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We study the domination monoid in various classes of structures arising from henselian valuations, including $\mathcal{RV}$-expansions of henselian valued fields of equicharacteristic 0 (and, more generally, of benign valued fields), $\mathfrak{p}$-adically closed fields, monotone D-henselian differential valued fields with many constants, regular ordered abelian groups, and pure short exact sequences of abelian structures. We obtain Ax–Kochen–Ershov-type reductions to suitable fully embedded families of sorts in quite general settings, and full computations in concrete ones.

In their seminal work [17] on stable domination, Haskell, Hrushovski and Macpherson introduced the *domination monoid* $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, and showed that in algebraically closed valued fields it decomposes as $\widetilde{\mathrm{Inv}}(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$, where k denotes the residue field, Γ the value group, and $\mathfrak{U}$ a *monster* model, that is, a sufficiently saturated and strongly homogeneous model. (Strictly speaking, Haskell et al. [17] work with $\overline{\mathrm{Inv}}(\mathfrak{U})$, which is in general different, but coincides with $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in their setting. See [21, Remark 2.1.14 and Theorem 5.2.22].) A similar result was proven in [12; 23] in the case of real closed fields with a convex valuation. This paper revolves around understanding $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in more general classes of valued fields, and expansions thereof. A special case of our results is the following.

Theorem A (Corollary 6.19). *Let T be the theory of a henselian valued field of equicharacteristic* 0, *or algebraically maximal Kaplansky*, *possibly enriched on* k *and* Γ. *If all* $\mathrm{k}^\times/(\mathrm{k}^\times)^n$ *are finite*, *then* $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$.

More generally, we obtain a two-step reduction, first to leading term structures, and then, using technology on pure short exact sequences recently developed in [2], to k and Γ, albeit in a form which, in general, is (necessarily) slightly more

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involved. We also compute $\widetilde{\operatorname{Inv}}(\Gamma(\mathfrak{U}))$ when the theory of Γ has an archimedean model, and prove several accessory statements.

Before stating our results in more detail, let us give an informal account of the context (see Section 1 for the precise definitions). The starting point is the space $S^{\mathrm{inv}}(\mathfrak{U})$ of *invariant types* over a monster model $\mathfrak{U}$: those which are invariant over a small subset. It is a dense subspace of $S(\mathfrak{U})$, whose points may be canonically extended to larger parameter sets. Such extensions allow to define the *tensor product*, or *Morley product*, obtaining a semigroup $(S^{\mathrm{inv}}(\mathfrak{U}), \otimes)$, in fact a monoid. The space $S^{\mathrm{inv}}(\mathfrak{U})$ also comes with a preorder $\geq_{\mathrm{D}}$, called *domination*: roughly, $p \geq_{\mathrm{D}} q$ means that q is recoverable from p plus a small amount of information. The quotient by the induced equivalence relation, *domination-equivalence* $\sim_{\mathrm{D}}$, is then a poset, denoted by $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}})$. If $\otimes$ respects $\geq_{\mathrm{D}}$, i.e., if $(S^{\mathrm{inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}})$ is a preordered semigroup, then $\sim_{\mathrm{D}}$ is a congruence with respect to $\otimes$ and we say that *the domination monoid is well defined*, and equip $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}})$ with the operation induced by $\otimes$. Compatibility of $\otimes$ and $\geq_{\mathrm{D}}$ in a given theory can be shown by using certain sufficient criteria, isolated in [22] and applied, e.g., in [24], or by finding a nice system of representatives for $\sim_{\mathrm{D}}$-classes (see Proposition 1.3). Nevertheless, in general, $\otimes$ may fail to respect $\geq_{\mathrm{D}}$ [22]. Hence, when dealing with $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ in a given structure, one needs to understand whether it is well defined as a monoid; and, when dealing with it in the abstract, the monoid structure cannot be taken for granted.

Recall that to a valued field K are associated certain abelian groups augmented by an absorbing element, fitting in a short exact sequence

$$1 \to (\mathrm{k}, \times) \to (\mathrm{K}, \times)/(1+\mathfrak{m}) \to \Gamma \cup \{\infty\} \to 0,$$

denoted by $\mathcal{RV}$. This sequence is interpretable in K, and this interpretation endows it with extra structure. The amount of induced structure clearly depends on whether K has extra structure itself, but at a bare minimum k will carry the language of fields and Γ that of ordered abelian groups. By [4] (see also [20], or [14; 15] for a more modern treatment), henselian valued fields of residue characteristic 0 eliminate quantifiers relatively to $\mathcal{RV}$, and the latter is fully embedded with the structure described above. This holds resplendently, in the sense that it is still true after arbitrary expansions of $\mathcal{RV}$. The same holds in the algebraically maximal Kaplansky case, by [20] (see also [15]).[1] These are known after [30] as classes of *benign* valued fields and, in several contexts, they turn out to be particularly amenable to model-theoretic investigation. One of our main results says the context of domination is no exception.

Theorem B (Theorem 6.18). *In every $\mathcal{RV}$-expansion of a benign theory of valued fields there is an isomorphism of posets $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \widetilde{\operatorname{Inv}}(\mathcal{RV}(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{RV}(\mathfrak{U})$, then $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}$, and the above is an isomorphism of monoids.*

[1] Note that these quantifier elimination results are already implicitly contained in [9].

Having reduced $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ to the short exact sequence $\mathcal{RV}$, the next step is to reduce it to its kernel k and quotient Γ. If we add an angular component map, the sequence $\mathcal{RV}$ splits and we obtain a product decomposition as in Theorem A (Remark 6.1). Without an angular component, a product decomposition is not always possible; yet, k and Γ still exert a tight control on $\mathcal{RV}$. This behaviour is not peculiar of $\mathcal{RV}$: it holds in short exact sequences of abelian structures, provided they satisfy a purity assumption, using the relative quantifier elimination from [2]. For reasons to be clarified later (Remark 4.17), here it is natural to look at types in infinitely many variables, say κ, and hence at the corresponding analogue $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$.

Theorem C (Corollary 4.9). *Let $\mathfrak{U}$ be a pure short exact sequence*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

of L-abelian structures, where $\mathcal{A}$ and $\mathcal{C}$ may carry extra structure. Let $\kappa \geq |L|$ be a small cardinal. There is an expansion $\mathcal{A}_{\mathcal{F}}$ of $\mathcal{A}$ by imaginary sorts yielding an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathcal{A}_{\mathcal{F}}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\kappa(\mathcal{C}(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in both $\mathcal{A}_{\mathcal{F}}(\mathfrak{U})$ and $\mathcal{C}(\mathfrak{U})$, then $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}$, and the above is an isomorphism of monoids.

In algebraically or real closed valued fields, the isomorphism

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$$

is complemented by a computation of the factors, carried out in [17; 23]. In particular, if $\Gamma(\mathfrak{U})$ is divisible, then $\widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$ is isomorphic to the upper semilattice of finite sets of invariant convex subgroups of $\Gamma(\mathfrak{U})$ (in the sense of Definition 3.16). A further contribution of this work is the computation of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in the next simplest class of theories of ordered abelian groups: those with an archimedean model, known as *regular*. Denote by $\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U})$ the set of invariant convex subgroups of $\mathfrak{U}$, by $\mathscr{P}_{\leq\kappa}(\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U}))$ the upper semilattice of its subsets of size at most κ, and by $\hat{\kappa}$ the ordered monoid of cardinals smaller or equal than κ with cardinal sum.

Theorem D (Corollary 3.33). *Let T be the theory of a regular ordered abelian group, κ a small infinite cardinal, and $\mathbb{P}_T$ the set of primes $\mathfrak{p}$ such that $\mathfrak{U}/\mathfrak{p}\mathfrak{U}$ is infinite. Then $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}})$ is well defined, and $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}}) \cong \mathscr{P}_{\leq\kappa}(\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U})) \times \prod_{\mathbb{P}_T} \hat{\kappa}$.*

Theorem D applies to Presburger arithmetic, the theory of $(\mathbb{Z}, +, <)$. Pairing this with a suitable generalisation of Theorem B, we obtain the following.

Theorem E (Corollary 7.7). *In the theory $\mathrm{Th}(\mathbb{Q}_\mathfrak{p})$ of $\mathfrak{p}$-adically closed fields, $\otimes$ respects $\geq_{\mathrm{D}}$, and $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \mathscr{P}_{<\omega}\big(\mathrm{CS}^{\mathrm{inv}}(\Gamma(\mathfrak{U}))\big)$.*

A similar statement (Corollary 7.5) holds for Witt vectors over $\mathbb{F}_\mathfrak{p}^{\mathrm{alg}}$. Finally, we move to monotone D-henselian differential valued fields with many constants.

While Theorem B does not generalise to this context (Remark 8.5), its analogue for $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ does (Theorem 8.2). We fully compute $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ in the model companion $\mathsf{VDF}_{\mathcal{EC}}$. Similar results hold for σ-henselian valued difference fields (Remark 8.6).

Theorem F (Theorem 8.4). *In $\mathsf{VDF}_{\mathcal{EC}}$, for every small infinite cardinal κ, the monoid $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ is well defined, and we have isomorphisms*

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\kappa(\Gamma(\mathfrak{U})) \cong \prod_{\delta(\mathfrak{U})}^{\leq\kappa} \hat{\kappa} \times \mathscr{P}_{\leq\kappa}\big(\mathrm{CS}^{\mathrm{inv}}(\Gamma(\mathfrak{U}))\big),$$

where $\delta(\mathfrak{U})$ is a certain cardinal, and $\prod_{\delta(\mathfrak{U})}^{\leq\kappa}\hat{\kappa}$ denotes the submonoid of $\prod_{\delta(\mathfrak{U})}\hat{\kappa}$ consisting of $\delta(\mathfrak{U})$-sequences with support of size at most κ.

The paper is structured as follows. In the first two sections we recall some preliminary notions and facts, and deal with some easy observations about orthogonality of invariant types. In Section 3 we prove Theorem D, while in Section 4 we study expanded pure short exact sequences of abelian structures, proving Theorem C. The results from these two sections are then combined in Section 5 to deal with the case of ordered abelian groups with finitely many definable convex subgroups. In Section 6 we prove Theorem B, and illustrate how it may be combined with Theorem C to obtain statements such as Theorem A. Section 7 deals with finitely ramified mixed characteristic henselian valued fields and includes a proof of Theorem E, and Section 8 deals with the differential case, proving Theorem F.

1. Preliminaries

Notation and conventions. We adopt the conventions and notations of [23, Section 1.1] (e.g., we usually (and tacitly) fix a *monster model* $\mathfrak{U}$, and *definable* means $\mathfrak{U}$-definable), with the following additions and differences. The set of prime natural numbers is denoted by $\mathbb{P}$. Sorts are denoted by upright letters, as in A, K, k, Γ, families of sorts by calligraphic letters such as $\mathcal{C}$, and $S_{\mathcal{C}^{<\omega}}(A)$ stands for the disjoint union of all spaces of types in finitely many variables, each with sort in $\mathcal{C}$. Terms may contain parameters, as in $t(x, d)$; we write $t(x)$ if they do not.

Domination. We assume familiarity with invariant types, and recall some basic definitions and facts about domination. See [23, Section 1.2], [21, Section 2.1.2] and [22] for a more thorough treatment.

If $p(x), q(y) \in S(\mathfrak{U})$, let $S_{pq}(A)$ be the set of types over A in variables xy extending $(p(x) \upharpoonright A) \cup (q(y) \upharpoonright A)$. We say that $p(x) \in S(\mathfrak{U})$ *dominates* $q(y) \in S(\mathfrak{U})$, and write $p \geq_{\mathrm{D}} q$, if there are a small $A \subset^+ \mathfrak{U}$ and $r \in S_{pq}(A)$ such that $p(x) \cup r(x, y) \vdash q(y)$. We say that $p, q \in S(\mathfrak{U})$ are *domination-equivalent*, and write $p \sim_{\mathrm{D}} q$, if $p \geq_{\mathrm{D}} q$ and $q \geq_{\mathrm{D}} p$. We denote the domination-equivalence class of p by $[\![p]\!]$. The *domination poset* $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is the quotient of $S^{\mathrm{inv}}(\mathfrak{U})$ by $\sim_{\mathrm{D}}$, equipped with the partial order induced by $\geq_{\mathrm{D}}$, denoted by the same symbol. In other words, domination is the semiisolation counterpart to $\mathrm{F}^{\mathrm{s}}_{\kappa(\mathfrak{U})}$-isolation in the sense of [29, Chapter IV]; the two notions are distinct, see [24, Example 3.3].

We will be mostly concerned with domination on $S^{\mathrm{inv}}(\mathfrak{U})$. When describing a witness to $p \geq_{\mathrm{D}} q$, we write, e.g., "let r contain $\varphi(x, y)$" with the meaning "let $r \in S_{pq}(A)$ contain $\varphi(x, y)$, for an A such that $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, A)$". By [22, Lemma 1.14], if $p_0, p_1 \in S^{\mathrm{inv}}(\mathfrak{U})$ and $p_0 \geq_{\mathrm{D}} p_1$, then $p_0 \otimes q \geq_{\mathrm{D}} p_1 \otimes q$. We say that $\otimes$ *respects* $\geq_{\mathrm{D}}$ if $q_0 \geq_{\mathrm{D}} q_1$ implies $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$. If this is the case, the *domination monoid* is the expansion of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ by the operation induced by $\otimes$, also denoted by $\otimes$. If we say $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ *is well defined* (as a partially ordered monoid) we mean "$\otimes$ respects $\geq_{\mathrm{D}}$". As $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is always well defined as a poset, this should cause no confusion.

Adding imaginary sorts to $\mathfrak{U}$ may result in an enlargement of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ [22, Corollary 3.8]. Yet, if T eliminates imaginaries, even just geometrically, then the natural embedding $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ is easily seen to be an isomorphism. By [22, Proposition 1.23], domination witnessed by algebraicity is compatible with $\otimes$: if $p, q_0, q_1 \in S^{\mathrm{inv}}(\mathfrak{U})$ and, for $i < 2$, there are realisations $a_i \models q_i$ such that $a_1 \in \mathrm{acl}(\mathfrak{U}a_0)$, then for all invariant p we have $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$. In particular, if T has geometric elimination of imaginaries, then $\widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ is well defined if and only if $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is.

Frequently, we will equip a family of sorts, say $\mathcal{A} = \{\mathrm{A}_s \mid s \in S\}$, with the traces of some $\varnothing$-definable relations, and consider it as a standalone structure. We call $\mathcal{A}$ *fully embedded* if, for each $s_0, \dots, s_n \in S$, every subset of $(\mathrm{A}_{s_0} \times \cdots \times \mathrm{A}_{s_n})(\mathfrak{U})$ is definable in $\mathfrak{U}$ if and only if it is definable in $\mathcal{A}(\mathfrak{U})$. When talking of a fully embedded $\mathcal{A}$ in the abstract, as below, we assume a structure on $\mathcal{A}$ to be fixed.

Fact 1.1 [21, Proposition 2.3.31]. Let $\mathcal{A}$ be a fully embedded family of sorts, and let $\iota : S_{\mathcal{A}^{<\omega}}(\mathcal{A}(\mathfrak{U})) \to S(\mathfrak{U})$ send a type of $\mathcal{A}(\mathfrak{U})$ to the unique type of $\mathfrak{U}$ it entails. The type p is invariant if and only if $\iota(p)$ is. The map $\iota \restriction S^{\mathrm{inv}}(\mathcal{A}(\mathfrak{U}))$ is an injective $\otimes$-homomorphism inducing an embedding of posets $\tilde{\iota} : \widetilde{\mathrm{Inv}}(\mathcal{A}(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$ which, if $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}$ (hence also in $\mathcal{A}(\mathfrak{U})$), is also an embedding of monoids.

Remark 1.2. With the notation and assumptions from Fact 1.1, if p is an invariant $\mathcal{A}(\mathfrak{U})$-type, $\mathfrak{U}_1 \succ \mathfrak{U}$, and $\mathcal{A}(\mathfrak{U}) \subseteq B \subseteq \mathcal{A}(\mathfrak{U}_1)$, then $(p \mid B) \vdash (\iota p \mid \mathfrak{U}B)$.

Proof. Suppose $\varphi(x, w, t) \in L(\varnothing)$, $d \in \mathfrak{U}$, $e \in B$, and $\iota p(x) \mid B \vdash \varphi(x, d, e)$. Since x, t are $\mathcal{A}$-variables, and $d \in \mathfrak{U}$, full embeddedness yields an $L_{\mathcal{A}}(\mathcal{A}(\mathfrak{U}))$-formula $\psi(x, t)$ equivalent to $\varphi(x, d, t)$. So $\psi(x, e) \in p \mid B$ and we are done. □

Proposition 1.3. *Assume for all $p \in S^{\mathrm{inv}}(\mathfrak{U})$ there is a tuple τ^p of definable functions with codomains in a fully embedded $\mathcal{A}$ such that $p \sim_{\mathrm{D}} \tau^p_* p$ and $p \otimes q \sim_{\mathrm{D}} \tau^p_* p \otimes \tau^q_* q$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{A}(\mathfrak{U})$, then $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}$.*

Proof. We need to show that if $q_0 \geq_{\mathrm{D}} q_1$ then $p \otimes q_0 \geq_{\mathrm{D}} p \otimes q_1$. By assumption, $p \otimes q_0 \sim_{\mathrm{D}} \tau_*^p p \otimes \tau_*^{q_0} q_0$ and $\tau_*^p p \otimes \tau_*^{q_1} q_1 \sim_{\mathrm{D}} p \otimes q_1$. Since $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{A}(\mathfrak{U})$, we obtain $\tau_*^p p \otimes \tau_*^{q_0} q_0 \geq_{\mathrm{D}} \tau_*^p p \otimes \tau_*^{q_1} q_1$, and we are done. □

Note that a map τ as above induces an inverse of $\tilde{\iota}$.

A word on* $*$*-types. We will deal with types in a small infinite number of variables, also known in the literature as $*$-types. We define $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ as the quotient of $S_{<\kappa^+}(\mathfrak{U})$ by $\sim_{\mathrm{D}}$. Note that, by padding with realised coordinates and permuting variables, every $\sim_{\mathrm{D}}$-class has a representative with variables indexed by κ. We leave to the reader easy tasks such as defining the α-th power $p^{(\alpha)}$, for α an ordinal, or such as convincing themselves that basic statements such as Fact 1.1 generalise.

