $(0, \infty)$  such that:*

- (i) *For every  $k \geq 0$ ,  $I_k$  and  $J_k$  are disjoint intervals of finite length.*
- (ii)  *$(I_k \cup J_k) \cap (I_l \cup J_l) = \emptyset$  for  $k \neq l$ .*
- (iii)  *$f$  and  $g$  are constant on  $I_k$  and on  $J_k$ .*
- (iv)  *$g|_{I_k \cup J_k} \prec_{hd} f|_{I_k \cup J_k}$  for every  $k \geq 0$ .*
- (v)  *$g \leq f$  on the complement of  $\bigcup_{k \geq 0} I_k \cup J_k$ .*

*If furthermore  $f, g \in L_1$  and  $g \prec_{hd} f$  then  $f = g$  on the complement of  $\bigcup_{k \geq 0} I_k \cup J_k$ .*

**Scholium 4.2.** *Let  $f, g \in \mathcal{X}$  be positive decreasing functions. Let  $\Delta \subset (0, \infty)$  be an arbitrary measurable set.*

- (i) *If  $f, g \in L_1 + L_\infty$  are such that*

$$\int_{[0,t] \cap \Delta} g \leq \int_{[0,t] \cap \Delta} f, \quad t > 0,$$

*then  $g\chi_\Delta \prec\prec_{hd} f\chi_\Delta$ .*

(ii) If  $f, g \in L_0 + L_1$  are such that

$$\int_{(t, \infty) \cap \Delta} g \leq \int_{(t, \infty) \cap \Delta} f, \quad t > 0,$$

then  $g \chi_{\Delta} \prec_{\lll} f \chi_{\Delta}$ .

The second partition lemma deals with describing the order  $\prec_{\lll}$  in terms of  $\prec_{\lll}$  and  $\leq$ .

**Lemma 4.3.** *Let  $f, g \in L_0 + L_1$  be such that  $f = \mu(f)$ ,  $g = \mu(g)$  and  $g \prec_{\lll} f$ . There exists a collection  $\{\Delta_k\}_{k \geq 0}$  of pairwise disjoint sets such that:*

- (i)  $f|_{\Delta_k} \prec_{\text{hd}} g|_{\Delta_k}$  for every  $k \geq 0$ .
- (ii)  $g \leq f$  on the complement of  $\bigcup_{k \geq 0} \Delta_k$ .

*Proof.* Consider the set  $\{g > f\}$ . Similarly to the previous proof, connected components of the set  $\{g > f\}$  are intervals (closed or not) not reduced to points. Let us enumerate these intervals as  $(a_k, b_k)$ ,  $k \geq 0$ .

We have

$$\int_t^\infty (f - g)_+ - \int_t^\infty (f - g)_- = \int_t^\infty (f - g) \geq 0.$$

Let

$$H(t) = \sup \left\{ u : \int_u^\infty (f - g)_+ = \int_t^\infty (f - g)_- \right\}.$$

Obviously,  $H$  is a monotone function,  $H(t) \geq t$  for all  $t > 0$  and

$$\int_{H(t)}^\infty (f - g)_+ = \int_t^\infty (f - g)_-.$$

Set

$$\Delta_k = (a_k, b_k) \cup ((H(a_k), H(b_k)) \cap \{g \leq f\}).$$

Note that

$$\int_{b_k}^\infty (f - g)_+ = \int_{a_k}^\infty (f - g)_+ \geq \int_{a_k}^\infty (f - g)_-$$

and therefore,  $H(a_k) \geq b_k$ .

We claim that  $\Delta_k \cap \Delta_l = \emptyset$  for  $k \neq l$ . Indeed, let  $a_k < b_k \leq a_l < b_l$ . We have  $H(a_k) \leq H(b_k) \leq H(a_l) \leq H(b_l)$ . Thus,  $(H(a_k), H(b_k)) \cap (H(a_l), H(b_l)) = \emptyset$ . We now have

$$\Delta_k \cap \Delta_l = ((\Delta_k \cap \{f < g\}) \cap (\Delta_l \cap \{f < g\})) \cup ((\Delta_k \cap \{f \geq g\}) \cap (\Delta_l \cap \{f \geq g\})).$$

Obviously,

$$(\Delta_k \cap \{f < g\}) \cap (\Delta_l \cap \{f < g\}) = (a_k, b_k) \cap (a_l, b_l) = \emptyset,$$

$$(\Delta_k \cap \{f \geq g\}) \cap (\Delta_l \cap \{f \geq g\}) = (H(a_k), H(b_k)) \cap (H(a_l), H(b_l)) \cap \{f \geq g\} = \emptyset.$$

This proves the claim.

We now claim that

$$\int_{(t, \infty) \cap \Delta_k} (f - g) \geq 0.$$

If  $t \geq b_k$ , then taking into account that  $H(a_k) \geq b_k$ , we infer that

$$(t, \infty) \cap \Delta_k \subset \{f \geq g\}$$

and the claim follows immediately. If  $t \in (a_k, b_k)$ , then

$$\begin{aligned} \int_{(t, \infty) \cap \Delta_k} (f - g) &= \int_{(H(a_k), H(b_k))} (f - g)_+ - \int_t^{b_k} (f - g)_- \\ &\geq \int_{(H(a_k), H(b_k))} (f - g)_+ - \int_{a_k}^{b_k} (f - g)_- = 0. \end{aligned}$$

This proves the claim.

It follows from the claim and Scholium 4.2 that  $g \chi_{\Delta_k} \prec\prec_{tl} f \chi_{\Delta_k}$ . Since

$$\int_{\Delta_k} g = \int_{\Delta_k} f,$$

it follows that  $g \chi_{\Delta_k} \prec_{tl} f \chi_{\Delta_k}$ , which immediately implies the first assertion.

By construction,  $(a_k, b_k) \subset \Delta_k$ . Thus,

$$\{g > f\} = \bigcup_{k \geq 0} (a_k, b_k) \subset \bigcup_{k \geq 0} \Delta_k.$$

The second assertion is now obvious. □

**Construction of operators.** We repeat the same structure as in the previous subsection, proving four lemmas, each one dealing with a certain order:  $\prec_{hd}$ ,  $\prec_{tl}$ ,  $\prec\prec_{hd}$ , and finally  $\prec\prec_{tl}$ .

