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It is known that spin cobordism can be determined by Stiefel–Whitney numbers and index theoretic invariants, namely KO-theoretic Pontryagin numbers. We show that string cobordism at dimension 24 can be determined by elliptic genus, a higher index theoretic invariant. We also compute the image of 24-dimensional string cobordism under elliptic genus. Using our results, we show that under certain curvature conditions, a compact 24-dimensional string manifold must bound a string manifold.

1. Introduction

Cobordism is a fundamental tool in geometry and topology. For the oriented cobordism ring Ω_*^{SO}, there are spin cobordism Ω_*^{Spin} and string cobordism $\Omega_*^{\mathrm{String}}$ as refinements through the Whitehead tower

$$\cdots \to \mathrm{String} \to \mathrm{Spin} \to \mathrm{SO}.$$

It is a classical problem to classify cobordism classes in terms of characteristic numbers. Historically, Wall [1960] showed that two closed oriented manifolds are oriented cobordant if and only if they have the same Stiefel–Whitney numbers and Pontryagin numbers. Anderson, Brown and Peterson [Anderson et al. 1967] showed that two closed spin manifolds are spin cobordant if and only if they have the same Stiefel–Whitney numbers and KO-theoretic characteristic numbers.

The problem for string manifolds is much more complicated. To our best knowledge, it is unknown yet which set of characteristic numbers classifies string cobordism. It is expected that TMF-theoretic characteristic numbers will play a similar role for string cobordism as KO-theoretic characteristic numbers do for spin cobordism. Here TMF stands for the topological modular form developed by Hopkins and Miller [Hopkins 2002]. The Witten genus [1987; 1988] plays a similar role in TMF as the $\widehat{A}$-genus does in KO and is refined to be the σ-orientation from the Thom spectrum of string cobordism to the spectrum TMF [Hopkins 2002; Ando et al. 2001].

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In this paper we show that the elliptic genus [Ochanine 1987], a higher index theoretic invariant, determines 24-dimensional string cobordism. As elliptic genus is a twisted Witten genus [Witten 1987; 1988; Liu 1995a; 1995b], it can be viewed as sort of TMF-theoretic characteristic numbers. This coincides with the expectation of the role that TMF-theoretic characteristic numbers should play for string cobordism. In the paper, we also compute the image of 24-dimensional string cobordism under elliptic genus as well as give some application of our results in geometry. It is worthwhile to remark that 24 is a dimension of special interest for string geometry. For instance, in this dimension, one has (see [Hirzebruch et al. 1994, pp. 85–87])

$$W(M) = \widehat{A}(M)\bar{\Delta} + \widehat{A}(M, T)\Delta,$$

where $W(M)$ is the Witten genus of M, $\widehat{A}(M)$ is the A-hat genus and $\widehat{A}(M, T)$ is the tangent bundle twisted A-hat genus of M, $\bar{\Delta} = E_4^3 - 744 \cdot \Delta$ with E_4 being the Eisenstein series of weight 4 and Δ being the modular discriminant of weight 12. Hirzebruch raised his prize question in [Hirzebruch et al. 1994] that whether there exists a 24-dimensional compact string manifold M such that $W(M) = \bar{\Delta}$ (or equivalently $\widehat{A}(M) = 1$, $\widehat{A}(M, T) = 0$) and the Monster group acts on M as self-diffeomorphisms. The existence of such a manifold was confirmed by Mahowald and Hopkins [2002]. They determined the image of Witten genus at this dimension via TMF. Based on their work, we [Han and Huang 2022] realized the kernel of Witten genus at dimension 24 and determined an integral basis of $\Omega_{24}^{\mathrm{String}}$. As applications, various Rokhlin type divisibility theorems were proved there, which significantly extend earlier relevant results of Chen and Han [2015] and Chen, Han and Zhang [Chen et al. 2012]. Additionally, Milivojević [2021] used rational homotopy theory to give a weak form solution to the Hirzebruch's prize question. However, the part of the question concerning the Monster group action is still open.