Nevertheless, it is not clear if well-definedness of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ implies well-definedness of $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ (the converse is easy): for instance, at least a priori, one could have a situation where the finitary $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well defined, but there are a 1-type q_0 and a κ-type q_1 such that $q_0 \geq_{\mathrm{D}} q_1$ but, for some p, we have $p \otimes q_0 \not\geq_{\mathrm{D}} p \otimes q_1$. In the rest of the paper we will say, e.g., "$\otimes$ respects $\geq_{\mathrm{D}}$" with the understanding that, whenever $*$-types are involved, this is to be read as "$\otimes$ respects $\geq_{\mathrm{D}}$ on $*$-types".

Question 1.4. If $\otimes$ respects $\geq_{\mathrm{D}}$ on finitary types, does $\otimes$ respect $\geq_{\mathrm{D}}$ on $*$-types?

2. Orthogonality

Definition 2.1. We say that $p, q \in S(A)$ are *weakly orthogonal*, and write $p \perp^{\mathrm{w}} q$, if $p(x) \cup q(y)$ implies a complete xy-type over A. We say that $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$ are *orthogonal*, and write $p \perp q$, if $(p \mid B) \perp^{\mathrm{w}} (q \mid B)$ for every $B \supseteq \mathfrak{U}$. Two definable sets φ, ψ are *orthogonal* if for every $n, m \in \omega$, every $p \in S_{\varphi^n}(\mathfrak{U})$ and $q \in S_{\psi^m}(\mathfrak{U})$, we have $p \perp^{\mathrm{w}} q$. Two families of sorts $\mathcal{A}, \mathcal{C}$ are *orthogonal* if every cartesian product of sorts in $\mathcal{A}$ is orthogonal to every cartesian product of sorts in $\mathcal{C}$.

It is easily seen that if $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, M)$ are weakly orthogonal and $\mathfrak{U}_1 \succ \mathfrak{U}$ is $|M|^+$-saturated and $|M|^+$-strongly homogeneous, then $(p \mid \mathfrak{U}_1) \perp^{\mathrm{w}} (q \mid \mathfrak{U}_1)$. This can fail for arbitrary $B \supseteq \mathfrak{U}$, i.e., weak orthogonality is indeed weaker than orthogonality. While this is folklore (Mennuni thanks E. Hrushovski for pointing this out), we could not find any example in print, so we record one.

Example 2.2. There is a theory with invariant p, q such that $p \perp^{\mathrm{w}} q$ but $p \not\perp q$.

Proof. Let L be a two-sorted language with sorts P, O (points, orders) and a relation symbol $x <_t y$ of arity $\mathrm{P}^2 \times \mathrm{O}$. The class K of finite L-structures where, for every $d \in \mathrm{O}$, the relation $x <_d y$ is a linear order, is a (strong) amalgamation class. Let T be the theory of the Fraïssé limit of K. Fix a small $M \vDash T$, and let p, q be the 1-types of sort P defined as $p(x) = \{m <_d x <_d e \mid d \in \mathrm{O}(\mathfrak{U}), m \in M, e \in \mathrm{P}(\mathfrak{U}), e > M\}$ and $q(y) := \{e <_d y \mid d \in \mathrm{O}(\mathfrak{U}), e \in \mathrm{P}(\mathfrak{U})\}$. By quantifier elimination p, q are complete, p is M-invariant, and q is $\varnothing$-definable, hence $\varnothing$-invariant.

Since M is small, for every $d \in \mathrm{O}(\mathfrak{U})$ it is $<_d$-bounded, hence $p \perp^{\mathrm{w}} q$. Let b be a point of sort O such that M is $\leq_b$-cofinal in $\mathfrak{U}$, and set $B := \mathfrak{U}b$. Then $(q(y) \mid B) \vdash y \geq_b \mathrm{P}(\mathfrak{U})$ and $(p(x) \mid B) \vdash x \geq_b \mathrm{P}(\mathfrak{U})$, and both $x <_b y$ and $y <_b x$ are consistent with $(p(x) \mid B) \cup (q(y) \mid B)$, which is therefore not complete. □

Remark 2.3. If $p \in S(A)$ is such that $p \perp^{\mathrm{w}} p$, then p is realised in $\mathrm{dcl}(A)$. If $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$ and $p \perp^{\mathrm{w}} q$, then $p(x) \otimes q(y) = q(y) \otimes p(x)$: they both coincide with (the unique completion of) $p(x) \cup q(y)$. Two definable sets φ, ψ are orthogonal if and only if every definable subset of $\varphi^m(x) \wedge \psi^n(y)$ can be defined by a finite disjunction of formulas of the form $\theta(x) \wedge \eta(y)$. If two M-definable sets are orthogonal, then the definition of orthogonality still holds after replacing $\mathfrak{U}$ with M. Adding imaginaries preserves orthogonality, in the following sense. Let $\mathcal{A}$ be a family of sorts, and let $\tilde{\mathcal{A}}$ be a larger family, consisting of $\mathcal{A}$ together with imaginary sorts obtained as definable quotients of products of elements of $\mathcal{A}$. Let $\tilde{\mathcal{C}}$ be obtained similarly from another family of sorts $\mathcal{C}$. If $\mathcal{A}$ and $\mathcal{C}$ are orthogonal, then so are $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{C}}$.

By [22, Proposition 3.13], if $p_0 \geq_{\mathrm{D}} p_1$ and $p_0 \perp^{\mathrm{w}} q$, then $p_1 \perp^{\mathrm{w}} q$. In particular, if $p_0 \geq_{\mathrm{D}} q$ and $p_0 \perp^{\mathrm{w}} q$, then q is realised. As a consequence, $\perp^{\mathrm{w}}$ induces a well-defined relation on the domination poset, which we may expand to $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}}, \perp^{\mathrm{w}})$. By [22, Proposition 2.3.31] the map $\tilde{\iota}$ from Fact 1.1 is a homomorphism for both $\perp^{\mathrm{w}}$ and $\not\perp^{\mathrm{w}}$. We prove the analogous statements for orthogonality.

Proposition 2.4. *Let $p_0, p_1, q \in S^{\mathrm{inv}}(\mathfrak{U})$. If $p_0 \perp q$ and $p_0 \geq_{\mathrm{D}} p_1$, then $p_1 \perp q$. In particular, $\perp$ induces a well-defined relation on $\widetilde{\mathrm{Inv}}(\mathfrak{U})$.*

Proof. Fix r witnessing $p_0 \geq_{\mathrm{D}} p_1$ and let $B \supseteq \mathfrak{U}$. Let $b \vDash p_1 \mid B$ and $c \vDash q \mid B$. By [22, Lemma 1.13], $(p_0 \mid B) \cup r \vdash (p_1 \mid B)$. Let a be such that $ab \vDash (p_0 \mid B) \cup r$. Since $p_0 \perp q$, we have $(p_0 \mid B) \perp^{\mathrm{w}} (q \mid B)$, and hence $a \vDash p_0 \mid Bc$. Again by [22, Lemma 1.13] we have $(p_0 \mid Bc) \cup r \vdash (p_1 \mid Bc)$, therefore $b \vDash p_1 \mid Bc$. □

Proposition 2.5. *In the setting of Fact 1.1, $\iota \restriction S^{\mathrm{inv}}_{\mathcal{A}^{<\omega}}(\mathcal{A}(\mathfrak{U}))$ is a $\perp$-homomorphism and a $\not\perp$-homomorphism, and so is the induced map $\tilde{\iota} : \widetilde{\mathrm{Inv}}(\mathcal{A}(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$.*

Proof. Let $p, q \in S^{\mathrm{inv}}_{\mathcal{A}^{<\omega}}(\mathcal{A}(\mathfrak{U}))$ be orthogonal and let $\mathfrak{U}_1 \succ \mathfrak{U}$ be $|\mathfrak{U}|^+$-saturated and $|\mathfrak{U}|^+$-strongly homogeneous. We show that, for $\varphi(x, y, z) \in L(\mathfrak{U})$ and $d \in \mathfrak{U}_1$, if $(\iota p(x) \otimes \iota q(y)) \mid \mathfrak{U}_1 \vdash \varphi(x, y, d)$ then $(\iota p \mid \mathfrak{U}d)(x) \cup (\iota q \mid \mathfrak{U}d)(y) \vdash \varphi(x, y, d)$. By full embeddedness, there are $\chi(x, y, w) \in L_{\mathcal{A}}(\mathcal{A}(\mathfrak{U}))$ and $e \in \mathcal{A}(\mathfrak{U}_1)$ such that $\mathfrak{U}_1 \vDash \forall x, y\ (\chi(x, y, e) \leftrightarrow \varphi(x, y, d))$. Because $(p \mid \mathcal{A}(\mathfrak{U})e) \perp^{\mathrm{w}} (q \mid \mathcal{A}(\mathfrak{U})e)$, there are $\theta_p(x, w), \theta_q(y, w) \in L_{\mathcal{A}}(\mathcal{A}(\mathfrak{U}))$ such that $(p \mid \mathcal{A}(\mathfrak{U})e) \vdash \theta_p(x, e)$, $(q \mid \mathcal{A}(\mathfrak{U})e) \vdash \theta_q(y, e)$, and $\mathcal{A}(\mathfrak{U}_1) \vDash \forall x, y\ ((\theta_p(x, e) \wedge \theta_q(y, e)) \rightarrow \chi(x, y, e))$. By invariance of p, q, we have

$$\pi_p(x) := \{\theta_p(x, e') \mid e' \in \mathfrak{U}_1, e \equiv_{\mathfrak{U}d} e'\} \subseteq \iota p \mid \mathfrak{U}_1,$$

$$\pi_q(y) := \{\theta_q(y, e') \mid e' \in \mathfrak{U}_1, e \equiv_{\mathfrak{U}d} e'\} \subseteq \iota q \mid \mathfrak{U}_1.$$

So π_p, π_q are consistent. As $\operatorname{Aut}(\mathfrak{U}_1/\mathfrak{U}d)$ fixes them, they are equivalent to partial types σ_p, σ_q over $\mathfrak{U}d$. But $\sigma_p \subseteq \iota p \mid \mathfrak{U}d$, $\sigma_q \subseteq \iota q \mid \mathfrak{U}d$, and $\sigma_p(x) \cup \sigma_q(y) \vdash \varphi(x, y, d)$, proving that $\perp$ is preserved.

Suppose there is B with $\mathcal{A}(\mathfrak{U}) \subseteq B \subseteq \mathcal{A}(\mathfrak{U}_1)$ such that $(p \mid B) \not\perp^{\mathrm{w}} (q \mid B)$. By Remark 1.2, this yields $(\iota p \mid \mathfrak{U}B) \not\perp^{\mathrm{w}} (\iota q \mid \mathfrak{U}B)$, proving that $\not\perp$ is preserved as well.

The statement for $\tilde{\iota}$ follows from Proposition 2.4. □

Lemma 2.6. *Let $p, q_0, q_1 \in S(\mathfrak{U})$, with $p \perp^{\mathrm{w}} q_0$ and $(p(x) \cup q_0(y)) \geq_{\mathrm{D}} q_1(z)$, witnessed by $r \in S_{p \otimes q_0, q_1}(M)$. If $(r \restriction x) \perp^{\mathrm{w}} (r \restriction yz)$, then $q_0 \geq_{\mathrm{D}} q_1$, witnessed by $r \restriction yz$. Hence, if $\mathcal{A}$, $\mathcal{C}$ are orthogonal families of sorts, $p \in S^{\mathrm{inv}}_{\mathcal{A}^{<\omega}}(\mathfrak{U})$, and $q_0, q_1 \in S^{\mathrm{inv}}_{\mathcal{C}^{<\omega}}(\mathfrak{U})$, if $(p \cup q_0) \geq_{\mathrm{D}} q_1$, then $q_0 \geq_{\mathrm{D}} q_1$.*

Proof. Routine, left to the reader. □

Recall that the product $\prod_{i \in I} P_i$ of a family of posets $(P_i, \leq_i)_{i \in I}$ is the cartesian product of the P_i partially ordered by $(p_i)_{i \in I} \leq (q_i)_{i \in I}$ if $\forall i \in I \;\; p_i \leq_i q_i$.

Corollary 2.7. *Suppose that $\mathcal{A}$, $\mathcal{C}$ are orthogonal, fully embedded families of sorts. Assume that for every $p \in S^{\mathrm{inv}}(\mathfrak{U})$ there are some $p_{\mathcal{A}} \in S^{\mathrm{inv}}_{\mathcal{A}^{<\omega}}(\mathfrak{U})$ and $p_{\mathcal{C}} \in S^{\mathrm{inv}}_{\mathcal{C}^{<\omega}}(\mathfrak{U})$ such that $p \sim_{\mathrm{D}} p_{\mathcal{A}} \cup p_{\mathcal{C}}$. Then the map $[\![p]\!] \mapsto ([\![p_{\mathcal{A}}]\!], [\![p_{\mathcal{C}}]\!])$ is an isomorphism of posets $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \to \widetilde{\operatorname{Inv}}(\mathcal{A}(\mathfrak{U})) \times \widetilde{\operatorname{Inv}}(\mathcal{C}(\mathfrak{U}))$. Moreover, if $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}$ (hence also in $\mathcal{A}(\mathfrak{U})$, $\mathcal{C}(\mathfrak{U})$), then this is also an isomorphism of monoids.*

Proof. Fact 1.1 yields embeddings of posets

$$\widetilde{\operatorname{Inv}}(\mathcal{A}(\mathfrak{U})) \hookrightarrow \widetilde{\operatorname{Inv}}(\mathfrak{U}) \quad \text{and} \quad \widetilde{\operatorname{Inv}}(\mathcal{C}(\mathfrak{U})) \hookrightarrow \widetilde{\operatorname{Inv}}(\mathfrak{U}).$$

We define a morphism of posets $\widetilde{\operatorname{Inv}}(\mathcal{A}(\mathfrak{U})) \times \widetilde{\operatorname{Inv}}(\mathcal{C}(\mathfrak{U})) \to \widetilde{\operatorname{Inv}}(\mathfrak{U})$ by setting $([\![p(x)]\!], [\![q(y)]\!]) \mapsto ([\![p(x) \cup q(y)]\!])$. It follows from orthogonality of $\mathcal{A}$ and $\mathcal{C}$ that this morphism is well defined: if $p' \sim_{\mathrm{D}} p$ and $q' \sim_{\mathrm{D}} q$, by just taking unions of domination witnesses we find that $p \cup q \sim_{\mathrm{D}} p' \cup q'$. As this map is injective by Lemma 2.6, it is enough to show that the natural candidate for its inverse, $[\![p]\!] \mapsto ([\![p_{\mathcal{A}}]\!], [\![p_{\mathcal{C}}]\!])$, is well defined and a morphism of posets. Both these statements follow from the observation that, if $(p_{\mathcal{A}} \cup p_{\mathcal{C}}) \sim_{\mathrm{D}} p \geq_{\mathrm{D}} q \sim_{\mathrm{D}} (q_{\mathcal{A}} \cup q_{\mathcal{C}})$, then by Lemma 2.6 we must have $p_{\mathcal{A}} \geq_{\mathrm{D}} q_{\mathcal{A}}$ and $p_{\mathcal{C}} \geq_{\mathrm{D}} q_{\mathcal{C}}$. The "moreover" part follows from Fact 1.1, and the fact that $\mathcal{AC}$ is fully embedded. □

Example 2.8. Let A, C be structures in disjoint languages, T the theory of their disjoint union, in families of sorts $\mathcal{A}$, $\mathcal{C}$. Then $\mathcal{A}$ and $\mathcal{C}$ are orthogonal, and invariant types in $\mathcal{A}$ are orthogonal to those in $\mathcal{C}$. Therefore, $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is isomorphic to $\widetilde{\operatorname{Inv}}(\mathcal{A}(\mathfrak{U})) \times \widetilde{\operatorname{Inv}}(\mathcal{C}(\mathfrak{U}))$, and is well defined as a monoid if and only if both factors are.

Orthogonality is preserved by the Morley product. The proof is folklore, and essentially the same as in the stable case, but we record it here for convenience.

Proposition 2.9. *If $p_0, p_1 \in S^{\mathrm{inv}}(\mathfrak{U})$ are orthogonal to q, then so is $p_0 \otimes p_1$.*

Proof. Let $ab \vDash p_0 \otimes p_1$ and $c \vDash q$. Because $p_1 \perp q$ we have $c \vDash q \mid \mathfrak{U}b$, and by definition of $\otimes$ we have $a \vDash p_0 \mid \mathfrak{U}b$. Since $p_0 \perp q$, this entails $c \vDash q \mid \mathfrak{U}ab$. □

3. Regular ordered abelian groups

In this section we study the domination monoid in certain theories of (linearly) ordered abelian groups, henceforth *oags*. Model-theoretically, the simplest oags are the (nontrivial) divisible ones. Their theory is o-minimal and their domination monoid was one of the first ones to be computed [17; 23]. It is isomorphic to the finite powerset semilattice $(\mathscr{P}_{<\omega}(\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U})), \cup, \subseteq)$ of the set of invariant convex subgroups of $\mathfrak{U}$, and weakly orthogonal classes of types correspond to disjoint finite sets. Divisible oags eliminate quantifiers in the language $L_{\mathrm{oag}} := \{+, 0, -, <\}$. In this section we compute the domination monoid in the next simplest case.

Definition 3.1. A (nontrivial) oag is *discrete* if it has a minimum positive element, and *dense* otherwise. We view an oag M as a structure in the *Presburger language* $L_{\mathrm{Pres}} := \{+, 0, -, <, 1, \equiv_n \mid n \in \omega\}$ by interpreting $+, 0, -, <$ in the natural way, 1 as the minimum positive element if M is discrete and as 0 otherwise, and $\equiv_n$ as congruence modulo nM. An oag is *regular* if it eliminates quantifiers in L_{Pres}.

Fact 3.2 [7; 8; 27; 33; 34]. For an oag M, the following are equivalent.

(1) M is regular.

(2) The only definable convex subgroups of M are $\{0\}$ and M.

(3) The theory of M has an archimedean model.

(4) For every $n > 1$, if the interval $[a, b]$ contains at least n elements, then it contains an element divisible by n.

(5) Every quotient of M by a nontrivial convex subgroup is divisible.

Fact 3.3 [27; 34]. Every discrete regular M is a model of *Presburger Arithmetic*, i.e., $M \equiv \mathbb{Z}$. If M, N are dense regular, then $M \equiv N$ if and only if, for each $\mathfrak{p} \in \mathbb{P}$, either $M/\mathfrak{p}M$ and $N/\mathfrak{p}N$ are both infinite or they have the same finite size.