**Lemma 4.4.** *Let  $p \in (0, \infty)$ . Let  $f, g \in L_p(0, \infty)$ , assume that  $|g|^p \prec_{hd} |f|^p$ ,  $f = \mu(f)$  and  $g = \mu(g)$ . There exists a linear operator  $T : \mathcal{X}(0, \infty) \rightarrow \mathcal{X}(0, \infty)$  such that  $g = T(f)$  and*

$$\|T\|_{L_p \rightarrow L_p} \leq 2 \cdot 3^{1/p}, \quad \|T\|_{L_\infty \rightarrow L_\infty} \leq 2 \cdot 2^{1/p}.$$

*Proof. Step 1.* First, let us assume that  $f$  and  $g$  are step functions.

Apply Lemma 4.1 to the functions  $f^p$  and  $g^p$  and let  $I_k$  and  $J_k$  be as in Lemma 4.1. Without loss of generality, the interval  $I_k$  is located to the left of the interval  $J_k$ .

For every  $k \geq 0$ , let's define the mapping  $S_k : \mathcal{X}(I_k \cup J_k) \rightarrow \mathcal{X}(I_k \cup J_k)$  as below. The construction of this mapping will depend on whether  $f^p|_{J_k} \leq \frac{1}{2}g^p|_{J_k}$  or  $f^p|_{J_k} > \frac{1}{2}g^p|_{J_k}$ .

If  $f^p|_{J_k} \leq \frac{1}{2}g^p|_{J_k}$ , then

$$\begin{aligned} g^p|_{J_k} \cdot m(J_k) &\leq g^p|_{I_k} \cdot m(I_k) + g^p|_{J_k} \cdot m(J_k) \\ &= f^p|_{I_k} \cdot m(I_k) + f^p|_{J_k} \cdot m(J_k) \leq f^p|_{I_k} \cdot m(I_k) + \frac{1}{2}g^p|_{J_k} \cdot m(J_k). \end{aligned}$$

Therefore,

$$g^p|_{J_k} \cdot m(J_k) \leq 2f^p|_{I_k} \cdot m(I_k).$$

Let  $l_k$  be a linear bijection from  $J_k$  to  $I_k$ . We set

$$S_k x = \frac{g|_{I_k}}{f|_{I_k}} \cdot x \chi_{I_k} + \frac{g|_{J_k}}{f|_{I_k}} \cdot (x \circ l_k) \chi_{J_k}.$$

Clearly,  $S_k$  is a contraction in the uniform norm.

Let  $x \in L_p$ . We have

$$\begin{aligned} \|S_k x\|_p^p &\leq \frac{g^p|_{I_k}}{f^p|_{I_k}} \cdot \|x \chi_{I_k}\|_p^p + \frac{g^p|_{J_k}}{f^p|_{I_k}} \cdot \|(x \circ l_k) \chi_{J_k}\|_p^p \\ &\leq \frac{g^p|_{I_k}}{f^p|_{I_k}} \cdot \|x\|_p^p + \frac{g^p|_{J_k}}{f^p|_{I_k}} \cdot \frac{m(J_k)}{m(I_k)} \cdot \|x\|_p^p \leq 3\|x\|_p^p. \end{aligned}$$

Also, we have

$$\|S_k x\|_\infty \leq \|x\|_\infty.$$

If  $f^p|_{J_k} > \frac{1}{2}g^p|_{J_k}$ , then we set  $S_k = M_{g^{f^{-1}}}$ . Clearly,  $\|S_k x\|_\infty \leq 2^{1/p}\|x\|_\infty$  and  $\|S_k x\|_p \leq 2^{1/p}\|x\|_p$ .

We define  $S : \mathcal{X} \rightarrow \mathcal{X}$  by

$$S = \bigoplus_{k \geq 0} S_k.$$

Remark that  $\|S\|_{r \rightarrow r} = \sup_{k \geq 0} \|S_k\|_{L_r \rightarrow L_r}$  for any  $r \in [0, \infty]$ . Hence,

$$\|S\|_{L_p \rightarrow L_p} \leq 3^{1/p}, \quad \|S\|_{L_\infty \rightarrow L_\infty} \leq 2^{1/p}.$$

It remains only to note that  $Sf = g$ .

**Step 2.** Now, only assume that  $f$  and  $g$  are positive and nonincreasing. Define, for any  $n \in \mathbb{Z}$ ,

$$a_n = \sup\{t \in (0, \infty) : f(t) \geq 2^{n/2}\} \quad \text{and} \quad b_n = \sup\{t \in (0, \infty) : g(t) \geq 2^{n/2}\}.$$

Let  $\mathcal{A}$  be the  $\sigma$ -algebra generated by the intervals  $(a_n, a_{n+1})$  and  $(b_n, b_{n+1})$ . Define

$$f_0^p = \mathbb{E}[f^p | \mathcal{A}] \quad \text{and} \quad g_0^p = \mathbb{E}[g^p | \mathcal{A}].$$

Note that  $f_0$  and  $g_0$  are step functions such that

$$g_0^p \prec_{\text{hd}} f_0^p, \quad 2^{-1/2}f \leq f_0 \leq 2^{1/2}f \quad \text{and} \quad 2^{-1/2}g \leq g_0 \leq 2^{1/2}g.$$

Apply Step 1 to  $f_0$  and  $g_0$  to obtain an operator  $S$  and set

$$T = M_{ff_0^{-1}} \circ S \circ M_{g_0g^{-1}}.$$

Clearly,  $Tf = g$  and

$$\|T\|_{L_p \rightarrow L_p} \leq 2\|S\|_{L_p \rightarrow L_p} \leq 2 \cdot 2^{1/p}, \quad \|T\|_{L_\infty \rightarrow L_\infty} \leq 2\|S\|_{L_\infty \rightarrow L_\infty} \leq 2^2 \cdot 2^{1/p}. \quad \square$$

**Lemma 4.5.** *Let  $f, g \in L_q(0, \infty)$  be positive nonincreasing functions such that  $g^q \prec_{\text{d}} f^q$ . Let  $d > 1$ . There exists a linear operator  $T : \mathcal{X}(0, \infty) \rightarrow \mathcal{X}(0, \infty)$  such that  $g = T(f)$  and*

$$\|T\|_{L_0 \rightarrow L_0} \leq 4, \quad \|T\|_{L_q \rightarrow L_q} \leq 2 \cdot 3^{1/q}.$$

*Proof.* Following Step 2 of Lemma 4.4, we are reduced to dealing with step functions.

Apply Lemma 4.1 to the functions  $g^q$  and  $f^q$  and let  $(I_k)_{k \geq 1}$  and  $(J_k)_{k \geq 1}$  be as in Lemma 4.1. Without loss of generality, the intervals  $I_k$  is located to the left of the intervals  $J_k$ .

