Pacific Journal of Mathematics

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Volume 328 No. 2 February 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 328, No. 2, 2024
https://doi.org/10.2140/pjm.2024.328.325

INVERSE SEMIGROUP FROM METRICS ON DOUBLES III: COMMUTATIVITY AND (IN)FINITENESS OF IDEMPOTENTS

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We have shown recently that, given a metric space X, the coarse equivalence classes of metrics on the two copies of X form an inverse semigroup $M(X)$. Here we study the property of idempotents in $M(X)$ of being finite or infinite, which is similar to this property for projections in C^*-algebras. We show that if X is a free group then the unit of $M(X)$ is infinite, while if X is a free abelian group then it is finite. As a by-product, we show that the inverse semigroup $M(X)$ is not a quasiisometry invariant. We also show that $M(X)$ is commutative if it is Clifford, and give a geometric description of spaces X for which $M(X)$ is commutative.

1. Introduction

Given metric spaces (X, d_X) and (Y, d_Y), a metric d on $X \sqcup Y$ that extends the metrics d_X on X and d_Y on Y, depends only on the values $d(x, y)$, $x \in X$, $y \in Y$, and it may be not easy to check which functions $d : X \times Y \to (0, \infty)$ determine a metric on $X \sqcup Y$. The problem of description of all such metrics is difficult due to the lack of a nice algebraic structure on the set of metrics, but, passing to coarse equivalence of metrics, we get an algebraic structure, namely, that of an inverse semigroup [Manuilov 2021a]. Recall that two metrics, b, d, on a space Z are coarsely equivalent, $b \sim d$, if there exist monotone functions $\varphi, \psi : [0, \infty) \to [0, \infty)$ such that

$$\lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \psi(t) = \infty$$

and

$$\varphi(d(z_1, z_2)) \le b(z_1, z_2) \le \psi(d(z_1, z_2))$$

for any $z_1, z_2 \in Z$. We denote by $[d]$ the coarse equivalence class of a metric d. Our standard reference on metric spaces is [Burago et al. 2001].

Let $\mathcal{M}(X, Y)$ denote the set of all metrics d on $X \sqcup Y$ such that:

- The restriction of d onto X and Y are d_X and d_Y respectively.
- $\inf_{x\in X, y\in Y} d(x, y) > 0$.

The research was supported by RSF, project No. 23-21-00068.
MSC2020: 20M18, 51F30.
Keywords: inverse semigroup, metric, finite projection.

Coarse equivalence classes of metrics in $\mathcal{M}(X, Y)$ can be considered as morphisms from X to Y [Manuilov 2019], where the composition $b \circ d$ of a metric d on $X \sqcup Y$ and a metric b on $Y \sqcup Z$ is given by the metric determined by

$$(b \circ d)(x, z) = \inf_{y \in Y}[d(x, y) + b(y, z)], \quad x \in X, z \in Z.$$

When $Y = X$, we call $X \sqcup X$ the double of X. In what follows we identify the double of X with $X \times \{0, 1\}$, and write X for $X \times \{0\}$ (resp., x for $(x, 0)$) and X' for $X \times \{1\}$ (resp., x' for $(x, 1)$). We also write $\mathcal{M}(X)$ for $\mathcal{M}(X, X)$.

The main result of [Manuilov 2021a] is that the semigroup $M(X) = \mathcal{M}(X)/\sim$ (with respect to this composition) of coarse equivalence classes of metrics on the double of X is an inverse semigroup with the unit element $\mathbf{1}$ and the zero element $\mathbf{0}$, and the unique pseudoinverse for $[d] \in M(X)$ is the coarse equivalence class of the metric d^* given by $d^*(x, y') = d(x', y)$, $x, y \in X$.

Recall that a semigroup S is an inverse semigroup if for any $s \in S$ there exists a unique $t \in S$ (denoted by s^* and called a pseudoinverse) such that $s = sts$ and $t = tst$ [Lawson 1998]. Philosophically, inverse semigroups describe local symmetries in a similar way as groups describe global symmetries, and technically, the construction of the (reduced) group C^*-algebra of a group generalizes to that of the (reduced) inverse semigroup C^*-algebra [Paterson 1999]. It is known that any two idempotents in an inverse semigroup S commute, and that there is a partial order on S defined by $s \leq t$ if $s = ss^*t$. Our standard references for inverse semigroups are [Lawson 1998] and [Howie 1995].

Close relation between inverse semigroups and C^*-algebras allows to use classification of projections in C^*-algebras for idempotents in inverse semigroups. Namely, as in C^*-algebra theory, we call two idempotents, $e, f \in E(S)$ von Neumann equivalent (and write $e \sim f$) if there exists $s \in S$ such that $s^*s = e$, $ss^* = f$. An idempotent $e \in E(S)$ is called *infinite* if there exists $f \in E(S)$ such that $f \preceq e$, $f \neq e$, and $f \sim e$. Otherwise e is *finite*. An inverse semigroup is *finite* if every idempotent is finite, and is *weakly finite* if it is unital and the unit is finite. A commutative unital inverse semigroup is patently finite.