Notation 3.4. For the rest of the section we adopt the following (not entirely standard) conventions. Let M be an oag and $A \subseteq M$. We denote by $A_{>0}$ the set $\{a \in A \mid a > 0\}$, by $\langle A\rangle$ the group generated by A, and by $\mathrm{div}(M)$ the divisible hull of M. We allow intervals to have endpoints in the divisible hull. In other words, an *interval* in M is a set of the form $\{x \in M \mid a \sqsubset_0 x \sqsubset_1 b\}$, for suitable $a, b \in \mathrm{div}(M) \cup \{\pm\infty\}$ and $\{\sqsubset_0, \sqsubset_1\} \subseteq \{<, \leq\}$.

A *cut* (L, R) is given by subsets $L, R \subseteq M$ such that $L \leq R$ and $L \cup R = M$. We call such a cut *realised* if $L \cap R \neq \varnothing$, and *nonrealised* otherwise. The *cut* (L, R) of $c \in N > M$ is given by $L = \{m \in M \mid c \geq m\}$ and $R = \{m \in M \mid c \leq m\}$. The *cut* of a type $p \in S_1(M)$ is the cut of any $c \vDash p$. We say that $c \in N > M$ *fills* a cut (L, R)

if the latter equals the cut of c. For $a \in M$, we denote by a^+ the cut (L, R) with $L = \{m \in M \mid m \leq a\}$ and $R = \{m \in M \mid a < m\}$, and similarly for a^-. Analogous notions are defined for $a \in \operatorname{div}(M)$.

Every interval is definable: e.g., $(a/n, +\infty)$ is defined by $a < n \cdot x$. If (L, R) is a cut then $|L \cap R| \leq 1$. A type is realised if and only if its cut is. Let $L_{\mathrm{ab}} := \{0, +, -\}$.

Remark 3.5. By regularity, a 1-type over $M \vDash T$ is determined by a cut in M and a choice of cosets modulo each nM (if M/nM is infinite a type may say that the coset $x + nM$ is not represented in M) consistent with the L_{ab}-theory of M.

Lemma 3.6. *If M is a dense regular oag then, for every $n > 0$, every coset of nM is dense in M. In particular, given any nonrealised $p \in S_1(M)$, and any nonrealised $q_0 \in S_1(M \restriction L_{\mathrm{ab}})$, there is $q \in S_1(M)$ restricting to q_0 and in the same cut as p.*

Proof. By density and point (4) of Fact 3.2, every nM is dense; as translations are homeomorphisms for the order topology, each coset of nM is dense. □

Imaginaries in regular ordered abelian groups. The first step to compute $\widetilde{\operatorname{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ is to take care of the reduct to a certain fully embedded family of imaginary sorts, that suffice for weak elimination of imaginaries by a result of Vicaría [32]. Recall that T has *weak elimination of imaginaries* if for every imaginary e there is a real tuple a such that $e \in \operatorname{dcl}^{\mathrm{eq}}(a)$ and $a \in \operatorname{acl}^{\mathrm{eq}}(e)$. For $\mathfrak{p} \in \mathbb{P}$ and $n \geq 1$, define $T_{\mathfrak{p}^n}$ as the L_{ab}-theory of $\bigoplus_{i \in \omega} \mathbb{Z}/\mathfrak{p}^n\mathbb{Z}$. The following is well known.

Fact 3.7. (1) Let A be an infinite abelian group. Then $A \vDash T_{\mathfrak{p}^n}$ if and only if $\mathfrak{p}A = \{a \in A \mid \mathfrak{p}^{n-1}a = 0\}$.

(2) $T_{\mathfrak{p}^n}$ has quantifier elimination and is totally categorical.

(3) If $A \vDash T_{\mathfrak{p}^n}$, then $\mathfrak{p}A$ is a model of $T_{\mathfrak{p}^{n-1}}$, and the induced structure on $\mathfrak{p}A$ is that of a pure abelian group.

(4) $T_{\mathfrak{p}^n}$ has weak elimination of imaginaries.

Proof sketch. For (4), as $T_{\mathfrak{p}^n}$ is stable, it suffices to show that canonical bases of types over models are interdefinable with real tuples [13, Proposition 3]. This is an application of the elementary divisor theorem, and is left to the reader. □

Let $T_{\mathfrak{p}^\infty}$ be the following multisorted theory:

- For every $n > 0$ there is a sort $\mathrm{Q}_{\mathfrak{p}^n}$, endowed with a copy of L_{ab}.
- For every $n > 0$ there is a function symbol $\rho_{\mathfrak{p}^{n+1}} : \mathrm{Q}_{\mathfrak{p}^{n+1}} \to \mathrm{Q}_{\mathfrak{p}^n}$.
- $M \vDash T_{\mathfrak{p}^\infty}$ if and only if, for all $n > 0$, $\mathrm{Q}_{\mathfrak{p}^n}(M) \vDash T_{\mathfrak{p}^n}$ and $\rho_{\mathfrak{p}^{n+1}} : \mathrm{Q}_{\mathfrak{p}^{n+1}}(M) \to \mathrm{Q}_{\mathfrak{p}^n}(M)$ is a surjective group homomorphism with kernel $\mathfrak{p}^n\, \mathrm{Q}_{\mathfrak{p}^{n+1}}(M)$.

Remark 3.8. In an earlier version of this manuscript, we had claimed that $T_{\mathfrak{p}^\infty}$ has quantifier elimination. This does not hold. But one may show that it is enough to add function symbols $\lambda_n : Q_{\mathfrak{p}^n} \to Q_{\mathfrak{p}^{n+1}}$ for all n, interpreted as the definable group isomorphism $Q_{\mathfrak{p}^n}(A) \to \mathfrak{p}\, Q_{\mathfrak{p}^{n+1}}(A)$ mapping a to $\mathfrak{p}\tilde{a}$ where $\tilde{a}$ is any element with $\rho_{\mathfrak{p}^{n+1}}(\tilde{a}) = a$. We thank A. Gehret for having pointed this out to us.

The quantifier elimination result above, which has been mentioned for the sake of completeness, will not be used below. Let $\hat{\kappa}$ be the monoid of cardinals not larger than κ, with the usual sum and order.

Corollary 3.9. (1) *The theory $T_{\mathfrak{p}^\infty}$ is complete, totally categorical,* 1*-based, and has weak elimination of imaginaries.*

(2) *In $T_{\mathfrak{p}^\infty}$, we have $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \mathbb{N}$ and, for each infinite cardinal κ, the monoid $\widetilde{\operatorname{Inv}}_\kappa(\mathfrak{U})$ is (well defined and) isomorphic to $\hat{\kappa}$.*

(3) *More precisely, if* $\operatorname{tp}(a/\mathfrak{U})$ *is M-invariant, then there is a basis $b \in$* $\operatorname{dcl}(Ma)$ *of the $\mathbb{F}_\mathfrak{p}$-vector space* $Q_\mathfrak{p}(\operatorname{dcl}(\mathfrak{U}a))$ *over $\mathfrak{U}$, and* $\operatorname{tp}(a/\mathfrak{U})$ *is domination-equivalent to* $\operatorname{tp}(b/\mathfrak{U})$, *witnessed by* $\operatorname{tp}(ab/M)$ *in both directions, and the isomorphism above sends its domination-equivalence class to the cardinality of b.*

Proof. Statement (1) is immediate from Fact 3.7 and the fact that abelian groups are 1-based. As for (2), each of the sorts $Q_{\mathfrak{p}^n}$ is stable unidimensional, that is, if $p \perp q$ then one of p, q is algebraic, and it follows easily that so is $T_{\mathfrak{p}^\infty}$. The conclusion for finitary types then follows from [22, Corollary 5.19], and the version for $*$-types is similar.

To prove (3), if $b \in \operatorname{dcl}(Ma)$ is a basis of $Q_\mathfrak{p}(\operatorname{dcl}(Ma))$ over M, by M-invariance it is also a basis of $Q_\mathfrak{p}(\operatorname{dcl}(\mathfrak{U}a))$ over $\mathfrak{U}$. Because in unidimensional theories the domination-equivalence class of a tuple is determined by its weight [22, Remark 5.12], it suffices to show that the cardinality κ of b equals the weight $w(\operatorname{tp}(a/\mathfrak{U}))$. For $T_{\mathfrak{p}^n}$ this is well known, and as $Q_{\mathfrak{p}^n}(\mathfrak{U})$ is a fully embedded model of $T_{\mathfrak{p}^n}$, the result is easily seen to transfer to $T_{\mathfrak{p}^\infty}$. □

We now consider a regular oag M. Since it is well known that Presburger arithmetic eliminates imaginaries (by definable choice), we may assume that M is dense.

We view M as a structure in the language with one sort for the oag itself, endowed with L_{oag}, one sort $Q_{\mathfrak{p}^n}$ for each prime $\mathfrak{p}$ and each $n > 0$, endowed with L_{ab} and interpreted as the group $M/\mathfrak{p}^n M$, functions $\pi_{\mathfrak{p}^n}$ for the quotient map from M to $M/\mathfrak{p}^n M$ and functions $\rho_{\mathfrak{p}^{n+1}}$ for the canonical surjections $M/\mathfrak{p}^{n+1}M \to M/\mathfrak{p}^n M$. Moreover, for every prime $\mathfrak{p}$ we definably expand the language on $(Q_{\mathfrak{p}^n})_{n>0}$ so that the multisorted structure $(Q_{\mathfrak{p}^n}(M))_{n>0}$ has quantifier elimination.

For every $\mathfrak{p} \in \mathbb{P}$, let $d_\mathfrak{p} \in \mathbb{N} \cup \{\infty\}$ be such that $(M : \mathfrak{p}M) = \mathfrak{p}^{d_\mathfrak{p}}$. Set $T := \operatorname{Th}(M)$. The proof of the following lemma is straightforward from Lemma 3.6 and quantifier elimination for the one-sorted theory of M in L_{Pres}, and we leave it to the reader.

Lemma 3.10. *The theory T eliminates quantifiers. For $\mathfrak{U} \vDash T$, the following holds. For every $\mathfrak{p}$ prime and $n > 0$, the sort $Q_{\mathfrak{p}^n}(\mathfrak{U})$ equipped with the natural L_{ab}-structure* *is fully embedded. If $d_\mathfrak{p} = \infty$, the structure given by $(\mathrm{Q}_{\mathfrak{p}^n}(\mathfrak{U}))_{n>0}$, together with the maps $\rho_{\mathfrak{p}^{n+1}}$ and the natural L_ab-structure on each sort, is fully embedded and a model of $T_{\mathfrak{p}^\infty}$. If $d_\mathfrak{p}$ is finite, every sort $\mathrm{Q}_{\mathfrak{p}^n}(\mathfrak{U})$ is finite. If $\mathfrak{p}$, $\mathfrak{q}$ are distinct primes, then $\mathrm{Q}_{\mathfrak{p}^n}(\mathfrak{U})_{n>0}$ and $\mathrm{Q}_{\mathfrak{q}^n}(\mathfrak{U})_{n>0}$ are orthogonal.*

Definition 3.11. Denote by $\mathcal{Q}$ the family of sorts $\{\mathrm{Q}_{\mathfrak{p}^n} \mid \mathfrak{p} \in \mathbb{P}, n > 0\}$. If $q = \mathrm{tp}(c/\mathfrak{U})$ is a $*$-type, possibly with coordinates in the sorts in $\mathcal{Q}$, for each $\mathfrak{p} \in \mathbb{P}$, let $\kappa_\mathfrak{p}(q)$ be the dimension of the $\mathbb{F}_\mathfrak{p}$-vector space $\mathrm{dcl}(\mathfrak{U}c)/\mathfrak{p}(\mathrm{dcl}(\mathfrak{U}c))$ over $\mathfrak{U}/\mathfrak{p}\mathfrak{U}$. Let $\mathbb{P}_T$ be the set of primes $\mathfrak{p}$ such that if $M \vDash T$ then $\mathfrak{p}M$ has infinite index, and denote by $\prod_{\mathbb{P}_T} \hat{\kappa}$ the monoid of $\mathbb{P}_T$-indexed sequences of cardinals smaller or equal than κ with pointwise cardinal sum, equipped with the product (partial) order.

Corollary 3.12. *The family of sorts $\mathcal{Q}$, equipped with the L_ab-structure on each sort and the maps $\rho_{\mathfrak{p}^{n+1}}$, is fully embedded. When viewed as a standalone structure, $\otimes$ respects $\geq_\mathrm{D}$ and $\widetilde{\mathrm{Inv}}_\kappa(\mathcal{Q}(\mathfrak{U})) \cong \prod_{\mathbb{P}_T} \hat{\kappa}$.*

Proof. This follows from Lemma 3.10, Corollary 3.9, and Fact 1.1. Compatibility of $\otimes$ with $\geq_\mathrm{D}$ is a consequence of stability, see [22, Propositions 1.21 and 1.25]. □

Fact 3.13 [32, Theorem 5.1]. The theory T has weak elimination of imaginaries.

Remark 3.14. Vicaría [32] proves a more general result, of which Fact 3.13 is a special case. Note that she adds sorts for quotients of the form M/nM for all $n > 0$. As M/nM is definably isomorphic to $\prod_{i=1}^m M/\mathfrak{p}_i^{n_i} M$, where $n = \prod_{i=1}^m \mathfrak{p}_i^{n_i}$ is the decomposition of n into prime powers, it suffices to add the sorts $\mathrm{Q}_{\mathfrak{p}^n}$.

Observe that, for the above to go through, we need to have in our language the sorts $\mathrm{Q}_{\mathfrak{p}^n}$ even when they are finite. Alternatively, one may dispense with the finite $\mathrm{Q}_{\mathfrak{p}^n}$ by naming enough constants, e.g., by naming a model.

Moving to the right of a convex subgroup.

Assumption 3.15. Until the end of the section, T is the complete L_Pres-theory of a regular oag. Imaginary sorts are not in our language until further notice.

Definition 3.16. Let $B \subseteq M$. A cut (L, R) is *right of B* if $R \cap B = \varnothing$ and B is cofinal in L. An element $c \in N > M$ is *right of B* if its cut is, and a type $q \in S_1(M)$ is *right of B* if any of its realisations is. A convex subgroup H of $\mathfrak{U}$ is called *(A-)invariant* if there is an (A-)invariant type to its right.

Remark 3.17. Let $p \in S_1(\mathfrak{U})$ be an M-invariant type. If its cut (L, R) is definable, then it is M-definable. If not, then exactly one between the cofinality of L and the coinitiality of R is small, and M contains a set cofinal in L or coinitial in R.

Proof. The case of a definable cut is clear, so let us assume (L, R) is a nondefinable cut of $\mathfrak{U}$. In particular, $L \neq \varnothing \neq R$. If $L \cap M$ is not cofinal in L, there is $\ell \in L$ with $L \cap M < \ell$, so by regularity of $\mathfrak{U}$ and saturation there is $\ell_0 \in L$ divisible by all

$n \geq 1$ such that $L \cap M < \ell_0 < \ell$. Similarly, if $R \cap M$ is not coinitial in M there is $r_0 \in R$, which is divisible by all $n \geq 1$, such that $r_0 < R \cap M$. By Remark 3.5 it follows that $\operatorname{tp}(\ell_0/M) = \operatorname{tp}(r_0/M)$, showing that (L, R) is not M-invariant. □

In particular, in a regular oag a nontrivial convex subgroup H of $\mathfrak{U}$ is invariant if and only if the cofinality of H or the coinitiality of $(\mathfrak{U} \setminus H)_{>0}$ is small, while the trivial subgroup $\{0\}$ is invariant if and only if $\mathfrak{U}$ is dense.

Lemma 3.18. *In the theory of a regular oag, suppose that $p \in S_1^{\mathrm{inv}}(\mathfrak{U})$ and f is a definable function such that $f_* p$ is not realised. Then $p \sim_{\mathrm{D}} f_* p$.*

Proof. Clearly $p \geq_{\mathrm{D}} f_* p$. By [7, Corollary 1.10], f is piecewise affine. As $f_* p$ is not realised, f cannot be constant at p, so it is invertible at p and

$$f_* p \geq_{\mathrm{D}} f_*^{-1}(f_* p) = p.$$ □

Proposition 3.19. *In Presburger arithmetic, every invariant 1-type is domination-equivalent to a type right of an invariant convex subgroup.*

Proof. By Lemma 3.18 it suffices to show that, for every nonrealised $p \in S_1^{\mathrm{inv}}(\mathfrak{U})$ there is a definable f such that $f_* p$ is right of an invariant convex subgroup. By Fact 3.2, $\mathfrak{U}/\mathbb{Z}$ is divisible, and it is easy to see that $\mathfrak{U}/\mathbb{Z}$ inherits saturation and strong homogeneity from $\mathfrak{U}$. The conclusion follows by lifting the analogous result [17, Corollary 13.11] (see also [23, Proposition 4.8]) from $\mathfrak{U}/\mathbb{Z}$. □

In the rest of the subsection we generalise the above to the regular case.

Assumption 3.20. Until the end of the subsection, M denotes a dense regular oag, and $\mathfrak{U}$ a monster model of $T := \operatorname{Th}(M)$.

Proposition 3.21. *Let $b \in \mathfrak{U} \setminus M$ be divisible by every $n > 1$ and let $B := \langle Mb \rangle = M + \mathbb{Q}b$. If $M_{>0}$ is coinitial in $B_{>0}$, then $M \prec B \prec \mathfrak{U}$.*

Proof. The inclusion $M \subseteq B$ is *pure*, i.e., for every $n > 1$ we have $nB \cap M = nM$. Moreover, if $c = a + \gamma b$, with $a \in M$ and $\gamma \in \mathbb{Q}$, then for every n we clearly have $c - a \in nB$, and hence B/nB may be naturally identified with M/nM.

Because M is dense and $M_{>0}$ is coinitial in $B_{>0}$, it follows that B is as well dense. Let $c < d \in B$ and $n > 1$. By assumption, $(0, d - c)$ intersects M, so it contains an interval I of M, and hence represents all elements of M/nM by Lemma 3.6. These can be identified with the elements of B/nB, as observed above, so there is $e \in I$ such that $c + e \in nB$. Clearly, $c + e \in (c, d)$, and hence B is regular by Fact 3.2.

By Fact 3.3 and the identification of M/nM with B/nB, we obtain $B \equiv M$. Since M is pure in B, it is an L_{Pres}-substructure of B, and the conclusion follows by quantifier elimination in L_{Pres}. □

Recall that an extension $A < B$ of oags is an *i-extension* if there is no $b \in B_{>0}$ such that the set $\{a \in A \mid a < b\}$ is closed under sum.

