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Sp(1)-SYMMETRIC HYPERKÄHLER QUANTISATION

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We provide a new general scheme for the geometric quantisation of Sp(1)-symmetric hyperkähler manifolds, considering Hilbert spaces of holomorphic sections with respect to the complex structures in the hyperkähler 2-sphere. Under properness of an associated moment map, or other finiteness assumptions, we construct unitary (super) representations of groups acting by Riemannian isometries preserving the 2-sphere, and we study their decomposition in irreducible components. We apply this scheme to hyperkähler vector spaces, the Taub–NUT metric on $\mathbb{R}^4$, moduli spaces of framed SU(r)-instantons on $\mathbb{R}^4$, and in part to the Atiyah–Hitchin manifold of magnetic monopoles in $\mathbb{R}^3$.

1. Introduction

The constructions of geometric quantisation offer a recipe for addressing problems related to the quantum mechanics of an object moving in an arbitrary, possibly curved, phase space [42; 89]. The process, abstracting canonical quantisation, is fundamentally based on the structure of a symplectic manifold. Two of the main goals are to obtain operators subject to commutation relations prescribed by the Poisson bracket, and unitary representations of groups associated to Hamiltonian flows. However, there are strong limitations to the extent to which these can be achieved in general. One of the most typical problems is the need of a polarisation, whose existence is generally not guaranteed, nor is its uniqueness ever satisfied.

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Furthermore, the choice of a particular polarisation poses serious constraints on which functions and Hamiltonian flows can be quantised.

A common approach to this issue consists in considering instead a *family* of polarisations, parametrised by some smooth manifold. One then attempts to assemble their corresponding quantum Hilbert spaces into a vector bundle and identify them via the holonomy of some appropriate connection. In this framework, the natural way to quantise Hamiltonian group actions is by automorphisms of the bundle as a whole rather than of the individual vector spaces. A group representation is then usually obtained by considering the space of (projectively) flat sections. The prototypical example of this is the Hitchin connection [16; 45], further discussed below. The latter also has a simple yet interesting adaptation to the case of a symplectic linear space, providing a quantisation of its full symplectic group in the form of a representation of a double cover of it, the metaplectic representation [89, Chapter 10].

Because polarisations on a symplectic manifold often arise as compatible complex structures, it is rather common in geometric quantisation to work with Kähler manifolds [30; 89]. This approach has been successfully applied to a number of moduli spaces arising from differential and algebraic geometry, representation theory, and mathematical physics. Notable examples include unitary flat connections [16; 45] and vector bundles on Riemann surfaces, compact coadjoint orbits [53], and polygons [29; 51]. Unitary flat connections in particular are a good example of the scheme sketched above, as the moduli space comes with a family of Kähler structures parametrised by the Teichmüller space. The construction of a projectively flat connection in that setting is due to Hitchin [45] and Axelrod, Della Pietra, and Witten [16] and was extended to a broader framework in later works [1; 3; 8]. The role of flat connections in Chern–Simons theory [38; 86] also motivated further study of the relation between geometric quantisation and other formulations of the theory, including deformation quantisation [2; 6; 24; 52; 63; 78; 79] and other approaches [9; 10; 11; 12; 59].

In many cases, spaces similar to those above, and related to interesting quantisation problems, come with natural *hyperkähler* structures rather than just Kähler. Some of these may be viewed as complexifications of those already mentioned, e.g., flat connections for complex groups [5; 7; 63; 73; 87], Higgs bundles [4; 31; 35; 44; 80], semisimple/nilpotent (co)adjoint orbits in (dual) complex Lie algebras [17; 55; 58], hyperpolygons, and Nakajima quiver varieties [41; 70]. Others, on the other hand, arise independently of an underlying "real" version, including for instance the Taub–NUT metrics on $\mathbb{R}^4$, moduli spaces of framed $\mathrm{SU}(r)$-instantons and magnetic monopoles, and the Nahm moduli spaces.

Crucially, complex structures on these spaces give rise to families of Kähler forms, whose parametrising spaces come with their own Kähler structures — isomorphic to that of $\mathbb{C}P^1$. Unlike in the Kähler case, geometric quantisation does not directly apply in this situation because no preferred symplectic structure is given in general. What is more, many interesting symmetries of such spaces act by hyperkähler rotations, i.e., by permuting the sphere of Kähler structures rather than fixing each of them individually. If one of the symplectic forms is fixed by the action, one may focus on that particular structure and apply quantisation with respect to it, an approach that was carried out by Andersen, Gukov, and Pei [4] in the case of the Hitchin moduli space. Nonetheless, one may still wish to obtain a version of quantisation with respect to other Kähler forms, or to the hyperkähler structure as a whole. In addition, the induced action on the sphere is in many cases transitive, suggesting again that a more "global" approach should be taken in that situation.

The latter is precisely the setup that we are going to address in this work. Namely, we shall consider a hyperkähler manifold M acted on by a compact Lie group G by hyperkähler isometries and assume that the induced action on $\mathbb{C}P^1$ is transitive. We will also assume that a smooth family of prequantum line bundles on M is given, parametrised by $\mathbb{C}P^1$, together with a lifted equivariant G-action. Carrying out geometric quantisation for each individual symplectic form will give rise to a family of Hilbert spaces, typically of infinite dimension. We will then attempt to use representation theory to "break down" these spaces into finite-dimensional components and assemble each family into a vector bundle over $\mathbb{C}P^1$. We study these objects explicitly and show that their structure is determined by the combinatorics of irreducible subrepresentations in the Hilbert spaces. In particular, we construct natural connections on these bundles and explicitly characterise their curvatures. While the resulting connection on the overall family fails to be projectively flat, we notice that it defines a *holomorphic* structure on it. Based on this, we propose a definition of the overall quantum Hilbert space as the supercohomology of this object, thus obtaining a natural G-representation as a space of sections of a bundle over $\mathbb{C}P^1$.

1A. ***Description of the main construction.*** Let us expand and detail the description sketched above. Suppose a hyperkähler manifold M is given and that G is a compact connected Lie group acting on it by hyperkähler rotations — by this we mean that G acts on M by isometries which permute the Kähler structures on M; we will additionally require that the induced action be transitive. Since Kähler forms are parametrised by $\mathbb{C}P^1$, this corresponds to a surjective group homomorphism $G \twoheadrightarrow \mathrm{SO}(3)$. As we shall see, this implies that G is covered by a product $\mathrm{Sp}(1) \times G_0$, with $\mathrm{Sp}(1)$ acting on $\mathbb{C}P^1$ in the usual way and G_0 fixing all Kähler structures.

Since no preferred symplectic form is given on M, it makes little sense to talk about a prequantum line bundle over the hyperkähler manifold. Instead, we will assume that M is equipped with a Hermitian line bundle (L, h) and a prequantum connection ∇_q for each symplectic form ω_q, depending smoothly on $q \in \mathbb{C}P^1$ in an appropriate sense (see Section 2B). We will further assume that the G-action lifts to L in such a way as to permute the connections equivariantly.

If M_q denotes the Kähler manifold corresponding to $q \in \mathbb{C}P^1$, its geometric quantisation consists of the Hilbert space $\mathcal{H}_q$ of L^2 holomorphic sections of the corresponding prequantum line bundle. The embedding into $\mathbb{C}P^1 \times \mathrm{L}^2(M, L)$ defines a Hermitian structure on this family, viewed informally as a vector bundle over $\mathbb{C}P^1$, together with a compatible connection and $\mathrm{Sp}(1)$-equivariant G-action. We study this object by decomposing each fibre $\mathcal{H}_q$ into isotypical components as a representation of (appropriate subgroups of) $G_q := \mathrm{Stab}_G(q)$. By transitivity of the $\mathrm{Sp}(1)$-action, all these stabilisers are conjugated, and the respective isotypical components form constant-rank subfamilies of $\mathcal{H}$. The following is the central theorem of our work.

Theorem 1.1 (see Theorem 2.11). *Suppose that*:

- *M is a hyperkähler manifold.*
- *G is a connected compact Lie group acting on M by fixing the metric and permuting the symplectic forms transitively.*
- *$L \to M$ is a Hermitian line bundle with a family of prequantum connections as in Section 2B and a G-action covering that on M.*
- *ρ is an irreducible representation of $G_q := \mathrm{Stab}_G(\omega_q)$ for ω_q one of the symplectic forms on M.*
- *ρ has finite multiplicity $m^{(\rho)}$ in the space $\mathcal{H}_q$ of L^2 holomorphic sections of $L \to M$ with respect to the structure associated to ω_q.*

For each other symplectic form $\omega_{q'}$, denote by $\mathcal{H}_{q'}^{(\rho)}$ the isotypical component in $\mathcal{H}_{q'}$ corresponding to ρ under the identification $G_q \simeq G_{q'}$ by conjugation in G. Then the collection of spaces $\mathcal{H}^{(\rho)}$ has a canonical structure of Hermitian vector bundle over $\mathbb{C}P^1$ with compatible connection. Moreover, for some integer $d = d_\rho$ there exists an isomorphism

$$\mathcal{H}^{(\rho)} \simeq (\mathcal{L}^d \otimes \underline{V_\rho})^{\oplus m^\rho} \tag{1}$$

*preserving the Hermitian structure and connection, where $\underline{V_\rho} = \mathbb{C}P^1 \times V_\rho$ carries the trivial connection and $\mathcal{L} \to \mathbb{C}P^1$ is the standard degree-*1 $\mathrm{Sp}(1)$*-equivariant Hermitian line bundle with connection.*

The result implies that, informally speaking, the family $\mathcal{H}$ decomposes as a sum of vector bundles with connections, as long as the appropriate multiplicities are finite. The holonomies of the various components may then be assembled to form parallel transport operators on $\mathcal{H}$. However, (1) also determines the curvature of the connection on each component, which is proportional to the degree d. Consequently, the parallel transport operators on $\mathcal{H}$ depend essentially on the choice of paths on the base and fail to unambiguously identify the different Hilbert spaces — even projectively.

Nonetheless, the components $\mathcal{H}^{(\rho)}$ may also be regarded as G-equivariant *holomorphic* bundles over $\mathbb{C}P^1$. We then obtain G-representations not as spaces of flat sections as customary, but as the cohomology of $\mathcal{H}^{(\rho)}$ as a super vector space. The following then descends from Theorem 1.1.

Theorem 1.2 (see Section 2F). *In the setting of Theorem 1.1, define*

$$H^{(\rho)} := H^*(\mathbb{C}P^1, \mathcal{H}^{(\rho)}) \tag{2}$$

as a super vector space. Then $H^{(\rho)}$ comes with a Hermitian structure and compatible G-action. With d as in Theorem 1.1, it is a direct sum of $|d+1|m^{(\rho)}$ copies of V_ρ, all in even (*resp. odd*) *degree if $d \geq 0$* (*resp. $d < 0$*). *In particular, the completed orthogonal sum*

$$H := \overline{\bigoplus_\rho H^{(\rho)}}, \tag{3}$$

with ρ ranging over all the isomorphism classes of irreducible G-representations, defines a Hilbert space G-representation, and (3) *is the isotypical decomposition.*

This viewpoint also lends itself to an approach in terms of rank-generating series and localisation formulæ, something which we address in Section 2H.

Again, the space H of (3) may informally be thought of as the cohomology of the sum of all the $\mathcal{H}^{(\rho)}$'s, regarded now as a *holomorphic* vector bundle over $\mathbb{C}P^1$. It is interesting to note how this is reminiscent of the description of M in terms of its twistor space, a holomorphic fibration $Z \to \mathbb{C}P^1$ (plus additional holomorphic data). It would be an interesting problem to investigate whether our setup can be obtained in terms of twistor data by purely holomorphic constructions, something which we would like to address in a separate work.

The most crucial assumptions in our construction, besides the surjectivity of $G \to \mathrm{SO}(3)$, is the finite-dimensionality of the isotypical components in $\mathcal{H}_q$. For that reason, we also investigate sufficient conditions to ensure it. They can be summarised as follows.

Theorem 1.3 (see Theorems 2.14 and 2.17). *Suppose one of the Kähler forms ω_q on M is fixed, $S \subseteq \mathrm{Stab}_G(\omega_q)$ is a connected Lie subgroup, and ρ is an isomorphism class of S-representations. Then each of the following is a sufficient condition for the corresponding S-isotypical component in $\mathcal{H}_q$ to have finite dimension.*

- *The Kostant moment map for S is proper on M_q, and its action extends holomorphically to the complexification of S.*

- *S is a torus, M_q has the structure of an affine scheme or Stein space, and the Kostant moment map for S is proper.*
- *S is a torus, M_q has the structure of an affine scheme or Stein space, and $M /\!\!/_w S$, w the weight of ρ, admits a compactification with rational singularities and boundary of codimension greater than or equal to 2.*

A further way to ensure finite dimensionality can be found in the discussion of meromorphic torus actions in [90].

1B. ***Applications and further directions.*** In Section 3 we showcase applications of the main construction. The first one is a hyperkähler vector space V of real dimension $4n$, with $n \in \mathbb{Z}_{\geq 1}$. In this case

$$\mathrm{Hk}(V) \simeq \mathrm{Sp}(n) \cdot \mathrm{Sp}(1),$$

identifying $V \simeq \mathbb{H}^n$ (see Remark 3.2). Indeed, under this isomorphism, Sp(1) acts on V via right multiplication of unit-norm imaginary quaternions, and commutes with the natural Sp(n)-action. Furthermore the norm associated to the hyperkähler metric provides a hyperkähler potential and we can apply the abstract construction (see Theorems 3.5 and 3.6).

Importantly, there are many more examples of (nonflat) Sp(1)-symmetric hyperkähler manifolds. These include moduli spaces of magnetic monopoles on $\mathbb{R}^3$ by the work of Atiyah and Hitchin [14] or equivalently, by the work of Donaldson [34], the moduli spaces of based rational maps from $\mathbb{C}P^1$ to itself; moduli spaces of framed SU(r)-instantons on $\mathbb{R}^4$, by the work of Maciocia [62]; the hyperkähler structure on nilpotent orbits, by the work of Kronheimer [57], and more generally the hyperkähler Swann bundle over any quaternionic Kähler manifold [82]. In four dimensions a complete classification of Sp(1)-symmetric hyperkähler manifolds is given (up to finite covers) by the work of Gibbons and Pope [40] and by Atiyah and Hitchin [14]. The three examples are the flat metric on $\mathbb{H}$, the Taub–NUT metric, and the hyperkähler metric on the moduli space of charge-2 monopoles, i.e., the Atiyah–Hitchin manifold.

We establish in Sections 3B and 3C that the Theorems 1.1, 1.2 and 2.17 (or slight modifications thereof) apply to some of these examples, producing a quantisation and corresponding irreducible unitary (super)representations of distinguished groups of hyperkähler isometries.

2. Abstract Sp(1)-symmetric hyperkähler quantisation

2A. ***Hyperkähler manifolds and their symmetry groups.*** Let n be a positive integer and M a smooth manifold of dimension $4n$.

Definition 2.1. A hyperkähler structure on M consists of a Riemannian metric g and an ordered triple (I, J, K) of covariant constant orthogonal automorphism of $\mathrm{T}M$ satisfying the quaternionic identities $I^2 = J^2 = K^2 = IJK = -\mathrm{Id}_{\mathrm{T}M}$.

It follows that I, J, K are g-skew-symmetric global sections of $\mathrm{End}(\mathrm{T}\,M) \to M$, and we denote by $\mathfrak{su}(2)_{\mathrm{M}}$ the three-dimensional real Lie algebra they span.

The hyperkähler 2-sphere of complex structures of (M, g, I, J, K) is

$$\mathbb{S}_{IJK} := \{I_q = aI + bJ + cK \mid q = (a, b, c) \in \mathbb{R}^3,\ a^2 + b^2 + c^2 = 1\} \subseteq \mathfrak{su}(2)_{\mathrm{M}}\,. \tag{4}$$

As customary, the structure on M identifies (4) with the 2-sphere of unit-norm imaginary quaternions, i.e., with $\mathbb{C}P^1$ as a Kähler manifold. In particular for $q \in \mathbb{C}P^1$ there is a (real) symplectic form on M defined by

$$\omega_q(v, w) := g(I_q v, w) \quad \text{for } v, w \in \mathrm{T}M.$$

The triple $M_q := (M, I_q, \omega_q)$ is a Kähler manifold, and for further use we denote by $\mu_q = \mathrm{d\,vol} \in \Omega^{\mathrm{top}}(M)$ the Liouville volume form — independent of $q \in \mathbb{C}P^1$ as it agrees with the Riemannian volume form of (M, g).

Remark 2.2. The above data can be encoded in a fibration $\pi_{\mathbb{C}P^1} : Z \to \mathbb{C}P^1$ of Kähler manifolds over the Riemann sphere, the *twistor space* of (M, g, I, J, K). Clearly this family comes with a natural global trivialisation $Z \simeq M \times \mathbb{C}P^1$ as a smooth fibre bundle, but not as fibre bundle with symplectic or complex fibres. Nonetheless the natural complex structure on Z makes $Z \to \mathbb{C}P^1$ into a *holomorphic* fibre bundle [46, pp. 141-142].

Now consider the group $\mathrm{Sp(M)} = \mathrm{Sp}(M, g, I, J, K) \subseteq \mathrm{Iso}(M, g)$ of Riemannian isometries of (M, g) preserving the Kähler forms ω_q (or equivalently the complex structures I_q) simultaneously for all $q \in \mathbb{C}P^1$. This group is sometimes referred to as the *hyperunitary* group. Denoting $\mathrm{Aut}_0(Z)$ the group of holomorphic automorphisms of $Z \to \mathbb{C}P^1$ over the identity, there is a natural group homomorphism

$$\mathrm{Sp(M)} \to \mathrm{Aut}_0(Z), \tag{5}$$

given by the fibrewise action of Sp(M).

We shall consider a group of isometries that preserve the hyperkähler structure in a looser sense, relaxing the condition that differentials should commute with I, J, and K individually.

Definition 2.3. Let $\mathrm{Hk(M)} \subseteq \mathrm{Iso}(M, g)$ be the subgroup stabilising the Lie algebra $\mathfrak{su}(2)_{\mathrm{M}}$:

$$\mathrm{Hk(M)} = \mathrm{Hk}(M, g, I, J, K) := \{\varphi \in \mathrm{Iso}(M, g) \mid \mathrm{Ad}_{\mathrm{d}\varphi}(\mathfrak{su}(2)_{\mathrm{M}}) = \mathfrak{su}(2)_{\mathrm{M}}\}.$$

Hence Hk(M) acts on $\mathfrak{su}(2)_{\mathrm{M}}$, and Sp(M) $\subseteq$ Hk(M) is the kernel of this action. Moreover the adjoint action Ad on $\mathfrak{su}(2)_{\mathrm{M}} \simeq \mathbb{R}^3$ is by positive isometries for the standard Euclidean structure, resulting in a group morphism

$$\mathrm{Ad} : \mathrm{Hk(M)} \to \mathrm{SO}(3), \quad \mathrm{Ad} : \varphi \mapsto \mathrm{Ad}_{\mathrm{d}\varphi},$$

and an action on the hyperkähler 2-sphere (4) — simply denoted by $q \mapsto \varphi.q$. The combination of the actions of $\mathrm{Hk}(M)$ on $\mathbb{C}P^1$ and M itself naturally extends (5) to a map

$$\mathrm{Hk}(M) \to \mathrm{Aut}(Z), \tag{6}$$

where $\mathrm{Aut}(Z)$ denotes the full group of biholomorphisms of Z compatible with the fibration map and covering arbitrary Kähler automorphisms of $\mathbb{C}P^1$.

Suppose now given a connected compact Lie group G, and a G-action

$$\rho : G \to \mathrm{Hk}(M)$$

on M by transformations in $\mathrm{Hk}(M)$. We will sometimes denote by

$$\rho^Z : G \to \mathrm{Aut}(Z)$$

the composition of ρ with (6); where unambiguous, we will often denote the G-action simply by $(\rho(g))(p) = gp$, and similarly for ρ^Z. As in the introduction, we require that the induced G-action on $\mathbb{C}P^1$ be transitive, or equivalently that the corresponding map $G \to \mathrm{SO}(3)$ be surjective. The kernel G_0 of this action is then also a compact Lie group, and by construction it acts on M by transformations in Sp(M).

Lemma 2.4. *The induced G-action on $\mathbb{C}P^1$ factors through a morphism*

$$\sigma : \mathrm{Sp}(1) \to G \tag{7}$$

from the universal cover Sp(1) *of* SO(3). *This, moreover, arises from a covering map $G_0 \times \mathrm{Sp}(1) \to G$.*

Proof. By compactness, the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ admits a nondegenerate invariant pairing. Once such a pairing is fixed, the orthogonal complement of $\mathfrak{g}_0 := \mathrm{Lie}(G_0)$ is a Lie subalgebra which maps isomorphically to $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$. This induces a section $\mathfrak{su}(2) \to \mathfrak{g}$ which integrates to the desired map σ. In fact, since $\mathfrak{g}_0$ and $\mathfrak{g}_0^\perp$ commute with each other, the splitting $\mathfrak{g} \simeq \mathfrak{g}_0 \oplus \mathfrak{g}_0^\perp$ is an isomorphism of Lie algebras. In particular, every element of G_0 commutes with $\sigma(\mathrm{Sp}(1))$, resulting in a map $G_0 \times \mathrm{Sp}(1) \to G$. □

In other words, for G as above, a G-action by transitive hyperkähler rotations always comes from an Sp(1)-action, and up to covers it splits as the product with an action by Sp(M). Henceforth we shall fix a group homomorphism σ as in (7).

2B. *Prequantum data.* As already noted, a notion of prequantum line bundle on M is ill-posed, since a hyperkähler manifold comes with a continuous family of incompatible prequantum conditions. Instead, we will assume given a Hermitian line bundle (L, h) on M together with a smooth family of compatible connections ∇_q, $q \in \mathbb{C}P^1$, each with curvature $F_q = -i\omega_q$. The smoothness in q may be expressed by the condition that, if a section ψ of $\pi_M^* L \to Z$ is smooth, then so is the family $\nabla_q \psi|_q$, as a section of $\pi_M^*(L \otimes T^*M)$. Equivalently, for every local trivialisation of L, the induced connection potentials should depend smoothly on $q \in \mathbb{C}P^1$. Together with the trivial derivative along the directions of $\mathbb{C}P^1$ in $Z \simeq M \times \mathbb{C}P^1$, these ∇_q's assemble to form a connection on $\pi^* L \to Z$. Additionally, we will require that L be equipped with a Hermitian G-action

$$\rho^L : G \to \operatorname{Aut}(L, h),$$

which lifts the one on M and permutes the connections equivariantly.

In practice, the G-action may not come with preferred prequantum data as above. Now we investigate criteria to determine whether such data exist for a given action.

A necessary condition for the existence of a prequantum line bundle on a symplectic manifold is that the symplectic form represent an integral class in cohomology. Conversely, in that case a prequantum line bundle can be constructed by a diagram chasing procedure on the Čech–de Rham complex [89].

In our situation, we will require that

$$[\omega_q] \in H^2(M, \mathbb{Z}) \quad \text{for all } q \in \mathbb{C}P^1.$$

In fact, if the condition holds for at least one q, then by the Sp(1)-action it does for all q, and it then follows by continuity that $[\omega_q]$ is independent of q. The diagram chasing procedure mentioned above may then be carried out with differential forms on M depending smoothly on q. Hence a family of prequantum line bundles exists if and only if $[\omega_q]$ is integral for some q.

Suppose such a family is fixed, with underlying Hermitian line bundle (L, h), and let $L_q := (L, h, \nabla_q)$ for each q. For every $g \in G$, the structure of $g^* L_{g.q} \otimes L_q^{-1}$ defines a family of *flat* Hermitian connections. Since such objects are classified up to isomorphism by $\Gamma := H^1\big(M, \mathrm{U}(1)\big)$, this defines a map

$$u : G \to C^\infty(\mathbb{C}P^1, \Gamma), \quad u : g \mapsto (q \mapsto [g^* L_{g.q} \otimes L_q^{-1}]). \tag{8}$$

Viewing the abelian group $\Gamma' := C^\infty(\mathbb{C}P^1, \Gamma)$ as a G-module under the pull-back action, u defines a cocycle in $C^1(G, \Gamma')$.

Lemma 2.5. *Suppose (L, h) is a Hermitian line bundle over M with a family of prequantum connections ∇_q smoothly parametrised by $\mathbb{C}P^1$. The cohomology class of the cocycle u from* (8) *vanishes in $H^1(G, \Gamma')$ if and only if there exist*:

- *a Hermitian line bundle B and*
- *a family of Hermitian flat connections ∇_q^B smoothly parametrised by $\mathbb{C}P^1$*

such that, for all $q \in \mathbb{C}P^1$ and $g \in G$, we have

$$g^*(L_{g.q} \otimes B_{g.q}^{-1}) \simeq L_q \otimes B_q^{-1} \tag{9}$$

as Hermitian line bundles with connection, where $B_q := (B, \nabla_q^B)$.

Proof. Suppose such a family exists. Then (9) is equivalent to

$$g^* L_{g.q} \otimes L_q^{-1} \simeq g^* B_{g.q} \otimes B_q^{-1},$$

i.e., $u = \delta\Lambda$ for $\Lambda(q) := [B_q] \in \Gamma$, and therefore $[u] = 0$.

Conversely, suppose that $u = \delta\Lambda$ for some $\Lambda \in \Gamma'$. It follows from the exact sequence

$$H^1(M, \mathbb{R}) \to H^1(M, \mathrm{U}(1)) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$$

that the components of $\Gamma = H^1(M, \mathrm{U}(1))$ are labelled by the torsion of $H^2(M, \mathbb{Z})$, while the identity component is covered by $H^1(M, \mathbb{R})$. Since $\Lambda : \mathbb{C}P^1 \to \Gamma$ is a continuous map, we may fix some $\Lambda_0 \in \Gamma$ so that $\Lambda - \Lambda_0$ takes values in the identity component. Since $\mathbb{C}P^1$ is simply connected, this lifts to a map $\tilde{\Lambda} : \mathbb{C}P^1 \to H^1(M, \mathbb{R})$. Choose a collection of 1-forms $\alpha_1, \ldots, \alpha_n$ on M whose de Rham cohomology classes form a basis of $H^1(M, \mathbb{R})$. Expressing $\tilde{\Lambda}$ as

$$\tilde{\Lambda}(q) = \sum_{i=1}^{n} c_i(q)[\alpha_i],$$

we see that each c_i is a smooth function of q and therefore $\alpha = \sum_{i=1}^{n} c_i \alpha_i$ is a smooth family of 1-forms on M parametrised by $\mathbb{C}P^1$. Choosing a representative (B, ∇_0^B) of $\Lambda_0 \in H^1(M, \mathrm{U}(1))$ and setting $\nabla_q^B := \nabla_0^B + \alpha(q)$, it follows by construction that

$$\Lambda(q) = [(B, \nabla_q^B)].$$

Expanding and manipulating the condition $u = \delta\Lambda$ leads to (9). □

Lemma 2.5 shows that, if $[u] = 0$, then L may be replaced by a new family of prequantum line bundles on which the action of every element of G admits an equivariant lift. In that case, the action of the group

$$G' := \{\varphi \in \mathrm{Aut}(L, h) \mid \varphi \text{ covers some } g \in G\}$$

covers that of G on M surjectively while permuting the connections equivariantly. Notice moreover that G' is also compact and connected, being a central extension of the image of G in $\mathrm{Hk}(M)$, so the discussion from the previous section also applies to it. In particular, by Lemma 2.4, there exists a map $\sigma^L : \mathrm{Sp}(1) \to G'$ lifting the $\mathrm{Sp}(1)$-action on M. Even though there may not be a lifted G-action on L, we obtain one by replacing the group with G', which does not essentially change the action on M.

The simplest vanishing $[u] = 0$ is obtained if Γ is trivial (which we will see in some examples) or by the existence of a hyperkähler potential (which we discuss in Section 2I).

Up to the necessary replacements, in what follows we thus assume to have fixed a family of prequantum connections with an equivariant action $\rho^L : G \to L$.

2C. ***Geometric quantisation.*** Following the prescription of geometric quantisation, for $q \in \mathbb{C}P^1$ consider the separable Hilbert space

$$\mathcal{H}_q := \left\{ \psi \in H^0(M_q, L_q) \,\middle|\, \int_M h(\psi, \psi)\,\mathrm{d\,vol} < \infty \right\} \subseteq \mathrm{L}^2(M, L), \tag{10}$$

using the holomorphic structure $\bar{\partial}_q = \nabla_q^{0,1}$ and the standard L^2 Hermitian product:

$$\langle \psi \mid \psi' \rangle := \int_M h(\psi, \psi')\,\mathrm{d\,vol}, \quad \psi, \psi' \in \mathcal{H}_q. \tag{11}$$

Let us denote by $\mathcal{H}$ the family of Hilbert spaces thus defined over $\mathbb{C}P^1$.

By construction there are unitary isomorphisms

$$\rho_g^{\mathcal{H}} : \mathcal{H}_q \to \mathcal{H}_{g.q}, \quad q \in \mathbb{C}P^1,\ g \in G,$$

explicitly given by

$$(\rho_g^{\mathcal{H}} \psi)(m) := \rho_g^L(\psi(\rho_{g^{-1}}^Z m)), \quad m \in M. \tag{12}$$

2D. ***Decomposition of $\mathcal{H}_q$.*** We will now consider the decompositions of the spaces (10) induced by viewing them as representations under the action (12). For a given $q \in \mathbb{C}P^1$, restricting $\rho^{\mathcal{H}}$ to

$$G_q := \mathrm{Stab}_G(q),$$

defines a group action on $\mathcal{H}_q$ by unitary operators, i.e., a Hilbert space representation. By the Peter–Weyl theorem [54, Theorem 1.12], $\mathcal{H}_q$ decomposes a completed orthogonal sum of irreducible components. Similarly, denoting by

$$T_q := \mathrm{Stab}_{\mathrm{Sp}(1)}(q)$$

the maximal torus in Sp(1) fixing q, its action on $\mathcal{H}_q$ gives a decomposition

$$\mathcal{H}_q = \overline{\bigoplus_{d \in \mathbb{Z}} \mathcal{H}_q^{(d)}},$$

where $\mathcal{H}_q^{(d)} \subseteq \mathcal{H}_q$ is the isotypical component corresponding to the character

$$T_q \simeq \mathrm{U}(1) \to \mathbb{C}^\times, \quad z \mapsto z^d,$$

under the natural identification with the standard torus $\mathrm{U}(1) \subseteq \mathbb{C}^\times$. Since T_q commutes with $G_0 \subseteq G_q$, each component $\mathcal{H}_q^{(d)}$ is a representation of G_0. Therefore, we obtain a refinement of the decomposition above as

$$\mathcal{H}_q^{(d)} = \overline{\bigoplus_{\lambda \in \Lambda} \mathcal{H}_{q,\lambda}^{(d)}}, \tag{13}$$

where Λ denotes the set of (analytically) integral weights of G_0 and $\mathcal{H}_{q,\lambda}^{(d)}$ is the isotypical component in $\mathcal{H}_q^{(d)}$ of maximal weight λ. In what follows we shall often denote by $\Lambda^{(d)} \subseteq \Lambda$ the subset of "active" representations.[1]

Remark 2.6. It is not difficult to see, given that G_q is covered by $T_q \times G_0$, that every irreducible representation of G_q has a single weight for T_q and is also irreducible for G_0. Therefore, every irreducible G_q-representation induces a pair (d, λ) of weights for T_q and G_0, by which the representation itself is unambiguously determined. In particular, the decomposition (13) is equivalent to the one into isotypical components under G_q.

In a similar way we may also consider a maximal torus $T \subseteq G_0$ and find

$$\mathcal{H}_q^{(d)} = \overline{\bigoplus_{a \in T^\vee} \mathcal{H}_{a,q}^{(d)}},$$

where $\mathcal{H}_{a,q}^{(d)} \subseteq \mathcal{H}_q^{(d)}$ is the isotypical component of the character $a : T \to \mathbb{C}^\times$. Again, the decomposition above is equivalent to the one we would obtain by considering the action of the maximal torus $T'_q := T_q \cdot T \subseteq G_q$ on $\mathcal{H}_q$.

We denote $\mathcal{H}^{(d)}$, $\mathcal{H}_a^{(d)}$ and $\mathcal{H}_\lambda^{(d)}$ the families of Hilbert spaces thus defined over $\mathbb{C}P^1$, so that we have L^2-completed orthogonal direct sums

$$\mathcal{H} = \overline{\bigoplus_{d \in \mathbb{Z}} \mathcal{H}^{(d)}}, \quad \overline{\bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda^{(d)}} = \mathcal{H}^{(d)} = \overline{\bigoplus_{a \in T^\vee} \mathcal{H}_a^{(d)}}.$$

2E. ***Structure of $\mathcal{H}_\lambda^{(d)}$.*** Our main assumption, unless otherwise stated, is as follows:

The spaces $\mathcal{H}_{q,\lambda}^{(d)}$ are finite-dimensional.

The goal of this section is to make each family $\mathcal{H}_\lambda^{(d)}$, under the finite-dimensionality condition above, into a vector bundle, and equip it with a connection. The simplest way to define a smooth structure on $\mathcal{H}_\lambda^{(d)}$ is to assume it is a Banach sumbanifold of the product $\mathbb{C}P^1 \times \mathrm{L}^2(M, L)$, with G acting smoothly on it. We shall now investigate the structure induced by this assumption, and later show that the same data can be obtained canonically from the group structure.

[1]The subset $\Lambda^{(d)}$ is independent of $q \in \mathbb{C}P^1$ since the Sp_0'(M)-modules $\mathcal{H}_q^{(d)}$ are isomorphic under the Sp(1)-action.

Now let (temporarily, see Remark 2.12 below) the family of Hilbert spaces $\mathcal{H}_\lambda^{(d)}$ forms a *smooth* Banach submanifold of the trivial Hilbert bundle $\overline{\mathrm{L}^2(M, L)} \to \mathbb{C}P^1$. We can then differentiate smooth local sections ψ of $\mathcal{H}_\lambda^{(d)} \to \mathbb{C}P^1$, viewed as maps $\mathbb{C}P^1 \to \mathrm{L}^2(M, L)$, along tangent vectors on $\mathbb{C}P^1$. Then, since $\mathcal{H}_{q,\lambda}^{(d)} \subseteq \mathrm{L}^2(M, L)$ is a closed subspace, there are orthogonal projections

$$\pi_{q,\lambda}^{(d)} : \mathrm{L}^2(M, L) \to \mathcal{H}_{q,\lambda}^{(d)}.$$

Definition 2.7. For any tangent vector $X \in T_q\mathbb{C}P^1$ set

$$\nabla_X^{\mathcal{H}_\lambda^{(d)}} \psi := \pi_{q,\lambda}^{(d)}(X[\psi]) \in \mathcal{H}_{q,\lambda}^{(d)}.$$

Remark 2.8. The same definition (of the standard L^2-connection) can be given verbatim in the case where the families $\mathcal{H}_a^{(d)} \subseteq \mathcal{H}^{(d)}$ also constitute smooth submanifolds.

Remark 2.9. This covariant derivative is characterised by the property that

$$\langle \nabla_X^{\mathcal{H}_\lambda^d} \psi | \psi' \rangle = \langle X[\psi] | \psi' \rangle$$

for all X, ψ, ψ' as appropriate.

Proposition 2.10. *The covariant-derivative operators of Definition 2.7 are compatible with the action $\rho^{\mathcal{H}}$ of* (12) *and with the Hermitian structure of $\mathcal{H}_\lambda^{(d)} \to \mathbb{C}P^1$.*

Proof. The operators $\nabla_X^{\mathcal{H}_\lambda^{(d)}}$ satisfy Leibniz and preserve the Hermitian pairing by construction. We need only show that they are $\rho^{\mathcal{H}}$-equivariant. Given $g \in G$, $q \in \mathbb{C}P^1$, a section ψ of $\mathcal{H}_\lambda^{(d)} \to \mathbb{C}P^1$, and a tangent vector $X \in \mathrm{T}_q\, \mathbb{C}P^1$, we have

$$X[\rho_g \psi] = \rho_g((g_*^{-1} X)[\psi]),$$

where the superscripts in the actions were removed for convenience. Combining the above with a change of variables in (11), one sees that

$$\langle X[\rho_g \psi] | \psi' \rangle = \langle (g_*^{-1} X)[\psi] | \rho_{g^{-1}} \psi' \rangle$$

for all $\psi' \in \mathcal{H}_{g.q,\lambda}^{(d)}$. By Remark 2.9, this shows that $\nabla_X(\rho_g \psi) = \rho_g(\nabla_{g_*^{-1} X}[\psi])$. $\square$

Recall now that for every integer d there exists an Sp(1)-equivariant Hermitian line bundle of degree d with compatible connection over $\mathbb{C}P^1$, unique up to isomorphism. This can be characterised as the holomorphic line bundle $\mathcal{O}(d) \to \mathbb{C}P^1$ together with the standard Hermitian metric and its corresponding Chern connection. Alternatively, it can also be described as the quotient of an appropriate line bundle over Sp(1) under the identification $\mathbb{C}P^1 \simeq \mathrm{Sp}(1)/\mathrm{U}(1)$. More precisely, consider the d-th character $\chi^{(d)} : \mathfrak{u}(1) \to \mathbb{R}$ and its unique $\mathrm{Ad}_{\mathrm{U}(1)}$-invariant extension to $\mathfrak{sp}(1)$.

Denoting by $\alpha^{(d)}$ the corresponding left-invariant 1-form on Sp(1), the connection $\mathrm{d}+2\pi i\alpha^{(d)}$ on $\mathrm{Sp}(1)\times\mathbb{C}$ is then invariant under the actions

$$A\cdot(x,z):=(Ax,z)\quad\text{and}\quad(x,z)\cdot h:=(xh,h^{-d}z)$$

for $A\in\mathrm{Sp}(1)$ and $h\in\mathrm{U}(1)$. Furthermore, the right U(1)-action is by construction horizontal for this connection. Therefore, the latter descends to a metric and Sp(1)-equivariant connection on $(\mathrm{Sp}(1)\times\mathbb{C})/\mathrm{U}(1)\to\mathrm{Sp}(1)/\mathrm{U}(1)\simeq\mathbb{C}P^1$.

Uniqueness can be established by noticing that the difference of two such line bundles comes with a connection whose curvature is Sp(1)-invariant and vanishes in cohomology, and is therefore zero. The space of flat sections is then a 1-dimensional Sp(1)-representation, so that choosing one unit element in this space gives an isomorphism of the line bundles intertwining the Hermitian structures and connections.

We will refer to this object as $\mathcal{L}^d$.

For each $d\in\mathbb{Z}$ and $\lambda\in\Lambda^{(d)}$, denote by $m_\lambda^{(d)}$ the multiplicity of V_λ in $\mathcal{H}_q^{(d)}$.[2] Finally, call $\underline{V_\lambda}\to\mathbb{C}P^1$ the trivial Hermitian bundle with fibre V_λ with ∇^{Tr} the trivial connection and Sp(1) acting on it trivially on the fibres.

Theorem 2.11. *Fix an integer d and a dominant weight λ of G_0. Suppose, for some $q\in\mathbb{C}P^1$, that the multiplicity $m_\lambda^{(d)}$ of the corresponding isotypical component in the Hilbert space $\mathcal{H}_q$ is finite. Consider the collection $\mathcal{H}_\lambda^{(d)}$ of corresponding isotypical components, and suppose it forms a Banach submanifold of $\mathbb{C}P^1\times\mathrm{L}^2(M,L)$ acted on smoothly by G. Then $\mathcal{H}_\lambda^{(d)}$ is a Hermitian vector bundle over $\mathbb{C}P^1$ and there exists a G-invariant isomorphism*

$$\mathcal{H}_\lambda^{(d)}\simeq(\mathcal{L}^d\otimes\underline{V_\lambda})^{\oplus m_\lambda^{(d)}}$$

of Hermitian vector bundles which intertwines the covariant derivative operators $\nabla^{\mathcal{H}_\lambda^{(d)}}$ of Definition 2.7 with the natural connection on the right-hand side.

Proof. Introducing for simplicity the notation $m:=m_\lambda^{(d)}$, fix $q\in\mathbb{C}P^1$, and identify T_q with U(1) by the orientation defined by q. Consider on Sp(1) the trivial vector bundle $\mathrm{Sp}(1)\times V_\lambda^{\oplus m}$ with the left and right actions

$$(A,g)\cdot(x,v):=(Ax,gv)\quad\text{and}\quad(x,v)\cdot h=(xh,h^{-d}v)$$

of $\mathrm{Sp}(1)\times G_0$ and U(1), respectively. Choose an isomorphism $\varphi:V_\lambda^{(\oplus m)}\to\mathcal{H}_{q,\lambda}^{(d)}$ as G_0-modules and define

$$\Phi:\mathrm{Sp}(1)\times V_\lambda^{\oplus m}\to\mathcal{H}_\lambda^{(d)},\quad\Phi:(x,v)\mapsto\rho^{\mathcal{H}}(x)(\varphi(v)).$$

By construction, Φ is invariant under the right U(1)-action and intertwines the $\mathrm{Sp}(1)\times G_0$-actions. It is also a surjective smooth map covering the projection

[2] The integer $m^{(d)}$ is independent of $q\in\mathbb{C}P^1$ (see the previous footnote).

$\pi : \mathrm{Sp}(1) \to \mathbb{C}P^1$, $\pi(x) := xq$ and restricts fibrewise to unitary isomorphisms. It follows that Φ is a submersion, and therefore the induced bijection

$$(\mathrm{Sp}(1) \times V_\lambda^{\oplus m})/\mathrm{U}(1) \to \mathcal{H}_\lambda^{(d)} \tag{14}$$

is a diffeomorphism, thus showing that $\mathcal{H}_\lambda^{(d)}$ is a vector bundle as claimed. It then follows from Proposition 2.10 that $\nabla^{\mathcal{H}_\lambda^{(d)}}$ is a Hermitian G-invariant connection.

The map Φ may also be regarded as a unitary isomorphism

$$\mathrm{Sp}(1) \times V_\lambda^{\oplus m} \simeq \pi^* \mathcal{H}_\lambda^{(d)}. \tag{15}$$

Both sides come with $\mathrm{Sp}(1) \times G_0$- and $\mathrm{U}(1)$-invariant Hermitian connections, both making the right $\mathrm{U}(1)$-action horizontal. Such a connection, however, is uniquely characterised by these properties. Indeed, left $\mathrm{Sp}(1)$-invariance implies that such a connection is determined by the potential over any element of $\mathrm{Sp}(1)$. On the other hand, combining the left and right $\mathrm{U}(1)$-invariance shows that the operation of lifting elements of $\mathrm{T}_{\mathrm{Id}}\,\mathrm{Sp}(1) \simeq \mathfrak{sp}(1)$ horizontally is $\mathrm{Ad}_{\mathrm{U}(1)}$-equivariant. The condition that the right $\mathrm{U}(1)$-action be horizontal, moreover, determines the lifts of vectors in $\mathfrak{u}(1)$, and therefore of those in $\mathfrak{sp}(1)$ by $\mathrm{Ad}_U(1)$-invariance. We conclude that the isomorphism (15) also identifies the connections on the two bundles, which is to say that the isomorphism (14) is also horizontal. The left-hand side of (14), however, is clearly isomorphic to $\mathcal{L}^d \otimes \underline{V_\lambda}^{\oplus m}$. Finally, since the kernel of the covering map $\mathrm{Sp}(1) \times G_0 \to G$ acts trivially on the right-hand side, it follows that the group action on the left-hand side descends to G. □

Remark 2.12. Theorem 2.11 yields an alternative definition of the bundles of isotypical components, without smoothness assumptions. Indeed, a map Φ constructed as above uniquely defines a smooth structure on $\mathcal{H}_\lambda^{(d)}$ making it a vector bundle which comes with an isomorphism with $\mathcal{L}^d \otimes \underline{V_\lambda}^m$, and therefore inducing also a connection with the desired properties. Given that the only ambiguity in the construction of Φ lies in the choice of φ, any two such maps are related by precomposition with a G_0-invariant automorphism of $V_\lambda^{\oplus m}$. Since this operation preserves the structure on $\mathrm{Sp}(1) \times V_\lambda^{\oplus m}$, the two choices induce the same data on $\mathcal{H}_\lambda^{(d)}$.

This yields finite-rank smooth G-equivariant Hermitian vector bundles over the Riemann sphere, equipped with Hermitian connections, defined from the combinatorial data of the multiplicities of $\mathcal{H}_q$ as a representation, as long as the main assumption that the $\mathcal{H}_{q,\lambda}^{(d)}$'s be finite-dimensional is verified.

Together with Remark 2.6, the content of this section proves Theorem 1.1.

2F. ***Quantum super Hilbert spaces and unitary representations.*** We now denote by $H_\lambda^{(d)}$ the super vector space obtained by taking the holomorphic cohomology of the bundles of isotypical components:

$$H_\lambda^{(d)} := H^*(\mathbb{C}P^1, \mathcal{H}_\lambda^d).$$

By Remark 2.6, the above is equivalent to the space $H^{(\rho)}$ of (2). Since $\mathcal{H}_\lambda^{(d)}$ is Hermitian and $\mathbb{C}P^1$ is Kähler, the L^2-pairing on harmonic representatives gives each of the above a natural Hermitian structure.

If $W^{(d)} = W_+^{(d)} \oplus W_-^{(d)}$ is the unitary super Sp(1)-representation defined by

$$W_+^{(d)} := H^0(\mathbb{C}P^1, \mathcal{L}^d), \quad W_-^{(d)} := H^1(\mathbb{C}P^1, \mathcal{L}^d),$$

then $H_\lambda^{(d)} \simeq W^{(d)} \otimes V_\lambda^{\oplus m_\lambda^{(d)}}$ as super G-representations, where V_λ is endowed with the trivial $\mathbb{Z}_2$-grading. Moreover, dim $W_+^{(d)}$ is equal to $d+1$ if $d \geq 0$ and 0 otherwise, while similarly dim $W_-^{(d)}$ vanishes for $d \geq 0$ and is equal to $-d-1$ otherwise.

Finally consider the nested L^2-completed orthogonal direct sums

$$H := \overline{\bigoplus_{d \in \mathbb{Z}} H^{(d)}}, \quad H^{(d)} := \overline{\bigoplus_{\lambda \in \Lambda^{(d)}} H_\lambda^{(d)}}.$$

This provides a G-representation quantising the G-action on (M, g, I, J, K), thus proving Theorem 1.2 from the introduction.

2G. *Finite-rank conditions.* We shall now consider conditions which entail finite-dimensionality for the isotypical components of Section 2D.

In general, if K is a compact Lie group with Lie algebra $\mathfrak{k} = \mathrm{Lie}(K)$, acting on a Kähler manifold X with a lifted K-action on a prequantum line bundle (L, ∇), there is a natural moment map $\mu : X \to \mathfrak{k}^\vee$ defined by Kostant's formula

$$2\pi i \langle \mu, \xi \rangle \frac{\partial}{\partial \theta} = \xi_X^H - \xi_L \tag{16}$$

for every $\xi \in \mathfrak{k}$, where ξ_L is the vector field corresponding to ξ on L, ξ_X^H the one on X lifted horizontally, and $\partial/\partial\theta$ is the fibrewise "angular" vector field. In this setup, we will make use of the following version of the general principle that "quantisation commutes with reduction".

Theorem 2.13 [43; 81]. *In the setup above, if the K-action extends holomorphically to the complexified group $K^{\mathbb{C}}$, and if the moment map* (16) *is proper, then for every dominant weight γ of K there is an identification*

$$\mathrm{Hom}_K(V_\gamma, H^0(X, L)) \simeq H^0(X_\gamma, L_\gamma),$$

where V_γ denotes a simple K-module of highest weight γ, $X_\gamma = X /\!\!/_\gamma K$ is the symplectic reduction of X at level γ, and L_γ is the induced (V-)bundle on X_γ.

This result was first established by Guillemin and Sternberg [43] in the case X is compact, with additional regularity conditions, and then extended by Sjamaar [81]. The statement has been subsequently generalised in various works including those of Meinrenken [67; 68], Meinrenken and Sjamaar [69], Vergne [84; 85], Ma [60], Ma and Zhang [61], and Hochs and Song [48].

We emphasise that this formulation of "quantisation commutes with reduction" requires no assumptions on γ being a regular value or the K-action being free on $\mu^{-1}(\gamma)$. In the statement of Theorem 2.13, X_γ and L_γ are regarded as a complex analytic space and a coherent sheaf, respectively. See [81] for further detail.

Returning to our setting, for any fixed q and Lie subgroup $S \subset G_q$ we have a moment map

$$\mu_S : M \to \mathrm{Lie}(S)^\vee,$$

given by Kostant's formula. Then Theorem 2.13 yields the following.

Theorem 2.14. *Fix $q \in \mathbb{C}P^1$ and a (connected) Lie subgroup $S \subseteq G_q$, and denote by μ_S the Kostant moment map of the S-action on M_q. Assume that μ_S is proper, and suppose that the S-action has a holomorphic extension to the complexified group $S^{\mathbb{C}}$ on M_q. Then every S-isotypical component in $\mathcal{H}_q$ has finite multiplicity.*

Proof. Properness of the moment map implies that, for any dominant weight γ of S, the symplectic reduction $M /\!\!/_\gamma S$ is a *compact* complex analytic space. On the other hand, L_γ is a coherent sheaf on it by [81, Section 2.2], and by compactness the space of sections is finite-dimensional [28].

It follows from Theorem 2.13 that the irreducible representation of S of highest weight γ has finite multiplicity inside $H^0(M_q, L_q)$, so *a fortiori* inside $\mathcal{H}_q$. □

Remark 2.15. Another way to ensure finite-dimensionality is to assume there are compactifications of the symplectic reductions, with rational singularities and boundary of (complex) codimension at least two; then Hartogs's theorem applies on the reduction (see, e.g., [83] for such generalisations, and see Theorem 2.17).

As briefly noted in the introduction, another approach to controlling the dimension of the isotypical components is offered by the results of [90]. Indeed, the cited work introduces a notion of meromorphicity for certain group actions which, under appropriate conditions (see Assumption 2.14 of the same work), ensures finite-dimensionality.

2H. ***Rank-generating series and localisation formulæ.*** If either $\mathcal{H}^{(d)}$, $\mathcal{H}^{(d)}_a$ or $\mathcal{H}^{(d)}_\lambda$ have finite rank, we consider the (formal) generating series:

$$H(t) = \sum_d \mathrm{rk}(\mathcal{H}^{(d)}) \cdot t^d, \tag{17}$$

and

$$H'(t, \tilde{t}) = \sum_{d,a} \mathrm{rk}(\mathcal{H}^{(d)}_a) \cdot t^d \, \tilde{t}^a, \tag{18}$$

as well as

$$G(t,\tilde{t}) = \sum_{d\in\mathbb{Z}} \sum_{\lambda\in\Lambda^{(d)}} m_\lambda^{(d)} \cdot t^d \tilde{t}^\lambda. \tag{19}$$

Note that if $\mathcal{H}_a^{(d)}$ and $\mathcal{H}_\lambda^{(d)}$ are both finite-rank then (18) can be obtained from (19) via the substitution $\tilde{t}^\lambda \mapsto \chi_\lambda(\tilde{t})$, where

$$\chi_\lambda(\tilde{t}) = \sum_{a\in E_\lambda} n_a^{(\lambda)} \cdot \tilde{t}^a,$$

and where E_λ is the set of weights of V_λ with multiplicities $n_a^{(\lambda)} \in \mathbb{Z}_{\geq 0}$.

If in particular G_0 is semisimple then the Weyl character formula yields

$$\chi_\lambda(\tilde{t}) = \frac{\sum_{w\in W} \epsilon(w)\, \tilde{t}^{w(\lambda+\rho)}}{\sum_{w\in W} \epsilon(w)\, \tilde{t}^{w(\rho)}},$$

where $W = N(T)/T$ is the Weyl group and $\rho \in \mathfrak{t}^\vee$ the half-sum of positive roots.

Conversely (19) can be recovered from (18) (when both are defined) as follows. Fix $d \in \mathbb{Z}$ and let $H_d(\tilde{t})$ be the coefficient of t^d in (18). Let λ be maximal among the weights such that $\tilde{t}^\lambda$ appears in $H_d(\tilde{t})$. In particular, the weight $\lambda_d^{(0)}$ can only appear in an irreducible component of $\mathcal{H}_q^{(d)}$ (as a G_0-module) if it is the highest. Therefore, the coefficient of $\tilde{t}^\lambda$ in $H_d(\tilde{t})$ is equal to $m_\lambda^{(d)}$. One may now consider $H_d(\tilde{t}) - m_\lambda^{(d)}\chi_\lambda(\tilde{t})$ and repeat the procedure inductively. Since each step strictly decreases one of the maximal weights the process terminates — exactly when the polynomial vanishes. This results in a decomposition

$$H_d(\tilde{t}) = \sum_{\lambda\in\Lambda^{(d)}} m_\lambda^{(d)} \chi_\lambda(\tilde{t}),$$

recovering all multiplicities and ultimately (19).

