Pacific Journal of Mathematics

LORENTZ–SHIMOGAKI AND ARAZY–CWIKEL THEOREMS REVISITED

LÉONARD CADILHAC, FEDOR SUKOCHEV AND DMITRIY ZANIN

Volume 328 No. 2 February 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 328, No. 2, 2024
https://doi.org/10.2140/pjm.2024.328.227

LORENTZ–SHIMOGAKI AND ARAZY–CWIKEL THEOREMS REVISITED

LÉONARD CADILHAC, FEDOR SUKOCHEV AND DMITRIY ZANIN

By using the space L_0 of finitely supported functions as a left endpoint on the interpolation scale of L_p-spaces, we present a new approach to the Lorentz–Shimogaki and Arazy–Cwikel theorems which covers the whole range of $p, q \in (0, \infty]$. In particular, we show that for $0 \le p < q < r < s \le \infty$,

$$\mathrm{Int}(L_q, L_r) = \mathrm{Int}(L_p, L_r) \cap \mathrm{Int}(L_q, L_s)$$

if the underlying space is $(0, \alpha)$, $\alpha \in (0, \infty]$ equipped with the Lebesgue measure. As a byproduct of our result, we solve a conjecture of Levitina, Sukochev and Zanin (2020).

1. Introduction

Descriptions of interpolation spaces for couples of L_p-spaces for $1 \le p \le \infty$ were extensively researched from the 1960s to the 80s, providing satisfying answers to most problems that were considered relevant at the time.

However, new questions arising from noncommutative analysis recently highlighted some gaps in our knowledge of this subject, especially for the case of $p < 1$ of quasi-Banach spaces. In this paper, we revisit some important results of the literature [1; 21; 27], generalizing them and thus filling some of the holes that were revealed in the theory. In particular, we answer a question asked by Levitina, Sukochev and Zanin in [20] and already partially studied in [11] regarding the interpolation theory of sequence spaces (see Theorem 1.2). Besides this new result, this paper introduces a general approach that covers the range of all $0 \le p \le \infty$ and is self-contained. It emphasizes the use of the space L_0 of all finitely supported measurable functions. As far as the authors know this space rarely appears in interpolation theory (however, see [2; 16; 24] and [15]). We provide evidence that L_0 is a suitable "left endpoint" on the interpolation scale of L_p-spaces, despite its possessing an atypical structure, that of a normed abelian group.

A function space E is an *interpolation space* for the couple (L_p, L_q) if any linear operator T bounded on L_p and L_q is also bounded on E (see Definition 2.4). This

MSC2020: 43A15, 46B70, 46M35.
Keywords: interpolation, symmetric function spaces, L_p-spaces, majorization.

notion provides a way of transferring inequalities well known in L_p-spaces to more exotic ones. To both understand the range of applicability of this technique and be able to check whether it applies to a given function space E, we are interested in simple descriptions of interpolation spaces for the couple (L_p, L_q).

This problem has a long history starting with the Calderón–Mityagin theorem (see [9; 22]) on the couple (L_1, L_∞) and followed by Lorentz and Shimogaki's [21] results on the couples (L_1, L_q) and (L_p, L_∞) with $1 \le p, q \le \infty$. A remarkable result of Arazy and Cwikel then states that a space E is an interpolation space for the couple (L_p, L_q), $1 < p < q < \infty$ if and only if it is an interpolation space for the couples (L_1, L_q) and (L_p, L_∞).

Describing interpolation spaces often comes down to understanding certain orders. In fact, at a very fundamental level, being an interpolation space can be understood as a monotonicity property. Indeed, given two compatible quasi-Banach spaces A, B, denote by $\mathbb{C}(A, B)$ the set of operators $A+B \to A+B$ that restrict to contractions on A and B. Consider the following order on $A+B$:

$$f \le_{p,q} g \Leftrightarrow \exists T \in \mathbb{C}(A, B), \quad T(g) = f.$$

With this definition in mind, E is an interpolation space for the couple (A, B) if and only if

$$\forall f \in E, \forall g \in L_p + L_q, \quad g \le_{p,q} f \Rightarrow g \in E.$$

In fact, the fundamental theorem of Calderón and Mityagin precisely describes the order $\le_{L_1,L_\infty}$ (from now on denoted by $\prec\prec_{\mathrm{hd}}$). It states that for $f, g \in L_1 + L_\infty$,

$$g \prec\prec_{\mathrm{hd}} f \Leftrightarrow g \le_{L_1,L_\infty} f \Leftrightarrow \forall t > 0, \quad \int_0^t \mu(s, g)\, ds \le \int_0^t \mu(s, f)\, ds,$$

where $\mu(g) : t \to \mu(t, g)$ denotes the right-continuous decreasing rearrangement of g. We will call this order *head majorization*. Moreover, if $f, g \in L_1$ and $\|f\|_1 = \|g\|_1$, then we write $g \prec\prec_{\mathrm{hd}} f$. Variants of this order allow to describe interpolation spaces for any couple (L_p, L_∞), $p \in (0, \infty)$ (see [21] for the Banach range and [8] for $p < 1$).

Another phenomenon, this time specific to the study of interpolation theory of L_p-spaces, is that to guarantee that a space E is an interpolation space for the couple (L_p, L_q), $p < q$ it is natural to impose two conditions: one which will impose that E is "on the right of L_p" and one that will impose that E is "on the left of L_q". An example of such a result is the above-mentioned Arazy–Cwikel theorem but one can think also of convexity/concavity conditions or Boyd indices (see [18] for an overview and [8, Theorem 1.4]).

In this spirit, the natural counterpart of head majorization is *tail majorization* defined on $L_0 + L_1$ by

$$g \prec\prec_{\text{tl}} f \Leftrightarrow \forall t > 0, \quad \int_t^\infty \mu(s, g)\, ds \le \int_t^\infty \mu(s, f)\, ds.$$

We'll show later that this order is in fact equivalent to $\le_{L_0,L_1}$. Moreover, if $f, g \in L_1$ and $\|f\|_1 = \|g\|_1$, then we write $g \prec_{\text{tl}} f$. Remark that $g \prec_{\text{tl}} f$ if and only if $f \prec_{\text{hd}} g$. Note that tail majorization coincides with the weak supermajorization of [13].

Let us now state our main theorem. Let $\mathcal{X}$ be the linear space of all measurable functions. If not specified otherwise, the *underlying measure space* we are working on is $(0, \infty)$ equipped with the Lebesgue measure m. We obtain:

Theorem 1.1. *Let $E \subset \mathcal{X}$ be a quasi-Banach function space (a priori, not necessarily symmetric). Let $p, q \in (0, \infty)$ such that $p < q$. Then:*

(a) *E is an interpolation space for the couple (L_p, L_∞) if and only if there exists $c_{p,E} > 0$ such that for any $f \in E$ and $g \in L_p + L_\infty$,*

$$|g|^p \prec\prec_{\text{hd}} |f|^p \Rightarrow g \in E \quad \textit{and} \quad \|g\|_E \le c_{p,E} \|f\|_E.$$

(b) *E is an interpolation space for the couple (L_0, L_q) if and only if there exists $c_{q,E} > 0$ such that for any $f \in E$ and $g \in L_0 + L_q$,*

$$|g|^q \prec\prec_{\text{tl}} |f|^q \Rightarrow g \in E \quad \textit{and} \quad \|g\|_E \le c_{q,E} \|f\|_E.$$

(c) *E is an interpolation space for the couples (L_0, L_q) and (L_p, L_∞) if and only if it is an interpolation space for the couple (L_p, L_q).*

This extends the results of Lorentz–Shimogaki and Arazy–Cwikel to the quasi-Banach setting and contributes to the two first questions asked by Arazy in [12, p. 232] in the particular case of L_p-spaces for $0 < p < \infty$. As mentioned before, our approach places L_0 as a left endpoint on the interpolation scale of L_p-spaces, in sharp contrast to earlier results which focused mostly on Banach spaces and had L_1 playing this part. An advantage of our approach is that it naturally encompasses every symmetric quasi-Banach space since they are all interpolation spaces for the couple (L_0, L_∞) (see [2; 16]). On the contrary, there exist some symmetric Banach spaces which are *not* interpolation spaces for the couple (L_1, L_∞) (see [26]). This led to some difficulties which were customarily circumvented with the help of various technical conditions such as the Fatou property (as appears, e.g., in [4]).

Compared to [8] where the first author investigates similar characterizations, the novelty of this theorem is statement (b) that deals with the space L_0. A deeper advantage of our new approach is that it no longer relies on Sparr's K-monotonicity result [27] for couples of L_p-spaces which was instrumental in [8].

Indeed, our strategy in this paper is different from the techniques used in [1; 2; 3; 4; 8; 9; 10; 11; 12; 16; 18; 21; 22; 27] and is based on partition lemmas, which were originally developed in a deep paper due to Braverman and Mekler [7] devoted to the study of the symmetric Banach function spaces E such that the set $\{f \in E : f \prec\prec_{\mathrm{hd}} g\}$ coincides with the closure of the convex hull of its extreme points.

The approach of Braverman and Mekler was subsequently revised and redeveloped in [28] and precisely this revision constitutes the core of our approach in this paper.

We restate partition lemmas based on [28, Proposition 19] in Section 4. These lemmas allow us to restrict head and tail majorizations to very simple situations and reduce the problem to functions taking at most two values. Then, we deduce interpolation results from those structural lemmas.

Note that this scheme of proof is quite direct and in particular, does not involve at any point duality related arguments which apply only to Banach spaces [21] or more generally to L-convex quasi-Banach spaces [17; 25].

In Section 6, we pursue the same type of investigation, but in the setting of sequence spaces. The nondiffuse aspect of the underlying measure generates substantial technical difficulties. In particular, we require a new partition lemma which is not as efficient as those in Section 4 (compare Lemmas 6.2 and 4.6). This deficiency has been first pointed out to the authors by Cwikel. However, we are still able to resolve the conjecture of [20] (in the affirmative) by combining Lemma 6.1 with a Boyd-type argument which we borrow from Montgomery and Smith [23]. In particular, we substantially strengthen the results in [11]. Here is the precise statement:

Theorem 1.2. *Let $E \subset \ell_\infty$ be a quasi-Banach sequence space and $q \geq 1$. The following conditions are equivalent*:

(a) *There exists $p < q$ such that E is an interpolation space for the couple (ℓ^p, ℓ^q).*

(b) *There exists $c > 0$ such that for any $u \in E$ and $v \in \ell_\infty$,*

$$|v|^q \prec\prec_{\mathrm{tl}} |u|^q \Rightarrow v \in E \quad \text{and} \quad \|v\|_E \leq c\|u\|_E.$$

(c) *For any $u \in E$ and $v \in \ell_\infty$,*

$$|v|^q \prec\prec_{\mathrm{tl}} |u|^q \Rightarrow v \in E.$$

In this section, we freely use results of Cwikel [10] and Cadilhac [8] to avoid repeating too many similar arguments.