Let  $k \geq 1$ . Define the mappings  $S_k : \mathcal{X}(I_k \cup J_k) \rightarrow \mathcal{X}(I_k \cup J_k)$  as below.

Note that since  $f^q|_{I_k \cup J_k} \prec_{\text{hd}} g^q|_{I_k \cup J_k}$ , we have

$$g|_{I_k} \geq f|_{I_k} \geq f|_{J_k} \geq g|_{J_k}.$$

The construction of  $S_k$  will depend on whether we have  $\|f\chi_{I_k}\|_q \leq \|f\chi_{J_k}\|_q$  or  $\|f\chi_{I_k}\|_q > \|f\chi_{J_k}\|_q$ .

If  $\|f\chi_{I_k}\|_q \leq \|f\chi_{J_k}\|_q$ , then  $m(I_k) \leq m(J_k)$  and

$$g^q|_{I_k} \cdot m(I_k) \leq 2f^q|_{J_k} \cdot m(J_k).$$

Let  $l_k : I_k \rightarrow J_k$  be a linear bijection. We set

$$S_k x = \frac{g|_{I_k}}{f|_{J_k}} (x \circ l_k) \chi_{I_k} + \frac{g}{f} x \chi_{J_k}.$$

Let  $x \in L_q$ . We have

$$\|S_k x\|_0 \leq \|x\chi_{J_k}\|_0 + \|(x \circ l_k) \chi_{I_k}\|_0 \leq \left(1 + \frac{m(I_k)}{m(J_k)}\right) \|x\|_0.$$

Thus,  $\|S_k\|_{L_0 \rightarrow L_0} \leq 2$ . We have

$$\begin{aligned} \|S_k x\|_q^q &= \frac{g^q|_{I_k}}{f^q|_{J_k}} \|(x \circ l_k) \chi_{I_k}\|_q^q + \frac{g^q|_{J_k}}{f^q|_{J_k}} \|x\chi_{J_k}\|_q^q \\ &\leq \frac{g^q|_{I_k}}{f^q|_{J_k}} \cdot \frac{m(I_k)}{m(J_k)} \cdot \|x\|_q^q + \frac{g^q|_{J_k}}{f^q|_{J_k}} \|x\|_q^q \leq 3\|x\|_q^q. \end{aligned}$$

If  $\|f\chi_{I_k}\|_q > \|f\chi_{J_k}\|_q$ , then

$$g^q|_{I_k} \cdot m(I_k) \leq 2f^q|_{I_k} \cdot m(I_k)$$

and therefore,  $g_0 \leq 2^{1/q} f_0$ . We now set  $S_k = M_{g f^{-1}}$ . Obviously,  $\|S_k\|_{L_0 \rightarrow L_0} \leq 1$  and  $\|S_k\|_{L_q \rightarrow L_q} \leq 2^{1/q}$ .

We set

$$S = \bigoplus_{k \geq 0} S_k.$$

Since  $S : \mathcal{X}(0, \infty) \rightarrow \mathcal{X}(0, \infty)$  is defined as a direct sum:

$$\|S\|_{L_0 \rightarrow L_0} = \sup_{k \geq 1} \|S_k\|_{L_0 \rightarrow L_0} \leq 2, \quad \|S\|_{L_q \rightarrow L_q} = \sup_{k \geq 1} \|S_k\|_{L_q \rightarrow L_q} \leq 3^{1/q},$$

it remains only to note that  $Sf = g$ .  $\square$

**Lemma 4.6.** *Let  $f, g \in (L_p + L_\infty)(0, \infty)$  be such that  $|g|^p \prec_{\text{hd}} |f|^p$ ,  $f = \mu(f)$  and  $g = \mu(g)$ . There exists a linear operator  $T : \mathcal{X} \rightarrow \mathcal{X}$  such that  $g = T(f)$  and*

$$\|T\|_{L_p \rightarrow L_p} \leq 2 \cdot 3^{1/p}, \quad \|T\|_{L_\infty \rightarrow L_\infty} \leq 2 \cdot 2^{1/p}.$$

*Proof.* Let  $(\Delta_k)_{k \geq 0}$  be as in Lemma 4.1 and let  $\Delta_\infty$  be the complement of  $\bigcup_{k \geq 0} \Delta_k$ . By Lemma 4.4, there exists  $T_k : \mathcal{X}(\Delta_k) \rightarrow \mathcal{X}(\Delta_k)$  such that  $T_k(f) = g$  on  $\Delta_k$  and

$$\|T_k\|_{L_p(\mathcal{X}_k) \rightarrow L_p(\mathcal{X}_k)} \leq 2 \cdot 9^{1/p}, \quad \|T_k\|_{L_\infty(\mathcal{X}_k) \rightarrow L_\infty(\mathcal{X}_k)} \leq 2 \cdot 4^{1/p}.$$

Set  $T_\infty = M_{g/f}$  on  $\mathcal{X}(\Delta_\infty)$ . We now set

$$T = T_\infty \bigoplus \left( \bigoplus_{k \geq 0} T_k \right).$$

Obviously,  $Tf = g$  on  $(0, \infty)$  and

$$\|T\|_{L_p \rightarrow L_p} \leq 2 \cdot 9^{1/p}, \quad \|T\|_{L_\infty \rightarrow L_\infty} \leq 2 \cdot 4^{1/p}. \quad \square$$

**Lemma 4.7.** *Let  $f, g \in (L_0 + L_q)(0, \infty)$  be such that  $|g|^q \prec_{\text{ul}} |f|^q$ ,  $f = \mu(f)$  and  $g = \mu(g)$ . There exists a linear operator  $T : \mathcal{X} \rightarrow \mathcal{X}$  such that  $g = T(f)$  and*

$$\|T\|_{L_0 \rightarrow L_0} \leq 4, \quad \|T\|_{L_q \rightarrow L_q} \leq 2 \cdot 3^{1/q}.$$

*Proof.* Without loss of generality,  $g = \mu(g)$  and  $f = \mu(f)$ . Let  $(\Delta_k)_{k \geq 0}$  be as in Lemma 4.3 and let  $\Delta_\infty$  be the complement of  $\bigcup_{k \geq 0} \Delta_k$ . By Lemma 4.5, there exists  $T_k : \mathcal{X}(\Delta_k) \rightarrow \mathcal{X}(\Delta_k)$  such that  $T_k(f) = g$  on  $\Delta_k$  and

$$\|T_k\|_{L_0(\mathcal{X}_k) \rightarrow L_0(\mathcal{X}_k)} \leq 8, \quad \|T_k\|_{L_q(\mathcal{X}_k) \rightarrow L_q(\mathcal{X}_k)} \leq 2 \cdot 9^{1/q}.$$

Set  $T_\infty = M_{g/f}$  on  $\mathcal{X}(\Delta_\infty)$ . We now set

$$T = T_\infty \bigoplus \left( \bigoplus_{k \geq 0} T_k \right).$$

Obviously,  $Tf = g$  on  $(0, \infty)$  and

$$\|T\|_{L_0 \rightarrow L_0} \leq 4, \quad \|T\|_{L_q \rightarrow L_q} \leq 2 \cdot 3^{1/q}. \quad \square$$

**5. Interpolation spaces for the couple  $(L_p, L_q)$**

In this section, we obtain characterizations of interpolation spaces for the couple  $(L_p, L_q)$ , in terms of the majorization notions studied earlier. The necessity of the condition we consider is a direct consequence of the constructions explained in the previous section.