Furthermore the generating series (17), (18), and (19) can sometimes be computed by localisation formulæ. We refer to [49] for general results, and we review here the simpler versions used in what follows.

Suppose the action of T_q on M_q has a finite number of fixed points $|M_q| \subseteq M_q$, and let $\overline{R}(T_q)$ be the formal completion of the character ring $R(T_q)$ of T_q.

Since the fixed points $p \in |M_q|$ are isolated we see that $\Lambda_{-1}(\mathrm{T}_p M_q) \in \overline{R}(T_q)$ is invertible. Suppose now we have a decomposition

$$H^i(M_q, L_q) = \overline{\bigoplus_{d\in\mathbb{Z}} H^i(M_q, L_q)^{(d)}},$$

such that T_q acts on $H^i(M_q, L_q)^{(d)}$ via the d-th power of the standard representation, and such that the spaces $H^i(M_q, L_q)^{(d)}$ are finite-dimensional.

Proposition 2.16 [25; 49]. *The following formula holds*:

$$\sum_{i=0}^{2n}(-1)^i \dim H^i(M_q, L_q)^{(d)} t^d = \sum_{p\in|M_q|} \frac{L_{q,p}}{\Lambda_{-1}(T_p M_q)}.$$

Hence if $H^i(M_q, L_q) = (0)$ for $i > 0$ then simply

$$H(t) = \sum_{p\in|M_q|} \frac{L_{q,p}}{\Lambda_{-1}(T_p M_q)}. \tag{20}$$

Considering the action of $T'_q = T_q \cdot T$ on M_q we get an analogous result, provided T'_q has finitely many fixed points and all spaces $H^i(M_q, L_q)^{(d)}_{a,q}$ are finite-dimensional, and interpreting the right-hand side as an element of $\bar{R}(T) \simeq \mathbb{Z}[\![t^{\pm 1}, \tilde{t}_i^{\pm 1}]\!]$. In particular,

$$H'(t, \tilde{t}) = \sum_{p\in|M_q|} \frac{L_{q,p}}{\Lambda_{-1}(T_p M_q)}. \tag{21}$$

Now recall that if M_q is a Stein space, or has the structure of an affine scheme, then Cartan's theorem yields the vanishing of higher cohomology groups [27]. Thus putting together the previous results we have established the following.

Theorem 2.17. *Suppose there exists $q \in \mathbb{C}P^1$ such that M_q is a Stein space*, *or has the structure of an affine scheme*, *and that the T_q-action* (*resp. T'_q-action*) *has finitely many fixed points. Assume further that one of the following holds*:

- *There is a proper moment map for the T_q-action* (*resp. T'_q-action*).
- *There exists a compactification of the symplectic reductions with rational singularities*, *with boundary of codimension at least two* (*see Remark 2.15*).

Then the family $\mathcal{H}^{(d)}$ (*resp. $\mathcal{H}^{(d)}_a$*) *has finite rank*, *and the associated localisation formula* (20) (*resp.* (21)) *holds for the rank-generating series* (17) (*resp.* (18)).

Remark 2.18. If the higher cohomology groups do not vanish one could replace (10) by the super space

$$\widetilde{\mathcal{H}}_q = H^{\text{even}}(M_q, L) \oplus H^{\text{odd}}(M_q, L),$$

in which case formulæ (20) and (21) hold for the super representations $\widetilde{\mathcal{H}}_q$ of T_q and T'_q. (In this setup one need not assume that M_q be a Stein space or an affine scheme.)

Remark 2.19. Alternatively, in the setting of [90], Wu's localisation results (Theorem 3.14 of the same work) yield the generating series (17) and (18) by an index computation of the fixed-point locus for the T_q- and T'_q-action, respectively.

2I. Sp(1)-*symmetric hyperkähler potentials.*

Definition 2.20. A hyperkähler potential on the hyperkähler manifold (M, g, I, J, K) is a smooth map $\mu : M \to \mathbb{R}$ such that $\omega_q = i\,\partial_q \bar{\partial}_q \mu$ for every $q \in \mathbb{C}P^1$.

One can also use such potentials to obtain equivariant prequantum data, as discussed below. Assume further that μ is Sp(1)-invariant and that it generates the T_q-actions, i.e., $i\mu : M_q \to i\mathbb{R} \simeq \mathfrak{t}_q^\vee$ is a moment map.

In this case we consider the trivial Hermitian line bundle, and lift the G-action by the identity on each fibre. Natural symplectic potentials are given by

$$\theta_q = \tfrac{1}{2}(\bar{\partial}_q \mu - \partial_q \mu) \in \Omega^1(M),$$

hence $\nabla_q = \mathrm{d} + (\theta_q/\hbar)$ is a prequantum connection for all $q \in \mathbb{C}P^1$, and the resulting prequantum data are G-equivariant since μ is Sp(1)-invariant.

Now if $\operatorname{grad}(\mu)$ is complete then each T_q-action extends holomorphically to $\mathbb{C}^\times$, and if in addition μ is proper then the subspaces $\mathcal{H}_q^{(d)}$ are finite-dimensional by Theorem 2.14.

Proposition 2.21. *Suppose that M admits a G-invariant hyperkähler potential μ which, for every $q \in \mathbb{C}P^1$, is also an ω_q-moment map for the T_q-action. Assume moreover that μ is bounded below and that it has finitely many critical values. Then for every $q \in \mathbb{C}P^1$ the function $\psi_0 := e^{-\mu/2\hbar}$ is square-integrable, and it is a holomorphic frame for the prequantum line bundle constructed above.*

Proof. Nonvanishing and holomorphicity are a straightforward consequence of the definition.

On the other hand, the L^2-square-norm of ψ_0 can be expressed as

$$\|\psi_0\|^2_{\mathrm{L}^2} = \int_M e^{-\mu/\hbar}\,\mathrm{d}\,\mathrm{vol} = \int_B^\infty e^{-\xi/\hbar}\,\mu_*(\mathrm{d}\,\mathrm{vol}), \tag{22}$$

where $B \in \mathbb{R}$ is a lower bound for μ and $\mu_*(\mathrm{d}\,\mathrm{vol})$ the push-forward of the Liouville measure. By the Duistermaat–Heckman theorem [36] the push-forward admits a density which restricts to a polynomial on every interval $I \subset \mathbb{R}$ not containing critical values for μ. Since there are finitely many such values, (22) splits as a finite sum of converging integrals. □

By construction, the compact torus $T_q \simeq \mathrm{U}(1)$ acts on the complex vector space of holomorphic functions on M_q — by (inverse) pullback. By definition, such a function is *d-homogeneous* if it transforms (under the T_q-action) in the irreducible representation corresponding to the character $z \mapsto z^d \in \mathrm{U}(1)$, where $d \in \mathbb{Z}$. Under the assumptions of Proposition 2.21 we thus get an isomorphism

$$\Psi : \mathrm{L}^2\, H^0(M_q, \mathcal{O}, e^{-\mu/\hbar}\,\mathrm{d}\,\mathrm{vol})^{(d)} \to \mathcal{H}_q^{(d)},$$

given by $\Psi(f) = f\psi_0$, where the left-hand side denotes the space of d-homogeneous holomorphic functions with finite L^2-norm with respect to $e^{-\mu/\hbar}\,\mathrm{d}\,\mathrm{vol}$.

3. Examples of applications

3A. ***Hyperkähler vector spaces.*** Let $n > 0$ be integer and V a real vector space of dimension $4n$.

Definition 3.1. A linear hyperkähler structure on V is a scalar product g and an ordered triple (I, J, K) of orthogonal automorphisms of V satisfying the quaternionic identities $I^2 = J^2 = K^2 = IJK = -\mathrm{Id}_V$.

Equivalently, a linear hyperkähler structure on V may be regarded as a Hermitian representation of the quaternion algebra

$$\mathbb{H} = \{q = d + ai + bj + ck \mid a, b, c, d \in \mathbb{R}\},$$

on V, where the quaternionic Hermitian form is

$$\boldsymbol{h} := g - i\omega_I - j\omega_J - k\omega_K, \qquad \text{with} \quad \omega_\bullet(v, w) := g(\bullet\, v, w) \quad \text{for } \bullet \in \{I, J, K\}.$$

It follows that I, J, K are g-skew-symmetric, and hence they span a real Lie subalgebra $\mathfrak{su}(2)_{\mathrm{V}} \subseteq \mathfrak{o}(V, g)$.

Attached to the hyperkähler vector space is the group $\mathrm{Sp}(V, \boldsymbol{h}) \subseteq O(V, g)$ of $\mathbb{R}$-linear endomorphisms of V preserving $\boldsymbol{h}$ — and hence g and each of the forms $\omega_I, \omega_J, \omega_K$. As above we are interested in transformations that preserve the hyperkähler structure in a looser sense, but here we restrict to *linear* ones:

$$\mathrm{Hk}(V) = \mathrm{Hk}(V, g, I, J, K) := \{A \in O(V, g) \mid \mathrm{Ad}_A(\mathfrak{su}(2)_{\mathrm{V}}) = \mathfrak{su}(2)_{\mathrm{V}}\}.$$

As a subgroup of $\mathrm{O}(V, g)$, the above is compact.

Remark 3.2. We are thus slightly abusing the notation from Section 2. Indeed if V is regarded as a smooth hyperkähler manifold then the group of *all* transformations preserving g and $\mathfrak{su}(2)_{\mathrm{V}}$ also contains the translations, and it is in fact generated by these two kinds of transformations. We shall still denote this subgroup $\mathrm{Hk}(V)$ in the linear case to simplify the notation.

Remark 3.3. In this case the twistor space is a rank-$2n$ holomorphic vector bundle $\pi_{\mathbb{C}P^1} : Z \to \mathbb{C}P^1$ isomorphic to $\underline{\mathbb{C}^{2n}} \otimes \mathscr{O}(1)$ (in the straightforward generalisation of the case $n = 1$ from [46, Example 2.4, p. 143]).

Lemma 3.4. *There is an exact sequence of Lie groups*:

$$1 \to \mathrm{Sp}(V, \boldsymbol{h}) \to \mathrm{Hk}(V) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(3) \to 1,$$

and an embedding $\sigma : \mathrm{Sp}(1) \to \mathrm{Hk}(V)$ *such that* $\mathrm{Ad} \circ \sigma : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ *is the natural surjection.*

Proof. The natural $\mathrm{Sp}(1)$-action on $\mathbb{H}$ by multiplication on the right induces the standard $\mathrm{Sp}(1)$-action on the unit sphere of complex structures $\mathbb{S}_{IJK}$. The conclusion follows from a choice of identification $V \simeq \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}^n$ as $\mathbb{H}$-module. □

Hence a choice of orthonormal basis for $(V, \boldsymbol{h})$ (as a left $\mathbb{H}$-module) yields an identification

$$\mathrm{Hk}(V) = \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \simeq (\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\mathbb{Z}_2.$$

Choosing $G = \mathrm{Hk}(V)$, we see that in the notation of the introduction we have

$$G_0 = \mathrm{Sp}(V, \boldsymbol{h}) \subseteq \mathrm{Hk}(V).$$

Geometric quantisation. Geometric quantisation on a Kähler vector space is straightforward and essentially unique up to the choice of a symplectic potential, which corresponds to a gauge choice on the prequantum line bundle. For $\hbar \in \mathbb{R}_{>0}$ one considers the triple (L, h, ∇_q), consisting of the trivial complex line bundle $L := V \times \mathbb{C} \to V$ with the tautological Hermitian metric h, and the connection $\nabla_q := d - \frac{i}{\hbar}\theta_q$ defined by the invariant symplectic potential

$$\theta_q(v)(X) = \tfrac{1}{2}\omega_q(v, X),$$

for $v \in V$ a point and X a tangent vector there. The above yields prequantum data for (V, ω_q) at level $\hbar^{-1}$. We may denote $L_q \to V$ the line bundle to emphasise the structure we are prequantising on V.

The bundle L_q comes endowed with a natural holomorphic frame

$$\psi_0(q, v) := \exp\left(-\frac{1}{4\hbar} g(v, v)\right),$$

which is manifestly independent of $q \in \mathbb{C}P^1$. For each q, the resulting quantum Hilbert space consists of sections $\psi = f\psi_0$, with $f : V \to \mathbb{C}$ an I_q-holomorphic *function* with finite L^2-norm with respect to the Gaussian measure. This space is well known to be densely generated by the polynomial functions, which induces a grading on each $\mathcal{H}_q$ — the Fock grading.

This setting is a particular case of the one discussed in Section 2I. Indeed, on a Kähler vector space, the function $\mu(v) = \frac{1}{2}\|v\|^2$ is a moment map for the $\mathrm{U}(1)$-action by scalar multiplication and a Kähler potential, and moreover

$$-\frac{i}{2}(\partial - \bar{\partial})\mu = \theta$$

is the invariant symplectic potential. Additionally, for each $q \in \mathbb{C}P^1$ the action of T_q is the standard one.

Furthermore d-homogeneous holomorphic functions on a complex vector space are d-homogeneous polynomials, whence the decomposition of $\mathcal{H}_q$ into isotypical components as a T_q-module reduces to the well-known Fock grading. By the identification of the space of such homogeneous polynomials with $\mathrm{Sym}^d V_q^\vee$, the finite-dimensional spaces $\mathcal{H}_q^{(d)}$ assemble into finite-rank Hermitian subbundles $\mathcal{H}^{(d)} \to \mathbb{C}P^1$ of the trivial $\mathrm{L}^2(V, L)$-bundle, with a natural isomorphism

$$\mathrm{Sym}^d Z^\vee \to \mathcal{H}^{(d)}$$

of vector bundles over the Riemann sphere.

Group action on quantum spaces. The action $\rho^Z : \mathrm{Hk}(V) \to \mathrm{Aut}(Z)$ has a natural lift to $L = Z \times \mathbb{C}$ as $\rho^Z \times \mathrm{Id}$. Since $A^*\theta_q = \theta_{A.q}$ for $A \in \mathrm{Hk}(V)$ and $q \in \mathbb{C}P^1$, it follows that this action preserves the structure of L as a family of prequantum line bundles. This defines an action $\rho^{\mathcal{H}}$ on sections of $\mathcal{H}^{(d)}$ by pull-back, as in (12), and it is easy to check this is a graded fibrewise unitary $\mathrm{Hk}(V)$-action — covering that on the hyperkähler 2-sphere.

Theorem 3.5. *For $q \in \mathbb{C}P^1$ there is a canonical isomorphism $\mathcal{H}_q^{(d)} \simeq \mathrm{Sym}^d(V)$ of simple $\mathrm{Sp}(V, \boldsymbol{h})$-modules, and the bundle with connection $(\mathcal{H}^{(d)}, \nabla^{\mathcal{H}^{(d)}})$ is $\mathrm{Hk}(V)$-equivariantly isomorphic to $\mathcal{L}^d \otimes \underline{\mathrm{Sym}^d(V)} \to \mathbb{C}P^1$.*

Proof. This follows directly from the above discussion and from Theorem 2.11: The metric g, and hence the section ψ_0, are fixed by $\mathrm{Sp}(V, \boldsymbol{h})$. It is known the natural action on $\mathrm{Sym}^d V_q^\vee$ is irreducible [76]. □

Altogether the statements of this section establish the assumptions needed to apply Theorem 1.2, which in this particular case yields the following.

Theorem 3.6 (see Theorem 1.2). *The $\mathrm{Sp}(1)$-symmetric geometric quantisation of the hyperkähler vector space V yields the super Hilbert space*

$$H = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^{(d)}},$$

in analogy with Section 2F. This carries a unitary $\mathrm{Hk}(V)$-representation preserving the splitting, and there is an isomorphism $H^{(d)} \simeq W^{(d)} \otimes \mathrm{Sym}^d(V)$ of simple $\mathrm{Hk}(V)$-modules.

For every $d \geq 0$ we thus have

$$\dim(W_+^{(d)}) = (d+1), \quad \dim(W_-^{(d)}) = 0, \quad \dim(H^{(d)}) = (d+1)\binom{2n+d}{d}.$$

The generating series (17) and (18) are obtained explicitly from the above:

$$H(t) = \frac{1}{(1-t)^{2n}}, \quad H'(t, \tilde{t}) = \frac{1}{\prod_{i=1}^n (1 - t\tilde{t}_i)(1 - t\tilde{t}_i^{-1})}.$$

On the other hand, since $V_q \simeq \mathbb{C}^{2n}$ is a Stein space, and since the actions of T_q and T'_q only fix the origin, Theorem 2.17 also applies, and the result from (20) and (21) yields the same formulæ. Now by Theorem 3.6 we see that $m^{(d)}_{\mathrm{Sym}^d(V)} = 1$ for $d \in \mathbb{Z}_{\geq 0}$, whence

$$G(t, \tilde{t}) = \sum_{d=0}^{\infty} t^d \, \tilde{t}^{\lambda_{\mathrm{Sym}^d(V)}}.$$

3B. *Four-dimensional examples.* As mentioned in the introduction, in dimension 4 there is a complete classification of Sp(1)-symmetric hyperkähler manifolds up to finite quotients. Besides $\mathbb{H}$ with its flat metric there are the Taub–NUT metrics on $\mathbb{R}^4$, and the hyperkähler metric on the moduli space of charge-2 monopoles, i.e., the Atiyah–Hitchin manifold M_{AH}.

Taub–NUT metrics. Consider the case of $M = \mathbb{R}^4$ with the Taub–NUT metric g^a corresponding to a positive real parameter a — the case $a = 0$ corresponds to the standard flat metric on $\mathbb{H}$, which we already discussed. We will denote ω^a_q the corresponding symplectic structures. It is well known (e.g., [39, Remark 1]) that

$$\mathrm{Hk(M)} \simeq (\mathrm{Sp}(1) \times \mathrm{U}(1))/\mathbb{Z}_2 \simeq \mathrm{U}(2).$$

In particular there is a faithful Sp(1)-action rotating the sphere of hyperkähler structures, while $\mathrm{Sp}(M) = \mathrm{U}(1)$ is compact and commutes with Sp(1). Furthermore there exists, unique up to isomorphism, a family of prequantum line bundles for M, since $H^2(M, \mathbb{Z}) = 0 = H^1(M, \mathrm{U}(1))$.

The action of $T'_q = (\mathrm{U}(1) \times \mathrm{U}(1))/\mathbb{Z}_2 \subseteq \mathrm{Hk}(M)$ is studied explicitly by Gauduchon in [39, Section 3.2] for the complex structure J_+ corresponding to $q = i$. The subgroup is identified in that context with $\mathrm{U}(1) \times \mathrm{U}(1)$ via the isomorphism $(t, s) \mapsto (ts, ts^{-1})$. From equations (3.10) and (3.19) of the same work one concludes that the action of $T_q = \mathrm{U}(1) \times \{1\}$ on M_q is Hamiltonian with moment map $\mu_q = \mu^+_1 + \mu^+_2$ (borrowing Gauduchon's notation), which is easily seen to be proper from the definitions. Finally [39, Proposition 1] provides a biholomorphism $\Phi^a_+ = \Phi : (M, J_+) \to \mathbb{C}^2$, and by a straightforward check this map intertwines the T_q-action on M_q with the standard U(1)-action on $\mathbb{C}^2$. In particular the T_q-action extends holomorphically to $\mathbb{C}^*$, and the hypotheses of Theorem 2.14 are verified.

Thus decomposing $\mathcal{H}_q$ with respect to the T_q-action yields

$$\mathcal{H}_q = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathcal{H}^{(d)}_q},$$

where the subspaces $\mathcal{H}^{(d)}_q \subseteq \mathcal{H}_q$ are finite-dimensional. Then we consider the action of the commuting compact group $\mathrm{Sp}_0(\mathrm{M}) = \{1\} \times \mathrm{U}(1)$ on $\mathcal{H}^{(d)}_q$ to refine:

$$\mathcal{H}^{(d)}_q = \bigoplus_{d' \in \Lambda^{(d)}} \mathcal{H}^{(d)}_{d',q},$$

where $\Lambda^{(d)} \subseteq \mathbb{Z}_{\geq 0}$ is finite. In addition, we also have the following statement.

Proposition 3.7. *For $q = i$, the prequantum line bundle L_q admits a T_q-invariant holomorphic frame ψ_q such that $\Phi^*(f) \cdot \psi_q$ is L^2 for every polynomial function f on $\mathbb{C}^2$.*

Proof. Recall that, again in the notations of [39], x_1, x_2, and x_3 are three real-valued functions on M whose span is preserved by Sp(1), which acts on them by rotations in the standard way. Furthermore, all three functions are fixed by the action of $\mathrm{U}(1) = \mathrm{Sp}(M)$. Writing $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, it follows from (3.19), (2.11), and (2.12) of [39] that the aforementioned moment map μ can be expressed as

$$\mu = r + a^2(x_2^2 + x_3^2).$$

Since $\mu_i := \mu$ is a moment map for T_i with respect to ω_i^a, it follows that for every $g \in \mathrm{Sp}(1)$ the function $(g^{-1})^*\mu$ generates the $T_{g.i}$-action with respect to $\omega_{g.i}^a$. In particular, if $g.i = j$, then the flow associated to

$$\mu_j := (g^{-1})^*\mu = r + a^2(x_1^2 + x_3^2)$$

with respect to ω_j^a rotates the circle spanned by ω_3^a and ω_1^a. It is therefore a Kähler potential for ω_i^a [47], and repeating the argument when $g.i = k$ so is $\mu_k := r + a^2(x_1^2 + x_2^2)$. Therefore the T_i-invariant function

$$\varphi = \varphi_i := \frac{\mu_j + \mu_k}{2} = r + \frac{a^2}{2}(r^2 + x_1^2)$$

is also a Kähler potential. It follows that for every $g \in \mathrm{Sp}(1)$ the function $(g^{-1})^*\varphi_i$ is completely determined by $q = g.i$, so that

$$\varphi_q := (g^{-1})^*\varphi_i$$

is well defined, and a potential for ω_q^a. From this we obtain an explicit realisation of the family of prequantum line bundles, for which the functions $\psi_{0,q} = e^{-\frac{1}{2\hbar}\varphi_q}$ define holomorphic frames.

Now note the function μ is bounded below, and its only critical point is the origin — the only fixed point of the induced action. We may then apply the Duistermaat–Heckman theorem [36] as in Proposition 2.21 to conclude that $e^{-\alpha\mu}$ is integrable with respect to the Taub–NUT volume $\mathrm{d}\,\mathrm{vol}^a$ for every parameter $\alpha \in \mathbb{R}_{>0}$. The same clearly applies to μ_j and μ_k, and from this it is easily deduced that

$$e^{-\alpha\varphi_q} \in \mathrm{L}^2(M, \mathrm{d}\,\mathrm{vol}^a)$$

for every $q \in \mathbb{C}P^1$ and $\alpha > 0$. In particular, the holomorphic frames constructed above are L^2.

To conclude we recall that the two components of Φ are defined as

$$w_1 = e^{a^2 x_1} z_1, \quad w_2 = e^{-a^2 x_2} z_2,$$

where z_1 and z_2 are the standard i-holomorphic coordinates on $M = \mathbb{H}$ with respect to the usual flat metric (see [39, (3.4)]). We need to show that, for every $n, m \in \mathbb{Z}$, the function $w_1^n\, w_2^m \psi_0$ is also L^2. Expanding the definition of φ yields

$$\begin{aligned}\frac{1}{\hbar}\varphi - 2a^2(n-m)\,x_1 &= \frac{r}{\hbar} + \frac{a^2}{2\hbar}(r^2 + x_1^2) - 2a^2(n-m)\,x_1 \\ &\geq \frac{r}{2\hbar} + \frac{a^2}{4\hbar}(r^2 + x_1^2) - C = \frac{1}{2\hbar}\varphi - C \geq \frac{1}{4\hbar}\mu_j - C,\end{aligned}$$

provided $C \in \mathbb{R}_{>0}$ is large enough. Furthermore, it is a simple consequence of the definitions in [39] that $|z_1|^2 + |z_2|^2 = 2r$, whence

$$|z_1^n\, z_2^m|^2 \leq (2r)^{2(n+m)} \leq \mu_j^{2(n+m)}.$$

Collecting the estimates and using again the Duistermaat–Heckman theorem we conclude

$$\int_M |w_1^n\, w_2^m|^2\, \psi_0^2\, \mathrm{d}\,\mathrm{vol}^a \leq e^C \int_M \mu_j^{2(m+n)} e^{-\frac{1}{4\hbar}\mu_j}\, \mathrm{d}\,\mathrm{vol}^a < \infty. \qquad \square$$

As a consequence of this result we have $\dim \mathcal{H}_{d'}^{(d)} = 1$ for all $d \in \mathbb{Z}_{\geq 0}$ and $d' = d - 2j$ with $j \in \{0, \dots, d\}$, and we conclude that $\mathcal{H}_{d'}^{(d)} \simeq \mathcal{L}^d$ for such values of d and d'.

Theorem 3.8. *The generating series* (18) *is the same as for the flat metric*, *namely*

$$H'(t, \tilde{t}) = \frac{1}{(1 - t\tilde{t})(1 - t\tilde{t}^{-1})}.$$

We thus have

$$H = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^{(d)}}, \quad H^{(d)} = \bigoplus_{d' \in \Lambda^{(d)}} H_{d'}^{(d)} = H^0(\mathbb{C}P^1, \mathcal{L}^d)^{\oplus(d+1)}.$$

The Atiyah–Hitchin manifold. Let us consider the Atiyah–Hitchin manifold M_{AH}, the last four-dimensional case. We shall discuss the extent to which our methods apply here.

The Atiyah–Hitchin manifold can be realised as the moduli space of charge-2 centred magnetic monopoles in $\mathbb{R}^3$, and it comes with a natural Riemannian metric preserved by the SO(3)-action induced by rotating monopoles. The quaternionic nature of the Bogomolny equation, of which the monopoles represented by M_{AH} are a particular class of solutions, induces a family of almost complex structures, which can be better understood via Donaldson's description in terms of rational maps [34]. More precisely, the choice of an oriented line through the origin in $\mathbb{R}^3$ induces an identification

$$\widetilde{M}_{\mathrm{AH}} = \left\{ S(z) = \frac{uz + v}{z^2 - w} \in \mathbb{C}(z) \;\middle|\; v^2 - wu^2 = 1 \right\} =: R_2^0,$$

where the left-hand side denotes the (two-fold) universal cover of M_{AH}. The Atiyah–Hitchin manifold is recovered from the monodromy action, generated by $(u, v, w) \mapsto (-u, -v, w)$. The resulting map is a biholomorphism with respect to one of the aforementioned almost complex structures, establishing that the latter is integrable and the former is Kähler. Rotations around the preferred direction induce a U(1)-action of R_2^0 by

$$t.(u, v, w) = (tu, v, t^{-2}w). \tag{23}$$

As the preferred direction changes across all possible choices, this results in a family of Kähler structures parametrised by $\mathbb{C}P^1$, which is clearly rotated by the SO(3)-action (see [15, Chapter 2]).

The above identification is not isometric with respect to the Riemannian embedding $R_2^0 \subseteq \mathbb{C}^3$; nonetheless, the Riemannian structure on M_{AH} can be described by studying the SO(3)-orbits [15, Chapters 8–11]. The generic stabiliser of a monopole is the Klein four-group K_4, while orbits are parametrised by $k = \sin(\alpha)$ for an angle $\alpha \in \left[0, \frac{\pi}{2}\right]$, resulting in a description of an open dense of M_{AH} as the product $(0, 1) \times \mathrm{SO}(3)/K_4$; furthermore, as $k \to 0$ the orbit degenerates to a diffeomorphic copy of $\mathbb{R}P^2$, onto which M_{AH} deformation-retracts.

According to Swann's work [82, Section 6, Four-manifolds], M_{AH} does not admit a hyperkähler potential. Furthermore, one sees from (23) that the stabiliser of each Kähler structure has exactly one fixed point, and since the manifold has the homotopy type of $\mathbb{R}P^2$ there can be no proper moment map. Nonetheless the above homotopy equivalence yields

$$H^1(M_{\mathrm{AH}}, \mathrm{U}(1)) \simeq H^2(M_{\mathrm{AH}}, \mathbb{Z}) \simeq \mathbb{Z}_2.$$

Hence by Section 2B there are exactly two inequivalent SO(3)-equivariant families of prequantum line bundles. They differ by a twist by a family of flat connections on the nontrivial complex line bundle on M_{AH}.

The family supported on the trivial bundle can be constructed by means of the Kähler potentials of Olivier [72]. Namely the metric on the Atiyah–Hitchin manifold is the completion of

$$\mathrm{d}s^2 = \frac{\beta^2\gamma^2\delta^2}{(4k^2(1-k^2)K^2)^2}\mathrm{d}m^2 + \beta^2\sigma_x^2 + \gamma^2\sigma_y^2 + \delta^2\sigma_z^2, \tag{24}$$

defined on $\left(0, \frac{\pi}{2}\right) \times \mathrm{SO}(3)/K_4$. We follow the conventions of [72]. Namely, $m = k^2$ is used as a coordinate in place of k, while $(\sigma_x, \sigma_y, \sigma_z)$ is an orthonormal frame of $\mathrm{T}^*\mathrm{SO}(3) \to \mathrm{SO}(3)$ and the coefficients β, γ, δ are functions of k determined by

$$\beta\gamma = -EK, \quad \gamma\delta = -EK + K^2, \quad \beta\delta = -EK + (1-k^2)K^2,$$

where

$$K := K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}}, \quad E := E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\phi}\,\mathrm{d}\phi$$

are the complete elliptic integrals of the first and second kind, respectively.

Oliver [72] then uses the Euler angles (φ, θ, ψ) as coordinates on SO(3) to give an explicit Kähler potential Ω for one of the complex structures, say I_3, preserved by rotations in the angle φ. This is given in of [72, (55)] and can be written explicitly using equations (6), (24), (25) and (36) therein, getting the formula

$$\Omega = \frac{\beta\gamma+\gamma\delta+\delta\beta}{8} + \tfrac{1}{8}(\gamma\delta\sin^2\theta\cos^2\psi + \delta\beta\sin^2\theta\sin^2\psi + \gamma\beta\cos^2\theta).$$

Note for $k \in (0, 1)$ this function extends continuously to the whole of SO(3), and the trigonometric functions of (θ, ψ) descend to the projective space at $k = 0$; hence the potential extends to the completion M_{AH}. Finally, we emphasise that this potential is independent of the variable φ, which is to say that it is invariant under the action of the I_3-stabiliser. It follows that Ω defines an equivariant family of potentials under the SO(3)-action, whence an equivariant family of prequantum line bundles by the usual construction, together with a holomorphic frame $\psi_0 = e^{-\frac{1}{2\hbar}\Omega}$ for I_3.

Proposition 3.9. *The function $e^{-\alpha\Omega}$ is integrable on M_{AH} for $\alpha \in \mathbb{R}_{>0}$.*

Proof. From (24) we obtain the following expression for the volume form on (the complement of a negligible set in) M_{AH}:

$$\mathrm{d}\,\mathrm{vol} = \frac{\beta^2\gamma^2\delta^2}{4k^2(1-k^2)K^2}\,\mathrm{d}m\sigma_x\sigma_y\sigma_z.$$

We need to show that

$$\int_{(0,1)\times\mathrm{SO}(3)} e^{-\alpha\Omega}\frac{\beta^2\gamma^2\delta^2}{4k^2(1-k^2)K^2}\,\mathrm{d}m\sigma_x\sigma_y\sigma_z < \infty.$$

Note that $\beta\gamma \leq 0$, $\gamma\delta \geq 0$, and $\beta\delta \leq 0$ yield

$$\Omega \geq \frac{\gamma\delta}{8}.$$

We may then use these bounds and the Fubini–Tonelli theorem to reduce the statement to

$$\int_0^1 e^{-\frac{\alpha}{8}\gamma\delta}\frac{\beta^2\gamma^2\delta^2}{k^2(1-k^2)K^2}\,\mathrm{d}m < \infty.$$

We will proceed by studying the asymptotic behaviour of the integrand in the limit $k \to 1$; the integral is necessarily regular for $k \to 0$. It is well known that

$$K \sim \tfrac{1}{2}\log(1-k^2),$$

and since $E(1) = 1$ we find that

$$\beta\gamma \sim -\tfrac{1}{2}\log(1-k^2), \quad \gamma\delta \sim \tfrac{1}{4}\log^2(1-k^2), \quad \beta\delta \sim -\tfrac{1}{2}\log(1-k^2),$$

and hence the integral converges by comparison with

$$\int_0^1 \exp\left(-\frac{\alpha}{32}\log^2(1-k^2)\right)\frac{\log^2(1-k^2)}{(1-k^2)}\mathrm{d}m = \int_0^\infty e^{-\frac{\alpha}{32}x^2}x^2\,\mathrm{d}x < \infty,$$

which concludes the proof. □

For $\alpha = 1/\hbar$ this implies the holomorphic frame ψ_0 is L^2, and hence an element of $\mathcal{H}^{(0)}_{I_3}$; in principle more L^2 holomorphic sections may be found considering functions of the holomorphic coordinates (u, v, w) on R^2_0. If all the monomials that descend to M_{AH} are L^2, then one concludes that $\mathcal{H}^{(d)}_q$ has infinite rank for every integer d, since $u^a v^b w^c$ is $(a-2c)$-homogeneous and well defined on M_{AH} if $a+b$ is even. We obtain a partial result in this direction, showing that all powers of w are L^2.

The problem of describing u, v, and w in terms of the setup above is addressed in [15, Chapter 6-7], by making use of the twistor description and spectral curves [50]. Introducing parameters

$$k_1 = \frac{\sqrt{k\sqrt{1-k^2}}K}{2}, \quad k_2 = \frac{1-2k^2}{3k\sqrt{1-k^2}},$$

consider the elliptic curve

$$y^2 = 4k_1^2(x^3 - 3k_2x^2 - x)$$

and let $\wp$, ζ be its corresponding Weierstrass functions, η the real period of ζ. Suppose that $a, b \in \mathbb{C}$ are the entries of a matrix in $\mathrm{SU}(2)$, thought of as a parametrisation of $\mathrm{SO}(3)/K_4$, and let $\xi \in \mathbb{C}$ be such that

$$\wp(\xi) = \frac{b}{\bar{a}} - k_2. \tag{25}$$

Then the corresponding point in M_{AH} has holomorphic coordinates

$$\begin{aligned}
u &= \frac{\sinh\left(2k_1\zeta(\xi) - \frac{\eta\xi}{2} + k_1\bar{a}\bar{b}\wp'(\xi)\right)}{k_1\bar{a}^2\wp'(\xi)},\\
v &= \cosh\left(2k_1\zeta(\xi) - \frac{\eta\xi}{2} + k_1\bar{a}\bar{b}\wp'(\xi)\right),\\
w &= k_1^2\bar{a}^4\wp'(\xi)^2,
\end{aligned}$$

up to the sign ambiguity resulting from the monodromy. Substituting (25) in the differential equation for $\wp$, and using g_2 and g_3 as given in [50], we obtain

$$w = k_1^2\bar{a}(-12\bar{a}b^2k_2 + 4b^3 - 4\bar{a}^2b).$$

Now since $|a|^2 + |b|^2 = 1$ a straightforward check shows that

$$|w|^2 \leq 16k_1^4(9k_2^2 + 2) \sim 4K^2 \sim \log^2(1 - k^2)$$

for $k \to 1$. Adapting the proof of Proposition 3.9 and using (23) we obtain:

Proposition 3.10. *For every integer $n \geq 0$ the holomorphic section $w^n \psi_0$ is* L^2 *and therefore an element of* $\mathcal{H}_{I_3}^{(-2n)}$.

The analysis is more delicate for the functions u and v. Using (25) one can express $\bar{a}\bar{b}$ in terms of ξ and write the argument of the hyperbolic functions as

$$\Phi(\xi) = 2k_1 \zeta(\xi) - \frac{\eta \xi}{2} + k_1 \wp'(\xi) \frac{k_2 + \overline{\wp(\xi)}}{1 + |k_2 + \wp(\xi)|^2}.$$

It follows from the definitions and the Legendre relation that this function is periodic for the real period of $\wp$ and quasiperiodic for the imaginary period, with step πi, whence the sign ambiguity of u and v. Moreover one can show the poles of the summands cancel out, leaving a nonholomorphic analytic function — hallmark of the fact that the SO(3)-action does not preserve the complex structure. In particular its real part is bounded for fixed “k”.

3C. ***Moduli spaces of framed* SU(r)*-instantons.*** Let $r \geq 2$ and $k \geq 0$ be integers, and consider the moduli space $M_{k,r}$ of charge-k framed SU(r)-instantons on $\mathbb{R}^4$, which is a hyperkähler manifold [13; 33]. Each of its complex structures can be described in terms of the ADHM construction as follows, after fixing an identification $\mathbb{R}^4 \simeq \mathbb{C}^2$. Consider the product

$$\mathbb{M} := \mathrm{End}(\mathbb{C}^k)^2 \times \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^r) \times \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k),$$

with GL($\mathbb{C}^k$)-action given by

$$g.(\alpha_0, \alpha_1, a, b) = (g\alpha_0 g^{-1}, g\alpha_1 g^{-1}, ga, bg^{-1}).$$

Remark 3.11. $\mathbb{M}$ is a space of representations of a quiver on two nodes and that the action naturally extends to $\mathrm{GL}(\mathbb{C}^k) \times \mathrm{GL}(\mathbb{C}^r)$ (which controls isomorphisms of representations).

Let $\mathbb{M}_0$ denote the set of elements of $\mathbb{M}$ satisfying the additional conditions:

(i) $[\alpha_0, \alpha_1] + ab = 0$.

(ii) For all $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{pmatrix} \alpha_0 + \lambda \\ \alpha_1 + \mu \\ a \end{pmatrix}$$

injective and $\begin{pmatrix} \lambda - \alpha_0 & \alpha_1 - \mu & b \end{pmatrix}$ surjective.

Then the restricted $\mathrm{U}(r)$-action is Hamiltonian with moment map

$$\mu(\alpha_0, \alpha_1, a, b) := [\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^*a,$$

and there is an identification

$$M_{k,r} \simeq \mathbb{M}_0 /\!\!/_{\mu} \mathrm{U}(k). \tag{26}$$

The rotation group SO(4) acts on $M_{k,r}$, and in particular the subgroup Sp(1), in the identification $\mathbb{R}^4 \simeq \mathbb{H}$, transitively permutes the complex structures. Furthermore, Maciocia [62] shows that for each $q \in \mathbb{C}P^1$ the T_q-action has moment map

$$m_2(A) = \frac{1}{16\pi^2} \int_{\mathbb{R}^4} \|x\|^2 \operatorname{tr} F_A^2.$$

This function is clearly Sp(1)-invariant, and therefore a hyperkähler potential, so one can construct an Sp(1)-invariant family of prequantum line bundles endowed with holomorphic frames as in Section 2I.

The function m_2 is not, however, a proper map. By [62], under the identification (26) it corresponds to the norm-squared function $f : \mathbb{M}_0 \to \mathbb{R}$, which is $\mathrm{U}(k)$-invariant but not proper, on account of the open condition (ii). However, Donaldson [33] identifies the symplectic reduction (26) with the GIT quotient of $\mathbb{M}_0$ by $\mathrm{GL}(k, \mathbb{C})$, whereupon (ii) translates into a stability condition. One may then include the semistable points to obtain a partial compactification

$$\overline{M}_{k,r} := \mathbb{M} /\!\!/_{\mathrm{GIT}} \mathrm{GL}(k, \mathbb{C}),$$

which is smooth by the work of Nakajima and Yoshioka [71, Corollary 2.2]. The map f descends then to a proper one on $\overline{M}_{k,r}$; it is also clear that its gradient is complete on the quotient, showing that geometric quantisation on this space yields finite-rank isotypical components by Theorem 2.14. On the other hand, the codimension of the boundary $\overline{M}_{k,r} \setminus M_{k,r}$ is greater than 2, so that Hartogs's theorem allows for the extension of holomorphic functions on $M_{k,r}$, which yields the finite-dimensionality of the isotypic components over this latter space.

4. Outlook and further perspectives

There are more spaces that fit some of the requirements for our quantisation scheme.

By the work of Kronheimer [57] the nilpotent (co)adjoint orbits of complex semisimple (1-connected) Lie groups are hyperkähler manifolds with transitively permuting SO(3)-actions, and by Swann's work [82] they admit hyperkähler potentials. Indeed, Proposition 5.5 of the same work states that such a potential exists on a hyperkähler manifold if it admits an Sp(1)-action permuting the complex structure and such that, denoting by X_q the vector field generating the T_q-action for each $q \in \mathbb{C}P^1$, the vector field $I_q X_q$ is independent of q. After [82, Proposition 6.5], Swann goes on to check that this condition is verified for Kronheimer's space, thus establishing the existence of a hyperkähler potential. This is a particular instance of hyperkähler moduli spaces of solutions of Nahm's equations, specifically on a half-line with nilpotent boundary conditions.[3] Since Nahm's equations come naturally with a quaternionic structure and Sp(1)-action, the resulting manifolds have symmetries of the kind considered in this paper, and different choices of domain and boundary conditions give rise to different hyperkähler structures. For instance, semisimple boundary conditions on a half-line result in orbits of semisimple elements [56], while the study of Nahm's equations on a compact interval leads to the cotangent bundle T^*G [32; 58]. By the works of Mayrand [64; 65; 66], the latter comes with natural Sp(1)-equivariant families of Kähler potentials and moment maps for the stabilizers T_q, rather than a hyperkähler one, and they enjoy interesting properties that might lead to a variation of our main construction.

Also, as mentioned in the introduction, many new interesting hyperkähler metrics can be defined on moduli spaces of irregular singular connections/Higgs bundles over (wild generalisations of) Riemann surfaces [18; 77; 88], with simple examples reviewed in [23]: the "multiplicative" versions of the Eguchi–Hanson space and Calabi's examples (whose standard "additive" versions are quiver varieties on two nodes). This fits into a more general (new) multiplicative theory of quiver varieties [22], involving a "fission" operation generalising the construction of moduli spaces of flat connections *à la* TFT [19; 21]; note that conjecturally this produces a lot more new hyperkähler manifolds [20], beyond (wild) nonabelian Hodge spaces. See [26; 37; 74; 75] about quantum moduli spaces of meromorphic connections.

Finally the example of Section 3C, i.e., the moduli spaces of framed SU(r)-instantons, opens the way for further discussion on the relation between the generating series produced by this new quantisation scheme and the well-known Nekrasov partition functions.

Appendix: Comparison with the standard approach

In this section we shall correct the family of quantum Hilbert spaces $\mathcal{H}_q$ to obtain finite-rank flat vector bundles of isotypical components (under the main assumption), as well as unitary equivalences between the quantisation of M with respect to the given Kähler polarisations.

Based on Theorem 2.11, we do this by a correcting twist of the finite-rank bundles $\mathcal{H}^{(d)}_\lambda \to \mathbb{C}P^1$; namely consider the tensor product

$$\widetilde{\mathcal{H}}^{(d)}_\lambda := \mathcal{H}^{(d)}_\lambda \otimes \mathcal{L}^{-d}, \quad d \in \mathbb{Z},\ \lambda \in \Lambda^{(d)}.$$

[3] The hyperkähler metric on general orbits was constructed in [17; 55].

This new vector bundle comes with a Hk_0'(M)-action, and we denote $\nabla^{\widetilde{\mathcal{H}}_\lambda^{(d)}}$ the resulting Hk_0'(M)-invariant flat connection.

Since $\mathbb{C}P^1$ is simply connected the parallel transport defines canonical unitary isomorphisms

$$\widetilde{\mathcal{H}}_{q,\lambda}^{(d)} \to \widetilde{\mathcal{H}}_{q',\lambda}^{(d)}, \quad q, q' \in \mathbb{C}P^1, \tag{27}$$

satisfying 1-cocycle identities. In analogy with the above we then define

$$\overline{\bigoplus_{\lambda \in \Lambda^{(d)}} \widetilde{\mathcal{H}}_{q,\lambda}^{(d)}} =: \widetilde{\mathcal{H}}_q^{(d)} \subseteq \widetilde{\mathcal{H}}_q := \overline{\bigoplus_{d \in \mathbb{Z}} \widetilde{\mathcal{H}}_q^{(d)}},$$

and these families of Hilbert spaces carry a 1-cocycle of unitary isomorphisms induced from (27): This is the usual geometric quantisation construction.

Now we can introduce super Hilbert spaces $\widetilde{H}_{\lambda,j}^{(d)}$ in analogy with Section 2F, taking the holomorphic cohomology of the twisted vector bundles $\widetilde{\mathcal{H}}_\lambda^{(d)} \to \mathbb{C}P^1$.

Theorem A.1 (see Theorem 1.2). *There is a unitary action* Hk_0'(M) $\to \mathrm{U}(\widetilde{H})$ *preserving the nested splittings*

$$\widetilde{H} := \overline{\bigoplus_{d \in \mathbb{Z}} \widetilde{H}^{(d)}}, \quad \widetilde{H}^{(d)} := \overline{\bigoplus_{\lambda \in \Lambda^{(d)}} \widetilde{H}_\lambda^{(d)}}, \quad \widetilde{H}_\lambda^{(d)} := \bigoplus_{j=1}^{m_\lambda^{(d)}} \widetilde{H}_{\lambda,j}^{(d)}.$$

Finally we can compare this representation with the one constructed in Section 2F, finding that twisting trivializes part of the action. Namely, the present super Hilbert space $\widetilde{H}_{\lambda,j}^{(d)} \simeq V_\lambda \otimes W^{(0)}$ replaces the original $H_{\lambda,j}^{(d)} \simeq V_\lambda \otimes W^{(d)}$ as a Hk_0'(M)-module, recalling that $W^{(0)}$ is the trivial one-dimensional Sp(1)-module. This should be compared with the (more) interesting irreducible representations of Hk_0'(M) obtained from Theorem 1.2.

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COMBINATORICS OF THE TAUTOLOGICAL LAMINATION

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The *tautological lamination* arises in holomorphic dynamics as a combinatorial model for the geometry of 1-dimensional slices of the shift locus. In each degree q the tautological lamination defines an iterated sequence of partitions of 1 (one for each integer n) into numbers of the form $2^m q^{-n}$. Denote by $N_q(n, m)$ the number of times $2^m q^{-n}$ arises in the n-th partition. We prove a recursion formula for $N_q(n, 0)$, and a gap theorem: $N_q(n, n) = 1$ and $N_q(n, m) = 0$ for $\lfloor n/2 \rfloor < m < n$.

1. Introduction

The *tautological lamination*, introduced in [Calegari 2022], is a combinatorially defined object which gives a holomorphic model for certain 1 complex dimensional slices of the *shift locus*, a fundamental object in the theory of holomorphic dynamics. There is a shift locus $\mathcal{S}_q$ for each degree q; it is the space of depressed monic polynomials $z^q + a_2 z^{q-2} + a_3 z^{q-3} + \cdots + a_q$ in a complex variable z (thought of as a subset of $\mathbb{C}^{q-1}$ with coordinates a_j) for which every critical point is in the attracting basin of infinity.

There is a tautological lamination $\Lambda_T(C)$ for each degree q and for each choice of *critical data* C (certain holomorphic parameters which determine the slice of $\mathcal{S}_q$). For the complex dynamics reader: the tautological lamination records the combinatorics of the 1 complex dimensional slices of the shift locus where $q-2$ critical Böttcher coordinates are fixed, and one critical point (with a smaller escape rate than any of the others) is allowed to vary.

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Each tautological lamination determines a sequence of operations, called pinching, which cut the unit circle S^1 up into pieces and reglue them into a collection of smaller circles, denoted $S^1 \bmod \Lambda_{T,n}(C)$. Subsequent operations refine the previous ones, so each component of $S^1 \bmod \Lambda_{T,n}(C)$ is cut up and reglued into a union of components of $S^1 \bmod \Lambda_{T,n+1}(C)$. The precise cut and paste operations depend on C, but the set of *lengths* of the components of $\Lambda_{T,n}(C)$ (counted with multiplicity) depends only on n and the degree q. These lengths are all of the form $2^m q^{-n}$ for various nonnegative integers m, and we can define $N_q(n,m)$ to be the number of components of $S^1 \bmod \Lambda_{T,n}(C)$ of length $2^m q^{-n}$.

The *short components* of $S^1 \bmod \Lambda_{T,n}(C)$ are those with length q^{-n}. The number of short components is $N_q(n,0)$. Our first main result is an exact recursive formula for $N_q(n,0)$ (which can be solved in closed form):

Theorem 3.10 (recursive formula). *$N_q(n,0)$ satisfies the recursion $N_q(0,0)=1$, $N_q(1,0)=(q-2)$ and*

$$N_q(2n,0)=qN_q(2n-1,0) \quad \textit{and} \quad N_q(2n+1,0)=qN_q(2n,0)-2N_q(n,0)$$

and has the generating function $(\beta(t)-1)/qt$, where a closed form for $\beta(t)$ is given in Proposition 2.2.

At the other extreme, there is a unique largest component of $S^1 \bmod \Lambda_{T,n}(C)$ of length $2^n q^{-n}$. Our second main result is a “gap” theorem:

Theorem 5.11 (gap theorem). *$N_q(n,m)=0$ for $\lfloor n/2 \rfloor < m < n$.*

Both the recursive formula for $N_q(n,0)$ and the existence of a gap were observed experimentally. Our main motivation in writing this paper was to give a rigorous proof of these observations.

One of the striking things about the tautological lamination is the existence of a rather mysterious bijection between the components of $S^1 \bmod \Lambda_{T,n}(C)$ and some seemingly unrelated objects called *tree polynomials*, introduced in Section 4. This bijection is a corollary of one of the main theorems of [Calegari 2022], and the proof there is topological. We know of no direct combinatorial proof of this bijection, and raise the question of whether one can be found.

2. Unbordered words

Some words end like they begin, such as `abra·cad·abra` and `b·aoba·b`. Such words are said to be *bordered*. Others (most) are *unbordered*. A *border* is a *nonempty*, *proper* suffix of some word which is equal to a prefix.

If a word contains a border, then it contains one of at most half the length (for, a border of more than half the length will itself be bordered and now we can apply induction).

If W is a word, let's denote its length by $|W|$. If W is an unbordered word of even length, we can write it as $W = W_1 W_2$ where $|W_1| = |W_2|$, and then for every letter c the word $W_1 \mathsf{c} W_2$ is also unbordered. If W is an unbordered word of odd length, we can write it as $W = W_1 W_2$ where $|W_1| + 1 = |W_2|$, and then for every letter c the word $W_1 \mathsf{c} W_2$ is unbordered *except* when $W_1 \mathsf{c} = W_2$. Thus: if a_n denotes the number of unbordered words of length n in a q-letter alphabet, then $a_0 = 1$ (there is one empty word) and

$$a_{2n+1} = q a_{2n} \quad \text{and} \quad a_{2n} = q a_{2n-1} - a_n.$$

Let's define a generating function $\alpha(t) := \sum_{n=0}^{\infty} a_n t^n$. Then the recurrence becomes the functional equation

$$\alpha(t) = \frac{2 - \alpha(t^2)}{1 - qt}.$$

Iteratively substituting t^2 for t and being careful about convergence, one obtains the following formula:

$$\alpha(t) = 1 + q \sum_{j=0}^{\infty} (-1)^j t^{2^j} \prod_{i=0}^{j} \frac{1}{(1 - q \cdot t^{2^i})}.$$

These facts are not new. Unbordered words have been studied by many authors. They are also called *bifix-free*, and *primary* (neither of these terms seem very descriptive to us). As far as we know they were first considered by Silberger [1971]; see also, e.g., [Lothaire 1997, p. 153].

A minor variation on this idea is as follows. Let's take for our q-letter alphabet the elements of $\mathbb{Z}/q\mathbb{Z}$. If W is a word in the alphabet, let W' denote the result of adding 1 to the first letter (digit). Say a word is 1-*unbordered* if no suffix S is equal to a prefix P or to P' (and say it is 1-bordered otherwise). Then reasoning as above gives:

Proposition 2.1 (recursion). *Let b_n denote the number of* 1*-unbordered words of length n in a q-letter alphabet. Then $b_0 = 1$ and*

$$b_{2n+1} = q b_{2n} \quad \textit{and} \quad b_{2n} = q b_{2n-1} - 2 b_n.$$

Define the generating function $\beta(t) := \sum_{n-0}^{\infty} b_n t^n$. Then

$$\beta(t) = \frac{3 - 2\beta(t^2)}{1 - qt}.$$

The following "closed form" for $\beta(t)$ (and the argument below) was kindly provided by Frank Calegari:

Proposition 2.2 (closed form solution). *The generating function $\beta(t)$ converges for small $|t|$, and can be meromorphically continued throughout the unit disk with a simple pole at every 2^k-th root of $1/q$.*

Define a sequence of integers $h(n)$ by

$$h(0) := 1 \quad \textit{and} \quad h(n) := (-q)^{s(n)}(1-(-2)^{k(n)}) \quad \textit{for } n > 0,$$

where $2^{k(n)}$ is the biggest power of 2 dividing n, and $s(n)$ is the sum of the binary digits of n. Then throughout the unit disk,

$$\beta(t) = \left(\sum_{n=0}^{\infty} h(n)t^n\right) \prod_{j=0}^{\infty} \frac{1}{(1-qt^{2^j})}.$$

Proof. From the growth rate of the coefficients it's clear that $\beta(t)$ has a pole at q^{-1} and converges uniformly throughout the open disk of radius q^{-1}. It follows that $\beta(t^2)$ converges uniformly throughout the open disk of radius $q^{-1/2}$. Using the identity $(1-qt)\beta(t) = 3-2\beta(t^2)$ and induction, the first claim is proved.