Note that Theorem 1.2 was since proved independently in [5] where a deeper analysis of the interpolation scale of sequence spaces ℓ_p, $0 \leq p \leq \infty$ is presented. In particular, it is shown in [5] that for any $0 < p < q < \infty$, E is an interpolation space for (ℓ_p, ℓ_q) if and only if it is an interpolation space for (ℓ_0, ℓ_q) and (ℓ_p, ℓ_∞), thus providing a counterpart to our main theorem in the sequence setting.

2. Preliminaries

Interpolation spaces. The reader is referred to [6] for more details on interpolation theory and to [19] for an introduction to symmetric spaces. In the remainder of this section, p and q will denote two nonnegative reals such that $p \le q$.

Let (Ω, m) be any measure space (in particular the following definitions apply to $\mathbb{N}$ equipped with the counting measure, i.e., sequence spaces). As previously mentioned, $L_0(\Omega) \subset \mathcal{X}(\Omega)$ denotes the set of functions whose supports have finite measures, it is naturally equipped with the group norm

$$\|f\|_0 = m(\text{supp } f), \quad f \in L_0(\Omega).$$

The "norm" of a linear operator $T : L_0(\Omega) \to L_0(\Omega)$, is defined as

$$\|T\|_{L_0 \to L_0} = \sup_{f \in L_0} \frac{m(\text{supp}(Tf))}{m(\text{supp}(f))}.$$

Definition 2.1. A linear space $E \subset \mathcal{X}(\Omega)$ becomes a quasi-Banach function space when equipped with a complete quasinorm $\|\cdot\|_E$ such that:

- If $f \in E$ and $g \in \mathcal{X}(\Omega)$ are such that $|g| \le |f|$, then $g \in E$ and $\|g\|_E \le \|f\|_E$.

Definition 2.2. A quasi-Banach function space $E \subset \mathcal{X}(\Omega)$ is called symmetric if

- $f \in E$ and $g \in \mathcal{X}(\Omega)$ are such that $\mu(f) = \mu(g)$, then $g \in E$ and $\|g\|_E = \|f\|_E$.

Definition 2.3 (bounded operator on a couple of quasi-Banach function spaces). Let X and Y be quasi-Banach function spaces. We say that a linear operator T is bounded on (X, Y) if T is defined from $X + Y$ to $X + Y$ and restricts to a bounded operator from X to X and from Y to Y. Set

$$\|T\|_{(X,Y)\to(X,Y)} = \max(\|T\|_{X\to X}, \|T\|_{Y\to Y}).$$

Let us recall the precise abstract definition of an interpolation space (see [6; 19]).

Definition 2.4 (interpolation space between function spaces). Let X, Y and Z be either quasi-Banach function spaces on Ω or $L_0(\Omega)$. We say that Z is an interpolation space for the couple (X, Y) if $X \cap Y \subset Z \subset X + Y$ and any bounded operator on (X, Y) restricts to a bounded operator on Z. Denote by $\text{Int}(X, Y)$ the set of interpolation spaces for the couple (X, Y).

For quasi-Banach spaces, the above definition is equivalent to a seemingly stronger quantitative property.

Proposition 2.5. *Let X, Y, Z be quasi-Banach function spaces. If Z is an interpolation space for the couple (X, Y), then there exists a constant $c(X, Y, Z) > 0$ such that for any bounded operator T on (X, Y),*

$$\|T\|_{Z\to Z} \le c(X, Y, Z) \cdot \|T\|_{(X,Y)\to(X,Y)}.$$

The best possible value of $c(X, Y, Z)$ is called interpolation constant of Z with respect to the couple (X, Y).

Proof. In [19, Lemma I.4.3], the assertion is proved for Banach spaces. The argument for quasi-Banach spaces is identical (because it relies on the closed graph theorem, which holds for F-spaces, and hence for quasi-Banach spaces). □

K-functional and E-functional. In the remainder of the subsection, X, Y and Z will denote function spaces which are either quasi-Banach, or L_0.

Definition 2.6. Let $f \in X+Y$ and $t>0$. Define

$$K_t(f, X, Y) := \inf_{g+h=f} \|g\|_X + t\|h\|_Y \quad \text{and} \quad E_t(f, X, Y) := \inf_{\|g\|_X \le t} \|f-g\|_Y.$$

These two notions are closely related to one another (see [24]) and the K-functional in particular plays a major role in the study of general interpolation spaces. Note that

$$K_t(f, L_1, L_\infty) = \int_0^t \mu(s, f)\,ds \quad \text{and} \quad E_t(f, L_0, L_1) = \int_t^\infty \mu(s, f)\,ds.$$

Thus the head and tail majorizations we consider can be in fact expressed in terms of K and E functionals. We say that Z is *K-monotone* with respect to the couple (X, Y) if $X \cap Y \subset Z \subset X+Y$ and for any $f \in Z$, $g \in X+Y$,

$$\forall t>0, \quad K_t(g, X, Y) \le K_t(f, X, Y) \Rightarrow g \in Z.$$

Similarly, Z is *E-monotone* with respect to the couple (X, Y) if $X \cap Y \subset Z \subset X+Y$ and for any $f \in Z$, $g \in X+Y$,

$$\forall t>0, \quad E_t(g, X, Y) \le E_t(f, X, Y) \Rightarrow g \in Z.$$

Remark 2.7. It is clear from the definitions that if Z is either E-monotone or K-monotone for the couple (X, Y) then Z is an interpolation space for (X, Y).

Symmetry of interpolation spaces. In this subsection, we show that a quasi-Banach interpolation space for a couple of symmetric spaces can always be renormed into a symmetric space. Note that similar results can be found in the literature, see, for example, [19, Theorem 2.1].

As usual, we will use the term *measure preserving* for a measurable map ω between measure spaces $(\Omega_1, \mathcal{A}_1, m_1)$ and $(\Omega_2, \mathcal{A}_2, m_2)$ verifying,

$$\forall A \in \mathcal{A}_1, \qquad \omega(A) \in \mathcal{A}_2 \quad \text{and} \quad m_2(\omega(A)) = m_1(A).$$

Lemma 2.8. *Assume that Ω is $(0, 1)$, $(0, \infty)$ or $\mathbb{N}$. Let $0 \le f, g \in L_0(\Omega)+L_\infty(\Omega)$ and let $\varepsilon>0$. Assume that $\mu(f)=\mu(g)$. There exists a measure preserving map $\omega : \mathrm{supp}(g) \to \mathrm{supp}(f)$ such that $(1+\varepsilon)(f \circ \omega) \ge g$.*

Proof. **Case 1**. Suppose first that $\mu(\infty, f) = \mu(\infty, g) = 0$.

Define, for any $n \in \mathbb{Z}$,

$$F_n = \{t : (1+\varepsilon)^n < f(t) \leq (1+\varepsilon)^{n+1}\}, \quad G_n = \{t : (1+\varepsilon)^n < g \leq (1+\varepsilon)^{n+1}\}.$$

By assumption, $m(F_n) = m(G_n)$ for every $n \in \mathbb{Z}$. Let $\omega_n : G_n \to F_n$ be an arbitrary measure preserving bijection.

Define the measure preserving map $\omega : \mathrm{supp}(g) \to \mathrm{supp}(f)$ by concatenating $\omega_n : G_n \to F_n,\ n \in \mathbb{Z}$. For every $t \in G_n$, we have

$$f(\omega(t)) \geq (1+\varepsilon)^n, \quad g \leq (1+\varepsilon)^{n+1}.$$

Thus,

$$(1+\varepsilon) f(\omega(t)) \geq g, \quad t \in \mathrm{supp}(g).$$

This completes the proof of Case 1.

Case 2. Let δ such that $(1+\delta)^2 = (1+\varepsilon)$. Let $a = \mu(\infty, f) = \mu(\infty, g) > 0$. Define, for any $n \geq 1$,

$$F_n = \{t : a(1+\delta)^n < f(t) \leq a(1+\delta)^{n+1}\}, \quad G_n = \{t : a(1+\delta)^n < g \leq a(1+\delta)^{n+1}\}$$

and

$$F_0 = \{t : (1+\delta)^{-1} a \leq f(t) \leq (1+\delta)\, a\}, \quad G_0 = \{t : 0 < g \leq (1+\delta)\, a\}.$$

By assumption, for any $n \geq 1$, $m(G_n) = m(F_n)$ and $m(G_0) = m(F_0) = \infty$. For any $n \geq 0$, choose a measure preserving bijection ω_n from G_n to F_n.

Define the measure preserving map $\omega : \mathrm{supp}(g) \to \mathrm{supp}(f)$ by concatenating the ω_n's. For any $n \geq 0$ and any $t \in G_n$,

$$f(\omega(t)) \geq a(1+\delta)^{n-1}, \quad g \leq a(1+\delta)^{n+1}.$$

Thus,

$$(1+\delta)^2 f(\omega(t)) = (1+\varepsilon) f(\omega(t)) \geq g, \quad t \in \mathrm{supp}(g). \qquad \square$$

Lemma 2.9. *Assume that* Ω *is* $(0, 1)$, $(0, \infty)$ *or* $\mathbb{N}$. *Let* $E, A, B \subset (L_0 + L_\infty)(\Omega)$ *be quasi-Banach function spaces. Assume that* A *and* B *are symmetric and that* E *is an interpolation space for the couple* (A, B). *Then* E *admits an equivalent symmetric quasinorm.*

Proof. Let $f \in E$ and $g \in L_0 + L_\infty$. Assume that $\mu(g) \leq \mu(f)$. By Lemma 2.8, there exists a map $\omega : \mathrm{supp}(g) \to \mathrm{supp}(f)$ such that for any $t \in \mathrm{supp}(g)$,

$$2|f \circ \omega(t)| \geq |g(t)|.$$

Define, for any $h \in \mathcal{X}(\Omega)$,

$$T(h) := \begin{cases} \frac{g}{f \circ \omega} h \circ \omega & \text{on } \mathrm{supp}(g), \\ 0, & \text{elsewhere.} \end{cases}$$

Since ω is measure preserving, T is bounded on A and B of norm less than 2. Let c_E be the interpolation constant of E for the couple (A, B) (as in Proposition 2.5). We know that $Tf = g \in E$ and

$$\|g\| \le 2c_E \|f\|. \tag{2-1}$$

Define, for any $f \in E$,

$$\|f\|_{E'} = \inf_{\mu(g) \ge \mu(f)} \|g\|_E.$$

By (2-1), $\|f\|_{E'} \le \|f\|_E \le 2c_E \|f\|_{E'}$ and $(E, \|\cdot\|_{E'})$ is a symmetric space. □

Remark 2.10. It is not difficult to see that if the underlying measure space Ω contains both a continuous part and atoms, then Lemma 2.9 is no longer true for $A = L_p(\Omega)$, $B = L_q(\Omega)$ and $p < 1$. However, one can observe that if A and B are *fully symmetric* (i.e., interpolation spaces between $L_1(\Omega)$ and $L_\infty(\Omega)$), Lemma 2.9 remains valid for any Ω. This is reminiscent of the conditions required in [27, Section 4].

