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DEFORMATION OF PAIRS AND SEMIREGULARITY

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We study relative deformation of a map into a Kähler manifold whose image is a divisor. We show that if the map satisfies a condition called semiregularity, then it allows relative deformations if and only if the cycle class of the image remains Hodge in the family. This gives a refinement of the so-called variational Hodge conjecture. We also show that the semiregularity of maps is related to classical notions such as Cayley–Bacharach conditions and d-semistability.

1. Introduction

Let $\pi : \mathfrak{X} \to D$ be a deformation of a compact Kähler manifold X_0 of dimension $n \geq 2$ over a disk D in the complex plane. Let C_0 be a compact reduced curve (when $n = 2$) or a compact smooth complex manifold of dimension $n - 1$ (when $n > 2$). Let $\varphi_0 : C_0 \to X_0$ be a map which is an immersion, that is, for any $p \in C_0$, there is an open neighborhood $p \in V_p \subset C_0$ such that $\varphi_0|_{V_p}$ is an embedding. Then, the image of φ_0 determines an integral cohomology class $[\varphi_0(C_0)]$ of type $(1, 1)$, that is, a Hodge class which is the Poincaré dual of the cycle $\varphi_0(C_0)$. Note that the class $[\varphi_0(C_0)]$ naturally determines an integral cohomology class of each fiber of π. Therefore, it makes sense to ask whether this class remains Hodge in these fibers or not. Clearly, the condition that the class $[\varphi_0(C_0)]$ remains Hodge is necessary for the existence of deformations of the map φ_0 to other fibers.

The notion of *semiregularity* plays a role in this context, though it was introduced [21] and developed [16] originally for submanifolds of codimension one in a *fixed* complex manifold. The main result of these studies is that if a submanifold of codimension one is semiregular, then the obstruction to deforming it in the ambient manifold vanishes. Bloch [5] generalized the notion of semiregularity to subvarieties of any codimension in a projective manifold which are local complete intersection. He generalized the results of [16; 21] to this case, and also related the notion of semiregularity to deformation of pairs. Namely, he proved that if C_0 is a subvariety of a projective manifold X_0 which is local complete intersection and semiregular, and if the class $[C_0]$ remains Hodge in an algebraic family $\mathfrak{X} \to \mathbb{C}$

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whose central fiber is X_0, then there is a deformation of C_0 relative to the base. In other words, a local complete intersection subvariety which is semiregular satisfies the *variational Hodge conjecture*. More precisely, the variational Hodge conjecture asks the existence of a family of cycles of the class $[C_0]$ which need not restrict to C_0 on the central fiber. Therefore, Bloch's theorem in fact shows that the semiregularity gives a result stronger than the variational Hodge conjecture. We also note that Ran [20] generalized Bloch's result to cases where a weaker version of semiregularity holds. However, although Bloch's and Ran's theorems guarantee the existence of a relative deformation of a cycle on the central fiber X_0, it gives little control of the geometry of the deformed cycle.

More recently, the notion of semiregularity has been generalized to maps between varieties [6; 11]. In [11], maps between compact Kähler manifolds were investigated, and it was shown that if the map is semiregular, then it deforms in a fixed target manifold. In [6], the notion of semiregularity was generalized to a very broad context using cotangent complexes, and many known results were generalized. The case of maps was also considered (see [6, Theorem 7.23]), but not in the context of the variational Hodge conjecture as we will do. See also [2; 3; 12; 17; 19] for recent developments related to semiregularity.

Our purpose is to show that the semiregularity in fact suffices to control the geometry of the deformed cycles when the cycle is of codimension one, and also that we can extend the result to maps to a family of Kähler manifolds. Recall that $\varphi_0 : C_0 \to X_0$ is an immersion where $\dim C_0 = \dim X_0 - 1$.

Theorem 1. *Assume that the map φ_0 is semiregular in the sense of Definition 4. If the class $[\varphi_0(C_0)]$ remains Hodge, then the map φ_0 deforms to other fibers.*

For example, if the image $\varphi_0(C_0)$ has normal crossing singularity, then there is a natural map $\tilde{\varphi}_0 : \tilde{C}_0 \to X_0$, where $\tilde{C}_0$ is the normalization of C_0 (when $n > 2$, $C_0 = \tilde{C}_0$). Then, if $\tilde{\varphi}_0$ is semiregular, Theorem 1 implies that it deforms to a general fiber and the singularity of the image remains the same (e.g., it gives a relative equigeneric deformation when $n = 2$).

On the other hand, if the image $\varphi_0(C_0)$ has normal crossing singularity, the semiregularity turns out to be related to some classical notions appeared in different contexts. Namely, we will prove the following (see Corollary 17).

Theorem 2. *Assume that the subvariety $\varphi_0(C_0)$ is semiregular in the classical sense. That is, the inclusion of $\varphi_0(C_0)$ into X_0 is semiregular in the sense of Definition 4. Then, if the map $H^0(\varphi_0(C_0), \mathcal{N}_\iota) \to H^0(\varphi_0(C_0), \mathcal{S})$ is surjective, the map φ_0 is semiregular. In particular, if the class $[\varphi_0(C_0)]$ remains Hodge on the fibers of $\mathfrak{X}$, the map φ_0 can be deformed to general fibers of $\mathfrak{X}$.*

Here, $\mathcal{N}_\iota$ is the normal sheaf of $\varphi_0(C_0)$ in X_0 and $\mathcal{S}$ is the *infinitesimal normal sheaf* of the variety $\varphi_0(C_0)$, see Section 6 for the definition. A variety with normal crossing singularity is called *d-semistable* if the infinitesimal normal sheaf is trivial, see [8]. The notion of d-semistability is known to be related to the existence of log-smooth deformations (see [13; 14]). By the above theorem, it turns out that it is also related to deformations of pairs, see Corollary 19.

In the case where $n = 2$, if X_0 is a K3 surface, any immersion φ_0 is semiregular, and Theorem 1 can be applied to any such φ_0. This result is well known and was proved, for example, using the twistor family associated with the hyperkähler structure of K3 surfaces. Theorem 1 gives a generalization of it to general surfaces. In general, we need to check whether a given map φ_0 is semiregular or not. For that purpose, Theorem 31 in [18] combined with Theorem 1 above implies the following. Let $\varphi_0 : C_0 \to X_0$ be an immersion such that the image $\varphi_0(C_0)$ is a reduced nodal curve. Let $p : C_0 \to \varphi_0(C_0)$ be the natural map (which is a partial normalization of $\varphi_0(C_0)$) and $P = \{p_i\}$ be the set of nodes of $\varphi_0(C_0)$ whose inverse image by p consists of two points.

Theorem 3. *Assume that $\varphi_0(C_0)$ is semiregular in the classical sense and the class $[\varphi_0(C_0)]$ remains Hodge on the fibers of $\mathfrak{X}$. Then, the map φ_0 deforms to general fibers of $\mathfrak{X}$ if for each $p_i \in P$, there is a first-order deformation of $\varphi_0(C_0)$ which smooths p_i, but does not smooth the other nodes of P.* □

The condition in Theorem 20 is related (in a sense opposite) to the classical *Cayley–Bacharach condition*, see [4], which requires that if a first-order deformation does not smooth the nodes $P \setminus \{p_i\}$, then it does not smooth p_i, either. Using this, we can also deduce a geometric criterion for the existence of deformations of pairs, see Corollary 21.

Notation. We will work in the complex analytic category. Later in the paper, we will study nonconstant maps $\varphi_0 : C_0 \to X_0$ from a variety C_0 to a Kähler manifold X_0 and their deformations. We denote by $\mathfrak{X}$ a family of compact Kähler manifolds over a disk $D \subset \mathbb{C}$ whose central fiber is X_0. A deformation of φ_0 over $\operatorname{Spec} \mathbb{C}[t]/t^{k+1}$ will be written as $\varphi_k : C_k \to X_k = \mathfrak{X} \times_D \operatorname{Spec} \mathbb{C}[t]/t^{k+1}$. By the image of a map φ_0 or φ_k, we mean the analytic locally ringed space with the annihilator structure, see [9, Chapter I, Definition 1.45]. That is, if U is an open subset of C_k with the induced structure of an analytic locally ringed space, and V is an open subset of X_k such that $\varphi_k(U)$ is closed in V, we associate the structure sheaf

$$\mathcal{O}_V / \mathcal{A}nn_{\mathcal{O}_V}((\varphi_k)_* \mathcal{O}_U)$$

to the image $\varphi_k(U)$.