Let's define $H(t) := \sum_{n=0}^{\infty} h(n)t^n$ and $B(t) := H(t)\prod_{j=0}^{\infty}(1-qt^{2^j})^{-1}$. Then the proposition will follow if we can show that $B(t)$ satisfies $B(t)(1-qt) = 3-2B(t^2)$. First observe that $h(n) = 0$ if n is odd; and furthermore,

$$\frac{2h(n)+h(2n)}{3} = \frac{(-q)^{s(n)}}{3}\big(3-2(-2)^{k(n)}+2(-2)^{k(n)}\big) = (-q)^{s(n)}.$$

The required identity is equivalent to

$$(1-qt)B(t)\prod_{k=1}^{\infty}(1-qt^{2^k}) = (3-2B(t^2))\prod_{k=1}^{\infty}(1-qt^{2^k})$$

or

$$B(t)\prod_{k=0}^{\infty}(1-qt^{2^k}) = (3-2B(t^2))\prod_{k=0}^{\infty}(1-q(t^2)^{2^k})$$

or

$$H(t)+2H(t^2) = 3\prod_{k=0}^{\infty}(1-q(t^2)^{2^k}).$$

Since $h(n) = 0$ for n odd this is equivalent to

$$\sum_{n=0}^{\infty} h(2n)t^{2n} + \sum_{n=0}^{\infty} 2h(n)t^{2n} = 3\prod_{k=0}^{\infty}(1-q(t^2)^{2^k}).$$

Replacing t^2 by t and using $h(2n)+2h(n) = 3(-q)^{s(n)}$ this is equivalent to

$$\sum_{n=0}^{\infty}(-q)^{s(n)}t^n = \prod_{k=0}^{\infty}(1-qt^{2^k})$$

which is clear. □

The definition of 1-unbordered words would seem utterly unmotivated — except that it just so happens that they arise naturally in an entirely different context which is the subject of the rest of the paper.

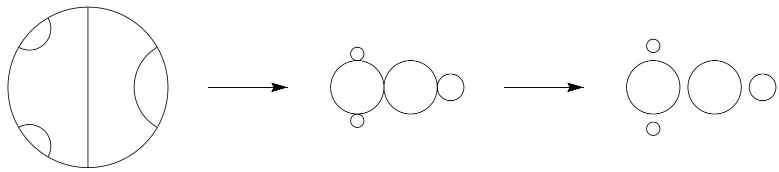

Figure 1. Pinching a circle along a finite lamination to obtain a collection of smaller circles.

3. Tautological laminations

3A. ***Laminations.*** A *leaf* is an unordered pair of distinct points in a circle S. Two leaves in S are *linked* if they are disjoint (as subsets of S) and each separates the other in S. A *lamination of* S is a set of leaves in S, no two of which are linked. A finite lamination is one with finitely many leaves.

If Λ is a finite lamination of S we may *pinch* S along Λ. This means that we quotient each leaf to a point, so that S collapses to a "tree" of smaller circles (sometimes called a *cactus*), and then split this tree apart into its constituent circles. We denote the result by $S \bmod \Lambda$. See Figure 1.

If there is a Riemannian metric on S then we get a Riemannian metric on $S \bmod \Lambda$, so it makes sense to talk about the *length* of the components of $S \bmod \Lambda$, and observe that the sum of these lengths is equal to the length of S.

Now suppose Λ is the increasing union of Λ_n (for $n = 1$ to ∞) where each Λ_n is finite. The *depth n leaves* are those in $\Lambda_n - \Lambda_{n-1}$ and for each n we can form $S \bmod \Lambda_n$ for each n and obtain in this way a sequence of increasingly refined partitions of $|S|$.

3B. ***Tautological elaminations and complex dynamics.*** We are interested in some naturally occurring laminations called *tautological laminations*. These objects were introduced in [Calegari 2022] to study the geometry and topology of the *shift locus* — a certain parameter space that arises naturally in holomorphic dynamics. For example, in degree 2, the shift locus is the complement (in $\mathbb{C}$) of the Mandelbrot set.

The tautological laminations in [Calegari 2022] have some extra structure — they are actually "extended laminations" or *elaminations*. If we identify the circle S^1 with the boundary of the closed unit disk $\overline{\mathbb{D}} \subset \mathbb{C}$, leaves in a lamination Λ corresponds to (infinite, unoriented) geodesics in $\mathbb{D}$ thought of as the hyperbolic plane in the Poincaré disk model. The unlinking property of leaves in a lamination corresponds to the condition that the geodesics in $\mathbb{D}$ they span are disjoint (except at their ideal "endpoints"). In an elamination these geodesics extend beyond S^1 to a pair of radial segments in $\mathbb{C} - \overline{\mathbb{D}}$. An elamination determines a lamination of S^1 (or equivalently, a geodesic lamination of $\mathbb{D}$) by forgetting these "extended" segments.

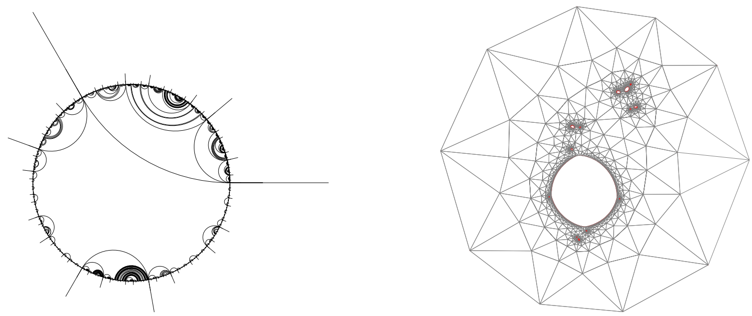

Figure 2. A finite elamination approximating the degree 3 tautological elamination for $z = 2$, and the result of pinching.

As mentioned in the introduction, the tautological lamination records the combinatorics of the 1 complex dimensional slices of the shift locus where $q-2$ critical Böttcher coordinates are fixed, and one critical point (with a smaller escape rate than any of the others) is allowed to vary. The extra structure of the tautological elamination records not only the combinatorics, but the holomorphic structure on these slices.

A finite elamination may be *pinched*, giving rise to a planar Riemann surface which may be (partially) compactified by a finite collection of circles, which are precisely the result of pinching the associated lamination of S^1. Figure 2 gives an example, approximating an infinite (tautological) elamination.

To orient the reader and to motivate the remainder of this paper, let us now describe the relationship between the tautological elamination and the holomorphic geometry of the shift locus, in the special case of degree 3. A depressed cubic polynomial $f(z) := z^3 + pz + q$ is in the shift locus $\mathcal{S}_3$ if the critical points c_1, c_2 (not necessarily distinct) are in the basin of attraction of infinity. These critical points have canonical *Böttcher coordinates* C_1, C_2, whose absolute value is well-defined and strictly greater than 1, and whose arguments are multivalued, where different values differ by multiples of $2\pi/3$. For $z \in \mathbb{C} - \overline{\mathbb{D}}$ let us define the *Böttcher slice* $B(z)$ of $\mathcal{S}_3$ to be the 1-complex dimensional subset where $C_1 = \{z, e^{2\pi i/3}z\}$ and $|C_1| > |C_2|$. The open dense subset of $\mathcal{S}_3$ for which the critical points are distinct and their Böttcher coordinates have distinct absolute values is foliated by such Böttcher slices, and in fact the Böttcher slices form the fibers of a (topological) fiber bundle over $\mathbb{C} - \overline{\mathbb{D}}$. Associated to each z is a tautological elamination $\Lambda_T(z)$, and the Böttcher slice $B(z)$ is obtained from $\Lambda_T(z)$ by pinching.

Figure 2 depicts the tautological elamination $\Lambda_T(z)$ for $z = 2$ and the Riemann surface obtained from $\Lambda_T(z)$ by pinching. The laminations of S^1 associated to $\Lambda_T(z)$ are the main objects of interest throughout this section; they depend only on the argument of z. We shall give them a precise definition in Section 3D.

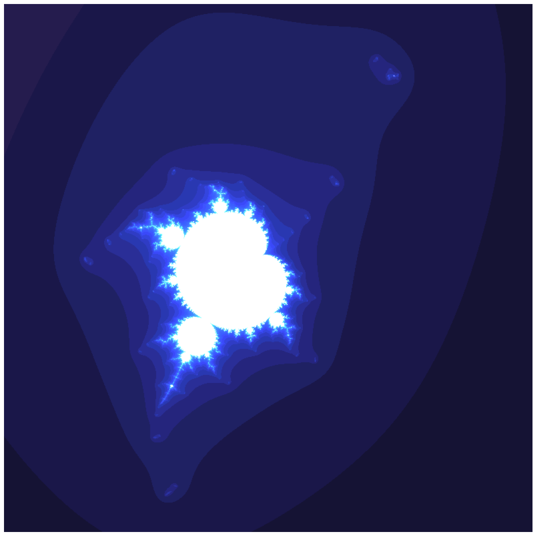

Figure 3. Part of the degree 3 shift locus (in blue) in a coordinate slice $f(z) = z^3 + pz + 1$.

It is computationally difficult to transform from Böttcher coordinates to polynomial coordinates. Fortunately, because the shift locus is more or less foliated by Böttcher slices, we may obtain a qualitatively reasonable picture of a Böttcher slice by instead giving (part of) a "coordinate slice" of $\mathcal{S}_3$ consisting of polynomials $z^3 + pz + q$ where the linear term q is fixed, at least in a region where such a coordinate slice lies close to a Böttcher slice. Figure 3 is (part of) a "coordinate slice" of $\mathcal{S}_3$ parametrizing shift polynomials of the form $z^3 + pz + 1$. There is one "large" continent, which resembles a lopsided Mandelbrot, surrounded by a few visible small islands; and there is a little archipelago to the northeast; compare with Figure 2.

There is a refinement of the tautological elamination (called the *completed* tautological elamination) which (conjecturally) parametrizes the cut points on the components of the complement of $\mathcal{S}_q$ in a Böttcher slice. When $q = 2$ this recovers Thurston's combinatorial model for the cut points in the Mandelbrot set. For a definition see [Calegari 2022, §8.7].

3C. ***Degree 3, a worked example.*** Tautological laminations are laminations of the unit circle, which we normalize as $S^1 = \mathbb{R}/\mathbb{Z}$ so that it has length 1. Tautological laminations depend on a degree $q \geq 2$ and a continuous parameter C (morally, a vector of $q - 2$ arguments of Böttcher coordinates) which, for $q = 3$, is encoded by a single angle $\theta \in \mathbb{R}/\mathbb{Z}$.

Although the laminations depend on the parameter, the result of pinching the circle to any finite depth does not. Thus, for each degree q and each depth n we obtain a *partition* of 1 into a vector of lengths of the components of $S^1 \bmod \Lambda_n$.

These lengths are all integer multiples of q^{-n} (in fact, they are of the form $2^m q^{-n}$ for various m). Our goal is to count the number of components of length exactly q^{-n} (the *short components*), and the main result of this section (Theorem 3.10) gives the generating function for the number of short components in degree n, and shows that it is related in a rather simple way to the function $\beta(t)$ from Proposition 2.2.

We shall first give an ad hoc (though precise) definition in the special case $q = 3$ and work out a few examples by hand. Multiplication by 3 gives a map from S^1 to itself. If λ is the leaf $\{p, q\}$ for distinct points $p, q \in S^1$ then 3λ is the leaf $\{3p, 3q\}$. For any $\theta \in S^1$ let $L(\theta)$ denote the leaf $\{\theta, \theta + 1/3\}$. By abuse of notation we let $L^-(\theta)$ be the limit of leaves $L(\theta - \epsilon)$ as $\epsilon \to 0$ from above, and we say that a leaf λ *links* $L^-(\theta)$ if it links $L(\theta - \epsilon)$ for all sufficiently small positive ϵ. Likewise, we let $L^+(\theta)$ be the limit of leaves $L(\theta + \epsilon)$ as $\epsilon \to 0$ from above.

Definition 3.1 (ad hoc definition, degree 3). There will be one depth n leaf of the tautological lamination Λ_T for each $x \in [1/3, 2/3)$ for which $3^n x = 0$. We claim there is a unique $y \in S^1$ such that

(1) $3^n y = 1/3$; and
(2) if λ denotes the leaf $\{x, y\}$ then $3^m\lambda$ does not link $L^-(0)$ or $L^+(x)$ for $m = 0, 1, 2, \dots, n-1$.

Then $\{3x, 3y\}$ is a depth n leaf of Λ_T, and all depth n leaves arise this way.

Example 3.2 (depth 1). The only $x \in [1/3, 2/3)$ with $3x = 0$ is $x = 1/3$. The only y with $3y = 1/3$ and for which $\{1/3, y\}$ does not link $L^-(0)$ or $L^+(1/3)$ is $y = 7/9$. Thus $\{x, y\} = \{1/3, 7/9\}$ and the leaf $3\{x, y\} = \{0, 1/3\}$ is the unique depth 1 leaf of Λ_T. Thus S^1 mod Λ_T to depth 1 has two components of length 1/3 and 2/3 respectively.

Example 3.3 (depth 2). For $x \in [1/3, 2/3)$ with $9x = 0$ we must have one of $x = 1/3, 4/9, 5/9$. For $x = 1/3$ we may check for $\lambda = \{1/3, 19/27\}$ that λ and 3λ do not link $L^-(0)$ or $L^+(1/3)$. Likewise for $\lambda = \{4/9, 22/27\}$ that λ and 3λ do not link $L^-(0)$ or $L^+(4/9)$, and for $\lambda = \{5/9, 25/27\}$ that λ and 3λ do not link $L^-(0)$ or $L^+(5/9)$. Thus the unique depth 2 leaves of Λ_T are $\{0, 1/9\}$, $\{1/3, 4/9\}$ and $\{2/3, 7/9\}$. Thus S^1 mod Λ_T to depth 2 has 3 components of length 1/9, 1 component of length 2/9, and 1 component of length 4/9.

Example 3.4 (depth 3). For $x \in [1/3, 2/3)$ with $27x = 0$ we must have one of $x = 1/3, 10/27, 11/27, \dots, 17/27$. Let's do one example. For $x = 11/27$ we want y with $27y = 1/3$ so for $\lambda = \{11/27, y\}$ that $\lambda, 3\lambda, 9\lambda$ do not link $L^-(0)$ or $L^+(11/27)$. One might naively guess (based on the examples in depth 1 and depth 2 in which $y = x + 3^{-1} + 3^{-n-1}$ is always the correct choice) that $y = 61/81$ would work, but $9\{11/27, 61/81\} = \{2/3, 7/9\}$ which links $L(11/27) = \{11/27, 20/27\}$. In fact $y = 58/81$, and $3\{11/27, 58/81\} = \{2/9, 4/27\}$ is a depth 3 leaf of Λ_T.

One may check that S^1 mod Λ_T to depth 3 has 7 components of length $1/27$, 6 components of length $2/27$, and 1 component of length $8/27$.

The set of leaves to depth 3 is

(1) $\{0, 1/3\}$;

(2) $\{0, 1/9\}$, $\{1/3, 4/9\}$, $\{2/3, 7/9\}$;

(3) $\{0, 1/27\}$, $\{1/9, 7/27\}$, $\{2/9, 4/27\}$, $\{1/3, 10/27\}$, $\{4/9, 13/27\}$, $\{5/9, 16/27\}$, $\{2/3, 19/27\}$, $\{7/9, 22/27\}$, $\{8/9, 25/27\}$.

See the left side of Figure 2.

Continuing out to greater depth, the number of components of S^1 mod Λ_T to depth n of length 3^{-n} is $1, 3, 7, 21, 57, 171, 499$ and so on.

3D. ***Tautological laminations.*** Let us now give a more precise definition. Fix a degree q which is an integer ≥ 2. Multiplication by q defines a degree q map from S^1 to itself; if λ is a leaf whose points do not differ by a multiple of $1/q$ then it makes sense to define the leaf $q\lambda$. Let C be a finite lamination C consisting of $q-2$ leaves $C_1, \ldots, C_{q-2}$ such that for each j the points of C_j differ by $1/q$. Thus we can write $C_j = \{\theta_j, \theta_j + 1/q\}$ for some $\theta_j \in S^1$. For simplicity we assume that C is *generic*, meaning that the θ_j are irrational and irrationally related.

The quotient S^1 mod C is a union of $q-1$ circles, $q-2$ of them of length $1/q$ and one of length $2/q$; we refer to this as the *big circle* and denote it B'. The preimage of B' in S^1 is a finite collection of arcs of S^1 bounded by points in leaves of C; denote this B. Note that the projection from B to B' is one-to-one away from points in leaves of C.

We shall now define the depth n leaves of the tautological lamination $\Lambda_T(C)$. Let $x \in B$ be a point for which $q^n x = \theta_j$; this x will be a point of a depth n leaf of *type* j. Let $z \in B$ be the unique point for which the projections $x', z' \in B'$ are antipodal (i.e., they are distance $1/q$ apart). Define a finite lamination $C(x)$ to be the union of C together with the leaf $\{x, z\}$.

Now, S^1 mod $C(x)$ is a union of q circles, all of length $1/q$. Furthermore if we denote by $\pi : S^1 \to S^1$ mod $C(x)$ the projection (which is well-defined away from the points of $C(x)$) the map $z \to q\pi^{-1}z$ extends from its domain of definition over the missing points to a homeomorphism from each component of S^1 mod $C(x)$ to S^1. By abuse of notation, we denote this map by $q : S^1$ mod $C(x) \to S^1$ and think of it as a q–1 map.

Lemma 3.5 (division by q). *If $\lambda := \{a, b\}$ is a generic leaf unlinked with $C(x)$, and $a' \in S^1$ satisfies $qa' = a$ then there is a unique b' with $qb' = b$ so that $\lambda' := \{a', b'\}$ is unlinked with $C(x)$.*

Proof. "Generic" is just to rule out boundary cases where, e.g., a' or b' is equal to a point in $C(x)$. In particular, if a' maps to a component S_i of S^1 mod $C(x)$ then we can pull back λ under the map $q : S_i \to S^1$ and then take its preimage in S^1 to obtain λ'. $\square$

Given x, we consider the sequence of points $x_i := q^i x$ for $0 \le i \le n$. By definition, $x_n = \theta_j$. Define $y_n = \theta_j + 1/q$ and $\lambda_n := \{x_n, y_n\}$ so that $\lambda_n = C_j$, and then inductively let λ_i be obtained from λ_{i+1} as in Lemma 3.5 so that $\lambda_i := \{x_i, y_i\}$ where $q y_i = y_{i+1}$ and λ_i is unlinked from $C(x)$. Finally we obtain the leaf λ_0 which, because it depends on x, we should really denote $\lambda_0(x)$.

Definition 3.6. With notation as above, the depth n leaves of $\Lambda_T(C)$ are the leaves $q\lambda_0(x)$ of S^1 as x ranges over the points in B with $q^n x = \theta_j$ and j ranges over $1, \dots, q-2$.

Notice that in this definition every leaf is enumerated exactly *twice*; if x and z in B have antipodal image in B' then the images of $\lambda_0(x)$ and $\lambda_0(z)$ are antipodal in B' so that $q\lambda_0(x) = q\lambda_0(z)$. So we only need to find a subset $A \subset B$ projecting to half of B' and add leaves $q\lambda_0(x)$ for $x \in A$ with $q^n x = \theta_j$. Thus the number of leaves of depth n is equal to $q^{n-1}(q-2)$. In particular $\Lambda_T(C)$ is empty if $q = 2$.

Proposition 3.7 (lamination). *The leaves of $\Lambda_T(C)$ are pairwise unlinked; thus $\Lambda_T(C)$ really is a lamination. Furthermore, if $\Lambda_{T,n}(C)$ denotes the leaves of $\Lambda_T(C)$ of depth at most n, the set of lengths of components of S^1* mod $\Lambda_{T,n}(C)$ (*counted with multiplicity*) *is independent of C and depends only on q.*

For a proof, see [Calegari 2022, §7]. Tautological laminations for $q = 3, 4, 5, 6$ (for a rather symmetric choice of C) are displayed in Figure 4.

Since the set of lengths of S^1 mod $\Lambda_{T,n}(C)$ (with multiplicity) is independent of C we can fix a normalization $\theta_j = (j-1)/q$ and suppress C in our notation in the sequel. This set of values is not generic; so we interpret the values of θ_j as limits as we approach $(j-1)/q$ from below. So we should interpret C_j as a "leaf" whose endpoints span the interval $[(j-1)/q, j/q)$, for the purposes of determining when leaves and their preimages are linked.

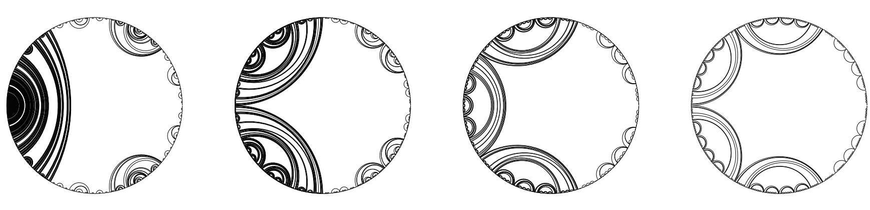

Figure 4. Tautological laminations for $q = 3, 4, 5, 6$.

Then every depth n leaf is of the form $q\lambda$, where $\lambda = \{x, y\}$, $q^n x = (j-1)/q$ and $q^n y = j/q$. It follows that every depth n leaf of Λ_T consists of a pair of points which are integer multiples of q^{-n}, and therefore every component of $S^1 \bmod \Lambda_{T,n}$ has length which is an integer multiple of q^{-n}. What is not obvious, but is nevertheless true, is that these integer multiples are all *powers of* 2 (we shall deduce this in the sequel). Write the length of a component as $\ell \cdot q^{-n}$ where ℓ is a power of 2, and define $N_q(n, m)$ to be the number of components of $S^1 \bmod \Lambda_{T,n}$ with $\ell = 2^m$.

Let's spell out Definition 3.6 in this normalization. We can take B and A to be the half-open intervals

$$B = [(q-2)/q, 1) \quad \text{and} \quad A = [(q-2)/q, (q-1)/q).$$

The base q expansion of $x \in A$ with $q^n x = (j-1)/q$ is a word of length $n+1$ in the alphabet $\{0, 1, \dots, (q-1)\}$ starting with the digit $(q-2)$ and ending with the digit $(j-1)$. If we denote the digits of x as $x_0 \cdots x_n$ then

$$x := \cdot(q-2)x_1x_2 \cdots x_{n-1}(j-1) \quad \text{and} \quad z := \cdot(q-1)x_1x_2 \cdots x_{n-1}(j-1).$$

Likewise, we denote the digits of y as $y_0 \cdots y_n$. Then

(1) $y_n = j$; and recursively,

(2) if $x_i \neq q-2$ or $q-1$ then $y_i = x_i$; and

(3) if $x_i = q-2$ or $q-1$ then y_i is the unique one of $q-1$ or $q-2$ so that $\cdot x_i \cdots x_n$ and $\cdot y_i \cdots y_n$ do not link x and z.

Although we are not able to give a simple formula for $N_q(n, m)$, it turns out there is a relatively simple formula for $N_q(n, 0)$ — i.e., the number of components of $S^1 \bmod \Lambda_{T,n}$ of length q^{-n}. These are the *short components*.

3E. *Short components.* One of the nice things about our normalization C is that there is a simple relationship between short components of $S^1 \bmod \Lambda_{T,n}$ and certain depth n leaves of $\Lambda_{T,n}$, a relationship which is substantially more complicated for longer components. Say a *short leaf* is a depth n leaf of $\Lambda_{T,n}$ whose points differ by exactly q^{-n} (this is the least it can be). Then:

Lemma 3.8 (short leaf). *There is a bijection between short components and short leaves of any fixed depth.*

Proof. Let S be a short component at depth n, and consider the preimage X in S^1. Then X is a union of finitely many disjoint arcs and isolated points bounded by leaves of depth $\leq n$. The total length of X is q^{-n} by the definition of short component. But leaves of depth k consist of points which are integer multiples of q^{-k} so the only possibility is that X consists of a single arc Y of S^1 together with finitely many (possibly zero) isolated points joined to the endpoints of Y by a chain of leaves $\gamma_0, \dots, \gamma_n$, each sharing one endpoint with the next.

We claim that in fact there are no isolated points, so that $X = Y$ is a single arc of S^1 cut off by a single (necessarily) short leaf. To see this, let's enlarge the circle by a factor of q^n so that depth k leaves with $k < n$ consist of points which are divisible by q, and each depth n leaf of type j joins a point congruent to $(j-1) \bmod q$ to a point congruent to $j \bmod q$. By the nature of their construction distinct depth n leaves of type j cannot share an endpoint, so a depth n leaf of type j must be followed by a depth n leaf of type $j+1$, and only a type 1 leaf of depth n can follow a depth $< n$ leaf and only in the positive (i.e., anticlockwise) direction around S^1 (remember our understanding of θ_j as the limit of a sequence approaching $(j-1)/q$ from below). It follows that if there is some intermediate point, the endpoints of Y differ by at least 2 mod q so that S is not short after all. This proves the claim. $\square$

Note that this lemma is *false* for generic C.

Let λ' be a short leaf of Λ_T of depth n of the form $q\lambda_0(x)$, where $x \in A$ and $q^n x = \theta_j$. For this normalization, $z = x + 1/q$ and $y = x + q^{-1-n}$, where $\lambda_0(x) = \{x, y\}$. The defining property of being a depth n leaf means that $\lambda_k(x) = q^k\lambda_0(x)$ does not link $C(x) = C \cup \{x, z\}$ for any $0 \le k \le n$. Actually, for *any* integer $m \bmod q^{n+1}$, setting $x = mq^{-1-n}$ and $y = (m+1)q^{-1-n}$, the leaf $\lambda_k(x)$ does not link any C_i for $0 \le i \le (q-2)$. So the short leaves are just the x for which $\lambda_k(x)$ does not link $\{x, z\}$ for $0 \le k \le n$.

Remember that the base q expansion of x is a word of length $n+1$ in the alphabet $\{0, 1, \dots, (q-1)\}$ starting with the digit $(q-2)$ and ending with the digit $(j-1)$. The base q expansion of y is the same as that of x with the last digit replaced by j. Similarly, the base q expansion of z is the same as that of x with the first digit replaced by $(q-1)$. We deduce:

Lemma 3.9 (short is 1-unbordered). *A word in the alphabet* $\{0, 1, \dots, (q-1)\}$ *of length* $(n+1)$ *starting with* $(q-2)$ *and ending with* $(j-1)$ *corresponds to a short leaf if and only if it is* 1*-unbordered.*

Proof. The leaf $\lambda_k(x) = \{q^k x, q^k y\}$, and the base q expansions of $q^k x$ and $q^k y$ are obtained from the base q expansions of x and y by the k-fold left shift. This leaf links $\{x, z\}$ if and only if the length k-suffix of x is either equal to a prefix of x, or to a prefix of z. But this is the definition of a 1-unbordered word. $\square$

Since $(j-1)$ is allowed to vary from 0 to $(q-3)$, and since a word that starts with $(q-2)$ and ends with $(q-2)$ or $(q-1)$ is already 1-bordered, it follows that $N_q(n, 0)$ is equal to the number of 1-unbordered words of length $(n+1)$ starting with $(q-2)$, which is just q^{-1} times the number of 1-unbordered words of length $(n+1)$. In other words:

Theorem 3.10 (recursive formula). *$N_q(n,0)$ satisfies the recursion $N_q(0,0)=1$, $N_q(1,0)=(q-2)$ and*

$$N_q(2n,0)=qN_q(2n-1,0) \quad \textit{and} \quad N_q(2n+1,0)=qN_q(2n,0)-2N_q(n,0)$$

and has the generating function $(\beta(t)-1)/qt$ where a closed form for $\beta(t)$ is given in Proposition 2.2.

4. Tree polynomials

We now discuss a rather different class of objects that turn out to be naturally in bijection with the components of S^1 mod $\Lambda_{T,n}$. These objects are called *tree polynomials*.

We give our definition in terms of rooted trees (with some auxiliary planar structure) and adopt the standard terminology of parents, children, siblings etc. Thus for every (nonroot) vertex there is a unique embedded path from that vertex to the root, and the *parent* of v is the unique vertex w on that path connected to v by an edge, and conversely v is the *child* of w; vertices are *siblings* if they share a common parent, and so on.

Definition 4.1. A *tree polynomial* is a finite rooted tree T together with the following data:

(1) **depth:** all leaves have a common depth n; we call this the *depth of T*;

(2) **critical:** all vertices are *critical* or *ordinary*;

   (a) the root is critical;
   (b) every nonleaf critical vertex has exactly one critical child;
   (c) every ordinary vertex has no critical children;

(3) **order:** the children of every vertex are *ordered*, and the critical child of the root is first among its siblings;

(4) **self-map:** there is a simplicial self-map $f : T \to T$ such that

   (a) $f(\text{root}) = \text{root}$;
   (b) $f(v) = \text{root}$ for all children v of the root; and
   (c) for all v with nonroot parent w, the image $f(v)$ is a child of $f(w)$;
   (d) if v is ordinary and not a leaf, then f maps the children of v bijectively and in an order-preserving way to the children of $f(v)$;
   (e) if v is critical and not a leaf or the root, then f maps the children of v in an order nondecreasing way to the children of $f(v)$; this map is onto and two-to-one except for the critical child of v which is the unique child mapping to its image;

(5) **length:** there is a *length* function ℓ from the vertices to $\mathbb{N}$;
   (a) $\ell(\text{root}) = 1$;
   (b) if v is ordinary, $\ell(v) = \ell(f(v))$;
   (c) if v is critical, $\ell(v) = 2\ell(f(v))$.

Another way of talking about the order structure on the children of each vertex is to say that T is a *planar* tree, and the map f is compatible with the planar structure.

4A. *Basic properties.*

Definition 4.2 (degree). Let T be a tree polynomial. The root has one critical child with $\ell = 2$ and some nonnegative number of ordinary children with $\ell = 1$. All children map to the root under f. Thus tree polynomials of depth 1 are classified by the number of children. The *degree* of a tree polynomial, denoted $q(T)$, is equal to the number of children of the root, plus one.

Example 4.3 (degree 2). There is a unique tree polynomial of degree 2 of any positive depth, since every vertex is critical and all but the leaf have a unique child.

Definition 4.4 (postcritical length). Let T be a tree polynomial and let c be the unique critical leaf. The *postcritical length* of T, denoted $\ell(T)$, is equal to $\ell(f(c)) = \ell(c)/2$.

By induction, $\ell(T)$ is always a power of 2.

The next proposition explains why we have introduced tree polynomials:

Proposition 4.5 (bijection). *There is a natural bijection between the set of degree q tree polynomials T of depth $(n+1)$ with $\ell(T) = \ell$ and the set of components of $S^1 \bmod \Lambda_{n,T}$ of length $\ell \cdot q^{-n}$ where $\Lambda_{n,T}$ are the leaves of depth $\le n$ in the tautological lamination from Section 3.*

Proof. This is a corollary of [Calegari 2022, Theorems 9.20 and 9.21]. The tree polynomials are combinatorial abstractions of the sausage polynomials defined in [Calegari 2022, Definition 9.4]. A sausage polynomial is a certain kind of infinite nodal genus 0 Riemann surface Σ together with a holomorphic self-map of degree q satisfying a number of properties. A tree polynomial records only the underlying combinatorics of Σ, which is enough to recover ℓ. □

It follows that for each n and each m, the number of degree q tree polynomials T of depth $(n+1)$ with $\ell(T) = 2^m$ is $N_q(n, m)$.

Lemma 4.6 (extension). *Let T be a tree polynomial of depth n and let c be the unique critical leaf. Then tree polynomials T' of depth $(n+1)$ that extend T are in bijection with the children of $f(c)$.*

Proof. To extend T to T' we just add children to each of the leaves of T. For each ordinary leaf v we add a copy of the children of $f(v)$. For the unique critical leaf c we must choose a child e of $f(c)$ and then add as children of c one copy of e, and two copies of every other child of $f(c)$. The copy of e becomes the unique critical child of c in T'. The functions f and ℓ extend to these new leaves uniquely. □

The next two lemmas give direct proofs in the language of tree polynomials of the identities $\sum_\ell N_q(n,m)\cdot 2^m = q^n$ and $\sum_\ell N_q(n,m) = 1+(q-2)(q^n-1)/(q-1)$. Both identities follow immediately from Proposition 4.5 since the first just says that the sum of the lengths of the components of S^1 mod $\Lambda_{T,n}$ is equal to 1, and the second just says that Λ_T has $(q-2)q^{n-1}$ leaves of depth n, both of which follow immediately from the definitions.

Lemma 4.7 (multiplication by q). *Let T have degree q. For each nonleaf vertex v with children w_i we have $q\cdot\ell(v)=\sum_i \ell(w_i)$. Consequently $\sum_\ell N_q(n,m)\cdot 2^m = q^n$.*

Proof. There is a unique tree polynomial of depth 1 and degree d. The root has $\ell(\text{root})=1$, and it has $q-1$ children with $\ell = 2, 1, \dots, 1$. Thus $q\cdot\ell(v)=\sum_i \ell(w_i)$ is true for the root vertex, and by induction on depth, it is true for each ordinary or critical nonleaf vertex.

For any T, the extensions T' of T are in bijection with the children of the postcritical vertex, and the formula we just proved shows $\sum \ell(T') = q\ell(T)$. □

Lemma 4.8 (number of children). *Let T have degree d. For each nonleaf vertex v the number of children of v is $(q-2)\ell(v)+1$. Consequently*

$$\sum_m N_q(n,m) = 1+(q-2)(q^n-1)/(q-1).$$

Proof. First we prove the formula relating $\ell(v)$ to the number of children of v. The formula is true for the root vertex. If v is ordinary then $\ell(v)=\ell(f(v))$ and v has the same number of children as $f(v)$, so if the formula is true for $f(v)$ it is true for v. If v is critical then $\ell(v)=2\ell(f(v))$ and if $f(v)$ has $(q-2)\ell(f(v))+1$ children then v has $2(q-2)\ell(f(v))+1$ children. So the formula is true by induction.

Since there are $N_q(n-1,m)$ depth n degree q tree polynomials of length $\ell = 2^m$, and since by Lemma 4.8 each has $(q-2)\ell(T)+1$ children, we obtain a recursion

$$\sum_m N_q(n,m) = \sum_m N_q(n-1,m)((q-2)2^m+1) = (q-2)q^{n-1} + \sum_m N_q(n-1,m)$$

Since $\sum_m N_q(0,m)=1$ the lemma follows. □

Lemma 4.9 (length subdoubles). *Every child w of v has $\ell(w)\le 2\ell(v)$ with equality if and only if every sibling of w has $\ell = 1$.*

Proof. By Lemmas 4.7 and 4.8 w has $(q-2)\ell(v)+1$ children, whose lengths sum to $q\ell(v)$. □

Remark 4.10. It is worth pointing out a close relationship between tree polynomials (as defined above) and the *polynomial-like tree maps* of [DeMarco and McMullen 2008]. The main difference seems to be that the latter objects forget the planar structure (i.e., the data of the ordering on each set of siblings). One should also mention that there is a close relationship between the *dynamical elaminations* (see [Calegari 2022, §4.4]) and the *pictographs* of [DeMarco and Pilgrim 2017] which are in turn closely related to the pattern and tableau of [Branner and Hubbard 1992]. The tautological elaminations we discuss in this article are (roughly speaking) related to dynamical elaminations as the shift locus is related to individual shift polynomials.

5. F-sequences

Definition 5.1 (critical vein). Let T be a tree polynomial of depth n. The *critical vein* is the segment of T containing all the critical vertices. We denote it by γ and label the critical points on γ as c_i, where c_0 is the root and c_n is the critical leaf.

For each vertex w of T, define $F(w)$ to be equal to $f^k(w)$ for the least positive k so that $f^k(w)$ is critical. Thus we can think of F as a map from the critical vein to itself, and by abuse of notation, for integers i, j we write $F(i) = j$ if $F(c_i) = c_j$ so that we can and do think of F as a function from $\{0, \dots, n\}$ to itself. We also write $\ell(i)$ for $\ell(c_i)$.

Lemma 5.2 (properties of F). *$F(0) = 0$ and for every positive i, $F(i) < i$ and $F(i+1) \le F(i) + 1$. Furthermore, $\ell(i) = 2\ell(F(i))$ for $i > 0$ so that $\ell(i) = 2^k$ where k is the least integer so that $F^k(i) = 0$*

Proof. Since $f(w)$ has smaller depth than w unless w is the root, $F(i) < i$. Moreover, $F(c_{i+1})$ is equal to $F(w)$ for some child w of $F(c_i)$, so $F(i+1) \le F(i) + 1$.

Finally, $\ell(w) = \ell(f(w))$ when w is ordinary or the root, and $\ell(w) = 2\ell(f(w))$ when w is critical and not the root. □

Let γ^+ denote the union of γ together with the siblings of every critical vertex. We may think of it as a digraph (i.e., a directed graph), where every edge points away from the root. Define Γ to be the quotient of γ^+ obtained by identifying every sibling w of a critical vertex with its image $F(w)$. Note that Γ is a digraph.

The next proposition gives a characterization of the functions F that can arise from tree polynomials.

Proposition 5.3 (F-sequence). *A function F from $\{0, 1, \dots, n\}$ to itself arises from some tree polynomial T of depth n if and only if it satisfies the following properties*:

(1) $F(0) = 0$ *and* $F(1) = 0$;

(2) *each j has a finite set of* ***options*** *which are the admissible values of $F(i+1)$ when $F(i)=j$;*
   (a) *the options of* 0 *are* 0 *and* 1;
   (b) *if $F(i)=0$ then the options of i are $i+1$ and whichever option of* 0 *is not equal to $F(i+1)$;*
   (c) *if $F(i)\neq 0$ and $F(i+1)$ is not equal to $F(i)+1$ then the options of i are $i+1$ together with the options of $F(i)$;*
   (d) *if $F(i)\neq 0$ and $F(i+1)=F(i)+1$ then the options of i are $i+1$ together with all the options of $F(i)$ except $F(i)+1$.*

A function F is an ***F-sequence*** *if it satisfies these properties.*

Proof. Imagine growing a tree T by iterated extensions from a tree of depth 1. The extensions at each stage are the children of the critical image $f(v)$, which in turn may be identified with the children of $F(v)$.

When we grow T of depth i to T' we grow Γ to Γ' by adding a new edge from i to a new vertex $i+1$, and adding two new edges from i to j for each edge from $F(i)$ to j *except* for the edge from $F(i)$ to $F(i+1)$ (outgoing edges at $F(i)$ may be identified with the *options* of $F(i)$ as above). The root 0 is joined by a single edge both to itself and to 1, but every other vertex i is joined by a single edge to $i+1$ and by an *even* number of edges to each of its options (this can be seen by induction). The proposition follows. □

Remark 5.4. The referee has pointed out that the F-function carries essentially the same information as the Yoccoz' τ-function derived from the tableau of Branner–Hubbard (see, e.g., [Branner and Hubbard 1992, §4.2; DeMarco and McMullen 2008, §11]). Proposition 5.3 is essentially equivalent to [DeMarco and Schiff 2010, Proposition 2.1].

The map from tree polynomials to F-sequences is many to one, since for every vertex i of Γ except the root, if there is an edge from i to $j\neq i+1$ then there are at least two such edges. Nevertheless, if F is an F-sequence corresponding to T of depth n, the extended sequence defined by $F(n+1)=F(n)+1$ corresponds to the unique extension of T for which $F(c_{n+1})$ is the critical child of $F(c_n)$.

Example 5.5. $\ell(T)=1$ if and only if $F(n)=0$, where n is the depth of the tree. The number of depth n tree polynomials with this property is $N_q(n-1,0)$ by definition.

Example 5.6 (maximal type). There is a unique depth n tree polynomial T of any degree with $2\ell(T)=\ell(n)=2^n$ namely the tree polynomial for which $F(i)=i-1$ for all positive i. Thus $N_q(n,n)=1$. We call these trees of *maximal type*.

The components of S^1 mod $\Lambda_{T,n}$ corresponding to the trees of maximal type are clearly evident in Figure 4 (they correspond to the large components of "white space").

Example 5.7 (degree 3 maximal component). Consider the tree polynomial sequence of maximal type of degree 3. Let's work, as in Section 3D in the normalization $C = \{0, 1/3\}$. For each n the result of pinching S^1 mod $\Lambda_{T,n}(C)$ has a unique component of length $2^n/3^n$, and this sequence of components corresponds precisely to the (degree 3) tree polynomial sequence of maximal type; we call this component, for each n, the *maximal component*.

For each n we can let $K_n(C)$ be the preimage of the maximal component in S^1. As a subset of S^1 this depends on C, but for the specific normalization $C = \{0, 1/3\}$ it does not, and we abbreviate $K_n(C) = K_n$. It turns out that there is a very explicit description of K_n: it consists of numbers in $[0, 1)$ whose base 3 expansion is of two types:

(1) the first n digits contain no 0; or

(2) numbers of the form $\cdot x\dot{0}$ where x is a string of $< n$ digits containing no 0.

In other words:

$$K_1 = 0 \cup [1/3, 1), \quad K_2 = 0 \cup 1/3 \cup [4/9, 2/3] \cup [7/9, 1), \quad \text{etc.}$$

If we denote $K := \cap_n K_n$, then K is the set of numbers in $[0, 1]$ whose base three expansion either contains no 0s, or is of the form $\cdot x\dot{0}$ for some finite string x containing no 0s. See Figure 5. Compare with the left side of Figure 2. This component corresponds to the "lopsided Mandelbrot" in Figure 3.

The "boundary" of the region K consists of a union of 2^{n-1} short components of each positive depth n, as follows from Lemma 3.9, and no other components (because otherwise the length of the maximal component would be strictly less than $2^n/3^{-n}$ for some n).

For each positive k let i_k denote the least index (if any) for which $\ell(i_k) = 2^k$. Note that $i_0 = 0$.

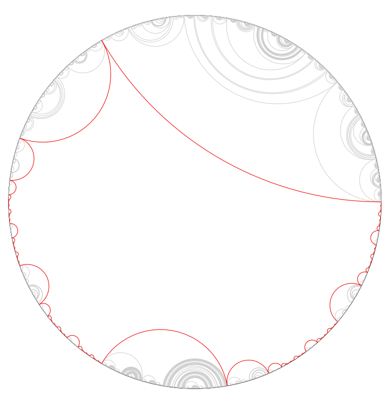

Figure 5. The maximal component K of S^1 mod Λ_T.

Lemma 5.8 (increments grow). *$F(i_k) = i_{k-1}$ for $k > 0$. Consequently $|i_{k+1} - i_k| \geq |i_k - i_{k-1}|$.*

Proof. By definition $F(i_k)$ is some value of j with $\ell(j) = 2^{k-1}$. But if $j > i_{k-1}$ there was some $i' < i_k$ with $F(i') = i_{k-1}$, contrary to the definition of i_k.

The inequality follows from $F(i+1) \leq F(i) + 1$. □

Definition 5.9 (S and B). Let F be an F-sequence. Let S be the set of indices i such that $F(i+1) = F(i) + 1$ and let B be the rest. Note that 0 is in B.

The *prior options* of i are the options other than $i + 1$. We denote these by $P(i)$. Thus, if $i \in S$ then $P(i) = P(F(i))$.

Lemma 5.10 (backslide). *Let F be an F-sequence and let $i \in B$. Then $F(i+1) < F(i)$ and $F(i+1) \in P(b)$, where $b = F^k(i)$ for some k and $b \in B$.*

Proof. Since $i \in B$ we must have $F(i+1) \in P(F(i))$ so that necessarily $F(i+1) < F(i)$. Furthermore, if $F(i) \in S$, then $P(F(i)) = P(F^2(i))$ and so on by induction until the first k so that $F^k(i) \in B$. □

Using F-sequences we may deduce the following "gap" theorem, that was observed experimentally.

Theorem 5.11 (gap). *$N_q(n, m) = 0$ for $\lfloor n/2 \rfloor < m < n$.*

Proof. Let T be a tree polynomial of depth $n+1$. If $m < n$ then T is not of maximal type, so there is some first positive index $k \in B$. Note that $i_k = k$ and in fact $i_j = j$ for all $j \leq k$. Since $k \in B$, by Lemma 5.10, $F(k+1) \in P(b)$, where $b \in B$ is $< k$. But then $b = 0$ so $F(k+1) = 0$. It follows that $i_{k+1} \geq 2k + 1$ and, successively, $i_{k+j} \geq k + j(k+1)$. From this the desired inequality follows. □

6. Tautological tree

Degree q tree polynomials of various depth can themselves be identified with the vertices of a (rooted, planar) *tautological tree* $\mathbb{T}_q$, whose vertices at depth n are the tree polynomials of degree q and depth n, and for each vertex T of $\mathbb{T}_q$, the children of T are the extensions of T.

Note that for each vertex T of $\mathbb{T}_q$ we can recover $\ell(T)$ from the number of children of T in $\mathbb{T}_q$, since this number is $(q-2)\ell(T) + 1$. So all the data of $N_q(n, m)$ can be read off from the abstract underlying tree of $\mathbb{T}_q$ (in fact, even the root can be recovered from the fact that it is the unique vertex of valence $q - 1$).

The tree $\mathbb{T}_3$ up to depth 4 (with vertices labeled by ℓ value, from which one could easily extend it another row as an unlabeled tree) is depicted in Figure 6

Every vertex labeled 1 has two children labeled 2 and 1. Every vertex labeled 2 has three children, but these might be labeled 4, 1, 1 or 2, 2, 2. Components of the complement of the shift locus $\mathcal{S}_q$ in a 1-dimensional slice are in bijection with the

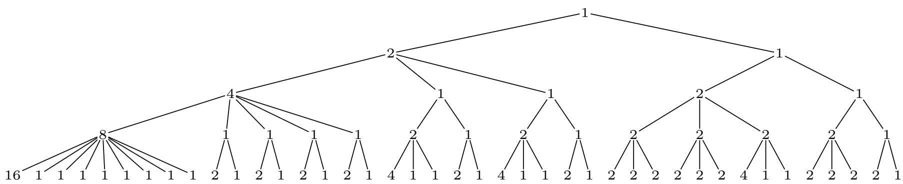

Figure 6. $\mathbb{T}_3$ up to depth 4.

ends of $\mathbb{T}_q$. Each such end gives rise to a sequence $\ell(n)$ of ℓ-values, and when $\sum 1/\ell(n)$ diverges, the corresponding component consists of a single point. Such ends are called *small*; those with $\sum 1/\ell(n) < \infty$ are *big*. Big ends are *dense* in the space of ends of $\mathbb{T}_q$:

Proposition 6.1 (big ends dense). *Big ends are dense. In other words, every finite rooted path in $\mathbb{T}_q$ can be extended to an infinite path converging to a big end.*

Proof. Let T be a tree polynomial of some finite depth n and let F be the associated F-sequence. Suppose $F(n) = i$. There is a unique infinite sequence of extensions of T defined recursively by $F(m+1) = F(m) + 1$ for all $m \geq n$. Then $F(m) = m + i - n$ for all sufficiently large m, so that $\ell(m) = 2\ell(m + i - n)$ and $\sum 1/\ell(m) < \infty$. □

Let's call an end *type S* if the associated F-sequence satisfies $F(m+1) = F(m)+1$ for all sufficiently large m (i.e., if it is of the sort constructed in Proposition 6.1). For example, the sequence of maximal type is of type S.

Example 6.2 (littlebrot). The right side of Figure 7 depicts the second biggest complementary component in a Böttcher's slice (this is a speck in the northeast corner in Figure 3). It corresponds to an end of type S with $\ell(n) = 2^{\lfloor n/2 \rfloor}$.

In the normalization $C = \{0, 1/3\}$ the base 3 decimal expansions of the points in the subset of S^1 associated to this component is a regular language in the alphabet $\{0, 1, 2\}$ (whose precise description is somewhat complicated and not very enlightening). Compare with Example 5.7.

Theorem 9.1 of [Branner and Hubbard 1992] implies (in degree 3, but the same result should hold in every degree) that every big end is of type S and a component of the complement of $\mathcal{S}_q$ in a slice has positive diameter if and only if it corresponds to a big end of $\mathbb{T}_q$.

Conjecture 6.3. *In the normalization $C = \{0, 1/3\}$, every big end corresponds to a subset of S^1 whose base* 3 *decimal expansion is a regular language in the alphabet* $\{0, 1, 2\}$.

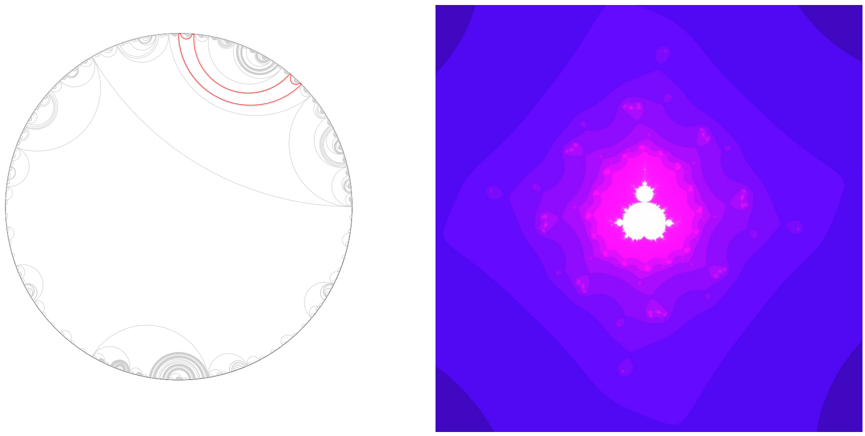

Figure 7. A slice through $z^3 + pz + 2$ of width 0.0003 centered at $1.72572 + 3.09778i$ and the corresponding component of the tautological lamination.

7. Tables of values

Values of $N_3(n, m)$ for $0 \le n, m \le 12$ are contained in Table 1. Values of $N_q(n, m)$ for $0 \le n, m \le 11$ and $q = 4, 5$ are in Tables 2 and 3. These tables were computed with the aid of the program `taut`, written by Alden Walker.

n	$m=0$	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1											
2	3	1	1										
3	7	6	0	1									
4	21	16	3	0	1								
5	57	51	13	0	0	1							
6	171	149	39	5	0	0	1						
7	499	454	117	23	0	0	0	1					
8	1497	1348	360	66	9	0	0	0	1				
9	4449	4083	1061	207	41	0	0	0	0	1			
10	13347	12191	3252	591	126	17	0	0	0	0	1		
11	39927	36658	9738	1799	370	81	0	0	0	0	0	1	
12	119781	109898	29292	5351	1125	240	33	0	0	0	0	0	1

Table 1. Values of $N_3(n, m)$.

n	$m=$ 0	1	2	3	4	5	6	7	8	9	10	11
0	1											
1	2	1										
2	8	2	1									
3	28	14	0	1								
4	112	52	6	0	1							
5	432	220	30	0	0	1						
6	1728	872	120	10	0	0	1					
7	6856	3540	472	54	0	0	0	1				
8	27424	14120	1924	204	18	0	0	0	1			
9	109472	56744	7620	828	98	0	0	0	0	1		
10	437888	226768	30752	3212	396	34	0	0	0	0	1	
11	1750688	908040	122852	12872	1556	194	0	0	0	0	0	1

Table 2. Values of $N_4(n,m)$.

n	$m=$ 0	1	2	3	4	5	6	7	8	9	10	11
0	1											
1	3	1										
2	15	3	1									
3	69	24	0	1								
4	345	114	9	0	1							
5	1695	597	51	0	0	1						
6	8475	2973	255	15	0	0	1					
7	42237	15018	1245	93	0	0	0	1				
8	211185	75012	6306	438	27	0	0	0	1			
9	1055235	375951	31287	2199	171	0	0	0	0	1		
10	5276175	1879269	157098	10767	858	51	0	0	0	0	1	
11	26377485	9400644	784596	53799	4230	339	0	0	0	0	0	1

Table 3. Values of $N_5(n,m)$.