3. Interpolation for the couple (L_0, L_q)

In this section, $\Omega = (0, \infty)$ (for brevity, we omit Ω in the notations). We investigate some basic properties of the interpolation couple (L_0, L_q). First, we provide a statement analogous to Proposition 2.5 and applicable to L_0.

Since the closed graph theorem does not apply to L_0 (it is not an F-space), our proof uses concrete constructions that rely on the structure of the underlying measure space.

For any $f \in \mathcal{X}$, denote by M_f the multiplication operator $g \mapsto f \cdot g$.

Theorem 3.1. *Let E be a quasi-Banach function space and $q \in (0, \infty]$. Assume that E is an interpolation space for the couple (L_0, L_q). Then, there exists a constant c such that for any contraction T on (L_0, L_q), $\|T\|_{E \to E} \le c$.*

Proof. Let $(A_n)_{n \ge 1}$ be a partition of $(0, \infty)$ such that $m(A_n) = \infty$ for every $n \ge 1$.

Let $\gamma_n : A_n \to A_n^c$ be a measure preserving bijective transform. Set

$$(U_n x)(t) = \begin{cases} x(\gamma_n(t)), & t \in A_n, \\ 0, & t \in A_n^c, \end{cases} \qquad (V_n x)(t) = \begin{cases} x(\gamma_n^{-1}(t)), & t \in A_n^c, \\ 0, & t \in A_n. \end{cases}$$

Obviously, U_n and V_n are bounded operators on the couple (L_0, L_q). By assumption, $U_n, V_n : E \to E$ are bounded mappings.

Note that

$$V_n U_n = M_{\chi_{A_n^c}}, \quad n \ge 1.$$

Let us argue by contradiction. For any $n \geq 1$, choose an operator T_n which is a contraction on (L_0, L_q) and such that

$$\|T_n\|_{E\to E} \geq 4^n \cdot \max\{\|U_n\|_{E\to E}, \|V_n\|_{E\to E}, 1\}^2. \tag{3-1}$$

It is immediate that

$$\begin{aligned} T_n &= M_{\chi_{A_n}} T_n\, M_{\chi_{A_n}} + M_{\chi_{A_n^c}} T_n\, M_{\chi_{A_n}} + M_{\chi_{A_n}} T_n\, M_{\chi_{A_n^c}} + M_{\chi_{A_n^c}} T_n\, M_{\chi_{A_n^c}} \\ &= T_{1,n} + V_n\, T_{2,n} + T_{3,n}\, U_n + V_n\, T_{4,n}\, U_n, \end{aligned}$$

where

$$T_{1,n} = M_{\chi_{A_n}} T_n\, M_{\chi_{A_n}}, \quad T_{2,n} = U_n T_n\, M_{\chi_{A_n}}, \quad T_{3,n} = M_{\chi_{A_n}} T_n V_n, \quad T_{4,n} = U_n T_n V_n.$$

By quasitriangle inequality, we have

$$\|T_n\|_{E\to E} \leq C_E^2 \cdot \left(\sum_{k=1}^{4} \|T_{k,n}\|_{E\to E}\right) \cdot \max\{\|U_n\|_{E\to E}, \|V_n\|_{E\to E}.1\}^2.$$

Let $k_n \in \{1, 2, 3, 4\}$ be such that

$$\|T_{k_n,n}\|_{E\to E} = \max_{1\leq k\leq 4} \|T_{k,n}\|_{E\to E}.$$

We, therefore, have

$$\|T_n\|_{E\to E} \leq 4C_E^2 \|T_{k_n,n}\|_{E\to E} \cdot \max\{\|U_n\|_{E\to E}, \|V_n\|_{E\to E}.1\}^2. \tag{3-2}$$

Set $S_n = T_{k_n,n}$. Note that $\|S_n\|_{L_0\to L_0} \leq 1$ and $\|S_n\|_{L_q\to L_q} \leq 1$. A combination of (3-1) and (3-2) yields

$$\|S_n\|_{E\to E} \geq 4^{n-1} C_E^{-2}, \quad n \geq 1.$$

Note that $S_n = M_{\chi_{A_n}} S_n\, M_{\chi_{A_n}}$. Set

$$S = \sum_{n\geq 1} S_n.$$

Since the S_n's are in direct sum, we have

$$\|S\|_{L_0\to L_0} = \sup_{n\geq 1} \|S_n\|_{L_0\to L_0} \leq 1 \quad \text{and} \quad \|S\|_{L_q\to L_q} = \sup_{n\geq 1} \|S_n\|_{L_q\to L_q} \leq 1.$$

Moreover, E is an interpolation space for the couple (L_0, L_q), it follows that $S : E \to E$ is bounded.

For any $n \geq 1$, choose $f_n \in E$ such that $\|f_n\|_E \leq 1$ and $\|S_n f_n\|_E \geq 4^{n-2}\, C_E^{-2}$. Recall that $S_n = S_n\, M_{\chi_{A_n}}$. Hence, we may assume without loss of generality that f_n is supported on A_n. Thus, $S(f_n) = S_n(f_n)$ and

$$\|S(f_n)\|_E = \|S_n(f_n)\|_E \geq 4^{n-2}\, C_E^{-2}.$$

This contradicts the boundedness of S. □

Remark 3.2. Theorem 3.1 above remains true for other underlying measure spaces:

- *For sequence spaces*. Indeed, in the proof of Theorem 3.1, we only use properties of the underlying measure space in the first sentence, namely when we consider a partition of $(0, \infty)$ into countably many sets, each of them isomorphic to $(0, \infty)$. Since a partition satisfying the same property exists for $\mathbb{Z}_+$, Theorem 3.1 remains true for interpolation spaces between ℓ_0 and ℓ_q.

- *For* $(0, 1)$. The same general idea applies in this case but some modifications have to be made because the maps γ_n introduced in the proof cannot be assumed to be measure-preserving. The details are left to the reader.

Lemma 3.3. *Let $E, Y \subset L_0 + L_\infty$ be quasi-Banach function spaces. Assume that Y is symmetric and that E is an interpolation space for the couple (L_0, Y). Then E admits an equivalent symmetric quasinorm.*

Proof. The argument follows that in Lemma 2.9 mutatis mutandi. □

The following assertion is a special case of Theorem 1.1 and an important ingredient in the proof of the latter theorem.

Corollary 3.4. *Let X be a quasi-Banach function space and $q \in (0, \infty)$. Assume that $L_0 \cap L_q \subset X \subset L_0 + L_q$ and that for any $f \in E$ and $g \in L_0 + L_q$,*

$$|g|^q \prec\prec_{\text{tl}} |f|^q \Rightarrow g \in E.$$

Then E is an interpolation space for the couple (L_0, L_q).

Proof. It is clear that the condition on X is equivalent to E-monotonicity with respect to the couple (L_0, L_q) so by Remark 2.7, E is an interpolation space for the couple (L_0, L_q). □

Corollary 3.4 applies in particular to L_p-spaces, $p \leq q$. We decided to add a more precise statement and to provide a direct proof of the latter.

Corollary 3.5. *Let $p, q \in (0, \infty)$ such that $p < q$. Then, L_p is an interpolation space for the couple (L_0, L_q). More precisely, if T is a contraction on (L_0, L_q), then T is a contraction on L_p.*

Proof. Let us first consider characteristic functions. Let E be a set with finite measure. Since T is a contraction on L_0, the measure of the support of $T(\chi_E)$ is less than $m(E)$. So by Hölder's inequality, setting $r = (p^{-1} - q^{-1})^{-1}$, we have

$$\|T(\chi_E)\|_p \leq \|T(\chi_E)\|_q \cdot m(E)^{1/r} \leq \|\chi_E\|_q \cdot m(E)^{1/r} = \|\chi_E\|_p.$$

First, consider the case $p \leq 1$. Let $f \in L_p$ be a step function, i.e.,

$$f = \sum_{i \in \mathbb{N}} a_i \chi_{E_i},$$

where $a_i \in \mathbb{C}$ and the sets E_i are disjoint sets with finite measure. By the p-triangular inequality we write

$$\|Tf\|_p^p \le \sum_{i\in\mathbb{N}} |a_i|^p \, \|T(\chi_{E_i})\|_p^p \le \sum_{i\in\mathbb{N}} |a_i|^p \, \|\chi_{E_i}\|_p^p = \|f\|_p^p.$$

Since $T : L_p \to L_p$ is bounded by Corollary 3.4 and since step functions are dense in L_p, it follows that $T : L_p \to L_p$ is a contraction (for $p \le 1$).

Now consider the case $p > 1$. Since $p < q$, it follows that $q > 1$. By the preceding paragraph, $T : L_1 \to L_1$ is a contraction. By complex interpolation, $T : L_p \to L_p$ is also contraction. □

4. Construction of contractions on (L_0, L_q) and (L_p, L_∞)

Let $p, q \in (0, \infty)$. In this section, we are interested in the following question. Given functions f and g in $L_0 + L_q$ (resp. $L_p + L_\infty$), does there exist a bicontraction T on (L_0, L_q) (resp. (L_p, L_∞)) such that $T(f) = g$? We show that such an operator exists provided that $|g|^q \prec\prec_{\mathrm{tl}} |f|^q$ (resp. $|g|^p \prec\prec_{\mathrm{hd}} |f|^p$. This directly implies a necessary condition for a symmetric space to be an interpolation space for the couple (L_0, L_q) (resp. (L_p, L_∞)) which will be exploited in the next section.

Our method of proof is very direct. We construct the bicontraction T as direct sums of very simple operators. This is made possible by two partition lemmas that enable us to understand the orders $\prec\prec_{\mathrm{tl}}$ and $\prec\prec_{\mathrm{hd}}$ as direct sums of simple situations.

Partition lemmas. We state our first lemma without proof since it essentially repeats that of Proposition 19 in [28].

Lemma 4.1. *Let $f, g \in L_1$ be positive decreasing step functions. Assume that $g \prec_{\mathrm{hd}} f$. There exists a sequence of intervals $\{I_k, J_k\}_{k\ge 0}$ of $(0, \infty)$ such that:*

(i) *For every $k \ge 0$, I_k and J_k are disjoint intervals of finite length.*

(ii) *$(I_k \cup J_k) \cap (I_l \cup J_l) = \varnothing$ for $k \ne l$.*

(iii) *f and g are constant on I_k and on J_k.*

(iv) *$g|_{I_k\cup J_k} \prec_{\mathrm{hd}} f|_{I_k\cup J_k}$ for every $k \ge 0$.*

(v) *$g \le f$ on the complement of $\bigcup_{k\ge 0} I_k \cup J_k$.*

If furthermore $f, g \in L_1$ and $g \prec_{\mathrm{hd}} f$ then $f = g$ on the complement of $\bigcup_{k\ge 0} I_k \cup J_k$.

Scholium 4.2. *Let $f, g \in \mathcal{X}$ be positive decreasing functions. Let $\Delta \subset (0, \infty)$ be an arbitrary measurable set.*

(i) *If $f, g \in L_1 + L_\infty$ are such that*

$$\int_{[0,t]\cap\Delta} g \le \int_{[0,t]\cap\Delta} f, \quad t > 0,$$

then $g\chi_\Delta \prec\prec_{\mathrm{hd}} f\chi_\Delta$.

