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FINITE AXIOMATIZABILITY OF THE RANK AND THE DIMENSION OF A PRO-π GROUP

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In memory of Avinoam Mann

The Prüfer rank $\mathrm{rk}(G)$ of a profinite group G is the supremum, across all open subgroups H of G, of the minimal number of generators $\mathrm{d}(H)$. It is known that, for any given prime p, a profinite group G admits the structure of a p-adic analytic group if and only if G is virtually a pro-p group of finite rank. The dimension $\dim G$ of a p-adic analytic profinite group G is the analytic dimension of G as a p-adic manifold; it is known that $\dim G$ coincides with the rank $\mathrm{rk}(U)$ of any uniformly powerful open pro-p subgroup U of G.

Let π be a finite set of primes, let $r \in \mathbb{N}$ and let $\mathbf{r} = (r_p)_{p\in\pi}$, $\mathbf{d} = (d_p)_{p\in\pi}$ be tuples in $\{0, 1, \ldots, r\}$. We show that there is a single sentence $\sigma_{\pi,r,\mathbf{r},\mathbf{d}}$ in the first-order language of groups such that for every pro-π group G the following are equivalent: (i) $\sigma_{\pi,r,\mathbf{r},\mathbf{d}}$ holds true in the group G, that is, $G \models \sigma_{\pi,r,\mathbf{r},\mathbf{d}}$; (ii) G has rank r and, for each $p \in \pi$, the Sylow pro-p subgroups of G have rank r_p and dimension d_p.

Loosely speaking, this shows that, for a pro-π group G of bounded rank, the precise rank of G as well as the ranks and dimensions of the Sylow subgroups of G can be recognized by a single sentence in the basic first-order language of groups.

1. Introduction

Nies, Segal and Tent [Nies et al. 2021] carried out an investigation of the model-theoretic concept of finite axiomatizability in the context of profinite groups. For instance, a profinite group G is finitely axiomatizable within a class $\mathcal{C}$ of profinite groups, with respect to the first-order language $\mathcal{L}_{\mathrm{gp}}$ of groups, if there is a sentence $\psi_{G,\mathcal{C}}$ in $\mathcal{L}_{\mathrm{gp}}$ such that the following holds: a profinite group H in $\mathcal{C}$ is isomorphic to G if and only if $\psi_{G,\mathcal{C}}$ holds true in H, in symbols $H \models \psi_{G,\mathcal{C}}$. More generally,

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one takes interest in whether specific properties or invariants of profinite groups, again within a given class $\mathcal{C}$, can be detected uniformly by a single sentence in $\mathcal{L}_{\mathrm{gp}}$.

Our main interest is in finitely generated profinite groups. Nikolov and Segal [2007] established that such groups are strongly complete; loosely speaking, this means that the topology of a finitely generated profinite group is already predetermined by the abstract group structure. Jarden and Lubotzky [2008] used Nikolov and Segal's finite width results for certain words to prove that every finitely generated profinite group is "first-order rigid", i.e., determined up to isomorphism by its first-order theory, within the class of profinite groups. By restricting to *finite* axiomatizability, we probe for more delicate first-order properties within suitable classes of finitely generated profinite groups.

In this paper we focus on the class of profinite groups of finite Prüfer rank, from now on "rank" for short. This invariant is connected to, but not to be confused with, the minimal number of generators: the rank of a profinite group G is defined as

$$\mathrm{rk}(G) = \sup\{\mathrm{d}(H) \mid H \le_{\mathrm{o}} G\} = \sup\{\mathrm{d}(H) \mid H \le_{\mathrm{c}} G\},$$

where $\mathrm{d}(H)$ denotes the minimal number of generators of a topological group H and, as indicated, H runs over all open or all closed subgroups of G. It is not difficult to see that the rank of G is the supremum of the ranks of its finite continuous quotients, i.e., $\mathrm{rk}(G) = \sup\{\mathrm{rk}(G/N) \mid N \trianglelefteq_{\mathrm{o}} G\}$. The rank plays a central role in the structure theory of p-adic Lie groups. It is known that, for any given prime p, a profinite group G admits the structure of a p-adic analytic group if and only if G is virtually a pro-p group of finite rank. The dimension $\dim G$ of a p-adic analytic profinite group G is the analytic dimension of G as a p-adic manifold; in fact, $\dim G \le \mathrm{rk}(G)$ and $\dim G$ coincides with the rank $\mathrm{rk}(U)$ of any uniformly powerful open pro-p subgroup U of G. Further details and related results about p-adic analytic pro-p groups can be found in [Dixon et al. 1999]; the concise introduction [Klopsch 2011] summarizes key aspects of the theory.

Loosely speaking, our aim is to show that, for every finite set of primes π, the precise rank r as well as the ranks $\mathbf{r} = (r_p)_{p\in\pi}$ and dimensions $\mathbf{d} = (d_p)_{p\in\pi}$ of the Sylow pro-p subgroups of any pro-π group G of finite rank can be recognized by a single sentence $\sigma_{\pi,r,\mathbf{r},\mathbf{d}}$ in the first-order language of groups $\mathcal{L}_{\mathrm{gp}}$. The starting point for our investigation is Proposition 5.1 in [Nies et al. 2021] which states: Given $r \in \mathbb{N}$, there is an $\mathcal{L}_{\mathrm{gp}}$-sentence $\rho_{p,r}$ such that for every pro-p group G, the following implications hold:

$$\mathrm{rk}(G) \le r \quad \Longrightarrow \quad G \models \rho_{p,r} \quad \Longrightarrow \quad \mathrm{rk}(G) \le r(2 + \log_2(r)).$$

Our first theorem both strengthens and generalizes this result. The p-rank $\mathrm{rk}_p(G)$ of a profinite group G is the common rank of all Sylow pro-p subgroups of G. A sentence ϕ in $\mathcal{L}_{\mathrm{gp}}$ is called an $\exists\forall\exists$-sentence if it results from a quantifier-free formula ϕ_0 by means of a sequence of existential, universal and existential quantifications (in this order), rendering the free variables of ϕ_0 to be bound in ϕ; compare with Example 3.1.

Theorem 1.1. *Let π be a finite set of primes. Let $r \in \mathbb{N}$ and let $\mathbf{r} = (r_p)_{p\in\pi}$ be a tuple in $\{0, 1, \dots, r\}$. Then there exists an $\exists\forall\exists$-sentence $\varrho_{\pi,r,\mathbf{r}}$ in $\mathcal{L}_{\mathrm{gp}}$ such that, for every pro-π group G, the following are equivalent*:

(i) $\mathrm{rk}(G) = r$, *and* $\mathrm{rk}_p(G) = r_p$ *for every* $p \in \pi$.

(ii) $\varrho_{\pi,r,\mathbf{r}}$ *holds in* G, *i.e.*, $G \models \varrho_{\pi,r,\mathbf{r}}$.

It is no coincidence that the sentences $\varrho_{\pi,r,\mathbf{r}}$ which we manufacture to prove the theorem depend on the given set of primes π. A standard ultraproduct construction reveals that, for every *infinite* set of primes $\tilde{\pi}$ and $r \in \mathbb{N}$, there is no $\mathcal{L}_{\mathrm{gp}}$-sentence $\vartheta_{\tilde{\pi},r}$ which could identify, uniformly across $p \in \tilde{\pi}$, among pro-p groups G those with rank $\mathrm{rk}(G) = r$; see Proposition 3.3.

In addition to Theorem 1.1 we establish a corresponding theorem which concerns the dimensions of the Sylow subgroups of a profinite group of finite rank.

Theorem 1.2. *Let π be a finite set of primes. Let $r \in \mathbb{N}$ and let $\mathbf{d} = (d_p)_{p\in\pi}$ be a tuple in $\{0, 1, \dots, r\}$. Then there exists an $\exists\forall\exists$-sentence $\tau_{\pi,r,\mathbf{d}}$ in $\mathcal{L}_{\mathrm{gp}}$ such that, for every pro-π group G with* $\mathrm{rk}(G) = r$, *the following are equivalent*:

(i) *For every $p \in \pi$, the Sylow pro-p subgroups of G have dimension d_p.*

(ii) $\tau_{\pi,r,\mathbf{d}}$ *holds in* G, *i.e.*, $G \models \tau_{\pi,r,\mathbf{d}}$.