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LIMIT THEOREMS AND WRAPPING TRANSFORMS IN BI-FREE PROBABILITY THEORY

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We characterize idempotent distributions with respect to the bi-free multiplicative convolution on the bi-torus. The bi-free analogous Lévy triplet of an infinitely divisible distribution on the bi-torus without nontrivial idempotent factors is obtained. This triplet is unique and generates a homomorphism from the bi-free multiplicative semigroup of infinitely divisible distributions to the classical one. Also, the relevances of the limit theorems associated with four convolutions, classical and bi-free additive convolutions and classical and bi-free multiplicative convolutions, are analyzed. The analysis relies on the convergence criteria for limit theorems and the use of push-forward measures induced by the wrapping map from the plane to the bi-torus.

1. Introduction

The main aim of the present paper is to build the association among various limit theorems and their convergence criteria in classical and bi-free probability theories.

Bi-free probability theory, introduced by Voiculescu in [20], is an outspread research field of free probability theory, which grew out to intend to simultaneously study the left and right actions of algebras over reduced free product spaces. Since its creation, a great deal of research work has been conducted to better understand this theory and its connections to other parts of mathematics [17; 19; 21; 22]. Aside from the combinatorial means, the utilization of analytic functions as transformations and the bond to classical probability theory also play crucial roles in the study and comprehension of this theory [12; 13]. Especially, recent developments of bi-free harmonic analysis enable one to investigate bi-free limit theorems and other related topics from the probabilistic point of view [11].

To work in the probabilistic framework, we thereby consider the family $\mathscr{P}_X$ of Borel probability measures on a complete separable metric space X and endow this family with a commutative and associative binary operation $\Diamond$. Classical and bi-free *convolutions*, respectively denoted by $*$ and $\boxplus\boxplus$, are two examples of such operations performed on $\mathscr{P}_{\mathbb{R}^2}$. In probabilistic terms, $\mu_1 * \mu_2$ is the probability

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distribution of the sum of two independent bivariate random vectors respectively having distributions μ_1 and μ_2. When restricted to compactly supported measures in $\mathscr{P}_{\mathbb{R}^2}$, $\mu_1 \boxplus\boxplus \mu_2$ is the distribution of the sum of two bi-free bipartite self-adjoint pairs with distributions μ_1 and μ_2, respectively [20]. This new notion of convolution was later extended, without any limitation, to the whole class $\mathscr{P}_{\mathbb{R}^2}$ by the continuity theorem of transforms [11]. The product of two independent random vectors having distributions on the bi-torus $\mathbb{T}^2$ gives rise to the classical multiplicative convolution $\circledast$, and the bi-free analog of multiplicative convolution $\boxtimes\boxtimes$ is defined in a similar manner [22].

In (noncommutative) probability theory, the limit theorem and its related subject, the notion of *infinite divisibility* of distributions, have attracted much attention. By saying that a distribution in $(\mathscr{P}_{\boldsymbol{X}}, \Diamond)$ is infinitely divisible we mean that it can be expressed as the operation $\Diamond$ of an arbitrary number of copies of identical distributions from $\mathscr{P}_{\boldsymbol{X}}$. The collection of measures having this infinitely divisible feature forms a semigroup and will be denoted by $\mathcal{ID}(\boldsymbol{X}, \Diamond)$, or simply by $\mathcal{ID}(\Diamond)$ if the identification of the metric space is unnecessary. Any measure satisfying $\mu = \mu \Diamond \mu$, known as *idempotent*, is an instance of infinitely divisible distributions. In the case of $\boldsymbol{X} = \mathbb{R}$, these topics have been thoroughly studied in classical probability by the efforts of de Finetti, Kolmogorov, Lévy and Khintchine (see [16]), and the same themes in the free contexts have also been deeply explored in the literature [5].

Bi-free probability, as expected, also parallels perfectly aspects of classical and free probability theories [3]. For example, the theory of bi-freely infinitely divisible distributions generalizes *bi-free central limit theorem* as they also serve as the limit laws for sums of bi-freely independent and identically distributed faces. Specifically, it was shown in [11] that for some infinitesimal triangular array $\{\mu_{n,k}\}_{n\geq 1, 1\leq k\leq n_k} \subset \mathscr{P}_{\mathbb{R}^2}$ and sequence $\{\boldsymbol{v}_n\} \subset \mathbb{R}^2$, the sequence

$$\delta_{\boldsymbol{v}_n} * \mu_{n1} * \cdots * \mu_{nk_n} \tag{1-1}$$

converges weakly if and only if so does the sequence

$$\delta_{\boldsymbol{v}_n} \boxplus\boxplus \mu_{n1} \boxplus\boxplus \cdots \boxplus\boxplus \mu_{nk_n}. \tag{1-2}$$

The limiting distributions in (1-1) and (1-2) respectively belong to the semigroups $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus\boxplus)$, and their classical and bi-free *Lévy triplets* agree. This conformity consequently brings out an isomorphism Λ between these two semigroups.

Same tasks are performed in the case of bi-free multiplicative convolution in this paper. We determine $\boxtimes\boxtimes$-idempotent elements and identify measures in $\mathscr{P}_{\mathbb{T}^2}$ bearing no nontrivial $\boxtimes\boxtimes$-idempotent factors. Specifically, we demonstrate that $\nu \in \mathcal{ID}(\boxtimes\boxtimes)$ has no nontrivial $\boxtimes\boxtimes$-idempotent factor if and only if it belongs to $\mathscr{P}_{\mathbb{T}^2}^{\times}$, the subcollection of $\mathscr{P}_{\mathbb{T}^2}$ with the attributes

$$\int_{\mathbb{T}^2} s_j \, d\nu(s_1, s_2) \neq 0, \quad j = 1, 2.$$

Fix an infinitesimal triangular array $\{\nu_{nk}\}_{n\geq 1, 1\leq k\leq k_n} \subset \mathscr{P}_{\mathbb{T}^2}$ and a sequence $\{\boldsymbol{\xi}_n\} \subset \mathbb{T}^2$. We also manifest that the weak convergence of the sequence

$$\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes\, \nu_{n1} \boxtimes\boxtimes \cdots \boxtimes\boxtimes\, \nu_{nk_n} \tag{1-3}$$

to some element in $\mathscr{P}_{\mathbb{T}^2}^{\times}$ yields the same property of the sequence

$$\delta_{\boldsymbol{\xi}_n} \circledast \nu_{n1} \circledast \cdots \circledast \nu_{nk_n}, \tag{1-4}$$

and that their limiting distributions are both infinitely divisible. This is done by distinct types of equivalent convergence criteria offered in the present paper. As in the case of addition, there exists a triplet concurrently serving as the classical and bi-free multiplicative Lévy triplets of the limiting distributions in (1-3) and (1-4). The consistency of their Lévy triplets, together with the description of $\mathcal{ID}(\boxtimes\boxtimes)\backslash\mathscr{P}_{\mathbb{T}^2}^{\times}$, consequently produces a homomorphism Γ from $\mathcal{ID}(\boxtimes\boxtimes)$ to $\mathcal{ID}(\circledast)$.

Because of the nature of $\mathcal{ID}(\boxtimes\boxtimes)\backslash\mathscr{P}_{\mathbb{T}^2}^{\times}$ and that the limit in (1-4) may generally not have a unique Lévy measure, the homomorphism stated above is neither surjective nor injective. However, postulating the uniqueness of the Lévy measure, the weak convergence of (1-4) derives that of (1-3).

In addition to the previously mentioned conjunctions, what we would like to point out is that measures in $\mathscr{P}_{\mathbb{R}^2}$ and $\mathscr{P}_{\mathbb{T}^2}$ can be linked through the wrapping map $W : \mathbb{R}^2 \to \mathbb{T}^2,\ (x, y) \mapsto (e^{ix}, e^{iy})$. This wrapping map induces a map $W_* : \mathscr{P}_{\mathbb{R}^2} \to \mathscr{P}_{\mathbb{T}^2}$ so that the measure $\nu_{nk} = W_*(\mu_{nk}) = \mu_{nk}W^{-1}$ enjoys the property: the weak convergence of (1-1) or (1-2) yields the weak convergence of (1-3) and (1-4) with $\boldsymbol{\xi}_n = W(\boldsymbol{v}_n)$. Furthermore, the synchronous convergence allows one to construct a homomorphism $W_{\boxplus\boxplus} : \mathcal{ID}(\boxplus\boxplus) \to \mathcal{ID}(\boxtimes\boxtimes)$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{ID}(*) \arrow[r, "W_*"] \arrow[d, "\Lambda"] & \mathcal{ID}(\circledast) \\
\mathcal{ID}(\boxplus\boxplus) \arrow[r, "W_{\boxplus\boxplus}"] & \mathcal{ID}(\boxtimes\boxtimes) \arrow[u, "\Gamma"']
\end{tikzcd} \tag{1-5}$$

This diagram is a two-dimensional analog of [6, Theorem 1].

The rest of the paper is organized as follows. In Section 2 we provide the necessary background in classical and noncommutative probability theories. In Section 3 we characterize $\boxtimes\boxtimes$-idempotent distributions. In Section 4 we make comparisons of the convergence criteria of limit theorems, as well as those through wrapping transforms. Section 5 is devoted to offering bi-free multiplicative Lévy triplets of infinitely divisible distributions and investigating the relationships among limit theorems in additive and multiplicative cases. Section 6 provides the derivation of the diagram in (1-5).

2. Preliminary

2A. ***Convergence of measures.*** Let $\mathscr{B}_{\boldsymbol{X}}$ be the collection of Borel sets on a complete separable metric space $(\boldsymbol{X}, \mathrm{d})$. A point is selected from $\boldsymbol{X}$ and fixed, named the *origin* and denoted by $\boldsymbol{x}_0$ in the following. In the present paper, we will be mostly concerned with the abelian groups $\boldsymbol{X} = \mathbb{R}^d$ and $\boldsymbol{X} = \mathbb{T}^d$ endowed with the relative topology from $\mathbb{C}^d$, where the origin is chosen to be the unit. They are respectively the d-dimensional Euclidean metric space and the d-dimensional torus (or the *d-torus* for short). The 1-torus is just the unit circle $\mathbb{T}$ on the complex plane. A set contained in $\{\boldsymbol{x} \in \boldsymbol{X} : \mathrm{d}(\boldsymbol{x}, \boldsymbol{x}_0) \geq r\}$ for some $r > 0$ is colloquially said to be *bounded away from the origin*.

Next, let us introduce several types of measures on $\boldsymbol{X}$ that will be discussed later. The first one is the collection $\mathscr{M}_{\boldsymbol{X}}$ of finite positive Borel measures on $\boldsymbol{X}$. We shall also consider the set $\mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ of all positive Borel measures that when confined to any Borel set bounded away from the origin yield a finite measure. Clearly, we have $\mathscr{M}_{\boldsymbol{X}} \subset \mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$. Another assortment concerned herein is the collection $\mathscr{P}_{\boldsymbol{X}}$ of elements in $\mathscr{M}_{\boldsymbol{X}}$ having unit total mass.

The set $C_b(\boldsymbol{X})$ of bounded continuous functions on $\boldsymbol{X}$ induces the weak topology on $\mathscr{M}_{\boldsymbol{X}}$. Likewise, $\mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ is equipped with the topology generated by $C_b^{\boldsymbol{x}_0}(\boldsymbol{X})$, bounded continuous functions having support bounded away from the origin. Concretely, basic neighborhoods of a $\tau \in \mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ are of the form

$$\bigcap_{j=1,\dots,n} \left\{ \tilde{\tau} \in \mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0} : \left| \int f_j \, d\tilde{\tau} - \int f_j \, d\tau \right| < \epsilon \right\},$$

where $\epsilon > 0$ and each $f_j \in C_b^{\boldsymbol{x}_0}(\boldsymbol{X})$. Putting it differently, a sequence $\{\tau_n\} \subset \mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ converges to some τ in $\mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$, written as $\tau_n \Rightarrow_{\boldsymbol{x}_0} \tau$, if and only if

$$\lim_{n\to\infty} \int f \, d\tau_n = \int f \, d\tau, \qquad f \in C_b^{\boldsymbol{x}_0}(\boldsymbol{X}).$$

We remark that τ is not unique as it may assign arbitrary mass to the origin. Nevertheless, any weak limit in $\mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ that comes across in our discussions will serve as the so-called *Lévy measure*, which does not charge the origin.

Portmanteau theorem and *continuous mapping theorem* in the framework of $\mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ are presented below (see [1; 14]). The push-forward measure $\tau h^{-1} : \mathscr{B}_{\boldsymbol{X}'} \to [0, +\infty]$ of $\tau \in \mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ provoked by a measurable mapping $h : (\boldsymbol{X}, \mathscr{B}_{\boldsymbol{X}}) \to (\boldsymbol{X}', \mathscr{B}_{\boldsymbol{X}'})$ is defined as

$$(\tau h^{-1})(B') = \tau(\{\boldsymbol{x} \in \boldsymbol{X} : h(\boldsymbol{x}) \in B'\}), \qquad B' \in \mathscr{B}_{\boldsymbol{X}'}. \tag{2-1}$$

Proposition 2.1. *The following statements* (1)–(3) *are equivalent for* $\{\tau_n\}$ *and* τ *in* $\mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$:

(1) *We have* $\tau_n \Rightarrow_{\boldsymbol{x}_0} \tau$.

(2) *For any* $f \in C_b(\boldsymbol{X})$ *and any* $B \subset \mathscr{B}_{\boldsymbol{X}}$, *which is bounded away from the origin and satisfies* $\tau(\partial B) = 0$, *we have*

$$\lim_{n\to\infty} \int_B f \, d\tau_n = \int_B f \, d\tau.$$

(3) *For every closed set* F *and open set* G *of* $\boldsymbol{X}$ *that are both bounded away from* $\boldsymbol{x}_0$, *we have*

$$\limsup_{n\to\infty} \tau_n(F) \le \tau(F) \quad \textit{and} \quad \liminf_{n\to\infty} \tau_n(G) \ge \tau(G).$$

If $h : (\boldsymbol{X}, \mathrm{d}) \to (\boldsymbol{X}', \mathrm{d}')$ *is measurable so that* h *is continuous at* $\boldsymbol{x}_0$, $h(\boldsymbol{x}_0) = \boldsymbol{x}_0'$, *and the set of discontinuities of* h *has* τ*-measure zero, then* $\tau_n \Rightarrow_{\boldsymbol{x}_0} \tau$ *implies* $\tau_n h^{-1} \Rightarrow_{\boldsymbol{x}_0'} \tau h^{-1}$.

Finally, let us introduce the subset $\widetilde{\mathscr{M}}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ consisting of measures in $\mathscr{M}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ that do not charge the origin $\boldsymbol{x}_0$. This set is metrizable and becomes a separable complete metric space [14, Theorem 2.2]. In particular, the relative compactness of a subset Y of $\widetilde{\mathscr{M}}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$ is equivalent to that any sequence of Y has a subsequence convergent in $\widetilde{\mathscr{M}}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$. We refer the reader to [14, Theorem 2.7] for an analog of Prokhorov's theorem, which characterizes the relative compactness of subsets in $\widetilde{\mathscr{M}}_{\boldsymbol{X}}^{\boldsymbol{x}_0}$.

2B. ***Notations.*** Below, we collect notations that will be commonly used in the sequel. The customary symbol $\arg s \in (-\pi, \pi]$ stands for the principal argument of a point $s \in \mathbb{T}$, while $\Re s$ and $\Im s$ respectively represent the real and imaginary parts of s. Here and elsewhere, points in a multidimensional space will be written in bold letters, for instance, $\boldsymbol{s} = (s_1, \dots, s_d) \in \mathbb{T}^d$ and $\boldsymbol{p} = (p_1, \dots, p_d) \in \mathbb{Z}^d$ with each $s_j \in \mathbb{T}$ and $p_j \in \mathbb{Z}$. For any $\epsilon > 0$, we shall use $\mathscr{V}_\epsilon = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| < \epsilon\}$ and $\mathscr{U}_\epsilon = \{\boldsymbol{s} \in \mathbb{T}^d : \|\arg \boldsymbol{s}\| < \epsilon\}$ to respectively express open neighborhoods of origins $\boldsymbol{0} \in \mathbb{R}^d$ and $\boldsymbol{1} \in \mathbb{T}^d$, where $\arg \boldsymbol{s} = (\arg s_1, \dots, \arg s_d) \in \mathbb{R}^d$. Analogous expressions also apply to vectors $\Re \boldsymbol{s} = (\Re s_1, \dots, \Re s_d)$ and $\Im \boldsymbol{s} = (\Im s_1, \dots, \Im s_d)$. Besides, we adopt the operational conventions in multidimensional spaces in the sequel, such as $\boldsymbol{s}^{\boldsymbol{p}} = s_1^{p_1} \cdots s_d^{p_d}$, $\boldsymbol{s}\boldsymbol{t} = (s_1 t_1, \dots, s_d t_d)$, $\boldsymbol{s}^{-1} = (1/s_1, \dots, 1/s_d)$, and $e^{i\boldsymbol{s}} = (e^{is_1}, \dots, e^{is_d})$.

The push-forward probabilities $\mu^{(j)} = \mu\pi_j^{-1}$, $j = 1, \dots, d$, on the real line induced by projections $\pi_j : \mathbb{R}^d \to \mathbb{R}$, $\boldsymbol{x} \mapsto x_j$, are called *marginals* of $\mu \in \mathscr{P}_{\mathbb{R}^d}$. Marginals of probability measures on $\mathbb{T}^d$ are defined and displayed in the same way. On $\mathbb{T}^2$, we shall also consider the (right) coordinate-flip transform $h_{\mathrm{op}} : \mathbb{T}^2 \to \mathbb{T}^2$ defined as $h_{\mathrm{op}}(\boldsymbol{s}) = (s_1, 1/s_2)$. Denote by $\boldsymbol{s}^\star = h_{\mathrm{op}}(\boldsymbol{s})$ and $B^\star = \{\boldsymbol{s}^\star : \boldsymbol{s} \in B\}$ if $\boldsymbol{s} \in \mathbb{T}^2$ and $B \subset \mathbb{T}^2$. By the (right) coordinate-flip measure of $\rho \in \mathscr{M}_{\mathbb{T}^2}^{\boldsymbol{1}}$, we mean the push-forward measure $\rho^\star = \rho h_{\mathrm{op}}^{-1}$, alternatively defined as $\rho^\star(B) = \rho(B^\star)$ for $B \in \mathscr{B}_{\mathbb{T}^2}$.

2C. ***Free probability and bi-free probability.*** Aside from the classical convolution on $\mathscr{P}_{\mathbb{R}^2}$, we shall also consider the bi-free convolution $\boxplus\boxplus$, where the *bi-free ϕ-transform* takes the place of Fourier transform [11]: for $\mu_1, \mu_2 \in \mathscr{P}_{\mathbb{R}^2}$, one has $\phi_{\mu_1 \boxplus\boxplus \mu_2} = \phi_{\mu_1} + \phi_{\mu_2}$. All information about marginals of the bi-free convolution is carried over to the free convolution: $(\mu_1 \boxplus\boxplus \mu_2)^{(j)} = \mu_1^{(j)} \boxplus \mu_2^{(j)}$ for $j = 1, 2$.

Now, we turn to probability measures on the d-torus. The sequence

$$m_{\boldsymbol{p}}(\nu) = \int_{\mathbb{T}^d} \boldsymbol{s}^{\boldsymbol{p}} \, d\nu(\boldsymbol{s}), \quad \boldsymbol{p} \in \mathbb{Z}^d,$$

is called the *d-moment sequence* of $\nu \in \mathscr{P}_{\mathbb{T}^d}$. In some circumstances, *characteristic function* and $\hat{\nu}(\boldsymbol{p})$ are the precise terminology and notation used for this sequence. Owing to Stone–Weierstrass theorem, we have $m_{\boldsymbol{p}}(\nu) \equiv m_{\boldsymbol{p}}(\nu')$ only when $\nu = \nu'$. The classical convolution $\circledast$ of distributions on $\mathbb{T}^d$ is characterized by $m_{\boldsymbol{p}}(\nu_1 \circledast \nu_2) = m_{\boldsymbol{p}}(\nu_1)\, m_{\boldsymbol{p}}(\nu_2)$ for $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^d}$.

The *bi-free multiplicative convolution* of $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}^{\times}$ is determined by its marginals $(\nu_1 \boxtimes\boxtimes \nu_2)^{(j)} = \nu_1^{(j)} \boxtimes \nu_2^{(j)}$ and the bi-free multiplicative formula

$$\Sigma_{\nu_1 \boxtimes\boxtimes \nu_2}(z, w) = \Sigma_{\nu_1}(z, w) \cdot \Sigma_{\nu_2}(z, w)$$

for points $(z, w) \in \mathbb{C}^2$ in a neighborhood of $(0, 0)$ and $(0, \infty)$. Here the free multiplicative convolution can be rephrased by means of the free Σ-transform $\Sigma_{\nu_1^{(j)} \boxtimes \nu_2^{(j)}} = \Sigma_{\nu_1^{(j)}} \cdot \Sigma_{\nu_2^{(j)}}$ valid in a neighborhood of the origin of the complex plane. The reader is referred to [4; 5; 12; 13; 17; 19; 21; 22] for more details along with properties of the transforms in (bi)-free probability theory. We remark that given a measure $\nu \in \mathscr{P}_{\mathbb{T}^2}^{\times}$, the transform Σ_ν is the identity map if and only if ν is a product measure, which leads to

$$(\nu_1^{(1)} \times \nu_1^{(2)}) \boxtimes\boxtimes (\nu_2^{(1)} \times \nu_2^{(2)}) = (\nu_1^{(1)} \boxtimes \nu_2^{(1)}) \times (\nu_1^{(2)} \boxtimes \nu_2^{(2)}), \tag{2-2}$$

whenever $\nu_1^{(1)} \times \nu_1^{(2)}, \nu_2^{(1)} \times \nu_2^{(2)} \in \mathscr{P}_{\mathbb{T}^2}^{\times}$. In fact, (2-2) holds for any $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$ by continuity arguments together with the facts that $m_{p,q}(\nu_1 \boxtimes\boxtimes \nu_2)$ can be expressed as a polynomial of $m_{k,l}(\nu_i)$ for $i = 1, 2$, $|k| \le |p|$, $|l| \le |q|$ and that $\nu \in \mathscr{P}_{\mathbb{T}^2}$ is a product measure if and only if $m_{p,q}(\nu) = m_p(\nu^{(1)})\, m_q(\nu^{(2)})$ for any $p, q \in \mathbb{Z}$.

Fix $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$, and let $\nu = \nu_1 \boxtimes\boxtimes \nu_2$. In order to analyze ν, it will be convenient to treat it as the distribution of a certain bipartite pair $(u_1 u_2, v_1 v_2)$, where (u_1, v_1) and (u_2, v_2) are bi-free bipartite unitary pairs in some C^*-probability space having distributions ν_1 and ν_2, respectively. Below, we briefly introduce the construction of such pairs carrying the mentioned properties. For more information, we refer the reader to [13; 20; 22]. Associating each ν_j with the Hilbert space $\mathcal{H}_j = L^2(\nu_j)$ with specified unit vector ξ_j, the constant function one in $\mathcal{H}_j$, consider the Hilbert space free product $(\mathcal{H}, \xi) = *_{j=1,2}(\mathcal{H}_j, \xi_j)$. The left and right factorizations of $\mathcal{H}_j$ from $\mathcal{H}$ can be respectively done via natural isomorphisms $V_j : \mathcal{H}_j \otimes \mathcal{H}(\ell, j) \to \mathcal{H}$

and $W_j : \mathcal{H}(r, j) \otimes \mathcal{H}_j \to \mathcal{H}$. Then for any $T \in B(\mathcal{H}_j)$, these isomorphisms induce the so-called *left* and *right operators*

$$\lambda_j(T) = V_j(T \otimes I_{\mathcal{H}(\ell,j)})V_j^{-1} \quad \text{and} \quad \rho_j(T) = W_j(I_{\mathcal{H}(r,j)} \otimes T)W_j^{-1} \qquad \text{on } \mathcal{H}.$$

For any $S_j, T_j \in B(\mathcal{H}_j)$, pairs $(\lambda_1(S_1), \rho_1(T_1))$ and $(\lambda_2(S_2), \rho_2(T_2))$ are, by definition, bi-free in the C^*-probability space $(B(\mathcal{H}), \varphi_\xi)$, where $\varphi_\xi(\,\cdot\,) = \langle \cdot\, \xi, \xi\rangle$. Particularly, the multiplication operators $(S_j f)(s,t) = sf(s,t)$ and $(T_j f)(s,t) = tf(s,t)$ for $f \in \mathcal{H}_j$ furnish the desired pairs (u_1, v_1) and (u_2, v_2), where $u_j = \lambda_j(S_j)$ and $v_j = \rho_j(T_j)$.

Recall from [13] that one can perform the *opposite bi-free multiplicative convolution* of ν_1 and ν_2:

$$\nu_1 \boxtimes\boxtimes^{\mathrm{op}} \nu_2 = (\nu_1^\star \boxtimes\boxtimes\, \nu_2^\star)^\star. \tag{2-3}$$

Then $\nu_1 \boxtimes\boxtimes^{\mathrm{op}} \nu_2$ is the distribution of (u_1u_2, v_2v_1), the pair obtained by performing the opposite multiplication on the right face $(u_1, v_1) \cdot^{\mathrm{op}} (u_2, v_2) = (u_1u_2, v_2v_1)$. The coordinate-flip map h_{op} gives rise to a homeomorphism from the semigroup $(\mathscr{P}_{\mathbb{T}^2}, \boxtimes\boxtimes)$ to another $(\mathscr{P}_{\mathbb{T}^2}, \boxtimes\boxtimes^{\mathrm{op}})$ satisfying

$$(\nu_1 \boxtimes\boxtimes\, \nu_2)h_{\mathrm{op}}^{-1} = (\nu_1 h_{\mathrm{op}}^{-1}) \boxtimes\boxtimes^{\mathrm{op}} (\nu_2 h_{\mathrm{op}}^{-1}),$$

which is the distribution of

$$h_{\mathrm{op}}((u_1, v_1)(u_2, v_2)) = (u_1u_2, v_2^{-1}v_1^{-1}) = h_{\mathrm{op}}((u_1, v_1)) \cdot^{\mathrm{op}} h_{\mathrm{op}}((u_2, v_2)).$$

Passing to the transform

$$\Sigma_\nu^{\mathrm{op}}(z, w) = \Sigma_{\nu^\star}(z, 1/w),$$

the equation (2-3) is translated into $\Sigma^{\mathrm{op}}_{\nu_1 \boxtimes\boxtimes^{\mathrm{op}} \nu_2}(z, w) = \Sigma^{\mathrm{op}}_{\nu_1}(z, w) \cdot \Sigma^{\mathrm{op}}_{\nu_1}(z, w)$.

2D. ***Limit theorem.*** Either in classical or in (bi-)free probability theory, one is concerned with the asymptotic behavior of the sequence

$$\delta_{\boldsymbol{x}_n} \Diamond \mu_{n1} \Diamond \cdots \Diamond \mu_{nk_n}, \quad n = 1, 2, \ldots, \tag{2-4}$$

where $\delta_{\boldsymbol{x}}$ is the Dirac measure concentrated at $\boldsymbol{x} \in \boldsymbol{X}$ and $\{\mu_{nk_n}\}_{n\geq 1, 1\leq k\leq k_n}$ is an infinitesimal triangular array in $\mathscr{P}_{\boldsymbol{X}}$. The *infinitesimality* of $\{\mu_{nk}\}$, by definition, means that $k_1 < k_2 < \cdots$ and that for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} \mu_{nk}(\{\boldsymbol{x} \in \boldsymbol{X} : \mathrm{d}(\boldsymbol{x}, \boldsymbol{x}_0) \geq \epsilon\}) = 0.$$

One phenomenon related to equation (2-4) is the concept of infinite divisibility: $\mu \in (\mathscr{P}_{\boldsymbol{X}}, \Diamond)$ is said to be *infinitely divisible* if for any $n \in \mathbb{N}$, it coincides with the n-fold $\Diamond$-operation $\mu_n^{\Diamond n}$ of some $\mu_n \in \mathscr{P}_{\boldsymbol{X}}$.

Commutative and associative binary operations to be considered throughout the paper are classical convolutions $*$ and $\circledast$ on $\mathscr{P}_{\mathbb{R}^d}$ and $\mathscr{P}_{\mathbb{T}^d}$, respectively, and bi-free additive and multiplicative convolutions $\boxplus\boxplus$ and $\boxtimes\boxtimes$ on $\mathscr{P}_{\mathbb{R}^2}$ and $\mathscr{P}_{\mathbb{T}^2}$, respectively. The following convergence criteria play an essential role in the asymptotic analysis of limit theorems of $\mathscr{P}_{\mathbb{R}^d}$.

Condition 2.2. Let $\{\tau_n\}$ be a sequence in $\mathscr{M}^{\mathbf{0}}_{\mathbb{R}^d}$.

(I) For $j = 1, \dots, d$, the sequence $\{\sigma_{nj}\}_{n\geq 1}$ defined by

$$d\sigma_{nj}(\boldsymbol{x}) = \frac{x_j^2}{1+x_j^2}\, d\tau_n(\boldsymbol{x})$$

belongs to $\mathscr{M}_{\mathbb{R}^d}$ and converges weakly to some $\sigma_j \in \mathscr{M}_{\mathbb{R}^d}$.

(II) For $j, \ell = 1, \dots, d$, the following limit exists in $\mathbb{R}$:

$$L_{j\ell} = \lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{x_j\, x_\ell}{(1+x_j^2)(1+x_\ell^2)}\, d\tau_n(\boldsymbol{x}).$$

Condition 2.3. Let $\{\tau_n\}$ be a sequence in $\mathscr{M}^{\mathbf{0}}_{\mathbb{R}^d}$.

(III) There is some $\tau \in \mathscr{M}^{\mathbf{0}}_{\mathbb{R}^d}$ with $\tau(\{\mathbf{0}\}) = 0$ (that is $\tau \in \widetilde{\mathscr{M}}^{\mathbf{0}}_{\mathbb{R}^d}$) so that $\tau_n \Rightarrow_{\mathbf{0}} \tau$.

(IV) For any vector $\boldsymbol{u} \in \mathbb{R}^d$, the following limits exist in $\mathbb{R}$:

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} \int_{\mathscr{V}_\epsilon} \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2\, d\tau_n(\boldsymbol{x}) = Q(\boldsymbol{u}) = \lim_{\epsilon\to 0}\liminf_{n\to\infty} \int_{\mathscr{V}_\epsilon} \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2\, d\tau_n(\boldsymbol{x}).$$

Although we describe the conditions in a higher dimension setup, the reader can effortlessly mimic the proof in [11] to obtain the equivalence of Conditions 2.2 and 2.3, and draw the following consequences:

(1) The function $Q(\,\cdot\,) = \langle \boldsymbol{A}\,\cdot\,, \cdot\,\rangle$ in (IV) defines a nonnegative quadratic form on $\mathbb{R}^d$, where the matrix $\boldsymbol{A} = (a_{j\ell})$ is given by

$$a_{j\ell} = L_{j\ell} - \int_{\mathbb{R}^d} \frac{x_j\, x_\ell}{(1+x_j^2)(1+x_\ell^2)}\, d\tau(\boldsymbol{x}), \qquad j, \ell = 1, \dots, d.$$

In particular, $a_{jj} = \sigma_j(\{\mathbf{0}\})$ for $j = 1, \dots, d$.

(2) Measures τ and $\sigma_1, \dots, \sigma_d$ are uniquely determined by the relations

$$d\sigma_j(\boldsymbol{x}) = \frac{x_j^2}{1+x_j^2}\, d\tau(\boldsymbol{x}) + Q(\boldsymbol{e}_j)\, \delta_{\mathbf{0}}(d\boldsymbol{x}),$$

where $\{\boldsymbol{e}_j\}$ is the standard basis of $\mathbb{R}^d$.

(3) The function $\boldsymbol{x} \mapsto \min\{1, \|\boldsymbol{x}\|^2\}$ is τ-integrable.

Now, let us briefly introduce the limit theorems of (1-1) and (1-2). Throughout our discussions in the paper,

$$\theta \in (0, 1) \tag{2-5}$$

is an arbitrary but fixed quantity. To meet the purpose, consider the shifted triangular array

$$\mathring{\mu}_{nk}(B) = \mu_{nk}(B + \boldsymbol{v}_{nk}), \quad B \in \mathscr{B}_{\mathbb{R}^d},$$

associated with an infinitesimal triangular array $\{\mu_{nk}\}_{n\geq 1, 1\leq k\leq k_n} \subset \mathscr{P}_{\mathbb{R}^d}$ and the vector

$$\boldsymbol{v}_{nk} = \int_{\mathscr{V}_\theta} \boldsymbol{x}\, d\mu_{nk}(\boldsymbol{x}). \tag{2-6}$$

Due to $\lim_{n\to\infty} \max_{k\leq k_n} \|\boldsymbol{v}_{nk}\| = 0$, $\{\mathring{\mu}_{nk}\}$ so obtained is also infinitesimal. In conjunction with this centered triangular array, we focus on the positive measures

$$\tau_n = \sum_{k=1}^{k_n} \mathring{\mu}_{nk}. \tag{2-7}$$

It turns out that the sequence in (1-1) converges weakly to a certain $\mu_* \in \mathscr{P}_{\mathbb{R}^d}$ if and only if τ_n defined in (2-7) meets Condition 2.3 (as well as Condition 2.2 since these two conditions are equivalent) and the limit

$$\boldsymbol{v} = \lim_{n\to\infty} \left[\boldsymbol{v}_n + \sum_{k=1}^{k_n} \left(\boldsymbol{v}_{nk} + \int_{\mathbb{R}^d} \frac{\boldsymbol{x}}{1 + \|\boldsymbol{x}\|^2}\, d\mathring{\mu}_{nk}(\boldsymbol{x}) \right) \right] \tag{2-8}$$

exists in $\mathbb{R}^d$. Additionally, μ_* is $*$-infinitely divisible and possesses the characteristic function read as

$$\hat{\mu}_*(\boldsymbol{u}) = \exp\left[i\langle \boldsymbol{u}, \boldsymbol{v}\rangle - \tfrac{1}{2}\langle \boldsymbol{A}\boldsymbol{u}, \boldsymbol{u}\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle \boldsymbol{u}, \boldsymbol{x}\rangle} - 1 - \frac{i\langle \boldsymbol{u}, \boldsymbol{x}\rangle}{1 + \|\boldsymbol{x}\|^2} \right) d\tau(\boldsymbol{x}) \right], \tag{2-9}$$

which is known as the *Lévy–Khintchine representation*. The limiting distribution is uniquely determined by the formula (2-9) and denoted by $\mu_*^{(\boldsymbol{v}, \boldsymbol{A}, \tau)}$, and $(\boldsymbol{v}, \boldsymbol{A}, \tau)$ is referred to as its *Lévy triplet*. The set $\mathcal{ID}(*)$ is completely parameterized by the triplets $(\boldsymbol{v}, \boldsymbol{A}, \tau)$, where

$$\boldsymbol{v} \in \mathbb{R}^d,\ \boldsymbol{A} \in M_d(\mathbb{R}) \text{ is positive semidefinite, and } \tau \text{ is a positive measure on } \mathbb{R}^d \text{ satisfying } \tau(\{\boldsymbol{0}\}) = 0 \text{ and } \min\{1, \|\boldsymbol{x}\|^2\} \in L^1(\tau). \tag{2-10}$$

As a matter of fact, when $d = 2$, the same convergence criteria are also necessary and sufficient to assure the weak convergence of (1-2). Paralleling to the classical case, the limiting distribution of (1-2) is $\boxplus\boxplus$-infinitely divisible and owns the bi-free ϕ-transform, called *bi-free Lévy–Khintchine* representation, of the form

$$\phi(z,w)=\frac{v_1}{z}+\frac{v_2}{w}+\left(\frac{a_{11}}{z^2}+\frac{a_{12}}{zw}+\frac{a_{22}}{w^2}\right)\\ +\int_{\mathbb{R}^2}\left[\frac{zw}{(z-x_1)(w-x_2)}-1-\frac{x_1z^{-1}+x_2w^{-1}}{1+\|\boldsymbol{x}\|^2}\right]d\tau(\boldsymbol{x}).$$

Analogically, this limiting distribution is always expressed as $\mu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{A},\tau)}$ and said to own the *bi-free Lévy triplet* $(\boldsymbol{v},\boldsymbol{A},\tau)$. Those triplets $(\boldsymbol{v},\boldsymbol{A},\tau)$ satisfying (2-10) also give a complete parametrization of the set $\mathcal{ID}(\boxplus\boxplus)$, and therefore output a bijective homomorphism Λ from $\mathcal{ID}(*)$ onto $\mathcal{ID}(\boxplus\boxplus)$, sending an element $\mu_*^{(\boldsymbol{v},\boldsymbol{A},\tau)}$ in the first set to the distribution $\mu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{A},\tau)}$ in the second one. No matter in the classical or bi-free probability, $*$- and $\boxplus\boxplus$-infinitely divisible distributions both appear as limiting distributions in the limit theorem.

Next, we turn our attention to the limit theorem on the d-torus, on which the Borel probability measures of interest are sometimes imposed the nonvanishing mean conditions:

$$\int_{\mathbb{T}^d}s_j\,d\nu(\boldsymbol{s})\neq 0,\quad j=1,\dots,d. \tag{2-11}$$

For convenience, we adopt the symbol $\mathscr{P}_{\mathbb{T}^d}^{\times}$ to signify the collection of probability measures carrying such features. As will be shown in Theorem 3.12, when $d=2$, these conditions (2-11) turn out to be necessary and sufficient for a $\boxtimes\boxtimes$-infinitely divisible distribution to contain no nontrivial $\boxtimes\boxtimes$-idempotent factors. We would also like to remind the reader that the symbol $\mathscr{P}_{\mathbb{T}^2}^{\times}$ introduced here is distinct from that in [13] as Theorem 3.10 of the present paper designates that the requirement $m_{1,1}(\nu)\neq 0$ in the limit theorem is redundant.

Given an infinitesimal triangular array $\{\nu_{nk}\}_{n\geq 1,1\leq k\leq k_n}$ in $\mathscr{P}_{\mathbb{T}^d}$, one works with the rotated probability measures $d\mathring{\nu}_{nk}(\boldsymbol{s})=d\nu_{nk}(\boldsymbol{b}_{nk}\boldsymbol{s})$, where

$$\boldsymbol{b}_{nk}=\exp\left[i\int_{\mathscr{U}_\theta}(\arg\boldsymbol{s})\,d\nu_{nk}(\boldsymbol{s})\right]. \tag{2-12}$$

Once again, $\{\mathring{\nu}_{nk}\}$ is infinitesimal because of $\lim_n\max_k\|\arg\boldsymbol{b}_{nk}\|=0$. Given a sequence $\{\boldsymbol{\xi}_n\}\subset\mathbb{T}^d$, further define vectors

$$\boldsymbol{\gamma}_n=\boldsymbol{\xi}_n\exp\left[i\sum_{k=1}^{k_n}\left(\arg\boldsymbol{b}_{nk}+\int_{\mathbb{T}^2}(\Im\boldsymbol{s})\,d\mathring{\nu}_{nk}(\boldsymbol{s})\right)\right]\in\mathbb{T}^d. \tag{2-13}$$

The bi-free multiplicative limit theorem on the bi-torus has been shown in [13, Theorem 3.4]:

Theorem 2.4. *The necessary and sufficient condition for the sequence* (1-3) *to converge weakly to a certain* $\nu_{\boxtimes\boxtimes}\in\mathscr{P}_{\mathbb{T}^2}^{\times}$ *is that the limit*

$$\lim_{n\to\infty}\boldsymbol{\gamma}_n=\boldsymbol{\gamma} \tag{2-14}$$

exists and the positive measures

$$\rho_n = \sum_{k=1}^{k_n} \mathring{\nu}_{nk} \tag{2-15}$$

satisfy Condition 2.5 *stated below with* $d = 2$.

Condition 2.5. Let $\{\rho_n\}$ be a sequence in $\mathscr{M}^{\mathbf{1}}_{\mathbb{T}^d}$.

(i) For $j = 1, \ldots, d$, the sequence $\{\lambda_{nj}\}_{n\geq 1}$ defined by

$$d\lambda_{nj}(\boldsymbol{s}) = (1 - \Re s_j)\, d\rho_n(\boldsymbol{s})$$

belongs to $\mathscr{M}_{\mathbb{T}^d}$ and converges weakly to some $\lambda_j \in \mathscr{M}_{\mathbb{T}^d}$.

(ii) For $1 \leq j, \ell \leq d$, the following limit exists in $\mathbb{R}$:

$$L_{j\ell} = \lim_{n\to\infty} \int_{\mathbb{T}^d} (\Im s_j)(\Im s_\ell)\, d\rho_n(\boldsymbol{s}).$$

The limiting distribution $\nu = \nu_{\boxtimes\boxtimes}$ in Theorem 2.4 is $\boxtimes\boxtimes$-infinitely divisible, as expected, and uniquely determined by the formulas [13]

$$\Sigma_{\nu^{(j)}}(\xi) = \exp[u_j(\xi)] \quad \text{and} \quad \Sigma_\nu(z, w) = \exp[u(z, w)]. \tag{2-16}$$

Here the functions u_j, $j = 1, 2$, are defined on $\mathbb{D}$ and given by

$$u_j(\xi) = -i \arg \gamma_j + \int_{\mathbb{T}^2} \frac{1 + \xi s_j}{1 - \xi s_j}\, d\lambda_j(\boldsymbol{s}),$$

and for $(z, w) \in (\mathbb{C}\backslash\mathbb{D})^2$, the function u satisfies

$$\begin{aligned}\frac{(1-z)(1-w)}{1-zw} u(z, w) = \int_{\mathbb{T}^2} \frac{1+zs_1}{1-zs_1} \frac{1+ws_2}{1-ws_2} (1 - \Re s_2)\, d\lambda_1(\boldsymbol{s})& \\ - i\int_{\mathbb{T}^2} \frac{1+zs_1}{1-zs_1} (\Im s_2)\, d\lambda_1(\boldsymbol{s})& \\ - i\int_{\mathbb{T}^2} \frac{1+ws_2}{1-ws_2} (\Im s_1)\, d\lambda_2(\boldsymbol{s}) - L_{12}&.\end{aligned}$$

In turn, any measure in $\mathcal{ID}(\boxtimes\boxtimes) \cap \mathscr{P}^\times_{\mathbb{T}^2}$ truly arises as a weak-limit point of (1-3).

Remark 2.6. Suppose that $\nu \in \mathcal{ID}(\boxtimes\boxtimes)\backslash\mathscr{P}^\times_{\mathbb{T}^2}$, and let $m_j = \int s_j\, d\nu^{(j)}$ for $j = 1, 2$. Then $\Sigma_{\nu^{(j)}}(0) = 1/m_j$, $\arg \gamma_j = \arg m_j$, and $\lambda_j(\mathbb{T}^2) = -\log |m_j| \in [0, \infty)$. We remind the reader that the parameter γ_j in $u_j(\xi)$ and that appearing in [13] are conjugate complex numbers. With the help of the equation

$$\frac{1+\xi s}{1-\xi s}(1 - \Re s) = i\Im s + \frac{(1-\xi)(1-s)}{1-\xi s}, \quad (\xi, s) \in \mathbb{D}\times\mathbb{T}, \tag{2-17}$$

one can see that

$$u_j(\xi) = -i \arg \gamma_j + \lim_{n\to\infty} \int_{\mathbb{T}^2} \frac{1+\xi s_j}{1-\xi s_j}(1-\Re s_j)\, d\rho_n(\boldsymbol{s})$$

and

$$u(z,w) = \lim_{n\to\infty} \int_{\mathbb{T}^2} \frac{(1-zw)(1-s_1)(1-s_2)}{(1-zs_1)(1-ws_2)}\, d\rho_n(\boldsymbol{s})$$

for some sequence $\{\rho_n\} \subset \mathscr{M}^{\mathbf{1}}_{\mathbb{T}^2}$ satisfying Condition 2.5.

3. ⊠⊠-Idempotent distributions

Let $\mu \in \mathscr{P}_{\boldsymbol{X}}$. A measure $\mu' \in \mathscr{P}_{\boldsymbol{X}}$ is called a $\Diamond$-factor of μ if $\mu = \mu' \Diamond \mu''$ for some $\mu'' \in \mathscr{P}_{\boldsymbol{X}}$. Particularly, μ is said to be $\Diamond$-idempotent when $\mu' = \mu = \mu''$. Idempotent distributions and other related subjects in classical probability have been extensively studied in [16]. It is to questions of these sorts in the bi-free probability theory that the present section is devoted.

The normalized Lebesgue measure $\mathrm{m} = d\theta/(2\pi)$ on $\mathbb{T}$ is the only $\boxtimes$-idempotent element except for the trivial one, the Dirac measure at 1. On $\mathbb{T}^2$, the probability measure

$$P(B) = \mathrm{m}(\{s \in \mathbb{T} : (s, \bar{s}) \in B\}), \quad B \in \mathscr{B}_{\mathbb{T}^2},$$

is $\circledast$-idempotent because $m_{p,q}(P) = 1$ for $p = q \in \mathbb{Z}$ and zero otherwise. As a matter of fact, this singularly continuous measure is also $\boxtimes\boxtimes$-idempotent proved below.

The following result is a direct consequence of Voiculescu's two-bands moment formula in [21, Lemma 2.1] and we provide its proof and notations for the later use.

Proposition 3.1. *A $\boxtimes\boxtimes$-idempotent distribution in $\mathscr{P}_{\mathbb{T}^2}$ is one of five types* $\delta_{(1,1)}$, $\mathrm{m}\times\delta_1$, $\delta_1\times\mathrm{m}$, $\mathrm{m}\times\mathrm{m}$, *and* P. *A measure in $\mathscr{P}_{\mathbb{T}^2}$ is $\boxtimes\boxtimes^{\mathrm{op}}$-idempotent if and only if it is* $\delta_{(1,1)}$, $\mathrm{m}\times\delta_1$, $\delta_1\times\mathrm{m}$, $\mathrm{m}\times\mathrm{m}$, *or* $P^\star$.

Proof. Let ν be $\boxtimes\boxtimes$-idempotent. Since each marginal satisfies $\nu^{(j)} = \nu^{(j)} \boxtimes \nu^{(j)}$, it follows that $\nu^{(j)}$ is $\boxtimes$-infinitely divisible. If $\nu^{(j)}$ has nonzero mean, then $\Sigma_{\nu^{(j)}}(0) = 1$, yielding $\nu^{(j)} = \delta_1$ by [4, Lemma 2.7]. Otherwise, we can infer from [4, Lemma 6.1] that $\nu^{(j)} = \mathrm{m}$. Thus, consideration given to the case $\nu^{(1)} = \mathrm{m} = \nu^{(2)}$ is sufficient to complete the proof. To continue the proof, we realize $\nu = \nu_1 \boxtimes\boxtimes \nu_2$ as the distribution of $(u, v) = (u_1u_2, v_1v_2)$, where $(u_j, v_j) = (\lambda_j(S_j), \rho_j(T_j))$, $j = 1, 2$, are bi-free unitary faces respectively following $\nu_j = \nu$ in the C^*-probability space $(B(\mathcal{H}), \varphi_\xi)$, as constructed in Section 2C.

From $\varphi_\xi(u_j) = 0$ for $j = 1, 2$, it follows that $S_j^{\pm 1}\xi_j \in \mathring{\mathcal{H}}_j = \mathcal{H}_j \ominus \mathbb{C}\xi_j$, which supplies a simplistic representation for $u^p\xi$ for any $p \in \mathbb{N}$, namely,

$$u^p\xi = ((S_1\xi_1)\otimes(S_2\xi_2))^{\otimes p} \quad \text{and} \quad u^{-p}\xi = ((S_2^{-1}\xi_2)\otimes(S_1^{-1}\xi_1))^{\otimes p} \tag{3-1}$$

lying in spaces $(\mathring{\mathcal{H}}_1 \otimes \mathring{\mathcal{H}}_2)^{\otimes p}$ and $(\mathring{\mathcal{H}}_2 \otimes \mathring{\mathcal{H}}_1)^{\otimes p}$. Similarly, $\varphi_\xi(v_1) = 0 = \varphi_\xi(v_2)$ implies that

$$v^q \xi = ((T_2\xi_2) \otimes (T_1\xi_1))^{\otimes q} \in (\mathring{\mathcal{H}}_2 \otimes \mathring{\mathcal{H}}_1)^{\otimes q}, \quad q \in \mathbb{N}. \tag{3-2}$$

We consequently arrive at that for $(p, q) \in (\mathbb{Z}\backslash\{0\}) \times (\mathbb{N} \cup \{0\})$,

$$m_{p,q}(\nu) = \varphi_\xi(u^p v^q) = \langle v^q \xi, u^{-p}\xi\rangle = \delta_{p,q}[\varphi_\xi(u_1v_1)\,\varphi_\xi(u_2v_2)]^p = \delta_{p,q}\, m_{1,1}(\nu)^{2p}$$

and that $m_{0,q}(\nu) = \varphi_\xi(v^q) = \delta_{0,q}$ for $q \in \mathbb{N} \cup \{0\}$. If $m_{1,1}(\nu) = 0$, then $m_{p,q}(\nu) = 0$ for any $(p, q) \in \mathbb{Z}^2\backslash\{(0, 0)\}$, which occurs only when $\nu = \mathrm{m} \times \mathrm{m}$. If $m_{1,1}(\nu) \neq 0$, then the equation $m_{1,1}(\nu) = m_{1,1}(\nu)^2$ results in $m_{1,1}(\nu) = 1$, yielding $\nu = P$ as they have a common 2-moment sequence.

The $\boxtimes\boxtimes^{\mathrm{op}}$-idempotent elements can be easily ascertained by formula (2-3) and established results. This finishes the proof. $\square$

It is known that for any $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$, $m_{p,q}(\nu_1 \boxtimes\boxtimes\, \nu_2) = m_{p,q}(\nu_1)\, m_{p,q}(\nu_2)$ holds when $(p, q) = (0, 1), (1, 0)$.

Lemma 3.2. *Identities*

$$m_{1,1}(\nu_1 \boxtimes\boxtimes\, \nu_2) = m_{1,1}(\nu_1)\, m_{1,1}(\nu_2)$$

and

$$m_{1,-1}(\nu_1 \boxtimes\boxtimes^{\mathrm{op}} \nu_2) = m_{1,-1}(\nu_1)\, m_{1,-1}(\nu_2)$$

hold for any $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$.

Proof. Following the notations in Section 2C, let $\alpha_j = \langle S_j^{-1}\xi_j, \xi_j\rangle$, $\beta_j = \langle T_j\xi_j, \xi_j\rangle$, $h_j = S_j^{-1}\xi_j - \alpha_j\xi_j$, and $k_j = T_j\xi_j - \beta_j\xi_j$ for $j = 1, 2$. Then

$$m_{1,1}(\nu_j) = \langle T_j\xi_j, S_j^{-1}\xi_j\rangle = \overline{\alpha_j}\beta_j + \langle k_j, h_j\rangle.$$

On the other hand, we have $u_2^{-1}u_1^{-1}\xi = \alpha_1\alpha_2\xi + \alpha_2 h_1 + \alpha_1 h_2 + h_2 \otimes h_1$ and $v_1v_2\xi = \beta_1\beta_2\xi + \beta_2 k_1 + \beta_1 k_2 + k_2 \otimes k_1$. Thus, the first desired result follows from the representation of $m_{1,1}(\nu_j)$ given above and the computations

$$\begin{aligned} m_{1,1}(\nu_1 \boxtimes\boxtimes\, \nu_2) &= \langle v_1v_2\xi, u_2^{-1}u_1^{-1}\xi\rangle \\ &= \overline{\alpha_1\alpha_2}\,\beta_1\beta_2 + \overline{\alpha_2}\,\beta_2\langle k_1, h_1\rangle + \overline{\alpha_1}\,\beta_1\langle k_2, h_2\rangle + \langle k_1, h_1\rangle\langle k_2, h_2\rangle. \end{aligned}$$

Thanks to (2-3) and the first result, we obtain

$$m_{1,-1}(\nu_1 \boxtimes\boxtimes^{\mathrm{op}} \nu_2) = m_{1,1}(\nu_1^\star \boxtimes\boxtimes\, \nu_2^\star) = m_{1,-1}(\nu_1)\, m_{1,-1}(\nu_2). \qquad \square$$

Remark 3.3. Results in Lemma 3.2 can also be easily derived by the moment-cumulant formula and vanishing of bi-free mixed cumulants [8].

In the sequel, except for $\delta_{(1,1)}$, the other four $\boxtimes\boxtimes$-idempotent distributions are called nontrivial. The abusing notation $0^0 = 1$ is used in the following proposition and elsewhere.

Proposition 3.4. *Let* $\nu \in \mathscr{P}_{\mathbb{T}^2}$.

(1) ν *has the* $\boxtimes\boxtimes$*-factor* $\mathrm{m} \times \delta_1$ *if and only if* $\nu = \mathrm{m} \times \nu^{(2)}$.

(2) ν *has the* $\boxtimes\boxtimes$*-factor* $\delta_1 \times \mathrm{m}$ *if and only if* $\nu = \nu^{(1)} \times \mathrm{m}$.