In [Manuilov 2021b] we gave a geometric description of idempotents in the inverse semigroup $M(X)$ (there are two types of idempotents, named type I and type II) and showed in Lemma 3.3 of [loc. cit.] that the type is invariant under the von Neumann equivalence. In Part I, we study the property of weak finiteness for $M(X)$ (i.e., finiteness of the unit element) and discuss its relation to geometric properties of X.

We start with several examples of finite or infinite idempotents, and then show that if X is a free group then $M(X)$ is not weakly finite, while if X is a free abelian group then it is weakly finite. We also show that the inverse semigroup $M(X)$ is not a quasiisometry invariant. The property of being weakly finite is also not a coarse invariant. We don't know if it is a quasiisometry invariant.

In Part II, we give a geometric description of spaces, for which the inverse semigroup $M(X)$ is commutative, and show that the condition of being a Clifford inverse semigroup (i.e., that $ss^* = s^*s$ for any $s \in S$) guarantees that $M(X)$ is commutative.

Part I. Weak finiteness of $M(X)$

2. Geometric description of weak finiteness

Two maps $f, g : X \to X$ are called *equivalent* if there exists $C > 0$ such that $d_X(f(x), g(x)) < C$ for any $x \in X$. A map $f : X \to X$ is an *almost isometry* if there exists $C > 0$ such that:

- $|d_X(f(x), f(y)) - d_X(x, y)| < C$ for any $x, y \in X$.
- For any $y \in X$ there exists $x \in X$ such that $d_X(f(x), y) < C$.

(The latter condition provides existence of an "inverse" map $g : X \to X$ such that $f \circ g$ and $g \circ f$ are equivalent to the identity map; this map is also an almost isometry, but with possibly greater constant C; if f is surjective then this property is superfluous.) We call f a C-almost isometry when we need an explicit value of the constant C.

In a metric space, it makes sense to define equivalence of subsets as follows: for $A, B \subset X$ we say that $A \sim B$ if there exists $C > 0$ such that $A \subset N_C(B)$ and $B \subset N_C(A)$, where $N_C(Y) = \{x \in X : d_X(x, Y) < C\}$ denotes the C-neighborhood of $Y \subset X$. In particular, a subset $A \subset X$ is equivalent to X if it is a C-net, i.e., if there exists $C > 0$ such that for any $x \in X$ there exists $y \in A$ with $d_X(x, y) < C$.

Theorem 2.1. *The following are equivalent*:

(1) *$M(X)$ is weakly finite.*

(2) *If there exists an almost isometry $X \to A \subset X$ then the subset A is equivalent to X.*

Proof. For $B \subset X$, set

$$d^B(x, y') = \inf_{u \in B} [d_X(x, u) + 1 + d_X(u, y)].$$

Then d_X is a metric on the double of X, and $[d^B]$ is an idempotent in $M(X)$ [Manuilov 2021a]. It was shown in Lemma 3.3 of [Manuilov 2021b] that if d is a metric on the double of X and $[d^*][d] = [d^B]$ then there exists $A \subset X$ such that $[d][d^*] = [d^A]$.

Suppose that there exists a C-almost isometry $f : X \to A$ for some $A \subset X$ and for some $C > 0$. Then set

$$d(x, y') = \inf_{u \in X} [d_X(x, u) + C + d_X(f(u), y)].$$

It was shown in Lemma 3.2 of [Manuilov 2019] that this defines a metric on the double of X. Then

$$\begin{aligned} d^* \circ d(x, x') &= \inf_{y \in X} [d(x, y') + d^*(y, x')] = 2 \inf_{y \in X} d(x, y') \\ &= 2 \inf_{u, y \in X} [d_X(x, u) + C + d_X(f(u), y)] \le 2C \end{aligned}$$

(we might take $y = f(u)$ and $u = x$), hence $[d^*][d] = \mathbf{1}$.

$$\begin{aligned} d \circ d^*(x, x') &= \inf_{y \in X} [d^*(x, y') + d(y, x')] = 2 \inf_{y \in X} d(y, x') \\ &= 2 \inf_{u, y \in X} [d_X(y, u) + C + d_X(f(u), x)] \\ &= 2C + 2 \inf_{u \in X} d_X(x, f(u)) = 2C + 2d_X(x, f(X)) \end{aligned}$$

(taking $u = y$), so, using that $f(X)$ is C-dense in A, we see that

$$|d \circ d^*(x, x') - d_X(x, A)| \le 4C,$$

hence $[d][d^*] = [d^A]$ by Proposition 3.2 of [Manuilov 2021a]. If $M(X)$ is weakly finite then $[d^A] = \mathbf{1}$, hence, by Proposition 4.2 of [Manuilov 2021a], X lies in a C-neighborhood of A for some $C > 0$.

In the opposite direction, let $M(X)$ be not weakly finite. Then there exists a metric d on the double of X such that $[d^*][d] = \mathbf{1}$, but $[d][d^*] \ne \mathbf{1}$. By Lemma 3.3 of [Manuilov 2021b], $[d \circ d^*] = [d^A]$, where $A \subset X$ is constructed as follows. As $[d^*][d] = \mathbf{1}$, there exists $C > 0$ such that

$$d^* \circ d(x, x') = 2d(x, X') < 2C$$

for any $x \in X$, i.e., for any $x \in X$ there exists $y \in X$ such that $d(x, y') < C$. Then $A = \{y \in X : d(X, y') < C\}$.