In combination, the two theorems provide the first-order sentences $\sigma_{\pi,r,\mathbf{r},\mathbf{d}}$ with the properties promised above. It is remarkable that such sentences exist in the basic language $\mathcal{L}_{\mathrm{gp}}$ of groups. In connection with p-adic analytic profinite groups, it is often necessary to employ suitably expanded languages in order to capture part of the topological or analytic structure; compare with [Macpherson and Tent 2016]. We do not need to enlarge the language at all. Moreover, the complexity of $\sigma_{\pi,r,\mathbf{r},\mathbf{d}}$ remains within three alternations of $\exists$- and $\forall$-quantifiers, even though the sentences that we manufacture depend strongly on the given set of primes π.

As we will show, the proofs of Theorems 1.1 and 1.2 reduce, in a certain sense, to the simpler setting of pronilpotent pro-π groups, termed $\mathcal{C}_\pi$-groups by [Nies et al. 2021, Section 5]. We recall that, even in the pronilpotent case, Sylow subgroups are not in general definable and there is no standard reduction to pro-p groups; this can be seen from relative quantifier elimination results (down to positive primitive formulas) for modules over rings; see [Prest 1988, Sections 2.4 and 2.$\mathbb{Z}$]. Part of our task is to develop appropriate tools to by-pass this obstacle.

Key to our approach for proving Theorems 1.1 and 1.2 are purely group-theoretic considerations leading to Theorem 2.1 and its corollary, about profinite groups which are virtually pronilpotent and of finite rank. Specialising to the setting of finite nilpotent groups to ease the exposition at this point, we can formulate the central insight as follows.

Theorem 1.3. *Let G be a finite nilpotent group of rank $r = \mathrm{rk}(G)$. Then*

$$\mathrm{rk}(G) = \mathrm{rk}\big(G/\Phi^{j(r)}(G)\big) \quad \textit{for } j(r) = 2r + \lceil \log_2(r) \rceil + 2,$$

where $\Phi^{j(r)}(G)$ denotes the $j(r)$-th iterated Frattini subgroup of G.

It is an open problem to identify, if at all possible, even smaller canonical quotients which witness the full rank of a finite nilpotent group.

Following a suggestion of González-Sánchez, we derive from a result of Héthelyi and Lévai [2003] a new description of the dimension of a finitely generated powerful pro-p group; this is useful for establishing Theorem 1.2, but also of independent interest.

Theorem 1.4. *Let G be a finitely generated powerful pro-p group with torsion subgroup T, and let $\Omega_{\{1\}}(G) = \{g \in G \mid g^p = 1\}$ denote the set of all elements of order 1 or p in G. Then*

$$\dim(G) = \mathrm{d}(G) - \log_p |\Omega_{\{1\}}(G)| = \mathrm{d}(G) - \mathrm{d}(T).$$

With a view toward possible future investigations, we add a final comment and a question. Naturally one wonders whether "being of finite rank" per se can be captured by a suitable first-order sentence. Results of Feferman and Vaught [1959] imply that, even for a fixed prime p, there is no set $\mathcal{T}_p$ of $\mathcal{L}_{\mathrm{gp}}$-sentences (and in particular no single sentence) which identifies among the collection of all pro-p groups those that possess finite rank. Indeed, the class of pro-p groups of finite rank is closed under taking finite cartesian products, but an infinite cartesian product of nontrivial pro-p groups of finite rank is not even finitely generated. Therefore [Feferman and Vaught 1959, Corollary 6.7] shows that no $\mathcal{T}_p$ with the desired property exists. However, a modified question suggests itself. Given $d \geq 2$, is there a set $\mathcal{T}_{p,d}$ of $\mathcal{L}_{\mathrm{gp}}$-sentences (possibly a single sentence) such that the following holds for pro-p groups G with $\mathrm{d}(G) \leq d$: the group G has finite rank if and only if G satisfies $\mathcal{T}_{p,d}$?

Remark. Our proofs for Theorems 1.1 and 1.2 involve results of Lucchini [1997] and an observation of Mazurov [1994] which currently rely on the classification of finite simple groups. However, in suitable circumstances, e.g., if we restrict attention to prosoluble groups, the required ingredients are known to hold without use of the classification; compare with [Lucchini 1989, Section 5]. If $2 \notin \pi$, the Odd Order Theorem guarantees that all pro-π groups are prosoluble.

Organization and Notation. In Section 2 we prove Theorem 2.1 and its corollary, which specialize to Theorem 1.3. In Example 2.3 we discuss limitations of our strategy; Proposition 3.3 shows that Theorem 2.1 does not generalize to groups involving infinitely many primes. In Section 3 we establish Theorem 1.1. In Section 4 we prove Theorem 1.4 and deduce Theorem 1.2.

Our notation is mostly standard and in line with current practice. For instance, $\mathrm{Z}(G)$ denotes the centre of a group G, and C_n denotes a cyclic group of order n. The meaning of possibly less familiar terms, such as $\Phi(G)$ for the Frattini subgroup and $\Phi_p(G)$ for the p-Frattini subgroup of a group G, are explained at their first occurrence. We deal exclusively with profinite groups. Accordingly, notions such as the Frattini subgroup, the commutator subgroup or the subgroup generated by a given set are tacitly understood in the topological sense: in each case we mean the topological closure of the corresponding abstract subgroup. Basic model-theoretic concepts which are employed without further reference are covered by standard texts such as [Hodges 1993].

2. Detecting the rank in bounded quotients

Every compact p-adic analytic group G has finite rank and contains an open normal powerful pro-p subgroup F. Since F is a pro-p group, its Frattini subgroup $\Phi(F)$ coincides with $[F, F]F^p$ and $F/\Phi(F)$ is elementary abelian. Since F is powerful, we know that $\mathrm{rk}(F) = \mathrm{d}(F) = \mathrm{rk}(F/\Phi(F))$; see [Dixon et al. 1999, Theorem 3.8]. Furthermore, the iterated Frattini series $\Phi^j(F)$, $j \in \mathbb{N}$, of F coincides with both the lower p-series and the iterated p-power series of F. It provides a base of neighbourhoods for 1 in G consisting of open normal subgroups. Consequently, the rank of G is given by

$$\mathrm{rk}(G) = \sup\{\mathrm{rk}(G/\Phi^j(F)) \mid j \in \mathbb{N}\} = \max\{\mathrm{rk}(G/\Phi^j(F)) \mid j \in \mathbb{N}\};$$

in other words, $\mathrm{rk}(G)$ is the terminal value of the nondecreasing, eventually constant sequence $\mathrm{rk}(G/\Phi^j(F))$, $j \in \mathbb{N}$.

It is natural to look for an upper bound for the smallest $j \in \mathbb{N}$ such that $\mathrm{rk}(G) = \mathrm{rk}(G/\Phi^j(F))$, a bound that is, as far as possible, independent of p and any special features of the pair $F \le G$. Based on our current knowledge, the strongest possible outcome could be that $\mathrm{rk}(G) = \mathrm{rk}(G/\Phi(F))$ holds without any exceptions. More modestly, one can ask for weaker bounds, possibly contingent on additional information regarding $\mathrm{rk}(G)$.

We establish a result of the latter kind, which applies more generally to profinite groups G of finite rank that admit a pronilpotent open normal subgroup F. We recall that the p-rank $\mathrm{rk}_p(G)$ of a profinite group G is simply the rank $\mathrm{rk}(P)$ of a Sylow pro-p subgroup P of G. Furthermore, we write $\Phi_p(G) = [G, G]G^p$ for the p-Frattini subgroup of G; the p-Frattini quotient $G/\Phi_p(G)$ is the largest elementary abelian pro-p quotient of the profinite group G.