**Theorem 5.1.** *Let  $0 \leq p < q \leq \infty$ . Let  $E$  be a quasi-Banach function space such that  $E \in \text{Int}(L_p, L_q)$ . There exist  $c_{p,E}$  and  $c_{q,E}$  in  $\mathbb{R}_{>0}$  such that:*

- (i) *Suppose  $p \neq 0$ . For any  $f \in E$  and  $g \in L_p + L_\infty$  if  $|g|^p \prec\prec_{\text{hd}} |f|^p$ , then  $g \in E$  and  $\|g\|_E \leq c_{p,E} \|f\|_E$ .*
- (ii) *Suppose  $q \neq \infty$ . For any  $f \in E$  and  $g \in L_0 + L_q$  if  $|g|^q \prec\prec_{\text{dl}} |f|^q$ , then  $g \in E$  and  $\|g\|_E \leq c_{q,E} \|f\|_E$ .*

*Proof.* By Lemma 2.9 (for  $p > 0$ ) or Lemma 3.3 (for  $p = 0$ ), we may assume without loss of generality that  $E$  is a symmetric function space.

Assume that  $p \neq 0$ . Let  $f \in E$  and let  $g \in L_p + L_\infty$  be such that  $|g|^p \prec\prec_{\text{hd}} |f|^p$ . Since  $E$  is symmetric, we may assume without loss of generality that  $f = \mu(f)$  and  $g = \mu(g)$ . By Lemma 4.6, there exists an operator  $T$  such that  $T(f) = g$  and

$$\|T\|_{(L_p, L_\infty) \rightarrow (L_p, L_\infty)} \leq 2 \cdot 3^{1/p}.$$

Recall that  $L_q$  is an interpolation space for the couple  $(L_p, L_\infty)$  (one can take, for example, real or complex interpolation method). Let  $c_{p,q}$  be the interpolation constant of  $L_q$  for the couple  $(L_p, L_\infty)$ . We have

$$\|T\|_{L_q \rightarrow L_q} \leq c_{p,q} \cdot 2 \cdot 3^{1/p}.$$

Let  $c_E$  be the interpolation constant of  $E$  for  $(L_p, L_q)$  (see Proposition 2.5). Then,

$$\|T\|_{E \rightarrow E} \leq c_E \cdot \max\{1, c_{p,q}\} \cdot 2 \cdot 3^{1/p}.$$

Thus,

$$\|g\|_E \leq \|T\|_{E \rightarrow E} \|f\|_E \leq c_E \cdot \max\{1, c_{p,q}\} \cdot 2 \cdot 3^{1/p} \|f\|_E.$$

This proves the first assertion. The proof of the second one follows mutatis mutandi using Corollary 3.5 instead of complex interpolation and (for  $p = 0$ ) Theorem 3.1 instead of Proposition 2.5. □

**Lemma 5.2.** *Assume that  $0 < p < q < \infty$ . Let  $f, g \in L_p + L_q$  such that  $f = \mu(f)$  and  $g = \mu(g)$ . Suppose that at every  $t > 0$ , one of the following inequalities holds:*

$$\int_0^t g^p ds \leq \int_0^t f^p ds \quad \text{or} \quad \int_t^\infty g^q ds \leq \int_t^\infty f^q ds.$$

*Then, there exist  $g_1, g_2 \in (L_p + L_q)^+$  which satisfies  $g = g_1 + g_2$ ,  $g_1^p \prec\prec_{\text{hd}} f^p$  and  $g_2^q \prec\prec_{\text{dl}} f^q$ .*

*Proof.* Set

$$A = \left\{ t > 0 : \int_0^t g(s)^p ds \leq \int_0^t f(s)^p ds \right\},$$

$$B = \left\{ t > 0 : \int_t^\infty g(s)^q ds \leq \int_t^\infty f(s)^q ds \right\}.$$

Let

$$u_+(t) = \inf\{s \in A : s \geq t\}, \quad u_-(t) = \sup\{s \in A : s \leq t\},$$

$$v_+(t) = \inf\{s \in B : s \geq t\}, \quad v_-(t) = \sup\{s \in B : s \leq t\}.$$

Note that  $f^p \chi_{(u_-(t), u_+(t))} \prec_{\text{hd}} g^p \chi_{(u_-(t), u_+(t))}$  for  $t \notin A$  and, therefore,

$$g(u_+(t) - 0) \leq f(u_+(t) - 0).$$

Set  $h_1(t) = g(u_+(t) - 0)$ ,  $t > 0$ . By definition,  $u_+(t) \geq t$  for all  $t > 0$ . Since  $g$  is decreasing, it follows that  $h_1 \leq g$ . Set  $h_2 = g \chi_B$ . Since  $u_+(t) = t$  for  $t \in A$ , it follows that  $h_1 = g$  on  $A$ . Thus,  $h_1 + h_2 \geq g \chi_A + g \chi_B \geq g$ .

We claim that

$$\int_0^t \mu(s, h_1)^p ds \leq \int_0^t f(s)^p ds.$$

Indeed, for  $t \in A$ , we have

$$\int_0^t h_1(s)^p ds \leq \int_0^t g(s)^p ds \leq \int_0^t f(s)^p ds.$$

For  $t \notin A$ , we have  $h_1(s) = g(u_+(t))$  for all  $s \in (u_-(t), u_+(t))$ . Thus,

$$\begin{aligned} \int_0^t h_1(s)^p ds &= \int_0^{u_-(t)} h_1(s)^p ds + \int_{u_-(t)}^t h_1(s)^p ds \\ &\leq \int_0^{u_-(t)} g(s)^p ds + \int_{u_-(t)}^t g(u_+(t))^p ds. \end{aligned}$$

Since

$$\int_0^{u_-(t)} g(s)^p ds = \int_0^{u_-(t)} f(s)^p ds, \quad g(u_+(t)) \leq f(u_+(t)),$$

it follows that

$$\int_0^t h_1(s)^p ds \leq \int_0^{u_-(t)} f(s)^p ds + \int_{u_-(t)}^t f(u_+(t))^p ds \leq \int_0^t f(s)^p ds.$$

Since  $h_1 = \mu(h_1)$ , the claim follows.