Lemma 3.22. *Let $H < M < N$, with M dense regular and H convex. The set of elements of N right of H is closed under sum. In particular, N is an i-extension if and only if $H \mapsto H \cap M$ is a bijection between the convex subgroups of N and M.*

Proof. If $H = M$, the statement is trivial. If $H = \{0\}$, let $0 < c, d < M_{>0}$ and pick $a \in M_{>0}$. By density, there is $b \in M$ with $0 < b < a$, and since b and $a - b$ are both in $M_{>0}$ we conclude $c + d < b + a - b = a$. If H is proper nontrivial, by Fact 3.2 the quotient M/H is divisible, and the conclusion follows from the previous case applied to M/H as a subgroup of the quotient of N by the convex hull of H. □

Proposition 3.23. *Every $M \vDash T$ has a maximal elementary i-extension.*

Proof. This is easy, see, e.g., [21, Proposition 4.2.17]. □

Proposition 3.24. *Suppose $M \vDash T$ has no proper elementary i-extension and let $p \in S_1(M)$ be nonrealised. Then there are $a \in M$ and $\beta \in \mathbb{Z} \setminus \{0\}$ such that, if $f(t) = a + \beta t$, then the pushforward $f_* p$ is right of a convex subgroup.*

Proof. Let $b \vDash p$, and suppose first that b is divisible by every n. Consider $B := \langle Mb \rangle = M + \mathbb{Q}b$. If there are $a' \in M$ and $\beta' \in \mathbb{Q}$ such that $0 < a' + \beta' b < M_{>0}$, by Lemma 3.22 multiplying by the denominator of β' yields a positive element smaller than $M_{>0}$, so we obtain the conclusion with the convex subgroup $\{0\}$. If instead there is no such $a' + \beta' b$, then $M_{>0}$ is coinitial in $B_{>0}$, and by Proposition 3.21 $B \succ M$. By maximality of M, there must be convex subgroups $H_0 \subsetneq H_1$ of B such that $H_0 \cap M = H_1 \cap M$. Hence any positive $a + \beta b \in H_1 \setminus H_0$ is right of $H_0 \cap M$. We conclude again by clearing the denominator of β and using Lemma 3.22.

This shows the conclusion when b is divisible by all n. In the general case, by Lemma 3.6, there is $c \in \mathfrak{U}$ with the same cut in M as b which is divisible by every n. As we just proved, there is $f(t) := a + \beta t$, with $\beta \in \mathbb{Z}$ and $a \in M$, such that the cut of $f(c)$ in M is that of a convex subgroup. Because $f(t)$ sends intervals to intervals, it sends cuts to cuts, and hence the cut of $f(b)$ equals that of $f(c)$. □

Corollary 3.25. *For every nonrealised $p(x) \in S_1^{\mathrm{inv}}(\mathfrak{U})$ there is a definable function f such that $(f_* p)(y)$ is right of an invariant convex subgroup, and domination-equivalent to p, witnessed by any small type containing $y = f(x)$.*

Proof. If p is M-invariant, up to enlarging M we may assume that it has no proper elementary i-extension. Let $f(t)$ be an M-definable function given by Proposition 3.24 applied to $p \restriction M$. Then $f_* p$ is M-invariant, and its cut is either the one to the left of $(f_* p \restriction M)(\mathfrak{U})$ or the one to its right, which are both cuts right of convex subgroups of $\mathfrak{U}$ by Lemma 3.22. Now apply Lemma 3.18. □

Computing the domination monoid. By Fact 3.13, regular oags weakly eliminate imaginaries after adding the sorts Q_{p^n}. As already remarked, this implies that passing to T^{eq} does not affect the poset $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, nor its well-definedness as a monoid. Hence, we will conflate the two settings, and refer to our theory in this language as T^{eq}, reserving T for the 1-sorted L_{Pres}-theory of a regular oag.

Assumption 3.26. Until the end of the section, we work in T^{eq}.

Lemma 3.27. *Let $H_0 \subsetneq H_1$ be convex subgroups of $M \vDash T$ and, for $i < 2$, let $q_i(x^i) \in S_1(M)$ be right of H_i. Suppose that there is no prime $\mathfrak{p} \in \mathbb{P}$ such that both $q_i(x^i)$ prove that x^i is in a new coset modulo some $\mathfrak{p}^{\ell_i}$ Then $q_0 \perp^{\mathrm{w}} q_1$.*

Proof. By Lemma 3.22 the cut of every $k_0 x^0 + k_1 x^1$ is determined by $q_0(x^0) \cup q_1(x^1)$, and we conclude by assumption and quantifier elimination. □

Proposition 3.28. *Suppose that $q_H(x) \in S_1^{\mathrm{inv}}(\mathfrak{U})$ is right of the convex subgroup H and prescribes realised cosets modulo every n for x. For an invariant $*$-type q with all coordinates in the home sort, the following are equivalent.*

(1) *For every (equivalently, some) $b \vDash q$, no type right of H is realised in $\langle \mathfrak{U} b\rangle$.*

(2) *$q_H \perp^{\mathrm{w}} q$.*

(3) *q_H commutes with q.*

(4) *$q_H \perp q$.*

Moreover, if q' is a $*$-type with no coordinates in the home sort, then $q_H \perp q'$.

Proof. To show (1) ⇒ (2), consider $q_H(x) \cup q(y)$. By assumption on q_H we only need to deal with inequalities of the form $kx + \sum_{i<|y|} k_i y_i + d \geq 0$, but (1) gives immediately that the cut of kx in $\langle \mathfrak{U} b\rangle$ is determined. If (1) fails, as witnessed by $f(b)$, say, then $q_H(x) \otimes q(y)$ and $q(y) \otimes q_H(x)$ disagree on the formula $f(y) < x$, proving (3) ⇒ (1), and (2) ⇒ (3) holds for every type in every theory.

We prove (2) ⇒ (4), the converse being trivial. Suppose that $B \supseteq \mathfrak{U}$ is such that $(q_H \mid B) \not\perp^{\mathrm{w}} (q \mid B)$. The cosets modulo every n of a realisation of q_H are all realised in $\mathfrak{U}$, so there must be some inequality of the form $kx + \sum_{i<|y|} k_i y_i + d \geq 0$, with $k_i \in \mathbb{Z}$ and $d \in \langle B\rangle$, that is not decided. Hence, if (4) fails, it fails for a 1-type $\tilde{q}$, namely the pushforward of q under the map $y \mapsto \sum_{i<|y|} k_i y_i$. By Corollary 3.25 and Proposition 2.4, we may assume $\tilde{q}$ is right of a convex subgroup. Therefore $q_H(x)$ and $\tilde{q}(z)$ are weakly orthogonal by (2) and [22, Proposition 3.13], to the right of distinct (by weak orthogonality) convex subgroups, but the cut in $\langle B\rangle$ of $kx + z$ is not determined by $(q_H \mid B)(x) \cup (\tilde{q} \mid B)(z)$. This contradicts Lemma 3.22.

Now we consider the "moreover" part. By Proposition 2.4 we may replace q' with any domination-equivalent type, so we may assume, using Corollary 3.12 and Proposition 2.5, that $q'(z)$ is the type of an independent tuple, with $z_i \in \mathrm{Q}_{\mathfrak{p}_i}$. Let H' be any invariant convex subgroup different from H, let p_i be the 1-type right of H' in a new coset modulo $\mathfrak{p}_i$ and congruent to 0 modulo every other prime, and let q be the tensor product, in any order, of the p_i. Clearly $q \geq_{\mathrm{D}} q'$ and, by construction, if $b \vDash q$ then no type right of H is realised in $\langle \mathfrak{U}b \rangle$, so we conclude by Proposition 2.4. $\square$

Definition 3.29. Let q be an invariant global $*$-type, and $c \vDash q$. Let $\mathcal{H}(q)$ be the set of cuts of convex subgroups of $\mathfrak{U}$ filled in $\langle \mathfrak{U}c \rangle$.

Theorem 3.30. *If p, q are invariant $*$-types, then $p \geq_{\mathrm{D}} q$ if and only if $\mathcal{H}(p) \supseteq \mathcal{H}(q)$ and $\forall \mathfrak{p} \in \mathbb{P}\ \kappa_{\mathfrak{p}}(p) \geq \kappa_{\mathfrak{p}}(q)$. Hence, $[\![q]\!]$ is determined by $\mathcal{H}(q)$ and $\mathfrak{p} \mapsto \kappa_{\mathfrak{p}}(q)$.*

Proof. Let $c \vDash q$, and write $c = c^0 c^1$, with c^0 a tuple in the home sort and c^1 a tuple from the sorts $\mathrm{Q}_{\mathfrak{p}^n}$. By enlarging c_1 with at most $|c|$ points of $\mathrm{dcl}(\mathfrak{U}c)$ if necessary, we may assume that it contains bases of all $\mathbb{F}_{\mathfrak{p}}$-vector spaces $\mathrm{Q}_{\mathfrak{p}}(\mathrm{dcl}(\mathfrak{U}c))$ over $\mathfrak{U}$. Observe that this is harmless domination-wise, and that it does not impact compatibility of $\otimes$ with $\geq_{\mathrm{D}}$ by [22, Proposition 1.23].

Index on a suitable cardinal κ, bounded by the cardinality of c^0, the (necessarily invariant) convex subgroups H_j whose cuts are filled in $\langle \mathfrak{U}c^0 \rangle$. Note that, by Corollary 3.25, we have $\kappa \neq 0$ unless c^0 is realised.

For $j < \kappa$, let $q_j(y_j)$ be the type right of H_j divisible by every nonzero integer. By Lemma 3.27 and Proposition 3.28, the q_j are orthogonal, and it follows from Proposition 2.9 and compactness that their union is a complete type; call it $q_{\mathrm{H}}(y)$. Let $q_{\mathrm{Q}}(z) := \mathrm{tp}(c_1/\mathfrak{U})$. By Propositions 3.28 and 2.9, $q_{\mathrm{H}} \perp q_{\mathrm{Q}}$. We prove that $q(x)$ is domination-equivalent to $q'(yz) := q_{\mathrm{H}}(y) \otimes q_{\mathrm{Q}}(z)$. If $c^0 \in \mathfrak{U}$, equivalently if q_{H} is realised, this is trivial, so we assume this is not the case.

To show $q'(yz) \geq_{\mathrm{D}} q(x)$, let $b \in \mathrm{dcl}(\mathfrak{U}c)$ be maximal amongst the tuples with each b_k in the cut of an invariant convex subgroup, and such that if $k < k'$ then $\langle b_k \rangle_{>0} < \langle b_{k'} \rangle_{>0}$. A maximal such b exists because the size of b is at most that of c^0, by looking at $\mathbb{Q}$-linear dimension over $\mathfrak{U}$ in the divisible hull. Since $c^0 \notin \mathfrak{U}$, by Corollary 3.25 there is a point of $\mathrm{dcl}(\mathfrak{U}c)$ in the cut of an invariant convex subgroup, and hence b is nonempty. By [7, Corollary 1.10] definable functions are piecewise affine and, by clearing denominators using Lemma 3.22, we may assume that $b \in \langle \mathfrak{U}c^0 \rangle$.

Write $b_k = f_k(c^0)$, for suitable affine functions f_k. Let $M \prec^+ \mathfrak{U}$ be large enough to contain the parameters of the f_k, such that q and q' are M-invariant, and such that M has no proper elementary i-extension. Let $r \in S_{qq'}(M)$ contain the following.

(1) For each k, by choice of q' there is $j < \kappa$ such that y_j is in the same cut as b_k according to q'. If the cut of b_k has small cofinality on the right, put in r the formula $f_k(x) > y_j$; if it has small cofinality on the left, put in r the formula $f_k(x) < y_j$.

(2) For each $j < |c^1|$, the formula $x_{|c^0|+j} = z_j$.

By Lemma 3.10, point (2) above, the fact that c^1 contains bases of all $\mathbb{F}_{\mathfrak{p}}$-vector spaces $\mathrm{Q}_{\mathfrak{p}}(\mathrm{dcl}(\mathfrak{U}c))$ over $\mathfrak{U}$, and Corollary 3.9, to prove $q' \geq_{\mathrm{D}} q$ it suffices to show that $q' \cup r$ decides the cut in $\mathfrak{U}$ of every $\sum_i \delta_i x_i$. We first prove a special case.

Claim. *$q' \cup r$ entails the quantifier-free $\{+, 0, -, <\}$-type of the $f_k(x)$ over $\mathfrak{U}$.*

Proof of Claim. It is enough to show that the cut of every $\sum_k \beta_k f_k(x)$ in $\mathfrak{U}$ is decided, where only finitely many $\beta_k \in \mathbb{Z}$ are nonzero. By choice of r and Remark 3.17, $q' \cup r$ determines the cut of each $f_k(x)$ over $\mathfrak{U}$. Moreover, r contains the information that $\langle f_k(x)\rangle_{>0} < \langle f_{k'}(x)\rangle_{>0}$ for $k < k'$. By this, the fact that the $f_k(x)$ are right of convex subgroups, and Lemma 3.22, the cut of $\sum_k \beta_k f_k(x)$ must be that of $\operatorname{sign}(\beta_k) f_k(x)$, with k the largest such that $\beta_k \neq 0$. □

As M has no proper elementary i-extension, given a term $\sum_i \delta_i x_i$, by Proposition 3.24 we can compose with an M-definable injective affine function and reduce to a term $\sum_i \gamma_i x_i + d$, with $d \in M$ and $\gamma_i \in \mathbb{Z}$, with cut in M right of a convex subgroup. As $\operatorname{tp}\bigl(\sum_i \gamma_i x_i + d/\mathfrak{U}\bigr)$ is M-invariant, $\sum_i \gamma_i x_i + d$ is in the cut of an M-invariant convex subgroup of $\mathfrak{U}$. By maximality of b, there must be k and positive integers n, m such that $nb_k \le m\bigl(\sum_i \gamma_i x_i + d\bigr) \le (n+1)b_k$. Thus $r \vdash nf_k(x) \le m\bigl(\sum_i \gamma_i x_i + d\bigr) \le (n+1) f_k(x)$, and by the Claim $q' \geq_{\mathrm{D}} q$.

Similar arguments show $q \geq_{\mathrm{D}} q'$ and that, if $\mathcal{H}(p) \supseteq \mathcal{H}(q)$ and $\kappa_{\mathfrak{p}}(p) \ge \kappa_{\mathfrak{p}}(q)$ for all $\mathfrak{p} \in \mathbb{P}$, and p' is defined analogously to q', then $p' \geq_{\mathrm{D}} q'$. That $\mathcal{H}(p) \supseteq \mathcal{H}(q)$ is necessary to have $p \geq_{\mathrm{D}} q$ follows from Proposition 3.28 and [22, Proposition 3.13]. As $\forall \mathfrak{p} \in \mathbb{P}\ \ \kappa_{\mathfrak{p}}(p) \ge \kappa_{\mathfrak{p}}(q)$, if for some $\mathfrak{p} \in \mathbb{P}$ we have $\kappa_{\mathfrak{p}}(q) > \kappa_{\mathfrak{p}}(p)$ then we easily find a type in the quotient sorts dominated by q but not by p, a contradiction. □

Proposition 3.31. *For all invariant $*$-types p, q and $\mathfrak{p} \in \mathbb{P}$, we have*

$$\mathcal{H}(p \otimes q) = \mathcal{H}(p) \cup \mathcal{H}(q) \quad \textit{and} \quad \kappa_{\mathfrak{p}}(p \otimes q) = \kappa_{\mathfrak{p}}(p) + \kappa_{\mathfrak{p}}(q).$$

Proof. By Proposition 3.28, $\mathcal{H}(q)$ is precisely the set of convex invariant subgroups H such that $q \not\perp q_H$. By Proposition 2.9, we therefore have the first statement. The second one is an easy consequence of the definition of $\otimes$. □

Note that if $q \in S^{\mathrm{inv}}_{<\kappa^+}(\mathfrak{U})$ then $|\mathcal{H}(q)|$ and each $\kappa_{\mathfrak{p}}(q)$ are at most κ.

Definition 3.32. We denote by $\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U})$ the set of invariant convex subgroups of $\mathfrak{U}$, and by $\mathscr{P}_{\le\kappa}(\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U}))$ the monoid of its subsets of size at most κ with union, partially ordered by inclusion.

Corollary 3.33 (Theorem D). *For T the theory of a regular oag and κ a small infinite cardinal, $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}})$ is well defined, and $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}}) \cong \mathscr{P}_{\le\kappa}(\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U})) \times \prod_{\mathbb{P}_T} \hat{\kappa}$.*

Proof. Compatibility of $\otimes$ and $\geq_{\mathrm{D}}$ follows from Theorem 3.30 and Proposition 3.31. The same results show that the map $[\![p]\!]$ to $(\mathcal{H}(p), \mathfrak{p} \mapsto \kappa_{\mathfrak{p}}(p))$ is well defined, an embedding of posets, and a morphism of monoids. Surjectivity is easily checked. □

In general, the embedding $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \hookrightarrow \widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}})$ is not surjective, although its image may be easily computed. We state the result of this computation, which we leave to the reader, and of the analogous ones for finitary types. Denote by $\prod^{\mathrm{bdd}}_{\mathbb{P}_T} \omega$ the submonoid of $\prod_{\mathbb{P}_T} \hat{\omega}$ consisting of bounded sequences of natural numbers.

Corollary 3.34. *The monoids* $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, $\widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}})$ *are all well defined, and*

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \left(\mathscr{P}_{\le\kappa}(\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U})) \times \prod_{\mathbb{P}_T} \hat{\kappa}\right) \setminus \{(a,b) \mid a = \varnothing, b \neq 0\},$$

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}^{\mathrm{eq}}) \cong \left(\mathscr{P}_{<\omega}(\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U})) \times \prod_{\mathbb{P}_T}^{\mathrm{bdd}} \omega\right) \setminus \{(a,b) \mid a = \varnothing, \operatorname{supp}(b) \text{ infinite}\},$$

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \left(\mathscr{P}_{<\omega}(\mathrm{CS}^{\mathrm{inv}}(\mathfrak{U})) \times \prod_{\mathbb{P}_T}^{\mathrm{bdd}} \omega\right) \setminus \{(a,b) \mid a = \varnothing, b \neq 0\}.$$

4. Pure short exact sequences

We study pure short exact sequences of abelian structures $0 \to \mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{\nu} \mathcal{C} \to 0$, where $\mathcal{A}$ and $\mathcal{C}$ may be equipped with extra structure. We view them as multisorted structures, and use the relative quantifier elimination results from [2] to describe the domination poset in terms of $\mathcal{A}$ and $\mathcal{C}$. A decomposition of the form $\widetilde{\mathrm{Inv}}(\mathcal{A}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\mathcal{C}(\mathfrak{U}))$ only holds in special cases; in general we will need to look at $*$-types and introduce a family of imaginaries of $\mathcal{A}$ which depends on $\mathcal{B}$.