Theorem 2.1. *Let $R \in \mathbb{N}$. Suppose that the profinite group G has an open normal subgroup $F \trianglelefteq_{\mathrm{o}} G$ which is pronilpotent and such that each Sylow subgroup of F is powerful.*

(i) *For every prime p such that $\mathrm{rk}_p(G) \le R$, the p-rank satisfies*

$$\mathrm{rk}_p(G) = \mathrm{rk}_p\big(G/\Phi^{2R+1}(F)\big).$$

(ii) *If $\mathrm{rk}(G) \le R$, then*

$$\mathrm{rk}(G) = \mathrm{rk}\big(G/\Phi^{2R+1}(F)\big).$$

Proof. It is convenient to write $F_i = \Phi^i(F)$ for $i \in \mathbb{N}$.

(i) Let p be a prime such that $r_p = \mathrm{rk}_p(G) \le R$. We show that $r_p = \mathrm{rk}_p(G/F_{2R+1})$. Since F is pronilpotent, its Hall pro-p' subgroup P' is normal in G; compare with [Ribes and Zalesskii 2010, Section 2.3]. Working modulo P', we may assume without loss of generality that F is a powerful pro-p group. In this situation G is virtually a pro-p group. Clearly, we have $r_p \ge \mathrm{rk}_p(G/F_{2R+1})$. For a contradiction, we assume that $r_p > \mathrm{rk}_p(G/F_{2R+1})$. Choose a pro-$p$ subgroup $H \le_{\mathrm{o}} G$ of minimal index among the open pro-p subgroups of G with $\mathrm{d}(H) = r_p$. In particular, this means that $\mathrm{d}(H) > \mathrm{d}(HF_{2R+1}/F_{2R+1})$.

The sequence $\mathrm{d}(HF_j/F_j)$, $j \in \mathbb{N}$, is nondecreasing and eventually constant, with final constant value $\mathrm{d}(H)$. Since $\mathrm{d}(H) = r_p < 2R+1$, we conclude that $\mathrm{d}(HF_j/F_j)$, $j \in \mathbb{N}$, cannot be strictly increasing until it becomes constant. Hence there exists $j = j(H) \in \mathbb{N}$ such that

$$\begin{aligned} \text{(2-1)} \qquad \mathrm{d}(HF_j/F_j) = \mathrm{d}(HF_{j+1}/F_{j+1}) &< \mathrm{d}(HF_{j+2}/F_{j+2}) \\ &< \cdots < \mathrm{d}(HF_{j+k+1}/F_{j+k+1}) = \mathrm{d}(H) \end{aligned}$$

for suitable $k = k(H)$ with $1 \le k \le r_p \le R$. In particular, this set-up implies that $j+k+1 > 2R+1$, hence $j > R$ and $2j \ge j+R+1 \ge j+k+1$. Consequently, we see that $[F_j, F_j] \subseteq F_{2j} \subseteq F_{j+k+1}$ and there is no harm in assuming that

$$[F_j, F_j] = F_{2j} = 1.$$

This reduction renders G finite, with abelian normal p-subgroups

$$A = F_j \quad \text{and} \quad B = F_{j+1} = \Phi(F_j) = A^p.$$

We set $l = \mathrm{d}(H/(H \cap B)) = \mathrm{d}(HB/B) < \mathrm{d}(H) = r_p$ and choose generators $y_1, \ldots, y_l$ for H modulo $H \cap B$ so that

$$L = \langle y_1, \ldots, y_l \rangle \le H$$

satisfies $LB = HB$. Put $m = \mathrm{d}(H) - l = r_p - l \ge 1$. A collection of elements generates H if and only if it generates the Frattini quotient $H/\Phi(H)$; the latter is elementary abelian, because H is a p-group. Thus the minimal generating set $y_1, \ldots, y_l$ modulo $H \cap B$ can be supplemented to a minimal generating set for H:

there are $b_1, \dots, b_m \in B$ such that

$$H = \langle y_1, \dots, y_l, b_1, \dots, b_m \rangle \quad \text{with } \mathrm{d}(H) = r_p = l + m.$$

We put $M = \langle b_1, \dots, b_m \rangle^H \trianglelefteq H$ so that $H = LM$.

Choose $a_1, \dots, a_m \in A$ with $b_i = a_i^{\,p}$ for $1 \le i \le m$ and set

$$\widetilde{H} = \langle y_1, \dots, y_l, a_1, \dots, a_m \rangle \le G.$$

We claim that $\widetilde{H}$ is a p-subgroup of G such that

$$|G : \widetilde{H}| < |G : H| \quad \text{and} \quad \mathrm{d}(\widetilde{H}) = r_p, \tag{2-2}$$

which yields the required contradiction.

Clearly, $\widetilde{H} \le HA$ is a p-group and $H \subseteq \widetilde{H}$. Moreover, we see that $HA = \widetilde{H}A = LA$. We may assume without loss of generality that $G = LA$. In this situation G is a p-group; furthermore, $L \cap A \trianglelefteq G$ is normal. By construction, compare with (2-1), we have $\mathrm{d}(L/(L \cap A)) = \mathrm{d}(HA/A) = \mathrm{d}(HB/B) = l = \mathrm{d}(L)$. Thus $L \cap A \subseteq \Phi(L) \subseteq \Phi(H)$ and there is no harm in assuming $L \cap A = 1$. This gives

$$G = L \ltimes A, \quad H = L \ltimes M \quad \text{and} \quad \widetilde{H} = L \ltimes \widetilde{M} \quad \text{for } \widetilde{M} = \langle a_1, \dots, a_m \rangle^{\widetilde{H}}.$$

We supplement $y_1, \dots, y_l$ to a minimal generating set $y_1, \dots, y_l, \tilde{a}_1, \dots, \tilde{a}_n$ for the p-group $\widetilde{H}$, for suitable $n \in \{0, 1, \dots, m\}$ and $\tilde{a}_1, \dots, \tilde{a}_n \in \widetilde{M}$. The p-power map $g \mapsto g^p$ induces a surjective L-invariant homomorphism $\alpha \colon \widetilde{M} \to M$ between finite abelian p-groups. This implies $|\widetilde{M}| > |M|$ and thus $|G : \widetilde{H}| < |G : H|$. Furthermore, using the identity map on L in combination with α, we obtain a surjective homomorphism from $\widetilde{H} = L \ltimes \widetilde{M}$ onto $L \ltimes M = H$. This shows that $r_p = \mathrm{d}(H) \le \mathrm{d}(\widetilde{H}) \le r_p$ and hence $\mathrm{d}(\widetilde{H}) = r_p$, which completes the proof of (2-2).

(ii) Now suppose that $\mathrm{rk}(G) \le R$. Clearly, the maximal local rank

$$\mathrm{mlr}(G) = \max\bigl(\{\mathrm{rk}_p(G) \mid p \text{ prime}\}\bigr)$$

is at most $\mathrm{rk}(G)$. Conversely, Lucchini [1997, Theorem 3 and Corollary 4] established that

$$\mathrm{rk}(G) \le \mathrm{mlr}(G) + 1,$$

with equality if and only if there are

- an odd prime p such that $r_p = \mathrm{rk}_p(G) = \mathrm{mlr}(G)$ and
- an open subgroup $H \le_{\mathrm{o}} G$ and $N \trianglelefteq_{\mathrm{o}} H$ such that

  $$H/\Phi_p(N) \cong H/N \ltimes N/\Phi_p(N) \cong C_q \ltimes C_p^{\,\mathrm{mlr}(G)},$$

  where $H/N \cong C_q$ is cyclic of prime order $q \mid (p-1)$, the p-Frattini quotient $N/\Phi_p(N) \cong C_p^{\,\mathrm{mlr}(G)}$ is elementary abelian of rank $\mathrm{mlr}(G)$, and H/N acts via conjugation faithfully on $N/\Phi_p(N)$ by power automorphisms (i.e., by nonzero homotheties if we regard $N/\Phi_p(N)$ as an $\mathbb{F}_p$-vector space).

For short let us refer, somewhat effusively, within this proof to such a pair (H, N) as a "runaway couple" for G with respect to p.