2. Semiregularity for local embeddings

Let n and p be positive integers with $p < n$. Let M be a complex variety (not necessarily smooth or reduced) of dimension $n - p$ and X a compact Kähler manifold of dimension n. Let $\varphi : M \to X$ be a map which is an immersion, that is, for any $p \in M$, there is an open neighborhood $p \in U_p \subset M$ such that $\varphi|_{U_p}$ is an embedding. We assume that the image is a local complete intersection. Then, the normal sheaf $\mathcal{N}_\varphi$ is locally free of rank p. Define the locally free sheaves $\mathcal{K}_\varphi$ and ω_M on M by

$$\mathcal{K}_\varphi = \wedge^p \mathcal{N}_\varphi^\vee \quad \text{and} \quad \omega_M = \mathcal{K}_\varphi^\vee \otimes \varphi^* \mathcal{K}_X,$$

where $\mathcal{K}_X$ is the canonical sheaf of X.

When φ is an inclusion, the natural inclusion

$$\varepsilon : \mathcal{N}_\varphi^\vee \to \varphi^* \Omega_X^1$$

gives rise to an element

$$\wedge^{p-1}\varepsilon \in \mathrm{Hom}_{\mathcal{O}_M}(\wedge^{p-1}\mathcal{N}_\varphi^\vee, \varphi^*\Omega_X^{p-1}) = \Gamma\big(M, (\varphi^*\Omega_X^{n-p+1})^\vee \otimes \varphi^*\mathcal{K}_X \otimes \mathcal{K}_\varphi^\vee \otimes \mathcal{N}_\varphi^\vee\big)$$
$$= \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X^{n-p+1}, \omega_M \otimes \mathcal{N}_\varphi^\vee).$$

This induces a map on cohomology:

$$\wedge^{p-1}\varepsilon : H^{n-p-1}(X, \Omega_X^{n-p+1}) \to H^{n-p-1}(M, \omega_M \otimes \mathcal{N}_\varphi^\vee).$$

When φ is not an inclusion, then $\Gamma\big(M, (\varphi^*\Omega_X^{n-p+1})^\vee \otimes \varphi^*\mathcal{K}_X \otimes \mathcal{K}_\varphi^\vee \otimes \mathcal{N}_\varphi^\vee\big)$ is not necessarily isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\Omega_X^{n-p+1}, \omega_M \otimes \mathcal{N}_\varphi^\vee)$, but the map

$$\wedge^{p-1}\varepsilon : H^{n-p-1}(X, \Omega_X^{n-p+1}) \to H^{n-p-1}(M, \omega_M \otimes \mathcal{N}_\varphi^\vee)$$

is still defined.

Definition 4. We call φ *semiregular* if the natural map $\wedge^{p-1}\varepsilon$ is surjective.

In this paper, we are interested in the case where $p = 1$ and M is reduced when $n = 2$, and M is smooth when $n > 2$. In this case, we have $\omega_M \otimes \mathcal{N}_\varphi^\vee \cong \varphi^*\mathcal{K}_X$ and the map $\wedge^{p-1}\varepsilon$ will be

$$H^{n-2}(X, \mathcal{K}_X) \to H^{n-2}(M, \varphi^*\mathcal{K}_X).$$

Remark 5. As we mentioned in the introduction, in [6; 11], Buchweitz–Flenner and Iacono also considered semiregularity of maps between varieties in broader contexts. In the case of maps we consider in this paper, their definitions coincide with ours.

3. Local calculation

Let $\pi : \mathfrak{X} \to D$ be a deformation of a compact Kähler manifold X_0 of dimension $n \geq 2$. Here, D is a disk on the complex plane centered at the origin. Let

$$\{(U_i, (x_{i,1}, \dots, x_{i,n})\}$$

be a coordinate system of X_0. Taking D small enough, the sets

$$\{\mathfrak{U}_i = U_i \times D, (x_{i,1}, \ldots, x_{i,n}, t)\}$$

gives a coordinate system of $\mathfrak{X}$. Precisely, we fix an isomorphism between $\mathfrak{U}_i$ and a suitable open subset of $\mathfrak{X}$ which is compatible with π and the inclusion $U_i \to X_0$. Here, t is a coordinate on D pulled back to $\mathfrak{U}_i$. The functions $x_{i,l}$ are also pulled back to $\mathfrak{U}_i$ from U_i by the natural projection.

Take coordinate neighborhoods $\mathfrak{U}_i, \mathfrak{U}_j$ and $\mathfrak{U}_k$. On the intersections of these open subsets, the coordinate functions on one of them can be written in terms of those on another. Namely, on $\mathfrak{U}_i \cap \mathfrak{U}_j$, $x_{i,l}$ can be written as $x_{i,l}(\boldsymbol{x}_j, t)$, here we write

$$\boldsymbol{x}_j = (x_{j,1}, \ldots, x_{j,n}).$$

Similarly, on $\mathfrak{U}_j \cap \mathfrak{U}_k$, we have $x_{j,l} = x_{j,l}(\boldsymbol{x}_k, t)$. Then, on $\mathfrak{U}_i \cap \mathfrak{U}_j \cap \mathfrak{U}_k$, we have

$$x_{i,l} = x_{i,l}(\boldsymbol{x}_k, t) = x_{i,l}(\boldsymbol{x}_j(\boldsymbol{x}_k, t), t).$$

For simplicity we often write $x_{i,l}(\boldsymbol{x}_k, t)$ as $x_{i,l}(\boldsymbol{x}_k)$ and $x_{i,l}(\boldsymbol{x}_j(\boldsymbol{x}_k, t), t)$ as $x_{i,l}(\boldsymbol{x}_j(\boldsymbol{x}_k))$.

Let $X_t = \pi^{-1}(t)$ be the fiber of the family π over $t \in D$. Assume that the map

$$\varphi_0 : C_0 \to X_0$$

exists from a variety C_0 of dimension $n-1$ to X_0, which is an immersion.

We can take an open covering $\{V_i\}$ of C_0 such that the restriction of φ_0 to V_i is an embedding, the image $\varphi_0(V_i)$ is contained in U_i and is defined by an equation $f_{i,0} = 0$ for some holomorphic function $f_{i,0}$. Moreover, we assume that if $V_i \cap V_j$ is nonempty, we have

$$\varphi_0(V_i \cup V_j) \cap (U_i \cap U_j) = \varphi_0(V_i \cap V_j).$$

Let $\operatorname{Spec} \mathbb{C}[t]/t^{m+1}$ be the m-th order infinitesimal neighborhood of the origin of D. Note that

$$\{U_{i,m} = \mathfrak{U}_i \times_D \operatorname{Spec} \mathbb{C}[t]/t^{m+1}\}$$

gives a covering by coordinate neighborhoods of $X_m = \mathfrak{X} \times_D \operatorname{Spec} \mathbb{C}[t]/t^{m+1}$. We write by $x_{i,l,m}$ the restriction of $x_{i,l}$ to $U_{i,m}$. Let us write

$$\boldsymbol{x}_{i,m} = \{x_{i,1,m}, \ldots, x_{i,n,m}\}.$$

Assume that we have constructed an m-th order deformation $\varphi_m : C_m \to X_m$ of φ_0. Here, m is a nonnegative integer and C_m is an m-th order deformation of C_0. Let $V_{i,m}$ be the locally ringed space obtained by restricting the structure of a locally ringed space on C_m to V_i.

Let $\{f_{i,m}(\boldsymbol{x}_{i,m}, t)\}$ be the set of local defining functions of $\varphi_m(V_{i,m})$ in $U_{i,m}$. We will often write $f_{i,m}(\boldsymbol{x}_{i,m}, t)$ as $f_{i,m}(\boldsymbol{x}_{i,m})$ for notational simplicity. In particular, on the intersection $U_{i,m} \cap U_{j,m}$, there is an invertible function $g_{ij,m}$ which satisfies

$$f_{i,m}(\boldsymbol{x}_{i,m}(\boldsymbol{x}_{j,m}, t), t) = g_{ij,m}(\boldsymbol{x}_{j,m}, t)\, f_{j,m}(\boldsymbol{x}_{j,m}, t) \mod t^{m+1}.$$

Define a holomorphic function $\nu_{ij,m}$ on $U_{i,m} \cap U_{j,m}$ by

$$\begin{aligned} t^{m+1}\,\nu_{ij,m}(\boldsymbol{x}_{j,m+1}) &= t^{m+1}\,\nu_{ij,m}(\boldsymbol{x}_{j,0}) \\ &= f_{i,m}(\boldsymbol{x}_{i,m+1}(\boldsymbol{x}_{j,m+1})) - g_{ij,m}(\boldsymbol{x}_{j,m+1})\, f_{j,m}(\boldsymbol{x}_{j,m+1}), \end{aligned}$$

which is an equality over $\mathbb{C}[t]/t^{m+2}$. Note that $\nu_{ij,m}$ can be regarded as a function on $U_i \cap U_j$.