The *elliptic genus*, which was first constructed by Ochanine [1987] and Landweber and Stong [1988], is a graded ring homomorphism

$$\phi : \Omega_*^{\mathrm{SO}} \to \mathbb{Z}\left[\tfrac{1}{2}\right][\delta, \varepsilon] \tag{1-1}$$

from the oriented cobordism ring to the graded polynomial ring $\mathbb{Z}\left[\frac{1}{2}\right][\delta, \varepsilon]$ with the degrees $|\delta| = 4$, $|\varepsilon| = 8$, such that the logarithm is given by the formal integral

$$g(z) = \int_0^z \frac{dt}{\sqrt{1 - 2\delta t^2 + \varepsilon t^4}}. \tag{1-2}$$

The background and the developments of the theory of elliptic genus can be found in [Landweber 1988b; Segal 1988; Kreck and Stolz 1993; Hirzebruch et al. 1994; Liu 1996a; Hopkins 2002; Witten 1988].

It is shown [Chudnovsky et al. 1988; Landweber 1988a] that the image of the elliptic genus is

$$\phi(\Omega_*^{\mathrm{SO}}) = \mathbb{Z}[\delta, 2\gamma, 2\gamma^2, \dots, 2\gamma^{2^s}, \dots], \tag{1-3}$$

where $\gamma = \frac{1}{4}(\delta^2 - \varepsilon)$; and when restricted to spin cobordism,

$$\phi(\Omega_*^{\mathrm{Spin}}) = \mathbb{Z}[16\delta, (8\delta)^2, \varepsilon]. \tag{1-4}$$

It follows that at dimension 24 the image is spanned over $\mathbb{Z}$ by

$$(8\delta)^6, \quad (8\delta)^4\varepsilon, \quad (8\delta)^2\varepsilon^2, \quad \varepsilon^3.$$

The map $\phi : \Omega_{24}^{\mathrm{Spin}} \to \mathbb{Z}[8\delta, \varepsilon]$ has nontrivial kernel. Actually $E - F \cdot B$ is in the kernel, where E is the total space of a fiber bundle of compact and connected structure group with F being spin manifold as fiber and B being the base. This comes from the multiplicativity of elliptic genus [Ochanine 1988], which is equivalent to the Witten–Bott–Taubes–Liu rigidity [Bott and Taubes 1989; Taubes 1989; Liu 1996b].

Our main result is stated as follows.

Theorem 1. *The elliptic genus*

$$\phi : \Omega_{24}^{\mathrm{String}} \to \mathbb{Z}[8\delta, \varepsilon]$$

is injective and its image is a subgroup of $\mathbb{Z}[8\delta, \varepsilon]$ *spanned by*

$$(8\delta)^6, \qquad 24(8\delta)^4\varepsilon, \qquad (8\delta)^2\varepsilon^2, \qquad 8\varepsilon^3.$$

The theorem shows us the following picture:

$$\begin{aligned}\Omega_{24}^{\mathrm{String}} \cong \phi(\Omega_{24}^{\mathrm{String}}) &\cong \mathbb{Z} \oplus 24\mathbb{Z} \oplus \mathbb{Z} \oplus 8\mathbb{Z} \\ &\le \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \cong \phi(\Omega_{24}^{\mathrm{Spin}}).\end{aligned}$$

In particular, it supports the expectation that TMF-theoretic characteristic numbers will play a similar role for string cobordism as KO-theoretic characteristic numbers do for spin cobordism.

The key to the proof of Theorem 1 is a result in [Han and Huang 2022], where we determine an integral basis of $\Omega_{24}^{\mathrm{String}}$, which consists of two explicitly constructed manifolds in the kernel of the Witten genus, and another two-manifolds constructed by Mahowald and Hopkins [2002] determining the image of the Witten genus. Then we can apply two concrete elliptic genera (2-9) to reduce the computations of the elliptic genus to those of classical twisted and untwisted genera on the generators of $\Omega_{24}^{\mathrm{String}}$. The details are carried out in Section 3.