By (i), we have $\mathrm{mlr}(G) = \mathrm{mlr}(G/F_{2R+1})$, and hence it suffices to show: if G admits a runaway couple, then so does G/F_{2R+1}, in fact, with respect to the same prime. Suppose that (H, N) is a runaway couple for G with respect to an odd prime p so that $H/\Phi_p(N) \cong C_q \ltimes C_p^{r_p}$ as detailed above, with the additional property that $|G : H|$ is as small as possible. Assume for a contradiction that G/F_{2R+1} does not admit a runaway couple.

As in the proof of (i) there is no harm in factoring out the Hall pro-p' subgroup P' of F, because $H \cap F \subseteq N$ and $H \cap P' \subseteq \Phi_p(N)$. Consequently we may as well assume that $F \trianglelefteq_{\mathrm{o}} G$ is a powerful pro-p group, which makes G virtually a pro-p group.

As in the proof of (i), the sequence

$$\mathrm{d}\big(H/\big((H \cap F_j)\Phi_p(N)\big)\big) = \mathrm{d}\big(HF_j/\Phi_p(N)F_j\big), \quad j \in \mathbb{N},$$

is nondecreasing and eventually constant, with final constant value

$$\mathrm{d}(H/\Phi_p(N)) = \mathrm{d}(H) = r_p + 1 < 2R + 1.$$

We use the same arguments as before to conclude that there exists $j = j(H)$ such that the analogue of (2-1) for $H/\Phi_p(N)$ holds and we reduce to the situation where $[F_j, F_j] = F_{2j} = 1$. This reduction renders G finite, with abelian normal p-subgroups

$$A = F_j \quad \text{and} \quad B = F_{j+1} = \Phi(F_j) = A^p;$$

furthermore, we have

$$l = \mathrm{d}\big(N/\big((H \cap A)\Phi_p(N)\big)\big) = \mathrm{d}\big(N/\big((H \cap B)\Phi_p(N)\big)\big) < \mathrm{d}(N/\Phi_p(N)) = r_p. \tag{2-3}$$

It suffices to produce a runaway couple $(\widetilde{H}, \widetilde{N})$ for the group HA with respect to p such that $|HA : \widetilde{H}| < |HA : H|$; thus we may assume that

$$G = HA.$$

This reduction allows us to conclude that $\Phi_p(N) \cap A \trianglelefteq G$ and there is no harm in assuming $\Phi_p(N) \cap A = 1$. Likewise $M = H \cap A \trianglelefteq G$, and reduction modulo $\Phi_p(N)$ induces an embedding of $M \le N$ into the elementary abelian group $N/\Phi_p(N) \cong C_p^{r_p}$. Using (2-3), we conclude that

$$M = H \cap A = H \cap B = \langle b_1, \ldots, b_m \rangle \cong C_p^m \quad \text{for } m = r_p - l \ge 1.$$

The normal subgroup $M\Phi_p(N) \trianglelefteq H$ decomposes as a direct product $M \times \Phi_p(N)$. Recall that $H/\Phi_p(N) \cong C_q \ltimes C_p^{r_p}$, with the action given by power automorphisms.

We build a minimal generating set $x, y_1, \dots, y_l, b_1, \dots, b_m$ for H modulo $\Phi_p(N)$ by choosing

$$x \in H \smallsetminus N \quad \text{and} \quad y_1, \dots, y_l \in N$$

which supplement $b_1, \dots, b_m$ suitably. We set

$$L_1 = \langle x, y_1, \dots, y_l \rangle \le H \quad \text{and} \quad L = L_1 \Phi_p(N) \le H.$$

In this situation $H = LM$ and we claim that $L \cap M = 1$ so that

$$H = L \ltimes M.$$

Indeed, our construction yields that the intersection in $H/\Phi_p(N) \cong C_q \ltimes C_p^{l+m}$ of the subgroups

$$L/\Phi_p(N) = \langle \bar{x} \rangle \ltimes \langle \bar{y_1}, \dots, \bar{y_l} \rangle \cong C_q \ltimes C_p^l \quad \text{and} \quad M\Phi_p(N)/\Phi_p(N) \cong M \cong C_p^m$$

is trivial. This gives $L \cap M \subseteq \Phi_p(N)$ and consequently $L \cap M \subseteq \Phi_p(N) \cap M = 1$.

Put $\widetilde{M} = \{a \in A \mid a^p \in M\} \trianglelefteq G$. Recall that $M = H \cap B$ and $B = A^p$. The p-power map constitutes a surjective G-equivariant homomorphism $\widetilde{M} \to M$ whose kernel $K \trianglelefteq G$, say, includes M. From $L \cap M = 1$ we conclude that $LK \cap \widetilde{M} = (L \cap \widetilde{M})K \subseteq K$. Moreover, we have $L \cap K \subseteq H \cap A = M$ and thus $L \cap K \subseteq L \cap M = 1$.

These considerations show that the group $\widetilde{H} = L\widetilde{M}$ maps onto

$$\widetilde{H}/K \cong LK/K \ltimes \widetilde{M}/K \cong L \ltimes M = H,$$

and hence onto $C_q \ltimes C_p^{r_p}$. Thus $\widetilde{H}$ gives rise to a runaway couple for G, with respect to the prime p, just as H does. To conclude the proof we observe that $|K| \ge |M| \ge p$ implies $|\widetilde{H}| > |\widetilde{H}|/|K| = |H|$ and hence $|G : \widetilde{H}| < |G : H|$. □

The following corollary yields in particular Theorem 1.3 about finite nilpotent groups, which was showcased in the introduction for its succinctness.

Corollary 2.2. *Let $R \in \mathbb{N}$. Suppose that the profinite group G has an open normal subgroup $F \trianglelefteq_o G$ which is pronilpotent.*

(i) *If $\mathrm{rk}_p(G) \le R$ for some prime p, then*

$$\mathrm{rk}_p(G) = \mathrm{rk}_p\big(G/\Phi^{2R+\lceil \log_2(R) \rceil+2}(F)\big).$$

(ii) *If $\mathrm{rk}(G) \le R$, then*

$$\mathrm{rk}(G) = \mathrm{rk}\big(G/\Phi^{2R+\lceil \log_2(R) \rceil+2}(F)\big).$$

Proof. As in the proof of Theorem 2.1, one reduces to the case in which F is a pro-p group for a single prime p. From $\mathrm{rk}(F) \le R$ it follows that $\Phi^{\lceil \log_2(R) \rceil+1}(F) \trianglelefteq_o G$ is powerful; compare with [Dixon et al. 1999, Chapter 2, Exercise 6]. Thus we can apply Theorem 2.1 to $\Phi^{\lceil \log_2(R) \rceil+1}(F)$ in place of F. □

The following example puts the basic idea behind the proof of Theorem 2.1 into perspective. It indicates that one would need to take a different approach or at least make more careful choices in order to eliminate the dependency on the parameter R. Indeed, the example yields, for $p > n \geq 2$, a pro-p group G, a powerful open normal subgroup $F \trianglelefteq_{\mathrm{o}} G$ and an open subgroup $H \leq_{\mathrm{o}} G$ such that $\mathrm{d}(H) = \mathrm{rk}(G)$ but $\mathrm{d}(\widetilde{H}\Phi^n(F)/\Phi^n(F)) < \mathrm{rk}(G)$ for all $\widetilde{H} \leq_{\mathrm{o}} G$ with $\widetilde{H} \supseteq H$.

Example 2.3. Let $n \in \mathbb{N}$ and consider the metabelian pro-p group

$$G = C \ltimes A, \qquad \text{where } C = \langle c \rangle \cong \mathbb{Z}_p, \quad A = \langle a_1, \dots, a_n \rangle \cong \mathbb{Z}_p^n$$

and the action of C on A is given by

$$a_i^c = a_i a_{i+1} \quad \text{for } 1 \leq i < n, \qquad \text{and} \qquad a_n^c = a_n.$$

Here $\mathbb{Z}_p$ denotes the additive group of the p-adic integers, viz. the infinite procyclic pro-p group. Then $G = \langle c, a_1 \rangle$ is 2-generated, nilpotent of class n and has rank $\mathrm{rk}(G) = n + 1$. For instance,

$$H = \langle c, a_1^{p^{n-1}}, a_2^{p^{n-2}}, \dots, a_{n-1}^{p}, a_n \rangle \leq_{\mathrm{o}} G$$

requires $n + 1$ generators.