We claim that

$$\int_t^\infty \mu(s, h_2)^q ds \leq \int_t^\infty f(s)^q ds.$$

For  $t \in B$ , we have

$$\int_t^\infty h_2(s)^q ds \leq \int_t^\infty g(s)^q ds \leq \int_t^\infty f(s)^q ds.$$

For  $t \notin B$ , we have

$$\int_t^\infty h_2(s)^q ds = \int_{v_+(t)}^\infty h_2(s)^q ds \leq \int_{v_+(t)}^\infty g(s)^q ds = \int_{v_+(t)}^\infty f(s)^q ds \leq \int_t^\infty f(s)^q ds.$$

In either case, we have

$$\int_t^\infty \mu(s, h_2)^q ds \leq \int_t^\infty h_2(s)^q ds \leq \int_t^\infty f(s)^q ds.$$

This proves the claim.

Setting

$$g_1 = \frac{h_1}{h_1 + h_2} g, \quad g_2 = \frac{h_2}{h_1 + h_2} g,$$

we complete the proof. □

**Theorem 5.3.** *Let  $0 \leq p < q \leq \infty$  with either  $p \neq 0$  or  $q \neq \infty$ . Let  $E$  be a quasi-Banach function space. Assume that there exist  $c_{p,E}$  and  $c_{q,E}$  in  $\mathbb{R}_{>0}$  such that:*

- (i) *Suppose  $p \neq 0$ . For any  $f \in E$  and  $g \in L_0 + L_p$ , if  $|g|^p \ll_{\text{hd}} |f|^p$ , then  $g \in E$  and  $\|g\|_E \leq c_{p,E} \|f\|_E$ .*
- (ii) *Suppose  $q \neq \infty$ . For any  $f \in E$  and  $g \in L_0 + L_q$ , if  $|g|^q \ll_{\text{cl}} |f|^q$ , then  $g \in E$  and  $\|g\|_E \leq c_{q,E} \|f\|_E$ .*

*Then  $E$  belongs to  $\text{Int}(L_p, L_q)$ .*

*Proof.* Assume that  $p \neq 0$ . Let us show that the first condition implies  $E \subset L_p + L_\infty$ . Indeed, assume the contrary and choose  $f \in E$  such that  $\mu(f) \chi_{(0,1)} \notin L_p$ . Let

$$f_n = \min \left\{ \mu \left( \frac{1}{n}, f \right), \mu(f) \chi_{(0,1)} \right\}, \quad n \geq 1.$$

Obviously,  $\|f_n\|_E \leq \|\mu(f) \chi_{(0,1)}\|_E \leq \|f\|_E$ . On contrary,  $\|f_n\|_p^p \chi_{(0,1)} \ll_{\text{hd}} f_n^p$ . By the first condition on  $E$ , we have  $\|f_n\|_p \|\chi_{(0,1)}\|_E \leq c_{p,E} \|f\|_E$ . However, we have  $\|f_n\|_p \uparrow \|\mu(f) \chi_{(0,1)}\|_p = \infty$ . This contradiction shows that our initial assumption was incorrect. Thus,  $E \subset L_p + L_\infty$ .

A similar argument shows that the second condition implies  $E \subset L_0 + L_q$ . Thus, a combination of both conditions implies  $E \subset L_p + L_q$ .

Let  $T$  be a contraction on  $(L_p, L_q)$  and  $f \in E$ . To conclude the proof, it suffices to show that  $Tf$  belongs to  $E$ . First, note that

$$K(t, Tf, L_p, L_q) \leq K(t, f, L_p, L_q).$$

Assume that  $p > 0$  and  $q < \infty$ . Let  $\alpha^{-1} = \frac{1}{p} - \frac{1}{q}$ . By the Holmstedt formula for the  $K$ -functional (see [14]), there exists a constant  $c_{p,q} > 0$  such that for any  $t \in \mathbb{R}_{>0}$ ,

$$\begin{aligned} \left(\int_0^{t^\alpha} \mu(s, Tf)^p ds\right)^{1/p} + t \left(\int_{t^\alpha}^\infty \mu(s, Tf)^q ds\right)^{1/q} \\ \leq c_{p,q} \left( \left(\int_0^{t^\alpha} \mu(s, f)^p ds\right)^{1/p} + t \left(\int_{t^\alpha}^\infty \mu(s, f)^q ds\right)^{1/q} \right). \end{aligned}$$

Hence, for any given  $t > 0$ , we have either

$$\int_0^{t^\alpha} \mu(s, Tf)^p ds \leq \int_0^{t^\alpha} \mu(s, c_{p,q} f)^p ds$$

or

$$\int_{t^\alpha}^\infty \mu(s, Tf)^q ds \leq \int_{t^\alpha}^\infty \mu(s, c_{p,q} f)^q ds.$$

By Lemma 5.2, one can write

$$(5-1) \quad \mu(Tf) = g_1 + g_2, \quad g_1^p \prec\prec_{\text{hd}} (c_{p,q} \mu(f))^p, \quad g_2 \prec\prec_{\text{tl}} (c_{p,q} \mu(f))^q.$$

By assumption, we have

$$\|g_1\|_E \leq c_{p,E} \|f\|_E, \quad \|g_2\|_E \leq c_{q,E} \|f\|_E.$$

By triangle inequality, we have

$$\|Tf\|_E \leq c_{p,q,E} \|f\|_E.$$

Assume now that  $p > 0$  and  $q = \infty$ . This case is simpler since by the Holmstedt formula (see [14]), there exists  $c_p \in \mathbb{R}_{>0}$  such that for any  $t \in \mathbb{R}_{>0}$ ,

$$\left(\int_0^{t^p} \mu(s, Tf)^p ds\right)^{1/p} \leq c_p \left(\int_0^{t^p} \mu(s, f)^p ds\right)^{1/p}.$$

This means that  $|Tf|^p \prec\prec_{\text{hd}} |c_p f|^p$  so by assumption (1),  $Tf$  belongs to  $E$  and

$$\|Tf\|_E \leq c_p c_{p,E} \|f\|_E.$$

The case of  $p = 0$  and  $q < \infty$  is given by Corollary 3.4. □

Theorem 1.1 claimed in the introduction compiles some results of this section.