(ii) *If* $f, g \in L_0 + L_1$ *are such that*

$$\int_{(t,\infty)\cap\Delta} g \le \int_{(t,\infty)\cap\Delta} f, \quad t > 0,$$

then $g\chi_\Delta \prec\prec_{\mathrm{tl}} f\chi_\Delta$.

The second partition lemma deals with describing the order $\prec\prec_{\mathrm{tl}}$ in terms of $\prec_{\mathrm{tl}}$ and $\le$.

Lemma 4.3. *Let* $f, g \in L_0 + L_1$ *be such that* $f = \mu(f)$, $g = \mu(g)$ *and* $g \prec\prec_{\mathrm{tl}} f$. *There exists a collection* $\{\Delta_k\}_{k\ge 0}$ *of pairwise disjoint sets such that*:

(i) $f|_{\Delta_k} \prec_{\mathrm{hd}} g|_{\Delta_k}$ *for every* $k \ge 0$.

(ii) $g \le f$ *on the complement of* $\bigcup_{k\ge 0}\Delta_k$.

Proof. Consider the set $\{g > f\}$. Similarly to the previous proof, connected components of the set $\{g > f\}$ are intervals (closed or not) not reduced to points. Let us enumerate these intervals as (a_k, b_k), $k \ge 0$.

We have

$$\int_t^\infty (f-g)_+ - \int_t^\infty (f-g)_- = \int_t^\infty (f-g) \ge 0.$$

Let

$$H(t) = \sup\left\{u : \int_u^\infty (f-g)_+ = \int_t^\infty (f-g)_-\right\}.$$

Obviously, H is a monotone function, $H(t) \ge t$ for all $t > 0$ and

$$\int_{H(t)}^\infty (f-g)_+ = \int_t^\infty (f-g)_-.$$

Set

$$\Delta_k = (a_k, b_k) \cup \big((H(a_k), H(b_k)) \cap \{g \le f\}\big).$$

Note that

$$\int_{b_k}^\infty (f-g)_+ = \int_{a_k}^\infty (f-g)_+ \ge \int_{a_k}^\infty (f-g)_-$$

and therefore, $H(a_k) \ge b_k$.

We claim that $\Delta_k \cap \Delta_l = \varnothing$ for $k \ne l$. Indeed, let $a_k < b_k \le a_l < b_l$. We have $H(a_k) \le H(b_k) \le H(a_l) \le H(b_l)$. Thus, $(H(a_k), H(b_k)) \cap (H(a_l), H(b_l)) = \varnothing$. We now have

$$\Delta_k \cap \Delta_l = \big((\Delta_k \cap \{f < g\}) \cap (\Delta_l \cap \{f < g\})\big) \cup \big((\Delta_k \cap \{f \ge g\}) \cap (\Delta_l \cap \{f \ge g\})\big).$$

Obviously,

$$(\Delta_k \cap \{f < g\}) \cap (\Delta_l \cap \{f < g\}) = (a_k, b_k) \cap (a_l, b_l) = \varnothing,$$

$$(\Delta_k \cap \{f \ge g\}) \cap (\Delta_l \cap \{f \ge g\}) = (H(a_k), H(b_k)) \cap (H(a_l), H(b_l)) \cap \{f \ge g\} = \varnothing.$$

This proves the claim.

We now claim that

$$\int_{(t,\infty)\cap\Delta_k}(f-g)\geq 0.$$

If $t\geq b_k$, then taking into account that $H(a_k)\geq b_k$, we infer that

$$(t,\infty)\cap\Delta_k\subset\{f\geq g\}$$

and the claim follows immediately. If $t\in(a_k,b_k)$, then

$$\begin{aligned}\int_{(t,\infty)\cap\Delta_k}(f-g)&=\int_{(H(a_k),H(b_k))}(f-g)_+-\int_t^{b_k}(f-g)_-\\&\geq\int_{(H(a_k),H(b_k))}(f-g)_+-\int_{a_k}^{b_k}(f-g)_-=0.\end{aligned}$$

This proves the claim.

It follows from the claim and Scholium 4.2 that $g\chi_{\Delta_k}\prec\prec_{\mathrm{tl}}f\chi_{\Delta_k}$. Since

$$\int_{\Delta_k}g=\int_{\Delta_k}f,$$

it follows that $g\chi_{\Delta_k}\prec_{\mathrm{tl}}f\chi_{\Delta_k}$, which immediately implies the first assertion.

By construction, $(a_k,b_k)\subset\Delta_k$. Thus,

$$\{g>f\}=\bigcup_{k\geq 0}(a_k,b_k)\subset\bigcup_{k\geq 0}\Delta_k.$$

The second assertion is now obvious. □

Construction of operators. We repeat the same structure as in the previous subsection, proving four lemmas, each one dealing with a certain order: $\prec_{\mathrm{hd}}$, $\prec_{\mathrm{tl}}$, $\prec\prec_{\mathrm{hd}}$, and finally $\prec\prec_{\mathrm{tl}}$.

Lemma 4.4. *Let* $p\in(0,\infty)$. *Let* $f,g\in L_p(0,\infty)$, *assume that* $|g|^p\prec_{\mathrm{hd}}|f|^p$, $f=\mu(f)$ *and* $g=\mu(g)$. *There exists a linear operator* $T:\mathcal{X}(0,\infty)\to\mathcal{X}(0,\infty)$ *such that* $g=T(f)$ *and*

$$\|T\|_{L_p\to L_p}\leq 2\cdot 3^{1/p},\quad \|T\|_{L_\infty\to L_\infty}\leq 2\cdot 2^{1/p}.$$

Proof. **Step 1**. First, let us assume that f and g are step functions.

Apply Lemma 4.1 to the functions f^p and g^p and let I_k and J_k be as in Lemma 4.1. Without loss of generality, the interval I_k is located to the left of the interval J_k.

For every $k\geq 0$, let's define the mapping $S_k:\mathcal{X}(I_k\cup J_k)\to\mathcal{X}(I_k\cup J_k)$ as below. The construction of this mapping will depend on whether $f^p|_{J_k}\leq\frac{1}{2}g^p|_{J_k}$ or $f^p|_{J_k}>\frac{1}{2}g^p|_{J_k}$.

If $f^p|_{J_k} \le \frac{1}{2} g^p|_{J_k}$, then

$$g^p|_{J_k} \cdot m(J_k) \le g^p|_{I_k} \cdot m(I_k) + g^p|_{J_k} \cdot m(J_k)$$
$$= f^p|_{I_k} \cdot m(I_k) + f^p|_{J_k} \cdot m(J_k) \le f^p|_{I_k} \cdot m(I_k) + \tfrac{1}{2} g^p|_{J_k} \cdot m(J_k).$$

Therefore,

$$g^p|_{J_k} \cdot m(J_k) \le 2 f^p|_{I_k} \cdot m(I_k).$$

Let l_k be a linear bijection from J_k to I_k. We set

$$S_k\, x = \frac{g|_{I_k}}{f|_{I_k}} \cdot x \chi_{I_k} + \frac{g|_{J_k}}{f|_{I_k}} \cdot (x \circ l_k)\, \chi_{J_k}.$$

Clearly, S_k is a contraction in the uniform norm.

Let $x \in L_p$. We have

$$\|S_k\, x\|_p^p \le \frac{g^p|_{I_k}}{f^p|_{I_k}} \cdot \|x \chi_{I_k}\|_p^p + \frac{g^p|_{J_k}}{f^p|_{I_k}} \cdot \|(x \circ l_k)\, \chi_{J_k}\|_p^p$$
$$\le \frac{g^p|_{I_k}}{f^p|_{I_k}} \cdot \|x\|_p^p + \frac{g^p|_{J_k}}{f^p|_{I_k}} \cdot \frac{m(J_k)}{m(I_k)} \cdot \|x\|_p^p \le 3\|x\|_p^p.$$

Also, we have

$$\|S_k\, x\|_\infty \le \|x\|_\infty.$$

If $f^p|_{J_k} > \frac{1}{2} g^p|_{J_k}$, then we set $S_k = M_{gf^{-1}}$. Clearly, $\|S_k\, x\|_\infty \le 2^{1/p}\|x\|_\infty$ and $\|S_k\, x\|_p \le 2^{1/p}\|x\|_p$.

We define $S : \mathcal{X} \to \mathcal{X}$ by

$$S = \bigoplus_{k \ge 0} S_k.$$

Remark that $\|S\|_{r \to r} = \sup_{k \ge 0} \|S_k\|_{L_r \to L_r}$ for any $r \in [0, \infty]$. Hence,

$$\|S\|_{L_p \to L_p} \le 3^{1/p}, \quad \|S\|_{L_\infty \to L_\infty} \le 2^{1/p}.$$

It remains only to note that $Sf = g$.

Step 2. Now, only assume that f and g are positive and nonincreasing. Define, for any $n \in \mathbb{Z}$,

$$a_n = \sup\{t \in (0, \infty) : f(t) \ge 2^{n/2}\} \quad \text{and} \quad b_n = \sup\{t \in (0, \infty) : g(t) \ge 2^{n/2}\}.$$

Let $\mathcal{A}$ be the σ-algebra generated by the intervals (a_n, a_{n+1}) and (b_n, b_{n+1}). Define

$$f_0^p = \mathbb{E}[f^p \mid \mathcal{A}] \quad \text{and} \quad g_0^p = \mathbb{E}[g^p \mid \mathcal{A}].$$

Note that f_0 and g_0 are step functions such that

$$g_0^p \prec_{\text{hd}} f_0^p, \quad 2^{-1/2} f \le f_0 \le 2^{1/2} f \quad \text{and} \quad 2^{-1/2} g \le g_0 \le 2^{1/2} g.$$

Apply Step 1 to f_0 and g_0 to obtain an operator S and set

$$T = M_{ff_0^{-1}} \circ S \circ M_{g_0 g^{-1}}.$$

Clearly, $Tf = g$ and

$$\|T\|_{L_p\to L_p} \le 2\|S\|_{L_p\to L_p} \le 2\cdot 2^{1/p}, \quad \|T\|_{L_\infty\to L_\infty} \le 2\|S\|_{L_\infty\to L_\infty} \le 2^2\cdot 2^{1/p}. \ \square$$

Lemma 4.5. *Let $f, g \in L_q(0,\infty)$ be positive nonincreasing functions such that $g^q \prec_{\mathrm{tl}} f^q$. Let $d > 1$. There exists a linear operator $T : \mathcal{X}(0,\infty) \to \mathcal{X}(0,\infty)$ such that $g = T(f)$ and*

$$\|T\|_{L_0\to L_0} \le 4, \quad \|T\|_{L_q\to L_q} \le 2\cdot 3^{1/q}.$$

Proof. Following Step 2 of Lemma 4.4, we are reduced to dealing with step functions.

Apply Lemma 4.1 to the functions g^q and f^q and let $(I_k)_{k\ge1}$ and $(J_k)_{k\ge1}$ be as in Lemma 4.1. Without loss of generality, the intervals I_k is located to the left of the intervals J_k.