Proposition 6. *Assume that the intersection $U_i \cap U_j \cap U_k$ is nonempty. Then, on $U_i \cap U_j \cap U_k \cap \varphi_0(V_i)$, the following identities hold*:

$$\nu_{ik,m}(\boldsymbol{x}_{k,m+1}) = \nu_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1})) + g_{ij,0}(\boldsymbol{x}_{j,0}(\boldsymbol{x}_{k,0}))\,\nu_{jk,m}(\boldsymbol{x}_{k,m+1})$$

and

$$\nu_{ij,m} = -g_{ij,0}\,\nu_{ji,m}.$$

Remark 7. The equality

$$U_i \cap U_j \cap U_k \cap \varphi_0(V_i) = U_i \cap U_j \cap U_k \cap \varphi_0(V_j) = U_i \cap U_j \cap U_k \cap \varphi_0(V_k)$$

holds by the way we took $\{V_i\}$.

Proof. We have

$$\boldsymbol{x}_{i,m+1}(\boldsymbol{x}_{k,m+1}) \equiv \boldsymbol{x}_{i,m+1}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1})) \mod t^{m+2},$$

on $U_{i,m+1} \cap U_{j,m+1} \cap U_{k,m+1}$. Then,

$$\begin{aligned} &t^{m+1}\,\nu_{ik,m}(\boldsymbol{x}_{k,m+1}) \\ &= f_{i,m}(\boldsymbol{x}_{i,m+1}(\boldsymbol{x}_{k,m+1})) - g_{ik,m}(\boldsymbol{x}_{k,m+1})\, f_{k,m}(\boldsymbol{x}_{k,m+1}) \\ &= f_{i,m}\big(\boldsymbol{x}_{i,m+1}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\big) - g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\, f_{j,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1})) \\ &\qquad + g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\, f_{j,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1})) - g_{ik,m}(\boldsymbol{x}_{k,m+1})\, f_{k,m}(\boldsymbol{x}_{k,m+1}) \\ &= t^{m+1}\,\nu_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1})) + g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))(f_{j,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1})) \\ &\qquad - g_{jk,m}(\boldsymbol{x}_{k,m+1})\, f_{k,m}(\boldsymbol{x}_{k,m+1})) \\ &\qquad + g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\, g_{jk,m}(\boldsymbol{x}_{k,m+1})\, f_{k,m}(\boldsymbol{x}_{k,m+1}) \\ &\qquad - g_{ik,m}(\boldsymbol{x}_{k,m+1})\, f_{k,m}(\boldsymbol{x}_{k,m+1}) \\ &= t^{m+1}\,\nu_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1})) + t^{m+1} g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\,\nu_{jk,m}(\boldsymbol{x}_{k,m+1}) \\ &\qquad + (g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\, g_{jk,m}(\boldsymbol{x}_{k,m+1}) - g_{ik,m}(\boldsymbol{x}_{k,m+1}))\, f_{k,m}(\boldsymbol{x}_{k,m+1}), \end{aligned}$$

mod t^{m+2}. Since

$$(g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\, g_{jk,m}(\boldsymbol{x}_{k,m+1}) - g_{ik,m}(\boldsymbol{x}_{k,m+1}))\, f_{k,m}(\boldsymbol{x}_{k,m+1}) \equiv 0 \mod t^{m+1},$$

we have

$$g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\, g_{jk,m}(\boldsymbol{x}_{k,m+1}) \equiv g_{ik,m}(\boldsymbol{x}_{k,m+1}) \mod t^{m+1}.$$

Therefore, we have

$$\begin{aligned}&(g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\, g_{jk,m}(\boldsymbol{x}_{k,m+1}) - g_{ik,m}(\boldsymbol{x}_{k,m+1}))\, f_{k,m}(\boldsymbol{x}_{k,m+1})\\ &\equiv (g_{ij,m}(\boldsymbol{x}_{j,m+1}(\boldsymbol{x}_{k,m+1}))\, g_{jk,m}(\boldsymbol{x}_{k,m+1}) - g_{ik,m}(\boldsymbol{x}_{k,m+1}))\, f_{k,0}(\boldsymbol{x}_{k,m+1}) \mod t^{m+2}.\end{aligned}$$

Since $f_{k,0}(\boldsymbol{x}_k) = 0$ on the image of $\varphi_0|_{V_k}$, we have the first identity. The second identity follows from this by taking $k = i$. □

Note that the pull back $\varphi_0^*|_{V_i \cap V_j} g_{ij,0}$ of the set of functions $\{g_{ij,0}\}$ is the set of transition functions for the normal sheaf of φ_0. Thus, the proposition shows that the pull back $\{\varphi_0^*|_{V_i \cap V_j} \nu_{ij,m+1}\}$ of the set of functions $\{\nu_{ij,m+1}\}$ behaves as a Čech 1-cocycle with values in the normal sheaf $\mathcal{N}_{\varphi_0}$ of φ_0. Then, the following is a straightforward generalization of an argument in [16, Section 3], whose proof we omit.

Lemma 8. *The cohomology class of the cocycle $\{\varphi_0^*|_{V_i \cap V_j} \nu_{ij,m+1}\}$ represents the obstruction to deforming φ_m one step further.* □

The assumption that $\varphi_0 : C_0 \to X_0$ is an immersion and $\dim C_0 = \dim X_0 - 1$ is crucial for this lemma. For simplicity, we will write $\{\varphi_0^*|_{V_i \cap V_j} \nu_{ij,m+1}\}$ as $\{\nu_{ij,m+1}\}$ if no confusion would occur.

4. Explicit description of the Kodaira–Spencer class

Let $\pi : \mathfrak{X} \to D$ be a deformation of a compact Kähler manifold X_0 as before. We have the exact sequence

$$0 \to \pi^* \Omega^1_D \to \Omega^1_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/D} \to 0$$

The Kodaira–Spencer class is, by definition, the corresponding class in

$$\mu \in \operatorname{Ext}^1(\Omega^1_{\mathfrak{X}/D}, \pi^* \Omega^1_D).$$

Lemma 9. *The class μ is represented by the Čech 1-cocycle*

$$\mu_{ij} = \sum_{l=1}^{n} \frac{\partial x_{i,l}(\boldsymbol{x}_j, t)}{\partial t} \partial_{x_{i,l}}\, dt.$$

Proof. See [10, Section II.1]. □

From now on, we drop dt from these expressions since it plays no role below. Restricting this to a presentation over $\mathbb{C}[t]/t^{m+1}$, we obtain the Kodaira–Spencer class for the deformation $X_{m+1} := \mathfrak{X} \times_D \operatorname{Spec} \mathbb{C}[t]/t^{m+2}$. We denote this class by μ_m.

Assume that we have constructed an m-th order deformation $\varphi_m : C_m \to X_m$ of φ_0. Let $\mathcal{N}_{m/D}$ be the relative normal sheaf of φ_m and

$$p_m : \varphi_m^* \mathcal{T}_{X_m/D} \to \mathcal{N}_{m/D}$$

be the natural map, where $\mathcal{T}_{X_m/D}$ is the relative tangent sheaf of X_m. Pulling μ_m back to C_m and taking the image by p_m, we obtain a class $\bar{\mu}_m \in H^1(C_m, \mathcal{N}_{m/D})$.

As before, let $\{f_{i,m}(\boldsymbol{x}_{i,m}, t)\}$ be the set of local defining functions of $\varphi_m(V_{i,m})$ on $U_{i,m}$.

Lemma 10. *The class $\bar{\mu}_m$ is represented by the pull back of*

$$\eta_{ij,m} = \sum_{l=1}^{n} \frac{\partial x_{i,l}(\boldsymbol{x}_j, t)}{\partial t} \partial_{x_{i,l}} f_{i,m}(\boldsymbol{x}_i, t)$$

to C_m.