Theorem 1 has interesting application in geometry. A closed manifold M is called almost flat if for any $\varepsilon > 0$, there is a Riemannian metric g_ε on M such that the diameter $\operatorname{diam}(M, g_\varepsilon) \le 1$ and g_ε is ε-flat, i.e., for the sectional curvature K_{g_ε}, we have $|K_{g_\varepsilon}| < \varepsilon$. Given n, there is a positive number $\varepsilon_n > 0$ such that if an n-dimensional manifold admits an ε_n-flat metric with diameter ≤ 1, then it is almost flat. The classical result of Gromov [1978] says that every almost flat manifold is finitely covered by a nilmanifold, and this was refined by Ruh [1982] by proving that an almost flat manifold is diffeomorphic to an infranilmanifold. It has been conjectured by Farrell and Zdravkovska [1983] and independently by Yau [1993] that every almost flat manifold is the boundary of a closed manifold. Davis and Fang [2016] showed that this conjecture holds under the assumption that the 2-Sylow subgroup of holonomy group is cyclic or generalized quaternionic. The general case of the conjecture remains open. Davis and Fang [2016] also pointed out that it is a difficult question whether every almost flat spin manifold (up to changing spin structures) bounds a spin manifold.

By Chern–Weil theory, it can be shown that the Pontryagin numbers of an oriented almost flat manifold M all vanish [Davis and Fang 2016]. Since the elliptic genus is determined by Pontryagin numbers, one can see from Theorem 1 that every 24-dimensional almost flat string manifold bounds a string manifold.

In [Chen and Han 2024], vanishing results for elliptic genus were proven under almost nonpositive Ricci curvature condition. Theorem 1.3 there shows that given $n \in \mathbb{N}$ and positive number λ, there exists some $\varepsilon = \varepsilon(n, \lambda) > 0$ such that if a compact $4n$-dimensional spin Riemannian manifold (M, g) satisfies $\operatorname{diam}(M, g) \le 1$, $\operatorname{Ric}(g) \le \varepsilon$, sectional curvature $\ge -\lambda$ and has infinite isometry group, then the elliptic genus of M vanishes. Combining with Theorem 1, we obtain:

Corollary 2. *Given positive number λ, there exists some $\varepsilon = \varepsilon(\lambda) > 0$ such that if a compact 24-dimensional string Riemannian manifold (M, g) satisfies* $\operatorname{diam}(M, g) \le 1$, $\operatorname{Ric}(g) \le \varepsilon$, *sectional curvature $\ge -\lambda$ and has infinite isometry group*, *then M bounds a string manifold.* □

2. Preliminaries

In this section we collect some necessary knowledge of elliptic genus used in the sequel. Details can be found in [Hirzebruch et al. 1994; Liu 1992; 1995a].

Let f be the formal inverse function of the logarithm g in (1-2). Then $Y = f'$, $X = f$ solve the Jacobi quadrics

$$Y^2 = 1 - 2\delta \cdot X^2 + \varepsilon X^4. \tag{2-1}$$

For concrete values of δ and ε, a solution f gives an elliptic genus with logarithm g. For instance, when $\delta = \varepsilon = 1$, $f(z) = \tanh z$ and ϕ reduces to the L-genus or the signature, and when $\delta = -\frac{1}{8}$, $\varepsilon = 0$, $f(z) = 2\sinh\frac{z}{2}$ and ϕ reduces to the $\widehat{A}$-genus.