Let $k \ge 1$. Define the mappings $S_k : \mathcal{X}(I_k\cup J_k) \to \mathcal{X}(I_k\cup J_k)$ as below.

Note that since $f^q|_{I_k\cup J_k} \prec_{\mathrm{hd}} g^q|_{I_k\cup J_k}$, we have

$$g|_{I_k} \ge f|_{I_k} \ge f|_{J_k} \ge g|_{J_k}.$$

The construction of S_k will depend on whether we have $\|f\chi_{I_k}\|_q \le \|f\chi_{J_k}\|_q$ or $\|f\chi_{I_k}\|_q > \|f\chi_{J_k}\|_q$.

If $\|f\chi_{I_k}\|_q \le \|f\chi_{J_k}\|_q$, then $m(I_k) \le m(J_k)$ and

$$g_0^q|_{I_k}\cdot m(I_k) \le 2f^q|_{J_k}\cdot m(J_k).$$

Let $l_k : I_k \to J_k$ be a linear bijection. We set

$$S_k x = \frac{g|_{I_k}}{f|_{J_k}}(x\circ l_k)\,\chi_{I_k} + \frac{g}{f}x\chi_{J_k}.$$

Let $x \in L_q$. We have

$$\|S_k x\|_0 \le \|x\chi_{J_k}\|_0 + \|(x\circ l_k)\,\chi_{I_k}\|_0 \le \left(1+\frac{m(I_k)}{m(J_k)}\right)\|x\|_0.$$

Thus, $\|S_k\|_{L_0\to L_0} \le 2$. We have

$$\begin{aligned}\|S_k x\|_q^q &= \frac{g^q|_{I_k}}{f^q|_{J_k}}\|(x\circ l_k)\,\chi_{I_k}\|_q^q + \frac{g^q|_{J_k}}{f^q|_{J_k}}\|x\chi_{J_k}\|_q^q \\ &\le \frac{g^q|_{I_k}}{f^q|_{J_k}}\cdot\frac{m(I_k)}{m(J_k)}\cdot\|x\|_q^q + \frac{g^q|_{J_k}}{f^q|_{J_k}}\|x\|_q^q \le 3\|x\|_q^q.\end{aligned}$$

If $\|f\chi_{I_k}\|_q > \|f\chi_{J_k}\|_q$, then

$$g^q|_{I_k}\cdot m(I_k) \le 2f^q|_{I_k}\cdot m(I_k)$$

and therefore, $g_0 \leq 2^{1/q} f_0$. We now set $S_k = M_{gf^{-1}}$. Obviously, $\|S_k\|_{L_0 \to L_0} \leq 1$ and $\|S_k\|_{L_q \to L_q} \leq 2^{1/q}$.

We set

$$S = \bigoplus_{k \geq 0} S_k.$$

Since $S : \mathcal{X}(0, \infty) \to \mathcal{X}(0, \infty)$ is defined as a direct sum:

$$\|S\|_{L_0 \to L_0} = \sup_{k \geq 1} \|S_k\|_{L_0 \to L_0} \leq 2, \quad \|S\|_{L_q \to L_q} = \sup_{k \geq 1} \|S_k\|_{L_q \to L_q} \leq 3^{1/q},$$

it remains only to note that $Sf = g$. ☐

Lemma 4.6. *Let $f, g \in (L_p + L_\infty)(0, \infty)$ be such that $|g|^p \prec\prec_{\mathrm{hd}} |f|^p$, $f = \mu(f)$ and $g = \mu(g)$. There exists a linear operator $T : \mathcal{X} \to \mathcal{X}$ such that $g = T(f)$ and*

$$\|T\|_{L_p \to L_p} \leq 2 \cdot 3^{1/p}, \quad \|T\|_{L_\infty \to L_\infty} \leq 2 \cdot 2^{1/p}.$$

Proof. Let $(\Delta_k)_{k \geq 0}$ be as in Lemma 4.1 and let Δ_∞ be the complement of $\bigcup_{k \geq 0} \Delta_k$. By Lemma 4.4, there exists $T_k : \mathcal{X}(\Delta_k) \to \mathcal{X}(\Delta_k)$ such that $T_k(f) = g$ on Δ_k and

$$\|T_k\|_{L_p(\mathcal{X}_k) \to L_p(\mathcal{X}_k)} \leq 2 \cdot 9^{1/p}, \quad \|T_k\|_{L_\infty(\mathcal{X}_k) \to L_\infty(\mathcal{X}_k)} \leq 2 \cdot 4^{1/p}.$$

Set $T_\infty = M_{g/f}$ on $\mathcal{X}(\Delta_\infty)$. We now set

$$T = T_\infty \bigoplus \Big(\bigoplus_{k \geq 0} T_k \Big).$$

Obviously, $Tf = g$ on $(0, \infty)$ and

$$\|T\|_{L_p \to L_p} \leq 2 \cdot 9^{1/p}, \quad \|T\|_{L_\infty \to L_\infty} \leq 2 \cdot 4^{1/p}.$$ ☐

Lemma 4.7. *Let $f, g \in (L_0 + L_q)(0, \infty)$ be such that $|g|^q \prec\prec_{\mathrm{tl}} |f|^q$, $f = \mu(f)$ and $g = \mu(g)$. There exists a linear operator $T : \mathcal{X} \to \mathcal{X}$ such that $g = T(f)$ and*

$$\|T\|_{L_0 \to L_0} \leq 4, \quad \|T\|_{L_q \to L_q} \leq 2 \cdot 3^{1/q}.$$

Proof. Without loss of generality, $g = \mu(g)$ and $f = \mu(f)$. Let $(\Delta_k)_{k \geq 0}$ be as in Lemma 4.3 and let Δ_∞ be the complement of $\bigcup_{k \geq 0} \Delta_k$. By Lemma 4.5, there exists $T_k : \mathcal{X}(\Delta_k) \to \mathcal{X}(\Delta_k)$ such that $T_k(f) = g$ on Δ_k and

$$\|T_k\|_{L_0(\mathcal{X}_k) \to L_0(\mathcal{X}_k)} \leq 8, \quad \|T_k\|_{L_q(\mathcal{X}_k) \to L_q(\mathcal{X}_k)} \leq 2 \cdot 9^{1/q}.$$

Set $T_\infty = M_{g/f}$ on $\mathcal{X}(\Delta_\infty)$. We now set

$$T = T_\infty \bigoplus \Big(\bigoplus_{k \geq 0} T_k \Big).$$

Obviously, $Tf = g$ on $(0, \infty)$ and

$$\|T\|_{L_0 \to L_0} \leq 4, \quad \|T\|_{L_q \to L_q} \leq 2 \cdot 3^{1/q}.$$ ☐

5. Interpolation spaces for the couple (L_p, L_q)

In this section, we obtain characterizations of interpolation spaces for the couple (L_p, L_q), in terms of the majorization notions studied earlier. The necessity of the condition we consider is a direct consequence of the constructions explained in the previous section.

Theorem 5.1. *Let $0 \le p < q \le \infty$. Let E be a quasi-Banach function space such that $E \in \mathrm{Int}(L_p, L_q)$. There exist $c_{p,E}$ and $c_{q,E}$ in $\mathbb{R}_{>0}$ such that*:

(i) *Suppose $p \ne 0$. For any $f \in E$ and $g \in L_p + L_\infty$ if $|g|^p \prec\prec_{\mathrm{hd}} |f|^p$, then $g \in E$ and $\|g\|_E \le c_{p,E}\|f\|_E$.*

(ii) *Suppose $q \ne \infty$. For any $f \in E$ and $g \in L_0 + L_q$ if $|g|^q \prec\prec_{\mathrm{tl}} |f|^q$, then $g \in E$ and $\|g\|_E \le c_{q,E}\|f\|_E$.*

Proof. By Lemma 2.9 (for $p > 0$) or Lemma 3.3 (for $p = 0$), we may assume without loss of generality that E is a symmetric function space.

Assume that $p \ne 0$. Let $f \in E$ and let $g \in L_p + L_\infty$ be such that $|g|^p \prec\prec_{\mathrm{hd}} |f|^p$. Since E is symmetric, we may assume without loss of generality that $f = \mu(f)$ and $g = \mu(g)$. By Lemma 4.6, there exists an operator T such that $T(f) = g$ and

$$\|T\|_{(L_p,L_\infty)\to(L_p,L_\infty)} \le 2 \cdot 3^{1/p}.$$

Recall that L_q is an interpolation space for the couple (L_p, L_∞) (one can take, for example, real or complex interpolation method). Let $c_{p,q}$ be the interpolation constant of L_q for the couple (L_p, L_∞). We have

$$\|T\|_{L_q\to L_q} \le c_{p,q} \cdot 2 \cdot 3^{1/p}.$$

Let c_E be the interpolation constant of E for (L_p, L_q) (see Proposition 2.5). Then,

$$\|T\|_{E\to E} \le c_E \cdot \max\{1, c_{p,q}\} \cdot 2 \cdot 3^{1/p}.$$

Thus,

$$\|g\|_E \le \|T\|_{E\to E}\|f\|_E \le c_E \cdot \max\{1, c_{p,q}\} \cdot 2 \cdot 3^{1/p}\|f\|_E.$$

This proves the first assertion. The proof of the second one follows mutatis mutandi using Corollary 3.5 instead of complex interpolation and (for $p = 0$) Theorem 3.1 instead of Proposition 2.5. □

Lemma 5.2. *Assume that $0 < p < q < \infty$. Let $f, g \in L_p + L_q$ such that $f = \mu(f)$ and $g = \mu(g)$. Suppose that at every $t > 0$, one of the following inequalities holds*:

$$\int_0^t g^p\, ds \le \int_0^t f^p\, ds \quad or \quad \int_t^\infty g^q\, ds \le \int_t^\infty f^q\, ds.$$

Then, there exist $g_1, g_2 \in (L_p + L_q)^+$ which satisfies $g = g_1 + g_2$, $g_1^p \prec\prec_{\mathrm{hd}} f^p$ and $g_2^q \prec\prec_{\mathrm{tl}} f^q$.

Proof. Set

$$A = \Big\{ t > 0 : \int_0^t g(s)^p\,ds \le \int_0^t f(s)^p\,ds \Big\},$$
$$B = \Big\{ t > 0 : \int_t^\infty g(s)^q\,ds \le \int_t^\infty f(s)^q\,ds \Big\}.$$

Let

$$u_+(t) = \inf\{s \in A : s \ge t\}, \quad u_-(t) = \sup\{s \in A : s \le t\},$$
$$v_+(t) = \inf\{s \in B : s \ge t\}, \quad v_-(t) = \sup\{s \in B : s \le t\}.$$

Note that $f^p\chi_{(u_-(t),u_+(t))} \prec_{\mathrm{hd}} g^p\chi_{(u_-(t),u_+(t))}$ for $t \notin A$ and, therefore,

$$g(u_+(t) - 0) \le f(u_+(t) - 0).$$

Set $h_1(t) = g(u_+(t) - 0),\ t > 0$. By definition, $u_+(t) \ge t$ for all $t > 0$. Since g is decreasing, it follows that $h_1 \le g$. Set $h_2 = g\chi_B$. Since $u_+(t) = t$ for $t \in A$, it follows that $h_1 = g$ on A. Thus, $h_1 + h_2 \ge g\chi_A + g\chi_B \ge g$.