Proof. We check the cocycle condition. Namely, we have

$$\begin{aligned}
&\eta_{ik,m} - \eta_{ij,m} - g_{ij,m}\,\eta_{jk,m} \\
&= \sum_{l=1}^{n} \frac{\partial x_{i,l}(\boldsymbol{x}_k, t)}{\partial t} \partial_{x_{i,l}} f_{i,m}(\boldsymbol{x}_i, t) - \sum_{l=1}^{n} \frac{\partial x_{i,l}(\boldsymbol{x}_j, t)}{\partial t} \partial_{x_{i,l}} f_{i,m}(\boldsymbol{x}_i, t) \\
&\qquad - g_{ij,m} \sum_{l=1}^{n} \frac{\partial x_{j,l}(\boldsymbol{x}_k, t)}{\partial t} \partial_{x_{j,l}} f_{j,m}(\boldsymbol{x}_i, t) \\
&= \sum_{l=1}^{n} \frac{\partial x_{i,l}(\boldsymbol{x}_k, t)}{\partial t} \partial_{x_{i,l}} f_{i,m}(\boldsymbol{x}_i, t) - \sum_{l=1}^{n} \frac{\partial x_{i,l}(\boldsymbol{x}_j, t)}{\partial t} \partial_{x_{i,l}} f_{i,m}(\boldsymbol{x}_j, t) \\
&\qquad - g_{ij,m} \sum_{l=1}^{n} \frac{\partial x_{j,l}(\boldsymbol{x}_k, t)}{\partial t} \partial_{x_{j,l}} (g_{ij,m}^{-1} f_{i,m}(\boldsymbol{x}_i(\boldsymbol{x}_j, t), t)) \\
&= (\mu_{ik} - \mu_{ij} - \mu_{jk}) f_{i,m} - g_{ij,m} f_{i,m}(\boldsymbol{x}_i(\boldsymbol{x}_j, t), t) \sum_{l=1}^{n} \frac{\partial x_{j,l}(\boldsymbol{x}_k, t)}{\partial t} \partial_{x_{j,l}} (g_{ij,m}^{-1}).
\end{aligned}$$

Since $\mu_{ik} - \mu_{ij} - \mu_{jk} = 0$ by the cocycle condition, and $f_{i,m}(\boldsymbol{x}_i(\boldsymbol{x}_j, t), t)$ is zero on the image of φ_m, we see that $\eta_{ik,m} = \eta_{ij,m} + g_{ij,m}\,\eta_{jk,m}$ on C_m. Also, note that $g_{ij,m}$ is the transition function of the normal sheaf $\mathcal{N}_{m/D}$. Then, it is clear that $\eta_{ij,m}$ represents the class $\bar{\mu}_m$. □

Recall that a complex analytic cycle of codimension r in a Kähler manifold determines a cohomology class of type (r, r), which is the Poincaré dual of the homology class of the cycle. Let $\zeta_{C_0} \in H^1(X_0, \Omega^1_{X_0/D})$ be the class corresponding to the image of φ_0. Note that since the family $\mathfrak{X}$ is differential geometrically trivial, the class ζ_{C_0} determines a cohomology class in $H^2(\mathfrak{X}, \mathbb{C})$. We denote it by $\tilde{\zeta}_{C_0}$. Then, we have:

Lemma 11. *When φ_0 is semiregular, the class $\tilde{\zeta}_{C_0}$ remains Hodge in X_{m+1} if and only if the class $\bar{\mu}_m$ is zero.*

Proof. Since we are assuming we have constructed $\varphi_m : C_m \to X_m$, the class $\tilde{\zeta}_{C_0}$ is Hodge on X_m. That is,

$$\tilde{\zeta}_{C_0}|_{X_m} \in H^1(X_m, \Omega^1_{X_m/D}).$$

Bloch [5, Proposition 4.2] showed that $\tilde{\zeta}_{C_0}$ remains Hodge on X_{m+1} if and only if the cup product

$$\tilde{\zeta}_{C_0}|_{X_m} \cup \mu_m \in H^2(X_m, \mathcal{O}_{X_m})$$

is zero. This is the same as the claim that the cup product $\tilde{\zeta}_{C_0}|_{X_m} \cup \mu_m \cup \alpha$ is zero for any $\alpha \in H^{2n-2}(X_m, \mathbb{C})$. On the other hand, we have:

Claim 12. *The cup product $\tilde{\zeta}_{C_0}|_{X_m} \cup \mu_m \cup \alpha$ is zero for any $\alpha \in H^{2n-2}(X_m, \mathbb{C})$ if and only if the cup product $\bar{\mu}_m \cup \varphi_m^* \alpha$ is zero on C_m.*

Proof of Claim 12. By definition of $\tilde{\zeta}_{C_0}|_{X_m}$, the class $\tilde{\zeta}_{C_0}|_{X_m} \cup \mu_m \cup \alpha$ is zero if and only if the class $\varphi_m^* \mu_m \cup \varphi_m^* \alpha$ is zero. Note that the cohomology group $H^{2n-2}(X_m, \mathbb{C})$ decomposes as

$$H^{2n-2}(X_m, \mathbb{C}) \cong H^n(X_m, \Omega^{n-2}_{X_m/D}) \oplus H^{n-1}(X_m, \Omega^{n-1}_{X_m/D}) \oplus H^{n-2}(X_m, \mathcal{K}_{X_m/D}),$$

here, $\mathcal{K}_{X_m/D}$ is the relative canonical sheaf. By dimensional reason, the cup product between $\varphi_m^* \mu_m$ and the pull back of the classes in

$$H^n(X_m, \Omega^{n-2}_{X_m/D}) \oplus H^{n-1}(X_m, \Omega^{n-1}_{X_m/D})$$

is zero. Therefore, we can assume that α belongs to $H^{n-2}(X_m, \mathcal{K}_{X_m/D})$, and so the class $\varphi_m^* \alpha$ belongs to $H^{n-2}(C_m, \varphi_m^* \mathcal{K}_{X_m/D})$. On the other hand, $\varphi_m^* \mu_m$ belongs to $H^1(C_m, \varphi_m^* \mathcal{T}_{X_m/D})$ and we have the natural map

$$H^1(C_m, \varphi^* \mathcal{T}_{X_m/D}) \to H^1(C_m, \mathcal{N}_{m/D}).$$

Here, $\bar{\mu}_m$ is the image of $\varphi_m^* \mu_m$ by this map. Recall that the dual of $H^1(C_m, \mathcal{N}_{m/D})$ is given by $H^{n-2}(C_m, \varphi_m^* \mathcal{K}_{X_m/D})$. So, it follows that the cup product $\varphi_m^* \mu_m \cup \varphi_m^* \alpha$ reduces to $\bar{\mu}_m \cup \varphi_m^* \alpha$. This proves the claim. □

It immediately follows that if $\bar{\mu}_m$ is zero, then $\tilde{\zeta}_{C_0}$ remains Hodge in X_{m+1}. For the converse, assume that $\tilde{\zeta}_{C_0}$ remains Hodge in X_{m+1}. There is a natural map

$$\iota : H^{2n-2}(X_m, \mathbb{C}) \to H^1(C_m, \mathcal{N}_{m/D})^\vee$$

as in the proof of the claim. Namely, for a class α of

$$H^{2n-2}(X_m, \mathbb{C}) = H^n(X_m, \Omega^{n-2}_{X_m/D}) \oplus H^{n-1}(\Omega^{n-1}_{X_m/D}) \oplus H^{n-2}(\Omega^n_{X_m/D})$$

and $\beta \in H^1(C_m, \mathcal{N}_{m/D})$, let $\iota(\alpha)(\beta)$ be the cup product $\beta \cup \varphi_m^* \alpha$ composed with the trace map $H^{n-1}(C_m, \omega_{C_m}) \to \mathbb{C}$. Here, ω_{C_m} is the dualizing sheaf of C_m, see [1]. The restriction of this map to X_0 is a surjection by the semiregularity of φ_0. Since the surjectivity is an open condition, ι is also a surjection. This shows that $\bar{\mu}_m \cup \varphi_m^* \alpha$ is zero for any $\alpha \in H^{2n-2}(X_m, \mathbb{C})$ is equivalent to the claim that $\bar{\mu}_m$ is zero. □

Thus, when the class $\tilde{\zeta}_{C_0}$ remains Hodge in X_{m+1}, we can write $\bar{\mu}_m$ as the coboundary of a Čech 0-cochain with values in $\mathcal{N}_{m/D}$ on C_m. We choose one such representative $\{\delta_i\}$ where $\delta_i \in \Gamma(V_{i,m}, \mathcal{N}_{m/D})$ such that

$$\delta_i - g_{ij,m}\, \delta_j = \eta_{ij,m}.$$

Here, $\{\eta_{ij,m}\}$ is a representative of $\bar{\mu}_m$ (see Lemma 10). Also, note that by the exact sequence

$$0 \to \mathcal{O}_{U_{i,m}} \to \mathcal{O}_{U_{i,m}}(\varphi_m(V_{i,m})) \to \mathcal{N}_{m/D}|_{V_{i,m}} \to 0,$$

there is a section $\tilde{\delta}_i$ of $\mathcal{O}_{U_{i,m}}(\varphi_m(V_{i,m}))$ which maps to δ_i. Then, we have a lift of $\eta_{ij,m}$ to an open subset of X_0 as follows.