Recall that the four Jacobi theta-functions (see [Chandrasekharan 1985]) defined by infinite multiplications are

$$\theta(v,\tau)=2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}\big[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)\big],$$
$$\theta_1(v,\tau)=2q^{1/8}\cos(\pi v)\prod_{j=1}^{\infty}\big[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^j)(1+e^{-2\pi\sqrt{-1}v}q^j)\big],$$
$$\theta_2(v,\tau)=\prod_{j=1}^{\infty}\big[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^{j-1/2})(1-e^{-2\pi\sqrt{-1}v}q^{j-1/2})\big],$$
$$\theta_3(v,\tau)=\prod_{j=1}^{\infty}\big[(1-q^j)(1+e^{2\pi\sqrt{-1}v}q^{j-1/2})(1+e^{-2\pi\sqrt{-1}v}q^{j-1/2})\big],$$

where $q=e^{2\pi\sqrt{-1}\tau}$. They are holomorphic functions for $(v,\tau)\in\mathbb{C}\times\mathbb{H}$, where $\mathbb{C}$ is the complex plane and $\mathbb{H}$ is the upper half plane. Write $\theta_j=\theta_j(0,\tau),\ 1\le j\le 3$, and $\theta'(0,\tau)=\frac{\partial}{\partial v}\theta(v,\tau)\big|_{v=0}$.

When

$$\begin{aligned}\delta=\delta_1(\tau)&=\frac{1}{8}(\theta_2^4+\theta_3^4)=\frac{1}{4}+6\sum_{n=1}^{\infty}\sum_{\substack{d|n\\ d\ \text{odd}}}dq^n=\frac{1}{4}+6q+6q^2+\cdots,\\ \varepsilon=\varepsilon_1(\tau)&=\frac{1}{16}\theta_2^4\theta_3^4=\frac{1}{16}+\sum_{n=1}^{\infty}\sum_{d|n}(-1)^d d^3q^n=\frac{1}{16}-q+7q^2+\cdots,\end{aligned}\tag{2-2}$$

equation (2-1) has the solution

$$f_1(z,\tau)=2\pi\sqrt{-1}\frac{\theta(z,\tau)}{\theta'(0,\tau)}\frac{\theta_1(0,\tau)}{\theta_1(z,\tau)}.$$

Similarly, when

$$\begin{aligned}\delta=\delta_2(\tau)&=-\frac{1}{8}(\theta_1^4+\theta_3^4)=-\frac{1}{8}-3\sum_{n=1}^{\infty}\sum_{\substack{d|n\\ d\ \text{odd}}}dq^{n/2}=-\frac{1}{8}-3q^{1/2}-3q+\cdots,\\ \varepsilon=\varepsilon_2(\tau)&=\frac{1}{16}\theta_1^4\theta_3^4=\sum_{n=1}^{\infty}\sum_{\substack{d|n\\ n/d\ \text{odd}}}d^3q^{n/2}=q^{1/2}+8q+\cdots,\end{aligned}\tag{2-3}$$

equation (2-1) has the solution

$$f_2(z,\tau)=2\pi\sqrt{-1}\frac{\theta(z,\tau)}{\theta'(0,\tau)}\frac{\theta_2(0,\tau)}{\theta_2(z,\tau)}.$$

Let M be a $4k$-dimensional closed smooth oriented manifold. Let $\{\pm 2\pi\sqrt{-1}x_i, 1 \le i \le 2k\}$ be the formal Chern roots of the complexification $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$. Consider the two characteristic numbers

$$\begin{aligned} \mathrm{Ell}_1(M,\tau) &= 2^{2k}\left\langle \prod_{i=1}^{2k} \frac{2\pi\sqrt{-1}x_i}{f_1(x_i,\tau)}, [M]\right\rangle \in \mathbb{Q}[\![q]\!], \\ \mathrm{Ell}_2(M,\tau) &= \left\langle \prod_{i=1}^{2k} \frac{2\pi\sqrt{-1}x_i}{f_2(x_i,\tau)}, [M]\right\rangle \in \mathbb{Q}[\![q^{1/2}]\!]. \end{aligned} \tag{2-4}$$