We claim that

$$\int_0^t \mu(s, h_1)^p\,ds \le \int_0^t f(s)^p\,ds.$$

Indeed, for $t \in A$, we have

$$\int_0^t h_1(s)^p\,ds \le \int_0^t g(s)^p\,ds \le \int_0^t f(s)^p\,ds.$$

For $t \notin A$, we have $h_1(s) = g(u_+(t))$ for all $s \in (u_-(t), u_+(t))$. Thus,

$$\int_0^t h_1(s)^p\,ds = \int_0^{u_-(t)} h_1(s)^p\,ds + \int_{u_-(t)}^t h_1(s)^p\,ds$$
$$\le \int_0^{u_-(t)} g(s)^p\,ds + \int_{u_-(t)}^t g(u_+(t))^p\,ds.$$

Since

$$\int_0^{u_-(t)} g(s)^p\,ds = \int_0^{u_-(t)} f(s)^p\,ds, \quad g(u_+(t)) \le f(u_+(t)),$$

it follows that

$$\int_0^t h_1(s)^p\,ds \le \int_0^{u_-(t)} f(s)^p\,ds + \int_{u_-(t)}^t f(u_+(t))^p\,ds \le \int_0^t f(s)^p\,ds.$$

Since $h_1 = \mu(h_1)$, the claim follows.

We claim that

$$\int_t^\infty \mu(s, h_2)^q\,ds \le \int_t^\infty f(s)^q\,ds.$$

For $t \in B$, we have

$$\int_t^\infty h_2(s)^q\,ds \le \int_t^\infty g(s)^q\,ds \le \int_t^\infty f(s)^q\,ds.$$

For $t \notin B$, we have

$$\int_t^\infty h_2(s)^q\,ds = \int_{v_+(t)}^\infty h_2(s)^q\,ds \le \int_{v_+(t)}^\infty g(s)^q\,ds = \int_{v_+(t)}^\infty f(s)^q\,ds \le \int_t^\infty f(s)^q\,ds.$$

In either case, we have

$$\int_t^\infty \mu(s,h_2)^q\,ds \le \int_t^\infty h_2(s)^q\,ds \le \int_t^\infty f(s)^q\,ds.$$

This proves the claim.

Setting

$$g_1 = \frac{h_1}{h_1+h_2}g, \quad g_2 = \frac{h_2}{h_1+h_2}g,$$

we complete the proof. □

Theorem 5.3. *Let $0 \le p < q \le \infty$ with either $p \ne 0$ or $q \ne \infty$. Let E be a quasi-Banach function space. Assume that there exist $c_{p,E}$ and $c_{q,E}$ in $\mathbb{R}_{>0}$ such that*:

(i) *Suppose $p \ne 0$. For any $f \in E$ and $g \in L_0 + L_p$, if $|g|^p \prec\prec_{\mathrm{hd}} |f|^p$, then $g \in E$ and $\|g\|_E \le c_{p,E}\|f\|_E$.*

(ii) *Suppose $q \ne \infty$. For any $f \in E$ and $g \in L_0 + L_q$, if $|g|^q \prec\prec_{\mathrm{tl}} |f|^q$, then $g \in E$ and $\|g\|_E \le c_{q,E}\|f\|_E$.*

Then E belongs to $\mathrm{Int}(L_p, L_q)$.

Proof. Assume that $p \ne 0$. Let us show that the first condition implies $E \subset L_p + L_\infty$. Indeed, assume the contrary and choose $f \in E$ such that $\mu(f)\,\chi_{(0,1)} \notin L_p$. Let

$$f_n = \min\Big\{\mu\Big(\frac{1}{n}, f\Big), \mu(f)\,\chi_{(0,1)}\Big\}, \quad n \ge 1.$$

Obviously, $\|f_n\|_E \le \|\mu(f)\,\chi_{(0,1)}\|_E \le \|f\|_E$. On contrary, $\|f_n\|_p^p \chi_{(0,1)} \prec\prec_{\mathrm{hd}} f_n^p$. By the first condition on E, we have $\|f_n\|_p \|\chi_{(0,1)}\|_E \le c_{p,E}\|f\|_E$. However, we have $\|f_n\|_p \uparrow \|\mu(f)\,\chi_{(0,1)}\|_p = \infty$. This contradiction shows that our initial assumption was incorrect. Thus, $E \subset L_p + L_\infty$.

A similar argument shows that the second condition implies $E \subset L_0 + L_q$. Thus, a combination of both conditions implies $E \subset L_p + L_q$.

Let T be a contraction on (L_p, L_q) and $f \in E$. To conclude the proof, it suffices to show that Tf belongs to E. First, note that

$$K(t, Tf, L_p, L_q) \le K(t, f, L_p, L_q).$$

Assume that $p > 0$ and $q < \infty$. Let $\alpha^{-1} = \frac{1}{p} - \frac{1}{q}$. By the Holmstedt formula for the K-functional (see [14]), there exists a constant $c_{p,q} > 0$ such that for any $t \in \mathbb{R}_{>0}$,

$$\Big(\int_0^{t^\alpha} \mu(s, Tf)^p\, ds\Big)^{1/p} + t\Big(\int_{t^\alpha}^\infty \mu(s, Tf)^q\, ds\Big)^{1/q} \le c_{p,q}\bigg(\Big(\int_0^{t^\alpha} \mu(s, f)^p\, ds\Big)^{1/p} + t\Big(\int_{t^\alpha}^\infty \mu(s, f)^q\, ds\Big)^{1/q}\bigg).$$

Hence, for any given $t > 0$, we have either

$$\int_0^{t^\alpha} \mu(s, Tf)^p\, ds \le \int_0^{t^\alpha} \mu(s, c_{p,q} f)^p\, ds$$

or

$$\int_{t^\alpha}^\infty \mu(s, Tf)^q\, ds \le \int_{t^\alpha}^\infty \mu(s, c_{p,q} f)^q\, ds.$$

By Lemma 5.2, one can write

$$\mu(Tf) = g_1 + g_2, \quad g_1^p \prec\prec_{\mathrm{hd}} (c_{p,q}\mu(f))^p, \quad g_2 \prec\prec_{\mathrm{tl}} (c_{p,q}\mu(f))^q. \tag{5-1}$$

By assumption, we have

$$\|g_1\|_E \le c_{p,E}\|f\|_E, \quad \|g_2\|_E \le c_{q,E}\|f\|_E.$$

By triangle inequality, we have

$$\|Tf\|_E \le c_{p,q,E}\|f\|_E.$$

Assume now that $p > 0$ and $q = \infty$. This case is simpler since by the Holmstedt formula (see [14]), there exists $c_p \in \mathbb{R}_{>0}$ such that for any $t \in \mathbb{R}_{>0}$,

$$\Big(\int_0^{t^p} \mu(s, Tf)^p\, ds\Big)^{1/p} \le c_p\Big(\int_0^{t^p} \mu(s, f)^p\, ds\Big)^{1/p}.$$

This means that $|Tf|^p \prec\prec_{\mathrm{hd}} |c_p f|^p$ so by assumption (1), Tf belongs to E and

$$\|Tf\|_E \le c_p\, c_{p,E}\|f\|_E.$$

The case of $p = 0$ and $q < \infty$ is given by Corollary 3.4. □

Theorem 1.1 claimed in the introduction compiles some results of this section.

Proof of Theorem 1.1. Assertion (a) is obtained by combining Theorems 5.1 and 5.3 with $q = \infty$.

Assertion (b) is derived similarly from Theorems 5.1 and 5.3 by applying them with $p = 0$.

Finally, using assertions (a) and (b) of Theorem 1.1, Theorems 5.1 and 5.3, for $0 < p < q < \infty$, one obtains Theorem 1.1(c). □

Remark 5.4. In the spirit of Corollary 3.4, we could have used a nonquantitative condition to deal with the case of $q = \infty$ in Theorem 5.3. Let E be a quasi-Banach function space and $p, q \in (0, \infty)$. This means that the following two conditions are equivalent:

(i) For any $f \in E,\ g \in L_p + L_\infty$,

$$|g|^p \prec\prec_{\mathrm{hd}} |f|^p \Rightarrow g \in E.$$

(ii) There exists $c > 0$ such that for any $f \in E,\ g \in L_p + L_\infty$,

$$|g|^p \prec\prec_{\mathrm{hd}} |f|^p \Rightarrow g \in E \quad \text{and} \quad \|g\|_E \le c\|f\|_E.$$

Similarly, the following two conditions are equivalent:

(i) For any $f \in E,\ g \in L_0 + L_q$,

$$|g|^q \prec\prec_{\mathrm{tl}} |f|^q \Rightarrow g \in E.$$

(ii) There exists $c > 0$ such that for any $f \in E,\ g \in L_0 + L_q$,

$$|g|^q \prec\prec_{\mathrm{tl}} |f|^q \Rightarrow g \in E \quad \text{and} \quad \|g\|_E \le c\|f\|_E.$$

6. Interpolation spaces for couples of ℓ^p-spaces

In this section, we show that our approach to the Lorentz–Shimogaki and Arazy–Cwikel theorems also applies to sequence spaces. We follow a structure similar to the previous sections, proving partition lemmas, then constructing bounded operators on couples (ℓ^p, ℓ^q) with suitable properties to finally conclude on the interpolation spaces of the couple (ℓ^p, ℓ^q). Additional arguments involving Boyd indices will be required to prove Theorem 1.2.

We identify sequences with bounded functions on $(0, \infty)$ which are almost constant on intervals of the form $(k, k+1),\ k \in \mathbb{Z}^+$ by

$$i : \ell^\infty \to L_\infty, \quad (u_k)_{k\in\mathbb{Z}^+} \mapsto \sum_{k=0}^{\infty} u_k\, \mathbf{1}_{(k,k+1)}.$$

An interpolation theorem for the couple (ℓ^p, ℓ^q). We start with a partition lemma playing, for sequence spaces, the role of Lemma 4.3.

Lemma 6.1. *Let $a = (a_n)_{n\in\mathbb{Z}^+},\ b = (b_n)_{n\in\mathbb{Z}^+}$ be two positive decreasing sequences such that $b \prec\prec_{\mathrm{tl}} a$. There exists a sequence $(\Delta_n)_{n\in\mathbb{Z}^+}$ of subsets of $\mathbb{Z}^+$ such that*:

(i) *For every $k \in \mathbb{Z}_+$, we have $|\{n \in \mathbb{Z}^+ : k \in \Delta_n\}| \le 3$.*

(ii) $\sum_{k\in\Delta_n} a_k \ge b_n$ *for any $n \in \mathbb{Z}^+$.*

Proof. Define $I = \{n \in \mathbb{Z}^+ : b_n > a_n\}$. For $n \notin I$, set $\Delta_n = \{n\}$.