*Proof of Theorem 1.1.* Assertion (a) is obtained by combining Theorems 5.1 and 5.3 with  $q = \infty$ .

Assertion (b) is derived similarly from Theorems 5.1 and 5.3 by applying them with  $p = 0$ .

Finally, using assertions (a) and (b) of Theorem 1.1, Theorems 5.1 and 5.3, for  $0 < p < q < \infty$ , one obtains Theorem 1.1(c). □

**Remark 5.4.** In the spirit of Corollary 3.4, we could have used a nonquantitative condition to deal with the case of  $q = \infty$  in Theorem 5.3. Let  $E$  be a quasi-Banach function space and  $p, q \in (0, \infty)$ . This means that the following two conditions are equivalent:

(i) For any  $f \in E, g \in L_p + L_\infty$ ,

$$|g|^p \prec\prec_{\text{hd}} |f|^p \Rightarrow g \in E.$$

(ii) There exists  $c > 0$  such that for any  $f \in E, g \in L_p + L_\infty$ ,

$$|g|^p \prec\prec_{\text{hd}} |f|^p \Rightarrow g \in E \quad \text{and} \quad \|g\|_E \leq c \|f\|_E.$$

Similarly, the following two conditions are equivalent:

(i) For any  $f \in E, g \in L_0 + L_q$ ,

$$|g|^q \prec\prec_{\text{tl}} |f|^q \Rightarrow g \in E.$$

(ii) There exists  $c > 0$  such that for any  $f \in E, g \in L_0 + L_q$ ,

$$|g|^q \prec\prec_{\text{tl}} |f|^q \Rightarrow g \in E \quad \text{and} \quad \|g\|_E \leq c \|f\|_E.$$

### 6. Interpolation spaces for couples of $\ell^p$ -spaces

In this section, we show that our approach to the Lorentz–Shimogaki and Arazy–Cwikel theorems also applies to sequence spaces. We follow a structure similar to the previous sections, proving partition lemmas, then constructing bounded operators on couples  $(\ell^p, \ell^q)$  with suitable properties to finally conclude on the interpolation spaces of the couple  $(\ell^p, \ell^q)$ . Additional arguments involving Boyd indices will be required to prove Theorem 1.2.

We identify sequences with bounded functions on  $(0, \infty)$  which are almost constant on intervals of the form  $(k, k + 1), k \in \mathbb{Z}^+$  by

$$i : \ell^\infty \rightarrow L_\infty, \quad (u_k)_{k \in \mathbb{Z}^+} \mapsto \sum_{k=0}^\infty u_k \mathbf{1}_{(k, k+1)}.$$

**An interpolation theorem for the couple  $(\ell^p, \ell^q)$ .** We start with a partition lemma playing, for sequence spaces, the role of Lemma 4.3.

**Lemma 6.1.** *Let  $a = (a_n)_{n \in \mathbb{Z}^+}, b = (b_n)_{n \in \mathbb{Z}^+}$  be two positive decreasing sequences such that  $b \prec\prec_{\text{tl}} a$ . There exists a sequence  $(\Delta_n)_{n \in \mathbb{Z}^+}$  of subsets of  $\mathbb{Z}^+$  such that:*

(i) For every  $k \in \mathbb{Z}_+,$  we have  $|\{n \in \mathbb{Z}^+ : k \in \Delta_n\}| \leq 3.$

(ii)  $\sum_{k \in \Delta_n} a_k \geq b_n$  for any  $n \in \mathbb{Z}^+.$

*Proof.* Define  $I = \{n \in \mathbb{Z}^+ : b_n > a_n\}$ . For  $n \notin I$ , set  $\Delta_n = \{n\}$ .

For any  $n \in \mathbb{Z}^+$ , define

$$i_n = \sup \left\{ i : \sum_{k=i}^{\infty} a_k \geq \sum_{k=n}^{\infty} b_k \right\}$$

and for  $n \in I$ , let  $\Delta_n = \{i_n, \dots, i_{n+1}\}$ .

From the definition of  $i_n$ , we have

$$\sum_{k \geq i_n} a_k \geq \sum_{k \geq n} b_k.$$

From the definition of  $i_{n+1}$ , we have

$$\sum_{k > i_{n+1}} a_k < \sum_{k \geq n+1} b_k.$$

Taking the difference of these inequalities, we infer that

$$\sum_{k \in \Delta_n} a_k \geq b_n.$$

This proves the second condition.

Note that since  $b \ll_{\text{ul}} a$ , we write  $i_n \geq n$  for any  $n \in \mathbb{Z}^+$ . Hence, if  $n \in I$ , then  $b_n > a_n \geq a_{i_{n+1}}$ . Furthermore, by definition of  $i_{n+1}$ , we have

$$\sum_{k > i_{n+1}} a_k < \sum_{k=n+1}^{\infty} b_k, \quad \text{so} \quad \sum_{k=i_{n+1}}^{\infty} a_k < \sum_{k=n}^{\infty} b_k.$$

Hence, by definition of  $i_n$ , we have  $i_{n+1} > i_n$  for  $n \in I$ .

Let us now check the first condition. Suppose there exist distinct numbers  $n_1, n_2, n_3 \in I$  such that  $k \in \Delta_{n_1}, \Delta_{n_2}, \Delta_{n_3}$ . Without loss of generality,  $n_1 < n_2 < n_3$ . Since  $k \in \Delta_{n_1}$ , it follows that  $k \leq i_{n_1+1} \leq i_{n_2}$ . Since  $k \in \Delta_{n_3}$ , it follows that  $k \geq i_{n_3} \geq i_{n_2+1}$ . Hence,  $i_{n_2+1} \leq k \leq i_{n_2}$  and, therefore,  $i_{n_2+1} = i_{n_2}$ . Since  $n_2 \in I$ , it follows  $i_{n_2+1} > i_{n_2}$ . This contradiction shows that  $|\{n \in I : k \in \Delta_n\}| \leq 2$ . By definition,  $k$  also belongs to at most one set  $\Delta_n$ ,  $n \notin I$ . Consequently,

$$|\{n \in \mathbb{Z}^+ : k \in \Delta_n\}| \leq 3. \quad \square$$

From the partition lemma, we deduce an operator lemma similar to Lemma 4.6. It extends Proposition 2 in [3], which is established there for the special case  $p = 1$  by a completely different method.

**Lemma 6.2.** *Let  $p \geq 1$ . Let  $a, b \in \ell^p$  such that  $|b|^p \ll_{\text{ul}} |a|^p$ . Then there exists an operator  $T : \ell^p \rightarrow \ell^p$  such that:*

- (i)  $T(a) = b$ .
- (ii)  $\|T\|_{p \rightarrow p} \leq 3^{1/p}$  and  $\|T\|_{0 \rightarrow 0} \leq 3$ .