Given $x \in X$, there may be several y's such that $d(x, y') < C$. Choose one of them and set $f(x) = y$. It follows from

$$d(X, f(x)') \le d(x, f(x)') < C$$

that $f(x) \in A$. If $x_1, x_2 \in X$ then the triangle inequality for the quadrangle $x_1, x_2, f(x_1)', f(x_2)'$ gives $|d_X(f(x_1), f(x_2)) - d_X(x_1, x_2)| < 2C$. If $z \in A$ then $d(X, z') < C$, hence there exists $x \in X$ such that $d(x, z') < C$. Then

$$d_X(z, f(x)) = d_X(z', f(x)') \le d(z', x) + d(x, f(x)') < 2C,$$

hence f is a $2C$-almost isometry. Finally, the condition $[d][d^*] \ne \mathbf{1}$ implies that A is not equivalent to X. $\square$

3. Some examples

The following example shows that in $M(X)$, for an appropriate X, we can imitate examples of partial isometries and projections in a Hilbert space.

Example 3.1. Let $l^1(\mathbb{N})$ be the space of infinite l^1 sequences, with the metric given by the l^1-norm, and let

$$X_n = \{(0, \dots, 0, t, 0, \dots) : t \in [0, \infty)\}$$

with t at the n-th place, $n \in \mathbb{N}$. Set

$$X = \bigcup_{n=1}^{\infty} X_n \subset l^1(\mathbb{N}), \quad A = \bigcup_{n=2}^{\infty} X_n \subset l^1(\mathbb{N}).$$

The set A is not equivalent to X, and there is an obvious isometry $f : X \to A$ that isometrically maps X_n to X_{n+1}, $n \in \mathbb{N}$. Thus, $\mathbf{1}$ is infinite. Let d be a metric on the double of X induced by f. Although d seems similar to a one-sided shift in a Hilbert space, it behaves differently: $h = [d \circ d^*]$ is orthogonally complemented, i.e., there exists $e \in E(M(X))$ such that $e \vee h = \mathbf{1}$, $e \wedge h = \mathbf{0}$ (recall that $E(M(X))$ is a lattice [Manuilov 2021b]), but the complement e is not a minimal idempotent, i.e., there exists a lot of idempotents $j \in E(M(X))$ such that $j \leq e$, $j \neq e$.

On the other hand, if $X \subset [0, \infty)$ with the standard metric then the inverse semigroup $M(X)$ is commutative [Manuilov 2021a, Proposition 7.1], hence any idempotent can be equivalent only to itself, hence is finite. In Part II, we shall give a geometric description of all metric spaces with commutative $M(X)$, which is then patently finite.

The next example shows that the picture may be more complicated.

Proposition 3.2. *There exists an amenable space X of bounded geometry and $s \in M(X)$ such that $s^*s = \mathbf{1}$, but $ss^* \neq \mathbf{1}$.*

Proof. Consider $l_\infty(\mathbb{N})$ with sup metric, and let

$$x_n = (\log 2, \log 3, \dots, \log(n-1), \log n, 0, 0, \dots) \in l_\infty(\mathbb{N}),$$
$$X = \{x_n : n \in \mathbb{N}\} \subset l_\infty(\mathbb{N}); \quad A = \{x_{2n} : n \in \mathbb{N}\}.$$

Set

$$f : X \to A; \quad f(x_n) = x_{2n}, n \in \mathbb{N}.$$

Given $n < m$, we have

$$d_X(x_n, x_m) = \log m, \quad d_X(f(x_n), f(x_m)) = d_X(x_{2n}, x_{2m}) = \log(2m),$$

hence

$$d_X(f(x_n), f(x_m)) - d_X(x_n, x_m) = \log(2m) - \log m = \log 2.$$

As f is surjective, it is an almost isometry.

Note that

$$d_X(x_{2n-1}, x_{2m}) = \begin{cases} \log(2m) & \text{if } 2n-1 < 2m, \\ \log(2n-1) & \text{if } 2n-1 > 2m, \end{cases}$$

hence

$$d_X(x_{2n-1}, A) = \inf_{m\in\mathbb{N}} d_X(x_{2n-1}, x_{2m}) = \log(2m),$$

thus $A \subset X$ is not equivalent to X, hence $M(X)$ is not weakly finite.

Note that X is amenable. Set $F_n = \{x_1, \dots, x_n\} \subset X$. Let $N_r(A)$ denote the r-neighborhood of the set A. Then $N_r(F_n) \setminus F_n$ is empty when $\log(n+1) > r$, hence $\{F_n\}_{n\in\mathbb{N}}$ is a Følner sequence. For $r = \log m$, the ball $B_r(x_n)$ of radius r centered at x_n contains either no other points besides x_n (if $n \geq m+1$), or it consists of the points $x_1, \dots, x_m$ (if $n \leq m$), hence the metric on X is of bounded geometry. In fact, this space is of asymptotic dimension zero. □

4. Case of free groups

In this section we show that $M(X)$ is not weakly finite for two classes of groups, both of which include free groups.

Let $X = \Gamma$ be a finitely generated group with the word length metric d_X. Consider the following Property (I):

(i1) $X = Y \sqcup Z$, and for any $D > 0$ there exists $z \in Z$ such that $d_X(z, Y) > D$.