We refer the reader to [2, Section 4.5] for definitions. We adopt almost identical notations, with the following differences. We write $\mathcal{A}$ for an abelian structure and L for its language. We denote by $\mathcal{F}$ a fundamental family of pp formulas for $\mathcal{B}$. The corresponding family of quotient sorts of $\mathcal{A}$ is denoted by $\mathcal{A}_\mathcal{F}$. An $\mathcal{A}$-*sort* is simply a sort in $\mathcal{A}$. We write, e.g., $t(x)$ for a tuple of terms, 0 for a tuple of zeroes of the appropriate length, etc. Tuples of the same length may be added, and tuples of appropriate lengths used as arguments, as in $f(t(x,0) - d) = 0$.

Example 4.1. In the simplest abelian structures, namely abelian groups, we have that $\mathcal{F} := \{\exists y\, x = n \cdot y \mid n \in \omega\}$ is always fundamental. In an arbitrary abelian structure, one may always resort to taking as $\mathcal{F}$ the trivially fundamental set of all pp formulas.

Remark 4.2. In an L-abelian structure, each L-term $t(x)$ is built from homomorphisms of abelian groups by taking $\mathbb{Z}$-linear combinations and compositions. Hence, $t(x)$ is itself a homomorphism of abelian groups.

A short exact sequence of abelian groups $0 \to \mathrm{A} \to \mathrm{B} \to \mathrm{C} \to 0$ is pure if and only if, for each n, we have $n\mathrm{B} \cap \mathrm{A} = n\mathrm{A}$. This holds, e.g., if C is torsion-free, and in particular in the two examples below. We may take as $\mathcal{F}$ that of Example 4.1.

Example 4.3. Suppose that the expansion L^*_{ac} endows A, C with the structure of oags. Note that one then recovers, definably, an oag structure on B, induced by declaring that $\iota(\mathrm{A})$ is convex. Because of this, and of the fact that the kernel of a morphism of oags is convex, this setting is equivalent to that of a short exact sequence of oags. This will be used in Section 5, with B an oag and A a suitably chosen convex subgroup. The sorts A_φ coincide with the quotients $\mathrm{A}/n\mathrm{A}$.

Example 4.4. In the valued field context (Section 6) we will deal with the sequence $1 \to \mathrm{k}^\times \to \mathrm{RV} \setminus \{0\} \to \Gamma \to 0$, which is pure since Γ is torsion-free. The extra structure in L^*_{ac} is induced by the field structure on k and the order on Γ. The sorts A_φ are in this case $\mathrm{k}^\times/(\mathrm{k}^\times)^n$.

We may and will assume that, for each variable x from an $\mathcal{A}$-sort A_s, the formula $\varphi := x = 0$ is in $\mathcal{F}$, and identify A_s with $\mathrm{A}_\varphi = \mathrm{A}_s/0\mathrm{A}_s$. In other words, $\mathcal{A} \subseteq \mathcal{A}_\mathcal{F}$.

Remark 4.5. As pp formulas commute with cartesian products, every split short exact sequence is pure. Since purity is first-order, a short exact sequence is pure in case some elementarily equivalent structure splits. Note that, even if a short exact sequence splits, it need not do so definably, and that the definition of expanded pure short exact sequence does *not* allow to add splitting maps. If we add one, then matters simplify considerably. For example, if in L^*_{ac} there is no symbol involving $\mathcal{A}$ and $\mathcal{C}$ jointly, a splitting map makes the short exact sequence interdefinable with the disjoint union of $\mathcal{A}$ and $\mathcal{C}$, where $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ decomposes as a product (Example 2.8).

Fact 4.6 [2, Remark 4.21]. Let $\varphi(x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}})$ be an L^*_{abcq}-formula with $x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}}$ tuples of variables from the $\mathcal{A}_\mathcal{F}$-sorts, $\mathcal{B}$-sorts and $\mathcal{C}$-sorts respectively. There are an L^*_{acq}-formula ψ and special terms σ_i such that, in the L^*_{abcq}-theory of all expanded pure short exact sequences, we have $\varphi(x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}}) \leftrightarrow \psi(x^{\mathrm{a}}, \sigma_1(x^{\mathrm{b}}), \dots, \sigma_m(x^{\mathrm{b}}), x^{\mathrm{c}})$.

Corollary 4.7. *The L^*_{acq}-reduct is fully embedded. In particular, $\mathcal{A}$ and $\mathcal{C}$ are orthogonal if and only if they are such in the L^*_{acq}-reduct.*

We show that expanded pure short exact sequences are controlled, domination-wise, by their L^*_{acq}-part, provided we pass to $*$-types. This is a necessity since, in general, there are finite tuples from $\mathcal{B}$ that cannot be domination-equivalent to any finitary tuple from the L^*_{acq}-reduct; see Remark 4.17.

Proposition 4.8. *In an expanded pure short exact sequence of L-abelian structures, let $\mathcal{F}$ be a fundamental family for $\mathcal{B}$, and let $\kappa \geq |L|$ be a small cardinal. There is a family of κ-tuples of definable functions $\{\tau^p \mid p \in S_\kappa(\mathfrak{U})\}$ such that:*

(1) *Each function in τ^p is defined at realisations of p.*

(2) *Each τ^p is partitioned as (ρ^p, ν^p), where each function in ρ^p is either the identity on some A_φ, or has domain a cartesian product of $\mathcal{B}$-sorts and codomain one of the A_φ, and each function in ν^p is either the identity on a $\mathcal{C}$-sort, or one of the ν_s.*

(3) *For each $p \in S_\kappa(\mathfrak{U})$ we have $p \sim_{\mathrm{D}} \tau^p_* p$.*

(4) *For each $p_0, p_1 \in S^{\mathrm{inv}}_\kappa(\mathfrak{U})$ we have $p_0 \otimes p_1 \sim_{\mathrm{D}} \tau^{p_0}_* p_0 \otimes \tau^{p_1}_* p_1$.*

Proof. Let $abc \vDash p(x^{\mathrm{a}}, x^{\mathrm{b}}, x^{\mathrm{c}})$, in the notation of Fact 4.6. Define the tuples ν^p and ρ^p as follows. For each coordinate in x^{c} of sort C_s, put in ν^p the corresponding identity map on C_s. For each coordinate in x^{b} of sort B_s, put in ν^p the corresponding map $\nu_s : \mathrm{B}_s \to \mathrm{C}_s$. For each coordinate in x^{a} of sort A_φ, put in ρ^p the corresponding identity map on A_φ. For each finite tuple of L_{b}-terms $t(x^{\mathrm{b}}, w)$ and $\varphi \in \mathcal{F}$, if there is $d \in \mathfrak{U}$ such that $p \vdash t(x^{\mathrm{b}}, 0) - d \in \nu^{-1}(\varphi(\mathcal{C}))$, choose such a d, call it $d_{p,\varphi,t,x^{\mathrm{b}}}$, and put in ρ^p the map $\rho_\varphi(t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t,x^{\mathrm{b}}})$.

Let τ^p be the concatenation of ρ^p and ν^p, let $q(y) := \tau^p_* p(x)$, let D_p be the set of all $d_{p,\varphi,t,x^{\mathrm{b}}}$ as above, and let $r(x, y) \in S_{pq}(D_p)$ contain $y = \tau(x)$. Clearly $p \cup r \vdash q$. By Fact 4.6, to show $q \cup r \vdash p$ it suffices to prove that $q \cup r$ recovers the formulas $\varphi(x^{\mathrm{a}}, d^{\mathrm{a}}, \sigma_1(x^{\mathrm{b}}, d^{\mathrm{b}}), \dots, \sigma_m(x^{\mathrm{b}}, d^{\mathrm{b}}), x^{\mathrm{c}}, d^{\mathrm{c}})$ implied by p, where the σ_i are special terms, φ is an $\mathcal{L}^*_{\mathrm{acq}}$-formula, and the $d^\bullet$ are tuples of parameters from the appropriate sorts of $\mathfrak{U}$. Let us say that $q \cup r$ *has access to* the term (with parameters) $\sigma(x^{\mathrm{b}}, d)$ if for some $\mathfrak{U}$-definable function f we have $q(y) \cup r(x, y) \vdash f(y) = \sigma(x^{\mathrm{b}}, d)$. We show that $q \cup r$ has access to all special terms with parameters, and hence $q \cup r \vdash p$.

By construction, $q \cup r$ has access to each $\nu_s(x^{\mathrm{b}}_i)$. Because ν is a homomorphism of L-structures, $q \cup r$ also has access to each $\nu(t_0(x^{\mathrm{b}}, d))$, for t_0 an L_{b}-term. In particular, $q \cup r$ decides whether a given tuple $t(x^{\mathrm{b}}, d)$ of L_{b}-terms with parameters is in $\nu^{-1}(\varphi(\mathcal{C}))$ or not. If not, then $q \cup r$ entails $\rho_\varphi(t(x^{\mathrm{b}}, d)) = 0$.

If instead $q \cup r \vdash t(x^{\mathrm{b}}, d) \in \nu^{-1}(\varphi(\mathcal{C}))$, by Remark 4.2 we have

$$t(x^{\mathrm{b}}, d) = t(x^{\mathrm{b}}, 0) + t(0, d),$$

and by construction and the fact that p is consistent with $q \cup r$ we have that p entails $t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t,x^{\mathrm{b}}} \in \nu^{-1}(\varphi(\mathcal{C}))$. As this formula is over D_p, it is in r. Hence

$$q \cup r \vdash t(0, d) + d_{p,\varphi,t,x^{\mathrm{b}}} = t(x^{\mathrm{b}}, 0) + t(0, d) - (t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t,x^{\mathrm{b}}}) \in \nu^{-1}(\varphi(\mathcal{C})).$$

But $t(0, d) + d_{p,\varphi,t,x^{\mathrm{b}}} \in \mathfrak{U}$, and $\rho_\varphi \restriction \nu^{-1}(\varphi(\mathcal{C}))$ is a homomorphism of L-structures. Because of this, and because $q \cup r$ has access to $\rho_\varphi(t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t,x^{\mathrm{b}}})$ by construction, it also has access to

$$\rho_\varphi(t(x^{\mathrm{b}}, 0) - d_{p,\varphi,t,x^{\mathrm{b}}}) + \rho_\varphi(t(0, d) + d_{p,\varphi,t,x^{\mathrm{b}}}) = \rho_\varphi(t(x^{\mathrm{b}}, d)).$$

We are left to prove (4). By definition of $\otimes$, if

$$p_0(x) \otimes p_1(y) \vdash t(x^{\mathrm{b}}, y^{\mathrm{b}}, d) \in \nu^{-1}(\varphi(\mathcal{C})),$$

then there is $\tilde{b} \in \mathfrak{U}$ with $p_0(x) \vdash t(x^{\mathrm{b}}, \tilde{b}, d) \in \nu^{-1}(\varphi(\mathcal{C}))$. Hence, by arguing as above, $p_0 \vdash t(x^{\mathrm{b}}, 0, 0) - d_{p_0,\varphi,t,x^{\mathrm{b}}} \in \nu^{-1}(\varphi(\mathcal{C}))$. So $p_0(x) \otimes p_1(y)$ entails

$$\nu^{-1}(\varphi(\mathcal{C})) \ni t(x^{\mathrm{b}}, y^{\mathrm{b}}, d) - t(x^{\mathrm{b}}, 0, 0) + d_{p_0,\varphi,t,x^{\mathrm{b}}} = t(0, y^{\mathrm{b}}, 0) + t(0, 0, d) + d_{p_0,\varphi,t,x^{\mathrm{b}}}$$

and because $t(0,0,d)+d_{p_0,\varphi,t}\in\mathfrak{U}$, by construction we have

$$p_1(y)\vdash t(0,y^{\mathrm{b}},0)-d_{p_1,\varphi,t,y^{\mathrm{b}}}\in\nu^{-1}(\varphi(\mathcal{C})).$$

Similar arguments show that, in order to have access to $\rho_\varphi(t(x^{\mathrm{b}},y^{\mathrm{b}},d))$, it is enough to have access to $\rho_\varphi(t(x^{\mathrm{b}},0,0)-d_{p_0,\varphi,t,x^{\mathrm{b}}})$ together with $\rho_\varphi(t(0,y^{\mathrm{b}},0)-d_{p_1,\varphi,t,y^{\mathrm{b}}})$, and the conclusion follows. □

Corollary 4.9 (Theorem C). *Suppose that $\mathfrak{U}$ is an expanded pure short exact sequence of L-abelian structures and $\kappa\geq|L|$ is a small cardinal.*

(1) *There is an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})\cong\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}\restriction L^*_{\mathrm{acq}})$.*

(2) *If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}\restriction L^*_{\mathrm{acq}}$, then the same is true in $\mathfrak{U}$, and the above is also an isomorphism of monoids.*

(3) *If $\mathcal{A}$ and $\mathcal{C}$ are orthogonal, then there is an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})\cong\widetilde{\mathrm{Inv}}_\kappa(\mathcal{A}_{\mathcal{F}}(\mathfrak{U}))\times\widetilde{\mathrm{Inv}}_\kappa(\mathcal{C}(\mathfrak{U}))$. Moreover, if $\otimes$ respects $\geq_{\mathrm{D}}$ in both $\mathcal{A}_{\mathcal{F}}(\mathfrak{U})$ and $\mathcal{C}(\mathfrak{U})$, then the same is true in $\mathfrak{U}$, and the above is also an isomorphism of monoids.*

Proof. By Fact 1.1 we have an embedding of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}\restriction L^*_{\mathrm{acq}})\hookrightarrow\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$. This embedding is surjective by Proposition 4.8, its inverse being induced by the maps τ, hence an isomorphism. For (2), by Proposition 4.8 we may apply Proposition 1.3 to the family of sorts $\mathcal{A}_{\mathcal{F}}\,\mathcal{C}$. We conclude by combining (2) with Corollary 2.7. □

Remark 4.10. Variants of Fact 4.6 for settings such as abelian groups augmented by an absorbing element are presented in [2, Section 4]. These yield variants of Proposition 4.8 and its consequences, with no significant difference in the proofs.

Specialised to abelian groups, the results above enjoy a form of local finiteness.

Notation 4.11. For the rest of the section, L is just the language of abelian groups, and $\mathcal{F}$ the family of formulas $\{\exists y\, x=n\cdot y\mid n\in\omega\}$. We will write $\rho_n:\mathrm{B}\to\mathrm{A}/n\mathrm{A}$ in place of $\rho_\varphi:\mathrm{B}\to\mathrm{A}_\varphi$, and identify A with A/0A for notational convenience.

Definition 4.12. A $*$-type $p(x)$ is *locally finitary* if x has finitely many coordinates of each sort.

Proposition 4.13. *Consider a pure short exact sequence of abelian groups equipped with an L^*_{abcq}-structure. Let $p(x)$ be a locally finitary global type. Then, in Proposition 4.8, we may choose τ^p in such a way that τ^p_*p is locally finitary.*

Proof. Write $p(x)=p(x^{\mathrm{a}},x^{\mathrm{b}},x^{\mathrm{c}})$ as in the proof of Proposition 4.8, and recall that an L-term is just a $\mathbb{Z}$-linear combination. For each $n\in\omega$, consider the subgroup

$$K^p_n:=\{k\in\mathbb{Z}^{|x^{\mathrm{b}}|}\mid\exists d\in\mathrm{B}(\mathfrak{U})\;p\vdash k\cdot x^{\mathrm{b}}-d\in\nu^{-1}(n\mathrm{C})\}\quad\text{of }\mathbb{Z}^{|x^{\mathrm{b}}|},$$

say generated by $k^n_0,\dots,k^n_{m(n)}$. Choose $d_{p,n,i,x^{\mathrm{b}}}$ witnessing $k^n_i\in K^p_n$. Proceed as in Proposition 4.8 but, instead of putting in ρ^p each $\rho_\varphi(t(x^{\mathrm{b}},0)-d_{p,\varphi,t,x^{\mathrm{b}}})$, use a locally finite ρ^p extending $(\rho_n(k_i^n \cdot x^{\mathrm{b}} - d_{p,n,i,x^{\mathrm{b}}}))_{n\in\omega, i\le m(n)}$. Besides this, τ^p contains a finite tuple of identity maps and finitely many ν, therefore $\tau^p_* p$ is locally finitary.

The proof of Proposition 4.8 now goes through, with a pair of modifications which we now sketch. The first one concerns proving access to each $\rho_n(t(x^{\mathrm{b}}, d))$. Fix n and $t(x^{\mathrm{b}}, d)$. Without loss of generality we have that d is a singleton and $t(x^{\mathrm{b}}, d) = \ell \cdot x^{\mathrm{b}} - d$. If $p \vdash t(x^{\mathrm{b}}, d) \in \nu^{-1}(n\mathrm{C})$, by definition we have $\ell \in K_n^p$, so we may write $\ell = \sum_{i\le m(n)} e_i k_i^n$ for suitable $e_i \in \mathbb{Z}$. This allows us to rewrite

$$\begin{aligned} t(x^{\mathrm{b}}, d) = \ell \cdot x^{\mathrm{b}} - d &= \Bigg(\sum_{i\le m(n)} e_i k_i^n\Bigg) \cdot x^{\mathrm{b}} - d \\ &= \sum_{i\le m(n)} e_i (k_i^n \cdot x^{\mathrm{b}} - d_{p,n,i,x^{\mathrm{b}}}) + \sum_{i\le m(n)} e_i\, d_{p,n,i,x^{\mathrm{b}}} - d. \end{aligned}$$

Since $\ell \cdot x^{\mathrm{b}} - d$ and all $k_i^n \cdot x^{\mathrm{b}} - d_{p,n,i,x^{\mathrm{b}}}$ are in $\nu^{-1}(n\mathrm{C})$, so is $\sum_{i\le m(n)} e_i\, d_{p,n,i,x^{\mathrm{b}}} - d$. Since $\rho_n \restriction \nu^{-1}(n\mathrm{C})$ is a homomorphism and $\sum_{i\le m(n)} e_i\, d_{p,n,i,x^{\mathrm{b}}} - d \in \mathfrak{U}$, we have that $q \cup r$ has access to $\rho_n(t(x^{\mathrm{b}}, d))$.

Finally, proving (4) of Proposition 4.8 boils down to showing $K_n^{p\otimes q} = K_n^p \times K_n^q$, where we identify, e.g., K_n^p with $K_n^p \times \{0\}$. Since by construction $K_n^p \cap K_n^q = \{0\}$, one only needs to show generation. We leave the easy proof to the reader. □

Remark 4.14. In the case of abelian groups, we therefore have an analogue of Corollary 4.9 where κ-types are replaced by locally finitary ω-types.