$\mathrm{Ell}_1(M,\tau)$, $\mathrm{Ell}_2(M,\tau)$ can be written as signature and $\widehat{A}$-genus twisted by the Witten bundles. More precisely, let

$$\widehat{A}(M) = \prod_{i=1}^{2k} \frac{\pi\sqrt{-1}x_i}{\sinh \pi\sqrt{-1}x_i} \tag{2-5}$$

be the $\widehat{A}$-class and

$$\widehat{L}(M) = \prod_{i=1}^{2k} \frac{2\pi\sqrt{-1}x_i}{\tanh \pi\sqrt{-1}x_i} \tag{2-6}$$

the $\widehat{L}$-class. Let E be a complex vector bundle on M. $\langle \widehat{L}(M)\mathrm{ch}E, [M]\rangle$ is equal to the index of the twisted signature operator $\mathrm{ind}(d_s \otimes E) = \mathrm{Sig}(M, E)$. When M is spin, $\langle \widehat{A}(M)\mathrm{ch}E, [M]\rangle$ is equal to the index of the twisted Atiyah–Singer Dirac operator $\mathrm{ind}(D \otimes E)$. When twisted by bundles naturally constructed from the tangent bundle TM of M, denote

$$\begin{aligned} \widehat{A}(M, T^i \otimes \Lambda^j \otimes S^k) &:= \widehat{A}(M, \otimes^i T_{\mathbb{C}}M \otimes \Lambda^j(T_{\mathbb{C}}M) \otimes S^k(T_{\mathbb{C}}M)), \\ \mathrm{Sig}(M, T^i \otimes \Lambda^j \otimes S^k) &:= \mathrm{Sig}(M, \otimes^i T_{\mathbb{C}}M \otimes \Lambda^j(T_{\mathbb{C}}M) \otimes S^k(T_{\mathbb{C}}M)), \end{aligned}$$

where $\Lambda^j(T_{\mathbb{C}}M)$ and $S^k(T_{\mathbb{C}}M)$ are the j-th exterior and k-th symmetric powers of $T_{\mathbb{C}}M$ respectively.

For any complex variable t, let

$$\Lambda_t(E) = \mathbb{C} + tE + t^2\Lambda^2(E) + \cdots, \qquad S_t(E) = \mathbb{C} + tE + t^2 S^2(E) + \cdots$$

denote respectively the total exterior and symmetric powers of E, which live in $K(M)[\![t]\!]$. Denote by

$$\Theta_1(T_{\mathbb{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M - \mathbb{C}^{4k}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(T_{\mathbb{C}}M - \mathbb{C}^{4k}), \tag{2-7}$$

$$\Theta_2(T_{\mathbb{C}}M) = \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}}M - \mathbb{C}^{4k}) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-(1/2)}}(T_{\mathbb{C}}M - \mathbb{C}^{4k}) \tag{2-8}$$

the Witten bundles, which are elements in $K(M)[[q^{1/2}]]$. Then one has

$$\begin{aligned}\mathrm{Ell}_1(M,\tau) &= \langle \widehat{L}(M)\mathrm{ch}(\Theta_1(T_{\mathbb{C}}M)), [M]\rangle,\\ \mathrm{Ell}_2(M,\tau) &= \langle \widehat{A}(M)\mathrm{ch}(\Theta_2(T_{\mathbb{C}}M)), [M]\rangle.\end{aligned} \tag{2-9}$$

3. Proof of Theorem 1

Let M be a $4k$-dimensional closed smooth oriented manifold. By (1-3),

$$\phi(M) = a_0(M)\delta^6 + a_1(M)\delta^4\varepsilon + a_2(M)\delta^2\varepsilon^2 + a_3(M)\varepsilon^3, \tag{3-1}$$

where $a_i(M) \in \mathbb{Z}\left[\frac{1}{2}\right]$, $0 \le i \le 3$. First we show that one can express the 4 Pontryagin numbers $a_i(M)$ $(0 \le i \le 3)$ in terms of $\widehat{A}$-genus, signature and their twists by the tangent bundle.