(3) ν *has the* $\boxtimes\boxtimes$*-factor* $\mathrm{m} \times \mathrm{m}$ *if and only if* $\nu = \mathrm{m} \times \mathrm{m}$.

(4) P *is a* $\boxtimes\boxtimes$*-factor of* ν *if and only if*

$$m_{p,q}(\nu) = \delta_{p,q}\, m_{1,1}(\nu)^p, \quad (p, q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\}), \tag{3-3}$$

where $\delta_{p,q}$ *is the Kronecker function of* p *and* q.

Statements (1)–(3) *remain true if the convolution* $\boxtimes\boxtimes$ *is replaced with* $\boxtimes\boxtimes^{\mathrm{op}}$. *Moreover*, $P^\star$ *is a* $\boxtimes\boxtimes^{\mathrm{op}}$*-factor of* ν *if and only if*

$$m_{p,q}(\nu) = \delta_{p,-q} m_{1,-1}(\nu)^p, \quad (p, q) \in \mathbb{Z} \times (-\mathbb{N} \cup \{0\}). \tag{3-4}$$

Remark 3.5. For negative integers q, by taking complex conjugate, formula (3-3) becomes $m_{p,q}(\nu) = \delta_{p,q}\, m_{-1,-1}(\nu)^{-p}$.

Proof. Write $\nu = \nu_1 \boxtimes\boxtimes \nu_2$, where neither ν_1 nor ν_2 is $\delta_{(1,1)}$. We shall stay employing the notations for ν_1, ν_2 introduced in Section 2C to accomplish the proof.

First, let $\nu_2 = \mathrm{m} \times \delta_1$. In order to obtain $\nu = \mathrm{m} \times \nu^{(2)}$ as desired in (1), it amounts to proving that $m_{p,q}(\nu) = 0$ for any $p \in \mathbb{Z}^\times = \mathbb{Z}\backslash\{0\}$ and $q \in \mathbb{Z}$ because a probability measure on the bi-torus is uniquely determined by its moments. To this end, we take operator models (u_1, v_1) and (u_2, v_2) as in the proof of Proposition 3.1. A consequence of [20, Lemma 5.3] is that $m_{p,q}(\nu) = \varphi_\xi((u_1u_2)^p(v_1v_2)^q)$ can be expressed as a sum of products of quantities from the set $\{\varphi_\xi(u_i^{m_i} v_i^{n_i}) : m_1, m_2, n_1, n_2 \in \mathbb{Z}\}$. Moreover, since $p \neq 0$, each product in the sum contains at least one factor $\varphi_\xi(u_2^{m_2} v_2^{n_2})$ with $m_2 \neq 0$, which vanishes because (u_2, v_2) follows $\mathrm{m} \times \delta_1$. This verifies the "only if" part of (1). The "if" part of (1) is a direct consequence of (2-2). Alternatively, one can obtain the result by considering the measure $\tilde{\nu} = \nu \boxtimes\boxtimes (\mathrm{m} \times \delta_1)$. Indeed, $\tilde{\nu}$ has the $\boxtimes\boxtimes$-factor $\mathrm{m} \times \delta_1$, and so $\tilde{\nu} = \mathrm{m} \times \tilde{\nu}^{(2)}$ by the result proved above. Since $\tilde{\nu}^{(2)} = \nu^{(2)} \boxtimes \delta_1 = \nu^{(2)}$, it follows that

$$m_{p,q}(\nu) = m_p(\mathrm{m})\, m_q(\nu^{(2)}) = m_p(\mathrm{m})\, m_q(\tilde{\nu}^{(2)}) = m_{p,q}(\tilde{\nu})$$

for any $p, q \in \mathbb{Z}$. Hence we have $\nu = \tilde{\nu}$, which proves the "if" part.

By similar reasonings, (2) holds. If $\mathrm{m} \times \mathrm{m}$ is a $\boxtimes\boxtimes$-factor of ν, then so are distributions $\mathrm{m} \times \delta_1$ and $\delta_1 \times \mathrm{m}$, from which we see that (3) holds by (1) and (2).

Finally, we suppose $\nu_2 = P$ and justify (4). In view of P being $\boxtimes\boxtimes$-idempotent, $\nu_1 \boxtimes\boxtimes P$ may take the place of ν_1, and we do assume so below, without affecting the convolution $\nu = \nu_1 \boxtimes\boxtimes P$. Since $m_{p,q}(\nu_1) = 0 = m_{p,q}(\nu_2)$ for $(p, q) = (0, 1), (1, 0)$, formulas (3-1) and (3-2), together with Lemma 3.2, allow one to see that

$$m_{p,q}(\nu) = \langle v^q \xi, u^{-p} \xi \rangle = \delta_{p,q} \langle S_1 T_1 \xi_1, \xi_1 \rangle^p \langle S_2 T_2 \xi_2, \xi_2 \rangle^p = \delta_{p,q}\, m_{1,1}(\nu)^p$$

for $(p, q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$. This furnishes all mixed moments (3-3) of ν.

That $\nu \boxtimes\boxtimes P$ has the $\boxtimes\boxtimes$-factor P and the established result implies that for any $(p, q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$, $m_{p,q}(\nu \boxtimes\boxtimes P) = \delta_{p,q}\, m_{1,1}(\nu \boxtimes\boxtimes P)^p = \delta_{p,q}\, m_{1,1}(\nu)^p$ by Lemma 3.2. Thus $m_{p,q}(\nu \boxtimes\boxtimes P) = m_{p,q}(\nu)$ or, equivalently, $\nu \boxtimes\boxtimes P = \nu$ if (3-3) holds, proving the converse of (4).

All assertions regarding $\boxtimes\boxtimes^{\mathrm{op}}$-idempotent factors are direct consequences of statements (1)–(4), equation (2-3), and the formula $m_{p,q}(\nu^\star) = m_{p,-q}(\nu)$. □

Remark 3.6. From $m_{1,1}(\mathrm{m} \times \mathrm{m}) = 0$, assertion (4) of Proposition 3.4 can be strengthened as that P is the only nontrivial $\boxtimes\boxtimes$-idempotent factor of $\nu \in \mathscr{P}_{\mathbb{T}^2}$ if and only if $m_{1,1}(\nu) \neq 0$ and (3-3) holds.

Remark 3.7. The notions of bi-R-diagonality and Haar bi-unitary elements were first introduced in [18, Example 4.7] and [7, Definition 10.1.2], respectively. A Haar bi-unitary element is a bipartite pair having distribution $P^\star$ [15, Definition 2.15]. The opposite multiplication plays a key role when characterizing bi-R-diagonal pairs in terms of Haar bi-unitary elements [15, Theorem 4.4]. Moreover, measures $\nu \in \mathscr{P}_{\mathbb{T}^2}$ satisfying (3-4) are bi-R-diagonal because of $\nu = \nu \boxtimes\boxtimes^{\mathrm{op}} P^\star$ according to Proposition 3.4 and because of [15, Theorem 4.4].

For any $c \in \mathbb{D}$, define

$$d\kappa_c(s) = \frac{1 - |c|^2}{|1 - \bar{c}s|^2}\, d\mathrm{m}(s),$$

which is the probability measure on $\mathbb{T}$ induced by the Poisson kernel. It is the normalized Haar measure on $\mathbb{T}$ in case $c = 0$. By taking the weak limit we define $\kappa_c = \delta_c$ for $c \in \mathbb{T}$. Alternatively, κ_c with $c \in \mathbb{D} \cup \mathbb{T}$ is the unique probability measure on $\mathbb{T}$ determined by the requirement $m_p(\kappa_c) = c^p$ for $p \in \mathbb{N}$. Also, we have $m_p(\kappa_c) = \bar{c}^{|p|}$ for $p \in -\mathbb{N}$.

Observe that for any $c, d \in \mathbb{D} \cup \mathbb{T}$, we have

$$\nu \boxtimes\boxtimes (\kappa_c \times \kappa_d) = \nu \circledast (\kappa_c \times \kappa_d), \quad \nu \in \mathscr{P}_{\mathbb{T}^2}. \tag{3-5}$$

To see this, consider ν and $\kappa_c \times \kappa_d$ as the distributions of two bi-free commuting unitary faces (u_1, v_1) and (u_2, v_2), respectively, in some C^*-probability space $(B(\mathcal{H}), \varphi_\xi)$. Observe that both pairs of faces (u_2, v_2) and $(cI_{B(\mathcal{H})}, dI_{B(\mathcal{H})})$ are commuting, bi-free from (u_1, v_1), and have the same (p, q)-moments $c^p d^q$ for $(p, q) \in (\mathbb{N} \cup \{0\})^2$. In view of the universal calculation formula for mixed moments

[20, Lemma 5.2], we may replace (u_2, v_2) with $(cI_{B(\mathcal{H})}, dI_{B(\mathcal{H})})$. This entails

$$\varphi_\xi((u_1u_2)^p(v_1v_2)^q) = c^p d^q \varphi_\xi(u_1^p v_1^q),$$

and hence $m_{p,q}(\nu \boxtimes\boxtimes (\kappa_c \times \kappa_d)) = m_{p,q}(\nu \circledast (\kappa_c \times \kappa_d))$ for $(p, q) \in (\mathbb{N} \cup \{0\})^2$. Similarly, one can obtain the same identity for $(p, q) \in (\mathbb{N} \cup \{0\}) \times (-\mathbb{N} \cup \{0\})$ by using that (u_2, v_2) and $(cI_{B(\mathcal{H})}, (1/\bar{d})I_{B(\mathcal{H})})$ have the same (p, q)-moments $c^p \bar{d}^{|q|}$. Therefore, we justify (3-5).

A special case of (3-5) is the validity of

$$(\kappa_{c_1} \times \kappa_{d_1}) \boxtimes\boxtimes (\kappa_{c_2} \times \kappa_{d_2}) = \kappa_{c_1c_2} \times \kappa_{d_1d_2} = (\kappa_{c_1} \times \kappa_{d_1}) \circledast (\kappa_{c_2} \times \kappa_{d_2})$$

for any $c_1, c_2, d_1, d_2 \in \mathbb{D} \cup \mathbb{T}$, yielding the following results.

Proposition 3.8. *The measure $\kappa_c \times \kappa_d$ is both $\circledast$- and $\boxtimes\boxtimes$-infinitely divisible for any $c, d \in \mathbb{D} \cup \mathbb{T}$.*

Proposition 3.9. *Any $\nu \in \mathscr{P}_{\mathbb{T}^2}$ with moments satisfying* (3-3) *can be expressed as $P \circledast (\kappa_c \times \delta_1)$, where $c = m_{1,1}(\nu)$. In particular, ν is both $\circledast$- and $\boxtimes\boxtimes$-infinitely divisible.*

Proof. Clearly, we have $m_{p,q}(P \circledast (\kappa_c \times \delta_1)) = \delta_{p,q}\, c^p$ for $(p, q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$, and hence $\nu = P \circledast (\kappa_c \times \delta_1)$. The $\circledast$-infinitely divisibility of P and Proposition 3.8 yield that ν is $\circledast$-infinitely divisible. Also, the identity $\nu = P \boxtimes\boxtimes (\kappa_c \times \delta_1)$ obtained by (3-5) proves the $\boxtimes\boxtimes$-infinite divisibility of ν. □

A consequence of (3-5) and Proposition 3.9 is that the following identity holds for every $\nu \in \mathscr{P}_{\mathbb{T}^2}$:

$$P \boxtimes\boxtimes (\kappa_{m_{1,1}(\nu)} \times \delta_1) = P \boxtimes\boxtimes \nu = P \circledast (\kappa_{m_{1,1}(\nu)} \times \delta_1). \tag{3-6}$$

The following is a bi-free multiplicative analog of the classical multiplicative limit theorem.

Theorem 3.10. *Let $\{\nu_{nk_n}\}_{n\geq,1\leq k\leq k_n}$ be an infinitesimal triangular array in $\mathscr{P}_{\mathbb{T}^2}$ and $\{\boldsymbol{\xi}_n\}$ be a sequence in $\mathbb{T}^2$. If the sequence in* (1-3) *has a weak limit ν, then ν is $\boxtimes\boxtimes$-infinitely divisible. If $m_{1,0}(\nu) \neq 0 \neq m_{0,1}(\nu)$, then $m_{1,1}(\nu) \neq 0$. Moreover, if $m_{1,0}(\nu) = 0$, then $\nu = \mathrm{m} \times \nu^{(2)}$ and if $m_{0,1}(\nu) = 0$, then $\nu = \nu^{(1)} \times \mathrm{m}$.*

Proof. We separately consider three possible statuses (i) $m_{1,0}(\nu) \neq 0 \neq m_{0,1}(\nu)$, (ii) $m_{1,0}(\nu) = 0 \neq m_{0,1}(\nu)$ (the case $m_{1,0}(\nu) \neq 0 = m_{0,1}(\nu)$ is treated similarly to (ii)), and (iii) $m_{1,0}(\nu) = 0 = m_{0,1}(\nu)$.

(i) Once we can prove that $m_{1,1}(\nu) \neq 0$, then the $\boxtimes\boxtimes$-infinite divisibility of ν will follow from [13, Theorem 4.2]. Assume to the contrary that $m_{1,1}(\nu) = 0$, which together with Lemma 3.2 implies that as $n \to \infty$,

$$m_{1,1}(\delta_{\boldsymbol{\xi}_n})\, m_{1,1}(\nu_{n1}) \cdots m_{1,1}(\nu_{nk_n}) = m_{1,1}(\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes \nu_{n1} \boxtimes\boxtimes \cdots \boxtimes\boxtimes \nu_{nk_n}) \to 0.$$

Then there exists a sequence $\{\ell_n\} \subset \mathbb{N}$ so that as $n \to \infty$, we have

$$m_{1,1}(\delta_{\xi_n})\, m_{1,1}(\nu_{n1}) \cdots m_{1,1}(\nu_{n\ell_n}) \to 0 \quad \text{and} \quad m_{1,1}(\nu_{n,\ell_n+1}) \cdots m_{1,1}(\nu_{nk_n}) \to 0,$$

namely, one sees from Lemma 3.2 that

$$m_{1,1}(\delta_{\xi_n} \boxtimes\!\!\boxtimes\, \nu_{n1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{n\ell_n}) \to 0 \quad \text{and} \quad m_{1,1}(\nu_{n,\ell_n+1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{nk_n}) \to 0$$

as $n \to \infty$. To obtain such a sequence $\{\ell_n\}$, one can select, for example,

$$\ell_n = \min\bigl\{k : |m_{1,1}(\delta_{\xi_n} \boxtimes\!\!\boxtimes\, \nu_{n1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{nk})| \le |m_{1,1}(\delta_{\xi_n} \boxtimes\!\!\boxtimes\, \nu_{n1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{nk_n})|^{1/2}\bigr\}.$$

One may assume, by passing to a subsequence if needed, that

$$\delta_{\xi_n} \boxtimes\!\!\boxtimes\, \nu_{n1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{n\ell_n} \Rightarrow \nu_1' \in \mathscr{P}_{\mathbb{T}^2}, \qquad \nu_{n,\ell_n+1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{nk_n} \Rightarrow \nu_1'' \in \mathscr{P}_{\mathbb{T}^2}.$$

Then we have $\nu = \nu_1' \boxtimes\!\!\boxtimes\, \nu_1''$ and $m_{1,1}(\nu_1') = 0 = m_{1,1}(\nu_1'')$. Also, the formula

$$m_{1,0}(\nu) = m_{1,0}(\nu_1')\, m_{1,0}(\nu_1'')$$

indicates that either $|m_{1,0}(\nu_1')| \ge |m_{1,0}(\nu)|^{1/2}$ or $|m_{1,0}(\nu_1'')| \ge |m_{1,0}(\nu)|^{1/2}$ must occur; assume, without loss of generality, that the first inequality is valid. Carrying out the same arguments on ν_1' allows us to obtain $\nu_2', \nu_2'' \in \mathscr{P}_{\mathbb{T}^2}$ fulfilling requirements $\nu_1' = \nu_2' \boxtimes\!\!\boxtimes\, \nu_2''$, $m_{1,1}(\nu_2') = 0 = m_{1,1}(\nu_2'')$, and $|m_{1,0}(\nu_2')| \ge |m_{1,0}(\nu_1')|^{1/2}$. Continuing this process then results in the existence of sequences $\{\nu_n'\}, \{\nu_n''\} \subset \mathscr{P}_{\mathbb{T}^2}$ for which $\nu_n' = \nu_{n+1}' \boxtimes\!\!\boxtimes\, \nu_{n+1}''$, $m_{1,1}(\nu_n') = 0 = m_{1,1}(\nu_n'')$, and $|m_{1,0}(\nu_{n+1}')| \ge |m_{1,0}(\nu_n')|^{1/2}$ hold.

One has $\nu = \nu_n' \boxtimes\!\!\boxtimes\, \nu_n'''$ for some $\nu_n''' \in \mathscr{P}_{\mathbb{T}^2}$ and $|m_{1,0}(\nu_n')| \ge |m_{1,0}(\nu)|^{1/2^n}$. Passing to subsequences if needed again, let $\nu_n' \Rightarrow \nu_1 \in \mathscr{P}_{\mathbb{T}^2}$ and $\nu_n''' \Rightarrow \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$, and so $\nu = \nu_1 \boxtimes\!\!\boxtimes\, \nu_2$, $m_{1,1}(\nu_1) = 0$, and $|m_{1,0}(\nu_1)| = 1$. The last identity reveals that $\nu_1 = \delta_\alpha \times \nu_1^{(2)}$, $\alpha = m_{1,0}(\nu_1) \in \mathbb{T}$. Also, using $0 \ne m_{0,1}(\nu) = m_{0,1}(\nu_1)\, m_{0,1}(\nu_2)$ we get $m_{0,1}(\nu_1) \ne 0$. However, these discussions would lead to $m_{1,1}(\nu_1) = \alpha m_{0,1}(\nu_1) \ne 0$, a contradiction. Hence we must have $m_{1,1}(\nu) \ne 0$, as desired.

(ii) Note that the marginal $\nu^{(2)}$ is $\boxtimes$-infinitely divisible by [2, Theorem 2.1]. The $\boxtimes\!\!\boxtimes$-infinite divisibility of ν will follow immediately if one can argue that $\nu = \mathrm{m} \times \nu^{(2)}$. The proof, presented below, is basically similar to that of (1).

First, applying the strategy employed in the first paragraph of (1) to $m_{1,0}(\nu) = 0$ indicates the presence of $\ell_n \in \mathbb{N}$ satisfying $m_{1,0}(\delta_{\xi_n} \boxtimes\!\!\boxtimes\, \nu_{n1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{n\ell_n}) \to 0$ and $m_{1,0}(\nu_{n,\ell_n+1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{nk_n}) \to 0$ as $n \to \infty$. Assume, dropping a subsequence if necessary, that

$$\delta_{\xi_n} \boxtimes\!\!\boxtimes\, \nu_{n1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{n\ell_n} \Rightarrow \nu_1' \in \mathscr{P}_{\mathbb{T}^2}, \qquad \nu_{n,\ell_n+1} \boxtimes\!\!\boxtimes \cdots \boxtimes\!\!\boxtimes\, \nu_{nk_n} \Rightarrow \nu_1'' \in \mathscr{P}_{\mathbb{T}^2}.$$

Thus we have $\nu = \nu_1' \boxtimes\!\!\boxtimes \nu_1''$ and $m_{1,0}(\nu_1') = 0 = m_{1,0}(\nu_1'')$. We may further assume $|m_{0,1}(\nu_1')| \geq |m_{0,1}(\nu)|^{1/2}$. Mimicking the arguments in (1) constructs sequences $\{\nu_n'\}, \{\nu_n''\} \subset \mathscr{P}_{\mathbb{T}^2}$ meeting conditions $\nu = \nu_n' \boxtimes\!\!\boxtimes \nu_n''$, $m_{1,0}(\nu_n') = 0 = m_{1,0}(\nu_n'')$, and $|m_{0,1}(\nu_n')| \geq |m_{0,1}(\nu)|^{1/2^n}$. Passing to subsequences if needed again, assume that $\nu_n' \Rightarrow \nu_1 \in \mathscr{P}_{\mathbb{T}^2}$ and $\nu_n'' \Rightarrow \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$. Then we come to that $\nu = \nu_1 \boxtimes\!\!\boxtimes \nu_2$, $m_{1,0}(\nu_1) = 0 = m_{1,0}(\nu_2)$, and $\nu_1^{(2)} = \delta_\alpha$ for some $\alpha \in \mathbb{T}$. To proceed the proof, we shall use notations introduced in Section 2C. Since $v_1 = \alpha I_{B(\mathcal{H})}$, it follows that $v^q \xi = \alpha^q v_2^q \xi \in \mathbb{C}\xi \oplus \mathring{\mathcal{H}}_2$ for any $q \in \mathbb{Z}$. Thus, equation (3-1) implies that $m_{p,q}(\nu) = \langle v^q \xi, u^{-p} \xi \rangle = 0$ for any $(p,q) \in \mathbb{N} \times \mathbb{Z}$, proving $\nu = \mathrm{m} \times \nu^{(2)}$.

(iii) In this case, we have $\nu^{(1)} = \mathrm{m} = \nu^{(2)}$ by [2, Theorem 2.1] and [4, Lemma 6.1]. Further, one can employ the proof in (ii) to show that there are $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$ so that $\nu = \nu_1 \boxtimes\!\!\boxtimes \nu_2$ and $m_{1,0}(\nu_1) = 0 = m_{1,0}(\nu_2)$. Then $m_{0,1}(\nu_1)\, m_{0,1}(\nu_2) = m_{0,1}(\nu) = 0$. If $m_{0,1}(\nu_1) = 0 = m_{0,1}(\nu_2)$, then (3-1) and (3-2) yield that $m_{p,q}(\nu) = \delta_{p,q}\, m_{1,1}(\nu)^p$ for $(p,q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$, whence ν is $\boxtimes\!\!\boxtimes$-infinitely divisible by Proposition 3.9. For the other case, say $m_{0,1}(\nu_1) \neq 0$, the established conclusion in (ii) then shows that $\nu_1 = \mathrm{m} \times \nu_1^{(2)}$. In such a situation, the measure ν_1, as well as ν, has the $\boxtimes\!\!\boxtimes$-factor $\mathrm{m} \times \delta_1$. Thus, Proposition 3.4 says that $\nu = \mathrm{m} \times \mathrm{m}$, which is clearly $\boxtimes\!\!\boxtimes$-infinitely divisible. □

Corollary 3.11. *The set $\mathcal{ID}(\boxtimes\!\!\boxtimes)$ is weakly closed.*

We are now in a position to characterize distributions in $\mathcal{ID}(\boxtimes\!\!\boxtimes)$ carrying no nontrivial $\boxtimes\!\!\boxtimes$-idempotent factors.

Theorem 3.12. *In order that a measure $\nu \in \mathcal{ID}(\boxtimes\!\!\boxtimes)$ contains no nontrivial $\boxtimes\!\!\boxtimes$-idempotent factor, it is necessary and sufficient that $m_{1,0}(\nu) \neq 0 \neq m_{0,1}(\nu)$, in which case $m_{1,1}(\nu) \neq 0$.*

Proof. According to Proposition 3.4, only the necessity requires a proof. We merely prove that ν has a nontrivial $\boxtimes\!\!\boxtimes$-idempotent factor when $m_{1,0}(\nu) = 0$, because the case $m_{0,1}(\nu) = 0$ can be handled in the same way. To do so, let $m_{1,0}(\nu) = 0$, and consider two possible cases (i) $m_{0,1}(\nu) = 0$ and (ii) $m_{0,1}(\nu) \neq 0$, which are discussed separately below. Note that $m_{p,0}(\nu) = 0$ for all $p \in \mathbb{N}$ since $\nu^{(1)} = m$.

Case (i): Since $\nu^{(j)} = \mathrm{m}$ for $j = 1, 2$, one can mimic the proof of Proposition 3.1, especially employ equations (3-1) and (3-2), to obtain $m_{p,q}(\nu) = \delta_{p,q}\, m_{1,1}(\nu)^p$ for $(p,q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$. Hence P is a $\boxtimes\!\!\boxtimes$-factor of ν by Proposition 3.4.

Case (ii): To treat this case, let $\nu_n' \in \mathscr{P}_{\mathbb{T}^2}$ be an n-th $\boxtimes\!\!\boxtimes$-convolution root of ν for any $n \in \mathbb{N}$, i.e., $(\nu_n')^{\boxtimes\!\!\boxtimes n} = \nu$. Then we have $\nu = \nu_n' \boxtimes\!\!\boxtimes \nu_n''$, where $\nu_n'' = (\nu_n')^{\boxtimes\!\!\boxtimes (n-1)}$, $m_{1,0}(\nu_n') = 0 = m_{1,0}(\nu_n'')$ and $|m_{0,1}(\nu_n')| = |m_{0,1}(\nu)|^{1/n}$. If ν' and ν'' are any weak limits of $\{\nu_n'\}$ and $\{\nu_n''\}$, respectively, then we further obtain $\nu = \nu' \boxtimes\!\!\boxtimes \nu''$, $m_{1,0}(\nu') = 0 = m_{1,0}(\nu'')$, and $|m_{0,1}(\nu')| = 1$. This leads to $(\nu')^{(2)} = \delta_\alpha$ for some $\alpha \in \mathbb{T}$, which is exactly the situation dealt in the last part of the proof (ii) of

Theorem 3.10. Thus we conclude that $\nu = \mathrm{m} \times \nu^{(2)}$, which has the $\boxtimes\boxtimes$-idempotent factor $\mathrm{m} \times \delta_1$ by Proposition 3.4.

Lastly, we turn to argue that $m_{1,1}(\nu) \neq 0$ if $m_{1,0}(\nu) \neq 0 \neq m_{0,1}(\nu)$. Any sequence $\{\nu_n\}$ satisfying $\nu = \nu_n^{\boxtimes\boxtimes 2^n}$ has a subsequence $\{\nu_{n_j}\}$ converging weakly to $\delta_{\boldsymbol{\xi}}$ for some $\boldsymbol{\xi} \in \mathbb{T}^2$ (see (i) of Theorem 3.10). Then Lemma 3.2 implies that $|m_{1,1}(\nu)|^{2^{-n_j}} = |m_{1,1}(\nu_{n_j})| \to |m_{1,1}(\delta_{\boldsymbol{\xi}})| = 1$, leading to the desired result. □

Propositions 3.4 and 3.9, and Theorem 3.12 readily imply the following.

Corollary 3.13. *Any measure ν in $\mathcal{ID}(\boxtimes\boxtimes)\backslash\mathscr{P}^{\times}_{\mathbb{T}^2}$ is either $\nu^{(1)} \times \mathrm{m}$, $\mathrm{m} \times \nu^{(2)}$, $\mathrm{m} \times \mathrm{m}$ or $P \circledast (\kappa_c \times \delta_1)$, where $\nu^{(1)}$ and $\nu^{(2)}$ are in $\mathcal{ID}(\boxtimes)$ with nonzero mean and $c \in (\mathbb{D} \cup \mathbb{T}) \setminus \{0\}$.*

4. Equivalent conditions on limit theorems

This section is devoted to exploring the associations among the conditions introduced in Section 2D and the following one.

Condition 4.1. Let $\{\rho_n\}$ be a sequence in $\mathscr{M}^{\mathbf{1}}_{\mathbb{T}^d}$.

(iii) There exists some $\rho \in \mathscr{M}^{\mathbf{1}}_{\mathbb{T}^d}$ with $\rho(\{\mathbf{1}\}) = 0$ (i.e., $\rho \in \tilde{\mathscr{M}}^{\mathbf{1}}_{\mathbb{T}^d}$) so that $\rho_n \Rightarrow_{\mathbf{1}} \rho$.

(iv) The following limits exist in $\mathbb{R}$ for any $\boldsymbol{p} \in \mathbb{Z}^d$:

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{\mathscr{U}_\epsilon}\langle \boldsymbol{p}, \Im\boldsymbol{s}\rangle^2\,d\rho_n(\boldsymbol{s}) = Q(\boldsymbol{p}) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\int_{\mathscr{U}_\epsilon}\langle \boldsymbol{p}, \Im\boldsymbol{s}\rangle^2\,d\rho_n(\boldsymbol{s}).$$

Condition 2.5 with $d = 2$ was used in [13, Theorem 3.4] to prove the limit theorem for the bi-free multiplicative convolution, while Condition 4.1 is beneficial for the corresponding classical limit theorem [10]. More properties regarding these two conditions are presented below.

Proposition 4.2. Condition 2.5 *is equivalent to* Condition 4.1, *in which*

$$d\lambda_j(\boldsymbol{s}) = (1 - \Re s_j)\,d\rho(\boldsymbol{s}) + \frac{Q(\boldsymbol{e}_j)}{2}\delta_{\mathbf{1}}(d\boldsymbol{s}), \quad j = 1, \ldots, d, \tag{4-1}$$

$$\int_{\mathbb{T}^d}\|\mathbf{1} - \Re\boldsymbol{s}\|\,d\rho(\boldsymbol{s}) < \infty, \tag{4-2}$$

and the quadratic form $Q(\cdot) = \langle \boldsymbol{A}\,\cdot, \cdot\rangle$ on $\mathbb{Z}^d$ is determined by the positive semidefinitive matrix $\boldsymbol{A} = (a_{j\ell})$ whose entries are

$$a_{j\ell} = L_{j\ell} - \int_{\mathbb{T}^d}(\Im s_j)(\Im s_\ell)\,d\rho(\boldsymbol{s}) \in \mathbb{R}, \quad j, \ell = 1, \ldots, d. \tag{4-3}$$

Moreover, $a_{jj} = 2\lambda_j(\{\mathbf{1}\})$ for $j = 1, \ldots, d$.

Proof. Suppose first that Condition 2.5 is satisfied. Then the relation

$$(1-\Re s_j)\,d\lambda_\ell = (1-\Re s_\ell)\,d\lambda_j, \quad j,\ell = 1,\dots,d,$$

guaranteed by item (i) of Condition 2.5 ensures that the measure

$$d\rho(\boldsymbol{s}) = \frac{1_{\{s_j\neq 1\}}(\boldsymbol{s})}{1-\Re s_j}\,d\lambda_j(\boldsymbol{s}) \tag{4-4}$$

is unambiguous and does not depend on j. In addition, it satisfies requirements $\rho(\mathbb{T}^d\backslash\mathscr{U}_\epsilon) < \infty$ for any $\epsilon > 0$ and (4-2). Hence the measure ρ that we just constructed belongs to $\mathscr{M}^{\mathbf{1}}_{\mathbb{T}^d}$.

To see $\rho_n \Rightarrow_{\mathbf{1}} \rho$, pick a continuous function f on $\mathbb{T}^d$ with support contained within $\mathbb{T}^d\backslash\mathscr{U}_\delta$ for some $\delta > 0$. Then this f produces d continuous functions on $\mathbb{T}^d$, which are

$$f_j(\boldsymbol{s}) = \frac{\operatorname{dist}(U_j,\boldsymbol{s})}{\operatorname{dist}(U_1,\boldsymbol{s})+\cdots+\operatorname{dist}(U_d,\boldsymbol{s})}f(\boldsymbol{s}),$$

where $U_j = \{\boldsymbol{u}\in\mathbb{T}^d : |\arg u_j| < \delta/\sqrt{2d}\}$ and $\operatorname{dist}(U_j,\boldsymbol{s}) = \inf\{\|\arg\boldsymbol{s}-\arg\boldsymbol{u}\| : \boldsymbol{u}\in U_j\}$ for $j = 1,\dots,d$. Obviously, the relation $f = f_1+\cdots+f_d$ holds and each $f_j/(1-\Re s_j)$ is continuous on $\mathbb{T}^d$. These observations and the weak convergence $\lambda_{nj} \Rightarrow \lambda_j$ then yield that

$$\begin{aligned}\int_{\mathbb{T}^d} f(\boldsymbol{s})\,d\rho_n(\boldsymbol{s}) &= \sum_{j=1}^d \int_{\mathbb{T}^d}\frac{f_j(\boldsymbol{s})}{1-\Re s_j}\,d\lambda_{nj}(\boldsymbol{s}) \xrightarrow[n\to\infty]{} \sum_{j=1}^d \int_{\mathbb{T}^d}\frac{f_j(\boldsymbol{s})}{1-\Re s_j}\,d\lambda_j(\boldsymbol{s})\\ &= \int_{\mathbb{T}^d} f(\boldsymbol{s})\,d\rho(\boldsymbol{s}).\end{aligned}$$

Therefore, we have completed the verification of item (iii) of Condition 4.1.

We next demonstrate the validity of the following identities for $1\le j,\ell\le d$,

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{\mathscr{U}_\epsilon}(\Im s_j)(\Im s_\ell)\,d\rho_n = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\int_{\mathscr{U}_\epsilon}(\Im s_j)(\Im s_\ell)\,d\rho_n, \tag{4-5}$$

which confirms that of Condition 4.1(iv). To continue, observe that the mapping $s\mapsto(\Im s)^2/(1-\Re s)$ is continuous on $\mathbb{T}$ and at the origin, it takes value

$$\lim_{\arg s\to 0}\frac{(\Im s)^2}{1-\Re s} = 2. \tag{4-6}$$

Then (4-2), (4-6), and the Hölder inequality imply that $(\Im s_j)(\Im s_\ell)\in L^1(\rho)$ for $j,\ell = 1,\dots,d$. In order to get results (4-3) and (4-5), we examine the following differences which are related to them:

$$D_n(\epsilon) = \int_{\mathbb{T}^d}(\Im s_j)(\Im s_\ell)\,d\rho_n - \int_{\mathbb{T}^d}(\Im s_j)(\Im s_\ell)\,d\rho - \int_{\mathscr{U}_\epsilon}(\Im s_j)(\Im s_\ell)\,d\rho_n,$$

which further splits into the sum of

$$I_{n1}(\epsilon)=\int_{\mathbb{T}^d\setminus\mathscr{U}_\epsilon}(\Im s_j)(\Im s_\ell)\,d\rho_n-\int_{\mathbb{T}^d\setminus\mathscr{U}_\epsilon}(\Im s_j)(\Im s_\ell)\,d\rho$$

and

$$I_2(\epsilon)=-\int_{\mathscr{U}_\epsilon}(\Im s_j)(\Im s_\ell)\,d\rho.$$

Apparently, we have $\lim_{\epsilon\to 0} I_2(\epsilon)=0$ owing to $(\Im s_j)(\Im s_\ell)\in L^1(\rho)$. Next, take an $\epsilon'\in(\epsilon,2\epsilon)$ and an $\epsilon''\in\left(\frac{\epsilon}{2},\epsilon\right)$ with the attributes that $\rho(\partial\mathscr{U}_{\epsilon'})=0$ and $\rho(\partial\mathscr{U}_{\epsilon''})=0$, the presence of which are insured by the finiteness of the measure $1_{\mathbb{T}^d\setminus\mathscr{U}_{\epsilon/2}}\,\rho$ on $\mathbb{T}^d$. Then applying Proposition 2.1 to the established result $\rho_n\Rightarrow_{\mathbf{1}}\rho$ results in

$$\lim_{n\to\infty}\int_{\mathbb{T}^d\setminus\mathscr{U}_{\epsilon'}}(\Im s_j)(\Im s_\ell)\,d\rho_n=\int_{\mathbb{T}^d\setminus\mathscr{U}_{\epsilon'}}(\Im s_j)(\Im s_\ell)\,d\rho.$$

On the other hand, working with the closed subset $F_\epsilon=\{\boldsymbol{s}\in\mathbb{T}^d:\epsilon''\le\|\arg\boldsymbol{s}\|\le\epsilon'\}$ and employing Proposition 2.1, we come to

$$\begin{aligned}\text{(4-7)}\qquad \left(\limsup_{n\to\infty}\int_{F_\epsilon}|\Im s_j|\,|\Im s_\ell|\,d\rho_n\right)^2&\le\limsup_{n\to\infty}\int_{F_\epsilon}(\Im s_j)^2\,d\rho_n\cdot\int_{F_\epsilon}(\Im s_\ell)^2\,d\rho_n\\&\le\int_{F_\epsilon}(\Im s_j)^2\,d\rho\cdot\int_{F_\epsilon}(\Im s_\ell)^2\,d\rho\to 0\end{aligned}$$

as $\epsilon\to 0$. With the help of the facts $\mathbb{T}^d\setminus\mathscr{U}_{\epsilon''}=(\mathbb{T}^d\setminus\mathscr{U}_\epsilon)\cup(\mathscr{U}_\epsilon\setminus\mathscr{U}_{\epsilon''})$ and $\mathscr{U}_\epsilon\setminus\mathscr{U}_{\epsilon''}\subset F_\epsilon$, we are able to conclude that $\lim_{\epsilon\to 0}\limsup_{n\to\infty}|I_{n1}(\epsilon)|=0$. Consequently, we have shown $\lim_{\epsilon\to 0}\limsup_{n\to\infty}|D_n(\epsilon)|=0$, which together with Condition 2.5(ii) accounts for (4-3) and (4-5) with any indices j and ℓ.

If ϵ' is also chosen so that $\lambda_j(\partial\mathscr{U}_{\epsilon'})=0$, then we draw once again from (4-6) that $a_{jj}=2\lambda_j(\{\mathbf{1}\})$ because

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathscr{U}_\epsilon}|2(1-\Re s_j)-(\Im s_j)^2|\,d\rho_n&\le\limsup_{n\to\infty}\int_{\mathscr{U}_{\epsilon'}}\left|2-\frac{(\Im s)^2}{1-\Re s}\right|\,d\lambda_{nj}\\&=\int_{\mathscr{U}_{\epsilon'}}\left|2-\frac{(\Im s_j)^2}{1-\Re s_j}\right|\,d\lambda_j\xrightarrow[\epsilon\to\infty]{}.\end{aligned}$$

This conclusion and (4-4) give (4-1). It is easy to see that the limits in (iv) of Condition 4.1 are equal to $\langle\boldsymbol{A}\boldsymbol{p},\boldsymbol{p}\rangle$ for any $\boldsymbol{p}\in\mathbb{Z}^d$ (in fact, for any $\boldsymbol{p}\in\mathbb{R}^d$ as well) with $\boldsymbol{A}=(a_{j\ell})$ and $a_{j\ell}$ the value of the limit given in (4-5). Also, it is clear that the quadratic form Q extends to $\mathbb{R}^d$ and is positive therein. Then the positivity of $\boldsymbol{A}\ge 0$ can be gained by that of Q on $\mathbb{R}^d$.

Next, we elaborate that Condition 4.1 implies Condition 2.5. Define λ_j's as in (4-1). These measures thus obtained are all in $\mathscr{M}_{\mathbb{T}^d}$, and the arguments for this go as follows. Select a sequence $\epsilon_m\downarrow 0$ as $m\to\infty$ and $\rho(\{\|\arg\boldsymbol{s}\|=\epsilon_m\})=0$

for each m. Then (iv), along with Proposition 2.1, indicates that for any numbers $m < m'$ both large enough, one has

$$\int_{\{\epsilon_{m'} < \|\arg \boldsymbol{s}\| < \epsilon_m\}} (\Im s_j)^2 \, d\rho(\boldsymbol{s}) \le 1 + Q(\boldsymbol{e}_j).$$

Thanks to monotone convergence theorem, (4-6), and the assumption $\rho(\{\mathbf{1}\}) = 0$, one further gets that for m large enough, $(1 - \Re s_j) 1_{\mathscr{U}_{\epsilon_m}} \in L^1(\rho)$ for any j. This proves that $\lambda_j(\mathbb{T}^d) < \infty$ and $(\Im s_j)^2 \in L^1(\rho)$ for any j.

After the previous preparations, we are in a position to justify the weak convergence $\lambda_{nj} \Rightarrow \lambda_j$. Given a continuous function f on $\mathbb{T}^d$, the difference

$$\left| \int_{\mathbb{T}^d} f \, d\lambda_{nj} - \int_{\mathbb{T}^d} f \, d\lambda_j \right|$$

is dominated by the sum of the following four terms:

$$\begin{aligned}
D_{n1}(m) &= \int_{\mathscr{U}_{\epsilon_m}} |f(\boldsymbol{s}) - f(\mathbf{1})| \, d\lambda_{nj}(\boldsymbol{s}), \\
D_{n2}(m) &= |f(\mathbf{1})| \left| \lambda_{nj}(\mathscr{U}_{\epsilon_m}) - \tfrac{1}{2} Q(\boldsymbol{e}_j) \right|, \\
D_3(m) &= \int_{\mathscr{U}_{\epsilon_m} \setminus \{\mathbf{1}\}} |f| \, d\lambda_j(\boldsymbol{s}), \\
D_{n4}(m) &= \left| \int_{\mathbb{T}^d \setminus \mathscr{U}_{\epsilon_m}} f \, d\lambda_{nj}(\boldsymbol{s}) - \int_{\mathbb{T}^d \setminus \mathscr{U}_{\epsilon_m}} f \, d\lambda_j(\boldsymbol{s}) \right|.
\end{aligned}$$

First, one can show that $\lim_{m\to\infty} \limsup_{n\to\infty} |D_{n2}(m)| = 0$ by applying (4-6) and item (iv) to

$$2\lambda_{nj}(\mathscr{U}_{\epsilon_m}) - Q(\boldsymbol{e}_j) = \int_{\mathscr{U}_{\epsilon_m}} [2(1 - \Re s_j) - (\Im s_j)^2] \, d\rho_n(\boldsymbol{s}) + \int_{\mathscr{U}_{\epsilon_m}} (\Im s_j)^2 \, d\rho_n(\boldsymbol{s}) - Q(\boldsymbol{e}_j).$$

Similarly, one can show

$$\lim_{m\to\infty} \limsup_{n\to\infty} |D_{n1}(m)| \le \tfrac{1}{2} Q(\boldsymbol{e}_j) \cdot \lim_{m\to\infty} \sup_{\boldsymbol{s} \in \mathscr{U}_{\epsilon_m}} |f(\boldsymbol{s}) - f(\mathbf{1})| = 0.$$

On the other hand, the finiteness of $\lambda_j(\mathbb{T}^d)$ leads to

$$\lim_{m\to\infty} D_3(m) \le \|f\|_\infty \lim_{m\to\infty} \lambda_j(\mathscr{U}_{\epsilon_m} \setminus \{\mathbf{1}\}) = 0.$$

That we have $\lim_{n\to\infty} D_{n4}(m) = 0$ for all m evidently follows from Condition 4.1(iii) and Proposition 2.1. Putting all these observations together illustrates $\lambda_{nj} \Rightarrow \lambda_j$.

It remains to deal with (ii) of Condition 2.5, in which the integral is rewritten as

$$\int_{\mathscr{U}_{\epsilon_m}} (\Im s_j)(\Im s_\ell) \, d\rho_n + \int_{\mathbb{T}^d \setminus \mathscr{U}_{\epsilon_m}} (\Im s_j)(\Im s_\ell) \, d\rho_n.$$

For any j, ℓ, taking the operations $\lim_{m\to\infty}\limsup_{n\to\infty}$ and $\lim_{m\to\infty}\liminf_{n\to\infty}$ of the first integral gives the same value $\frac{1}{2}[Q(\boldsymbol{e}_j+\boldsymbol{e}_\ell)-Q(\boldsymbol{e}_j)-Q(\boldsymbol{e}_\ell)]$, while doing the same thing to the second integral yields the value $\int_{\mathbb{T}^d}(\Im s_j)(\Im s_\ell)\,d\rho$ because of $\rho_n \Rightarrow_{\mathbf{1}} \rho$ and $(\Im s_j)^2+(\Im s_\ell)^2 \in L^1(\rho)$. This finishes the proof of the proposition. □

An intuitive thought is that measures on $\mathbb{T}^d$ obtained by rotating measures within controllable angles maintain the same structural properties, such as Condition 4.1, as the original ones. The statement and its rigorous proof are given below.

Proposition 4.3. *Suppose that $\{\nu_{nk}\} \subset \mathscr{P}_{\mathbb{T}^d}$ is a triangular array for which the measure $\rho_n = \sum_{k=1}^{k_n}\nu_{nk}$ satisfies* Condition 4.1. *If an array $\{\boldsymbol{\theta}_{nk}\} \subset (-\pi,\pi]^d$ fulfills the condition*

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}(\mathbf{1}-\cos\boldsymbol{\theta}_{nk}) = \mathbf{0}, \tag{4-8}$$

then Condition 4.1 *is still applicable to measures $\tilde{\rho}_n(\,\cdot\,) = \sum_{k=1}^{k_n}\nu_{nk}(\,\cdot\,e^{i\boldsymbol{\theta}_{nk}})$, in which $\tilde{\rho}_n \Rightarrow_{\mathbf{1}} \rho$ and ρ_n and $\tilde{\rho}_n$ define the same quadratic form in* Condition 4.1(iv).

Proof. First of all, (4-8) reveals that $\lim_n \max_k \|\boldsymbol{\theta}_{nk}\| = 0$. We now argue that $\tilde{\rho}_n \Rightarrow_{\mathbf{1}} \rho$ as well by using Proposition 2.1. To do so, pick a closed subset $F \subset \mathbb{T}^d\backslash\mathscr{U}_r$ for some $r>0$. Since $\rho(F)<\infty$, it follows that given any $\delta>0$, there exists a closed set $F' \subset \mathbb{T}^d\backslash\mathscr{U}_{r/2}$ such that $e^{i\boldsymbol{\theta}_{nk}}F \subset F'$ for all sufficiently large n and for all $1\le k\le k_n$, and $\rho(F'\backslash F)<\delta$. Then

$$\tilde{\rho}_n(F) = \sum_k \nu_{nk}(e^{i\boldsymbol{\theta}_{nk}}F) \le \sum_k \nu_{nk}(F') = \rho_n(F')$$

implies that $\limsup_{n\to\infty}\tilde{\rho}_n(F) \le \limsup_{n\to\infty}\rho_n(F') \le \rho(F') \le \rho(F)+\delta$. Consequently, we arrive at the inequality $\limsup_{n\to\infty}\tilde{\rho}_n(F) \le \rho(F)$. In the same vein, one can show that $\liminf_{n\to\infty}\tilde{\rho}_n(G) \ge \rho(G)$ for any set G which is open and bounded away from $\mathbf{1}$. Hence $\tilde{\rho}_n \Rightarrow_{\mathbf{1}} \rho$ by Proposition 2.1.

Next, we turn to demonstrate that both ρ_n and $\tilde{\rho}_n$ bring out the tantamount quantities in (4-5), which asserts that the quadratic form in (iv) output by them is unchanged on $\mathbb{Z}^d$. Any index n considered below is always sufficiently large. In the case $j=\ell$, we have the estimate

$$\begin{aligned}\int_{\mathscr{U}_\epsilon}(\Im s_j)^2\,d\tilde{\rho}_n(\boldsymbol{s}) &= \sum_{k=1}^{k_n}\int_{e^{i\boldsymbol{\theta}_{nk}}\mathscr{U}_\epsilon}(\Im(e^{-i\theta_{nkj}}s_j))^2\,d\nu_{nk}(\boldsymbol{s}) \\ &\le \sum_{k=1}^{k_n}\int_{\mathscr{U}_{2\epsilon}}(\Im(e^{-i\theta_{nkj}}s_j))^2\,d\nu_{nk}(\boldsymbol{s}),\end{aligned}$$

where we express $\boldsymbol{\theta}_{nk} = (\theta_{nk1},\dots,\theta_{nkd})$. The inequality

$$(\Im(e^{-i\theta_{nkj}}s_j))^2 \le (\Im s_j)^2 + 2|\sin\theta_{nkj}|\,|\Im s_j| + \sin^2(\theta_{nkj}) \tag{4-9}$$

will help us to continue with the arguments. Consideration given to the first term on the right-hand side of (4-9) gives

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\sum_{k=1}^{k_n}\int_{\mathscr{U}_{2\epsilon}}(\Im s_j)^2\,d\nu_{nk}(\boldsymbol{s})=a_{jj}$$

by the hypothesis, while analyzing the second term results in

$$\sum_{k=1}^{k_n}|\sin\theta_{nkj}|\cdot\int_{\mathscr{U}_{2\epsilon}}|\Im s_j|\,d\nu_{nk}(\boldsymbol{s})\le\biggl(\sum_{k=1}^{k_n}\sin^2\theta_{nkj}\biggr)^{1/2}\biggl(\int_{\mathscr{U}_{2\epsilon}}(\Im s_j)^2\,d\rho_n(\boldsymbol{s})\biggr)^{1/2}$$

by the Cauchy–Schwarz inequality. The simple fact $\sin^2 x\le 2(1-\cos x)$ for $x\in\mathbb{R}$ and the assumption (4-8) immediately yield that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\sum_{k=1}^{k_n}\int_{\mathscr{U}_{2\epsilon}}|\sin\theta_{nkj}|\,|\Im s_j|\,d\nu_{nk}(\boldsymbol{s})=0$$

and

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\sum_{k=1}^{k_n}\int_{\mathscr{U}_{2\epsilon}}\sin^2\theta_{nkj}\,d\nu_{nk}(\boldsymbol{s})=0.$$

These estimates then lead to $\lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{\mathscr{U}_\epsilon}(\Im s_j)^2\,d\tilde{\rho}_n(\boldsymbol{s})\le a_{jj}$. Employing the opposite inclusion $\mathscr{U}_{\epsilon/2}\subset e^{-i\boldsymbol{\theta}_{nk}}\mathscr{U}_\epsilon$ and inequality

$$(\Im(e^{-i\theta_{nkj}}s_j))^2\ge(\Im s_j)^2-2(1-\cos\theta_{nkj})-2|\sin\theta_{nkj}|\,|\Im s_j|-\sin^2\theta_{nkj}$$

allows us to obtain $\lim_{\epsilon\to 0}\liminf_{n\to\infty}\int_{\mathscr{U}_\epsilon}(\Im s_j)^2\,d\rho_n(\boldsymbol{s})\ge a_{jj}$.

Now we deal with the situation $j\ne\ell$ in (4-5). After careful consideration of all available information, the focus is only needed on the summand

$$\sum_{k=1}^{k_n}\int_{e^{i\boldsymbol{\theta}_{nk}}\mathscr{U}_\epsilon}(\Im s_j)(\Im s_\ell)\,d\nu_{nk}(\boldsymbol{s})$$

and justifying that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\sum_{k=1}^{k_n}\int_{(e^{i\boldsymbol{\theta}_{nk}}\mathscr{U}_\epsilon)\triangle\mathscr{U}_\epsilon}|\Im s_j|\,|\Im s_\ell|\,d\nu_{nk}(\boldsymbol{s})=0,\tag{4-10}$$

where $\triangle$ denotes the operation of symmetric difference on sets. Using the fact $(e^{i\boldsymbol{\theta}_{nk}}\mathscr{U}_\epsilon)\triangle\mathscr{U}_\epsilon\subset\bigl\{\frac{\epsilon}{2}\le\|\arg\boldsymbol{s}\|\le 2\epsilon\bigr\}$ and mimicking the proof of (4-7) allow us to get (4-10) done. □

Recall from (2-1) that the push-forward measure $\tau W^{-1}\in\mathscr{M}^{\mathbf{1}}_{\mathbb{T}^d}$ of a given $\tau\in\mathscr{M}^{\mathbf{0}}_{\mathbb{R}^d}$ via the wrapping map $W(\boldsymbol{x})=e^{i\boldsymbol{x}}$ from $\mathbb{R}^d$ to $\mathbb{T}^d$ is defined as

$$(\tau W^{-1})(B)=\tau(\{\boldsymbol{x}\in\mathbb{R}^d:e^{i\boldsymbol{x}}\in B\}),\quad B\in\mathscr{B}_{\mathbb{T}^d}.\tag{4-11}$$

A useful and frequently used result regarding W is the change-of-variables formula stating that a Borel function f on $\mathbb{T}^d$ belongs to $L^1(\tau W^{-1})$ if and only if the function $\boldsymbol{x} \mapsto f(e^{i\boldsymbol{x}})$ lies in $L^1(\tau)$, and the equation

$$\int_{\mathbb{T}^d} f(\boldsymbol{s})\, d(\tau W^{-1})(\boldsymbol{s}) = \int_{\mathbb{R}^d} f(e^{i\boldsymbol{x}})\, d\tau(\boldsymbol{x}) \tag{4-12}$$

holds in either case. In the following, we will translate conditions introduced in Section 2D accordingly via the wrapping map W.

Proposition 4.4. *Assume that* $\{\tau_n\}$ *and* τ *are in* $\mathscr{M}^{\mathbf{0}}_{\mathbb{R}^d}$ *satisfying* Condition 2.3 (*or* Condition 2.2). *Then* Condition 4.1, *as well as* Condition 2.5, *applies to* $\rho_n = \tau_n W^{-1}$ *and* $\rho = 1_{\mathbb{T}^d \setminus \{\mathbf{1}\}} \tau W^{-1}$. *Moreover*, τ_n *and* ρ_n *determine the same quadratic form on* $\mathbb{Z}^d$, *in particular*, *the same matrix in* (IV) *and* (iv), *respectively.*

Proof. Suppose that Condition 2.3 holds for τ_n and τ, and let $\boldsymbol{A} = (a_{j\ell})$ represent the matrix produced by these measures in (IV). According to Proposition 4.2, we shall only elaborate that Condition 4.1 is applicable to ρ_n and ρ.