Corollary 4.15. *Let $\mathfrak{U}$ be an expanded pure short exact sequences of abelian groups where, for all $n > 0$, the sort $\mathrm{A}/n\mathrm{A}$ is finite. If A and C are orthogonal, there is an isomorphism of posets $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathrm{A}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\mathrm{C}(\mathfrak{U}))$. If $\otimes$ respects $\ge_{\mathrm{D}}$ in A and C, then $\otimes$ respects $\ge_{\mathrm{D}}$, and the above is an isomorphism of monoids.*

Proof. Use Proposition 4.13 and observe that for each p we may replace τ^p by its composition with the projection on the nonrealised coordinates of $\tau^p_* p$ and still have the same results. If $\mathrm{A}/n\mathrm{A}$ is finite for all $n > 0$ and p is finitary, this yields another finitary type. The conclusion now follows as in the proof of Corollary 4.9. □

Remark 4.16. The A/nA are in general necessary to obtain a product decomposition. For example, let A be a regular oag divisible by all $\mathfrak{p} \in \mathbb{P} \setminus \{2\}$, and with $[\mathrm{A} : 2\mathrm{A}]$ infinite, and let C be a nontrivial divisible oag. The expanded short exact sequence $0 \to \mathrm{A} \to \mathrm{B} \to \mathrm{C} \to 0$ induces a group ordering on B (Example 4.3). Let $p(y)$ concentrate on B, at $+\infty$, in a new coset modulo 2B. For every nonrealised 1-type q of an element of sort A divisible by all n, we have $p \perp^{\mathrm{w}} q$. It follows that p cannot dominate any nonrealised p' in a cartesian power of A: such a p' must have a coordinate in a nonrealised cut, and hence dominate a type q as above. Hence, if we had a product decomposition as in Corollary 4.15, then p would be domination-equivalent to a type in a cartesian power of C. This is a contradiction, because C is orthogonal to $(\mathrm{A}/n\mathrm{A})_{n<\omega}$, while p dominates a nonrealised type in A/2A.

Remark 4.17. Analogously, ω-types are a necessity: let A be a regular oag with each $[\mathrm{A} : n\mathrm{A}]$ infinite, C a nontrivial divisible oag, and take as $p \in S_{\mathrm{B}}(\mathfrak{U})$ the type at $+\infty$ in a new coset of each $n\mathrm{A}$. For each $n > 1$, there is a nonrealised 1-type q_n of sort $\mathrm{A}/n\mathrm{A}$ such that $p \geq_{\mathrm{D}} q_n$. One shows that the only way for a finitary type in $((\mathrm{A}/n\mathrm{A})_{n\in\omega}, \mathrm{C})$ to dominate all of the q_n is to have a nonrealised coordinate in the sort A, hence to dominate a type orthogonal to p.

5. Finitely many definable convex subgroups

Using the previous two sections we may describe $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in oags with finitely many L_{oag}-definable convex subgroups. The arguments still work if the subgroups are defined "by fiat" using additional predicates, so we work in this setting.

Definition 5.1. Let G be an oag with unary predicates $H_0, \dots, H_s$, each defining a convex subgroup, with $0 = H_0 \subsetneq H_1 \subsetneq \dots \subsetneq H_{s-1} \subsetneq H_s = G$, and such that G has no other definable convex subgroup. Denote by $\mathbb{T}$ the union of the set of prime powers with $\{0\}$ and work with the following sorts. For $0 \leq i < s$, a sort S_i for G/H_i, carrying L_{oag} together with predicates for H_j/H_i for $i < j < s$. For $1 \leq i \leq s$ and $n \in \mathbb{T}$, sorts $\mathrm{Q}_{i,n}$ for $H_i/(nH_i + H_{i-1})$, carrying L_{ab} if $n \neq 0$ and L_{oag} if $n = 0$. We denote by $\mathcal{Q}_i$ the family of sorts $(\mathrm{Q}_{i,n})_{n\in\mathbb{T}}$. We include the canonical projection and inclusion maps together with, for each $n \in \mathbb{T}$ and $1 \leq i \leq s-1$, the maps $\rho_{n,i} : \mathrm{S}_{i-1} \to \mathrm{Q}_{i,n}$ as in Notation 4.11, relative to the short exact sequence $0 \to \mathrm{Q}_{i,0} \to \mathrm{S}_{i-1} \to \mathrm{S}_i \to 0$.

For $1 \leq i < s$ the short exact sequence $0 \to H_i/H_{i-1} \to G/H_{i-1} \to G/H_i \to 0$ is pure and, as pointed out in Example 4.4, interdefinable with an expanded pure short exact sequence of abelian groups.

Lemma 5.2. *Every H_{i+1}/H_i is regular. For each $i \neq j$, the sort S_i is fully embedded as an oag, the family $\mathcal{Q}_i$ (with L_{oag}-structure on $\mathrm{Q}_{i,0}$, L_{ab}-structure on other sorts, and projection maps) is fully embedded, orthogonal to S_i, and orthogonal to $\mathcal{Q}_j$.*

Proof. Apply Fact 3.2 to H_{i+1}/H_i, whose only definable convex subgroups are itself and $\{0\}$. The rest is by Corollary 4.7, Remark 2.3, and induction on i. ☐

Theorem 5.3. *Let G be as in Definition 5.1, and κ a small infinite cardinal. Then $\otimes$ respects $\geq_{\mathrm{D}}$, and $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}}) \cong \prod_{i=1}^{s} \widetilde{\mathrm{Inv}}_\kappa(\mathcal{Q}_i(\mathfrak{U}))$.*

Proof. By the previous lemma, Corollaries 4.9, 3.33 and induction we get that $\otimes$ respects $\geq_{\mathrm{D}}$, and $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \prod_{i=1}^{s} \widetilde{\mathrm{Inv}}_\kappa(\mathcal{Q}_i(\mathfrak{U}))$.

If the H_i are L_{oag}-definable,[2] a result of Vicaría [32, Theorem 5.1] yields weak elimination of imaginaries in the language with sorts $\mathrm{S}_i/n\mathrm{S}_i$ for $0 \le i < s$ and $n \in \mathbb{T}$,[3] and one may check that her proof goes through also in the case where the H_i are explicitly named by predicates, i.e., not necessarily L_{oag}-definable.

After adding the sorts from Vicaría's result, for $1 \le i \le s$ the short exact sequences $0 \to \mathrm{Q}_{i,n} \to \mathrm{S}_{i-1}/n\mathrm{S}_{i-1} \to \mathrm{S}_i/n\mathrm{S}_i \to 0$ are fully embedded, and Corollary 4.9 may thus be applied to these. From this, we obtain an embedding $\prod_{i=1}^{s} \widetilde{\mathrm{Inv}}_\kappa(\mathcal{Q}_i(\mathfrak{U})) \hookrightarrow \widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}^{\mathrm{eq}})$. As $\mathrm{Q}_{s,n} = \mathrm{S}_{s-1}/n\mathrm{S}_{s-1}$, by induction on i one obtains surjectivity of this embedding. We leave to the reader to check this, along with the proof of transfer of compatibility of $\otimes$ and $\geq_{\mathrm{D}}$, by showing that every $*$-type is dominated by its image among a suitable tuple of definable maps. □

6. Benign valued fields

In this section T is a complete $\mathcal{RV}$-expansion of a theory of henselian valued fields with elimination of K-quantifiers and "enough saturated maximal models" (see below for the precise definitions). We show the existence of an isomorphism $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathcal{RV}(\mathfrak{U}))$. In particular, our results hold in any *benign* valued field in the sense of [30],[4] i.e., in any henselian valued field which is of equicharacteristic 0, or algebraically closed, or algebraically maximal Kaplansky of characteristic $\mathfrak{p} > 0$.

Associate to a valued field K the pure (Example 4.4) short exact sequence $1 \to \mathrm{k}^\times \to \mathrm{K}^\times/(1+\mathfrak{m}) \to \Gamma \to 0$. Add absorbing elements 0, 0, ∞, and view it as a short exact sequence of abelian monoids $1 \to \mathrm{k} \to \mathrm{K}/(1+\mathfrak{m}) \to \Gamma \cup \{\infty\} \to 0$. We may harmlessly conflate the two settings (Remark 4.10) and write Γ for $\Gamma \cup \{\infty\}$.

The middle term $\mathrm{K}/(1+\mathfrak{m})$ is called the *leading term structure* RV, and comes with a natural map $\mathrm{rv} : \mathrm{K} \to \mathrm{K}/(1+\mathfrak{m}) = \mathrm{RV}$ through which the valuation $v : \mathrm{K} \to \Gamma$ factors. Besides the structure of a (multiplicatively written) monoid, RV is equipped with a "partially defined sum": a ternary relation defined by

$$\oplus(x_0, x_1, x_2) \overset{\mathrm{def}}{\iff} \exists y_0, y_1, y_2 \in \mathrm{K}\left(y_2 = y_0 + y_1 \wedge \bigwedge_{i<3} \mathrm{rv}(y_i) = x_i\right).$$

When there is a unique x_2 such that $\oplus(x_0, x_1, x_2)$, we write $x_0 \oplus x_1 = x_2$, and say that $x_0 \oplus x_1$ *is well defined.* It turns out that $\mathrm{rv}(x) \oplus \mathrm{rv}(y)$ is well defined if and only if $v(x + y) = \min\{v(x), v(y)\}$. If we say that $\bigoplus_{i<\ell} x_i$ *is well defined*, we mean that, regardless of the choice of parentheses and order of the summands, the "sum" is well defined and always yields the same result.

[2]Oags with finitely many definable convex subgroups are known as the oags *of finite regular rank.* Note that every H_i must be fixed by every automorphism, and is therefore $\varnothing$-definable.

[3]Vicaría uses sorts indexed by $n \in \omega$; as in Remark 3.14, it suffices to work with $n \in \mathbb{T}$.

[4]Definition 1.57 of [30] allows $\{k\}$-$\{\Gamma\}$-expansions in the definition of benign. Since we are shortly going to allow more general expansions, the difference is immaterial for our purposes.

Let $\mathcal{RV}$ be the expansion of $1 \to \mathrm{k} \xrightarrow{\iota} \mathrm{RV} \xrightarrow{v} \Gamma \to 0$ by the field structure on k and the order on Γ. This induces an expansion of RV, which is precisely that given by multiplication and ⊕ [2, Lemma 5.17], is biinterpretable with $\mathcal{RV}$, and can be axiomatised independently [30, Appendix B]. Hence, we may view RV as a standalone structure $(\mathrm{RV}, \cdot, \oplus)$, fully embedded in $(\mathrm{K}, \mathrm{RV}, \mathrm{rv})$, and in $\mathcal{RV}$.

By the (short) five lemma, an extension of valued fields is immediate, i.e., does not change k nor Γ, if and only if it does not change $\mathcal{RV}$.

In this section, L has sorts K, k, RV, Γ, function symbols $\mathrm{rv}: \mathrm{K} \to \mathrm{RV}$, $\iota: \mathrm{k} \to \mathrm{RV}$, $v : \mathrm{RV} \to \Gamma$. We abuse the notation and also write v for the composition $v \circ \mathrm{rv}$. The sorts K and k carry disjoint copies of the language of rings, $\Gamma = \Gamma \cup \{\infty\}$ carries the (additive) language of ordered groups, together with an absorbing element ∞ and an extra constant symbol $v(\mathrm{Char}(\mathrm{K}))$, and RV carries the (multiplicative) language of groups, together with an absorbing element 0 and a ternary relation symbol ⊕. We denote by $\mathcal{RV}$ the reduct to the sorts k, RV, Γ. There may be other arbitrary symbols on $\mathcal{RV}$, i.e., as long as they do not involve K. An *$\mathcal{RV}$-expansion* of a theory T' of valued fields is a complete L-theory $T \supseteq T'$. Until the end of the section, T denotes such a theory. We identify k with the image of its embedding ι in RV.

Remark 6.1. Angular components factor through the map rv, yielding a splitting of $\mathcal{RV}$. Therefore, the Denef–Pas language (and each of its {k, Γ}-expansions)[5] may be seen as an $\mathcal{RV}$-expansion. In that case $\mathcal{RV}$ is definably isomorphic to $\mathrm{k} \times \Gamma$.

Fact 6.2. Fix a language L as above. The theory of all $\mathcal{RV}$-expansions of benign valued fields eliminates K-sorted quantifiers.

Proof. In equicharacteristic this follow from [9, Théorème 2.1]. The residue characteristic 0 case is explicitly done in [4, Theorem B] (see also [20, Corollary 2.2]), the algebraically maximal Kaplansky case in [20, Theorem 2.6] (see [15, Corollary A.3] for a modern treatment). The algebraically closed case is folklore (see, e.g., [18, Fact 2.4]). □

Remark 6.3. If T eliminates K-quantifiers, then every formula is equivalent to one of the form $\varphi\big(x, \mathrm{rv}(f_0(y)), \dots, \mathrm{rv}(f_m(y))\big)$, where $\varphi(x, z_0, \dots, z_m)$ is a formula in $\mathcal{RV}$, x and z tuples of $\mathcal{RV}$-variables, y a tuple of K-variables, and the f_i polynomials over $\mathbb{Z}$. In particular, $\mathcal{RV}$ (with the restriction of L to its sorts) is fully embedded.

Proof. By inspecting the formulas without K-sort quantifiers and observing that, for example, if y is of sort K then $T \vdash y = 0 \leftrightarrow \mathrm{rv}(y) = 0$. □

Definition 6.4. Let $K_0 \subseteq K_1$ be an extension of valued fields. A basis $(a_i)_i$ of a K_0-vector subspace of K_1 is *separating* if for all finite tuples d from K_0^ℓ and pairwise distinct i_j, we have $v\big(\sum_{j<\ell} d_j a_{i_j}\big) = \min_{j<\ell}(v(d_j) + v(a_{i_j}))$.

[5] A {k, Γ}-expansion is one where the new symbols only involve the sorts k and Γ, possibly simultaneously. If we want to exclude the latter possibility, we speak of {k}-{Γ}-expansions.

Fact 6.5. A basis $(a_i)_i$ is separating if and only if each sum $\bigoplus_{j<\ell} \mathrm{rv}(d_j)\,\mathrm{rv}(a_{i_j})$ is well defined. If this is the case, it equals $\mathrm{rv}\big(\sum_{j<\ell} d_j\, a_{i_j}\big)$.

Lemma 6.6. *Let $p \in S^{\mathrm{inv}}_{\mathrm{K}^{\leq\omega}}(\mathfrak{U}, M_0)$, $M_0 \preceq M \prec^+ \mathfrak{U} \subseteq B$, $a \vDash p \mid B$, and $(f_i)_{i\in I}$ a family of M-definable functions $\mathrm{K}^\omega \to \mathrm{K}$ such that $(f_i(a))_{i\in I}$ is a separating basis of the $\mathrm{K}(M)$-vector space they generate. If M is $|M_0|^+$-saturated, or p is definable, then $(f_i(a))_{i\in I}$ is a separating basis of the $\mathrm{K}(B)$-vector space they generate.*

Proof. Towards a contradiction, suppose there are an $L(M)$-formula

$$\varphi(x, w) := v\left(\sum_{i<\ell} w_i\, f_i(x)\right) > \min_{i<\ell}\{v(w_i) + v(f_i(x))\}$$

and $d \in B^{|w|}$ such that $a \vDash \varphi(x, d)$. Let H be the set of parameters appearing in $\varphi(x, w)$. Choose $\tilde{d} \in M$ with $\tilde{d} \equiv_{M_0 H} d$ if M is $|M_0|^+$-saturated, or in $d_p\,\varphi$ if p is definable. Then $a \vDash \varphi(x, \tilde{d})$ contradicts that $(f_i(a))_{i\in I}$ is separating over M. $\square$

Hence, saturation of M allows to lift separating bases. As maximality of M guarantees their existence (see Lemma 6.13 below), we give the following definition.

Definition 6.7. We say that T has *enough saturated maximal models* if for every $\kappa > |L|$, for every $M_0 \vDash T$ of size at most κ there is $M \succ M_0$ of size at most 2^{2^κ} which is maximally complete and $|M_0|^+$-saturated.

Remark 6.8. If we restrict to definable types, saturation is not necessary to lift separating bases (see Lemma 6.6), and it is enough to assume only "enough maximal models" for weak versions of the results of this section to go through.

Proposition 6.9. *Let T be an $\mathcal{RV}$-expansion of a theory of henselian valued fields eliminating K-quantifiers, where every $M \vDash T$ has a unique maximal immediate extension up to isomorphism over M. If $M' \vDash T$ is maximal, $\kappa > |L|$, and $\mathcal{RV}(M')$ is κ-saturated, then M' is κ-saturated.*

The proposition above is folklore, but we include a proof for convenience. As pointed out to us by the referee, uniqueness of the maximal immediate extension is not needed, and maximality of M' may be relaxed to requiring that chains of balls of length smaller than κ have nonempty intersection; the result then follows by using Swiss cheese decomposition. Nevertheless, the proof below has the advantage that it can be adapted to more general contexts, which we will need in Proposition 8.1.

Proof. If κ is limit κ-saturation equals λ-saturation for all $\lambda < \kappa$, so we may assume κ is successor, and hence regular. It suffices to prove that if $M \equiv M'$ is κ-saturated, then the set $\mathcal{S}$ of partial elementary maps between M and M' with domain of size less than κ has the back-and-forth property. In fact, we only need the "forth" part (and the "back" part is true by κ-saturation of M). So assume $f \in \mathcal{S}$, with

$$f : A = (\mathrm{K}(A), \mathcal{RV}(A)) \to A' = (\mathrm{K}(A'), \mathcal{RV}(A'))$$

and suppose that $A \subseteq B \subseteq M$, with $|B| < \kappa$. In order to extend f to some $g \in \mathcal{S}$ with domain containing B, consider the following two constructions.

Construction 1. Enlarge A to an elementary substructure. That is, there are $A_1 \supseteq A$ and $f_1 : A_1 \to A_1'$ extending f such that $f_1 \in \mathcal{S}$ and $A_1 \preceq M$. To do this, we find A_1' with $A' \subseteq A_1' \preceq M'$ and $|A_1'| < \kappa$ using the Löwenheim–Skolem theorem, and invoke κ-saturation of M to obtain the desired A_1, f_1.