Proposition 3.1. *Let M be a $4k$-dimensional closed smooth oriented manifold. One has*

$$\begin{aligned}
a_0(M) &= 2^{18}\widehat{A}(M),\\
a_1(M) &= -2^{15}\cdot 3\cdot 5\widehat{A}(M) - 2^{12}\widehat{A}(M,T),\\
a_2(M) &= 2^{16}\cdot 3\widehat{A}(M) + 2^{13}\widehat{A}(M,T) + \frac{1}{2^5}\,\mathrm{Sig}(M,T),\\
a_3(M) &= 2^{15}\widehat{A}(M) - 2^{12}\widehat{A}(M,T) - \frac{1}{2^5}\,\mathrm{Sig}(M,T) + \mathrm{Sig}(M).
\end{aligned}$$

Proof. From the preliminary in Section 2, we see that

$$\mathrm{Ell}_1(M) = 2^{12}\big(a_0(M)\delta_1^6 + a_1(M)\delta_1^4\varepsilon_1 + a_2(M)\delta_1^2\varepsilon_1^2 + a_3(M)\varepsilon_1^3\big), \tag{3-2}$$

$$\mathrm{Ell}_2(M) = a_0(M)\delta_2^6 + a_1(M)\delta_2^4\varepsilon_2 + a_2(M)\delta_2^2\varepsilon_2^2 + a_3(M)\varepsilon_2^3. \tag{3-3}$$

On the other hand, by (2-7), (2-8) and (2-9), it is not hard to compute that

$$\begin{aligned}\mathrm{Ell}_1(M,\tau) &= \mathrm{Sig}(M) + (2\,\mathrm{Sig}(M,T) - 48\,\mathrm{Sig}(M))q + \cdots,\\ \mathrm{Ell}_2(M,\tau) &= \widehat{A}(M) - (\widehat{A}(M,T) - 24\widehat{A}(M))q^{1/2} + \cdots.\end{aligned} \tag{3-4}$$

With the help of (2-2) and (2-3), we can compare (3-4) with (3-3) and (3-2). For instance, by modulo higher terms $(q^{1/2})^i$ with $i \ge 2$,

$$\begin{aligned}\mathrm{Ell}_2(M) &\equiv \frac{a_0(M)}{8^6}(-1-24q^{1/2})^6 + \frac{a_1(M)}{8^4}(-1-24q^{1/2})^4(q^{1/2})\\ &\equiv \frac{a_0(M)}{2^{18}} + \left(\frac{3^2a_0(M)}{2^{14}} + \frac{a_1(M)}{2^{12}}\right)q^{1/2}.\end{aligned}$$

Combining the above formula with (3-4), we have

$$\frac{a_0(M)}{2^{18}} = \widehat{A}(M), \qquad \frac{3^2a_0(M)}{2^{14}} + \frac{a_1(M)}{2^{12}} = -\widehat{A}(M,T) + 24\widehat{A}(M). \tag{3-5}$$

Similarly, by modulo higher terms q^i with $i \geq 2$,

$$\begin{aligned}\mathrm{Ell}_1(M) \equiv 2^{12}\biggl(&\frac{a_0(M)}{8^6}(2+48q)^6 + \frac{a_1(M)}{8^4}(2+48q)^4\Bigl(\frac{1}{16}-q\Bigr)\\ &+\frac{a_2(M)}{8^2}(2+48q)^2\Bigl(\frac{1}{16}-q\Bigr)^2 + a_3(M)\Bigl(\frac{1}{16}-q\Bigr)^3\biggr)\\ \equiv{}& (a_0(M)+a_1(M)+a_2(M)+a_3(M))\\ &+(144a_0(M)+80a_1(M)+16a_2(M)-48a_3(M))q.\end{aligned}$$