Lemma 13. *When the class $\tilde{\zeta}_{C_0}$ remains Hodge in X_{m+1}, the section*

$$\tilde{\eta}_{ij,m} = \tilde{\delta}_i(\boldsymbol{x}_i(\boldsymbol{x}_j, t), t) - g_{ij,m}(\boldsymbol{x}_j, t)\, \tilde{\delta}_j(\boldsymbol{x}_j, t)$$

of $\mathcal{O}_{U_{i,m} \cap U_{j,m}}(\varphi_m(V_{i,m} \cap V_{j,m}))$ coincides with $\eta_{ij,m}$ when restricted to $V_{i,m}$. □

5. Proof of Theorem 1

As we mentioned in the introduction, in [16], it was shown that if $C_0 \subset X_0$ is a submanifold of codimension one that is semiregular, then the obstruction to deforming C_0 in X_0 vanishes. The point of their proof is to construct a Čech 1-cocycle on X_0 with values in the sheaf $\mathcal{O}_X(C_0)$, whose restriction to C_0 is the relevant obstruction class. Then, the vanishing of such a class in cohomology is a straightforward consequence of the definition of semiregularity. Thus, it is important to represent the obstruction as a restriction of a Čech cocycle on the *ambient* space. In the case which was studied in [16], the construction of such a cocycle on the ambient space can be done by a direct calculation. In our case of maps where $\varphi_0(C_0)$ may be singular, we need an additional argument which is a variant of that in [18]. Also, we need to take account of the effect of the deformation of the ambient space, but it is covered by Lemma 13. In this section, we unify these arguments and complete the proof of the main theorem.

Recall that the obstruction to deforming φ_m is given by a cocycle

$$\{\varphi_0^*|_{V_i \cap V_j} \nu_{ij,m+1}\} \quad \text{on } C_0,$$

where $\nu_{ij,m}$ is defined by

$$t^{m+1}\nu_{ij,m}(\boldsymbol{x}_j) = f_{i,m}(\boldsymbol{x}_i(\boldsymbol{x}_j,t),t) - g_{ij,m}(\boldsymbol{x}_j,t)\, f_{j,m}(\boldsymbol{x}_j,t).$$

For the explicit calculation of the obstruction, we eliminate $g_{ij,m}(\boldsymbol{x}_j,t)$ from this expression as follows.

Lemma 14. *On $U_{i,m}\cap U_{j,m}$, we have*

$$(*)\quad (m+1)\,t^m\frac{\nu_{ij,m}(\boldsymbol{x}_j)}{f_{i,m}(\boldsymbol{x}_i,t)} = \frac{1}{f_{i,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t}+\tilde{\delta}_i\right) - \frac{1}{f_{j,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{j,m}(\boldsymbol{x}_j,t)}{\partial t}+\tilde{\delta}_j\right),$$

modulo functions holomorphic on the image of $\varphi_m|_{V_{i,m}\cap V_{j,m}}$.

Proof. First, by differentiating the equation

$$t^{m+1}\nu_{ij,m}(\boldsymbol{x}_j) = f_{i,m}(\boldsymbol{x}_i(\boldsymbol{x}_j,t),t) - g_{ij,m}(\boldsymbol{x}_j,t)\, f_{j,m}(\boldsymbol{x}_j,t)$$

with respect to t, we have

$$\begin{aligned}(m+1)\,t^m\,\nu_{ij,m}(\boldsymbol{x}_j) &= \frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t} + \sum_{l=1}^{n}\frac{\partial x_{i,l}(\boldsymbol{x}_j,t)}{\partial t}\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial x_{i,l}} \\ &\qquad - g_{ij,m}(\boldsymbol{x}_j,t)\frac{\partial f_{j,m}(\boldsymbol{x}_j,t)}{\partial t} - \frac{\partial g_{ij,m}(\boldsymbol{x}_j,t)}{\partial t} f_{j,m}(\boldsymbol{x}_j,t) \\ &= \frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t} - g_{ij,m}(\boldsymbol{x}_j,t)\frac{\partial f_{j,m}(\boldsymbol{x}_j,t)}{\partial t} + \eta_{ij,m} - \frac{\partial g_{ij,m}(\boldsymbol{x}_j,t)}{\partial t} f_{j,m}(\boldsymbol{x}_j,t)\end{aligned}$$

on $V_{i,m}\cap V_{j,m}$. Since $f_{j,m}$ is zero on the image of $\varphi_m|_{V_{j,m}}$, we can ignore the last term. By the same reason, we can replace $\eta_{ij,m}$ by $\tilde{\eta}_{ij,m}$ introduced in Lemma 13, and we can regard the above equation as an equation on $U_{i,m}\cap U_{j,m}$.

Dividing this by $f_{i,m}(\boldsymbol{x}_i,t)$, we have

$$\begin{aligned}(*)\quad (m+1)\,t^m\frac{\nu_{ij,m}(\boldsymbol{x}_j)}{f_{i,m}(\boldsymbol{x}_i,t)} &= \frac{1}{f_{i,m}(\boldsymbol{x}_i,t)}\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t} - \frac{g_{ij,m}(\boldsymbol{x}_j,t)}{f_{i,m}(\boldsymbol{x}_i,t)}\frac{\partial f_{j,m}(\boldsymbol{x}_j,t)}{\partial t} + \frac{\eta_{ij,m}}{f_{i,m}(\boldsymbol{x}_i,t)} \\ &= \frac{1}{f_{i,m}(\boldsymbol{x}_i,t)}\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t} - \frac{g_{ij,m}(\boldsymbol{x}_j,t)}{f_{i,m}(\boldsymbol{x}_i,t)}\frac{\partial f_{j,m}(\boldsymbol{x}_j,t)}{\partial t} + \frac{\tilde{\delta}_i}{f_{i,m}(\boldsymbol{x}_i,t)} - \frac{g_{ij,m}\tilde{\delta}_j}{f_{i,m}(\boldsymbol{x}_i,t)} \\ &= \frac{1}{f_{i,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t}+\tilde{\delta}_i\right) - \frac{1}{f_{j,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{j,m}(\boldsymbol{x}_j,t)}{\partial t}+\tilde{\delta}_j\right)\end{aligned}$$

modulo functions holomorphic on C_m. Note that this is an equation over $\mathbb{C}[t]/t^{m+1}$, and so we have

$$\frac{g_{ij,m}(\boldsymbol{x}_j,t)\,f_{j,m}(\boldsymbol{x}_j,t)}{f_{i,m}(\boldsymbol{x}_i,t)}=1. \qquad \square$$

Let

$$\left[\frac{1}{f_{i,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t}+\tilde{\delta}_i\right)\right]_m$$

be the coefficient of t^m in

$$\frac{1}{f_{i,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t}+\tilde{\delta}_i\right).$$

Note that the above equation still holds when we replace

$$\frac{1}{f_{i,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t}+\tilde{\delta}_i\right) \quad\text{and}\quad \frac{1}{f_{j,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{j,m}(\boldsymbol{x}_j,t)}{\partial t}+\tilde{\delta}_j\right)$$

by

$$\left[\frac{1}{f_{i,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t}+\tilde{\delta}_i\right)\right]_m \quad\text{and}\quad \left[\frac{1}{f_{j,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{j,m}(\boldsymbol{x}_j,t)}{\partial t}+\tilde{\delta}_j\right)\right]_m,$$

respectively. Also, we can think of

$$\left[\frac{1}{f_{i,m}(\boldsymbol{x}_i,t)}\left(\frac{\partial f_{i,m}(\boldsymbol{x}_i,t)}{\partial t}+\tilde{\delta}_i\right)\right]_m$$

as a function on U_i by forgetting t^m.