That $\rho_n \Rightarrow_{\mathbf{1}} \rho$ is clearly valid according to the continuous mapping theorem, Proposition 2.1. It remains to argue that in Condition 4.1(iv), ρ_n also outputs $\boldsymbol{A}$. The simple observation that $e^{i\boldsymbol{x}} \in \mathscr{U}_\epsilon$ if and only if $\boldsymbol{x}$ belongs to the set

$$\widetilde{\mathscr{V}_\epsilon} = \bigcup_{\boldsymbol{p} \in \mathbb{Z}^d} \{\boldsymbol{x} + 2\pi \boldsymbol{p} : \boldsymbol{x} \in \mathscr{V}_\epsilon\} \tag{4-13}$$

and formula (4-12) help us to establish that for $j, \ell = 1, \ldots, d$,

$$\begin{aligned}\int_{\mathscr{U}_\epsilon} (\Im s_j)(\Im s_\ell)\, d\rho_n(\boldsymbol{s}) &= \int_{\mathbb{T}^d} 1_{\mathscr{U}_\epsilon}(\boldsymbol{s})(\Im s_j)(\Im s_\ell)\, d\rho_n(\boldsymbol{s}) \\ &= \int_{\mathbb{R}^d} 1_{\mathscr{U}_\epsilon}(e^{i\boldsymbol{x}})(\Im e^{ix_j})(\Im e^{ix_\ell})\, d\tau_n(\boldsymbol{x}) \\ &= \int_{\widetilde{\mathscr{V}_\epsilon}} \sin(x_j)\sin(x_\ell)\, d\tau_n(\boldsymbol{x}).\end{aligned}$$

Observe next that we have $\widetilde{\mathscr{V}_\epsilon} \setminus \mathscr{V}_\epsilon = \bigcup_{m=1}^d \mathscr{D}_{\epsilon m}$, where $\mathscr{D}_{\epsilon m} = \widetilde{\mathscr{V}_\epsilon} \cap \{|x_m| \geq \pi\}$, provided that $\epsilon < \pi$. If we temporarily impose the requirement $\sigma_m(\partial \mathscr{D}_{\epsilon m}) = 0$ for some $m \in \{1, \ldots, d\}$, then the weak convergence $\sigma_{nm} \Rightarrow \sigma_m$ implies that

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathscr{D}_{\epsilon m}} |\sin(x_j)\sin(x_\ell)|\, d\tau_n &= \limsup_{n\to\infty} \int_{\mathscr{D}_{\epsilon m}} |\sin(x_j)\sin(x_\ell)| \cdot \frac{1+x_m^2}{x_m^2}\, d\sigma_{nm} \\ &= \int_{\mathscr{D}_{\epsilon m}} |\sin(x_j)\sin(x_\ell)| \cdot \frac{1+x_m^2}{x_m^2}\, d\sigma_m \xrightarrow[\epsilon\to\infty]{} 0.\end{aligned}$$

This, along with facts $x - \sin x = o(|x|^2)$ as $|x| \to 0$ and $x_j^2 \in L^1(\sigma_j)$, leads to

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{\mathscr{U}_\epsilon}(\Im s_j)(\Im s_\ell)\,d\rho_n(\boldsymbol{s}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{\mathscr{V}_\epsilon}\sin(x_j)\sin(x_\ell)\,d\tau_n(\boldsymbol{x}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{\mathscr{V}_\epsilon}x_j\,x_\ell\,d\tau_n(\boldsymbol{x}).$$

The same arguments also elaborate the identity

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\int_{\mathscr{U}_\epsilon}(\Im s_j)(\Im s_\ell)\,d\rho_n(\boldsymbol{s}) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\int_{\mathscr{V}_\epsilon}x_j\,x_\ell\,d\tau_n(\boldsymbol{x}).$$

Apparently, the selection of ϵ does not vary the validity of these identities, and so we have established that ρ_n generates the matrix $\boldsymbol{A}$ in (iv) as well. □

Measures in $\mathscr{M}^{\mathbf{0}}_{\mathbb{R}^d}$ can be wrapped either clockwise or counterclockwise (see equation (4-11)) in all variables, and consequences, such as Proposition 4.4, are not affected at all by this slight change. As a matter of fact, it is also the case when one wraps some variables counterclockwise and others clockwise. Without loss of generality, we shall use the simplest circumstance, the 2-dimensional opposite wrapping map $W_2^\star : \mathbb{R}^2 \to \mathbb{T}^2$, $(x_1, x_2) \mapsto (e^{ix_1}, e^{-ix_2})$, to illustrate these features. The following result is merely an easy consequence of the continuous mapping theorem, the relations $(\tau(W_2^\star)^{-1})(B) = (\tau W_2^{-1})(B^\star) = (\tau W_2^{-1})^\star(B)$ for any $B \in \mathscr{B}_{\mathbb{T}^2}$, and Proposition 4.4.

Proposition 4.5. *If* $\{\rho_n\}$ *and* ρ *in* $\mathscr{M}^{\mathbf{1}}_{\mathbb{T}^2}$ *fulfill* Condition 4.1, *then*

(1) $\rho_n^\star \Rightarrow_{\mathbf{1}} \rho^\star$, *and*

(2) *for any* $\boldsymbol{p} = (p_1, p_2) \in \mathbb{Z}^2$, *denoting by* $\boldsymbol{p}^\star = (p_1, -p_2)$, *we have*

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{\mathscr{U}_\epsilon}\langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle^2\,d\rho_n^\star(\boldsymbol{s}) = Q(\boldsymbol{p}^\star) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\int_{\mathscr{U}_\epsilon}\langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle^2\,d\rho_n^\star(\boldsymbol{s}).$$

Particularly, *if* $\{\tau_n\}$ *and* τ *in* $\mathscr{M}^{\mathbf{0}}_{\mathbb{R}^2}$ *satisfy* Condition 2.3 (*or* Condition 2.2), *then statements* (1) *and* (2) *above apply to* $\rho_n^\star = \tau_n(W_2^\star)^{-1}$ *and* $\rho^\star = \tau(W_2^\star)^{-1}$.

We add one remark on item (2) of the preceding proposition: if $Q(\boldsymbol{p}) = \langle \boldsymbol{A}\boldsymbol{p}, \boldsymbol{p}\rangle$, then $Q(\boldsymbol{p}^\star) = \langle \boldsymbol{A}^{\mathrm{op}}\boldsymbol{p}, \boldsymbol{p}\rangle$, where the (i, j)-entry of $\boldsymbol{A}^{\mathrm{op}}$ is $(-1)^{i+j}\boldsymbol{A}_{ij}$.

5. Limit theorems and bi-free multiplicative Lévy triplet

5A. ***Bi-free multiplicative Lévy–Khintchine representation.*** Thanks to Proposition 4.2, one can correlate the quantity L_{12} and measures λ_j given in the formulas (2-16) with the matrix $\boldsymbol{A}$ and measure $\rho \in \mathscr{M}^{\mathbf{1}}_{\mathbb{T}^2}$ determined by (4-1), (4-2), and (4-3). Therefore, instead of working with the parametrization $(\boldsymbol{\gamma}, \lambda_1, \lambda_2, L_{12})$

for measures in $\mathcal{ID}(\boxtimes\boxtimes)\cap\mathscr{P}^{\times}_{\mathbb{T}^2}$, one may take another parametrization $(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)$ (with the same $\boldsymbol{\gamma}$) having the following properties with $d=2$:

$$\text{(5-1)}\quad \boldsymbol{\gamma}\in\mathbb{T}^d,\ \boldsymbol{A} \text{ is a positive semidefinite } d\times d \text{ symmetric matrix, and } \rho \text{ is a positive measure on } \mathbb{T}^d \text{ so that } \rho(\{\mathbf{1}\})=0 \text{ and } \|\mathbf{1}-\Re \boldsymbol{s}\|\in L^1(\rho).$$

We shall refer to $(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)$ as the *bi-free multiplicative Lévy triplet* of the measure in $\mathcal{ID}(\boxtimes\boxtimes)\cap\mathscr{P}^{\times}_{\mathbb{T}^2}$ having (bi-)free Σ-transforms presented in (2-16), and signify this measure by $\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$ to comply with the correspondence. This triplet plays the role of the classical multiplicative Lévy triplet. We will clarify this in more details in Corollary 5.3, where limit theorems between classical and bi-free multiplicative convolutions are examined and in Section 6, where the commutativity of diagram (1-5) is verified.

A measure ν belongs to $\mathcal{ID}(\boxtimes\boxtimes^{\mathrm{op}})\cap\mathscr{P}^{\times}_{\mathbb{T}^2}$ if and only if $\nu^\star\in\mathcal{ID}(\boxtimes\boxtimes)\cap\mathscr{P}^{\times}_{\mathbb{T}^2}$ by (2-3) and Theorem 3.12. Thus, we shall denote by $\nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$ the measure ν satisfying $\nu^\star=\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}^\star,\boldsymbol{A}^{\mathrm{op}},\rho^\star)}$ and refer to $(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)$ as its *opposite bi-free multiplicative Lévy triplet*. Passing to analytic transforms, we have

$$\Sigma_\nu^{\mathrm{op}}(z,w)=\Sigma_{\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}^\star,\boldsymbol{A}^{\mathrm{op}},\rho^\star)}}(z,1/w)\quad\text{for }(z,w)\in\mathbb{D}\times(\mathbb{T}\cup\{0\})^c.$$

In terms of notations introduced above, we reformulate the basic limit theorem [13, Theorem 3.4] on the bi-free multiplicative convolution, including statements for $\boxtimes\boxtimes^{\mathrm{op}}$.

Theorem 5.1. *Given an infinitesimal array* $\{\nu_{nk}\}\subset\mathscr{P}^{\times}_{\mathbb{T}^2}$ *and a sequence* $\{\boldsymbol{\xi}_n\}\subset\mathbb{T}^2$, *define* $\boldsymbol{\gamma}_n$ *as in* (2-13). *The following are equivalent.*

(1) *The sequence*

$$\text{(5-2)}\qquad \delta_{\boldsymbol{\xi}_n}\boxtimes\boxtimes\,\nu_{n1}\boxtimes\boxtimes\cdots\boxtimes\boxtimes\,\nu_{nk}$$

converges weakly to some $\nu_{\boxtimes\boxtimes}\in\mathscr{P}^{\times}_{\mathbb{T}^2}$.

(2) *The sequence*

$$\text{(5-3)}\qquad \delta_{\boldsymbol{\xi}_n}\boxtimes\boxtimes^{\mathrm{op}}\nu_{n1}\boxtimes\boxtimes^{\mathrm{op}}\cdots\boxtimes\boxtimes^{\mathrm{op}}\nu_{nk}$$

converges weakly to some $\nu_{\boxtimes\boxtimes^{\mathrm{op}}}\in\mathscr{P}^{\times}_{\mathbb{T}^2}$.

(3) *The measure* $\rho_n=\sum_{k=1}^{k_n}\mathring{\nu}_{nk}$ *satisfies* Condition 4.1 (*or* Condition 2.5) *with* $d=2$ *and* $\lim_n\boldsymbol{\gamma}_n=\boldsymbol{\gamma}$ *exists.*

If (1)–(3) *hold, then* $\nu_{\boxtimes\boxtimes}=\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$ *and* $(\nu_{\boxtimes\boxtimes^{\mathrm{op}}})^\star=\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}^\star,\boldsymbol{A}^{\mathrm{op}},\rho^\star)}$, *where* ρ *and* $\boldsymbol{A}$ *are as in* Condition 4.1 *and* Proposition 4.2, *respectively.*

Proof. We only prove (2)⇔(3). With $\{\boldsymbol{b}_{nk}\}$ defined in (2-12), the equality

$$\exp\left[i\int_{\mathscr{U}_\theta}(\arg\boldsymbol{s})\,d\nu_{nk}^\star(\boldsymbol{s})\right]=\boldsymbol{b}_{nk}^\star$$

shows that $(\nu_{nk}^\star)^\circ(B) = \nu_{nk}^\star(\boldsymbol{b}_{nk}^\star B) = \nu_{nk}(\boldsymbol{b}_{nk} B^\star) = \mathring{\nu}_{nk}(B^\star) = (\mathring{\nu}_{nk})^\star(B)$ for any Borel set B on $\mathbb{T}^2$. Since the operations $\star$ and $\circ$ acting on ν_{nk} are interchangeable in order, we adopt the notation $\mathring{\nu}_{nk}^\star$ instead of $(\nu_{nk}^\star)^\circ = (\mathring{\nu}_{nk})^\star$ if no confusions arise.

Item (2) holds if and only if

$$\delta_{\boldsymbol{\xi}_n^\star} \boxtimes\boxtimes\, \nu_{n1}^\star \boxtimes\boxtimes \cdots \boxtimes\boxtimes\, \nu_{nk}^\star = (\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes^{\mathrm{op}} \nu_{n1} \boxtimes\boxtimes^{\mathrm{op}} \cdots \boxtimes\boxtimes^{\mathrm{op}} \nu_{nk})^\star \Rightarrow (\nu_{\boxtimes\boxtimes^{\mathrm{op}}})^\star$$

according to (2-3). This happens if and only if Condition 4.1 applies to the measure $\sum_{k=1}^n \mathring{\nu}_{nk}^\star = \left(\sum_{k=1}^n \mathring{\nu}_{nk}\right)^\star$ and the vector

$$\boldsymbol{\gamma}_n^\star = \boldsymbol{\xi}_n^\star \exp\left[i \sum_{k=1}^{k_n} \left(\arg \boldsymbol{b}_{nk}^\star + \int_{\mathbb{T}^2} (\Im \boldsymbol{s})\, d\mathring{\nu}_{nk}^\star(\boldsymbol{s})\right)\right]$$

has a limit by Theorem 2.4. Then Proposition 4.5 proves the equivalence (2)$\Leftrightarrow$(3) and the last assertion. □

Recall from [16] that a measure ν in $\mathcal{ID}(\circledast)$ has no nontrivial $\circledast$-idempotent factor if and only if its characteristic function takes the form

$$\hat{\nu}(\boldsymbol{p}) = \boldsymbol{\gamma}^{\boldsymbol{p}} \exp\left(-\tfrac{1}{2}\langle \boldsymbol{A}\boldsymbol{p}, \boldsymbol{p}\rangle + \int_{\mathbb{T}^d} (\boldsymbol{s}^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle)\, d\rho(\boldsymbol{s})\right), \quad \boldsymbol{p} \in \mathbb{Z}^d$$

for certain triplet $(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)$ fulfilling the conditions in (5-1). We shall write $\nu_\circledast^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$ for this measure, and refer to ρ and $(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)$ as its multiplicative Lévy measure and multiplicative Lévy triplet, respectively. A known phenomenon is that a $\circledast$-infinitely divisible distribution on $\mathbb{T}^d$ has unique $\boldsymbol{\gamma}$ and $\boldsymbol{A}$, but may have various Lévy measures. For example, it was pointed out in [6] that when $d = 1$, one has $\nu_\circledast^{(1,0,\pi\delta_i)} = \nu_\circledast^{(1,0,\pi\delta_{-i})}$. The uniqueness of multiplicative Lévy measures will be more systematically studied in [10]. This observation leads to the following definition.

Definition 5.2. Let ρ be a multiplicative Lévy measure on $\mathbb{T}^d$. The symbol $\mathcal{L}(\rho)$ stands for the collection of those measures serving as multiplicative Lévy measures for $\nu_\circledast^{(\boldsymbol{1},\boldsymbol{0},\rho)}$.

The following corollary, derived from Theorem 2.4 and [10], supplies the link between classical and bi-free limit theorems on the bi-torus. The attentive reader can also notice that the hypothesis $\mathcal{L}(\rho) = \{\rho\}$ is redundant in the implication (2) $\Rightarrow$ (1).

Corollary 5.3. *Let $\{\nu_{nk}\} \subset \mathscr{P}_{\mathbb{T}^2}$ be infinitesimal, $\{\boldsymbol{\xi}_n\} \subset \mathbb{T}^2$, and $(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)$ be a multiplicative Lévy triplet such that $\mathcal{L}(\rho) = \{\rho\}$. With the notations in* (2-13) *and* (2-15) *for $d = 2$, the following statements are equivalent*:

(1) $\delta_{\boldsymbol{\xi}_n} \circledast \nu_{n1} \circledast \cdots \circledast \nu_{nk_n} \Rightarrow \nu_\circledast^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$.

(2) $\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes\, \nu_{n1} \boxtimes\boxtimes \cdots \boxtimes\boxtimes\, \nu_{nk_n} \Rightarrow \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$.

(3) $\lim_{n\to\infty} \boldsymbol{\gamma}_n = \boldsymbol{\gamma}$, $\rho_n \Rightarrow_{\mathbf{1}} \rho$, *and*

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \int_{\mathscr{U}_\epsilon} \langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle^2 \, d\rho_n(\boldsymbol{s}) = \langle \boldsymbol{A}\boldsymbol{p}, \boldsymbol{p}\rangle = \lim_{\epsilon\to 0} \liminf_{n\to\infty} \int_{\mathscr{U}_\epsilon} \langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle^2 \, d\rho_n(\boldsymbol{s}), \qquad \boldsymbol{p} \in \mathbb{Z}^2.$$

The one-dimensional multiplicative limit theorem, which was pointed out in the remark to [23, Corollary 4.2], is a consequence of Corollary 5.3, e.g., by considering product measures.

Corollary 5.4. *Let* $\{\nu_{nk}\} \subset \mathscr{P}_{\mathbb{T}}$ *be infinitesimal*, $\{\xi_n\} \subset \mathbb{T}$, *and* (γ, a, ρ) *be a multiplicative Lévy triplet such that* $\mathcal{L}(\rho) = \{\rho\}$. *With the notations in* (2-13) *and* (2-15) *for* $d = 1$, *the following statements are equivalent*:

(1) $\delta_{\xi_n} \circledast \nu_{n1} \circledast \cdots \circledast \nu_{nk_n} \Rightarrow \nu_{\circledast}^{(\gamma,a,\rho)}$.

(2) $\delta_{\xi_n} \boxtimes \nu_{n1} \boxtimes \cdots \boxtimes \nu_{nk_n} \Rightarrow \nu_{\boxtimes}^{(\gamma,a,\rho)}$.

(3) $\lim_{n\to\infty} \gamma_n = \gamma$, $\rho_n \Rightarrow_1 \rho$, *and*

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \int_{\mathscr{U}_\epsilon} (\Im s)^2 d\rho_n(s) = a = \lim_{\epsilon\to 0} \liminf_{n\to\infty} \int_{\mathscr{U}_\epsilon} (\Im s)^2 d\rho_n(s).$$

Apparently, the nonuniqueness of Lévy measures is the exclusive obstruction for reaching the equivalence of limit theorems, thus complementing the work of Chistyakov and Götze [9, Theorems 2.3 and 2.4].

The goal of this section is to provide an alternative description for the Σ-transform of a measure in $\mathcal{ID}(\boxtimes\boxtimes) \cap \mathscr{P}_{\mathbb{T}^2}^{\times}$ in terms of its bi-free multiplicative Lévy triplets. To achieve this, we need some basics. For any $p \in \mathbb{N}$, the function

$$\mathcal{K}_p(s) = \frac{s^p - 1 - ip\Im s}{1 - \Re s}$$

is continuous on $\mathbb{T}$ and equal to $-p^2$ at $s = 1$.

Lemma 5.5. *For any* $p \in \mathbb{N}$, *we have* $\|\Im\mathcal{K}_p\|_\infty \le p^3$ *and* $\int_{-\pi}^{\pi} \mathcal{K}_p(e^{i\theta})\, d\theta = -2p\pi$.

Proof. In the following arguments, we shall make use of the basic formula:

$$\frac{1 - \cos(p\theta)}{1 - \cos\theta} = e^{i(1-p)\theta} \sum_{j,k=0}^{p-1} e^{i(j+k)\theta}. \tag{5-4}$$

Clearly, we have $\Im\mathcal{K}_1 \equiv 0$. If $\|\Im\mathcal{K}_p\|_\infty \le p^3$ for some $p \ge 2$, then for $s \ne 1$, the inequality $|(1 - \Re s^p)/(1 - \Re s)| \le p^2$ following from (5-4) implies that

$$|\Im\mathcal{K}_{p+1}(s)| = \left|\Im s^p - \Im\mathcal{K}_p(s) + \frac{1 - \Re s^p}{1 - \Re s} \cdot \Im s\right| \le 1 + p^3 + p^2 \le (p+1)^3.$$

By induction, this finishes the proof of the first assertion. To prove the second assertion, it suffices to show $\int_{-\pi}^{\pi}(1-\cos(p\theta))/(1-\cos\theta)\,d\theta = 2p\pi$, which can be easily obtained by using (5-4) again. $\square$

Fix a measure $\nu \in \mathscr{P}_{\mathbb{T}^2}^{\times} \cap \mathcal{ID}(\boxtimes\boxtimes)$, and suppose that its (bi-)free Σ-transforms are given as in (2-16). Due to the integral representations, both u_1 and u_2 are analytic in $\Omega = (\mathbb{C}\backslash\mathbb{T}) \cup \{\infty\}$ and u is analytic in Ω^2. Hence the function

$$U_\nu(z, w) = \frac{zw}{1-zw}u(z, w) - \frac{z}{1-z}u_1(z) - \frac{w}{1-w}u_2(w)$$

is analytic in Ω^2. If $\nu \in \mathcal{ID}(\boxtimes\boxtimes^{\mathrm{op}}) \cap \mathscr{P}_{\mathbb{T}^2}^{\times}$, then we define

$$U_\nu^{\mathrm{op}}(z, w) = U_{\nu^\star}(z, 1/w),$$

which is also an analytic function in Ω^2.

When $\nu \in \mathcal{ID}(\boxtimes\boxtimes^{\mathrm{op}}) \cap \mathscr{P}_{\mathbb{T}^2}^{\times}$, one can obtain an equivalent formula for U_ν in terms of the bi-free multiplicative Lévy triplet, which we call the *bi-free multiplicative Lévy–Khintchine representation*. Note that we acquire the following proof with the help of limit theorems, in spite of the algebraic nature of the statement. Also, it is simpler even though there exists an algebraic proof.

Theorem 5.6. *Letting* $\nu = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)}$, *we have*

$$U_\nu(z, w) = \frac{iz}{1-z}\arg\gamma_1 + \frac{iw}{1-w}\arg\gamma_2 - N_\nu(z, w) + P_\nu(z, w), \tag{5-5}$$

where

$$N_\nu(z, w) = \frac{a_{11}}{2}\cdot\frac{z(1+z)}{(1-z)^2} + \frac{a_{12}\,zw}{(1-z)(1-w)} + \frac{a_{22}}{2}\cdot\frac{w(1+w)}{(1-w)^2}$$

and

$$P_\nu(z, w) = (1-z)(1-w)\sum_{\boldsymbol{p}=\boldsymbol{0}}^{\infty}\left[\int_{\mathbb{T}^2}(\boldsymbol{s}^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im\boldsymbol{s}\rangle)\,d\rho(\boldsymbol{s})\right]z^{p_1}w^{p_2}.$$

Further, letting $\tilde{\nu} = \nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)}$, *we have* $U_{\tilde{\nu}}^{\mathrm{op}}(z, w) = U_{\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}^\star, \boldsymbol{A}^{\mathrm{op}}, \rho^\star)}}(z, 1/w)$.

Proof. First of all, using Remark 2.6 and the function

$$f(z, w, \boldsymbol{s}) = \frac{zw(1-s_1)(1-s_2)}{(1-zs_1)(1-ws_2)} - \frac{z(1+zs_1)(1-\Re s_1)}{(1-z)(1-zs_1)} - \frac{w(1+ws_2)(1-\Re s_2)}{(1-w)(1-ws_2)},$$

one can rewrite U_ν as

$$U_\nu(z, w) = \frac{iz}{1-z}\arg\gamma_1 + \frac{iw}{1-w}\arg\gamma_2 + \lim_{n\to\infty}\int_{\mathbb{T}^2} f(z, w, \boldsymbol{s})\,d\rho_n(\boldsymbol{s}).$$

Below, $r > 0$ is taken so that $\rho(\partial\mathscr{U}_r) = 0$. The continuity of $\boldsymbol{s} \mapsto f(z, w, \boldsymbol{s})$ on $\mathbb{T}^2$ for any fixed $(z, w) \in \mathbb{D}^2$ and Proposition 2.1 imply that

$$\lim_{n\to\infty} \int_{\mathbb{T}^2\setminus\mathscr{U}_r} f(z, w, \boldsymbol{s})\, d\rho_n = \int_{\mathbb{T}^2\setminus\mathscr{U}_r} f(z, w, \boldsymbol{s})\, d\rho.$$

Using dominated convergence theorem, we arrive at

$$\lim_{r\to 0} \lim_{n\to\infty} \int_{\mathbb{T}^2\setminus\mathscr{U}_r} f(z, w, \boldsymbol{s})\, d\rho_n = \int_{\mathbb{T}^2} f(z, w, \boldsymbol{s})\, d\rho.$$

On the other hand, thanks to weak convergence $\lambda_{nj} = (1-\Re s_j)\,\rho_n \Rightarrow \lambda_j,\ j = 1, 2$, we see that for $\xi \in \mathbb{D}$,

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\int_{\mathscr{U}_r} \frac{1+\xi s_j}{1-\xi s_j}(1-\Re s_j)\, d\rho_n - \frac{a_{jj}}{2}\cdot\frac{1+\xi}{1-\xi}\right| \\
&\qquad\le \limsup_{n\to\infty}\left(\left|\lambda_{nj}(\mathscr{U}_r) - \tfrac{1}{2}a_{jj}\right|\left|\frac{1+\xi}{1-\xi}\right| + \int_{\mathscr{U}_r}\left|\frac{1+\xi s_j}{1-\xi s_j} - \frac{1+\xi}{1-\xi}\right| d\lambda_{nj}\right) \\
&\qquad\le \frac{2}{(1-|\xi|)^2}\limsup_{n\to\infty}\left(\left|\lambda_{nj}(\mathscr{U}_r) - \tfrac{1}{2}a_{jj}\right| + \int_{\mathscr{U}_r}|1-s_j|\, d\lambda_{nj}\right) \xrightarrow[r\to\infty]{} 0.
\end{aligned}$$

Similarly, one can show that

$$\lim_{r\to 0}\lim_{n\to\infty}\int_{\mathbb{T}^2\setminus\mathscr{U}_r} \frac{(1-s_1)(1-s_2)}{(1-zs_1)(1-ws_2)}\, d\rho_n = \int_{\mathbb{T}^2} \frac{(1-s_1)(1-s_2)}{(1-zs_1)(1-ws_2)}\, d\rho$$

and

$$\lim_{r\to 0}\limsup_{n\to\infty}\left|\int_{\mathscr{U}_r} \frac{(1-s_1)(1-s_2)}{(1-zs_1)(1-ws_2)}\, d\rho_n + \frac{a_{12}}{(1-z)(1-w)}\right| = 0.$$

Next, we shall make use of the equation (2-17). After some algebraic manipulations, we come to the result

$$\lim_{n\to\infty}\int_{\mathbb{T}^2} f(z, w, \boldsymbol{s})\, d\rho_n(\boldsymbol{s}) = -N(z, w) + (1-z)(1-w)\int_{\mathbb{T}^2}\tilde{f}(z, w, \boldsymbol{s})\, d\rho(\boldsymbol{s}),$$

where

$$\begin{aligned}
&\tilde{f}(z, w, \boldsymbol{s}) \\
&\quad= \frac{1}{(1-zs_1)(1-ws_2)} - \frac{1}{(1-z)(1-w)} - \frac{iz\Im s_1}{(1-z)^2(1-w)} - \frac{iw\Im s_2}{(1-z)(1-w)^2}.
\end{aligned}$$

Lastly, the use of the power series expansion

$$\xi_j(1-\xi_1)^{-2}(1-\xi_2)^{-1} = \sum_{\boldsymbol{p}\ge\boldsymbol{0}} p_j\, \xi_1^{p_1}\xi_2^{p_2} \quad \text{for } \xi_1, \xi_2 \in \mathbb{D},$$

allows us to get

$$\int_{\mathbb{T}^2}\tilde{f}(z, w, \boldsymbol{s})\, d\rho(\boldsymbol{s}) = \int_{\mathbb{T}^2}\sum_{\boldsymbol{p}\ge\boldsymbol{0}}(\boldsymbol{s}^{\boldsymbol{p}} - 1 - i\langle\boldsymbol{p}, \Im\boldsymbol{s}\rangle)\, z^{p_1}w^{p_2}\, d\rho(\boldsymbol{s}).$$

The operations of integration and summation performed above are interchangeable due to Lemma 5.5. Indeed, one can utilize the uniform convergence of the summands to obtain

$$\begin{aligned}\int \sum_{\boldsymbol{p}\geq \boldsymbol{0}} (s_j^{p_j} - 1 - ip_j \Im s_j)\, z^{p_1} w^{p_2}\, d\rho &= \sum_{\boldsymbol{p}\geq \boldsymbol{0}} \int \mathcal{K}_{p_j}(s)\, d\lambda_j\, z^{p_1} w^{p_2} \\ &= \sum_{\boldsymbol{p}\geq \boldsymbol{0}} \int (s_j^{p_j} - 1 - ip_j \Im s_j)\, d\rho\, z^{p_1} w^{p_2}\end{aligned}$$

and similarly

$$\int \sum_{\boldsymbol{p}\geq \boldsymbol{0}} (s_1^{p_1} - 1)(s_2^{p_2} - 1)\, z^{p_1} w^{p_2}\, d\rho = \sum_{\boldsymbol{p}\geq \boldsymbol{0}} \int (s_1^{p_1} - 1)(s_2^{p_2} - 1)\, d\rho\, z^{p_1} w^{p_2}.$$

Putting all these findings together yields the desired result.

According to the definition of $\tilde{\nu}$, which is characterized by $(\tilde{\nu})^\star = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}^\star, \boldsymbol{A}^{\mathrm{op}}, \rho^\star)}$, the last assertion follows from the definition of U_ν^{op}. □

Performing the power series expansion to $N_\nu(z, w)$ in Theorem 5.6 further yields that

$$\begin{aligned}&\frac{U_\nu(z, w)}{(1-z)(1-w)} \\ &\qquad = \sum_{\boldsymbol{p}=\boldsymbol{0}}^{\infty} \left[i\langle \boldsymbol{p}, \arg \boldsymbol{\gamma}\rangle - \tfrac{1}{2}\langle \boldsymbol{A}\boldsymbol{p}, \boldsymbol{p}\rangle + \int_{\mathbb{T}^2} (\boldsymbol{s}^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle)\, d\rho(\boldsymbol{s}) \right] z^{p_1} w^{p_2},\end{aligned}$$

which offers the generating series for the exponent of the characteristic function

$$\hat{\nu}(\boldsymbol{p}) = \boldsymbol{\gamma}^{\boldsymbol{p}} \exp\left[-\tfrac{1}{2}\langle \boldsymbol{A}\boldsymbol{p}, \boldsymbol{p}\rangle + \int_{\mathbb{T}^2} (\boldsymbol{s}^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle)\, d\rho(\boldsymbol{s}) \right], \quad \boldsymbol{p} \in \mathbb{Z}^2 \tag{5-6}$$

of a measure in $\mathcal{ID}(\mathbb{T}^2, \circledast) \cap \mathscr{P}_{\mathbb{T}^2}^{\times}$ (cf. Corollary 5.3.)

5B. ***Limit theorems via wrapping transformations.*** We next present the limit theorems through the wrapping transformations.

Theorem 5.7. *Let* $(\boldsymbol{v}, \boldsymbol{A}, \tau)$ *be a triplet satisfying* (2-10) *with* $d = 2$, *and let* $\{\mu_{nk}\} \subset \mathscr{P}_{\mathbb{R}^2}$ *be an infinitesimal triangular array and* $\{\boldsymbol{v}_n\}$ *a sequence of vectors in* $\mathbb{R}^2$. *If the sequence in* (1-2) *converges weakly to* $\mu_{\boxplus\boxplus}^{(\boldsymbol{v}, \boldsymbol{A}, \tau)}$, *then the sequences in* (5-2) *and* (5-3) *generated by* $\nu_{nk} = \mu_{nk} W^{-1}$ *and* $\boldsymbol{\xi}_n = e^{i\boldsymbol{v}_n}$ *converge weakly to* $\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)}$ *and* $\nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)}$, *respectively, where*

$$\rho = 1_{\mathbb{T}^2 \setminus \{\boldsymbol{1}\}}(\tau W^{-1}) \tag{5-7}$$

and

$$\boldsymbol{\gamma} = \exp\left[i\boldsymbol{v} + i\int_{\mathbb{R}^2}\left(\sin(\boldsymbol{x}) - \frac{\boldsymbol{x}}{1+\|\boldsymbol{x}\|^2}\right)d\tau(\boldsymbol{x})\right]. \tag{5-8}$$

Proof. Before carrying out the main proof, let us record some properties instantly inferred from the hypotheses for the later utilization. Because the index n goes to infinity ultimately, it is always big enough whenever mentioned in the proof.

Firstly, observe that ν_{nk} belongs to $\mathscr{P}_{\mathbb{T}^2}^{\times}$ and the vector

$$\boldsymbol{\theta}_{nk} = \sum_{\boldsymbol{p}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} \int_{\mathscr{V}_\theta} \boldsymbol{x}\, d\mu_{nk}(\boldsymbol{x} + 2\pi\boldsymbol{p}) \tag{5-9}$$

satisfies $\lim_{n\to\infty}\max_k\|\boldsymbol{\theta}_{nk}\| = 0$ by the infinitesimality of $\{\mu_{nk}\}$. Secondly, following the notations in (2-6) and (4-13), an application of (4-12) gives

$$\begin{aligned}
\boldsymbol{v}_{nk} + \boldsymbol{\theta}_{nk} &= \int_{\mathbb{R}^d} 1_{\{e^{i\boldsymbol{x}}:\boldsymbol{x}\in\widetilde{\mathscr{V}_\theta}\}}(e^{i\boldsymbol{x}})\arg(e^{i\boldsymbol{x}})\, d\mu_{nk}(\boldsymbol{x}) \\
&= \int_{\mathbb{T}^d} 1_{\{\boldsymbol{s}:\|\arg\boldsymbol{s}\|<\theta\}}(\boldsymbol{s})\arg(\boldsymbol{s})\, d\nu_{nk}(\boldsymbol{s}) = \int_{\mathscr{U}_\theta}\arg(\boldsymbol{s})\, d\nu_{nk}(\boldsymbol{s}).
\end{aligned}$$

This and equation (2-12) provide us with the relations $\arg\boldsymbol{b}_{nk} = \boldsymbol{v}_{nk} + \boldsymbol{\theta}_{nk}$ and $d\mathring{\nu}_{nk}(\boldsymbol{s}) = d(\mathring{\mu}_{nk}W^{-1})(e^{i\boldsymbol{\theta}_{nk}}\boldsymbol{s})$ as for any $B\in\mathscr{B}_{\mathbb{T}^2}$, we have

$$(\mathring{\mu}_{nk}W^{-1})(B) = \mu_{nk}(\{e^{i\boldsymbol{x}}\in e^{i\boldsymbol{v}_{nk}}B\}) = \nu_{nk}(e^{i\boldsymbol{v}_{nk}}B) = \mathring{\nu}_{nk}(e^{-\boldsymbol{\theta}_{nk}}B).$$

Except for the beforehand mentioned results, the array $\{\boldsymbol{\theta}_{nk}\}$ in (5-9) also fulfills the condition in (4-8), which will play a dominant role in our arguments. Its proof, provided below, is based on the convergence $\tau_n = \sum_k \mathring{\mu}_{nk} \Rightarrow_{\boldsymbol{0}} \tau$ and some estimates. For convenience, denote $\boldsymbol{\theta}_{nk} = (\theta_{nk1}, \theta_{nk2})$ and $\boldsymbol{v}_{nk} = (v_{nk1}, v_{nk2})$, and consider the positive Borel measure $\varrho_{nk}(\,\cdot\,) = \sum_{\boldsymbol{p}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}}\mathring{\mu}_{nk}(\cdot + 2\pi\boldsymbol{p})1_{\overline{\mathscr{V}_{2\theta}}}$ on the closure of $\mathscr{V}_{2\theta}$. The infinitesimality of $\{\mathring{\mu}_{nk}\}$ indicates that $\lim_{n\to\infty}\max_{1\le k\le k_n}\varrho_{nk}(\mathscr{V}_{2\theta}) = 0$ and the assumption $\theta\in(0,1)$ in (2-5) shows that

$$\varrho_{nk}(\mathscr{V}_\theta - \boldsymbol{v}_{nk}) \le \sum_{\boldsymbol{p}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}}\mathring{\mu}_{nk}(\mathscr{V}_{2\theta} + 2\pi\boldsymbol{p}) = \mathring{\mu}_{nk}(\widetilde{\mathscr{V}_{2\theta}}\setminus\mathscr{V}_{2\theta}).$$

This, together with Cauchy–Schwarz inequality, enables us to obtain

$$\begin{aligned}
\sum_{k=1}^{k_n}\theta_{nkj}^2 &= \sum_{k=1}^{k_n}\left(\int_{\mathscr{V}_\theta - \boldsymbol{v}_{nk}}(x_j + v_{nkj})\, d\varrho_{nk}(\boldsymbol{x})\right)^2 \\
&\le \sum_{k=1}^{k_n}\varrho_{nk}(\mathscr{V}_\theta - \boldsymbol{v}_{nk})\int_{\mathscr{V}_\theta - \boldsymbol{v}_{nk}}(x_j + v_{nkj})^2\, d\varrho_{nk}(\boldsymbol{x}) \\
&\le \theta^2\tau_n(\widetilde{\mathscr{V}_{2\theta}}\setminus\mathscr{V}_{2\theta})\max_{1\le k\le k_n}\varrho_{nk}(\mathscr{V}_{2\theta}).
\end{aligned}$$

Since $\widetilde{\mathscr{V}}_{2\theta}\backslash\mathscr{V}_{2\theta}$ is bounded away from $\mathbf{1} \in \mathbb{T}^2$, the relation $\tau_n \Rightarrow_{\mathbf{0}} \tau$ leads us to $\limsup_n \tau_n(\widetilde{\mathscr{V}}_{2\theta}\backslash\mathscr{V}_{2\theta}) < \infty$. Thus, we are able to conclude that $\sum_{k=1}^{k_n} \theta_{nkj}^2 \to 0$ as $n \to \infty$, yielding (4-8) by the inequality $1 - \cos x \le \frac{x^2}{2}$ on $\mathbb{R}$.

After these preparations, we are ready to present the proof of the theorem. Since (1-2) converges weakly, τ_n meets Condition 2.3, and thus $\rho_n = \tau_n W^{-1}$ satisfies Condition 2.5 according to Proposition 4.4. Then Proposition 4.3 consequently yields that Condition 2.5 also applies to $\tilde{\rho}_n = \sum_{k=1}^{k_n} \mathring{\nu}_{nk}$.

To finish the proof, we just need to verify (2-14) due to Theorem 5.1. The existence of the limit in (2-8) implies that the vector

$$\boldsymbol{E}_n = i\left[\boldsymbol{v}_n + \sum_{k=1}^{k_n}\left(\boldsymbol{v}_{nk} + \int \sin(\boldsymbol{x})\, d\mathring{\mu}_{nk}\right)\right]$$

also has a limit when $n \to \infty$. Indeed, the limit $-i \lim_{n\to\infty} \boldsymbol{E}_n$ disintegrates into the sum of that in (2-8) and

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{\mathbb{R}^2}\left(\sin(\boldsymbol{x}) - \frac{\boldsymbol{x}}{1+\|\boldsymbol{x}\|^2}\right) d\mathring{\mu}_{nk}(\boldsymbol{x}) = \int_{\mathbb{R}^2}\left(\sin(\boldsymbol{x}) - \frac{\boldsymbol{x}}{1+\|\boldsymbol{x}\|^2}\right) d\tau(\boldsymbol{x}).$$

The validity of the equality displayed above is just because of that the integrand is $O(\|\boldsymbol{x}\|^3)$ as $\|\boldsymbol{x}\| \to 0$ and the function $\min\{1, \|\boldsymbol{x}\|^2\}$ is τ-integrable.

In order to go further, we analyze the difference

$$\left(\arg \boldsymbol{b}_{nk} + \int_{\mathbb{T}^2} \Im \boldsymbol{s}\, d\mathring{\nu}_{nk}(\boldsymbol{s})\right) - \left(\boldsymbol{v}_{nk} + \int_{\mathbb{R}^2} \sin(\boldsymbol{x})\, d\mathring{\mu}_{nk}(\boldsymbol{x})\right),$$

which, along with the help of equation $\int \sin(\boldsymbol{x})\, d\mathring{\mu}_{nk} = \int \Im(e^{i\boldsymbol{\theta}_{nk}}\boldsymbol{s})\, d\mathring{\nu}_{nk}$, becomes

$$(\boldsymbol{\theta}_{nk} - \sin\boldsymbol{\theta}_{nk}) + \sin(\boldsymbol{\theta}_{nk})\int_{\mathbb{T}^2}(1 - \Re\boldsymbol{s})\, d\mathring{\nu}_{nk}(\boldsymbol{s}) + (1 - \cos\boldsymbol{\theta}_{nk})\int_{\mathbb{T}^2} \Im\boldsymbol{s}\, d\mathring{\nu}_{nk}(\boldsymbol{s}). \tag{5-10}$$

Using the elementary inequality

$$|x - \sin x| \le 1 - \cos x, \quad |x| \le \frac{\pi}{4}, \tag{5-11}$$

we see from the established result that

$$\sum_{k=1}^{k_n} |\theta_{nkj} - \sin\theta_{nkj}| \le \sum_{k=1}^{k_n}(1 - \cos\theta_{nkj}) \to 0 \quad \text{as } n \to \infty.$$

For the second term in (5-10), $\lambda_{nj} = (1 - \Re s_j)\,\tilde{\rho}_n \Rightarrow \lambda_j \in \mathscr{M}_{\mathbb{T}^2}$ yields that

$$\sum_{k=1}^{k_n}\left|\sin(\theta_{nkj})\int_{\mathbb{T}^2}(1 - \Re s_j)\, d\mathring{\nu}_{nk}(\boldsymbol{s})\right| \le \left(\max_{1\le k\le k_n} |\sin\theta_{nkj}|\right) \lambda_{nj}(\mathbb{T}^2) \xrightarrow[n\to\infty]{} 0.$$

As for the last term, we then have

$$\sum_{k=1}^{k_n}(1-\cos\theta_{nkj})\left|\int_{\mathbb{T}^2}(\Im s_j)\,d\mathring{\nu}_{nk}(\boldsymbol{s})\right| \le \sum_{k=1}^{k_n}(1-\cos\theta_{nkj}) \xrightarrow[n\to\infty]{} 0.$$

Consequently, we have arrived at that the limit in (2-14) exists and equals the vector in (5-8). ☐

The employment of the wrapping limit theorem with $\boldsymbol{v}_n=\boldsymbol{0}$ gives the following identically distributed limit theorem, which is the bi-free version of [6, Theorem 3.9].

Corollary 5.8. *Let* $(\boldsymbol{v},\boldsymbol{A},\tau)$ *be a triplet satisfying* (2-10) *with* $d=2$, $\{\mu_n\}$ *a sequence in* $\mathscr{P}_{\mathbb{R}^2}$, *and* $\{k_n\}$ *a strictly increasing sequence in* $\mathbb{N}$. *If* $\mu_n^{\boxplus\boxplus k_n}\Rightarrow\mu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{A},\tau)}$, *then* $(\mu_n W^{-1})^{\boxtimes\boxtimes k_n}\Rightarrow\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$ *and* $(\mu_n W^{-1})^{\boxtimes\boxtimes^{\mathrm{op}}k_n}\Rightarrow\nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$, *where* $\boldsymbol{\gamma}$ *and* ρ *are as in* Theorem 5.7.

Example 5.9. Given a 2×2 real matrix $\boldsymbol{A}=(a_{ij})\ge 0$ with $a_{11}\ge a_{22}>0$, consider planar probability measures $\mu_n=\frac{1}{4}(\delta_{\boldsymbol{\alpha}_n}+\delta_{-\boldsymbol{\alpha}_n}+\delta_{\boldsymbol{\beta}_n}+\delta_{-\boldsymbol{\beta}_n})$, where $\boldsymbol{\alpha}_n=(\sqrt{2\det\boldsymbol{A}},0)/\sqrt{na_{22}}$ and $\boldsymbol{\beta}_n=(\sqrt{2}a_{12},\sqrt{2}a_{22})/\sqrt{na_{22}}$. Clearly, $\mathring{\mu}_n=\mu_n$ for all n and $\tau_n:=n\mu_n\Rightarrow_{\boldsymbol{0}}0$ as $n\to\infty$. Furthermore, for any $\theta>0$, if n is large enough, then $\int_{\mathscr{V}_\theta}x_j^2\,d\tau_n=a_{jj}$ and $\int_{\mathscr{V}_\theta}x_1x_2\,d\tau_n=a_{12}$. Hence the identically distributed limit theorem introduced in Section 2D indicates that $\mu_n^{\boxplus\boxplus n}$ converges weakly to $\mu_{\boxplus\boxplus}^{(\boldsymbol{0},\boldsymbol{A},0)}$, which is known as the bi-free Gaussian distribution with bi-free Lévy triplet $(\boldsymbol{0},\boldsymbol{A},0)$. For the measures

$$\nu_n=\mu_n W^{-1}=\tfrac{1}{4}(\delta_{e^{i\boldsymbol{\alpha}_n}}+\delta_{e^{-i\boldsymbol{\alpha}_n}}+\delta_{e^{i\boldsymbol{\beta}_n}}+\delta_{e^{-i\boldsymbol{\beta}_n}})\in\mathscr{P}_{\mathbb{T}^2},$$

a direct verification or an application of Corollary 5.8 shows that $\nu_n^{\boxtimes\boxtimes n}\Rightarrow\nu_{\boxtimes\boxtimes}^{(\boldsymbol{1},\boldsymbol{A},0)}$ and $\nu_n^{\boxtimes\boxtimes^{\mathrm{op}}n}\Rightarrow\nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(\boldsymbol{1},\boldsymbol{A},0)}$. Analogically, $\nu_{\boxtimes\boxtimes}=\nu_{\boxtimes\boxtimes}^{(\boldsymbol{1},\boldsymbol{A},0)}$ is called the *bi-free multiplicative Gaussian distribution* with Lévy triplet $(\boldsymbol{1},\boldsymbol{A},0)$. Note that the component $P_{\nu_{\boxtimes\boxtimes}}$ in the representation (5-5), called the bi-free multiplicative compound Poisson part (see Example 5.10), vanishes.

Example 5.10. Given any $r>0$ and $\mu\in\mathscr{P}_{\mathbb{R}^2}$, let $\mu_n=(1-r/n)\,\delta_{\boldsymbol{0}}+r/n\mu$, $\tau_n=n\mu_n$, and $\tau=r1_{\mathbb{R}^2\setminus\{\boldsymbol{0}\}}\mu$. A straightforward verification reveals that Condition 2.3 applies to τ_n, τ, and $Q\equiv 0$. Hence [11, Theorem 5.6] shows that $\mu_n^{\boxplus\boxplus n}$ converges weakly to the so-called bi-free compound Poisson distribution $\mu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{0},\tau)}$ with rate r and jump distribution μ, where $\boldsymbol{v}=r\int\boldsymbol{x}(1+\|\boldsymbol{x}\|^2)^{-1}d\mu$. Applying Corollary 5.8 shows that

$$\big((1-r/n)\,\delta_{\boldsymbol{1}}+r/n(\mu W^{-1})\big)^{\boxtimes\boxtimes n}\Rightarrow\nu_{\boxtimes\boxtimes}^{(e^{i\boldsymbol{u}},\boldsymbol{0},\rho)},$$

as well as $(\mu_n W^{-1})^{\boxtimes\boxtimes^{\mathrm{op}}n}\Rightarrow\nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(e^{i\boldsymbol{u}},\boldsymbol{0},\rho)}$, where

$$\rho=r1_{\mathbb{T}^2\setminus\{\boldsymbol{1}\}}(\mu W^{-1})\quad\text{and}\quad\boldsymbol{u}=r\int\sin\boldsymbol{x}\,d\mu.$$

Analogous to the planar case, we refer to measures of the form $\nu_{\boxtimes\boxtimes} = \nu_{\boxtimes\boxtimes}^{(e^{i\boldsymbol{u}},\mathbf{0},r\nu)}$, where $r > 0$, $\nu \in \mathscr{P}_{\mathbb{T}^2}$ with $\nu(\{\mathbf{1}\}) = 0$, and $\boldsymbol{u} = r\int \Im\boldsymbol{s}\,d\nu$ as the *bi-free multiplicative compound Poisson distribution* with rate r and jump distribution ν. In (5-5), we have the bi-free Gaussian component $N_{\nu_{\boxtimes\boxtimes}} \equiv 0$.

5C. ***Limit theorems for identically distributed case.*** The following is a special case of the limit theorem in the context of identically distributed random vectors on the bi-torus.

Proposition 5.11. *Let $\rho_n = k_n\nu_n$, where $\{\nu_n\} \subset \mathscr{P}_{\mathbb{T}^2}^{\times}$ and $\{k_n\} \subset \mathbb{N}$ with $k_1 < k_2 < \dots$. If ρ_n satisfies* Condition 4.1 (*or* Condition 2.5) *and the limit*

$$\boldsymbol{v} = \lim_{n\to\infty} \int_{\mathbb{T}^2} \Im\boldsymbol{\xi}\, d\rho_n(\boldsymbol{\xi})$$

exists, then $\nu_n^{\boxtimes\boxtimes k_n} \Rightarrow \nu_{\boxtimes\boxtimes}^{(e^{i\boldsymbol{v}},\boldsymbol{A},\rho)}$ and $\nu_n^{\boxtimes\boxtimes^{\mathrm{op}} k_n} \Rightarrow \nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(e^{i\boldsymbol{v}},\boldsymbol{A},\rho)}$, where ρ and $\boldsymbol{A}$ are as in Condition 4.1 *and* Proposition 4.2, *respectively.*

Proof. Let $h : \mathbb{T}^2 \to (-\pi, \pi]^2$ be the inverse of the wrapping map $W(\boldsymbol{x}) = e^{i\boldsymbol{x}}$ restricted to $(-\pi, \pi]^2$, namely, $h(\boldsymbol{\xi}) = \arg\boldsymbol{\xi}$. Further let $\mu_n = \nu_n h^{-1} \in \mathscr{P}_{\mathbb{R}^2}$ and $\tau = \rho h^{-1} \in \mathscr{M}_{\mathbb{R}^2}^{\mathbf{0}}$, whose supports are all contained in $[-\pi, \pi]^2$. Then $\nu_n = \mu_n W^{-1}$, and $\tau_n = \rho_n h^{-1} \Rightarrow_{\mathbf{0}} \tau$ by the continuous mapping theorem. Also, (4-2) and (4-12) show that $\min\{1, \|\boldsymbol{x}\|^2\} \in L^1(\tau)$. One can utilize (5-11) to justify

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty} \int_{\mathscr{U}_\epsilon} (\arg s_j)(\arg s_\ell)\, d\rho_n(\boldsymbol{s}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty} \int_{\mathscr{U}_\epsilon} (\arg s_j)(\arg s_\ell)\, d\rho_n(\boldsymbol{s}).$$

On the other hand, one has the equation $\int_{\mathscr{V}_\epsilon} x_j\, x_\ell\, d\tau_n = \int_{\mathscr{U}_\epsilon} (\arg s_j)(\arg s_\ell)\, d\rho_n$ by the change-of-variables formula (4-12), which implies that τ_n satisfies (IV) of Condition 2.3. Ultimately, observe that

$$\int_{\mathbb{R}^2} \frac{\boldsymbol{x}}{1 + \|\boldsymbol{x}\|^2}\, d\tau_n(\boldsymbol{x}) = \int_{\mathbb{T}^2} \Im\boldsymbol{s}\, d\rho_n(\boldsymbol{s}) + \int_{\mathbb{R}^2} \left(\frac{\boldsymbol{x}}{1 + \|\boldsymbol{x}\|^2} - \sin(\boldsymbol{x}) \right) d\tau_n(\boldsymbol{x})$$

has a limit when $n \to \infty$ owing to $\boldsymbol{x}/(1 + \|\boldsymbol{x}\|^2) - \sin(\boldsymbol{x}) = O(\|\boldsymbol{x}\|^3)$ as $\|\boldsymbol{x}\| \to 0$ and $\min\{1, \|\boldsymbol{x}\|^2\} \in L^1(\tau)$. Thus, $\mu_n^{\boxplus\boxplus k_n} \Rightarrow \mu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{A},\tau)}$ by [11, Theorem 5.6], and so we accomplish the proof by Corollary 5.8. □

We shall also consider the rotated probabilities

$$d\tilde{\nu}_n(\boldsymbol{s}) = d\nu_n(\boldsymbol{\omega}_n\, \boldsymbol{s})$$

associated with a sequence $\{\nu_n\} \subset \mathscr{P}_{\mathbb{T}^2}^{\times}$, where $\boldsymbol{\omega}_n = (\omega_{n1}, \omega_{n2}) \in \mathbb{T}^2$ has components

$$\omega_{nj} = \int_{\mathbb{T}^2} s_j\, d\nu_n(\boldsymbol{s}) \Big/ \left| \int_{\mathbb{T}^2} s_j\, d\nu_n(\boldsymbol{s}) \right|.$$

Through this sort of rotated distributions, we next present the bi-freely identically distributed limit theorem, which is the bi-free analog of [6, Proposition 3.6].