For any $n \in \mathbb{Z}^+$, define

$$i_n = \sup\Big\{i : \sum_{k=i}^{\infty} a_k \ge \sum_{k=n}^{\infty} b_k\Big\}$$

and for $n \in I$, let $\Delta_n = \{i_n, \dots, i_{n+1}\}$.

From the definition of i_n, we have

$$\sum_{k \ge i_n} a_k \ge \sum_{k \ge n} b_k.$$

From the definition of i_{n+1}, we have

$$\sum_{k > i_{n+1}} a_k < \sum_{k \ge n+1} b_k.$$

Taking the difference of these inequalities, we infer that

$$\sum_{k \in \Delta_n} a_k \ge b_n.$$

This proves the second condition.

Note that since $b \prec\prec_{\mathrm{tl}} a$, we write $i_n \ge n$ for any $n \in \mathbb{Z}^+$. Hence, if $n \in I$, then $b_n > a_n \ge a_{i_{n+1}}$. Furthermore, by definition of i_{n+1}, we have

$$\sum_{k > i_{n+1}} a_k < \sum_{k=n+1}^{\infty} b_k, \quad \text{so} \quad \sum_{k=i_{n+1}}^{\infty} a_k < \sum_{k=n}^{\infty} b_k.$$

Hence, by definition of i_n, we have $i_{n+1} > i_n$ for $n \in I$.

Let us now check the first condition. Suppose there exist distinct numbers $n_1, n_2, n_3 \in I$ such that $k \in \Delta_{n_1}, \Delta_{n_2}, \Delta_{n_3}$. Without loss of generality, $n_1 < n_2 < n_3$. Since $k \in \Delta_{n_1}$, it follows that $k \le i_{n_1+1} \le i_{n_2}$. Since $k \in \Delta_{n_3}$, it follows that $k \ge i_{n_3} \ge i_{n_2+1}$. Hence, $i_{n_2+1} \le k \le i_{n_2}$ and, therefore, $i_{n_2+1} = i_{n_2}$. Since $n_2 \in I$, it follows $i_{n_2+1} > i_{n_2}$. This contradiction shows that $|\{n \in I : k \in \Delta_n\}| \le 2$. By definition, k also belongs to at most one set Δ_n, $n \notin I$. Consequently,

$$|\{n \in \mathbb{Z}^+ : k \in \Delta_n\}| \le 3. \qquad \square$$

From the partition lemma, we deduce an operator lemma similar to Lemma 4.6. It extends Proposition 2 in [3], which is established there for the special case $p = 1$ by a completely different method.

Lemma 6.2. *Let $p \ge 1$. Let $a, b \in \ell^p$ such that $|b|^p \prec\prec_{\mathrm{tl}} |a|^p$. Then there exists an operator $T : \ell^p \to \ell^p$ such that:*

(i) $T(a) = b$.

(ii) $\|T\|_{p \to p} \le 3^{1/p}$ *and* $\|T\|_{0 \to 0} \le 3$.

Proof. We can assume that both sequences are nonnegative and decreasing. Apply Lemma 6.1 to $|a|^p$ and $|b|^p$. For every $n \in \mathbb{Z}^+$, choose a linear form φ_n on ℓ^p of norm less than 1, supported on Δ_n and such that $\varphi_n(a) = \varphi_n(a\mathbf{1}_{\Delta_n}) = b_n$. Define

$$T : x \in \ell^p \mapsto (\varphi_n(x))_{n\in\mathbb{Z}^+}.$$

By construction, $T(a) = b$. Let us check the norm estimates. Let $x \in \ell^p$, then

$$\begin{aligned}\|T(x)\|_p^p = \sum_{n\in\mathbb{Z}^+} |\varphi_n(x)|^p = \sum_{n\in\mathbb{Z}^+} |\varphi_n(x\mathbf{1}_{\Delta_n})|^p \\ \le \sum_{n\in\mathbb{Z}^+}\sum_{k\in\Delta_n} |x_k|^p = \sum_{k\in\mathbb{Z}^+} |\{n : k \in \Delta_n\}|\,|x_k|^p \le 3\|x\|_p^p.\end{aligned}$$

The second estimate is clear, using once again the fact that an integer k belongs to at most three Δ_n's. □

The following remarks were communicated to the authors by Cwikel and Nilsson.

Remark 6.3. A bounded linear operator on ℓ^p, $p \le 1$ extends automatically to a bounded linear operator on ℓ^1.

Proof. Indeed, let $(e_n)_{n\in\mathbb{Z}^+}$ be the canonical basis of ℓ^∞. Let T be a contraction on ℓ^p, $p < 1$. Then by Hölder's inequality $\|T(e_n)\|_1 \le \|T(e_n)\|_p \le \|T\|_{p\to p}$. By the triangle inequality, for any finite sequence $a = (a_n)_{n\in\mathbb{Z}^+}$,

$$\|T(a_n)\|_1 \le \sum_{n\in\mathbb{Z}^+} |a_n|\|T(e_n)\|_1 \le \|a\|_1\|T\|_{p\to p}.$$

Hence, T extends to a contraction on ℓ^1. □

Remark 6.4. The condition $p \ge 1$ in Lemma 6.2 is necessary.

Proof. Let us show that Lemma 6.2 cannot be true for $p < 1$. Assume by contradiction that there exists $c > 0$ such that for any finite sequences a and b in ℓ^p such that $|b|^p \prec\prec_{\mathrm{tl}} |a|^p$ there exists T with $\|T\|_{p\to p} \le c$ and $T(a) = b$. By Remark 6.3 above, we also have $\|T\|_{1\to 1} \le c$. In particular, $\|b\|_1 \le c\|a\|_1$. By considering $b = e_1$ and $a = \frac{1}{N^{1/p}} \sum_{i=1}^N e_i$ for N large enough, one obtains a contradiction. □

We will not prove a sequence version of Lemma 4.5 to avoid the repetition of too many similar arguments. Fortunately, the expected result already appears in the literature, see [10, Theorem 3].

Lemma 6.5. *Let $p > 0$. Let $a, b \in \ell^\infty$ such that $|b|^p \prec\prec_{\mathrm{hd}} |a|^p$. Then there exists an operator $T : \ell^p \to \ell^p$ such that*:

(i) $T(a) = b$.

(ii) $\|T\|_{p\to p} \le 8^{1/p}$ *and* $\|T\|_{\infty\to\infty} \le 2^{1/p}$.

We conclude this subsection with a new interpolation theorem.

Theorem 6.6. *Let $p < q \in (0, \infty]$ such that $q \geq 1$. Let E be a quasi-Banach sequence space. Then E belongs to $Int(\ell^p, \ell^q)$ if and only if there exists $c_{p,E}$ and $c_{q,E}$ in $\mathbb{R}_{>0}$ such that*:

(i) *For any $u \in E$ and $v \in \ell^\infty$, if $|v|^p \prec\prec_{\mathrm{hd}} |u|^p$, then $v \in E$ and $\|v\|_E \leq c_{p,E}\|u\|_E$.*

(ii) *Suppose $q < \infty$. For any $u \in E$ and $v \in \ell^\infty$, if $|v|^q \prec\prec_{\mathrm{tl}} |u|^q$, then $v \in E$ and $\|v\|_E \leq c_{q,E}\|u\|_E$.*

Proof. The proof of the "only if" implication is identical to the proof of Theorem 5.1 using Lemmas 6.2 and 6.5 instead of Lemmas 4.5 and 4.4. The "if" implication is given by [8, Theorem 4.7]. □

Upper Boyd index. Let us now recall the definition of the upper Boyd index, in the case of sequence spaces. For any $n \in \mathbb{N}$ define the dilation operator

$$D_n : \ell^\infty \to \ell^\infty, \quad (u_k)_{k\in\mathbb{Z}^+} \mapsto (u_{\lfloor k/n \rfloor})_{k\in\mathbb{Z}^+}.$$

Let E be a symmetric function space. Define the Boyd index associated to E by

$$\beta_E = \lim_{k\to\infty} \frac{\log\|D_k\|_{E\to E}}{\log k}.$$

Note that since E is a quasi-Banach space, $\beta_E < \infty$.

In the next proposition, we relate the upper Boyd index to an interpolation property. We follow [23, Theorem 2].

Proposition 6.7. *Assume that E is a quasi-Banach symmetric sequence space. Let $p < 1/\beta_E$. There exists a constant C such that for any $u \in E$ and $v \in \ell^\infty$, satisfying $|v|^p \prec\prec_{\mathrm{hd}} |u|^p$, we have $v \in E$ and $\|v\|_E \leq C\|u\|_E$.*

Define the map $V : \ell_\infty \to \ell_\infty$ by setting

$$Vu = \sum_{n=0}^{\infty} 2^{-n} D_{2^n} u$$

and the map $C : \ell_\infty \to \ell_\infty$ by

$$(Cu)(n) = \frac{1}{n+1} \sum_{i=0}^{n} u_n.$$

Lemma 6.8. *If $p < 1/\beta_E$, then*

$$\|(V(u^p))^{1/p}\|_E \leq c_{p,E}\|u\|_E, \quad 0 \leq u \in E.$$

Proof. Let E_p be the p-concavification of E, that is,

$$E_p = \{f : |f|^{1/p} \in E\}, \quad \|f\|_{E_p} = \||f|^{1/p}\|_E^p.$$

Obviously, E_p is a quasi-Banach space. Apply the Aoki–Rolewicz theorem to the space E_p and fix $q = q_{p,E} > 0$ such that

$$\Big\|\sum_{n\geq 0} x_n\Big\|_{E_p}^q \leq C_{p,E}\sum_{n\geq 0}\|x_n\|_{E_p}^q.$$

For every $u \in E$, we have

$$\begin{aligned}\|(V(u^p))^{1/p}\|_E^{qp} = \|V(u^p)\|_{E_p}^q &= \Big\|\sum_{n=0}^{\infty}\frac{1}{2^n}(D_{2^n}u)^p\Big\|_{E_p}^q\\ &\leq C_{p,E}\sum_{n=0}^{\infty}\Big\|\frac{1}{2^n}(D_{2^n}u)^p\Big\|_{E_p}^q\\ &= C_{p,E}\sum_{n=0}^{\infty}2^{-nq}\|D_{2^n}u\|_E^{qp}.\end{aligned}$$

Let $r \in (p, \beta_E^{-1})$. By the definition of β_E, there exists $c_{p,E} > 0$ such that $\|D_n\|_{E\to E} \leq c_{p,E}\, n^{1/r}$ for any $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\|(V(u^p))^{1/p}\|_E^{qp} &\leq C_{p,E}\cdot c_{p,E}^q\cdot\sum_{n=0}^{\infty}2^{-nq}\,2^{nqp/r}\|u\|_E^{qp}\\ &= C_{p,E}\cdot c_{p,E}^q\cdot\frac{2^q}{2^q-2^{qp/r}}\cdot\|u\|_E^{qp}.\end{aligned}$$