Construction 2. For a given $\hat{B}$ such that $A \subseteq \hat{B} \subseteq M$ and $|\hat{B}| < \kappa$, enlarge $\mathcal{RV}(A)$ so that it contains $\mathcal{RV}(\hat{B})$. That is, there are $A_1 \supseteq A$ and $f_1 : A_1 \to A_1'$ extending f such that $f_1 \in \mathcal{S}$ and $\mathcal{RV}(A_1) \supseteq \mathcal{RV}(\hat{B})$. To do this, it suffices to set $A_1 = (\mathrm{K}(A), \mathcal{RV}(\hat{B}))$ and extend f on $\mathcal{RV}$ using κ-saturation of $\mathcal{RV}(M')$; by elimination of K-quantifiers, the extension is still an elementary map.

By repeated applications of the constructions above, we find an elementary chain $(M_n)_{n\in\omega}$ of elementary submodels of M, with $A \subseteq M_0$, and $f_n \in \mathcal{S}$ with domain M_n such that $f_0 \supseteq f$, $f_{n+1} \supseteq f_n$, and that if B_n is the structure generated by $M_n B$ then $\mathcal{RV}(B_n) \subseteq \mathcal{RV}(M_{n+1})$. Let $M_\omega := \bigcup_{n\in\omega} M_n$ and let $f_\omega := \bigcup_{n\in\omega} f_n$. Since κ is regular and uncountable we have $f \in \mathcal{S}$, and by construction the structure B_ω generated by $M_\omega B$ is K-generated and an immediate extension of M_ω. Since M' is maximal and the maximal immediate extension of M_ω is uniquely determined up to M_ω-isomorphism, we may extend f_ω to a map $g \in \mathcal{S}$ with domain $B_\omega \supseteq B$. □

Remark 6.10. Above (and in {k}-{Γ}-expansions of the Denef–Pas language), if k and Γ are orthogonal it suffices to assume that $\mathrm{k}(M')$ and $\Gamma(M')$ are κ-saturated.

Corollary 6.11. *Suppose that T satisfies the assumptions of Proposition 6.9, and furthermore that every maximal immediate extension of every $M \vDash T$ is an elementary extension. Then T has enough saturated maximal models.*

Proof. Given $\kappa > |L|$ and $M_0 \vDash T$ of size $|M_0| \le \kappa$, find $M_1 \succ M_0$ which is $|M_0|^+$-saturated of size $|M_1| \le 2^{|M_0|}$. Let M be a maximal immediate extension of M_1. Then $\mathcal{RV}(M) = \mathcal{RV}(M_1)$, and the latter is $|M_0|^+$-saturated because M_1 is. By assumption, $M \succ M_1$, and by Proposition 6.9 M is $|M_0|^+$-saturated. To conclude, observe that, since by Krull's inequality [10, Proposition 3.6] we have $|\mathrm{K}| \le \mathrm{k}^\Gamma$, we obtain

$$|M| \le |\mathrm{k}(M)|^{|\Gamma(M)|} = |\mathrm{k}(M_1)|^{|\Gamma(M_1)|} \le (2^{|M_0|})^{2^{|M_0|}} = 2^{2^{|M_0|}}. \qquad \square$$

Corollary 6.12. *Every $\mathcal{RV}$-expansion of a benign T has enough saturated maximal models.*

Proof. Since the assumptions of Fact 6.2 are preserved by taking maximal immediate extensions (which are unique by [19, Theorem 5]) elementarity follows from elimination of K-quantifiers. We conclude by Corollary 6.11. □

Lemma 6.13. *Let $p, q \in S^{\mathrm{inv}}_{\mathrm{K}^{<\omega}}(\mathfrak{U}, M_0)$, let $(a, b) \vDash p \otimes q$ and $M_0 \prec M \prec^+ \mathfrak{U}$.*

(1) *If M is maximally complete, then there are polynomials $(f_i)_{i<\omega}$ in $\mathrm{K}(M)[x]$ such that $(f_i(a))_{i<\omega}$ is a separating basis of $\mathrm{K}(M)[a]$ as a $\mathrm{K}(M)$-vector space.*

(2) *If M is $|M_0|^+$-saturated then, for each $(f_i)_{i<\omega}$ as above, $\{f_i(a) \mid i < \omega\}$ is a separating basis of $\mathrm{K}(\mathfrak{U})[a]$.*

(3) *If $(f^p_i(a))_{i<\omega}$, $(f^q_j(b))_{j<\omega}$ are separating bases of $\mathrm{K}(\mathfrak{U})[a]$ and $\mathrm{K}(\mathfrak{U})[b]$, then $(f^p_i(a) \cdot f^q_j(b))_{i,j<\omega}$ is a separating basis of $\mathrm{K}(\mathfrak{U})[ab]$.*

Proof. Part (1) is by [5, Lemma 3] (see also [17, Lemma 12.2]) and does not require saturation, and part (2) is by Lemma 6.6 applied to $(f_i)_{i<\omega}$. So we only need to prove (3). By the definition of $\otimes$, the tuple $(f^p_i(a) \cdot f^q_j(b))_{i,j<\omega}$ is linearly independent, and clearly it generates $\mathrm{K}(\mathfrak{U})[ab]$ as a $\mathrm{K}(\mathfrak{U})$-vector space. Let us check that this basis is separating. Let B be the structure generated by $\mathfrak{U}b$. By Lemma 6.6, $(f^p_i(a))_{i<\omega}$ is a separating basis of the $\mathrm{K}(B)$-vector space $\mathrm{K}(B)[a]$, so we have

$$\begin{aligned}
v\Big(\sum_{i,j} d_{ij} f^p_i(a) f^q_j(b)\Big) &= v\Big(\sum_i \Big(\sum_j d_{ij} f^q_j(b)\Big) f^p_i(a)\Big) \\
&= \min_i \Big(v\Big(\sum_j d_{ij} f^q_j(b)\Big) + v(f^p_i(a))\Big) \\
&= \min_i \Big(\min_j \big(v(d_{ij}) + v(f^q_j(b))\big) + v(f^p_i(a))\Big) \\
&= \min_{i,j} \big(v(d_{ij}) + v(f^q_j(b)) + v(f^p_i(a))\big) \\
&= \min_{i,j} \big(v(d_{ij}) + v(f^q_j(b) \cdot f^p_i(a))\big).
\end{aligned}$$ □

Proposition 6.14. *Suppose that T eliminates K-quantifiers and has enough saturated maximal models. For every $p \in S^{\mathrm{inv}}(\mathfrak{U})$ there is $q \in S^{\mathrm{inv}}_{\mathcal{RV}^\omega}(\mathfrak{U})$ such that $p \sim_{\mathrm{D}} q$. More precisely, let $p(x, z) \in S^{\mathrm{inv}}(\mathfrak{U}, M_0)$, where x is a tuple of K-variables and z a tuple of $\mathcal{RV}$-variables. Let $(a, c) \vDash p(x, z)$, let $M \succ M_0$ be $|M_0|^+$-saturated and maximally complete, and let $(f_i)_{i<\omega}$ be given by Lemma 6.13 applied to a and M. Then p is domination-equivalent to the $*$-type $q(y, t) := \mathrm{tp}(\mathrm{rv}(f_i(a))_{i<\omega}, c/\mathfrak{U})$, witnessed by $r(x, z, y, t) := \mathrm{tp}(a, c, \mathrm{rv}(f_i(a))_{i<\omega}, c/M)$.*

Proof. That $p \cup r \vdash q$ is trivial. By elimination of K-quantifiers (Fact 6.2), to prove $q \cup r \vdash p$ it is enough to show that $q \cup r$ has access to every $\mathrm{rv}(f(x))$, that is, that for every $f \in \mathrm{K}(\mathfrak{U})[x]$, there is a $\mathfrak{U}$-definable function g such that $q \cup r \vdash \mathrm{rv}(f(x)) = g(y)$. Write $f(x) = \sum_{i<\ell} d_i f_i(x)$. By Fact 6.5, we have $\mathrm{rv}(f(a)) = \bigoplus_{i<\ell} \mathrm{rv}(d_i)\,\mathrm{rv}(f_i(a))$, and we only need to ensure that this information is in $q \cup r$. But by Fact 6.5 whether the $(f_i(a))_{i<\omega}$ form a separating basis or not only depends on the type of their images in RV, which is part of q by definition. □

The work done so far is enough to obtain an infinitary version of Theorem B. After stating such a version, we will proceed to finitise it.

Remark 6.15. Separating bases of vector spaces of uncountable dimension need not exist. Nevertheless, a $*$-type version of Lemma 6.13 still holds, with the $f_i(a)$ now enumerating separating bases of all finite dimensional subspaces of $\mathrm{K}(M)[a]$.

Corollary 6.16. *If κ is a small infinite cardinal, there is an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathcal{RV}(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ on $*$-types in $\mathcal{RV}(\mathfrak{U})$, then the same holds in $\mathfrak{U}$, and the above is also an isomorphism of monoids.*

Proof. By the $*$-type versions of Lemma 6.13 and Propositions 6.14 and 1.3. □

Lemma 6.17. *Let $M_0 \prec^+ M \prec^+ \mathfrak{U}$, let $e \vDash q \in S^{\mathrm{inv}}_{\mathrm{RV}^\omega}(\mathfrak{U}, M_0)$. Let $I \subseteq \omega$ be such that $(v(e_i))_{i\in I}$ generates $\mathbb{Q}\langle\Gamma(\mathfrak{U})v(e)\rangle$ over $\mathbb{Q}\,\Gamma(\mathfrak{U})$ as $\mathbb{Q}$-vector spaces. Let $G \subseteq \mathrm{RV}$ be the multiplicative group generated by $\mathrm{RV}(\mathfrak{U})e$. Let $(g_j)_{j\in J} \subseteq \mathrm{k} \cap G$ be such that $\mathrm{k} \cap G \subseteq \mathrm{acl}(\mathfrak{U}(g_j)_{j\in J})$ and J is countable. Let $b := (e_i, g_j \mid i \in I, j \in J)$. Then there is $M \prec N \prec^+ \mathfrak{U}$ such that e and b are interalgebraic over N.*

Proof. By assumption, for $\ell \in \omega \setminus I$ there are $n_\ell > 0$, $d_\ell \in \mathfrak{U}$, a finite $I_0 \subseteq I$ and, for $i \in I_0$, integers $n_{\ell,i} \in \mathbb{Z}$, with $n_\ell v(e_\ell) = v(d_\ell) + \sum_{i\in I_0} n_{\ell,i} v(e_i)$. By M_0-invariance, we may assume $d_\ell \in M$. Let $h_\ell(x)$ be the M-definable function $h_\ell(y) := (y_\ell^{n_\ell})/\big(d_\ell \prod_{i\in I_0} y_i^{n_{\ell,i}}\big)$. By construction, we have $v(h_\ell(e)) = 0$, and hence $h_\ell(e) \in G \cap \mathrm{k}^\times$, so by assumption $h_\ell(e) \in \mathrm{acl}(\mathfrak{U}(g_j)_{j\in J})$. Let $N \succ M$ be small such that $\{h_\ell(e) \mid \ell \in \omega \setminus I\} \subseteq \mathrm{acl}(N(g_j)_{j\in J})$ and $\{g_j \mid j \in J\}$ is contained in the group generated by $\mathrm{RV}(N)e$. By definition of h_ℓ, for each $\ell \in \omega \setminus I$, we therefore have $e_\ell^{n_\ell} \in \mathrm{acl}(Nb)$. As Γ is ordered and the kernel of $v : \mathrm{RV} \to \Gamma$ is the multiplicative group of a field, RV has finite n-torsion for each n, so e_ℓ is algebraic over $e_\ell^{n_\ell}$, and hence $e \in \mathrm{acl}(Nb)$. □

Theorem 6.18 (Theorem B). *For T an $\mathcal{RV}$-expansion of a theory of valued fields with enough saturated maximal models eliminating* K*-quantifiers (e.g., a benign one), there is an isomorphism of posets $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathcal{RV}(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{RV}(\mathfrak{U})$, then $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathfrak{U}$, and the above is an isomorphism of monoids.*

Proof. Fix $p(x, z) \in S^{\mathrm{inv}}(\mathfrak{U})$ and $ac \vDash p$, where x is a tuple of K-variables and z a tuple of $\mathcal{RV}$-variables. Let $(f_i)_{i<\omega}$ be given by Lemma 6.13. As usual, denote by $\mathfrak{U}(a)$ the field generated by a over $\mathfrak{U}$. As $\mathrm{trdeg}(\mathfrak{U}(a)/\mathfrak{U})$ is finite, by the Abhyankar inequality so is $\dim_{\mathbb{Q}}(\mathbb{Q}\,\Gamma(\mathfrak{U}(a))/\mathbb{Q}\,\Gamma(\mathfrak{U}))$. Let m be such that $v(f_i(a))_{i<m}$ generates $\mathbb{Q}\,\Gamma(\mathfrak{U}(a))$ over $\mathbb{Q}\,\Gamma(\mathfrak{U})$. Again by the Abhyankar inequality, $\mathrm{trdeg}(\mathrm{k}(\mathfrak{U}(a))/\mathrm{k}(\mathfrak{U}))$ is finite. By the choice of the f_j and Fact 6.5, we may choose a transcendence basis $(g_j \mid j < n)$ of $\mathrm{k}(\mathfrak{U}(a))$ over $\mathrm{k}(\mathfrak{U})$, which is contained in the group generated by $\mathrm{RV}(\mathfrak{U})\big(\mathrm{rv}(f_i(a))\big)_{i<\omega}$. Write each g_j as $h_j(a)$, for suitable definable functions h_j. We may now apply Lemma 6.17 to $e = \big(\mathrm{rv}(f_i(a))\big)_{i<\omega}$, the g_j defined above, and $I = \{i \in \omega \mid i < m\}$. Together with Proposition 6.14, we obtain

$$p \sim_{\mathrm{D}} p' := \operatorname{tp}\big(\operatorname{rv}(f_i(a))_{i<m}, (h_j(a))_{j<n}, c/\mathfrak{U}\big) \tag{1}$$

Therefore, every (finitary) type is equivalent to one in $\mathcal{RV}$. By full embeddedness of $\mathcal{RV}$, and Fact 1.1, we obtain the required isomorphism of posets.

By Proposition 1.3 it is enough to show that if p', q' are obtained from p, q as in (1) above, then $p \otimes q \sim_{\mathrm{D}} p' \otimes q'$. Denote by

$$\rho^p(x, z) := \big(\operatorname{rv}(f_i^p(x))_{i<m_p}, (h_j^p(x))_{j<n_p}, \operatorname{id}^p(z)\big)$$

the tuple of definable functions from (1), and similarly for q and $p \otimes q$. By point (3) of Lemma 6.13 we may take as $(f_i^{p\otimes q})_{i<\omega}$ (a reindexing on ω of) the concatenation of $(f_i^p)_{i<\omega}$ with $(f_i^q)_{i<\omega}$. By the properties of $\otimes$, the concatenation of $(f_i^p(a))_{i<m_p}$ and $(f_i^q(b))_{i<m_q}$ is a basis of the vector space

$$\mathbb{Q}\big\langle\Gamma(\mathfrak{U})\big(v(f_i^p(a))\big)_{i<\omega}\big(v(f_i^q(b))\big)_{i<\omega}\big\rangle$$

over $\mathbb{Q}\,\Gamma(\mathfrak{U})$, and so as $(f_i^{p\otimes q})_{i<m_{p\otimes q}}$ we may take the concatenation of $(f_i^p)_{i<m_p}$ with $(f_i^q)_{i<m_q}$. Similarly, as $(h_j^{p\otimes q})_{j<n_{p\otimes q}}$ we may take the concatenation of the respective tuples for p and q, and ultimately we obtain that as $\rho^{p\otimes q}$ we may take the concatenation of ρ^p with ρ^q. By (1), we have $p \otimes q \sim_{\mathrm{D}} p' \otimes q'$ and we are done. □

For $\{\mathrm{k}, \Gamma\}$-expansions, we are in the setting of Section 4, so we may combine the above with, e.g., Theorem C or Corollary 4.15. We spell out two nice cases; the special subcases of ACVF and RCVF were previously known (see the introduction).

Corollary 6.19 (Theorem A). *Let T be a complete $\{\mathrm{k}\}$-$\{\Gamma\}$-expansion of a benign theory of valued fields where, for all $n > 1$, the group $\mathrm{k}^\times/(\mathrm{k}^\times)^n$ is finite. There is an isomorphism of posets $\widetilde{\operatorname{Inv}}(\mathfrak{U}) \cong \widetilde{\operatorname{Inv}}(\mathrm{k}(\mathfrak{U})) \times \widetilde{\operatorname{Inv}}(\Gamma(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in k and Γ, then $\otimes$ respects $\geq_{\mathrm{D}}$, and the above is an isomorphism of monoids.*

Proof. Apply Theorem 6.18. By Fact 6.2, if the extra structure on $\mathcal{RV}$ involves only k and Γ, and never both at the same time, then the sorts k and Γ are orthogonal. As $\mathcal{RV}$ is an expanded pure short exact sequence, we conclude by Corollary 4.15. □

Corollary 6.20. *Let T be a complete $\{\mathrm{k}\}$-$\{\Gamma\}$-expansion of a benign theory of valued fields, and let $\mathcal{A}_{\mathrm{k}}$ denote the family of sorts $(\mathrm{k}^\times/(\mathrm{k}^\times)^n)_{n\in\omega}$. For $\kappa \geq |L|$, there is an isomorphism of posets $\widetilde{\operatorname{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\operatorname{Inv}}_\kappa(\mathcal{A}_{\mathrm{k}}(\mathfrak{U})) \times \widetilde{\operatorname{Inv}}_\kappa(\Gamma(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{A}_{\mathrm{k}}$ and Γ, then $\otimes$ respects $\geq_{\mathrm{D}}$, and the above is an isomorphism of monoids.*

Proof. As in Corollary 6.19, but using Corollary 4.9 instead of Corollary 4.15. □

In special cases, results such as the previous corollaries may also be obtained by using domination by a family of sorts in the sense of [12, Definition 1.7] (see [23, Section 6]). This kind of domination was proven in the algebraically closed case in [17], in the real closed case in [12], and in the equicharacteristic zero case in [31].

In algebraically or real closed valued fields, the decomposition

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$$

remains valid after passing to T^{eq}, as can be shown using resolutions [12; 17; 23]. A natural question is whether Theorem 6.18 generalises to T^{eq}, or at least to $T^{\mathcal{G}}$, the expansion of T by the geometric sorts of [16].