Theorem 5.12. *The following are equivalent for a sequence* $\{\nu_n\}$ *in* $\mathscr{P}_{\mathbb{T}^2}^{\times}$ *and a strictly increasing sequence* $\{k_n\}$ *in* $\mathbb{N}$.

(1) *The sequence* $\nu_n^{\boxtimes\boxtimes k_n}$ *converges weakly to some* $\nu_{\boxtimes\boxtimes} \in \mathscr{P}_{\mathbb{T}^2}^{\times}$.

(2) *The sequence* $\nu_n^{\boxtimes\boxtimes^{\mathrm{op}} k_n}$ *converges weakly to some* $\nu_{\boxtimes\boxtimes^{\mathrm{op}}} \in \mathscr{P}_{\mathbb{T}^2}^{\times}$.

(3) Condition 4.1 *holds for* $\rho_n = k_n \tilde{\nu}_n$ *and the limit* $\boldsymbol{\gamma} = \lim_{n\to\infty}(\omega_{n1}^{k_n}, \omega_{n2}^{k_n})$ *exists in* $\mathbb{T}^2$.

If (1)–(3) *hold*, *then* $\nu_{\boxtimes\boxtimes} = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$ *and* $(\nu_{\boxtimes\boxtimes^{\mathrm{op}}})^{\star} = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}^{\star},\boldsymbol{A}^{\mathrm{op}},\rho^{\star})}$, *where* ρ *and* $\boldsymbol{A}$ *are respectively as in* Condition 4.1 *and* Proposition 4.2.

Proof. Only the equivalence (1) $\Leftrightarrow$ (3) needs a proof, which relies on Proposition 4.3. First of all, the weak convergence of $\nu_n^{\boxtimes\boxtimes k_n}$ to $\nu \in \mathscr{P}_{\mathbb{T}^2}^{\times}$ yields that $\tilde{\nu}_n \Rightarrow \delta_{(1,1)}$. Indeed, $m_{1,0}(\nu_n)^{k_n} = \Sigma_{\nu_n^{(1)}}(0)^{-k_n} \to \Sigma_{\nu^{(1)}}(0)^{-1} = m_{1,0}(\nu)$ shows that

$$\omega_{n1}^{k_n} \to m_{1,0}(\nu)/|m_{1,0}(\nu)| := \omega_1.$$

Since $\Sigma_{\tilde{\nu}_n^{(1)}}(z)^{k_n} = \omega_{n1}^{k_n}\Sigma_{\nu_n^{(1)}}(z)^{k_n} \to \omega_1\Sigma_{\nu^{(1)}}(z) = \Sigma_{\tilde{\nu}^{(1)}}(z)$ uniformly for z in a neighborhood of zero as $n \to \infty$ by [4, Proposition 2.9], it follows from [4, Lemma 2.7] that $\tilde{\nu}_n^{(1)} \Rightarrow \delta_1$. In the same vein, one can obtain $\tilde{\nu}_n^{(2)} \Rightarrow \delta_1$, giving the desired weak convergence. On other hand, the $\mathscr{M}_{\mathbb{T}^2}^{\mathbf{1}}$-weak convergence of $\rho_n = k_n\tilde{\nu}_n$ also implies $\tilde{\nu}_n \Rightarrow \delta_{\mathbf{1}}$. In other words, $\tilde{\nu}_n$ is infinitesimal if assertion (1) or (3) holds.

Write $\nu_n^{\boxtimes\boxtimes k_n} = \delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes\, \tilde{\nu}_n^{\boxtimes\boxtimes k_n}$ and consider measures $d\mathring{\tilde{\nu}}_n(\boldsymbol{s}) = d\tilde{\nu}_n(\tilde{\boldsymbol{b}}_n\,\boldsymbol{s})$, where $\boldsymbol{\xi}_n = \boldsymbol{\omega}_n^{k_n}$ and $\tilde{\boldsymbol{b}}_n = \exp[i\int_{\mathscr{U}_\theta}(\arg \boldsymbol{s})\,d\tilde{\nu}_n]$. Then as indicated in Theorem 5.1, assertion (1) holds if and only if $\rho_n' = k_n\mathring{\tilde{\nu}}_n$ satisfies Condition 2.5 and $\boldsymbol{\gamma}_n = \boldsymbol{\xi}_n \exp(i\boldsymbol{E}_n)$ has a finite limit, where $\boldsymbol{E}_n = k_n[\arg\tilde{\boldsymbol{b}}_n + \int(\Im\boldsymbol{s})\,d\mathring{\tilde{\nu}}_n]$. The infinitesimality of $\tilde{\nu}_n$ reveals that $\boldsymbol{\theta}_n = (\theta_{n1}, \theta_{n2}) \to \boldsymbol{0}$ as $n \to \infty$, where

$$\theta_{nj} = \arg\tilde{b}_{nj} = \int_{\mathscr{U}_\theta} \arg s_j\, d\tilde{\nu}_n(\boldsymbol{s}). \tag{5-12}$$

This simple fact will be often utilized in the following proof, and all the indices n considered below are sufficiently large. The equivalence of Condition 2.5 and Condition 4.1 is employed below as well. With a view toward applying Proposition 4.3 to ρ_n and ρ_n', we shall prove that $\lim_{n\to\infty} k_n\|\boldsymbol{\theta}_n\|^2 = 0$.

Now, we argue that $\rho_n'(\,\cdot\,) = k_n\mathring{\tilde{\nu}}_n(\,\cdot\,) = \rho_n(e^{i\boldsymbol{\theta}_n}\cdot)$ satisfies Condition 2.5 if the same condition applies to $\rho_n = k_n\tilde{\nu}_n$. Let $\lambda_{nj} = (1 - \Re s_j)\,\rho_n$. Using the fact

$$\int_{\mathbb{T}^2} \Im s_j\, d\tilde{\nu}_n(\boldsymbol{s}) = 0, \quad j = 1, 2, \tag{5-13}$$

we have

$$k_n\theta_{nj} = \int_{\mathscr{U}_\theta} \arg s_j \, d\rho_n(\boldsymbol{s}) - \int_{\mathbb{T}^2} \Im s_j \, d\rho_n(\boldsymbol{s})$$
$$= \int_{\mathbb{T}^2} \frac{\arg s_j - \Im s_j}{1-\Re s_j} \, d\lambda_{nj}(\boldsymbol{s}) - \int_{\mathbb{T}^2\setminus\mathscr{U}_\theta} \arg s_j \, d\rho_n(\boldsymbol{s}).$$

Then the continuity of $s \mapsto (\arg s - \Im s)/(1-\Re s)$ on $\mathbb{T}$ implies that

$$\limsup_{n\to\infty} k_n|\theta_{nj}| < \infty,$$

and so $\lim_{n\to\infty} k_n\|\boldsymbol{\theta}_n\|^2 = 0$. Thus, ρ_n' meets Condition 2.5 by Proposition 4.3.

Conversely, suppose that ρ_n' satisfies Condition 2.5. We first rewrite (5-12) as $\arg\tilde{b}_{nj} = \int_{\tilde{\boldsymbol{b}}_n^{-1}\mathscr{U}_\theta}(\arg s_j + \arg\tilde{b}_{nj})\, d\mathring{\tilde{\nu}}_n$. On the other hand, the integral in (5-13) can be decomposed into the sum

$$\Im\tilde{b}_{nj} - (\Im\tilde{b}_{nj})\int_{\mathbb{T}^2}(1-\Re s_j)\, d\mathring{\tilde{\nu}}_n(\boldsymbol{s}) - (1-\Re\tilde{b}_{nj})\int_{\mathbb{T}^2}(\Im s_j)\, d\mathring{\tilde{\nu}}_n(\boldsymbol{s}) + \int_{\mathbb{T}^2}(\Im s_j)\, d\mathring{\tilde{\nu}}_n(\boldsymbol{s}).$$

Since $\tilde{b}_{nj} = \cos\theta_{nj} + i\sin\theta_{nj}$, some simple calculations allow us to obtain

$$\theta_{nj} = \arg\tilde{b}_{nj} - \int_{\mathbb{T}^2}(\Im s_j)\, d\tilde{\nu}_n(\boldsymbol{s}) = B_{nj} + R_{nj},$$

where

$$R_{nj} = (\theta_{nj} - \sin\theta_{nj}) + (1-\cos\theta_{nj})\int(\Im s_j)\, d\mathring{\tilde{\nu}}_n$$

and

$$B_{nj} = -\theta_{nj}\mathring{\tilde{\nu}}_n(\mathbb{T}^2\setminus\tilde{\boldsymbol{b}}_n^{-1}\mathscr{U}_\theta) - \int_{\mathbb{T}^2\setminus\tilde{\boldsymbol{b}}_n^{-1}\mathscr{U}_\theta}(\arg s_j)\, d\mathring{\tilde{\nu}}_n(\boldsymbol{s})$$
$$+ \sin(\theta_{nj})\int_{\mathbb{T}^2}(1-\Re s_j)\, d\mathring{\tilde{\nu}}_n(\boldsymbol{s}) + \int_{\mathbb{T}^2}\frac{\arg s_j - \Im s_j}{1-\Re s_j}\, d(1-\Re s_j)\mathring{\tilde{\nu}}_n(\boldsymbol{s}).$$

Note that sets $\mathbb{T}^2\setminus\tilde{\boldsymbol{b}}_n^{-1}\mathscr{U}_\theta$ are uniformly bounded away from $\mathbf{1}$, whence we see that $\limsup_{n\to\infty} k_n|B_{nj}| < \infty$ by the $\mathscr{M}_{\mathbb{T}^2}^{\mathbf{1}}$-convergence assumption of ρ_n'. Then $|R_{nj}| \le |\theta_{nj}|^3 + |\theta_{nj}|^2$ leads to

$$\limsup_{n\to\infty} k_n|\theta_{nj}|[1-|\theta_{nj}|-|\theta_{nj}|^2] \le \limsup_{n\to\infty} k_n|B_{nj}| < \infty.$$

We thus obtain $\limsup_n k_n|\theta_{nj}| < \infty$, and so $\lim_n k_n\|\boldsymbol{\theta}_n\|^2 = 0$. Consequently, ρ_n satisfies Condition 2.5 by Proposition 4.3 again.

Finally, by using (5-13), one can express components of $\boldsymbol{E}_n = (E_{n1}, E_{n2})$ as

$$E_{nj} = k_n\theta_{nj} + k_n(\Im\tilde{b}_{nj}^{-1})\int_{\mathbb{T}^2}(\Re s_j)\, d\tilde{\nu}_n(\boldsymbol{s})$$
$$= k_n(\theta_{nj} - \sin\theta_{nj}) + \sin(\theta_{nj})\int_{\mathbb{T}^2}(1-\Re s_j)\, d\rho_n(\boldsymbol{s}).$$

As noted above that ρ_n meets Condition 2.5 if and only if so does ρ_n' and that $\lim_{n\to\infty} k_n|\theta_{nj}|^2 = 0$ in either case. Consequently, we have shown $\lim_{n\to\infty} E_{nj} = 0$ for $j = 1, 2$ and arrived at $\boldsymbol{\gamma} = \lim_{n\to\infty} \boldsymbol{\gamma}_n$ if (1) or (3) holds. $\square$

Remark 5.13. In spite of $\delta_{-\mathbf{1}}^{\boxtimes\boxtimes 2n} = \delta_{\mathbf{1}}$, $2n\delta_{-\mathbf{1}}$ fails to converge in $\mathscr{M}_{\mathbb{T}^2}^{\mathbf{1}}$. This example demonstrates that in Theorem 5.12, the rotated probabilities $\tilde{\nu}_n$ are a necessary medium in the convergence criteria of the bi-free multiplicative limit theorem. For the same inference, the converse statement of Proposition 5.11 does not hold, yet it does in the additive setting [11, Theorem 5.6].

6. Homomorphisms between infinitely divisible distributions

This section will provide explanations for the diagram (1-5). The bijection

$$\Lambda : \mathcal{ID}(*) \to \mathcal{ID}(\boxplus\boxplus)$$

was already defined in [11], specifically,

$$\Lambda(\mu_*^{(\boldsymbol{v},\boldsymbol{A},\tau)}) = \mu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{A},\tau)}.$$

If $\nu = \mu_*^{(\boldsymbol{v},\boldsymbol{A},\tau)} W^{-1}$, then (2-9) and (4-12) show that

$$\begin{aligned}\hat{\nu}(\boldsymbol{p}) &= \exp\left[i\langle \boldsymbol{p}, \boldsymbol{v}\rangle - \tfrac{1}{2}\langle \boldsymbol{A}\boldsymbol{p}, \boldsymbol{p}\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle \boldsymbol{p},\boldsymbol{x}\rangle} - 1 - \frac{i\langle \boldsymbol{p}, \boldsymbol{x}\rangle}{1+\|\boldsymbol{x}\|^2}\right) d\tau(\boldsymbol{x})\right] \\ &= \boldsymbol{\gamma}^{\boldsymbol{p}} \exp\left[-\tfrac{1}{2}\langle \boldsymbol{A}\boldsymbol{p}, \boldsymbol{p}\rangle + \int_{\mathbb{T}^d}(\boldsymbol{s}^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im \boldsymbol{s}\rangle)\, d\rho(\boldsymbol{s})\right],\end{aligned}$$

where ρ and $\boldsymbol{\gamma}$ are respectively given in (5-7) and (5-8). Putting it differently, the wrapping map induces a homomorphism $W_* : \mathcal{ID}(*) \to \mathcal{ID}(\circledast)$ satisfying

$$W_*(\mu_*^{(\boldsymbol{v},\boldsymbol{A},\tau)}) = \nu_{\circledast}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}. \tag{6-1}$$

Motivated by (6-1), we analogously define $W_{\boxplus\boxplus} : \mathcal{ID}(\boxplus\boxplus) \to \mathcal{ID}(\boxtimes\boxtimes)$ as

$$W_{\boxplus\boxplus}(\nu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{A},\tau)}) = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)},$$

where $\boldsymbol{\gamma}$ and ρ are given as before. It was shown in Theorem 5.7 that the weak convergence of (1-2) to some $\nu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{A},\tau)}$ implies that equation (1-3) converges weakly to $W_{\boxplus\boxplus}(\nu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{A},\tau)})$.

For the last ingredient $\Gamma : \mathcal{ID}(\boxtimes\boxtimes) \to \mathcal{ID}(\circledast)$, recall from Proposition 3.9 that $\boxtimes\boxtimes$-idempotent elements also belong to $\mathcal{ID}(\circledast)$. Also, [6, Definition 3.3] introduced a homomorphism $\Gamma_1 : \mathcal{ID}(\mathbb{T}, \boxtimes) \to \mathcal{ID}(\mathbb{T}, \circledast)$ (which was denoted by Γ therein), which leads to the following definition.

Definition 6.1. Let $\nu \in \mathcal{ID}(\boxtimes\boxtimes)$. Define $\Gamma(\nu) = \nu_{\circledast}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$ if $\nu = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$. For $\nu \in \mathscr{P}_{\mathbb{T}^2}\backslash\mathscr{P}_{\mathbb{T}^2}^{\times}$, define $\Gamma(\nu) = \nu$ if $\nu = P\boxtimes\boxtimes(\kappa_c \times \delta_1)$, and let $\Gamma(\nu) = \mathrm{m} \times \Gamma_1(\nu^{(2)})$ if $\nu = \mathrm{m} \times \nu^{(2)}$ and $\Gamma(\nu) = \Gamma_1(\nu^{(1)}) \times \mathrm{m}$ if $\nu = \nu^{(1)} \times \mathrm{m}$.

One can check that $\Gamma : \mathcal{ID}(\boxtimes\boxtimes) \to \mathcal{ID}(\circledast)$ is a homomorphism and that the diagram (1-5) commutes. The latter result comes from the definition, while the former one requires convolution identities in Section 3. For example, if we write $\mu = P\boxtimes\boxtimes(\kappa_c \times \delta_1)$ and $\nu = \mathrm{m} \times \nu^{(2)}$ with $\nu^{(2)} \in \mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$, then we have $\mu \boxtimes\boxtimes \nu = P \boxtimes\boxtimes (\mathrm{m} \times \nu^{(2)}) = \mathrm{m} \times \mathrm{m}$, where the last equality can be confirmed by the use of (3-6) and computing moments. On the other hand,

$$\Gamma(\mu) \circledast \Gamma(\nu) = \mu \circledast (\mathrm{m} \times \Gamma_1(\nu^{(2)})) = P \circledast (\mathrm{m} \times \Gamma_1(\nu^{(2)})) = \mathrm{m} \times \mathrm{m},$$

where the last equality is again obtained by computing moments. Consequently, we arrive at $\Gamma(\mu \boxtimes\boxtimes \nu) = \Gamma(\mu) \circledast \Gamma(\nu)$.

This map Γ is neither injective nor surjective as we have

$$\nu_{\circledast}^{((1,0),\mathbf{0},\pi\delta_{(i,0)})} = \nu_{\circledast}^{((1,0),\mathbf{0},\pi\delta_{(-i,0)})}$$

and $P \circledast (\mu \times \delta_1)$ lies in $\mathcal{ID}(\circledast) \backslash \Gamma(\mathcal{ID}(\boxtimes\boxtimes))$ for any $\mu \in \mathcal{ID}(\mathbb{T}, \circledast) \backslash \{\kappa_c : c \in \mathbb{D} \cup \mathbb{T}\}$. Further, Γ is not weakly continuous. More strongly, we prove the following.

Proposition 6.2. (1) *The restriction of* Γ_1 *to the set* $\mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$ *has no weakly continuous extension to* $\mathcal{ID}(\boxtimes)$.

(2) *The restriction of* Γ *to the set* $\mathcal{ID}(\boxtimes\boxtimes) \cap \mathscr{P}_{\mathbb{T}^2}^{\times}$ *has no weakly continuous extension to* $\mathcal{ID}(\boxtimes\boxtimes)$.

Proof. Since $\Gamma(\mu^{(1)} \times \mu^{(2)}) = \Gamma_1(\mu^{(1)}) \times \Gamma_1(\mu^{(2)})$ for $\mu^{(1)}, \mu^{(2)} \in \mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$, assertion (2) follows immediately from (1).

Suppose that $\Gamma_1^0 := \Gamma_1|_{\mathcal{ID}(\boxtimes)\cap\mathscr{P}_{\mathbb{T}}^{\times}}$ has a weakly continuous extension $\tilde{\Gamma}_1$ to $\mathcal{ID}(\boxtimes)$. Observe that $\kappa_c \in \mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$ and $\Gamma_1^0(\kappa_c) = \kappa_c$ for any $c \in (\mathbb{D} \cup \mathbb{T}) \backslash \{0\}$. The latter identity is shown below. From the moments $m_p(\kappa_c) = c^p$ for $p \in \mathbb{N}$, the formula

$$\Sigma_{\kappa_c}(z) = \frac{1}{c} = \frac{1}{c/|c|} \exp\left[(-\log|c|) \int_{\mathbb{T}^\times} \frac{1+sz}{1-sz} (1 - \Re s) \frac{d\mathrm{m}(s)}{1 - \Re s}\right]$$

yields that κ_c has $(c/|c|, 0, \rho)$, where $\rho(ds) = [-\log|c|/(1 - \Re s)]\, \mathrm{m}(ds)$ on $\mathbb{T}^\times$, as its free multiplicative Lévy triplet (also known as $\boxtimes$-characteristic triplet in [6, p. 2437]). On the other hand, Lemma 5.5 says that the same triplet $(c/|c|, 0, \rho)$ also serves as the classical multiplicative Lévy triplet of κ_c. Thus we have shown that $\Gamma_1^0(\kappa_c) = \kappa_c$. That $\kappa_c \Rightarrow \mathrm{m}$ as $c \to 0$ allows us to further obtain $\tilde{\Gamma}_1(\mathrm{m}) = \mathrm{m}$.

Next, denote by ν_n the probability distribution in $\mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$ having the free multiplicative Lévy triplet $(1, 0, n\delta_{-1})$, and let $\mu_n = \Gamma_1^0(\nu_n)$. Then (5-6) shows that for any $p \in \mathbb{Z}$,

$$\hat{\mu}_n(p) = \exp[n((-1)^p - 1)] = \begin{cases} 1, & p \text{ is even,} \\ e^{-2n}, & p \text{ is odd,} \end{cases}$$

which readily implies that $\mu_n \Rightarrow \frac{1}{2}(\delta_{-1} + \delta_1)$. However, we will explain in the next paragraph that $\nu_n \Rightarrow \mathrm{m}$, which apparently leads to a contradiction.

To see why $\nu_n \Rightarrow \mathrm{m}$, select a weakly convergent subsequence of $\{\nu_n\}$ (still denoted by $\{\nu_n\}$ in the remaining arguments) and denote the weak limit by ν. Let ν_n' be the probability measure having the free multiplicative Lévy triplet $\left(1, 0, \left(\frac{n}{2}\right)\delta_{-1}\right)$. Passing to a further subsequence we may assume that ν_n' weakly converge to ν'. Then letting $n \to \infty$ in the identity $\nu_n = \nu_n' \boxtimes \nu_n'$ gives $\nu = \nu' \boxtimes \nu'$. On the other hand, we see from (2-16) or from [6, Section 2.5] that $\Sigma_{\nu_n'}(0) = e^n$, i.e., $m_1(\nu_n') = e^{-n} \to 0$ as $n \to \infty$ by Remark 2.6, whence $m_1(\nu') = 0$. By the definition of freeness, we can further conclude that $m_p(\nu) = 0$ for all $p \in \mathbb{Z} \setminus \{0\}$ or, equivalently, $\nu = \mathrm{m}$. $\square$

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TAME QUASICONFORMAL MOTIONS AND MONODROMY

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The concept of *tame quasiconformal motions* was first introduced by Jiang et al. (2018). The concept of *monodromy* of holomorphic motions was first introduced by Beck et al. (2012). In this paper, we will show that the concept of *monodromy* of tame quasiconformal motions can be defined, whereas it cannot be defined for quasiconformal motions, in the sense of Sullivan and Thurston (1986). We also study some other properties of tame quasiconformal motions.

1. Introduction

The concept of quasiconformal motions was first introduced by Sullivan and Thurston [12]. Theorem 3 of [12] claimed that every quasiconformal motion of any set over an interval can be extended to the Riemann sphere. Jiang et al. [7] presented a counterexample to Theorem 3 of [12]. They [7] introduced a new concept, called *tame quasiconformal motions*, and showed that Theorem 3 of [12] holds for tame quasiconformal motions over any simply connected Hausdorff space. They also showed that this extension can be done in a conformally natural way, for tame quasiconformal motions. The crucial idea was to show that tame quasiconformal motions have a certain "universal property" that quasiconformal motions (in the sense of Sullivan and Thurston) do not have.

Beck et al. [2] introduced the concept of *monodromy* associated with a holomorphic motion of a closed subset of the Riemann sphere over a hyperbolic Riemann surface. Jiang and Mitra [6] proved that the triviality of the monodromy for this holomorphic motion is a necessary and sufficient condition for the given holomorphic motion to be extended to the whole Riemann sphere over the same hyperbolic Riemann surface. However, the concept of monodromy cannot be defined for a quasiconformal motion of a closed subset of the Riemann sphere over a hyperbolic Riemann surface, due to the counterexample in [7]. In the present paper, we show that the concept of monodromy can be defined for a tame quasiconformal motion

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of a closed subset of the Riemann sphere over any connected Hausdorff space. We prove that the triviality of the monodromy for a tame quasiconformal motion of a closed subset of the Riemann sphere over a path-connected Hausdorff space is a necessary and sufficient condition for this tame quasiconformal motion to be extended to a quasiconformal motion of the whole Riemann sphere over the same path-connected Hausdorff space. We also study some other properties of tame quasiconformal motions.

This paper is organized as follows. In Section 2, we give all basic definitions and note the various facts that are needed in this paper, and then state the two main theorems. In Section 3 we present three lemmas and in Sections 4 and 5, we prove the two main theorems.

2. Basic definitions and statements of the main theorems

Throughout this paper, $\mathbb{C}$ denotes the complex plane, $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere and $E \subset \hat{\mathbb{C}}$ is a closed subset such that $0, 1, \infty \in E$.

When we write $\widetilde{V}$ or $\widetilde{W}$ or $\widetilde{X}$ is "simply connected", we mean that it is a path-connected topological space and that its fundamental group is trivial.

We begin with some definitions.

Definition 1. Let $E \subset \hat{\mathbb{C}}$ and let X be a connected Hausdorff space with basepoint x_0. A *motion of E over X* is a map $\phi : X \times E \to \hat{\mathbb{C}}$ satisfying

(i) $\phi(x_0, z) = z$ for all $z \in E$, and

(ii) for all $x \in X$, the map $\phi(x, \cdot) : E \to \hat{\mathbb{C}}$ is injective.

We say that X is the *parameter space* of the motion ϕ. We will assume that 0, 1, and ∞ belong to E and that the motion ϕ is *normalized*, i.e., 0, 1, and ∞ are fixed points of the map $\phi(x, \cdot)$ for every x in X.

Let $E \subset \hat{E}$, $\phi : X \times E \to \hat{\mathbb{C}}$ and $\hat{\phi} : X \times \hat{E} \to \hat{\mathbb{C}}$ be two motions. We say that $\hat{\phi}$ *extends* ϕ if $\hat{\phi}(x, z) = \phi(x, z)$ for all $(x, z) \in X \times E$.

For any motion $\phi : X \times E \to \hat{\mathbb{C}}$, x in X, and any quadruplet of distinct points a, b, c, d of points in E, let $\phi_x(a, b, c, d)$ denote the cross-ratio of the values $\phi(x, a), \phi(x, b), \phi(x, c)$ and $\phi(x, d)$. We will often write $\phi(x, z)$ as $\phi_x(z)$ for x in X and z in E. So we have

$$\phi_x(a, b, c, d) = \frac{(\phi_x(a) - \phi_x(c))(\phi_x(b) - \phi_x(d))}{(\phi_x(a) - \phi_x(d))(\phi_x(b) - \phi_x(c))} \quad \text{for each } x \text{ in } X.$$

It is obvious that condition (ii) in Definition 1 holds if and only if $\phi_x(a, b, c, d)$ is a well-defined point in the thrice-punctured sphere $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ for all x in X and all quadruplets a, b, c, d of distinct points in E.

Let ρ be the Poincaré distance on $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Sullivan and Thurston [12] introduced the following definition.

Definition 2. A *quasiconformal motion* is a motion $\phi : X \times E \to \hat{\mathbb{C}}$ of E over X with the following additional property:

(iii) Given any x in X and any $\epsilon > 0$, there exists a neighborhood U_x of x such that for any quadruplet of distinct points a, b, c, d in E, we have

$$\rho\big(\phi_y(a, b, c, d), \phi_{y'}(a, b, c, d)\big) < \epsilon \quad \text{for all } y \text{ and } y' \text{ in } U_x.$$

Definition 3. A *continuous motion* of $\hat{\mathbb{C}}$ over X is a motion $\phi : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that the map ϕ is continuous.

Remark. If ϕ is a continuous motion of $\hat{\mathbb{C}}$, then each ϕ_x, x in X, is a map from $\hat{\mathbb{C}}$ to itself that fixes 0, 1, and ∞. Since ϕ_x is injective and continuous, it is a homeomorphism of $\hat{\mathbb{C}}$ onto itself, by invariance of domain.

Recall that a homeomorphism of $\hat{\mathbb{C}}$ is called *normalized* if it fixes the points 0, 1, and ∞. We use $M(\mathbb{C})$ to denote the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\hat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

We will need the following properties that were proved in [11].

Proposition 4. *A motion $\phi : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is quasiconformal if and only if it satisfies*:

(i) *The map $\phi_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is quasiconformal for each x in X.*

(ii) *The map from X to $M(\mathbb{C})$ that sends x to the Beltrami coefficient of ϕ_x for each x in X is continuous.*

Part (ii) means that the map $x \mapsto \mu_x = (\phi_x)_{\bar{z}}/(\phi_x)_z$, $x \in X$, is continuous.

Proposition 5. *Every quasiconformal motion of $\hat{\mathbb{C}}$ is a continuous motion.*

Definition 6. Assume that W is a connected complex manifold with basepoint x_0. A *holomorphic motion* of E over W is a motion $\phi : W \times E \to \hat{\mathbb{C}}$ of E over W such that the map $\phi(\,\cdot\,, z) : W \to \hat{\mathbb{C}}$ is holomorphic for each z in E.

Definition 7. Let X be a connected Hausdorff space with a basepoint x_0, and E be a set in $\hat{\mathbb{C}}$ (containing the points 0, 1, and ∞). A *tame quasiconformal motion* is a motion $\phi : X \times E \to \hat{\mathbb{C}}$ of E over X with the following additional property:

(iii) Given any x in X, there exists a quasiconformal map $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, a neighborhood $N(x)$, with basepoint x, and a quasiconformal motion $\psi : N(x) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ over $N(x)$ such that $\phi(y, z) = \psi(y, w(z))$ for all $(y, z) \in N(x) \times E$.

The lemma below was proved in [7].

Lemma 8. *A motion $\phi : X \times E \to \hat{\mathbb{C}}$ is a tame quasiconformal motion if and only if given any $x \in X$, there exists a neighborhood $N(x)$, and a continuous map $g_x : N(x) \to M(\mathbb{C})$ such that $\phi(y, z) = w^{g_x(y)}(z)$ for all $(y, z) \in N(x) \times E$.*

Definition 9. Let X and Y be connected Hausdorff spaces with basepoint, and f be a continuous basepoint preserving map of X into Y. If ϕ is a motion of E over Y its *pullback* by f is the motion

$$f^*(\phi)(x,z) = \phi(f(x),z) \quad \text{for all } (x,z) \in X \times E$$

of E over X.

Remark. If the motion ϕ is continuous, or tame quasiconformal, $f^*(\phi)$ has the same property. If X and Y are complex manifolds, f holomorphic and ϕ is a holomorphic motion, then so is $f^*(\phi)$.

Proposition 10. *If $\phi : X \times E \to \hat{\mathbb{C}}$ is a holomorphic motion where X is a connected complex Banach manifold with a basepoint x_0. Then ϕ is a tame quasiconformal motion.*

See Proposition 6 in [7].

Remark. In [7] it was shown that holomorphic motions $\Rightarrow$ tame quasiconformal motions $\Rightarrow$ quasiconformal motions $\Rightarrow$ continuous motions.

Definition 11. Let $\phi : X \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion. Let G be a group of Möbius transformations, and suppose that E is invariant under G (which means, $g(E) = E$ for all g in G). We say that ϕ is *G-equivariant* if and only if for each g in G, and x in X, there is a Möbius transformation $\theta_x(g)$ such that

$$\phi(x, g(z)) = (\theta_x(g))(\phi(x,z)) \quad \text{for all } z \in E. \tag{2-1}$$

Definition 12. Let G be a subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and suppose that E is invariant under G. An isomorphism $\eta : G \to \mathrm{PSL}(2,\mathbb{C})$ is said to be induced by an injection $f : E \to \hat{\mathbb{C}}$ if $f(g(z)) = \eta(g)(f(z))$ for all $g \in G$ and for $z \in E$. An isomorphism induced by a quasiconformal self-map of $\hat{\mathbb{C}}$ is called a *quasiconformal deformation* of G.

Definition 13. Let X be a connected Hausdorff space and let G be a subgroup of $\mathrm{PSL}(2,\mathbb{C})$. A *continuous family $\{\theta_x\}$ of isomorphisms of G* is such that:

(i) For each $x \in X$, $\theta_x : G \to \mathrm{PSL}(2,\mathbb{C})$ is an isomorphism.

(ii) The map $x \mapsto \theta_x(g)$ is continuous for each $g \in G$, and for each $x \in X$.

We will need the following result; see Corollaries 1 and 2 of [7].

Theorem 14. *Let $\widetilde{V}$ be a simply connected Hausdorff space with a basepoint, and let $\phi : \widetilde{V} \times E \to \hat{\mathbb{C}}$ be a G-equivariant tame quasiconformal motion. Then, there exists a G-equivariant quasiconformal motion $\tilde{\phi} : \widetilde{V} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends ϕ.*

This means the following:

(i) For each x in $\widetilde{V}$, the map $\tilde{\phi}_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map; let its Beltrami coefficient be μ_x.

(ii) The map $x \mapsto \mu_x$ is continuous for x in $\widetilde{V}$.

(iii) $\tilde{\phi}(x, z) = \phi(x, z)$ for all $(x, z) \in \widetilde{V} \times E$.

(iv) $\tilde{\phi}_x \circ g \circ \tilde{\phi}_x^{-1} = \theta_x(g)$ for each g in G.

We also need the following result; see Remark 4 in [7].

Lemma 15. *Assume that $\phi : X \times E \to \hat{\mathbb{C}}$ is a tame quasiconformal motion where X is a connected Hausdorff space with a basepoint x_0. For each z in E, the map $\phi(\,\cdot\,, z) : X \to \hat{\mathbb{C}}$ is continuous.*

2A. *Teichmüller space of a closed set E.* Two normalized quasiconformal self-mappings f and g of $\hat{\mathbb{C}}$ are said to be E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The *Teichmüller space* $T(E)$ is the set of all E-equivalence classes of normalized quasiconformal self-mappings of $\hat{\mathbb{C}}$. The basepoint of $T(E)$ is the E-equivalence class of the identity map.

Recall that $M(\mathbb{C})$ denotes the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\hat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

We can define the quotient map $P_E : M(\mathbb{C}) \to T(E)$ by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ, written as $[w^\mu]_E$. Clearly, P_E maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$.

G. Lieb [8] proved that $T(E)$ is a complex Banach manifold such that the projection map P_E from $M(\mathbb{C})$ to $T(E)$ is a holomorphic split submersion. (The result was also proved in [3].)

2B. *Changing the basepoint.* Let w be a normalized quasiconformal self-mapping of $\hat{\mathbb{C}}$, and let $\hat{E} = w(E)$. By definition, the *allowable map* g from $T(\hat{E})$ to $T(E)$ maps the $\hat{E}$-equivalence class of f (written as $[f]_{\hat{E}}$) to the E-equivalence class of $f \circ w$ (written as $[f \circ w]_E$) for every normalized quasiconformal self-mapping f of $\hat{\mathbb{C}}$.

Proposition 16. *The allowable map $g : T(\hat{E}) \to T(E)$ is biholomorphic. If μ is the Beltrami coefficient of w, then g maps the basepoint of $T(\hat{E})$ to the point $P_E(\mu)$ in $T(E)$.*

See Proposition 7.20 in [3] or Proposition 6.7 in [9].

2C. *Universal holomorphic motion of E.* The *universal holomorphic motion* Ψ_E of E over $T(E)$ is defined as

$$\Psi_E(P_E(\mu), z) = w^\mu(z) \quad \text{for } \mu \in M(\mathbb{C}) \text{ and } z \in E.$$

The definition of P_E in Section 2A guarantees that Ψ_E is well defined. It is a holomorphic motion since P_E is a holomorphic split submersion and $\mu \mapsto w^\mu(z)$ is a holomorphic map from $M(\mathbb{C})$ to $\hat{\mathbb{C}}$ for every fixed z in $\hat{\mathbb{C}}$ (by Theorem 11 in [1]). This holomorphic section is "universal" in the following sense.

Theorem 17. *Let $\phi : \widetilde{W} \times E \to \hat{\mathbb{C}}$ be a holomorphic motion where $\widetilde{W}$ is a simply connected complex Banach manifold with a basepoint x_0, there exists a unique basepoint preserving holomorphic map $f : \widetilde{W} \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

For a proof, see Section 14 in [9].

By Proposition 10, every holomorphic motion is also a tame quasiconformal motion. Hence, $\Psi_E : T(E) \times E \to \hat{\mathbb{C}}$ is also a tame quasiconformal motion. In [7], it was proved that this is the universal tame quasiconformal motion of the closed set E over a simply connected Hausdorff space. Here is the precise statement:

Theorem 18. *Let $\phi : \widetilde{X} \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion where $\widetilde{X}$ is a simply connected Hausdorff space with a basepoint x_0. There exists a unique basepoint preserving continuous map $f : \widetilde{X} \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

See Theorem II in [7].

2D. *Douady–Earle section.* Below we present some important facts.

Proposition 19. *There is a continuous basepoint preserving map s from $T(E)$ to $M(\mathbb{C})$ such that $P_E \circ s$ is the identity map on $T(E)$.*

See [3] or [5] for a proof. It immediately implies that:

Corollary 20. *The Teichmüller space $T(E)$ is contractible.*

Definition 21. The map s from $T(E)$ to $M(\mathbb{C})$ is called the *Douady–Earle section of P_E* for the Teichmüller space $T(E)$.

2E. *Monodromy associated with a tame quasiconformal motion.* We now discuss the concept of *monodromy* of a tame quasiconformal motion. Let $\phi : X \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion, where X is a connected Hausdorff space with a basepoint x_0. Let $\pi : \widetilde{X} \to X$ be a universal covering, with the group of deck transformations Γ. We choose a point $\tilde{x}_0$ in $\widetilde{X}$ such that $\pi(\tilde{x}_0) = x_0$. Let $\pi_1(X, x_0)$ denote the fundamental group of X with basepoint x_0.

Let $\Phi = \pi^*(\phi)$. Then, $\Phi : \widetilde{X} \times E \to \hat{\mathbb{C}}$ is a tame quasiconformal motion of E over $\widetilde{X}$ with $\tilde{x}_0$ as the basepoint. By Theorem 18, there exists a unique basepoint preserving continuous map $f : \widetilde{X} \to T(E)$ such that $f^*(\Psi_E) = \phi$. Then by Proposition 19, there is a continuous basepoint preserving map $\tilde{f} = s \circ f$ from $\widetilde{X} \to M(\mathbb{C})$ such that

$$\Phi(x, z) = w^{\tilde{f}(x)}(z) \quad \text{for each } x \in \widetilde{X} \text{ and each } z \in E.$$

For each $z \in E$ and each $\gamma \in \Gamma$, we have

$$w^{\tilde{f} \circ \gamma(\tilde{x}_0)}(z) = \Phi(\gamma(\tilde{x}_0), z) = \phi(\pi \circ \gamma(\tilde{x}_0), z) = \phi(x_0, z) = z.$$

Therefore, $w^{\tilde{f}\circ\gamma(\tilde{x}_0)}$ keeps every point of E fixed. Since s may not be unique, $\tilde{f}$ is not necessarily unique. So we need the next lemma.

Lemma 22. *The homotopy class of $w^{\tilde{f}\circ\gamma(\tilde{x}_0)}$ relative to E does not depend on the choice of the continuous map $\tilde{f}$.*

Proof. Let $\tilde{f}_1, \tilde{f}_2 : \widetilde{X} \to M(\mathbb{C})$ be basepoint preserving continuous maps which are obtained from the given tame quasiconformal motion $\phi : X \times E \to \hat{\mathbb{C}}$. For each $\gamma \in \Gamma$, take a path $c_\gamma : [0, 1] \to \widetilde{X}$ which connects $\tilde{x}_0$ and $\gamma(\tilde{x}_0)$ and write

$$H(z, t) := w^{\tilde{f}_1\circ\gamma(\tilde{x}_0)} \circ \{w^{\tilde{f}_1\circ c_\gamma(t)}\}^{-1} \circ w^{\tilde{f}_2\circ c_\gamma(t)}(z)$$

for $(z, t) \in \hat{\mathbb{C}} \times [0, 1]$. Then, we see that $H(\,\cdot\,,\cdot\,)$ gives a homotopy from $w^{\tilde{f}_1\circ\gamma(\tilde{x}_0)}$ to $w^{\tilde{f}_2\circ\gamma(\tilde{x}_0)}$ relative to E. Hence, we conclude that $w^{\tilde{f}_1\circ\gamma(\tilde{x}_0)}$ and $w^{\tilde{f}_2\circ\gamma(\tilde{x}_0)}$ belong to the same homotopy class relative to E, as claimed. □

We now assume that E' is a finite set containing n points where $n \geq 4$; as usual, 0, 1, and ∞ are in E'. Let $\phi : X \times E' \to \hat{\mathbb{C}}$ be a tame quasiconformal motion. The map $w^{\tilde{f}\circ\gamma(\tilde{x}_0)}$ is quasiconformal self-map of the hyperbolic Riemann surface $X'_E := \hat{\mathbb{C}} \setminus E'$. Therefore, it represents a mapping class of X'_E, and by Lemma 22, we have a homomorphism $\rho_\phi : \pi_1(X, x_0) \to \mathrm{Mod}(0, n)$ given by

$$\rho_\phi(c) = [w^{\tilde{f}\circ\gamma(\tilde{x}_0)}],$$

where $\mathrm{Mod}(0, n)$ is the mapping class group of the n-times punctured sphere, $\gamma \in \Gamma$ is the element corresponding to $c \in \pi_1(X, x_0)$, and $[w]$ denotes the mapping class group of X'_E for w.

Definition 23. Suppose $\phi : X \times E \to \hat{\mathbb{C}}$ is a tame quasiconformal motion where X is a connected Hausdorff space. We say ϕ has trivial monodromy if for every finite subset $\{0, 1, \infty\} \subset E' \subseteq E$, the homomorphism ρ_ϕ for the tame quasiconformal motion $\phi : X \times E' \to \hat{\mathbb{C}}$ is *trivial*, that is, it maps every element of $\pi_1(X, x_0)$ to the identity of $\mathrm{Mod}(0, n)$.

We now state the two main theorems of this paper.

Theorem A. *Let $\phi : V \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion where V is a path-connected Hausdorff space. Then the following are equivalent.*

(i) *There exists a continuous motion $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends ϕ.*

(ii) *There exists a quasiconformal motion $\hat{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\hat{\phi}$ extends ϕ.*

(iii) *There exists a unique basepoint preserving continuous map $F : V \to T(E)$ such that $F^*(\Psi_E) = \phi$.*

(iv) *The monodromy of ϕ is trivial.*

Remark. For (ii) ⇔ (iii), X does not have to be path-connected; a connected Hausdorff space with basepoint is sufficient.

Theorem B. *Let G be a subgroup of* $\mathrm{PSL}(2, \mathbb{C})$, *and suppose E is a closed set in $\hat{\mathbb{C}}$ which is invariant under G. Let $\phi : X \times E \to \hat{\mathbb{C}}$ be a G-equivariant tame quasiconformal motion where X is a connected Hausdorff space. Then*

(i) $\{\theta_x\}$ *is a continuous family of isomorphisms of G, and*

(ii) θ_x *is a quasiconformal deformation of G for every x in X.*

3. Three lemmas

In what follows, V is a path-connected Hausdorff space with a basepoint x_0. Let $\mathcal{H}(\hat{\mathbb{C}})$ be the group of homeomorphisms of $\hat{\mathbb{C}}$ onto itself, with the topology of uniform convergence in the spherical metric.

Lemma 24. *Let $h : V \to \mathcal{H}(\hat{\mathbb{C}})$ be a continuous map such that $h(x)(z) = z$ for all x in V and for all z in E. If $h(x_0)$ is isotopic to the identity* rel E *for some fixed x_0 in V, then $h(x)$ is isotopic to the identity* rel E *for all x in V.*

See Lemma 12.1 in [9].

Lemma 25. *Let $s : T(E) \to M(\mathbb{C})$ be the Douady–Earle section, and let $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be any homeomorphism. There is at most one point t in $T(E)$ such that ψ is isotopic to $w^{s(t)}$* rel E.

Proof. If $w^{s(t)}$ and $w^{s(t')}$ are both isotopic to ψ rel E, then they are E-equivalent, and hence $t = P_E(s(t)) = P_E(s(t')) = t'$. □

Lemma 26. *If the continuous maps f and g from V into $T(E)$ satisfy*

(1) $\Psi_E(f(x), z) = \Psi_E(g(x), z)$ *for all x in V, and for all z in E, and*

(2) $f(p) = g(p)$ *for some p in V,*

then $f(x) = g(x)$ for all x in V.

See Lemma 12.2 in [9].

4. Proof of Theorem A

We first prove the following theorem. The proof is similar to that given in [10] (which was for holomorphic motions). We include the details for the reader's convenience, and also to make our paper self-contained.

Theorem 27. *Let V be a path-connected Hausdorff space with a basepoint x_0, and let $\phi : V \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion. If there exists a continuous motion $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends ϕ, then there exists a unique basepoint preserving continuous map $F : V \to T(E)$ such that $F^*(\Psi_E) = \phi$.*

Proof. Let S be the set of points x in V with the following property: there exists a neighborhood N of x and a continuous map $h : N \to T(E)$ such that $w^{s(h(x'))}$ is isotopic to $\tilde{\phi}_{x'}$ rel E for all x' in N. We claim that $S = V$.

It is clear that S is an open set. We first show that S is nonempty; in fact, $x_0 \in S$. Choose a simply connected neighborhood N of x_0 in V, and give N the basepoint x_0. By Theorem 18, there exists a basepoint preserving continuous map $h : N \to T(E)$ such that $h^*(\Psi_E) = \phi$ on $N \times E$. Define

$$H(x) = (w^{s(h(x))})^{-1} \circ \tilde{\phi}_x \quad \text{for each } x \text{ in } N.$$

Clearly, $H(x_0)$ is the identity. Also, for all x in N, and for all z in E, we have

$$\tilde{\phi}_x(z) = \tilde{\phi}(x, z) = \phi(x, z) = \Psi_E(h(x), z) = w^{s(h(x))}(z).$$

Hence, for all z in E, $H(x)(z) = z$. Since $H(x)$ is continuous in x, it follows from Lemma 24 that $H(x)$ is isotopic to the identity rel E. Hence, for each x in N, $w^{s(h(x))}$ is isotopic to $\tilde{\phi}_x$ rel E. This shows that x_0 belongs to S.

Now we shall prove that S is closed. Let y be a limit point of S; choose a simply connected neighborhood B of y. Since y is a limit point of S, B contains a point p in S. Choose p to be the basepoint of B. Let

$$\hat{E} = \phi_p(E) = \{\phi(p, z) : z \in E\}$$

and define $\hat{\phi} : B \times \hat{E} \to \hat{\mathbb{C}}$ as

$$\hat{\phi}(x, \phi_p(z)) = \phi(x, z), \quad (x, z) \in B \times E.$$

It is easy to see that $\hat{\phi} : B \times \hat{E} \to \hat{\mathbb{C}}$ is a tame quasiconformal motion of $\hat{E}$ over B with basepoint p. By Theorem 18, there exists a basepoint preserving continuous map $f : B \to T(\hat{E})$ such that $f^*(\Psi_{\hat{E}}) = \hat{\phi}$ on $B \times \hat{E}$ (where $\Psi_{\hat{E}} : T(\hat{E}) \times \hat{E} \to \hat{\mathbb{C}}$ is the universal tame quasiconformal motion of $\hat{E}$).

This means

$$\Psi_{\hat{E}}(f(x), \phi_p(z)) = \hat{\phi}(x, \phi_p(z)) \tag{4-1}$$

for all x in B and for all z in E.

Since $p \in S$, there is a point t in $T(E)$ such that $\tilde{\phi}_p$ is isotopic to $w^{s(t)}$ rel E. Thus, $w^{s(t)}$ maps E onto $\hat{E}$; so it induces a biholomorphic map $g : T(\hat{E}) \to T(E)$ as in Section 2B. Define $\hat{h} : B \to T(E)$ by $\hat{h} = g \circ f$. We will show that $w^{s(\hat{h}(x))}$ is isotopic to $\tilde{\phi}_x$ rel E for all x in B.

Note that f maps p to the basepoint of $T(\hat{E})$ and by Proposition 16, g maps $f(p)$ to the point $P_E(s(t))$ in $T(E)$. So, $\hat{h}(p) = P_E(s(t))$ and since $\hat{h}(p) = P_E\big(s(\hat{h}(p))\big)$, we have $P_E(s(t)) = P_E\big(s(\hat{h}(p))\big)$. That means, $w^{s(t)}$ is isotopic to $w^{s(\hat{h}(p))}$ rel E;

so $\tilde{\phi}_p$ is isotopic to $w^{s(\hat{h}(p))}$ rel E. Let

$$\hat{H}(x) = (w^{s(\hat{h}(x))})^{-1} \circ \tilde{\phi}_x \quad \text{for all } x \text{ in } B. \tag{4-2}$$

By the above discussion, $\hat{H}(p)$ is isotopic to the identity rel E.

We have the standard projection map

$$P_{\hat{E}} : M(\mathbb{C}) \to T(\hat{E}),$$

and $\hat{s} : T(\hat{E}) \to M(\mathbb{C})$ is a continuous basepoint preserving map such that $P_{\hat{E}} \circ \hat{s}$ is the identity map on $T(\hat{E})$. Since $\tilde{\phi}_p$ is isotopic to $w^{s(t)}$ rel E, and $\tilde{\phi}_p(z) = \phi_p(z)$ for all z in E, it follows that

$$\phi_p(z) = w^{s(t)}(z) \tag{4-3}$$

for all z in E. Furthermore, for all $x \in B$, and $z \in E$, we have

$$\tilde{\phi}_x(z) = \phi_x(z) = \hat{\phi}_x(\phi_p(z)) = \Psi_{\hat{E}}(f(x), \phi_p(z))$$

by (4-1). Also,

$$\Psi_{\hat{E}}(f(x), \phi_p(z)) = w^{\hat{s}(f(x))}(\phi_p(z)) = w^{\hat{s}(f(x))}(w^{s(t)}(z))$$

by (4-3). We conclude that

$$\tilde{\phi}_x(z) = w^{\hat{s}(f(x))}(w^{s(t)}(z)) \tag{4-4}$$

for all x in B, and for all z in E.

For all x in B, we have $\hat{h}(x) = g(f(x))$. Also, $f(x) = P_{\hat{E}}\big(\hat{s}(f(x))\big) = [w^{\hat{s}(f(x))}]_{\hat{E}}$ and by Section 2B, we have

$$g : [w^{\hat{s}(f(x))}]_{\hat{E}} \mapsto [w^{\hat{s}(f(x))} \circ w^{s(t)}]_E.$$

Therefore,

$$\hat{h}(x) = [w^{\hat{s}(f(x))} \circ w^{s(t)}]_E.$$

We also have $\hat{h}(x) = P_E\big(s(\hat{h}(x))\big) = [w^{s(\hat{h}(x))}]_E$ for all x in B. Hence, for all x in B, and for all z in E, we have

$$w^{\hat{s}(f(x))}(w^{s(t)}(z)) = w^{s(\hat{h}(x))}(z). \tag{4-5}$$

Therefore, by (4-4) and (4-5), we get $\tilde{\phi}_x(z) = w^{s(\hat{h}(x))}(z)$ for all x in B and for all z in E. Hence, by (4-2), $\hat{H}(x)(z) = z$ for all x in B, and for all z in E. Since $\hat{H}$ is continuous in x, it follows from Lemma 24 that $\hat{H}(x)$ is isotopic to the identity rel E for all x in B. Therefore, $w^{s(\hat{h}(x))}$ is isotopic to $\tilde{\phi}_x$ rel E for all x in B. Hence B is contained in S. In particular, $y \in S$, so S is closed. As S is also open and nonempty, $S = V$.

We now define a continuous map $F : V \to T(E)$ as follows: Given any x in V, choose a neighborhood N of x and a continuous map $h : N \to T(E)$ such that

$w^{s(h(x'))}$ is isotopic to $\tilde{\phi}_{x'}$ rel E for all x' in N. Set $F = h$ in N. By Lemma 25, F is well defined on all of V. It is obviously continuous, and $w^{s(F(x))}$ is isotopic to $\hat{\phi}_x$ rel E for all x in V.

Finally, for all x in V, and for all z in E, we have

$$F^*(\Psi_E)(x,z) = \Psi_E(F(x),z) = \Psi_E\big(P_E\big(s(F(x))\big),z\big) = w^{s(F(x))}(z)$$

and $\phi(x,z) = \tilde{\phi}(x,z) = \tilde{\phi}_x(z) = w^{s(F(x))}(z)$ (since $w^{s(F(x))}$ is isotopic to $\tilde{\phi}_x$ rel E for all x in V). Therefore, $F^*(\Psi_E)(x,z) = \phi(x,z)$ for all x in V and for all z in E.