*Proof.* We can assume that both sequences are nonnegative and decreasing. Apply Lemma 6.1 to  $|a|^p$  and  $|b|^p$ . For every  $n \in \mathbb{Z}^+$ , choose a linear form  $\varphi_n$  on  $\ell^p$  of norm less than 1, supported on  $\Delta_n$  and such that  $\varphi_n(a) = \varphi_n(a\mathbf{1}_{\Delta_n}) = b_n$ . Define

$$T : x \in \ell^p \mapsto (\varphi_n(x))_{n \in \mathbb{Z}^+}.$$

By construction,  $T(a) = b$ . Let us check the norm estimates. Let  $x \in \ell^p$ , then

$$\begin{aligned} \|T(x)\|_p^p &= \sum_{n \in \mathbb{Z}^+} |\varphi_n(x)|^p = \sum_{n \in \mathbb{Z}^+} |\varphi_n(x\mathbf{1}_{\Delta_n})|^p \\ &\leq \sum_{n \in \mathbb{Z}^+} \sum_{k \in \Delta_n} |x_k|^p = \sum_{k \in \mathbb{Z}^+} |\{n : k \in \Delta_n\}| |x_k|^p \leq 3 \|x\|_p^p. \end{aligned}$$

The second estimate is clear, using once again the fact that an integer  $k$  belongs to at most three  $\Delta_n$ 's. □

The following remarks were communicated to the authors by Cwikel and Nilsson.

**Remark 6.3.** A bounded linear operator on  $\ell^p$ ,  $p \leq 1$  extends automatically to a bounded linear operator on  $\ell^1$ .

*Proof.* Indeed, let  $(e_n)_{n \in \mathbb{Z}^+}$  be the canonical basis of  $\ell^\infty$ . Let  $T$  be a contraction on  $\ell^p$ ,  $p < 1$ . Then by Hölder's inequality  $\|T(e_n)\|_1 \leq \|T(e_n)\|_p \leq \|T\|_{p \rightarrow p}$ . By the triangle inequality, for any finite sequence  $a = (a_n)_{n \in \mathbb{Z}^+}$ ,

$$\|T(a_n)\|_1 \leq \sum_{n \in \mathbb{Z}^+} |a_n| \|T(e_n)\|_1 \leq \|a\|_1 \|T\|_{p \rightarrow p}.$$

Hence,  $T$  extends to a contraction on  $\ell^1$ . □

**Remark 6.4.** The condition  $p \geq 1$  in Lemma 6.2 is necessary.

*Proof.* Let us show that Lemma 6.2 cannot be true for  $p < 1$ . Assume by contradiction that there exists  $c > 0$  such that for any finite sequences  $a$  and  $b$  in  $\ell^p$  such that  $|b|^p \ll_{\text{hd}} |a|^p$  there exists  $T$  with  $\|T\|_{p \rightarrow p} \leq c$  and  $T(a) = b$ . By Remark 6.3 above, we also have  $\|T\|_{1 \rightarrow 1} \leq c$ . In particular,  $\|b\|_1 \leq c \|a\|_1$ . By considering  $b = e_1$  and  $a = \frac{1}{N^{1/p}} \sum_{i=1}^N e_i$  for  $N$  large enough, one obtains a contradiction. □

We will not prove a sequence version of Lemma 4.5 to avoid the repetition of too many similar arguments. Fortunately, the expected result already appears in the literature, see [10, Theorem 3].

**Lemma 6.5.** Let  $p > 0$ . Let  $a, b \in \ell^\infty$  such that  $|b|^p \ll_{\text{hd}} |a|^p$ . Then there exists an operator  $T : \ell^p \rightarrow \ell^p$  such that:

- (i)  $T(a) = b$ .
- (ii)  $\|T\|_{p \rightarrow p} \leq 8^{1/p}$  and  $\|T\|_{\infty \rightarrow \infty} \leq 2^{1/p}$ .

We conclude this subsection with a new interpolation theorem.

**Theorem 6.6.** *Let  $p < q \in (0, \infty]$  such that  $q \geq 1$ . Let  $E$  be a quasi-Banach sequence space. Then  $E$  belongs to  $\text{Int}(\ell^p, \ell^q)$  if and only if there exists  $c_{p,E}$  and  $c_{q,E}$  in  $\mathbb{R}_{>0}$  such that:*

- (i) *For any  $u \in E$  and  $v \in \ell^\infty$ , if  $|v|^p \prec\prec_{\text{hd}} |u|^p$ , then  $v \in E$  and  $\|v\|_E \leq c_{p,E} \|u\|_E$ .*
- (ii) *Suppose  $q < \infty$ . For any  $u \in E$  and  $v \in \ell^\infty$ , if  $|v|^q \prec\prec_{\text{tl}} |u|^q$ , then  $v \in E$  and  $\|v\|_E \leq c_{q,E} \|u\|_E$ .*

*Proof.* The proof of the “only if” implication is identical to the proof of Theorem 5.1 using Lemmas 6.2 and 6.5 instead of Lemmas 4.5 and 4.4. The “if” implication is given by [8, Theorem 4.7].  $\square$

**Upper Boyd index.** Let us now recall the definition of the upper Boyd index, in the case of sequence spaces. For any  $n \in \mathbb{N}$  define the dilation operator

$$D_n: \ell^\infty \rightarrow \ell^\infty, \quad (u_k)_{k \in \mathbb{Z}^+} \mapsto (u_{\lfloor k/n \rfloor})_{k \in \mathbb{Z}^+}.$$

Let  $E$  be a symmetric function space. Define the Boyd index associated to  $E$  by

$$\beta_E = \lim_{k \rightarrow \infty} \frac{\log \|D_k\|_{E \rightarrow E}}{\log k}.$$

Note that since  $E$  is a quasi-Banach space,  $\beta_E < \infty$ .

In the next proposition, we relate the upper Boyd index to an interpolation property. We follow [23, Theorem 2].