(i2) There exist $g, h \in \Gamma$ such that $gY \subset Y$, $hZ \subset Y$ and $gY \cap hZ = \varnothing$.

(i3) There exists $C > 0$ such that $|d_X(gy, hz) - d_X(y, z)| < C$ for any $y \in Y$, $z \in Z$.

Property (I) looks similar to nonamenability, but, at least formally, is neither stronger nor weaker than nonamenability.

Lemma 4.1. *The free group $\mathbb{F}_2$ on two generators satisfies Property (I).*

Proof. Let a and b be the generating elements of $\mathbb{F}_2$, and let $Y \subset X$ be the set of all reduced words in a, a^{-1}, b and b^{-1} that begin with a or a^{-1}, $Z = X \setminus Y$. Let $g = ab$, $h = a^2$. Clearly, $gY \subset Y$ and $hZ \subset Y$.

If z begins with a^n, $n > D$, then $d_X(z, Y) \geq n$.

If $y \in Y$, $z \in Z$ then

$$d_X(aby, a^2z) = |y^{-1}b^{-1}a^{-1}a^2z| = |y^{-1}b^{-1}az| = |y^{-1}z| + 2 = d_X(y, z) + 2,$$

as the word $y^{-1}b^{-1}az$ cannot be reduced any further (y^{-1} ends with $a^{\pm}$, and z either begins with $b^{\pm}$, or is an empty word). □

Theorem 4.2. *Let $X = \Gamma$ be a group with Property (I). Then X is not weakly finite.*

Proof. We shall prove that there exists an almost isometry $f : X \to A \subset X$, where A is not equivalent to X.

Let $X = Y \sqcup Z$, $g, h \in \Gamma$ satisfy the conditions of Property (I). Define a map $f : X \to X$ by setting

$$f(x) = \begin{cases} gx & \text{if } x \in Y; \\ hx & \text{if } x \in Z. \end{cases}$$

The maps $f|_Y$ and $f|_Z$ are left multiplications by g and h, respectively, hence are isometries. If $y \in Y$, $z \in Z$ then (i3) holds for some $C > 0$, hence

$$|d_X(f(x), f(y)) - d_X(x, y)| < C$$

holds for any $x, y \in X$. Set $A = f(X)$, then f is an almost isometry from X to A. By (i1), A is not equivalent to X. □

Our next argument also works for free groups, but refers to non-co-Hopfian groups, i.e., groups isomorphic to a proper subgroup.

Theorem 4.3. *Let $X = G$ be a finitely generated group with the word length metric, and let $A = H \subset G$ be an infinite index subgroup. Suppose that there exists a map $f : G \to H$ that is both an isomorphism and an almost isometry. Then X is not weakly finite.*

Proof. We need only to check that A is not equivalent to X. Suppose it is, i.e., there exists $C > 0$ such that for any $x \in X$ there exists $y \in H$ with $d_X(x, y) < C$. As H is of infinite index, there are infinitely many different cosets Hg_i, $g_i \in G$, $i \in \mathbb{N}$. Let $h_i \in H$ satisfy $d_X(g_i, h_i) < C$, $i \in \mathbb{N}$, which means that $|g_i^{-1}h_i| < C$. As G is finitely generated, the set of group elements g with $|g| < C$ is finite, so there exist $i \neq j$ such that $g_i^{-1}h_i = g_j^{-1}h_j$, or, equivalently, $h_i^{-1}g_i = h_j^{-1}g_j$, hence $Hg_i = Hg_j$ — a contradiction. □

Remark 4.4. It is easy to find examples of isomorphisms that are also almost isometries. Indeed, if $\gamma \in G$ then the map $f(g) = \gamma^{-1}g\gamma$ is an example: it follows from $d_X(f(g_1), f(g_2)) = |\gamma^{-1}g_1^{-1}g_2\gamma|$ and $d_X(g_1, g_2) = |g_1^{-1}g_2|$ that

$$|d_X(f(g_1), f(g_2)) - d_X(g_1, g_2)| \leq 2|\gamma|$$

for any $g_1, g_2 \in G$. There are many examples when the subgroup $H = \gamma^{-1}G\gamma$ is of infinite index in G, e.g., if G is a free group, and γ is not a generator.

5. Case of abelian groups

A positive result is given by the following theorem.

Theorem 5.1. *Let $X = \mathbb{R}^n$, with a norm $\|\cdot\|$, and let the metric d_X be determined by the norm $\|\cdot\|$. Then $M(X)$ is weakly stable.*

Proof. We have to show that if $f : X \to X$ is a C-almost isometry for some $C > 0$ then $f(X)$ is equivalent to X. Suppose the contrary: for any $n \in \mathbb{N}$ there exists $x_n \in X$ such that $d_X(x_n, f(X)) > n$.

First, note that we can replace f by another almost isometry g, which is continuous and close to f. Namely, choose a triangulation of X by simplices with length of edges greater than C and with a uniform lower bound for their volumes. Then set $g(v) = f(v)$ for all vertices and extend this map to the inner points of the simplices by linearity. Then $g : X \to X$ is continuous and there exists $C' > 0$ depending on the dimension of X and on the norm $\|\cdot\|$, such that $d_X(f(x), g(x)) < C'$ for any $x \in X$. As f was a C-almost isometry, g is a D-almost isometry, where $D = 2C' + C$.