Now, introduce any Riemannian metric on X_0. Recall that we fixed an open covering $\{V_i\}$ of C_0. If V_i does not contain a singular point of C_0, we write $\mathring{V}_i=V_i$. If V_i contains a singular point of C_0, we write by $\mathring{V}_i$ the complement of a small closed disk around the singular point in V_i. For each $\mathring{V}_i$, let $N_{\varphi_0}|_{\mathring{V}_i}$ be the normal bundle of φ_0 restricted to $\mathring{V}_i$. Let $S_\delta|_{\mathring{V}_i}$ be the circle bundle of radius δ in $N_{\varphi_0}|_{\mathring{V}_i}$. Here, δ is a small positive real number. If δ is small enough, the exponential map gives an embedding of $S_\delta|_{\mathring{V}_i}$ into a small neighborhood of the image $\varphi_0(\mathring{V}_i)$. We can assume that the image of $S_\delta|_{\mathring{V}_i}$ is disjoint from $\varphi_0(V_i)$ even if V_i contains a singular point of C_0. Note that the bundles $S_\delta|_{\mathring{V}_i}$ on each $\mathring{V}_i$ glue and give a circle bundle S_δ on the open subset $\bigcup_i \mathring{V}_i$ of C_0. When $n\geq 3$, this is actually a bundle over C_0.

Note that the obstruction class

$$[\varphi_0^*|_{V_i\cap V_j}\nu_{ij,m+1}]\in H^1(C_0,\mathcal{N}_{\varphi_0})$$

is zero if and only if the pairing of it with any class in $H^{n-2}(C_0,\varphi^*\mathcal{K}_{X_0})$ is zero. By semiregularity, any class in $H^{n-2}(C_0,\varphi^*\mathcal{K}_{X_0})$ is a restriction of an element of $H^{n-2}(X_0, \mathcal{K}_{X_0})$. Let Θ be any closed C^∞ $(2n-2)$-form on X_0. In particular, Θ represents a class in

$$H^{2n-2}(X_0, \mathbb{C}) = H^{n-2}(X_0, \mathcal{K}_{X_0}) \oplus H^{n-1}(X_0, \Omega^{n-1}_{X_0}) \oplus H^n(X_0, \Omega^{n-2}_{X_0}).$$

Here, $\Omega^i_{X_0}$ is the sheaf of holomorphic i-forms on X_0. Integrating the restriction of the singular $(2n-2)$-form

$$\left[\frac{1}{f_{i,m}(\boldsymbol{x}_i, t)}\left(\frac{\partial f_{i,m}(\boldsymbol{x}_i, t)}{\partial t} + \tilde{\delta}_i\right)\right]_m \Theta$$

to the circle bundle along the fibers, we obtain a closed $(2n-3)$-forms γ_i on $\mathring{V}_i$. Then, we have:

Lemma 15. *On $\mathring{V}_i \cap \mathring{V}_j$, the limit $\lim_{\delta\to 0} \gamma_i - \gamma_j$ exists, and is $m+1$ times the fiberwise pairing between $\varphi_0^*|_{V_i\cap V_j} \nu_{ij,m}(\boldsymbol{x}_j)$ and $\varphi_0^*\Theta$.*

Proof. Note that $\nu_{ij,m}(\boldsymbol{x}_j)$ is a local section of the normal sheaf $\mathcal{N}_{\varphi_0}$ of $\varphi_0 : C_0 \to X_0$. Thus, it naturally pairs with the pull back of Θ and gives a $(2n-3)$-form on $\mathring{V}_i \cap \mathring{V}_j$. Then, the claim is a consequence of the equation $(*)$ and standard estimates of integrations.

Now, if C_0 is nonsingular (in particular if $n \geq 3$), let $\mathcal{C}^{2n-3}(C_0)$ be the sheaf of smooth closed $(2n-3)$-forms on C_0. It has a resolution

$$0 \to \mathcal{C}^{2n-3}(C_0) \to \mathcal{A}^{2n-3}(C_0) \to \mathcal{A}^{2n-2}(C_0) \to 0$$

by flabby sheaves. Here, $\mathcal{A}^i(C_0)$ is the sheaf of complex valued smooth i-forms on C_0. Thus, the cohomology group $H^1(C_0, \mathcal{C}^{2n-3})$ is naturally isomorphic to $H^{2n-2}(C_0, \mathbb{C}) \cong H^{n-1}(C_0, \mathcal{K}_{C_0})$.

By Lemma 15, as the radius δ goes to zero, the Čech 1-cocycle $\{\gamma_{ij}\}$ with values in closed $(2n-3)$-forms obtained as the differences of $\{\gamma_i\}$ converges to the obstruction class $[\nu_{ij,m}]$ paired with the pull back of Θ by φ_0, considered as a class in $H^1(C_0, \mathcal{C}^{2n-3})$. However, by the above isomorphism between $H^1(C_0, \mathcal{C}^{2n-3})$ and $H^{n-1}(C_0, \mathcal{K}_{C_0})$, this class is the same as the obstruction class paired with $\varphi_0^*\Theta$. Thus, the obstruction to deforming φ_m vanishes if and only if the limit class in Lemma 15 vanishes for any $\Theta \in H^{n-2}(X_0, \mathcal{K}_{X_0})$.

If C_0 is nonsingular, $\{\gamma_i\}$ is defined on a genuine open covering of C_0. Thus, the Čech cocycle $\{\gamma_{ij}\}$ vanishes for all δ. So, the limit also vanishes. This finishes the proof of Theorem 1 for C_0 nonsingular.

When $n = 2$ and C_0 is singular, $\bigcup_i \mathring{V}_i$ covers only an open subset of C_0. However, one can show that the Čech 1-cocycle defined by $\gamma_{ij} = \gamma_i - \gamma_j$ still does not depend on the radius δ, and $\lim_{\delta\to 0} \gamma_i - \gamma_j$ gives the obstruction class paired with $\varphi_0^*\Theta$. Thus, it suffices to prove the vanishing of the class $[\gamma_{ij}]$ for a small δ. This can be reduced to an application of the Stokes theorem. See [18] for full details. □

6. Criterion for semiregularity

In this section, we give necessary conditions for a map $\varphi_0 : C_0 \to X_0$ to be semiregular. It turns out that some classical notions which appeared in different contexts such as Cayley–Bacharach condition and d-semistability are related to relative deformations of maps.

The case n > 2. First, we consider the case $n > 2$. Let $\pi : \mathfrak{X} \to D$ be a family of n-dimensional Kähler manifolds. Let $\varphi_0 : C_0 \to X_0$ be a map from a compact smooth complex manifold of dimension $n-1$ which is an immersion. We also assume that the image $\varphi_0(C_0)$ has normal crossing singularity.

Consider the exact sequence on $\varphi_0(C_0)$ given by

$$0 \to \iota^* \mathcal{K}_{X_0} \to p_* \varphi_0^* \mathcal{K}_{X_0} \to \mathcal{Q} \to 0,$$

where $\iota : \varphi_0(C_0) \to X_0$ is the inclusion, and $p : C_0 \to \varphi_0(C_0)$ is the normalization. The sheaf $\mathcal{Q}$ is defined by this sequence. It is supported on the singular locus $\mathrm{sing}(\varphi_0(C_0))$ of $\varphi_0(C_0)$. We have an associated exact sequence of cohomology groups

$$\begin{aligned} \cdots \to H^{n-2}(\varphi_0(C_0), \iota^* \mathcal{K}_{X_0}) &\to H^{n-2}(\varphi_0(C_0), p_* \varphi_0^* \mathcal{K}_{X_0}) \\ &\to H^{n-2}(\varphi_0(C_0), \mathcal{Q}) \\ &\to H^{n-1}(\varphi_0(C_0), \iota^* \mathcal{K}_{X_0}) \\ &\to H^{n-1}(\varphi_0(C_0), p_* \varphi_0^* \mathcal{K}_{X_0}) \\ &\to H^{n-1}(\varphi_0(C_0), \mathcal{Q}). \end{aligned} \tag{1}$$

By dimensional reason, we have $H^{n-1}(\varphi_0(C_0), \mathcal{Q}) = 0$. Also, note that

$$H^i(\varphi_0(C_0), p_* \varphi_0^* \mathcal{K}_{X_0}) \cong H^i(C_0, \varphi_0^* \mathcal{K}_{X_0})$$

for $i = n-2, n-1$, by the Leray spectral sequence. Therefore, if $\varphi_0(C_0)$ is semiregular in the classical sense, that is, the natural map

$$H^{n-2}(X_0, \mathcal{K}_{X_0}) \to H^{n-2}(\varphi_0(C_0), \iota^* \mathcal{K}_{X_0})$$

is surjective, then the map φ_0 is semiregular if and only if the map

$$H^{n-2}(\varphi_0(C_0), \iota^* \mathcal{K}_{X_0}) \to H^{n-2}(\varphi_0(C_0), p_* \varphi_0^* \mathcal{K}_{X_0})$$

is surjective.