Suppose that $p > n \geq 2$. Then $F = \langle c^p \rangle \ltimes A \trianglelefteq_{\mathrm{o}} G$ is powerful, and $\Phi^j(F) = \langle c^{p^j} \rangle \ltimes A^{p^{j-1}}$ for $j \in \mathbb{N}$. Thus any subgroup $\widetilde{H} \leq_{\mathrm{o}} G$ with $\widetilde{H}F = HF = \langle c \rangle F$ and $\mathrm{d}(\widetilde{H}) = \mathrm{d}(\widetilde{H}\Phi^n(F)/\Phi^n(F))$ requires less than $\mathrm{d}(H) = n + 1$ generators, but nevertheless $\mathrm{rk}(G) = \mathrm{rk}(G/\Phi(F))$. The group

$$K = \langle c^p, a_1, \dots, a_n \rangle,$$

which is unrelated to H, requires $n + 1$ generators, even modulo $\Phi(F)$.

3. Finite axiomatizability of the rank

In this section we establish Theorem 1.1. We begin with a basic example which illustrates the concept of an $\exists\forall\exists$-sentence in $\mathcal{L}_{\mathrm{gp}}$ and related constructions which we use frequently; compare with [Nies et al. 2021, Sections 2 and 5]. Despite its simplicity, the example is a key building block in later proofs, where we need to control the quantifier complexity of more involved first-order formulae.

Example 3.1. Let G be a profinite group and let $N \subseteq G$. Suppose that N is definable in G; this means that there is an $\mathcal{L}_{\mathrm{gp}}$-formula $\varphi(x)$, with a single free variable x, such that $N = \{g \in G \mid \varphi(g)\}$.

Let $B = \{b_1, \dots, b_n\}$ be a finite group of order n, with multiplication "table"

$$b_i b_j = b_{m(i,j)}$$

encoded by a suitable function $m \colon \{1, \dots, n\} \times \{1, \dots, n\} \to \{1, \dots, n\}$.

Then the sentence

$$\exists a_1, \dots a_n \, \forall x, y, z : \varphi(1) \wedge \big((\varphi(x) \wedge \varphi(y)) \to \varphi(x^{-1}y)\big) \wedge \big(\varphi(x) \to \varphi(y^{-1}xy)\big) \wedge \Big(\bigwedge_{1 \le i < j \le n} \neg\varphi(a_i^{-1}a_j)\Big) \wedge \Big(\bigvee_{1 \le i \le n} \varphi(a_i^{-1}y)\Big) \wedge \Big(\bigwedge_{1 \le i, j \le n} \varphi(a_{m(i,j)}^{-1}a_i a_j)\Big)$$

can be used to express that $N \trianglelefteq G$ and $G/N \cong B$. The quantifier complexity of this sentence is the same as the quantifier complexity of φ increased by $\exists\forall$. In particular, if $N \subseteq_{\mathrm{c}} G$ is $\exists$-definable as a closed set, i.e., definable by means of an $\exists$-formula which implicitly ensures that N is topologically closed, we obtain an $\exists\forall\exists$-sentence to express that $N \trianglelefteq_{\mathrm{c}} G$ and $G/N \cong B$.

For instance, if we know or suspect that the commutator word has a certain finite width in G, we may consider the $\exists$-definable set

$$N = \big\{[x_1, y_1] \cdots [x_r, y_r] \mid x_1, y_1, \dots, x_r, y_r \in G\big\} \subseteq_{\mathrm{c}} G,$$

for a given parameter $r \in \mathbb{N}$, and formulate an $\exists\forall\exists$-sentence in $\mathcal{L}_{\mathrm{gp}}$ which expresses that, indeed, N is equal to the entire commutator subgroup $[G, G]$ and that the abelianization $G/[G, G]$ is isomorphic to a given finite group.

Sometimes we want to express, by means of an $\mathcal{L}_{\mathrm{gp}}$-sentence, extra features of a definable subgroup $H \le_{\mathrm{c}} G$. This process typically involves quantification over elements of H rather than G which, in general, may increase the quantifier complexity of the resulting sentences. However, if $H = \{g \in G \mid \varphi(g)\}$ is $\exists$-definable, where $\varphi(x)$ takes the form $\exists \underline{z} : \varphi_0(x, \underline{z})$ with φ_0 quantifier-free in free variables x and $z_1, \dots, z_m$, say, then H is "quantifier-neutral" in the following sense. First-order assertions about H can be translated into assertions of the same quantifier complexity about G, simply by expressing universal quantification over elements of H as $\forall x, \underline{z} : (\varphi_0(x, \underline{z}) \to \cdots)$ and existential quantification over elements of H as $\exists x, \underline{z} : (\varphi_0(x, \underline{z}) \wedge \cdots)$. $\diamond$

It is convenient to establish the assertions of Theorem 1.1 first for pronilpotent groups before considering the general situation.

Proposition 3.2. *Let π be a finite set of primes, let $r \in \mathbb{N}$ and let $\mathbf{r} = (r_p)_{p \in \pi}$ be a tuple in $\{0, 1, \dots, r\}$. Then there exists an $\exists\forall\exists$-sentence $\omega_{\pi,r,\mathbf{r}}$ in $\mathcal{L}_{\mathrm{gp}}$ such that, for every pronilpotent pro-π group H, the following are equivalent*:

(i) $\mathrm{rk}(H) = r$, *and* $\mathrm{rk}_p(H) = r_p$ *for every* $p \in \pi$.

(ii) $\omega_{\pi,r,\mathbf{r}}$ *holds in* H, *i.e.*, $H \models \omega_{\pi,r,\mathbf{r}}$.

Proof. We set $k = |\pi|$, write $\pi = \{p_1, \dots, p_k\}$ and put $q = q(\pi) = p_1 \cdots p_k$. As H is pronilpotent, it is the direct product $H = \prod_{i=1}^{k} H_i$ of its Sylow pro-p_i subgroups H_i. We set $m = m(r) = \lceil \log_2(r) \rceil + 1$.

Similar to Example 3.1, there is an $\exists\forall\exists$-sentence β_1 in $\mathcal{L}_{\mathrm{gp}}$ to express that there are elements $a_1, \dots, a_r$ in H such that every element $h \in H$ can be written as $h = \prod_{j=1}^{r} a_j^{e_j} b$, for suitable choices for $e_j \in \{0, 1, \dots, q-1\}$ and

$$b \in B(H) = \left\{[x_1, y_1] \cdots [x_r, y_r] z^q \mid x_1, y_1, \dots, x_r, y_r, z \in H\right\} \subseteq_{\mathrm{c}} \Phi(H).$$

We recall that $\mathrm{d}(H) = \mathrm{d}(H/\Phi(H))$ and that $\mathrm{d}(H) \leq r$ implies $B(H) = \Phi(H)$; see [Dixon et al. 1999, Lemma 1.23]. Thus β_1 holds for H if and only if $\mathrm{d}(H) \leq r$. Moreover, in this case $\Phi(H) = B(H)$ is $\exists$-definable in H and hence quantifier-neutral in the sense of Example 3.1. By recursion, there is an $\exists\forall\exists$-sentence β_{m+1} such that β_{m+1} holds for H if and only if

$$\mathrm{rk}(\Phi^j(H)/\Phi^{j+1}(H)) \leq r \quad \text{for } 0 \leq j \leq m; \tag{3-1}$$

in this case the subgroup $F = \Phi^m(H)$ is $\exists$-definable in H and hence quantifier-neutral, moreover it satisfies $\mathrm{d}(F) \leq r$. Furthermore, there is an $\forall\exists$-sentence γ which expresses that every Sylow subgroup of F is powerful, viz. that F is semipowerful in the terminology introduced in [Nies et al. 2021, Section 5]. Indeed, by [Dixon et al. 1999, Proposition 2.6], it suffices to express that every commutator $[x, y]$ of elements $x, y \in F$ is a $(2q)$-th power z^{2q} of a suitable $z \in F$.