Let x_0 denote the origin of X. Without loss of generality, we may assume that $f(x_0) = x_0$ (we may compose f with a translation).

Denote by S_R the sphere of radius R centered at x_0. Then $g(x)$ lies between S_{R-D} and S_{R+D} for any $x \in S_R$. Let $d_X(x_0, x_n) = R_n$. Clearly, $\lim_{n\to\infty} R_n = \infty$. Passing to a subsequence, we may assume that $\lim_{n\to\infty} R_{n+1} - R_n = \infty$. Then, once again, we can replace g by a continuous D'-almost isometry $h : X \to X$ with $\sup_{x\in X} d_X(f(x), h(x)) < D'$ for some $D' > 0$ such that $h(S_{R_n}) \subset S_{R_n}$.

As $d_X(x_n, f(X)) > n$, $d_X(x_n, h(X)) > n - D'$, hence $x_n \notin h(S_{R_n})$ when $n > D'$. Thus, the map $h|_{S_{R_n}} : S_{R_n} \to S_{R_n}$ is not surjective. Then, by the Borsuk–Ulam theorem, there exists a pair of antipodal points $y_1, y_2 \in S_{R_n}$ such that $h(y_1) = h(y_2) = z$. But this contradicts the almost isometricity of h:

$$|d_X(h(y_1), h(y_2)) - d_X(y_1, y_2)| = |d_X(z, z) - d_X(y_1, y_2)| = |0 - 2R_n| = 2R_n$$

is not bounded. □

Corollary 5.2. *Let $X = \mathbb{Z}^n$ with an l_p-metric, $1 \le p \le \infty$. Then $M(X)$ is weakly finite.*

Proof. By Proposition 9.2 of [Manuilov 2021a], $M(\mathbb{Z}^n) = M(\mathbb{R}^n)$. □

Corollary 5.3. *$M(X)$ is weakly finite for any finitely generated free abelian group X with a word length metric with respect to any finite set of generators.*

6. $M(X)$ doesn't respect equivalences

Proposition 6.1. *The inverse semigroup $M(X)$ is not a coarse invariant.*

Proof. The space X from Proposition 3.2 is coarsely equivalent to the space $Y = \{n^2 : n \in \mathbb{N}\}$ with the standard metric, which we denote by b_X. Indeed, for $n < m$, we have $b_X(x_n, x_m) = m^2 - n^2$ and $d_X(x_n, x_m) = \log(m + 1)$. As $m^2 - (m - 1)^2 = 2m - 1 > \log(m + 1)$ for $m > 1$, we have $d_X(x, y) \le b_X(x, y)$ for any $x, y \in X$, and taking $f(t) = 2e^t$, we have $b_X(x, y) \le f(d_X(x, y))$ for any $x, y \in X$.

For the metric d_X from Proposition 3.2, the inverse semigroup $M(X, d_X)$ is not commutative ($[d^*d] \neq [dd^*]$), while the inverse semigroup $M(X, b_X)$ is commutative by Proposition 7.1 of [Manuilov 2021b]. □

Theorem 6.2. *The inverse semigroup $M(X)$ is not a quasiisometry invariant.*

Proof. Let $X = \mathbb{N}$ be endowed with the metric b_X given by $b_X(n,m) = |2^n - 2^m|$, $n, m \in \mathbb{N}$, and let $y_n = s(n)4^{[n/2]}$, where $s(n) = (-1)^{[(n-1)/2]}$ and $[t]$ is the greatest integer not exceeding t. Let d_X be the metric on X given by $d_X(n,m) = |y_n - y_m|$, $n, m \in \mathbb{N}$. The two metrics are quasiisometric. Indeed, suppose that $n > m$. If $s(n) = -s(m)$ then

$$d_X(n,m) = 4^{[n/2]} + 4^{[m/2]} \leq 4^{n/2+1} + 4^{m/2+1} = 4(2^n + 2^m) \leq 12 b_X(n,m);$$
$$d_X(n,m) = 4^{[n/2]} + 4^{[m/2]} \geq 4^{n/2} + 4^{m/2} \geq 2^n - 2^m = b_X(n,m).$$

We use here that $(2^r + 1)/(2^r - 1) \leq 3$ for any $r = n - m \in \mathbb{N}$. If $s(n) = s(m)$ then

$$d_X(n,m) = 4^{[n/2]} - 4^{[m/2]} \leq 4^{n/2+1} - 4^{m/2} = 4 \cdot 2^n - 2^m \leq 7 b_X(n,m).$$

We use here that $(4 \cdot 2^r - 1)/(2^r - 1) \leq 7$ for any $r = n - m \in \mathbb{N}$. To obtain an estimate in other direction, note that $s(n) = s(m)$ implies that $[n/2] \geq [m/2] + 1$, and that $n - m \neq 2$. If $n = m + 1$ then

$$d_X(m+1,m) = 3 \cdot 4^{[m/2]} \geq \tfrac{3}{2} \cdot 2^m = \tfrac{3}{2} b_X(m+1,m),$$

If $n \geq m + 3$ then

$$d_X(n,m) = 4^{[n/2]} - 4^{[m/2]} \geq 4^{n/2} - 4^{m/2+1} = 2^n - 4 \cdot 2^m \geq \tfrac{4}{7} b_X(n,m).$$

We use here that $(2^r - 4)/(2^r - 1) \geq \frac{4}{7}$ for any $r = n - m \geq 3$. Thus,

$$\tfrac{3}{7} b_X(n,m) \leq d_X(n,m) \leq 12 \cdot b_X(n,m)$$

for any $n, m \in \mathbb{N}$, so the two metrics are quasiisometric.