Corollary 16. *Assume that $\varphi_0(C_0)$ is semiregular in the classical sense and the class $[\varphi_0(C_0)]$ remains Hodge on the fibers of $\mathfrak{X}$. Then, if the map*

$$H^{n-2}(\varphi_0(C_0), \iota^* \mathcal{K}_{X_0}) \to H^{n-2}(\varphi_0(C_0), p_* \varphi_0^* \mathcal{K}_{X_0})$$

is surjective, φ_0 can be deformed to general fibers of $\mathfrak{X}$. □

On the other hand, consider the exact sequence

$$0 \to p_* \mathcal{N}_{\varphi_0} \to \mathcal{N}_\iota \to \mathcal{S} \to 0,$$

of sheaves on $\varphi_0(C_0)$, where $\mathcal{S}$ is defined by this sequence. The associated exact sequence of cohomology groups is

$$\begin{aligned} 0 \to H^0(\varphi_0(C_0), p_* \mathcal{N}_{\varphi_0}) &\to H^0(\varphi_0(C_0), \mathcal{N}_\iota) \\ &\to H^0(\varphi_0(C_0), \mathcal{S}) \\ &\to H^1(\varphi_0(C_0), p_* \mathcal{N}_{\varphi_0}) \\ &\to H^1(\varphi_0(C_0), \mathcal{N}_\iota) \to \cdots \end{aligned} \tag{2}$$

We have

$$H^i(\varphi_0(C_0), p_* \mathcal{N}_{\varphi_0}) \cong H^i(C_0, \mathcal{N}_{\varphi_0})$$

again by the Leray spectral sequence. Note that the group $H^i(C_0, \mathcal{N}_{\varphi_0})$ is isomorphic to the dual of $H^{n-1-i}(C_0, \varphi_0^* \mathcal{K}_{X_0})$, $i = 0, 1$. Similarly, the group $H^i(\varphi_0(C_0), \mathcal{N}_\iota)$ is isomorphic to the dual of $H^{n-1-i}(\varphi_0(C_0), \iota^* \mathcal{K}_{X_0})$, $i = 0, 1$.

Comparing the dual of the cohomology exact sequence (1) with (2), we obtain $H^{n-2}(\varphi_0(C_0), \mathcal{Q})^\vee \cong H^0(\varphi_0(C_0), \mathcal{S})$. In particular, we can restate Corollary 16 as follows.

Corollary 17. *Assume that $\varphi_0(C_0)$ is semiregular in the classical sense and the class $[\varphi_0(C_0)]$ remains Hodge on the fibers of $\mathfrak{X}$. Then, if the map*

$$H^0(\varphi_0(C_0), \mathcal{N}_\iota) \to H^0(\varphi_0(C_0), \mathcal{S})$$

is surjective, φ_0 can be deformed to general fibers of $\mathfrak{X}$. □

The sheaf $\mathcal{S}$ is the *infinitesimal normal sheaf* of the singular locus of $\varphi_0(C_0)$, as we will see below. Recall that we assume that the image $\varphi_0(C_0)$ has normal crossing singularity. Then, for any point $p \in \varphi_0(C_0)$, we can take a coordinate system $(z_1, \dots, z_n)$ on a neighborhood U of p in X_0 so that $U \cap \varphi_0(C_0)$ is given by $x_1 \cdots x_k = 0$, $1 \le k \le n$. Let $\mathcal{I}_j$ be the ideal of $\mathcal{O}_U$ generated by x_j and let $\mathcal{I}$ be the ideal defining $\varphi_0(C_0) \cap U$ in U. Then,

$$\mathcal{I}_1/\mathcal{I}_1\mathcal{I} \otimes \cdots \otimes \mathcal{I}_k/\mathcal{I}_k\mathcal{I}$$

gives an invertible sheaf on the singular locus of $\varphi_0(C_0) \cap U$. Globalizing this construction, we obtain an invertible sheaf on the singular locus of $\varphi_0(C_0)$. Then, the dual invertible sheaf of this is called the infinitesimal normal sheaf of the singular locus of $\varphi_0(C_0)$, see [8]. We note that the infinitesimal normal sheaf is canonically isomorphic to the sheaf (see [8, Proposition 2.3])

$$\mathrm{Ext}^1_{\mathcal{O}_{\varphi_0(C_0)}}(\Omega^1_{\varphi_0(C_0)}, \mathcal{O}_{\varphi_0(C_0)}).$$

Lemma 18. *The sheaf $\mathcal{S}$ is isomorphic to the infinitesimal normal sheaf.*

Proof. Note that the sheaf $\mathcal{I}_1/\mathcal{I}_1\mathcal{I}\otimes\cdots\otimes\mathcal{I}_k/\mathcal{I}_k\mathcal{I}$ is generated by the element $x_1\otimes\cdots\otimes x_k$. The sheaf $p_*\mathcal{N}_{\varphi_0}$ is given by

$$\bigoplus_{i=1}^{k}\mathrm{Hom}(\mathcal{I}_i/\mathcal{I}_i^2,\mathcal{O}_U)\quad\text{on } U.$$

The sheaf $\mathcal{N}_\iota$ is given by $\mathrm{Hom}(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_U)$. The sheaf $\mathcal{N}_\iota$ is an invertible sheaf and generated by the morphism which maps $x_1\cdots x_k$ to $1\in\mathcal{O}_U$. In particular, by multiplying any $x_1\cdots\check{x}_i\cdots x_k$, the generator is mapped into the image of $p_*\mathcal{N}_{\varphi_0}\to\mathcal{N}_\iota$, namely, the image of the generator of $\mathrm{Hom}(\mathcal{I}_i/\mathcal{I}_i^2,\mathcal{O}_U)$. Also, note that the ideal of the singular locus of $\varphi_0(C_0)$ is generated by $x_1\cdots\check{x}_i\cdots x_k$, $i=1,\dots,k$. From these, it is easy to see that the cokernel of the map $p_*\mathcal{N}_{\varphi_0}\to\mathcal{N}_\iota$ is isomorphic to the dual of $\mathcal{I}_1/\mathcal{I}_1\mathcal{I}\otimes\cdots\otimes\mathcal{I}_k/\mathcal{I}_k\mathcal{I}$. □

Recall that the infinitesimal normal sheaf is related to deformations of $\varphi_0(C_0)$ which smooth the singular locus, see [8]. In particular, $\varphi_0(C_0)$ is called *d-semistable* if the infinitesimal normal sheaf is trivial, and d-semistable variety carries a log structure log smooth over a standard log point, so that one can study its deformations via log smooth deformation theory [13; 14; 15].

By Corollary 17, the infinitesimal normal sheaf plays a crucial in the deformation theory even if it is not d-semistable.

On the other hand, the notion of d-semistability gives a sufficient condition for the existence of deformations in this situation, too, as follows.

Corollary 19. *Let the image $\varphi_0(C_0)$ be very ample and $H^1\big(X_0,\mathcal{O}_{X_0}(\varphi_0(C_0))\big)=0$. Let $\varphi_0(C_0)$ be d-semistable and the singular locus of $\varphi_0(C_0)$ is connected. Then, the map φ_0 is semiregular.*

Proof. First, we note that the subvariety $\varphi_0(C_0)$ is semiregular in the classical sense. Namely, consider the cohomology exact sequence

$$\cdots\to H^1\big(X_0,\mathcal{O}_{X_0}(\varphi_0(C_0))\big)\to H^1(\varphi_0(C_0),\mathcal{N}_\iota)\to H^2(X_0,\mathcal{O}_{X_0})\to\cdots,$$

here $\iota:\varphi_0(C_0)\to X_0$ is the inclusion and $\mathcal{N}_\iota$ is the normal sheaf of it. When $H^1\big(X_0,\mathcal{O}_{X_0}(\varphi_0(C_0))\big)=0$, the map $H^1(\varphi_0(C_0),\mathcal{N}_\iota)\to H^2(X_0,\mathcal{O}_{X_0})$ is injective. Since this map is the dual of the semiregularity map

$$H^{n-2}(X_0,\mathcal{K}_{X_0})\to H^{n-2}(\varphi_0(C_0),\iota^*\mathcal{K}_{X_0}),$$

it follows that $\varphi_0(C_0)$ is semiregular.