Once F is r-generated and semipowerful, we know that $\mathrm{rk}(F) \leq r$. If, in addition, the rank bounds specified in (3-1) hold, we deduce that $\mathrm{rk}(H/F) \leq mr$ and hence $\mathrm{rk}(H) \leq R$ for $R = (m+1)r$. Furthermore, the group

$$\Phi^{2R+1}(F) = \left\{x^{q^{2R+1}} \mid x \in F\right\}$$

is $\exists$-definable in H and hence quantifier-neutral; in particular, $H/\Phi^{2R+1}(F)$ is interpretable in H. Finally, $|H/\Phi^{2R+1}(F)|$ is bounded by $q^{(2R+m+1)r}$ and there is an $\exists\forall\exists$-sentence θ which expresses that $H/\Phi^{2R+1}(F)$ is one of the finitely many finite π-groups of suitable order which has rank r and whose p-ranks are in agreement with the prescribed $\mathbf{r}$; compare with Example 3.1.

With the backing of Theorem 2.1, we form the conjunction of the sentences $\beta_{m+1}, \gamma, \theta$ to arrive at an $\exists\forall\exists$-sentence $\omega_{\pi,r,\mathbf{r}}$ with the desired property. □

Proof of Theorem 1.1. We analyse the structure of a pro-π group G of rank $\mathrm{rk}(G) = r$ to build step-by-step a first-order sentence $\eta_{\pi,r}$ that is satisfied by any such group G. Following that we check that, conversely, every pro-π group satisfying $\eta_{\pi,r}$ has rank at most $2r$. Applying Theorem 2.1, we extend $\eta_{\pi,r}$ to a sentence $\varrho_{\pi,r,\mathbf{r}}$ which pins down precisely the rank as being r and the ranks of the Sylow subgroups as being given by $\mathbf{r}$.

Our discussion involves upper bounds for certain integer parameters that depend on π and r, but not on the specific group G used in our discussion; for short, we say that such parameters are (π, r)-bounded. The proof proceeds in four steps along the following plan of action. In Step 1 we produce a pronilpotent open normal subgroup $K \trianglelefteq_o G$ of (π, r)-bounded index. This is used in Step 2 to describe an $\exists$-definable pronilpotent open normal subgroup $H \trianglelefteq_o G$ of (π, r)-bounded index. In Step 3 we show that the fact that H is pronilpotent can be expressed by an $\exists\forall\exists$-sentence. This uses a simple but effective trick: we would like to express that H is a direct product of its Sylow subgroups, but in general the latter fail to be definable; to overcome this problem we work modulo the centre $\mathrm{Z}(H)$ which is sufficient for our purposes. In Step 4 we use the tools that we already prepared in Example 3.1 and in Proposition 3.2 to conclude the argument.

Step 1. The classification of finite simple groups implies that, up to isomorphism, there are only finitely many finite simple π-groups; see [Mazurov 1994, Remark following Lemma 2]. A fortiori there is a finite set

$$\mathcal{S} = \mathcal{S}_{\pi,r}$$

of representatives for the isomorphism classes of finite simple π-groups S such that $\mathrm{rk}(S) \leq r$. Consequently, the cardinality of the set

$$\Psi = \Psi_{G,\pi,r} = \left\{ \psi \mid \psi \colon G \to \mathrm{Aut}(S^l) \text{ a homomorphism for } S \in \mathcal{S} \text{ and } 0 \leq l \leq r \right\}$$

is (π, r)-bounded, because G can be generated by at most r elements and any homomorphism between groups is determined by its effect on a chosen set of generators. From this we observe that the index of

$$K = K_{G,\pi,r} = \bigcap_{\psi \in \Psi} \ker \psi \trianglelefteq_o G$$

in G is (π, r)-bounded. Thus there exists $f(\pi, r) \in \mathbb{N}$, depending on π and r, but not on the specific group G, such that $|G : K|$ divides $f(\pi, r)$.

We claim that K is pronilpotent. For this it suffices to show that $K/(K \cap L)$ is nilpotent for each $L \trianglelefteq_o G$. Let $L \trianglelefteq_o G$. By pulling back a chief series for the finite group G/L to G, we obtain a normal series

$$L = G_{n+1} \trianglelefteq G_n \trianglelefteq \cdots \trianglelefteq G_1 = G$$

of finite length n such that, for each $i \in \{1, \ldots, n\}$, the group G_i/G_{i+1} is a minimal normal subgroup of G/G_{i+1} and thus isomorphic to $S_i^{m(i)}$ for suitable choices of $S_i \in \mathcal{S}$ and $m(i) \in \mathbb{N}$. Since each of the groups $S_i^{m(i)}$ contains an elementary abelian p-subgroup of rank $m(i)$, for primes p dividing $|S_i|$, we obtain $m(i) \leq \mathrm{rk}(S_i^{m(i)}) \leq \mathrm{rk}(G) = r$ for all $i \in \{1, \ldots, n\}$. Intersecting with K, we obtain a series

$$K \cap L = K \cap G_{n+1} \trianglelefteq K \cap G_n \trianglelefteq \ldots \trianglelefteq K \cap G_1 = K \tag{3-2}$$

consisting of G-invariant subgroups with factors $(K \cap G_i)/(K \cap G_{i+1}) \cong S_i^{l(i)}$ satisfying $0 \leq l(i) \leq m(i) \leq r$, for $i \in \{1, \ldots, n\}$. By construction, K acts trivially on each of these factors so that $[K\cap G_i, K]\subseteq K\cap G_{i+1}$ for $i\in\{1,\dots,n\}$. Thus (3-2) constitutes a central series for $K/(K\cap L)$, and $K/(K\cap L)$ is nilpotent (of class at most n).

Step 2. Next we consider the group

$$H = G^{f(\pi,r)} = \langle g^{f(\pi,r)} \mid g\in G\rangle \trianglelefteq_{\mathrm{o}} G \quad \text{with } H\subseteq K;$$

the index $|G:H|$ is (π,r)-bounded, by the positive solution to the restricted Burnside problem. In fact, we do not require the general result, but a rather special case, which is easy to establish. Indeed, assume for the moment that the pro-π group G of rank r is finite of exponent $f(\pi,r)$. We need to show that $|G|$ is (π,r)-bounded. In Step 1 we established that G has a nilpotent normal subgroup K of (π,r)-bounded index. Thus there is no harm in assuming that $G=K$. Furthermore, K is a direct product of its Sylow p-subgroups, where p ranges over the finite set π. Hence we may even assume that G is a p-group of rank at most r, for some $p\in\pi$, and that $f(\pi,r)$ is a p-power, p^e say. In this situation, G contains a powerful normal subgroup of (p,r)-bounded index (see [Dixon et al. 1999, Theorem 2.13]), and we may assume that G itself is powerful. The p-power series of a powerful p-group coincides with its lower p-series, and we obtain the bound $|G|\le p^{re}$.

Next we observe that the verbal subgroup H is an $\exists$-definable subgroup of G and hence quantifier-neutral, in the sense discussed in Example 3.1. Indeed, by [Nikolov and Segal 2011, Theorem 1], every element of H can be written as a product of a (π,r)-bounded number of $f(\pi,r)$-th powers. But again we only require the bound in a rather special case which is much easier to handle. Indeed, descending without loss of generality to a subgroup of (π,r)-bounded index, as above, it suffices to recall that in a powerful pro-p group every product of p^e-th powers is itself a p^e-th power; see [Dixon et al. 1999, Corollary 3.5].