We already know that $M(X, b_X)$ is commutative, so it remains to expose two noncommuting elements in $M(X, d_X)$.

Let

$$X = \{(y_n, 0) : n \in \mathbb{N}\}, \quad X' = \{(-y_n, 1) : n \in \mathbb{N}\},$$

and let d be the metric on $X \sqcup X'$ induced from the standard metric on the plane $\mathbb{R}^2$, $s = [d]$. Note that $-y_n = y_{n-1}$ if $y_n > 0$ and $n > 1$, and $-y_n = y_{n+1}$ if $y_n < 0$. Hence, $d^* = d$ and $s^2 = \mathbf{1}$.

Let

$$A_+ = \{y_n : n \in \mathbb{N};\ y_n > 0\}, \quad A_- = \{y_n : n \in \mathbb{N};\ y_n < 0\},$$

$X = A_+ \sqcup A_-$, and let the metrics d_+ and d_- on $X \sqcup X'$ be given by

$$d_\pm(n, m') = \inf_{k \in A_\pm} [d_X(n, k) + 1 + d_X(k, m)],$$

$e = [d_+]$, $f = [d_-]$. Then $es = \mathbf{0}$, while $se = f$, i.e., e and s do not commute. □

Note that, unlike $M(X)$, the set $E(M(X))$ of idempotents of $M(X)$ is a coarse invariant. This follows from the geometric description of idempotents in [Manuilov 2021b].

Part II. When $M(X)$ is commutative

7. R-spaces

Definition 7.1. A metric space X is an R-*space* (R for rigid) if, for any $C > 0$ and any two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ of points in X satisfying

$$|d_X(x_n, x_m) - d_X(y_n, y_m)| < C \quad \text{for any } n, m \in \mathbb{N} \tag{7-1}$$

there exists $D > 0$ such that $d_X(x_n, y_n) < D$ for any $n \in \mathbb{N}$.

Example 7.2. As $M(X)$ is commutative for any $X \subset [0, \infty)$, it would follow from Theorem 8.2 below that such X is an R-space. A less trivial example is a planar spiral X given by $r = e^\varphi$ in polar coordinates with the metric induced from the standard metric on the plane. Indeed, take any two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, in X. Without loss of generality we may assume that $x_1 = y_1 = 0$ is the origin. If these sequences satisfy (7-1) then

$$|d_X(0, x_n) - d_X(0, y_n)| < C$$

for some fixed $C > 0$ (we take $m = 1$). If $x_n = (r_n, \varphi_n)$, $y_n = (s_n, \psi_n)$ then $d_X(0, x_n) = r_n$, $d_X(0, y_n) = s_n$, and we have $|r_n - s_n| < C$. Then x_n and y_n lie in a ring of width C, say $R \le r \le R + C$. If R is sufficiently great then

$$d_X(x_n, y_n) \le (\log(R + C) - \log R)(R + C),$$

which is bounded as a function of R.

Consider the set $AI(X)$ of all equivalence classes of almost isometries of X. It is easy to see that it is a group with respect to the composition. A metric space X is called AI-rigid [Kar et al. 2016] if the group $AI(X)$ is trivial.

Proposition 7.3. *A countable R-space X is AI-rigid.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of all points of X, and let $f : X \to X$ be an almost isometry. Set $y_n = f(x_n)$. Then there exists $C > 0$ such that

$$|d_X(f(x_n), f(x_m)) - d_X(x_n, x_m)| < C$$

for any $n, m \in \mathbb{N}$, hence there exists $D > 0$ such that

$$d_X(x_n, f(x_n)) = d_X(x_n, y_n) < D$$

for any $n \in \mathbb{N}$, i.e., f is equivalent to the identity map, hence X is AI-rigid. □

Example 7.4. Euclidean spaces $\mathbb{R}^n$, $n \geq 1$, are not R-spaces, as they have a nontrivial symmetry. The Archimedean spiral $r = \varphi$ is not an R-space, as it is π-dense in $\mathbb{R}^2$.

8. Criterion of commutativity

Let $a, b : T \to [0, \infty)$ be two functions on a set T. We say that $a \preceq b$ if there exists a monotone increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\lim_{s\to\infty} \varphi(s) = \infty$ (we call such functions *reparametrizations*) such that $a(t) \leq \varphi(b(t))$ for any $t \in T$.

The following lemma should be known, but we could not find a reference.