Question 6.21. Let T be an $\mathcal{RV}$-expansion of a theory of valued fields with enough saturated maximal models eliminating K-quantifiers. Are there conditions guaranteeing that the isomorphism $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\mathcal{RV}(\mathfrak{U}))$ holds in $T^{\mathcal{G}}$, or even in T^{eq}? Does compatibility of $\geq_{\mathrm{D}}$ with $\otimes$ transfer?

7. Mixed characteristic henselian valued fields

Let K be henselian of characteristic $(0, \mathfrak{p})$ for $\mathfrak{p} \in \mathbb{P}$. For $n \in \omega$, we define $\mathfrak{m}_n := \{x \in \mathrm{K} \mid v(x) > v(\mathfrak{p}^n)\}$. Let RV_n be the multiplicative monoid $\mathrm{RV}_n := \mathrm{K}/(1+\mathfrak{m}_n)$ and $\mathrm{RV}_n^{\times} := \mathrm{RV}_n \setminus \{0\}$. For each n, denote by $\mathrm{rv}_n : \mathrm{K} \to \mathrm{RV}_n$ the quotient map. For $m > n$, we have natural maps $\mathrm{rv}_{m,n} : \mathrm{RV}_m \to \mathrm{RV}_n$, and the valuation $v : \mathrm{K} \to \Gamma$ induces maps $\mathrm{RV}_n \to \Gamma$, still denoted by v. The kernel k_n of v fits in a short exact sequence $1 \to \mathrm{k}_n \to \mathrm{RV}_n \xrightarrow{v} \Gamma \to 0$. We have relations $\oplus_n$, defined analogously to $\oplus$, and again well defined precisely when $v(x+y) = \min\{v(x), v(y)\}$. For $n = 0$ we recover the notions from the previous section. The following generalises Fact 6.5.

Fact 7.1. A basis $(a_i)_i$ is separating if and only if, for each $n \in \omega$, each sum $\mathrm{rv}_n(d_0)\,\mathrm{rv}_n(a_{i_0}) \oplus_n \ldots \oplus_n \mathrm{rv}_n(d_\ell)\,\mathrm{rv}_n(a_{i_\ell})$ is well defined, if and only if this happens for $n = 0$. If this is the case, then the sum equals $\mathrm{rv}_n\big(\sum_{j \leq \ell} d_j\, a_{i_j}\big)$.

In this section, L is a language as follows. We have sorts K, Γ and, for each $n \in \omega$, sorts $\mathrm{k}_n, \mathrm{RV}_n$. There are function symbols $\mathrm{rv}_n : \mathrm{K} \to \mathrm{RV}_n$, $\iota : \mathrm{k}_n \to \mathrm{RV}_n$, $v : \mathrm{RV}_n \to \Gamma$. The sort K carries a copy of the language of rings, while the sort $\Gamma = \Gamma \cup \{\infty\}$ carries the (additive) language of ordered groups, together with an absorbing element ∞ and an extra constant symbol $v(\mathfrak{p})$. Each RV_n and k_n carries the (multiplicative) language of groups, together with an absorbing element 0 and a ternary relation symbol $\oplus_n$. We denote by $\mathcal{RV}_*$ the reduct to the sorts $\mathrm{k}_n, \mathrm{RV}_n, \Gamma$. There may be other arbitrary symbols on $\mathcal{RV}_*$, i.e., as long as they do not involve K.

An $\mathcal{RV}_*$*-expansion* of a theory T' of henselian valued fields of characteristic $(0, \mathfrak{p})$ is a complete L-theory $T \supseteq T'$, with the sorts and symbols above interpreted in the natural way. Until the end of the section, T denotes such a theory. We will freely confuse the sort k_n with the image of its embedding in RV_n. By [4, Theorem B] (see also [14, Proposition 4.3]) T eliminates K-quantifiers, so $\mathcal{RV}_*$ is fully embedded.

Proposition 7.2. *Suppose T eliminates* K*-quantifiers and has enough saturated maximal models. For every $p\in S^{\mathrm{inv}}(\mathfrak{U})$ there is $q\in S^{\mathrm{inv}}_{\mathcal{RV}_*^{\omega}}(\mathfrak{U})$ such that $p\sim_{\mathrm{D}} q$. More precisely, let $p(x,z)\in S^{\mathrm{inv}}(\mathfrak{U},M_0)$, where x is a tuple of* K*-variables and z a tuple of $\mathcal{RV}_*$-variables. Let $(a,c)\vDash p(x,z)$, let $M\succ M_0$ be $|M_0|^+$-saturated and maximally complete, and let $(f_i)_{i<\omega}$ be given by the $*$-type version of Lemma 6.13 applied to a and M (see Remark 6.15). Then $p\sim_{\mathrm{D}} q(y,t):=\mathrm{tp}(\mathrm{rv}_n(f_i(a))_{i,n<\omega},c/\mathfrak{U})$, witnessed by*

$$r(x,z,y,t):=\mathrm{tp}(a,c,\mathrm{rv}_n(f_i(a))_{i,n<\omega},c/M).$$

If $\kappa\geq|L|$ is small, there is an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})\cong\widetilde{\mathrm{Inv}}_\kappa(\mathcal{RV}_*(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{RV}_*(\mathfrak{U})$, then the same holds in $\mathfrak{U}$, and the above is an isomorphism of monoids.

Proof. Adapt the proofs of Lemma 6.13, Proposition 6.14 and Corollary 6.16, replacing Facts 6.2 and 6.5 by [4, Theorem B] and Fact 7.1 respectively. □

The assumptions of Proposition 7.2 are satisfied in a number of cases of interest. Besides the algebraically closed case, we note the following.

Remark 7.3. Every $\mathcal{RV}_*$-expansion of a finitely ramified henselian valued field has enough saturated maximal models.

Proof. Finite ramification ensures immediate extensions are precisely those where $\mathcal{RV}_*$ does not change. By this and [14, Proposition 4.3], maximal immediate extensions are elementary, and by [10, Corollary 4.29] they are also unique. We may therefore adapt the proof of Proposition 6.9, replacing $\mathcal{RV}$ with $\mathcal{RV}_*$. □

Remark 7.4. $\mathcal{RV}_*$ may be viewed as a short exact sequence of abelian structures, each consisting of an inverse system of abelian groups. Since Γ is torsion-free, this sequence is pure.[6] Hence, the results from Section 4 apply to this setting, e.g., by taking as $\mathcal{F}$ the family of all pp formulas.

If k eliminates imaginaries, we can get rid of those arising from $\mathcal{F}$ and obtain a product decomposition. We state a special case as an example application of the results above. We thank the referee for pointing out the "moreover" part.

Corollary 7.5. *In the theory of the Witt vectors over $\mathbb{F}_p^{\mathrm{alg}}$, the domination monoid is well defined. If κ is a small infinite cardinal, then*

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})\cong\widetilde{\mathrm{Inv}}_\kappa(\mathrm{k}(\mathfrak{U}))\times\widetilde{\mathrm{Inv}}_\kappa(\Gamma(\mathfrak{U}))\cong\hat{\kappa}\times\mathscr{P}_{\leq\kappa}\big(\mathrm{CS}^{\mathrm{inv}}(\Gamma(\mathfrak{U}))\big).$$

Moreover, $\widetilde{\mathrm{Inv}}(\mathfrak{U})\cong\hat{\omega}\times\mathscr{P}_{<\omega}\big(\mathrm{CS}^{\mathrm{inv}}(\Gamma(\mathfrak{U}))\big)$.

[6] Another way of seeing this is that, in a saturated enough model of T, the valuation map has a section, inducing a compatible system of angular components, i.e., a splitting of $\mathcal{RV}_*$.

Proof. The residue field k is fully embedded. Moreover, $\mathrm{k}_n = W_n(\mathrm{k})^\times$ for each n, where $W_n(\mathrm{k})$ is the truncated ring of Witt vectors over k, and k_n is in definable bijection with $\mathrm{k}^{n-1} \times \mathrm{k}^\times$ (see [30, Corollary 1.62 and Proposition 1.67]). The computation of $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ follows. As for $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, using discreteness of the value group it is possible to build a prodefinable surjection $\mathrm{K} \to \mathrm{k}^\omega$ [30, proof of Remark 3.23]; together with the argument above, this gives the "moreover" part. □

Remark 7.6. The product decomposition fails for finitary types: the surjection $\mathrm{K} \to \mathrm{k}^\omega$ yields a 1-type in K dominating the type of an infinite independent k-tuple.

However, finitisation is possible in the case of the $\mathfrak{p}$-adics.

Corollary 7.7 (Theorem E). *Let T be a complete* $\{\Gamma\}$*-expansion of* $\mathrm{Th}(\mathbb{Q}_\mathfrak{p})$*. There is an isomorphism of posets* $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}(\Gamma(\mathfrak{U}))$*. If* $\otimes$ *respects* $\geq_\mathrm{D}$ *in* $\Gamma(\mathfrak{U})$*, then the same holds in* $\mathfrak{U}$*, and the above is also an isomorphism of monoids. In particular, in* $\mathrm{Th}(\mathbb{Q}_\mathfrak{p})$*,* $\otimes$ *respects* $\geq_\mathrm{D}$*, and* $(\widetilde{\mathrm{Inv}}(\mathfrak{U}), \otimes, \geq_\mathrm{D}) \cong \big(\mathscr{P}_{<\omega}\big(\mathrm{CS}^{\mathrm{inv}}(\Gamma(\mathfrak{U}))\big), \cup, \supseteq\big)$.

Proof. By Remark 7.3 we may apply Proposition 7.2. Since each k_n is finite, each RV_n is a finite cover of Γ, so each element of RV_n is interalgebraic with an element of Γ. Thus if $p(x, z) \in S^{\mathrm{inv}}(\mathfrak{U}, M_0)$, where x is a tuple of K-variables and z a tuple of $\mathcal{RV}_*$-variables, and if $ac \vDash p$, then $\dim_\mathbb{Q}\big(\mathbb{Q}\,\Gamma\big(\mathrm{dcl}(\mathfrak{U}(ac))\big)/\mathbb{Q}\,\Gamma(\mathfrak{U})\big) \leq |xz|$ by the Abhyankar inequality, so there is a finitary invariant type in Γ which is interalgebraic with the type $q(y, t) \sim_\mathrm{D} p$ found in Proposition 7.2. We conclude by Proposition 1.3. The "in particular" part then follows from Corollary 3.34. □

The infinite ramification case remains open.

Problem 7.8. Compute $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in an infinitely ramified mixed characteristic henselian valued field that is not algebraically closed.

8. D-henselian valued fields with many constants

Here we deal with certain differential valued fields. As the proofs are adaptations of those in Section 6, we give sketches and leave it to the reader to fill in the details.

We let T be a complete theory with sorts K, k, Γ, RV, as in Section 6, naturally interpreted, and use the notation $\mathcal{RV}$. The fields k and K have characteristic 0 and both carry a derivation ∂ (denoted by the same symbol), commuting with the residue map. The valued differential field K is *monotone*, i.e., $v(\partial x) \geq v(x)$, has *many constants*,[7] i.e., for every $\gamma \in \Gamma$ there is $x \in \mathrm{K}$ with $\partial x = 0$ and $v(x) = \gamma$, and is *D-henselian*, i.e., the following holds. If $P(X) \in \mathcal{O}\{X\} = \mathcal{O}[\partial^i X]_{i\in\omega}$ is a differential polynomial over the valuation ring $\mathcal{O}$, and $a \in \mathcal{O}$ is such that $v(P(a)) > 0$ and for some i we have $v(\mathrm{d}P/\mathrm{d}(\partial^i X))(a) = 0$, then there is $b \in \mathcal{O}$ such that $P(b) = 0$ and $v(a - b) > 0$. The family of sorts $\mathcal{RV}$ may carry additional structure.

[7]Here we follow the terminology of [1]. In [28], this condition is called having *enough constants*.

The derivation ∂ on K induces a map ∂_{RV} on RV which, for all $\gamma \in \Gamma$, fixes $v^{-1}(\gamma) \cup \{0\}$ setwise, defined by $\partial_{\mathrm{RV}}(\mathrm{rv}(x)) = \mathrm{rv}(\partial(x))$ if $v(\partial(x)) = v(x)$, and $\partial_{\mathrm{RV}}(\mathrm{rv}(x)) = 0$ otherwise, which extends the derivation ∂ on k.

By [28, Theorem 6.4 and Corollary 5.8] (see also [1, Corollary 8.3.3]) the theory T given by the list of properties above (in a fixed language) eliminates K-quantifiers.

Proposition 8.1. *The theory T has enough saturated maximal models.*

Proof sketch. By [28, Remark 6.2], k is linearly surjective in the terminology of [1], so by [1, Theorem 7.4.3] T has uniqueness of maximal immediate extensions. The maximal immediate extension N of M is monotone and D-henselian by [1, Lemma 6.3.5 and Theorem 7.4.3] with many constants. As T eliminates K-quantifiers, $M \prec N$, so the proofs of Proposition 6.9 and Corollary 6.11 may be adapted. □

Theorem 8.2. *Let κ be a small infinite cardinal. There is an isomorphism of posets $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathcal{RV}(\mathfrak{U}))$. If $\otimes$ respects $\geq_{\mathrm{D}}$ in $\mathcal{RV}(\mathfrak{U})$, then the same holds in $\mathfrak{U}$, and the above is also an isomorphism of monoids.*

Proof sketch. By elimination of K-quantifiers, $\mathcal{RV}(M)$ is fully embedded in M. If we replace "polynomial" by "differential polynomial", $\mathrm{K}(M)[a]$ by $\mathrm{K}(M)\{a\}$, and so on, in the statements of Lemma 6.13 and Proposition 6.14, essentially the same proofs go through. We can then conclude as in the proof of Corollary 6.16. □

Lemma 8.3. *∂_{RV} is definable from the short exact sequence structure, the differential field structure on* k, *and a predicate for $C := \{c \in \mathrm{RV} \mid \partial_{\mathrm{RV}}(c) = 0\}$.*

Proof. Suppose $a \in \mathrm{RV}$ and $v(a) \notin \{0, \infty\}$. Since K has many constants, there is $c \in \mathrm{RV}(M)$ with $\partial_{\mathrm{RV}}(c) = 0$ and $v(c) = v(a)$. Then we have $a/c \in \mathrm{k}(\mathfrak{U})$ and $\partial_{\mathrm{RV}}(a) = c\partial(a/c)$. Because this does not depend on the choice of c, the function $y = \partial_{\mathrm{RV}}(x)$ is $\varnothing$-definable by the formula

$$\varphi(x, y) := \exists z \in C\big((v(z) = v(x)) \wedge (y = z\partial(x/z))\big).$$ □

If L had a section of the valuation, or an angular component compatible with ∂, we could recover C from the constant field of k, and conclude by (the $*$-type version of) Remark 4.5. Yet, the absence of definable splitting is not a serious obstacle. For simplicity, we only give a result in the model companion $\mathsf{VDF}_{\mathcal{EC}}$.

Theorem 8.4 (Theorem F). *In $\mathsf{VDF}_{\mathcal{EC}}$, for every small infinite cardinal κ, the monoid $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U})$ is well defined, and we have isomorphisms*

$$\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\kappa(\Gamma(\mathfrak{U})) \cong \prod_{\delta(\mathfrak{U})}^{\leq\kappa} \hat{\kappa} \times \mathscr{P}_{\leq\kappa}\big(\mathrm{CS}^{\mathrm{inv}}(\Gamma(\mathfrak{U}))\big),$$

where $\delta(\mathfrak{U})$ is a cardinal, and $\prod_{\delta(\mathfrak{U})}^{\leq\kappa}\hat{\kappa}$ denotes the submonoid of $\prod_{\delta(\mathfrak{U})}\hat{\kappa}$ consisting of $\delta(\mathfrak{U})$-sequences with support of size at most κ.

Proof. By Theorem 8.2 we reduce to $\mathcal{RV}$. Let $L_C := L_{\mathrm{ab}} \cup \{C\}$, with C a unary predicate. Expand the language of $\mathcal{RV}$ by a predicate C on each sort, interpreted as the constants in both k and RV and as the full Γ in Γ, obtaining a short exact sequence of L_C-abelian structures (to be precise, of abelian structures augmented by an absorbing element, see Remark 4.10) expanded by the differential field structure on k and the order on Γ. By Lemma 8.3, we may apply the material from Section 4, say, by taking as a fundamental family that of all pp L_C-formulas, provided we show that $\mathcal{RV}$ is pure. If $M \models \mathsf{VDF}_{\mathcal{EC}}$ is $\aleph_1$-saturated then, since M has many constants, we may find a section $s : \Gamma(M) \to \mathrm{RV}(M)$ of the valuation with image included in $C(\mathrm{RV}(M))$. Hence the short exact sequence $\mathcal{RV}(M)$ of L_C-abelian structures splits, so is pure by Remark 4.5. Since k is a model of DCF_0, which eliminates imaginaries, we may get rid of the auxiliary sorts A_φ. We conclude by Corollary 3.33 and the fact that DCF_0 is ω-stable multidimensional (see [22, Section 5] for the relation between our setting and that of domination via forking in stable theories). □

Remark 8.5. In $\mathsf{VDF}_{\mathcal{EC}}$, finitisation is not to be expected (e.g., by [26, Proposition 4.2]), and in fact not possible: one may construct a 1-type $p \in S_{\mathrm{K}}^{\mathrm{inv}}(\mathfrak{U})$ with $((v \circ \partial^n)_* p)_{n\in\omega}$ nonalgebraic and pairwise weakly orthogonal, and hence not domination-equivalent.

Computing the image of the home sort in finitely many variables seems difficult.

Remark 8.6. Most arguments in this section may be adapted to σ-henselian valued difference fields of residue characteristic 0. An analogue of Theorem 8.2 goes through, using quantifier reduction to $\mathcal{RV}$ and a σ-Kaplansky theory yielding uniqueness and elementarity of maximal immediate extensions [11, Theorems 5.8 and 7.3]. In every completion of the model companion of the *isometric* case (see [6]), in sufficiently saturated models there is a section of the valuation with values in the fixed field, and hence one may obtain the decomposition $\widetilde{\mathrm{Inv}}_\kappa(\mathfrak{U}) \cong \widetilde{\mathrm{Inv}}_\kappa(\mathrm{k}(\mathfrak{U})) \times \widetilde{\mathrm{Inv}}_\kappa(\Gamma(\mathfrak{U}))$, by regarding $\mathcal{RV}$ as a pure short exact sequence of $\mathbb{Z}[\sigma]$-modules, and using elimination of imaginaries in ACFA_0. The same goes through in the *multiplicative* setting, provided that, in the notation of [25], ρ is transcendental. This applies, e.g., to the model companion of the *contractive* case (see [3]).

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