Step 3. Since K is pronilpotent, so is H. In the situation at hand, this fact can be expressed by an $\exists\forall\exists$-sentence. Indeed, H is pronilpotent if and only if $H/\mathrm{Z}(H)$ is pronilpotent. Hence it suffices to express the assertion that $H/\mathrm{Z}(H)$ is pronilpotent. Clearly, $\mathrm{Z}(H)$ is $\forall$-definable in H and hence in G. We set $k=|\pi|$ and write $\pi=\{p_1,\dots,p_k\}$. As H is pronilpotent, $H=\prod_{i=1}^k H_i$ is the direct product of its Sylow pro-p_i subgroups H_i and $\mathrm{Z}(H)=\prod_{i=1}^k \mathrm{Z}(H_i)$ so that $H/\mathrm{Z}(H)\cong\prod_{i=1}^k H_i/\mathrm{Z}(H_i)$. From

$$C_i = \mathrm{C}_H(H_i) = \textstyle\prod_{j=1}^{i-1} H_j \times \mathrm{Z}(H_i)\times \prod_{j=i+1}^{k} H_j, \quad \text{for } i\in\{1,\dots,k\},$$

we deduce that

$$D_i = \textstyle\bigcap\{C_j \mid 1\le j\le k \text{ and } j\ne i\} = \prod_{j=1}^{i-1}\mathrm{Z}(H_j)\times H_i\times\prod_{j=i+1}^{k}\mathrm{Z}(H_j)$$

and thus

$$D_i/\mathrm{Z}(H)\cong H_i/\mathrm{Z}(H_i), \quad \text{for } i\in\{1,\dots,k\}.$$

As $\mathrm{rk}(G) \le r$, there exist, for each $i \in \{1, \dots, k\}$, elements $x_{i,1}, \dots, x_{i,r} \in H_i$ such that $H_i = \langle x_{i,1}, \dots, x_{i,r} \rangle$ and thus

$$C_i = \mathrm{C}_H(\{x_{i,1}, \dots, x_{i,r}\}).$$

Subject to the kr parameters $x_{1,1}, \dots, x_{k,r}$, this makes $\mathrm{Z}(H) = \bigcap_{i=1}^k C_i$ and each of the groups D_i quantifier-free definable, by suitable centralizer conditions; moreover $Q_i = D_i / \mathrm{Z}(H)$ becomes interpretable in H, for $1 \le i \le k$.

We conclude that it suffices to express in an $\forall\exists$-sentence, subject to the (π, r)-bounded number of parameters $x_{s,t}$, that

(a) $\bigcap_{i=1}^k C_i = \mathrm{Z}(H)$, hence $\mathrm{Z}(H) \subseteq D_i$, for $i \in \{1, \dots k\}$;

(b) $D_i / \mathrm{Z}(H)$ is a pro-p_i group for $i \in \{1, \dots k\}$;

(c) $[D_i, D_j] \subseteq \mathrm{Z}(H)$ for $i, j \in \{1, \dots, k\}$ with $i \ne j$;

(d) $H = D_1 \cdot D_2 \cdot \ldots \cdot D_k$, where the right-hand side denotes the set of all products $y_1 \cdots y_k$ with factors $y_i \in D_i$ for $i \in \{1, \dots, k\}$;

for this implies that $H / \mathrm{Z}(H) = \prod_{i=1}^k D_i / \mathrm{Z}(H)$ is the direct product of its Sylow subgroups and thus pronilpotent. Turning the parameters $x_{s,t}$ into variables bound by an extra existential quantifier at the front, we arrive at an $\exists\forall\exists$-sentence without parameters which verifies that H is pronilpotent.

Subject to the parameters $x_{s,t}$, the assertions in (a), (c) can be expressed by an $\forall$-sentence, and (d) can be achieved by means of an $\forall\exists$-sentence. The only tricky part occurs in (b) where we need to express that the group $Q_i = D_i / \mathrm{Z}(H)$ is a pro-p_i group. Since we know a priori that Q_i is a pro-π group, this is achieved by demanding that every element of Q_i is a q_i-th power, for $q_i = p_1 \cdots p_{i-1} p_{i+1} \cdots p_k$. This can be expressed by an $\forall\exists$-sentence at the level of H, because $\mathrm{Z}(H) = \bigcap_{i=1}^k C_i$ is quantifier-free definable subject to the parameters $x_{s,t}$.

Step 4. By Step 2, the group G/H is interpretable in G and finite of (π, r)-bounded order. There is an $\exists\forall\exists$-sentence that expresses that the factor group G/H is among the finitely many finite groups of rank at most r and exponent dividing $f(\pi, r)$; compare with Example 3.1. Using our results from Step 2, Step 3 and Proposition 3.2, we produce an $\exists\forall\exists$-sentence that expresses that the power word $x^{f(\pi,r)}$ has (π, r)-bounded width in G and that $H = G^{f(\pi,r)}$ is pronilpotent of rank at most r.

The conjunction of these two sentences yields an $\exists\forall\exists$-sentence $\eta_{\pi,r}$ such that

- every pro-π group G of rank $\mathrm{rk}(G) = r$ satisfies $\eta_{\pi,r}$;
- conversely, if a pro-π group $\widetilde{G}$ satisfies $\eta_{\pi,r}$, then $\widetilde{H} = \widetilde{G}^{f(\pi,r)} \trianglelefteq_{\mathrm{o}} \widetilde{G}$ is pronilpotent and both $\widetilde{H}$ and $\widetilde{G}/\widetilde{H}$ have rank at most r; in particular, this ensures that $\mathrm{rk}(\widetilde{G}) \le R$ for $R = 2r$.

We put $m = m(R) = \lceil \log_2(R) \rceil + 1$. As in the proof of Proposition 3.2 we see that $F = \Phi^{m(R)}(H) \trianglelefteq_o G$ is $\exists$-definable, hence quantifier-neutral, and semipowerful. Furthermore, $\Phi^{2R+1}(F)$ is $\exists$-definable, hence quantifier-neutral, and, by Theorem 2.1,

$$\mathrm{rk}(G) = \mathrm{rk}\big(G/\Phi^{2R+1}(F)\big) \quad \text{and} \quad \mathrm{rk}_p(G) = \mathrm{rk}_p\big(G/\Phi^{2R+1}(F)\big) \quad \text{for every } p \in \pi.$$

Just as in the proof of Proposition 3.2 we find an $\exists\forall\exists$-sentence which in conjunction with $\eta_{\pi,r}$ produces an $\exists\forall\exists$-sentence $\varrho_{\pi,r,\mathbf{r}}$ with the desired property. □

The next result complements Theorem 1.1. It illustrates that the rank of a pro-p group cannot be detected by a first-order sentence uniformly across all primes p, even if the language $\mathcal{L}_{\mathrm{gp}}$ was to be enlarged by an extra function to be interpreted as the p-power map $x \mapsto x^p$ in pro-p groups. (Note that regarding elementary abelian p-groups it is futile to enlarge the language in this way.) We sketch a proof for completeness; it relies on a standard ultraproduct construction and a well-known quantifier elimination result in model theory.

Proposition 3.3. *Let $\tilde{\pi}$ be an infinite set of primes and let $r \in \mathbb{N}$. Then there is no $\mathcal{L}_{\mathrm{gp}}$-sentence $\vartheta_{\tilde{\pi},r}$ such that, for every $p \in \tilde{\pi}$ and every finite elementary abelian p-group G, the following are equivalent:*

(i) $\mathrm{rk}(G) = r$.

(ii) *$\vartheta_{\tilde{\pi},r}$ holds in G, i.e., $G \models \vartheta_{\tilde{\pi},r}$.*

Proof. For a contradiction, assume that the $\mathcal{L}_{\mathrm{gp}}$-sentence $\vartheta = \vartheta_{\tilde{\pi},r}$ has the desired property. Then $C_p^r \models \vartheta$ and $C_p^{r+1} \models \neg\vartheta$ for all $p \in \tilde{\pi}$. We regard C_p^r and C_p^{r+1} as the additive groups of the vector spaces $\mathbb{F}_p^r$ and $\mathbb{F}_p^{r+1}$ over the prime field $\mathbb{F}_p$.