Lemma 8.1. *Let $a, b : T \to [0, \infty)$ be two functions. If $a \preceq b$ is not true then there exists $C > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ of points in T such that $b(t_n) < C$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a(t_n) = \infty$.*

Proof. If $a \preceq b$ is not true then for any reparametrization φ there exists $t \in T$ such that $a(t) > \varphi(b(t))$. Suppose that for any $C > 0$, the value $\max\{a(t) : b(t) \leq C\}$ is finite. Then set

$$f(C) = \max(\max\{a(t) : b(t) \leq C\}, C).$$

This gives a reparametrization f. If $b(t) = C$ then $a(t) \leq f(C) = f(b(t))$ — a contradiction. Thus, there exists $C > 0$ such that $\max\{a(t) : b(t) \leq C\} = \infty$. It remains to choose a sequence $(t_n)_{n\in\mathbb{N}}$ in the set $\{t \in T : b(t) \leq C\}$ such that $a(t_n) > n$. □

Theorem 8.2. *X is an R-space if and only if $M(X)$ is commutative.*

Proof. Let X be an R-space. We shall show that any $s \in M(X)$ is a projection. It would follow that $M(X)$ is commutative. First, we shall show that any $s \in M(X)$ is selfadjoint. Let $d \in \mathcal{M}(X)$, $[d] = s$. Suppose that $[d^*] \neq [d]$. This means that either $d^* \preceq d$ or $d \preceq d^*$ is not true, where d and d^* are considered as functions on $T = X \times X'$. Without loss of generality we may assume that $d^* \preceq d$ is not true. Then there exist sequences $(x_n)_{n\in\mathbb{N}}$ in X and $(y')_{n\in\mathbb{N}}$ in X' and $L > 0$ such that $d(x_n, y'_n) < L$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(y_n, x'_n) = \infty$ (recall that $d^*(x, y') := d(y, x')$).

Take $n, m \in \mathbb{N}$. Since $d(x_n, y'_n) < L$, $d(x_m, y'_m) < L$, we have

$$|d_X(x_n, x_m) - d_X(y_n, y_m)| = |d_X(x_n, x_m) - d_X(y'_n, y'_m)| < 2L,$$

and, since X is an R-space, there exists $D > 0$ such that $d_X(x_n, y_n) < D$ for any $n \in \mathbb{N}$.

Then, using the triangle inequality for the quadrangle $x_n y_n x'_n y'_n$, we get

$$\begin{aligned} d(y_n, x'_n) &\le d_X(y_n, x_n) + d(x_n, y'_n) + d_X(y'_n, x'_n) \\ &= d_X(y_n, x_n) + d(x_n, y'_n) + d_X(y_n, x_n) < D + L + D, \end{aligned}$$

which contradicts the condition $\lim_{n\to\infty} d(y_n, x'_n) = \infty$.

Now, let us show that $[d] \in M(X)$ is idempotent if X is an R-space. Let $a(x) = d(x, X')$, $b(x) = d(x, x')$. It was shown in [Manuilov 2021a, Theorem 3.1 and remark at the end of Section 11] that if $[d]$ is selfadjoint then it is idempotent if and only if $b \preceq a$. Suppose that the latter is not true. Then there exists $L > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in X such that $d(x_n, X') < L$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(x_n, x'_n) = \infty$. In particular, this means that there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ of points in X such that $d(x_n, y'_n) < L$ for any $n \in \mathbb{N}$. Since $[d]$ is selfadjoint, for any $L > 0$ there exists $R > 0$ such that if $d(x, y') < L$ then $d(x', y) < R$.

It follows from the triangle inequality for the quadrangle $x_n x_m y'_n y'_m$ that

$$|d_X(x_n, x_m) - d_X(y_n, y_m)| = |d_X(x_n, x_m) - d_X(y'_n, y'_m)| \le d(x_n, y'_n) + d(x_m, y'_m) < 2L$$

for any $n, m \in \mathbb{N}$, hence, the property of being an R-space implies that there exists $D > 0$ such that $d_X(x_n, y_n) < D$ for any $n \in \mathbb{N}$. Therefore,

$$d(x_n, x'_n) \le d_X(x_n, y_n) + d(y_n, x'_n) < D + R$$

for any $n \in \mathbb{N}$ — a contradiction with $\lim_{n\to\infty} d(x_n, x'_n) = \infty$.

In the opposite direction, suppose that X is not an R-space. i.e., that there exists $C > 0$ and sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ of points in X such that (7-1) holds and $\lim_{n\to\infty} d_X(x_n, y_n) = \infty$.

Note that these sequences cannot be bounded. Indeed, if there exists $R > 0$ such that $d_X(x_1, x_n) < R$ for any $n \in \mathbb{N}$ then

$$d_X(y_1, y_n) \le d_X(x_1, x_n) + C = R + C$$

for any $n \in \mathbb{N}$, but then

$$d_X(x_n, y_n) \le d_X(x_n, x_1) + d_X(x_1, y_1) + d_X(y_1, y_n) < R + d_X(x_1, y_1) + R + C,$$

which contradicts $\lim_{n\to\infty} d_X(x_n, y_n) = \infty$. Passing to a subsequence, we may assume that

$$d_X(x_k, x_n) > k, \quad d_X(x_k, y_n) > k, \quad d_X(y_k, x_n) > k, \quad d_X(y_k, y_n) > k$$

for any $n < k$, and $d_X(x_k, y_k) > k$ for any $k \in \mathbb{N}$. In particular, this means that

$$d_X(x_k, y_n) > k \quad \text{for any } k, n \in \mathbb{N}. \tag{8-1}$$

Let us define two metrics on the double of X and show that they don't commute. For $x, y \in X$ set

$$d_1(x, y') = \min_{n\in\mathbb{N}} [d_X(x, x_n) + C + d_X(y_n, y)];$$

$$d_2(x, y') = \min_{n\in\mathbb{N}} [d_X(x, y_n) + C + d_X(x_n, y)]$$

(it is clear that the minimum is attained on some $n \in \mathbb{N}$ as $x_n, y_n \to \infty$). Let us show that d_1 is a metric on $X \sqcup X'$ (the case of d_2 is similar).