Combining the above formula with (3-4), we have

$$\begin{aligned} a_0(M)+a_1(M)+a_2(M)+a_3(M) &= \mathrm{Sig}(M),\\ 144a_0(M)+80a_1(M)+16a_2(M)-48a_3(M) &= 2\,\mathrm{Sig}(M,T)-48\,\mathrm{Sig}(M).\end{aligned} \tag{3-6}$$

The equalities in (3-5) and (3-6) can be organized to result in a matrix equation

$$\begin{pmatrix} \frac{1}{2^{18}} & 0 & 0 & 0\\ \frac{3^2}{2^{14}} & \frac{1}{2^{12}} & 0 & 0\\ 1 & 1 & 1 & 1\\ 144 & 80 & 16 & -48\end{pmatrix}\cdot\begin{pmatrix} a_0(M)\\ a_1(M)\\ a_2(M)\\ a_3(M)\end{pmatrix} = \begin{pmatrix} \widehat{A}(M)\\ -\widehat{A}(M,T)+24\widehat{A}(M)\\ \mathrm{Sig}(M)\\ 2\,\mathrm{Sig}(M,T)-48\,\mathrm{Sig}(M)\end{pmatrix}. \tag{3-7}$$

We can solve $a_i(M)$ from (3-7), and then the proposition is proved. □

Now suppose M is further a string manifold. With the string condition, we can rewrite the equalities of $a_i(M)$ in Proposition 3.1 in terms of a new family of (twisted) genera, which is helpful for proving Theorem 1.

Proposition 3.2. *Let M be a 24-dimensional closed smooth string manifold. Then*

$$\begin{pmatrix} a_0(M)\\ a_1(M)\\ a_2(M)\\ a_3(M)\end{pmatrix} = \begin{pmatrix} 2^{18} & 0 & 0 & 0\\ -2^{15}\cdot 3\cdot 5 & -2^{15}\cdot 3 & 0 & 0\\ 2^8\cdot 3\cdot 331 & 2^9\cdot 3^5 & 2^6 & 0\\ -2^8\cdot 97 & -2^9\cdot 3\cdot 17 & -2^6 & 2^3\end{pmatrix}\cdot\begin{pmatrix} \widehat{A}(M)\\ \frac{1}{24}\widehat{A}(M,T)\\ \widehat{A}(M,\Lambda^2)\\ \frac{1}{8}\,\mathrm{Sig}(M)\end{pmatrix}. \tag{3-8}$$

Proof. Under the string condition, the twisted and untwisted genera in Proposition 3.1 possess intrinsic relations. Indeed, by combining modularity of the Witten genus and a modular form constructed in [Liu and Wang 2013], Chen and Han [2015] showed that, when M is a 24-dimensional closed smooth string manifold, one has

$$\mathrm{Sig}(M,T) = 2^{11}\bigl(\widehat{A}(M,\Lambda^2)-47\widehat{A}(M,T)+900\widehat{A}(M)\bigr). \tag{3-9}$$

With (3-9) we can rewrite the equalities of $a_i(M)$ in Proposition 3.1 as displayed in this proposition. □

In [Han and Huang 2022] we determined an integral basis of $\Omega_{24}^{\mathrm{String}}$, which is crucial for the proof of Theorem 1.

Theorem 3.3 [Han and Huang 2022, Theorem 1 and Corollary 3]. *The correspondence $\kappa : \Omega_{24}^{\mathrm{String}} \to \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ defined by*

$$\kappa(M) = \left(\widehat{A}(M),\, \frac{1}{24}\widehat{A}(M,T),\, \widehat{A}(M,\Lambda^2),\, \frac{1}{8}\,\mathrm{Sig}(M)\right)$$

is an isomorphism of abelian groups. Moreover, there exists a basis $\{M_i\}_{1\le i\le 4}$ of $\Omega_{24}^{\mathrm{String}}$ such that