Let $\mathfrak{U}$ be a nonprincipal ultrafilter on the infinite index set $\tilde{\pi}$. By Łoś's theorem,

$$\mathcal{K} = \big(\textstyle\prod_{p \in \tilde{\pi}} \mathbb{F}_p\big)/\sim_{\mathfrak{U}}$$

is a field of characteristic 0, and

$$\mathcal{V} = \big(\textstyle\prod_{p \in \tilde{\pi}} \mathbb{F}_p^r\big)/\sim_{\mathfrak{U}} \quad \text{and} \quad \mathcal{W} = \big(\textstyle\prod_{p \in \tilde{\pi}} \mathbb{F}_p^{r+1}\big)/\sim_{\mathfrak{U}}$$

are nonzero $\mathcal{K}$-vector spaces. Let $\mathcal{L}_{\mathcal{K}\text{-vs}}$ denote the language of $\mathcal{K}$-vector spaces, which comprises the language of groups (for the additive group of vectors) and, for each scalar $c \in \mathcal{K}$, a 1-ary operation f_c (to denote scalar multiplication by c). Clearly, the $\mathcal{L}_{\mathrm{gp}}$-sentence ϑ gives rise to an $\mathcal{L}_{\mathcal{K}\text{-vs}}$-sentence θ, not involving scalar multiplication at all, such that by Łoś's theorem

$$\mathcal{V} \models \theta \quad \text{and} \quad \mathcal{W} \models \neg\theta,$$

in contradiction to the known fact that the infinite $\mathcal{K}$-vector spaces $\mathcal{V}$ and $\mathcal{W}$ have the same theory, due to quantifier elimination; see [Hodges 1993, Section 8.4]. □

4. Finite axiomatizability of the dimension

In this section we establish Theorems 1.4 and 1.2. We derive the former from a result of Héthelyi and Lévai [2003] about finite powerful p-groups; compare with [Wilson 2002; Fernández-Alcober 2007]. We recall from [Dixon et al. 1999, Theorem 4.20] that the elements of finite order in a finitely generated powerful pro-p group form a powerful finite subgroup, its torsion subgroup.

Proof of Theorem 1.4. The torsion subgroup T is finite and characteristic in G so that $\mathrm{C}_G(T) \trianglelefteq_{\mathrm{o}} G$. We choose a uniformly powerful open normal subgroup $U \trianglelefteq_{\mathrm{o}} G$ such that $U \subseteq \mathrm{C}_G(T)$ and $U \subseteq \Phi(G)$. Since U is torsion-free, this implies that

$$N = U \times T \trianglelefteq_{\mathrm{o}} G \quad \text{and} \quad \mathrm{d}(G) = \mathrm{d}(G/U).$$

We show below that there exists $k \in \mathbb{N}$ such that $U^{p^k} = \Phi^k(U) \trianglelefteq_{\mathrm{o}} G$ satisfies

$$\Omega_{\{1\}}(G/U^{p^k}) = \Omega_{\{1\}}(N/U^{p^k}). \tag{4-1}$$

Since $N/U^{p^k} \cong U/U^{p^k} \times T$ and because U is uniformly powerful, $\Omega_{\{1\}}(N/U^{p^k})$ is in bijection with the cartesian product of sets

$$\Omega_{\{1\}}(U/U^{p^k}) \times \Omega_{\{1\}}(T) = U^{p^{k-1}}/U^{p^k} \times \Omega_{\{1\}}(G)$$

and furthermore $\log_p|U^{p^{k-1}}/U^{p^k}| = \mathrm{d}(U)$. Put $s(G) = \log_p|\Omega_{\{1\}}(G)|$. Stringing all pieces together, we see that the finite powerful p-group $P = G/U^{p^k}$ satisfies

$$\log_p|\Omega_{\{1\}}(P)| = \mathrm{d}(U) + s(G) = \dim(G) + s(G).$$

The theorem of Héthelyi and Lévai [2003] yields $\log_p|\Omega_{\{1\}}(P)| = \mathrm{d}(P)$ and $s(G) = \log_p|\Omega_{\{1\}}(T)| = \mathrm{d}(T)$ so that

$$\dim(G) = \log_p|\Omega_{\{1\}}(P)| - s(G) = \mathrm{d}(P) - s(G) = \mathrm{d}(G) - s(G) = \mathrm{d}(G) - \mathrm{d}(T).$$

It remains to establish (4-1). Since U^{p^k}, $k \in \mathbb{N}$, is a base for the neighbourhoods of 1 in G, it suffices to show that there exists an open normal subgroup $W \trianglelefteq_{\mathrm{o}} G$ such that for every $x \in G \smallsetminus N \subseteq_{\mathrm{c}} G$ we have $x^p \notin W$, or in other words $x^p \not\equiv_W 1$. From $T \subseteq N$ we see that $G \smallsetminus N$ does not contain any elements of finite order. Hence for every $x \in G \smallsetminus N$ there exists $W_x \trianglelefteq_{\mathrm{o}} G$ such that $x^p \not\equiv_{W_x} 1$, and consequently $y^p \not\equiv_{W_x} 1$ for all $y \in xW_x \subseteq_{\mathrm{o}} G$. Since $G \smallsetminus N$ is compact, it is covered by a finite union of such cosets xW_x, i.e., $G \smallsetminus N \subseteq \bigcup_{x \in X} xW_x$ with $|X| < \infty$. This implies that $W = \bigcap_{x \in X} W_x \trianglelefteq_{\mathrm{o}} G$ has the required property. $\square$

Proof of Theorem 1.2. Let $p \in \pi$ and put $d = d_p$. It suffices to explain how one can build an $\exists\forall\exists$-sentence $\tau_{\pi,r,p,d}$ in $\mathcal{L}_{\mathrm{gp}}$ which expresses that a pro-π group G of rank $\mathrm{rk}(G) = r$ has Sylow pro-p subgroup dimension d. As in the proof of Theorem 1.1 we work with a general pro-π group G with $\mathrm{rk}(G) = r$ to concoct $\tau_{\pi,r,p,d}$.

Using the same approach as in the proof of Theorem 1.1, we find an $\exists$-definable and hence quantifier-neutral subgroup $H \trianglelefteq_{\mathrm{o}} G$ that is pronilpotent and has (π, r)-bounded index in G; moreover the arrangement can be expressed by means of a suitable $\exists\forall\exists$-sentence. We put $m = m(r) = \lceil \log_2(r) \rceil + 1$. In the proof of Proposition 3.2 we saw that we can use an $\exists\forall\exists$-sentence to describe that $\Phi^m(H)$ is semipowerful and of (π, r)-bounded index in H; in parallel we can realize $\Phi^m(H)$ as an $\exists$-definable and hence quantifier-neutral subgroup. The Sylow subgroup dimensions do not change if we pass from G to an open subgroup. Replacing G by $\Phi^m(H)$, we may therefore assume without loss of generality that G itself is pronilpotent and semipowerful.

As G is pronilpotent, G is the direct product of its powerful Sylow subgroups; let G_p denote the Sylow pro-p subgroup and T_p its torsion subgroup. By Theorem 1.4 it suffices to produce an $\exists\forall\exists$-sentence which pins down within the finite range $\{0, 1, \ldots, r\}$ the invariants

$$\mathrm{d}(G_p) = \log_p |G_p : \Phi(G_p)| \quad \text{and} \quad \mathrm{d}(T_p) = \log_p |\Omega_{\{1\}}(G_p)|,$$

where $\Omega_{\{1\}}(G_p) = \{g \in G_p \mid g^p = 1\}$ is the set of all elements of order 1 or p. We observe that $G_p/\Phi(G_p) \cong G/\Phi_p(G)$ is essentially the p-Frattini quotient of G and that $\Omega_{\{1\}}(G_p) = \{g \in G \mid g^p = 1\}$.

The Frattini quotient $G/\Phi(G)$ has (π, r)-bounded order and maps onto the p-Frattini quotient $G/\Phi_p(G)$. As in the proof of Proposition 3.2, the group $G/\Phi(G)$ is interpretable in G. There is an $\exists\forall\exists$-sentence which detects any prescribed isomorphism type of $G/\Phi(G)$ among a (π, r)-bounded number of possibilities; compare with Example 3.1. Forming a suitable disjunction, we can also detect the isomorphism type of the p-Frattini quotient $G/\Phi_p(G)$ and hence the minimal numbers of generators $\mathrm{d}(G_p)$.

Clearly, the closed subset $\{g \in G \mid g^p = 1\} \subseteq_{\mathrm{c}} G$ is quantifier-free definable in G. Moreover, its size equals $p^{\mathrm{d}(T_p)}$ and is thus at most p^r. We can easily identify by means of an $\exists\forall$-sentence its precise size and hence the invariant $\mathrm{d}(T_p)$. □

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