Due to symmetry, it suffices to check the two triangle inequalities for the triangle xzy', $z \in X$:

$$\begin{aligned}
&d_1(x, y') + d_1(z, y') \\
&\qquad = \min_{n\in\mathbb{N}}[d_X(x, x_n) + C + d_X(y_n, y)] + \min_{m\in\mathbb{N}}[d_X(z, x_m) + C + d_X(y_m, y)] \\
&\qquad = d_X(x, x_{n_x}) + d_X(y_{n_x}, y) + d_X(y, y_{n_z}) + d_X(z, x_{n_z}) + 2C \\
&\qquad \geq d_X(x, x_{n_x}) + d_X(y_{n_x}, y_{n_z}) + d_X(z, x_{n_z}) + 2C \\
&\qquad \geq d_X(x, x_{n_x}) + (d_X(x_{n_x}, x_{n_z}) - C) + d_X(z, x_{n_z}) + 2C \\
&\qquad = d_X(x, x_{n_x}) + d_X(x_{n_x}, x_{n_z} + d_X(z, x_{n_z}) + C \\
&\qquad \geq d_X(x, z) + C \geq d_X(x, z).
\end{aligned}$$

and

$$\begin{aligned}
d_1(x, y') &= \min_{n\in\mathbb{N}}[d_X(x, x_n) + C + d_X(y_n, y)] \\
&\leq d_X(x, x_{n_z}) + d_X(y_{n_z}, y) + C \\
&\leq d_X(x, z) + d_X(z, x_{n_z}) + d_X(y_{n_z}, y) + C = d_X(x, z) + d_1(z, y').
\end{aligned}$$

Let us evaluate $(d_2 \circ d_1)(x_k, x'_k)$ and $(d_1 \circ d_2)(x_k, x'_k)$.

Taking fixed values $n = m = k$, $u = y_k$, we get

$$\begin{aligned}
&(d_2 \circ d_1)(x_k, x'_k) \\
&\quad = \inf_{u\in X}\{\min_{n\in\mathbb{N}}[d_X(x_k, x_n) + C + d_X(y_n, u)] + \min_{m\in\mathbb{N}}[d_X(u, y_m) + C + d_X(x_m, x_k)]\} \\
&\quad \leq \inf_{u\in X}\{[d_X(x_k, x_k) + C + d_X(y_k, u)] + [d_X(u, y_k) + C + d_X(x_k, x_k)]\} \\
&\quad = [d_X(x_k, x_k) + C] + [C + d_X(x_k, x_k)] \\
&\quad = 2C.
\end{aligned}$$

Using the triangle inequality for the triangle $x_n x_m u$ and (8-1), we get

$$\begin{aligned}
&(d_1 \circ d_2)(x_k, x'_k) \\
&\quad = \inf_{u\in X}\{\min_{n\in\mathbb{N}}[d_X(x_k, y_n) + C + d_X(x_n, u)] + \min_{m\in\mathbb{N}}[d_X(u, x_m) + C + d_X(y_m, x_k)]\} \\
&\quad \geq \inf_{u\in X}\{\min_{n\in\mathbb{N}}[d_X(x_k, y_n) + d_X(x_n, u)] + \min_{m\in\mathbb{N}}[d_X(u, x_m) + d_X(y_m, x_k)]\} \\
&\quad \geq \min_{n,m\in\mathbb{N}}[d_X(x_k, y_n) + d_X(x_n, x_m) + d_X(y_m, x_k)] > k + d_X(x_n, x_m) + k > 2k.
\end{aligned}$$

Thus, for the sequence $\{x_k\}_{k\in\mathbb{N}}$ of points in X, the distances $(d_2 \circ d_1)(x_k, x'_k)$ are uniformly bounded, while $\lim_{k\to\infty}(d_1 \circ d_2)(x_k, x'_k) = \infty$, hence the metrics $d_2 \circ d_1$ and $d_1 \circ d_2$ are not equivalent, i.e., $[d_2][d_1] \neq [d_1][d_2]$. □

Recall that an inverse semigroup S is Clifford (see [Howie 1995], Theorem 4.2.1) if $s^*s = ss^*$ for any $s \in S$. If S is commutative then it is patently Clifford, but not the other way. Nevertheless, for inverse semigroups of the form $M(X)$ these two properties are the same.

Corollary 8.3. *If $M(X)$ is Clifford then X is an R-space* (*and $M(X)$ is commutative*).

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in X satisfying (7-1), and let d_1, d_2 are metrics on the double of X defined above. Note that $d_1^* = d_2$, and let $s = [d_1]$. We have $s^*s \neq ss^*$, which contradicts that $M(X)$ is Clifford. □

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Received January 15, 2024.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 328 No. 2 February 2024