$$K := \begin{pmatrix} \kappa(M_1) \\ \kappa(M_2) \\ \kappa(M_3) \\ \kappa(M_4) \end{pmatrix}^{\tau} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2^3\cdot 3^3\cdot 5 & 2^2\cdot 3\cdot 17\cdot 1069 & -1 & 0 \\ 2^8\cdot 3\cdot 61 & 2^8\cdot 5\cdot 37 & 2^2\cdot 7 & 1 \end{pmatrix}. \qquad \square$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Suppose M satisfies that $\phi(M)=0$. Then by (3-1) $a_i(M)=0$ for $0\le i\le 3$. Notice that in (3-8) the coefficient matrix is invertible. Then by Proposition 3.2, the 4 index numbers $\widehat{A}(M)$, $\frac{1}{24}\widehat{A}(M,T)$, $\widehat{A}(M,\Lambda^2)$ and $\frac{1}{8}\,\mathrm{Sig}(M)$ vanish. This means that $\kappa(M)=(0,0,0,0)$. Since by Theorem 3.3 κ is an isomorphism, $[M]=0\in\Omega_{24}^{\mathrm{String}}$. Hence ϕ is injective.

To compute the image of the elliptic genus, we need to compute the elliptic genus of the generators M_i $(1\le i\le 4)$ in Theorem 3.3. By Proposition 3.2 and Theorem 3.3, they can be computed by the matrix multiplication

$$\begin{pmatrix} 2^{18} & 0 & 0 & 0 \\ -2^{15}\cdot 3\cdot 5 & -2^{15}\cdot 3 & 0 & 0 \\ 2^8\cdot 3\cdot 331 & 2^9\cdot 3^5 & 2^6 & 0 \\ -2^8\cdot 97 & -2^9\cdot 3\cdot 17 & -2^6 & 2^3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2^3\cdot 3^3\cdot 5 & 2^2\cdot 3\cdot 17\cdot 1069 & -1 & 0 \\ 2^8\cdot 3\cdot 61 & 2^8\cdot 5\cdot 37 & 2^2\cdot 7 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 2^{18} & 0 & 0 \\ 2^{15}\cdot 3 & -2^{15}\cdot 3\cdot 5 & 0 & 0 \\ -2^{11}\cdot 3^3 & 2^{11}\cdot 3^3\cdot 257 & -2^6 & 0 \\ 2^{12}\cdot 3^4 & -2^{12}\cdot 3^4\cdot 41 & 2^5\cdot 3^2 & 2^3 \end{pmatrix} = (a_j(M_i))_{j\times i}.$$

Combining (3-1), the above matrix gives the 4 generators of the image $\phi(\Omega_*^{\mathrm{String}})$ as

$$\begin{aligned} 2^3\cdot\varepsilon^3 &= \phi(M_4), \\ -(8\delta)^2\varepsilon^2 + 2^5\cdot 3^2\cdot\varepsilon^3 &= \phi(M_3), \\ 2^3\cdot 3\cdot(8\delta)^4\varepsilon - 2^5\cdot 3^3\cdot(8\delta)^2\varepsilon^2 + 2^{12}\cdot 3^4\cdot\varepsilon^3 &= \phi(M_1), \\ (8\delta)^6 - 2^3\cdot 3\cdot 5\cdot(8\delta)^4\varepsilon + 2^5\cdot 3^3\cdot 257\cdot(8\delta)^2\varepsilon^2 - 2^{12}\cdot 3^4\cdot 41\cdot\varepsilon^3 &= \phi(M_2). \end{aligned}$$

It follows that $\phi(\Omega_*^{\mathrm{String}})$ is generated by $2^3 \cdot \varepsilon^3$, $(8\delta)^2\varepsilon^2$, $2^3 \cdot 3 \cdot (8\delta)^4\varepsilon$ and $(8\delta)^6$. This completes the proof of the theorem. □

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