**Proposition 6.7.** *Assume that  $E$  is a quasi-Banach symmetric sequence space. Let  $p < 1/\beta_E$ . There exists a constant  $C$  such that for any  $u \in E$  and  $v \in \ell^\infty$ , satisfying  $|v|^p \prec\prec_{\text{hd}} |u|^p$ , we have  $v \in E$  and  $\|v\|_E \leq C \|u\|_E$ .*

Define the map  $V: \ell_\infty \rightarrow \ell_\infty$  by setting

$$Vu = \sum_{n=0}^{\infty} 2^{-n} D_{2^n} u$$

and the map  $C: \ell_\infty \rightarrow \ell_\infty$  by

$$(Cu)(n) = \frac{1}{n+1} \sum_{i=0}^n u_i.$$

**Lemma 6.8.** *If  $p < 1/\beta_E$ , then*

$$\|(V(u^p))^{1/p}\|_E \leq c_{p,E} \|u\|_E, \quad 0 \leq u \in E.$$

*Proof.* Let  $E_p$  be the  $p$ -concavification of  $E$ , that is,

$$E_p = \{f: |f|^{1/p} \in E\}, \quad \|f\|_{E_p} = \||f|^{1/p}\|_E^p.$$

Obviously,  $E_p$  is a quasi-Banach space. Apply the Aoki–Rolewicz theorem to the space  $E_p$  and fix  $q = q_{p,E} > 0$  such that

$$\left\| \sum_{n \geq 0} x_n \right\|_{E_p}^q \leq C_{p,E} \sum_{n \geq 0} \|x_n\|_{E_p}^q.$$

For every  $u \in E$ , we have

$$\begin{aligned} \|(V(u^p))^{1/p}\|_E^{qp} &= \|V(u^p)\|_{E_p}^q = \left\| \sum_{n=0}^{\infty} \frac{1}{2^n} (D_{2^n} u)^p \right\|_{E_p}^q \\ &\leq C_{p,E} \sum_{n=0}^{\infty} \left\| \frac{1}{2^n} (D_{2^n} u)^p \right\|_{E_p}^q \\ &= C_{p,E} \sum_{n=0}^{\infty} 2^{-nq} \|D_{2^n} u\|_E^{qp}. \end{aligned}$$

Let  $r \in (p, \beta_E^{-1})$ . By the definition of  $\beta_E$ , there exists  $c_{p,E} > 0$  such that  $\|D_n\|_{E \rightarrow E} \leq c_{p,E} n^{1/r}$  for any  $n \in \mathbb{N}$ . Therefore,

$$\begin{aligned} \|(V(u^p))^{1/p}\|_E^{qp} &\leq C_{p,E} \cdot c_{p,E}^q \cdot \sum_{n=0}^{\infty} 2^{-nq} 2^{nqp/r} \|u\|_E^{qp} \\ &= C_{p,E} \cdot c_{p,E}^q \cdot \frac{2^q}{2^q - 2^{q/p/r}} \cdot \|u\|_E^{qp}. \end{aligned} \quad \square$$

**Lemma 6.9.** *If  $x = \mu(x)$ , then  $Cx \leq 3Vx$  for every  $x \in \ell_\infty$ .*

*Proof.* Let  $k \geq 0$ . Since  $x$  is decreasing, it follows that

$$(Cx)(2^k - 1) = \frac{1}{2^k} \left( x(0) + \sum_{i=0}^{k-1} \sum_{j=2^i}^{2^{i+1}-1} x(j) \right) \leq \frac{1}{2^k} \left( x(0) + \sum_{i=0}^{k-1} 2^i x(2^i) \right).$$

On the other hand, we have

$$\begin{aligned} (Vx)(2^{k+1} - 1) &= \sum_{n \geq 0} 2^{-n} x \left( \left\lfloor \frac{2^{k+1} - 1}{2^n} \right\rfloor \right) \\ &= \sum_{n=0}^k 2^{-n} x(2^{k+1-n} - 1) + \sum_{n=k+1}^{\infty} 2^{-n} x(0) \\ &= \frac{1}{2^k} \left( x(0) + \sum_{i=0}^k 2^i x(2^{i+1} - 1) \right). \end{aligned}$$

Again using the fact that  $x$  is decreasing, we obtain

$$\begin{aligned} \sum_{i=0}^{k-1} 2^i x(2^i) &= x(1) + \sum_{i=0}^{k-2} 2^{i+1} x(2^{i+1}) \\ &\leq x(1) + 2 \sum_{i=0}^{k-2} 2^i x(2^{i+1} - 1) \leq 3 \sum_{i=0}^k 2^i x(2^{i+1} - 1). \end{aligned}$$

Combining the three previous inequalities, we have just shown that for any  $k \geq 0$ ,

$$(Cx)(2^k - 1) \leq 3(Vx)(2^{k+1} - 1).$$

Now let  $n \geq 0$  and choose  $k$  such that  $n \in [2^k - 1, 2^{k+1} - 1]$ . Since  $Cx$  and  $Vx$  are decreasing, we have

$$(Cx)(n) \leq (Cx)(2^k - 1) \leq 3(Vx)(2^{k+1} - 1) \leq 3(Vx)(n). \quad \square$$

*Proof of Proposition 6.7.* Without loss of generality,  $u = \mu(u)$  and  $v = \mu(v)$ . Since  $v^p \ll_{\text{hd}} u^p$ , it follows that

$$|v|^p \leq C(|v|^p) \leq C(|u|^p) \leq 3V(|u|^p),$$

where we used Lemma 6.9 to obtain the last inequality. By Lemma 6.8, we have

$$\|v\|_E \leq 3^{1/p} \|(V(|u|^p))^{1/p}\|_E \leq c_{p,E} \|u\|_p. \quad \square$$

We are now ready to deliver a complete resolution of the conjecture stated by Levitina et al. in [20].

*Proof of Theorem 1.2.* Let  $E$  be a quasi-Banach sequence space. Let  $q \geq 1$ . Recall that Theorem 1.2 states that the following two conditions are equivalent:

- (a) There exists  $p < q$  such that  $E$  is an interpolation space for the couple  $(\ell^p, \ell^q)$ .
- (b) There exists  $c > 0$  such that for any  $u \in E$  and  $|v|^q \ll_{\text{cl}} |u|^q$ , then  $v \in E$  and  $\|v\|_E \leq c\|u\|_E$ .

Note that by Lemma 2.9, we may assume that  $E$  is a symmetric space.

(a)  $\Rightarrow$  (b). This is immediate by Theorem 6.6.

(b)  $\Rightarrow$  (a). Let  $p < 1/\beta_E$ . By Proposition 6.7, for any sequence  $u \in E$  and  $v \in \ell^\infty$ , if  $|v|^p \ll_{\text{hd}} |u|^p$ ,  $v \in E$  and  $\|v\|_E \leq c_{p,E} \|u\|_E$ . Applying Theorem 6.6 for indices  $p$  and  $q$ , we obtain that  $E$  belongs to  $\text{Int}(\ell^p, \ell^q)$ .  $\square$

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
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