□

Lemma 6.9. *If $x = \mu(x)$, then $Cx \leq 3Vx$ for every $x \in \ell_\infty$.*

Proof. Let $k \geq 0$. Since x is decreasing, it follows that

$$(Cx)(2^k-1) = \frac{1}{2^k}\Big(x(0) + \sum_{i=0}^{k-1}\sum_{j=2^i}^{2^{i+1}-1}x(j)\Big) \leq \frac{1}{2^k}\Big(x(0)+\sum_{i=0}^{k-1}2^i\,x(2^i)\Big).$$

On the other hand, we have

$$\begin{aligned}(Vx)(2^{k+1}-1) &= \sum_{n\geq 0}2^{-n}x\Big(\Big\lfloor\frac{2^{k+1}-1}{2^n}\Big\rfloor\Big)\\ &= \sum_{n=0}^{k}2^{-n}x(2^{k+1-n}-1) + \sum_{n=k+1}^{\infty}2^{-n}x(0)\\ &= \frac{1}{2^k}\Big(x(0)+\sum_{i=0}^{k}2^i\,x(2^{i+1}-1)\Big).\end{aligned}$$

Again using the fact that x is decreasing, we obtain

$$\sum_{i=0}^{k-1} 2^i x(2^i) = x(1) + \sum_{i=0}^{k-2} 2^{i+1} x(2^{i+1})$$
$$\leq x(1) + 2\sum_{i=0}^{k-2} 2^i x(2^{i+1}-1) \leq 3\sum_{i=0}^{k} 2^i x(2^{i+1}-1).$$

Combining the three previous inequalities, we have just shown that for any $k \geq 0$,

$$(Cx)(2^k - 1) \leq 3(Vx)(2^{k+1} - 1).$$

Now let $n \geq 0$ and choose k such that $n \in [2^k - 1, 2^{k+1} - 1]$. Since Cx and Vx are decreasing, we have

$$(Cx)(n) \leq (Cx)(2^k - 1) \leq 3(Vx)(2^{k+1} - 1) \leq 3(Vx)(n). \qquad \square$$

Proof of Proposition 6.7. Without loss of generality, $u = \mu(u)$ and $v = \mu(v)$. Since $v^p \prec\prec_{\mathrm{hd}} u^p$, it follows that

$$|v|^p \leq C(|v|^p) \leq C(|u|^p) \leq 3V(|u|^p),$$

where we used Lemma 6.9 to obtain the last inequality. By Lemma 6.8, we have

$$\|v\|_E \leq 3^{1/p} \|(V(|u|^p))^{1/p}\|_E \leq c_{p,E}\|u\|_p. \qquad \square$$

We are now ready to deliver a complete resolution of the conjecture stated by Levitina et al. in [20].

Proof of Theorem 1.2. Let E be a quasi-Banach sequence space. Let $q \geq 1$. Recall that Theorem 1.2 states that the following two conditions are equivalent:

(a) There exists $p < q$ such that E is an interpolation space for the couple (ℓ^p, ℓ^q).

(b) There exists $c > 0$ such that for any $u \in E$ and $|v|^q \prec\prec_{\mathrm{tl}} |u|^q$, then $v \in E$ and $\|v\|_E \leq c\|u\|_E$.

Note that by Lemma 2.9, we may assume that E is a symmetric space.

(a) $\Rightarrow$ (b). This is immediate by Theorem 6.6.

(b) $\Rightarrow$ (a). Let $p < 1/\beta_E$. By Proposition 6.7, for any sequence $u \in E$ and $v \in \ell^\infty$, if $|v|^p \prec\prec_{\mathrm{hd}} |u|^p$, $v \in E$ and $\|v\|_E \leq c_{p,E}\|u\|_E$. Applying Theorem 6.6 for indices p and q, we obtain that E belongs to $\mathrm{Int}(\ell^p, \ell^q)$. $\square$

References

[1] J. Arazy and M. Cwikel, "A new characterization of the interpolation spaces between L^p and L^q", *Math. Scand.* **55**:2 (1984), 253–270. MR Zbl

[2] S. V. Astashkin, "Interpolation of operators in quasinormed groups of measurable functions", *Sibirsk. Mat. Zh.* **35**:6 (1994), 1215–1222. MR Zbl

[3] S. V. Astashkin, "A version of Calderón–Mityagin theorem for the class of rearrangement invariant groups", *Nonlinear Anal.* **200** (2020), art. id. 112063. MR Zbl

[4] S. V. Astashkin and L. Maligranda, "Interpolation between L_1 and L_p, $1 < p < \infty$", *Proc. Amer. Math. Soc.* **132**:10 (2004), 2929–2938. MR Zbl

[5] S. V. Astashkin, M. Cwikel, and P. G. Nilsson, "Arazy–Cwikel and Calderón–Mityagin type properties of the couples (ℓ^p, ℓ^q), $0 \le p < q \le \infty$", *Ann. Mat. Pura Appl.* (4) **202**:4 (2023), 1643–1678. MR Zbl

[6] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundl. Math. Wissen. **223**, Springer, Berlin, 1976. MR Zbl

[7] M. Š. Braverman and A. A. Mekler, "The Hardy–Littlewood property for symmetric spaces", *Sibirsk. Mat. Ž.* **18**:3 (1977), 522–540, 717. MR Zbl

[8] L. Cadilhac, "Majorization, interpolation and noncommutative Khinchin inequalities", *Stud. Math.* **258**:1 (2021), 1–26. Zbl

[9] A.-P. Calderón, "Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz", *Studia Math.* **26** (1966), 273–299. MR Zbl

[10] M. Cwikel, "Monotonicity properties of interpolation spaces, II", *Ark. Mat.* **19**:1 (1981), 123–136. MR Zbl

[11] M. Cwikel and P. G. Nilsson, "An alternative characterization of normed interpolation spaces between ℓ^1 and ℓ^q", preprint, 2017. Zbl arXiv 1711.04809

[12] M. Cwikel and J. Peetre (editors), *Interpolation spaces and allied topics in analysis*, Lecture Notes in Mathematics **1070**, Springer, Berlin, 1984. MR Zbl

[13] P. G. Dodds, T. K. Dodds, and F. A. Sukochev, "On p-convexity and q-concavity in non-commutative symmetric spaces", *Integral Equations Operator Theory* **78**:1 (2014), 91–114. MR Zbl

[14] T. Holmstedt, "Interpolation of quasi-normed spaces", *Math. Scand.* **26** (1970), 177–199. MR Zbl

[15] J. Huang and F. Sukochev, "Interpolation between $L_0(\mathcal{M}, \tau)$ and $L_\infty(\mathcal{M}, \tau)$", *Math. Z.* **293**:3-4 (2019), 1657–1672. MR Zbl

[16] H. Hudzik and L. Maligranda, "An interpolation theorem in symmetric function F-spaces", *Proc. Amer. Math. Soc.* **110**:1 (1990), 89–96. MR Zbl

[17] N. J. Kalton, "Convexity conditions for nonlocally convex lattices", *Glasgow Math. J.* **25**:2 (1984), 141–152. MR Zbl

[18] N. Kalton and S. Montgomery-Smith, "Interpolation of Banach spaces", pp. 1131–1175 in *Handbook of the geometry of Banach spaces*, vol. 2, North-Holland, 2003. MR Zbl

[19] S. G. Kreĭn, Y. I. Petunīn, and E. M. Semënov, *Interpolation of linear operators*, Translations of Mathematical Monographs **54**, American Mathematical Society, Providence, RI, 1982. MR Zbl

[20] G. Levitina, F. Sukochev, and D. Zanin, "Cwikel estimates revisited", *Proc. Lond. Math. Soc.* (3) **120**:2 (2020), 265–304. MR Zbl

[21] G. G. Lorentz and T. Shimogaki, "Interpolation theorems for the pairs of spaces (L^p, L^∞) and (L^1, L^q)", *Trans. Amer. Math. Soc.* **159** (1971), 207–221. MR Zbl

[22] B. S. Mityagin, "An interpolation theorem for modular spaces", *Mat. Sb.* (*N.S.*) **66(108)**:4 (1965), 473–482. MR Zbl

[23] S. J. Montgomery-Smith, "The Hardy operator and Boyd indices", pp. 359–364 in *Interaction between functional analysis, harmonic analysis, and probability* (Columbia, MO, 1994), Lecture Notes in Pure and Appl. Math. **175**, Dekker, New York, 1996. MR Zbl

[24] J. Peetre and G. Sparr, "Interpolation of normed abelian groups", *Ann. Mat. Pura Appl.* (4) **92** (1972), 217–262. MR Zbl

[25] N. Popa, "Rearrangement invariant p-spaces on [0, 1], $0 < p < 1$, containing a complemented subspace isomorphic to l_2", *Rev. Roumaine Math. Pures Appl.* **28**:2 (1983), 153–160. MR Zbl

[26] A. A. Sedaev, E. M. Semenov, and F. A. Sukochev, "Fully symmetric function spaces without an equivalent Fatou norm", *Positivity* **19**:3 (2015), 419–437. MR Zbl

[27] G. Sparr, "Interpolation of weighted L_p-spaces", *Studia Math.* **62**:3 (1978), 229–271. MR Zbl

[28] F. Sukochev and D. Zanin, "Orbits in symmetric spaces", *J. Funct. Anal.* **257**:1 (2009), 194–218. MR Zbl

Received December 15, 2021. Revised March 6, 2024.

LÉONARD CADILHAC
SORBONNE UNIVERSITÉ
INSTITUT DE MATHÉMATIQUES DE JUSSIEU
PARIS
FRANCE
cadilhac@imj-prg.fr

FEDOR SUKOCHEV
SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES
KENSINGTON
AUSTRALIA
f.sukochev@unsw.edu.au

DMITRIY ZANIN
SCHOOL OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NEW SOUTH WALES
KENSINGTON
AUSTRALIA
d.zanin@unsw.edu.au

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2024 is US $645/year for the electronic version, and $875/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 328 No. 2 February 2024

Hankel operators on $L^p(\mathbb{R}_+)$ and their p-completely bounded multipliers 193
LORIS ARNOLD, CHRISTIAN LE MERDY and SAFOURA ZADEH

Stable functorial equivalence of blocks 217
SERGE BOUC and DENIZ YILMAZ

Lorentz–Shimogaki and Arazy–Cwikel theorems revisited 227
LÉONARD CADILHAC, FEDOR SUKOCHEV and DMITRIY ZANIN

Finite axiomatizability of the rank and the dimension of a pro-π group 255
MARTINA CONTE and BENJAMIN KLOPSCH

Elliptic genus and string cobordism at dimension 24 275
FEI HAN and RUIZHI HUANG

The domination monoid in henselian valued fields 287
MARTIN HILS and ROSARIO MENNUNI

Inverse semigroup from metrics on doubles, III: Commutativity and (in)finiteness of idempotents 325
VLADIMIR MANUILOV

The number of $\mathbb{F}_q$-points on diagonal hypersurfaces with monomial deformation 339
DERMOT MCCARTHY

Deformation of pairs and semiregularity 361
TAKEO NISHINOU