To prove that φ_0 is semiregular, it suffices to show the map

$$H^0(\varphi_0(C_0),\mathcal{N}_\iota)\to H^0(\varphi_0(C_0),\mathcal{S})$$

is surjective. When $\varphi_0(C_0)$ is d-semistable, the sheaf $\mathcal{S}$ is the trivial line bundle on the singular locus of $\varphi_0(C_0)$. Since we assume that the singular locus is connected, it suffices to show that the map $H^0(\varphi_0(C_0), \mathcal{N}_\iota) \to H^0(\varphi_0(C_0), \mathcal{S})$ is not the zero map. This in turn is equivalent to the claim that the injection

$$H^0(\varphi_0(C_0), p_*\mathcal{N}_{\varphi_0}) \to H^0(\varphi_0(C_0), \mathcal{N}_\iota)$$

is not an isomorphism. Since $\varphi_0(C_0)$ is very ample, there is a section s of $\mathcal{O}_X(\varphi_0(C_0))$ which does not entirely vanish on the singular locus of $\varphi_0(C_0)$. Then, if σ is a section of $\mathcal{O}_X(\varphi_0(C_0))$ defining $\varphi_0(C_0)$, the sections $\sigma + \tau s$, where $\tau \in \mathbb{C}$ is a parameter, deforms $\varphi_0(C_0)$, and the nonvanishing of s on the singular locus of $\varphi_0(C_0)$ implies that this smooths a part of the singular locus of $\varphi_0(C_0)$. Since the sections of $H^0(\varphi_0(C_0), p_*\mathcal{N}_{\varphi_0})$ give first-order deformations which does not smooth the singular locus, it follows that the map

$$H^0(\varphi_0(C_0), p_*\mathcal{N}_{\varphi_0}) \to H^0(\varphi_0(C_0), \mathcal{N}_\iota)$$

is not an isomorphism. This proves the claim. □

The case n = 2. Now, let us consider the case $n = 2$. Although we can work in a more general situation, we assume $\varphi_0(C_0)$ is a reduced nodal curve for simplicity. However C_0 need not be smooth. Let $p : C_0 \to \varphi_0(C_0)$ be the natural map, which is a partial normalization. In this case, we can deduce very explicit criterion for the semiregularity. Again, we have the exact sequence

$$\begin{aligned} 0 \to H^0(\varphi_0(C_0), p_*\mathcal{N}_{\varphi_0}) &\to H^0(\varphi_0(C_0), \mathcal{N}_\iota) \\ &\to H^0(\varphi_0(C_0), \mathcal{S}) \\ &\to H^1(\varphi_0(C_0), p_*\mathcal{N}_{\varphi_0}) \to H^1(\varphi_0(C_0), \mathcal{N}_\iota) \to \cdots, \end{aligned}$$

and if $\varphi_0(C_0)$ is semiregular in the classical sense, then φ_0 is semiregular if and only if the map $H^0(\varphi_0(C_0), \mathcal{N}_\iota) \to H^0(\varphi_0(C_0), \mathcal{S})$ is surjective. Let $P = \{p_i\}$ be the set of nodes of $\varphi_0(C_0)$ whose inverse image by p consists of two points. Then, the sheaf $\mathcal{S}$ is isomorphic to $\oplus_i \mathbb{C}_{p_i}$, where $\mathbb{C}_{p_i}$ is the skyscraper sheaf at p_i. By an argument similar to the one in the previous subsection, we proved Theorem 20 below in [18].

Theorem 20. *Assume that $\varphi_0(C_0)$ is semiregular in the classical sense. Then, the map φ_0 is semiregular if and only if for each $p_i \in P$, there is a first-order deformation of $\varphi_0(C_0)$ which smooths p_i, but does not smooth the other nodes of P.* □

For applications, it will be convenient to write this in a geometric form. Consider the exact sequence

$$0 \to \mathcal{O}_{X_0} \to \mathcal{O}_{X_0}(\varphi_0(C_0)) \to \mathcal{N}_\iota \to 0$$

of sheaves on X_0 and the associated cohomology sequence

$$0 \to H^0(X_0, \mathcal{O}_{X_0}) \to H^0\big(X_0, \mathcal{O}_{X_0}(\varphi_0(C_0))\big) \to H^0(\varphi_0(C_0), \mathcal{N}_\iota) \to H^1(X_0, \mathcal{O}_{X_0}) \to \cdots.$$

Let V be the image of the map $H^0(\varphi(C_0), \mathcal{N}_\iota) \to H^1(X_0, \mathcal{O}_{X_0})$. Since we are working in the analytic category, we have the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_0} \to \mathcal{O}^*_{X_0} \to 0$$

of sheaves on X. Let $\overline{V}$ be the image of V in $Pic^0(X_0) = H^1(X_0, \mathcal{O}^*_{X_0})$. In [18], we proved the following.

Corollary 21. *In the situation of Theorem 20, the map φ_0 is unobstructed if for each $p_i \in P$, there is an effective divisor D such that $\mathcal{O}_X(\varphi_0(C_0) - D) \in \overline{V}$ which avoids p_i but passes through all points in $P \setminus \{p_i\}$.*

A particularly nice case is when the map $H^0(\varphi_0(C_0), \mathcal{N}_\iota) \to H^1(X_0, \mathcal{O}_{X_0})$ is surjective. This is the case when $\varphi_0(C_0)$ is sufficiently ample. Then, if for each $p_i \in P$ there is an effective divisor D which is algebraically equivalent to $\varphi_0(C_0)$ which avoids p_i but passes through all points in $P \setminus \{p_i\}$, the map φ_0 is semiregular. This is, in a sense, the opposite to the classical *Cayley–Bacharach property*, see, for example, [4].

Combined with Theorem 1, we have:

Corollary 22. *Assume that $\varphi_0(C_0)$ is reduced, nodal and semiregular in the classical sense and the class $[\varphi_0(C_0)]$ remains Hodge on the fibers of $\mathfrak{X}$. Then, the map φ_0 deforms to general fibers of $\mathfrak{X}$ if the condition in Theorem 20 or Corollary 21 is satisfied.* □

In the case of $n = 2$, the original exact sequence

$$\cdots \to H^0(\varphi_0(C_0), \iota^*\mathcal{K}_{X_0}) \to H^0(\varphi_0(C_0), p_*\varphi_0^*\mathcal{K}_{X_0}) \to H^0(\varphi_0(C_0), \mathcal{Q}) \to H^1(\varphi_0(C_0), \iota^*\mathcal{K}_{X_0}) \to H^1(\varphi_0(C_0), p_*\varphi_0^*\mathcal{K}_{X_0}) \to H^1(\varphi_0(C_0), \mathcal{Q})$$

before taking the dual is sometimes also useful. In this case, if $\varphi_0(C_0)$ is semiregular in the classical sense, then φ_0 is semiregular if and only if the map

$$H^0(\varphi_0(C_0), \iota^*\mathcal{K}_{X_0}) \to H^0(\varphi_0(C_0), p_*\varphi_0^*\mathcal{K}_{X_0}) \cong H^0(C_0, \varphi_0^*\mathcal{K}_X)$$

is surjective. For example, when the canonical sheaf $\mathcal{K}_{X_0}$ is trivial, then it is clear that this map is surjective and also $\varphi_0(C_0)$ is semiregular in the classical sense. In fact, in this case it is not necessary to assume that the image $\varphi_0(C_0)$ is nodal or reduced, and any immersion φ_0 from a reduced curve C_0 is semiregular. It is known that when X_0 is a K3 surface and the image $\varphi_0(C_0)$ is reduced, then the map φ_0 deforms to general fibers if the class $[\varphi_0(C_0)]$ remains Hodge. This claim is proved using the twistor family associated with the hyperkähler structure of K3 surfaces, see, for example, [7]. Corollary 22 gives a generalization of this fact to general surfaces.

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