The uniqueness of F follows from Lemma 26. This completes the proof. □

Proof of Theorem A. Theorem 27 proved the direction (i) ⇒ (iii).

For (iii) ⇒ (ii), define $\widetilde{F} : V \to M(\mathbb{C})$ by $\widetilde{F} = s \circ F$. Then, $\widetilde{F} : V \to M(\mathbb{C})$ is a basepoint preserving continuous map. Define $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$\tilde{\phi}(x,z) = w^{\widetilde{F}(x)}(z) \quad \text{for all } x \text{ in } V \text{ and for all } z \text{ in } \hat{\mathbb{C}}.$$

By Proposition 4, $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal motion, and for all z in E,

$$\begin{aligned}\phi(x,z) = F^*(\Psi_E)(x,z) = \Psi_E(F(x),z) &= \Psi_E\big(P_E\big(s(F(x))\big),z\big) \\ &= w^{s(F(x))}(z) = w^{\widetilde{F}(x)}(z) = \tilde{\phi}(x,z).\end{aligned}$$

Hence $\tilde{\phi}$ extends ϕ.

The direction (ii) ⇒ (i) is obvious by Proposition 5.

Finally, we prove (i) ⇔ (iv).

Let $\pi : \widetilde{V} \to V$ be a universal covering with the group Γ of deck transformations, so that $V = \widetilde{V}/\Gamma$ and $\pi(\tilde{x}_0) = x_0$.

Suppose ϕ can be extended to a continuous motion $\tilde{\phi}$ of $\hat{\mathbb{C}}$ over V. Then, by Theorem 27, there exists a continuous map $f : V \to M(\mathbb{C})$ such that

$$\tilde{\phi}(x,z) = w^{f(x)}(z) \quad \text{for all } (x,z) \in V \times \hat{\mathbb{C}}.$$

Let $\tilde{f} = f \circ \pi$. Then, for any $c \in \pi_1(X, x_0)$ with corresponding $\gamma \in \Gamma$, we have

$$\rho_\phi(c) = [w^{\tilde{f}\circ\gamma(\tilde{x}_0)}] = [w^{f\circ\pi\circ\gamma(\tilde{x}_0)}] = [w^{f(x_0)}] = [Id].$$

This shows that the monodromy ρ_ϕ is trivial.

Let $\phi : V \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion with trivial monodromy. Let $\Phi = \pi^*(\phi)$ be the tame quasiconformal motion of E over $\widetilde{V}$. By Theorem 18, there exists a unique basepoint preserving continuous map $\tilde{f} : \widetilde{V} \to T(E)$ such that $\tilde{f}^*(\Psi_E) = \Phi$. For any element $\gamma \in \Gamma$, we also have $\tilde{f} \circ \gamma : \widetilde{V} \to T(E)$. Note that

$$\begin{aligned}\Psi_E(\tilde{f}\circ\gamma(x),z) &= (\tilde{f}\circ\gamma)^*(\Psi_E)(x,z) \\ &= \Phi(\gamma(x),z) \\ &= \phi(\pi\circ\gamma(x),z) \\ &= \phi(\pi(x),z) = \Phi(x,z) = (\tilde{f})^*(\Psi_E)(x,z) = \Psi_E(\tilde{f}(x),z).\end{aligned}$$

By the triviality of the monodromy, we have $\tilde{f} \circ \gamma(x_0) = \tilde{f}(x_0) = [Id]$ for all $\gamma \in \Gamma$. Lemma 26 implies that $\tilde{f} \circ \gamma = \tilde{f}$ for all $\gamma \in \Gamma$. Thus, $\tilde{f}$ defines a unique basepoint preserving continuous map $f : V \to T(E)$ such that $\phi = f^*(\Psi_E)$. Thus there exists a continuous motion of $\hat{\mathbb{C}}$ over V that extends ϕ. □

5. Proof of Theorem B

Part (i). The proof is similar to the one given in [4]. We include the arguments for reader's convenience. Since E has at least three points, for each x in X and g in G, the Möbius transformation $\theta_x(g)$ is completely determined by (2-1). It easily follows that θ_x is a homomorphism for each x in X. Also, for x in X, θ_x is injective. For $\theta_x(g_1) = \theta_x(g_2)$, we have $\phi_x(g_1(z)) = \phi_x(g_2(z))$ which implies $g_1(z) = g_2(z)$ for all z in E (by injectivity). We conclude that $g_1 = g_2$. Hence, for each x in X, the map θ_x is an isomorphism.

Choose three distinct points z_1, z_2, z_3 in E. For x in X, let h_x be the unique Möbius transformation such that

$$h_x(z_i) = \phi_x(z_i) \qquad \text{for all } i = 1, 2, 3,$$
$$\theta_x(g)(h_x(z_i)) = \phi_x(g(z_i)) \quad \text{for all } i = 1, 2, 3.$$

By Lemma 15, for each i, the right-hand sides of the above equations depend continuously on x. Therefore, $x \mapsto h_x$ and $x \mapsto \theta_x(g) \circ h_x$ are continuous maps. Hence so is $x \mapsto \theta_x(g)$ for each g in G.

Part (ii). Let Ω be the set of all x in X with the following property: for each x in Ω, there exists a neighborhood $N(x)$ such that θ_t is a quasiconformal deformation of G for every t in $N(x)$.

Clearly, Ω is open. Also, Ω is nonempty, for the basepoint x_0 is in Ω. To see this, choose a simply connected neighborhood V of x_0 and use Theorem 14.

We will show that Ω is closed. Let k be a limit point of Ω. Choose a simply connected neighborhood B of k. Then, B contains a point p in Ω. So, θ_p is a quasiconformal deformation of G. Choose p to be the basepoint of B. Let $\theta_p(G) = \hat{G}$ and $\phi_p(E) = \hat{E}$.

Define $\hat{\phi} : B \times \hat{E} \to \hat{\mathbb{C}}$ as

$$\hat{\phi}_x(\phi_p(z)) = \phi_x(z) \quad \text{for } x \in B \text{ and } z \in E.$$

Since $\phi : X \times E \to \hat{\mathbb{C}}$ is a tame quasiconformal motion, for p in B, there exists a neighborhood $N(p)$ and a continuous map $f_p : N(p) \to M(\mathbb{C})$ such that $\phi_x(z) = w^{f_p(x)}(z)$ for x in $N(p)$ and z in E (see Lemma 8). Set $w = w^{f_p(p)}$. Then, $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map and $\phi_p(z) = w(z)$ for all z in E.

Now, assume $t \in B$. There exists a neighborhood $N(t)$ and a continuous map $f_t : N(t) \to M(\mathbb{C})$ such that $\phi_x(z) = w^{f_t(x)}(z)$ for x in $N(t)$ and z in E. This means there exists a quasiconformal motion $w^{f_t} : N(x) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ over $N(x)$ such that

$$\hat{\phi}_x(\phi_p(z)) = \phi_x(z) = w^{f_t(x)}(z)$$

for all x in $N(t)$ and z in E.

Let $\phi_p(z) = \hat{z} \in \hat{E}$. Then, we have

$$\hat{\phi}_x(\hat{z}) = w^{f_t(x)}(w^{-1}(\hat{z})) \quad \text{for all } x \in N(t).$$

It follows that $\hat{\phi} : B \times \hat{E} \to \hat{\mathbb{C}}$ is a tame quasiconformal motion with basepoint p.

Next, note that $\hat{E}$ is $\hat{G}$-invariant. In fact, for $\hat{g}$ in $\hat{G}$, we have

$$\hat{g}(\hat{E}) = \theta_p(g)(\phi_p(E)) = \phi_p(g(E)) = \phi_p(E) = \hat{E}.$$

Recall that $\phi_p(z) = \hat{z}$ and $\theta_p(g) = \hat{g}$. By (2-1), we have

$$\begin{aligned}\hat{\phi}_x\big(\theta_p(g)(\phi_p(z))\big) = \hat{\phi}_x\big(\phi_p(g(z))\big) = \phi_x(g(z)) &= \theta_x(g)(\phi_x(z)) \\ &= \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)(\phi_x(z)) \\ &= \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)\big(\hat{\phi}_x(\phi_p(z))\big) \\ &= \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)(\hat{\phi}_x(\hat{z})).\end{aligned}$$

It follows that $\hat{\phi} : B \times \hat{E} \to \hat{\mathbb{C}}$ is a tame quasiconformal motion with the property

$$\hat{\phi}_x(\hat{g}(\hat{z})) = \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)(\hat{\phi}(x, \hat{z})) \quad \text{for all } x \text{ in } B \text{ and } \hat{z} \text{ in } \hat{E}.$$

Therefore, by Theorem 14, there exists a quasiconformal motion $\tilde{\hat{\phi}} : B \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\hat{\phi}}$ extends $\hat{\phi}$, and for each x in B, $\tilde{\hat{\phi}}_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map. We also have

$$\tilde{\hat{\phi}}(\hat{g}(z)) = \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)(\tilde{\hat{\phi}}(z)) \quad \text{for all } x \text{ in } B \text{ and for all } z \text{ in } \hat{\mathbb{C}}.$$

This implies that

$$\tilde{\hat{\phi}} \circ \hat{g} \circ \tilde{\hat{\phi}}^{-1} = \theta_x \circ \theta_p^{-1}(\hat{g}).$$

Using $\theta_p(g) = \hat{g}$, it follows that

$$\tilde{\hat{\phi}} \circ \theta_p(g) \circ \tilde{\hat{\phi}}^{-1} = \theta_x(g).$$

Recall that θ_p is a quasiconformal deformation of G. Hence, there exists a quasiconformal map $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $w \circ g \circ w^{-1} = \theta_p(g)$. By the above equation, we get

$$\tilde{\hat{\phi}} \circ w \circ g \circ w^{-1} \circ \tilde{\hat{\phi}}^{-1} = \theta_x(g) \quad \text{for } x \in B.$$

Let $\tilde{f}_x = \tilde{\hat{\phi}} \circ w$; so, for each x in B, $\tilde{f}_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map and

$$\tilde{f}_x \circ g \circ \tilde{f}_x^{-1} = \theta_x(g) \quad \text{for each } x \in B.$$

Hence, k is in Ω, and therefore, Ω is closed. Since X is connected, it follows that $\Omega = X$. $\square$

Below is a direct and short proof of part (ii) of Theorem B. We thank one of the referees for bringing this to our attention.

An alternative proof of Theorem B(ii). Let $\tilde{\phi} : \widetilde{X} \times E \to \hat{\mathbb{C}}$ be the lift of $\phi : X \times E \to \hat{\mathbb{C}}$ to the universal covering $\widetilde{X}$ of X. Namely, $\tilde{\phi}$ is defined as

$$\tilde{\phi}(\tilde{x}, z) = \phi(\pi(\tilde{x}), z) \quad (\tilde{x}, z) \in \widetilde{X} \times E,$$

where $\pi : \widetilde{X} \to X$ is the canonical projection. It is a tame quasiconformal motion of E over $\widetilde{X}$ because tame quasiconformal motion is a local property. Moreover, we have

$$\tilde{\phi}(\tilde{x}, g(z)) = \phi(\pi(\tilde{x}), g(z)) = \theta_{\pi(\tilde{x})}(g)(\phi(\pi(\tilde{x}), z)) = \theta_{\pi(\tilde{x})}(g)(\tilde{\phi}(\tilde{x}, z)) \quad \text{for } g \in G.$$

Hence, it is G-equivariant with isomorphisms $\theta_{\pi(\tilde{x})} : G \to \mathrm{PSL}(2, \mathbb{C})$ where $\tilde{x} \in \widetilde{X}$. Since $\widetilde{X}$ is simply connected, it follows from Theorem 14 that $\theta_{\pi(\tilde{x})}(G)$ is a quasiconformal deformation of G and so is $\theta_x(G)$. $\square$

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A CHARACTERIZATION AND SOLVABILITY OF QUASIHOMOGENEOUS SINGULARITIES

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Let $(V, 0)$ be an isolated hypersurface singularity defined by the holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. A local k-th $(0 \le k \le n+1)$ Hessian algebra $H_k(V)$ of isolated hypersurface singularity $(V, 0)$ is a finite-dimensional $\mathbb{C}$-algebra and it depends only on the isomorphism class of the germ $(V, 0)$. It is a natural question to ask for a necessary and sufficient condition for a complex analytic isolated hypersurface singularity to be quasihomogeneous in terms of its local k-th Hessian algebra $H_k(f)$. Xu and Yau proved that $(V, 0)$ admits a quasihomogeneous structure if and only if $H_0(f)$ is isomorphic to a finite-dimensional nonnegatively graded algebra in the early 1980s. In this paper, on the one hand, we generalize Xu and Yau's result to $H_{n+1}(f)$. On the other hand, a new series of finite-dimensional Lie algebras $L_k(V)$ (resp. $L^k(V)$) was defined to be the Lie algebra of derivations of the k-th $(0 \le k \le n+1)$ Hessian algebra $H_k(V)$ (resp. $A^k(V) := \mathcal{O}_{n+1}/(f, m^k J_f)$) and is finite-dimensional. We prove that $(V, 0)$ is quasihomogeneous singularity if $L_{n+1}(V)$ (resp. $L^k(V) := \mathrm{Der}(A^k(V))$) satisfies certain conditions. Moreover, we investigate whether the Lie algebras $L_k(V)$ (resp. $L^k(V)$) are solvable.

1. Introduction

A polynomial $f(z_0, \dots, z_n)$ is weighted homogeneous of type $(q_0, \dots, q_n; d)$, where $q_0, \dots, q_n$ and d are fixed positive integers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $q_0 i_0 + q_1 i_1 + \cdots + q_n i_n = d$. In this case, we say that z_i has weight q_i and f has weight d. Recall that an isolated hypersurface singularity $(V, 0) = \{(z_0, \dots, z_n) : f(z_0, \dots, z_n) = 0 \subset \mathbb{C}^{n+1}\}$ is quasihomogeneous if f is in the Jacobian ideal J_f, i.e., $f \in J_f = \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)$. By a theorem of Saito [1971], if f is quasihomogeneous with isolated singularity at 0, then after a biholomorphic change of coordinates, f becomes a weighted homogeneous polynomial.

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Let $(V, 0)$ be an isolated hypersurface singularity defined by the holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Let $\mathcal{O}_{n+1}$ denote the $\mathbb{C}$-algebra of germs of analytic functions defined at the origin of $\mathbb{C}^{n+1}$. Recall that the moduli algebra is $A(V) := \mathcal{O}_{n+1}/\big(f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\big)$. Mather and Yau [1982] proved that two germs of complex analytic hypersurfaces of the same dimension with isolated singularities are contact equivalent if and only if their moduli algebras are isomorphic. Therefore the moduli algebra $A(V)$ is important in the study of the complex structures of $(V, 0)$. In 1983,Yau introduced the Yau algebra $L(V)$ which was defined as the Lie algebra of derivations of the moduli algebra $A(V)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$ [Seeley and Yau 1990]. It plays an important role in singularity theory [Chen 1995]. In a beautiful paper, Elashvili and Khimshiashvili [2006] first used it to characterize ADE singularities. It is known that $L(V)$ is a finite-dimensional Lie algebra and its dimension $\lambda(V)$ is called Yau number [Khimshiashvili 2006; Yu 1996]. Yau, Zuo and their collaborators have been systematically studying various Lie algebras of isolated singularities [Benson and Yau 1990; Chen et al. 1995; 2019; 2020a; 2020b; Hussain et al. 2018; 2020; 2021b; Yau and Zuo 2016a; 2016b]. In this article, we study two kinds of new derivation Lie algebra arising from the isolated hypersurface singularity $(V, 0)$ as follows.

Hussain, Yau and Zuo [Hussain et al. 2020; 2021b], introduced the new series of k-th Yau algebras $L^k(V)$ which was defined to be the Lie algebra of derivations of the moduli algebra $A^k(V) = \mathcal{O}_{n+1}/(f, m^k J_f)$, $k \ge 0$, where m is the maximal ideal of $\mathcal{O}_n$, i.e., $L^k(V) := \mathrm{Der}(A^k(V), A^k(V))$. Its dimension was denoted as $\lambda^k(V)$. This series of integers $\lambda^k(V)$ are new numerical analytic invariants of singularities. It is natural to call it the k-th Yau number. In particular, when $k = 0$, these are exactly the previous Yau algebra and Yau number, i.e., $L(V) = L^0(V)$, $\lambda^0(V) = \lambda(V)$.

Let $\mathrm{Hess}(f)$ be the Hessian matrix (f_{ij}) of the second order partial derivatives of f, and $h(f)$ (the Hessian of f) be the determinant of $\mathrm{Hess}(f)$. More generally, for each k satisfying $0 \le k \le n+1$ we denote by $h_k(f)$ the ideal in $\mathcal{O}_{n+1}$ generated by all $k \times k$-minors in the matrix $\mathrm{Hess}(f)$. In particular, the ideal $h_{n+1}(f) = (h(f))$ is a principal ideal. For each k as above, consider the graded k-th Hessian algebra of the polynomial f defined by

$$H_k(f) = \mathcal{O}_{n+1}/((f) + J_f + h_k(f)).$$

In particular, $H_0(f)$ is exactly the well-known moduli algebra $A(V)$. It is easy to check that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is a contact invariant of f, i.e., $H_k(f)$ depends only on the isomorphism class of the germ $(V, 0)$ [Dimca and Sticlaru 2015].

Hussain, Yau and Zuo [Hussain et al. 2021a] defined a series of new derivation Lie algebras

$$L_k(V) := \mathrm{Der}(H_k(f), H_k(f)), \quad 0 \le k \le n+1.$$

Since $H_0(f) = A(V)$, so $L_k(V)$ is also a generalization of Yau algebra $L(V)$ and $L_0(V) = L(V)$. $L_k(V)$ is a finite-dimensional Lie algebra and the dimension of $L_k(V)$ is denoted by $\lambda_k(V)$ which is new numerical analytic invariant of isolated hypersurface singularities. It is natural to ask how to use $H_k(f)$ (resp. $L_{n+1}(V)$) to characterize the quasihomogeneity of an isolated hypersurface singularity. In this paper, we shall answer this question partially and prove that $(V, 0)$ admits a quasihomogeneous structure if and only if $H_{n+1}(f)$ (resp. $L_{n+1}(V)$) is isomorphic to a finite-dimensional nonnegatively graded algebra (resp. nonnegatively graded Lie algebra). We propose the following two conjectures.

Conjecture 1.1. *Let* $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be an isolated hypersurface singularity. Then the following are equivalent*:

(1) $(V, 0)$ *is quasihomogeneous.*

(2) *There exists a* k, $0 \le k \le n+1$, *such that the* k*-th Hessian algebra* $H_k(f)$ *is isomorphic to a finite-dimensional graded commutative local algebra* $\bigoplus_{i\ge 0} A_i$ *with* $A_0 = \mathbb{C}$.

(3) *For all* k, $0 \le k \le n+1$, *the* k*-th Hessian algebra* $H_k(f)$ *is isomorphic to a finite-dimensional graded commutative local algebra* $\bigoplus_{i\ge 0} A_i$ *with* $A_0 = \mathbb{C}$.

Conjecture 1.2. *Let* $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be an isolated hypersurface singularity with* $n \ge 1$. *Then* $(V, 0)$ *is a quasihomogeneous singularity if there exists* k, $0 \le k \le n+1$, *such that the following conditions are satisfied*:

(1) $L_k(V)$ *(resp.* $L^k(V)$*) is isomorphic to a nonnegatively graded Lie algebra* $\bigoplus_{i=0}^{\ell}(L_k(V))_i$ *without center.*

(2) *There exists* $E \in (L_k(V))_0$ *(resp.* $(L^k(V))_0$*) such that* $[E, D_i] = i(D_i)$ *for any* $D_i \in (L_k(V))_i$.

(3) *For any element* $\alpha \in m - m^2$, *where* m *is the maximal ideal of* $H_k(V)$ *(resp.* $A^k(V)$*),* αE *is not in* $(L_k(V))_0$ *(resp.* $(L^k(V))_0$*).*

Remark 1.1. For Conjecture 1.1, the implication (3) $\Rightarrow$ (2) is obvious. Meanwhile, (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are immediate corollaries of the well-known theorem of Saito [1971]. Thus the key point to prove Conjecture 1.1 is the implication (2) $\Rightarrow$ (1) (see Theorem A). Conjectures 1.1 and 1.2 are verified in [Xu and Yau 1996] when $k = 0$. One of our main goals in this paper is to verify these two conjectures for the case of $k = n+1$. We obtain the following two main results.

Theorem A. *Let* $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be an isolated hypersurface singularity. Then* $(V, 0)$ *is quasihomogeneous if and only if its* $(n+1)$*-th Hessian algebra* $H_{n+1}(f)$ *is isomorphic to a finite-dimensional graded commutative local algebra* $\bigoplus_{i\ge 0} A_i$ *with* $A_0 = \mathbb{C}$.

Theorem B. *Let* $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be an isolated hypersurface singularity with* $n \geq 1$. *Then* $(V, 0)$ *is a quasihomogeneous singularity if the following conditions are satisfied*:

(1) $L_{n+1}(V)$ *is isomorphic to a nonnegatively graded Lie algebra* $\bigoplus_{i=0}^{k} L_i$ *without center.*

(2) *There exists* $E \in L_0$ *such that* $[E, D_i] = i D_i$ *for any* $D_i \in L_i$.

(3) *For any element* $\alpha \in m - m^2$ *where* m *is the maximal ideal of* $H_{n+1}(f)$, αE *is not in* L_0 *(For brevity, we use* L_i *to denote* $(L_{n+1}(V))_i$*).*

Remark 1.2. We can only prove Conjectures 1.1 and 1.2 for $k = n+1$. The reason is that the proof of Theorem B depends on Theorem A. In our proof of Theorem A, we use a beautiful result of Saito [1974, Corollary 3.8], which cannot be generalized to general k. As for $L^k(V)$, we can only verify the conjectures when k is sufficiently large (see Theorem C), $k = 1$ is still a open problem.

Theorem C. *Let* $(V, 0)$ *be an isolated hypersurface singularity defined by* f *with multiplicity of at least three. Then* $(V, 0)$ *is quasihomogeneous if there exists* $k_0 \in \mathbb{N}$ *such that for all* $k \geq k_0$:

(1) $L^k(V) \cong \bigoplus_{i=0}^{j} L_i$ *which is nonnegatively graded and without center.*

(2) *There exists* $E \in L_0$ *such that* $[E, D_i] = i D_i$ *for all* $D_i \in L_i$.

(3) *For any element* $\alpha \in m - m^2$ *where* m *is the maximal ideal of* $A(V)$, αE *is not in* L_0.

In [Yau 1991], the Lie algebra $L_0(V) = L^0(V)$ was shown to be solvable. Thus a necessary condition for a commutative local Artinian algebra to be a moduli algebra is that its algebra of derivations is a solvable Lie algebra. Naturally one expects that $L_k(V)$ and $L^k(V)$ are also solvable. We prove that $L^k(V)$ $(k \geq 2)$ is indeed solvable for any dimension n, and $k = 1$ is solvable for some special cases. For the sake of convenience to the readers, we abuse the notations of x and z. The subscript of x we shall use in the following theorem begins with 1 instead of 0 which is slightly different with the above two main theorems. We do this in order to be consistent with the symbols in [Yau 1983; 1986; 1991], so that the reader can easily refer to them.

Theorem D. *Let* f *be a homogeneous isolated singularity in* n *variables* $x_1, \dots, x_n$ *of degree* $d \geq 4$. *Then* $L^k(V)$ *is solvable for* $k \geq 2$ *or* $k = 1, n = 4$.

Remark 1.3. In Theorem D, the condition $d \geq 4$ cannot be omitted. In fact, there is a counterexample when $d = 3$.

Let $f = x^2y + xy^2$, then the $A^1(V)$ is $\mathcal{O}_2$ module the following relations:

$$x^2y + xy^2 = 0, \quad 2x^2y + xy^2 = 0, \quad 2xy^2 + y^3 = 0,$$
$$2xy^2 + x^2y = 0, \quad 2x^2y + x^3 = 0.$$

The monomial basis for $A^1(V)$ is

$$1, \quad x, \quad y, \quad x^2, \quad xy, \quad y^2.$$

It is easy to check that $x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y} \in L^1(V)$, Hence $L^1(V)$ is not solvable.

2. The derivation Lie algebra of a graded commutative Artinian algebra

We first state some elementary properties of the derivation Lie algebra of a graded commutative Artinian local algebra.

Lemma 2.1. *Let $A=\bigoplus_{i=0}^{t} A_i$ be a graded commutative Artinian local algebra. Then the derivation algebra of A denoted by $L(A)$ is a graded Artinian Lie algebra.* (*Here Artinian Lie algebra means $L(A)$ is finite dimension as $\mathbb{C}$-vector space.*)

Proof. See Lemma 2.1 in [Xu and Yau 1996]. □

Definition 2.1. The socle of a local Artinian algebra A with maximal ideal m is the complex vector subspace $\operatorname{Soc} A=\{a\in A : a\cdot m=0\}$ in A. The type of A is the complex dimension of $\operatorname{Soc} A$ as a vector space. The algebra A is Gorenstein when its type is one.

Lemma 2.2. *Let A be a commutative Artinian local algebra. Let $D\in L(A)$ be any derivation of A. Then D preserves the m-adic filtration of A, i.e., $D(m)\subset m$, where m is the maximal ideal of A.*

Proof. See Lemma 2.5 in [Xu and Yau 1996]. □

Proposition 2.1. *Let $A=\bigoplus_{i=0}^{k} A_i$ be a graded commutative Artinian local algebra with $A_0=\mathbb{C}$. Suppose the maximal ideal of A is generated by A_j for some $j>0$. Then $L(A)$ is a graded Lie algebra without negative weight.*

Proof. See Proposition 2.6 in [Xu and Yau 1996]. □

Lemma 2.3. *Let f be a weighted homogeneous polynomial with isolated singularity in $z_0,\dots,z_n$ variables of type $(\alpha_0,\dots,\alpha_n;d)$. Assume $\mathrm{wt}(z_0)=\alpha_0\ge \mathrm{wt}(z_1)=\alpha_1\ge\cdots\ge \mathrm{wt}(z_n)=\alpha_n$. Then f must be of either the form*

$$f=z_0^m+a_1(z_1,\dots,z_n)z_0^{m-1}+\cdots+a_{m-1}(z_1,\dots,z_n)z_0+a_m(z_1,\dots,z_n),$$

or

$$f=z_0^m z_i+a(z_1,\dots,z_n)z_0^{m-1}+\cdots+a_{m-1}(z_1,\dots,z_n)z_0+a_m(z_1,\dots,z_n).$$

Proof. See Lemma 2.1 in [Chen et al. 1995]. □

3. Proof of Theorems A and B

We first recall the following useful lemma.

Lemma 3.1 (Rossi). *Let* $(V,0)=\{(z_0,\dots,z_n): f(z_0,\dots,z_n)=0\}\subset\mathbb{C}^{n+1}$ *be an isolated hypersurface singularity. Let* $\theta=\sum_{i=0}^{n}a_i(z)\frac{\partial}{\partial z_i}$ *be a holomorphic vector field of* $(V,0)$. *Then* $a_i(0)=0$ *for* $0\le i\le n$.

Proof. See [Rossi 1963]. □

Proof of Theorem A. If $(V,0)$ is a quasihomogeneous singularity, then by the theorem of Saito, we can assume that f is a weight homogeneous polynomial after a biholomorphic change if necessary. So the moduli ideal $(f)+J_f+h_{n+1}(f)=J_f+h(f)$ is a graded ideal and $H_{n+1}(f)=\mathcal{O}_{n+1}/((f)+J_f+h_{n+1}(f))=\bigoplus_{i\ge 0}A_i$ with $A_0=\mathbb{C}$.

On the other side, we assume that $H_{n+1}(f)=\bigoplus_{i\ge 0}A_i$ with $A_0=\mathbb{C}$. Let $m=\bigoplus_{i\ge 1}A_i$ be the maximal ideal of $H_{n+1}(f)$. It is not difficult to find a $\mathbb{C}$-basis of m/m^2, denoted by $\{x_0,\dots,x_n\}$, with $x_i\in A_{q_i}$ for $0\le i\le n$. Let $E: H_{n+1}(f)\to H_{n+1}(f)$ be the linear map such that the restriction of E on A_i is just multiplication by i. Then it is easy to see E satisfies Leibniz rule on $H_{n+1}(f)$, i.e., E is a derivation of $H_{n+1}(f)$. E can be viewed as a derivation of $\mathbb{C}[x_0,\dots,x_n]$ which leaves the moduli ideal $(f)+J_f+h_{n+1}(f)$ in $\mathcal{O}_{n+1}$ invariant. E is of the form $\sum_{i=0}^{n}q_ix_i\frac{\partial}{\partial x_i}$. If we let the degree of x_i be q_i for $0\le i\le n$, then $\mathbb{C}[x_0,\dots,x_n]$ is graded and the natural map $\mathbb{C}[x_0,\dots,x_n]\to H_{n+1}(f)$ is a graded homomorphism of degree 0. Let $\bigoplus_{r>0}J_r$ be the grading of the moduli ideal $(f)+J_f+h(f)$. As E is a graded derivation of degree 0, E leaves J_r invariant for all $r>0$. Since $\ker(E|_{J_r})=0$ and $\dim_{\mathbb{C}}J_r<\infty$, we obtain that $E|_{J_r}$ is surjective for all $r>0$. Hence $E:(f)+J_f+h_{n+1}(f)\to(f)+J_f+h_{n+1}(f)$ is bijective. Let b_i, r_i and $a_{i0},a_{i1},\dots,a_{in}$ be such that

$$E\left(\frac{\partial f}{\partial x_i}\right)=b_if+\sum_{j=0}^{n}a_{ij}\frac{\partial f}{\partial x_j}+r_ih(f)$$

for all $0\le i\le n$. Let e, h and p_j be such that

$$E(h(f))=ef+\sum_{j=0}^{n}p_j\frac{\partial f}{\partial x_j}+h\cdot h(f).$$

By the surjectivity of $E:(f)+J_f+h_{n+1}(f)\to(f)+J_f+h_{n+1}(f)$, there exist c_i, s_i and $d_{i0},d_{i1},\dots,d_{in}$ such that

$$(1)\quad \frac{\partial f}{\partial x_i}=E\left(c_if+\sum_{j=0}^{n}d_{ij}\frac{\partial f}{\partial x_j}+s_ih(f)\right)$$

$$
\begin{aligned}
&= E(c_i)f + c_i\sum_{j=0}^{n} q_j x_j \frac{\partial f}{\partial x_j} + \sum_{j=0}^{n} E(d_{ij})\frac{\partial f}{\partial x_j} \\
&\quad + \sum_{j=0}^{n} d_{ij}\left(b_j f + \sum_{l=0}^{n} a_{jl}\frac{\partial f}{\partial x_l} + r_j h(f)\right) \\
&\quad + E(s_i)h(f) + s_i\left(ef + \sum_{j=0}^{n} p_j\frac{\partial f}{\partial x_j} + h\cdot h(f)\right) \\
&= \left(E(c_i) + \sum_{j=0}^{n} d_{ij}b_j + s_i e\right)f + c_i\sum_{j=0}^{n} q_j x_j\frac{\partial f}{\partial x_j} + \sum_{j=0}^{n} E(d_{ij})\frac{\partial f}{\partial x_j} \\
&\quad + \sum_{j=0}^{n} E(d_{ij})\frac{\partial f}{\partial x_j} + \sum_{j=0}^{n} d_{ij}\sum_{l=0}^{n} a_{jl}\frac{\partial f}{\partial x_l} + s_i\sum_{j=0}^{n} p_j\frac{\partial f}{\partial x_j} \\
&\quad + (E(s_i) + s_i h)h(f)
\end{aligned}
$$

$$
\begin{aligned}
\text{(2)} \qquad &= \left(E(c_i) + \sum_{j=0}^{n} d_{ij}b_j + s_i e\right)f \\
&\quad + \sum_{j=0}^{n}\left[c_i q_j x_j + E(d_{ij}) + \sum_{l=0}^{n} d_{il}a_{lj} + s_i p_j\right]\frac{\partial f}{\partial x_j} + (E(s_i) + s_i h)h(f).
\end{aligned}
$$

Now we assume that f is not quasihomogeneous. Recall the beautiful result of Saito [1974, Corollary 3.8]: Let $f \in \mathcal{O}_{n+1}$ be a germ of a holomorphic function which defines a hypersurface with an isolated singularity at 0, then f is not quasihomogeneous, precisely when

$$
h(f) = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{0\le i,j\le n} \in (f) + J_f.
$$

Without loss of generality, we assume that $r_i = 0$, $s_i = 0$ for $0 \le i \le n$ and $h = 0$. Thus

$$
\begin{aligned}
\text{(3)} \qquad \frac{\partial f}{\partial x_i} &= E\left(c_i f + \sum_{j=0}^{n} d_{ij}\frac{\partial f}{\partial x_j}\right) \\
&= \left(E(c_i) + \sum_{j=0}^{n} d_{ij}b_j\right)f + \sum_{j=0}^{n}\left[c_i q_j x_j + E(d_{ij}) + \sum_{l=0}^{n} d_{il}a_{lj}\right]\frac{\partial f}{\partial x_j}.
\end{aligned}
$$

Let

$$
\theta_i = \sum_{j=0}^{n}\left[c_i q_j x_j + E(d_{ij}) + \sum_{l=0}^{n} d_{il}a_{lj} - \delta_{ij}\right]\frac{\partial}{\partial x_j}.
$$

Then $\theta_i(f) = \big(E(c_i) + \sum_{j=0}^{n} d_{ij}b_j\big)f$. So θ_i is a holomorphic vector field of $\{f(x_0, \dots, x_n) = 0\}$. By Lemma 3.1, $\theta_{ij}(0) = 0$ for all $0 \le j \le n$, where we write $\theta_i = \sum_{j=0}^{n} \theta_{ij} \frac{\partial}{\partial x_j}$. Observe that for any $g \in \mathbb{C}[x_0, \dots, x_n]$, $E(g)$ vanishes at 0. Therefore we conclude that

$$\left(\sum_{l=0}^{n} d_{il}a_{lj} - \delta_{ij}\right)(0) = 0$$

for all $0 \le i \le n$. This means that

$$\begin{bmatrix} d_{00}(0) & d_{01}(0) & \dots & d_{0n}(0) \\ d_{10}(0) & d_{11}(0) & \dots & d_{1n}(0) \\ \dots & \dots & \dots & \dots \\ d_{n0}(0) & d_{n1}(0) & \dots & d_{nn}(0) \end{bmatrix} \cdot \begin{bmatrix} a_{00}(0) & a_{01}(0) & \dots & a_{0n}(0) \\ a_{10}(0) & a_{11}(0) & \dots & a_{1n}(0) \\ \dots & \dots & \dots & \dots \\ a_{n0}(0) & a_{n1}(0) & \dots & a_{nn}(0) \end{bmatrix} = I,$$

where I is the identity matrix. On the other hand, by the surjectivity of

$$E : (f) + J_f + h(f) \to (f) + J_f + h(f),$$

there exist c and $d_0, \dots, d_n$ such that

$$\begin{aligned} f &= E\left(cf + \sum_{i=0}^{n} d_i \frac{\partial f}{\partial x_i}\right) \\ &= E(c)f + c\sum_{j=0}^{n} q_j x_j \frac{\partial f}{\partial x_j} + \sum_{i=0}^{n} E(d_i)\frac{\partial f}{\partial x_i} + \sum_{i=0}^{n} d_i\left(b_i f + \sum_{j=0}^{n} a_{ij}\frac{\partial f}{\partial x_j}\right) \\ &= \left(E(c) + \sum_{i=0}^{n} b_i d_i\right) f + \sum_{j=0}^{n}\left(cq_j x_j + E(d_j) + \sum_{i=0}^{n} d_i a_{ij}\right)\frac{\partial f}{\partial x_j}. \end{aligned} \tag{4}$$

Let

$$H = \sum_{j=0}^{n}\left(cq_j x_j + E(d_j) + \sum_{i=0}^{n} d_i a_{ij}\right)\frac{\partial}{\partial x_j}.$$

Then $H(f) = [1 - E(c) - b_0 d_0 - b_1 d_1 - \dots - b_n d_n]f$. So H is a vector field of $\{f(x_0, \dots, x_n) = 0\}$. By Lemma 3.1, $H_i(0) = 0$ for $0 \le i \le n$, where $H = \sum_{i=0}^{n} H_i \frac{\partial}{\partial x_i}$. Since $E(d_i)$ vanishes at the origin for $i = 0, 1, \dots, n$, we conclude that

$$\left(\sum_{i=0}^{n} d_i a_{ij}\right)(0) = 0$$

for all $0 \le j \le n$, i.e.,

$$\begin{bmatrix} d_0(0) & d_1(0) & \cdots & d_n(0) \end{bmatrix} \cdot \begin{bmatrix} a_{00}(0) & a_{01}(0) & \cdots & a_{0n}(0) \\ a_{10}(0) & a_{11}(0) & \cdots & a_{1n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0}(0) & a_{n1}(0) & \cdots & a_{nn}(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since the matrix

$$\begin{bmatrix} a_{00}(0) & a_{01}(0) & \dots & a_{0n}(0) \\ a_{10}(0) & a_{11}(0) & \dots & a_{1n}(0) \\ \dots & \dots & \dots & \dots \\ a_{n0}(0) & a_{n1}(0) & \dots & a_{nn}(0) \end{bmatrix}$$

is nonsingular, we deduce that $\begin{bmatrix} d_0(0) & d_1(0) & \cdots & d_n(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$. It follows that $1-E(c)-b_0d_0-b_1d_1-\cdots-b_nd_n$ is a unit in $\mathcal{O}_{n+1} = \mathbb{C}\{x_0, \dots, x_n\}$ since $E(c)$ vanishes at the origin. Because $(1 - E(c) - b_0d_0 - b_1d_1 - \cdots - b_nd_n)f = H(f)$, we conclude that $f \in \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)\mathcal{O}_{n+1}$. By definition of quasihomogeneity, $(V, 0)$ is quasihomogeneous which is contradict to our assumption. Hence f is quasihomogeneous, i.e., $(V, 0)$ is quasihomogeneous. □

Theorem 3.1. *Let $(V, 0)$ be a hypersurface singularity defined by a weighted homogeneous polynomial $f(z_0, \dots, z_n)$ which has an isolated singularity at the origin with multiplicity at least three. Suppose that $n \ge 1$. When the multiplicity is equal to three, we also need to suppose that $n > 1$. Then the Lie algebra $L_{n+1}(V)$ is graded and without center.*

Proof. Since f is a weighted homogeneous polynomial, the moduli ideal

$$(f) + J_f + h_{n+1}(f) = J_f + h_{n+1}(f)$$

is graded and hence

$$H_{n+1}(f) := \mathbb{C}[z_0, \dots, z_n]/((f) + J_f + h_{n+1}(f))$$

is graded. By Lemma 2.1, $L_{n+1}(V)$ is graded. Let D be an element in the center of $L_{n+1}(V)$. Write $D = \sum_i D_i$ where D_i is a derivation with weight i. Let

$$E = \sum_{i=0}^{n} q_i z_i \frac{\partial}{\partial z_i}$$

be the Euler derivation where $q_i = wt(z_i)$. Then

$$0 = [E, D] = \left[E, \sum_i D_i\right] = \sum_i i D_i$$

which implies $D_i = 0$ for $i \ne 0$. Hence D is a homogeneous element of weight 0.

If we write

$$D(f_i) = \sum c_j f_j + c \cdot h(f),$$

then by comparing weight of both sides, we get $c = 0$. This shows $D(f_i) \in J_f$. From now on we consider D as a derivation on $\mathcal{O}_{n+1}/J_f$. Let $D = \sum_{i=0}^{n} b_i \frac{\partial}{\partial z_i}$. Then

$$0 = [z_i E, D] = z_i [E, D] + [z_i, D]E = -b_i E.$$

This implies that $b_i \in$ Socle of $\mathcal{O}_{n+1}/J_f$ for all $0 \le i \le n$, i.e., $z_j \cdot b_i \in J_f$ for any $0 \le j \le n$. By local duality, we know that the socle of $\mathcal{O}_{n+1}/J_f$ is the highest degree nonzero subspace of $\mathcal{O}_{n+1}/J_f$. We shall assume without loss of generality that $d \ge 2q_0 \ge 2q_1 \ge \cdots \ge 2q_n$. By Lemma 2.1 in [Chen et al. 1995], we obtain that f must satisfy one of the following two cases:

$$f = \begin{cases} z_0^m + a_1(z_1, \ldots, z_n) z_0^{m-1} + \cdots + a_m(z_1, \ldots, z_n), & \text{Case (1)}, \\ z_0^{m-1} z_i + a_1(z_1, \ldots, z_n) z_0^{m-2} + \cdots + a_m(z_1, \ldots, z_n). & \text{Case (2)}. \end{cases}$$

Hence

$$\begin{aligned} wt\ h(f) &= (d - 2q_0) + (d - 2q_1) + \cdots + (d - 2q_n) \\ &= \begin{cases} m(n+1)q_0 - 2\sum_{j=0}^{n} q_j, & \text{Case (1)}, \\ (m-1)(n+1)q_0 + (n+1)q_i - 2\sum_{j=0}^{n} q_j, & \text{Case (2)}. \end{cases} \end{aligned}$$

If the multiplicity of f is at least four, we have $wt\ h(f) > 2q_0$ and $wt\big(\frac{\partial f}{\partial z_n}\big) \ge \cdots \ge wt\big(\frac{\partial f}{\partial z_0}\big) > 2q_0$. The fact that D is a homogeneous element of weight 0 implies that $wt(b_i) = wt(z_i) = q_i$ for all $0 \le i \le n$. Hence $wt(z_j \cdot b_i) \le 2q_0$. This would lead to a contradiction unless $b_i = 0$ for all $0 \le i \le n$. Hence $D = 0$.

Now we consider the case of $\operatorname{mult}(f) = 3$.

Case (1) $\qquad f = z_0^3 + a_1(z_1, \ldots, z_n) z_0^2 + a_2(z_1, \ldots, z_n) z_0 + a_3(z_1, \ldots, z_n).$

In this case $wt\ h(f) = 3(n+1)q_0 - 2\sum_{i=0}^{n} q_i$ which implies that

$$wt\ h(f) > 3q_0 - q_n = wt\left(\frac{\partial f}{\partial z_n}\right) \ge \cdots \ge wt\left(\frac{\partial f}{\partial z_0}\right)$$

for all n. Since D is a homogeneous element of weight 0, we obtain that $D\big(\frac{\partial f}{\partial z_j}\big) \in J_f$ for all $0 \le j \le n$, i.e., D is a derivation of the algebra $\mathbb{C}[z_0, \ldots, z_n]/((f) + J_f)$. By Proposition 3.1 in [Xu and Yau 1996], we obtain that $D = 0$.

Case (2) $\qquad f = z_0^2 z_i + a_1(z_1, \ldots, z_n) z_0 + a_2(z_1, \ldots, z_n).$

In this case $wt\ h(f) = 2(n+1)q_0 + nq_i - 2\sum_{j=0}^{n} q_j$, which implies that

$$wt\ h(f) > 2q_0 - q_n = wt\left(\frac{\partial f}{\partial z_n}\right) \ge \cdots \ge wt\left(\frac{\partial f}{\partial z_0}\right)$$

when $n \geq 2$. Since D is a homogeneous element of weight 0, we obtain that $D\big(\frac{\partial f}{\partial z_j}\big) \in J_f$ for all $0 \leq j \leq n$, i.e., D is a derivation of the algebra $\mathbb{C}[z_0, \dots, z_n]/(f) + J_f$. By Proposition 3.1 in [Xu and Yau 1996] we obtain that $D = 0$. $\square$

Notice that L_0 has no center for $\mathrm{mult}(f) \geq 3$ and $n \geq 1$ [Xu and Yau 1996]. However, for L_{n+1}, some interesting new phenomena have been discovered, e.g., the following remark.

Remark 3.2. A counterexample when $\mathrm{mult}(f) = 3$ and $n = 1$ is as follows:

$$f = z_0^2 z_1 + a_1(z_1) z_0 + a_2(z_1).$$

Let $q_0 = s q_1$, then $a_1(z_1) = a z_1^{s+1}$, $a_2(z_1) = b z_1^{2s+1}$.

If $b = 0$ and $s = 1$, then $f = z_0^2 z_1 + a z_0 z_1^2$. Hence $\frac{\partial f}{\partial z_0} = 2 z_0 z_1 + a z_1^2$, $\frac{\partial f}{\partial z_1} = z_0^2 + 2 a z_0 z_1$ and $h(f) = -4(z_0^2 + a z_0 z_1 + a^2 z_1^2)$. It is obvious that D is a linear combination of $z_0 \frac{\partial}{\partial z_0}$, $z_0 \frac{\partial}{\partial z_1}$, $z_1 \frac{\partial}{\partial z_0}$ and $z_1 \frac{\partial}{\partial z_1}$.

It is easy to verify that $\big(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, h(f)\big) = (z_0^2, z_1^2, z_0 z_1)$. Hence for any derivation $D' = (a_0 z_0 + a_1 z_1) \frac{\partial}{\partial z_0} + (b_0 z_0 + b_1 z_1) \frac{\partial}{\partial z_1}$, we obtain that

$$\begin{aligned}
\Big[z_0 \frac{\partial}{\partial z_0}, D'\Big] &= b_0 z_0 \frac{\partial}{\partial z_1} - a_1 z_1 \frac{\partial}{\partial z_0};\\
\Big[z_1 \frac{\partial}{\partial z_0}, D'\Big] &= a_0 z_1 \frac{\partial}{\partial z_0} + b_0 z_1 \frac{\partial}{\partial z_1} - b_0 z_0 \frac{\partial}{\partial z_0} - b_1 z_1 \frac{\partial}{\partial z_0};\\
\Big[z_0 \frac{\partial}{\partial z_1}, D'\Big] &= a_1 z_0 \frac{\partial}{\partial z_0} + b_1 z_0 \frac{\partial}{\partial z_1} - a_0 z_0 \frac{\partial}{\partial z_1} - a_1 z_1 \frac{\partial}{\partial z_1};\\
\Big[z_1 \frac{\partial}{\partial z_1}, D'\Big] &= a_1 z_1 \frac{\partial}{\partial z_0} - b_0 z_0 \frac{\partial}{\partial z_1}.
\end{aligned} \tag{5}$$

Let $D = z_0 \frac{\partial}{\partial z_0} + z_1 \frac{\partial}{\partial z_1}$, then $[D, D'] = 0$ for all derivations D', i.e., D is in the center.

Proof of Theorem B. By conditions (1) and (2), the adjoint representation of $L_{n+1}(V)$ is faithful and $ad\ E$ is semisimple. Take the Jordan decomposition of $E = S + N$, where S is semisimple and N is nilpotent. In view of the theorem on page 99 of [Humphreys 1975], we know that $N = 0$. Therefore, there exists a coordinate $x_0, \dots, x_n$ such that

$$E = \alpha_0 x_0 \frac{\partial}{\partial x_0} + \alpha_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n x_n \frac{\partial}{\partial x_n}.$$

Observe that

$$[E, x_i E] = -x_i [E, E] + [E, x_i] E = \alpha_i x_i E. \tag{6}$$

Write $x_i E = D_0 + D_1 + \cdots + D_k$ where $D_i \in L_i$ for all $0 \le i \le k$. Then

$$[E, x_i E] = \sum_{j=0}^{k} [E, D_j] = \sum_{j=0}^{k} j D_j. \tag{7}$$

On the other hand, equation (6) says that

$$[E, x_i E] = \alpha_i \sum_{j=0}^{k} D_j. \tag{8}$$

If $\alpha_i = 0$, equations (7) and (8) imply $D_j = 0$ for all $1 \le j \le k$, i.e., $x_i E \in L_0$. This contradicts hypothesis (3) of the Theorem A. Therefore, $\alpha_i = j$ for some positive integer j between 1 and k in the view of equations (7) and (8). Since E acts on $H_{n+1}(f)$, $H_{n+1}(f)$ is graded according to the eigenspace of E. $H_{n+1}(f)$ is nonnegatively graded because all the α_i's are positive integers. Notice that the kernel of E on $H_{n+1}(f)$ is precisely $\mathbb{C}$. Hence we can apply Theorem A to conclude that $(V, 0)$ is a quasihomogeneous singularity. $\square$

For the proof of Theorem C, it is much simpler:

Proof of Theorem C. By the proof of Theorem B, we know there is an Euler derivation in $L^k(V)$, written as $E = \sum_i \alpha_i x_i \frac{\partial}{\partial x_i}$. Notice that

$$E(f) = \sum_i \alpha_i x_i \frac{\partial f}{\partial x_i} \in (f, m^k J_f).$$

Take $k_0 \in \mathbb{N}$ such that $\mathrm{mult}(f) - 1 + k_0 > \deg(f)$. For $k \ge k_0$,

$$\deg(E(f)) = \deg\left(\sum_i \alpha_i x_i \frac{\partial f}{\partial x_i}\right) < \mathrm{mult}(m^k J_f).$$

(Here, $\mathrm{mult}(m^k J_f) := \min\{\mathrm{mult}(g) \mid g \in (m^k J_f) \text{ and } g \neq 0\}$, $\deg(f)$ means the degree of the highest degree monomial in f.) This means $E(f)$ can only be some multiple of f:

$$E(f) = \sum_i \alpha_i x_i \frac{\partial f}{\partial x_i} = af.$$

Comparing degrees of both sides shows that a is a nonzero constant. This tells us that $f \in (J_f)$, thus f is quasihomogeneous. $\square$

The following theorem tells us that the condition "without center" is necessary:

Theorem 3.3. *Let f be weight homogeneous of multiplicity at least three, with weights given in Theorem C, then $L^k(V)$ is without center.*

Proof. Let D be in the center of $L^k(V)$ written as $D = \sum_i D_i$, where D_i is a derivation of weight i. Let

$$E = \sum \alpha_i x_i \frac{\partial}{\partial x_i}$$

be the Euler derivation. Then

$$0 = [E, D] = \left[E, \sum_i D_i\right] = \sum_i i D_i$$

which implies only $D_0 \neq 0$. Hence, D is homogeneous of weight 0. If we write

$$D = \sum a_i \frac{\partial}{\partial x_i}.$$

Then

$$0 = [x_i E, D] = x_i [E, D] + [x_i, D] E = -a_i E.$$

This means if we regard $a_i E$ as a derivation of $\mathbb{C}\{x_0, \dots, x_n\}$, then for all $g \in \mathbb{C}\{x_0, \dots, x_n\}$,

$$a_i E(g) \in (m^k J_f, f).$$

Since $(m^l J_f, f) \supset (m^k J_f, f)$ for all $l \leq k$, we know $a_i E$ maps any $g \in \mathbb{C}\{x_0, \dots, x_n\}$ into $(m^l J_f, f)$. Let $l = 0$, therefore $a_i E$ can be regarded as a zero derivation of $A^0(V)$. This leads to a_i is in the socle of $A^0(V)$. By Lemma 2.3, we obtain that

$$d \geq wt(x_n) + 2wt(x_0) = \alpha_n + 2\alpha_0.$$

Since the socle of $A^0(V)$ is generated by Hess(f), we have

$$wt(a_i) = (d - 2\alpha_0) + \cdots + (d - 2\alpha_n) > \alpha_0.$$

However, D with weight 0 means $wt(a_i) = wt(x_i) \leq \alpha_0$, which is a contradiction. Hence, D must be zero as a derivation of $A^0(V)$, which implies that $a_i \in J_f$. Again, since f is of multiplicity at least three, $a_i \in J_f$ implies that $wt(a_i) \geq wt(f) - wt(x_0) \geq \alpha_0 + \alpha_n > wt(x_i)$. This is a contradiction. Therefore, $a_i = 0$. □

4. Solvability of $L^k(V)$

Firstly, we recall two classical results given in [Yau 1986; 1991].

Theorem 4.1 [Yau 1991]. *Let $sl(2, \mathbb{C})$ act on the formal power series ring $\mathbb{C}[[x_1, \dots, x_n]]$, preserving the m-adic filtration where m is the maximal ideal* *in* $\mathbb{C}[[x_1, \ldots, x_n]]$. *Then there exists a coordinate system*

$$\begin{aligned}
&x_1, \quad x_2, \quad \ldots, \quad x_{l_1};\\
&x_{l_1+1}, \quad x_{l_1+2}, \quad \ldots, \quad x_{l_1+l_2};\\
&\qquad\qquad\vdots\\
&x_{l_1+l_2+\cdots+l_{r-1}+1}, \quad \ldots, \quad x_{l_1+l_2+\cdots+l_r};\\
&x_{l_1+l_2+\cdots+l_r+1}, \quad \ldots, \quad x_n
\end{aligned} \tag{9}$$

via

$$\begin{aligned}
H &= H_1 + \cdots + H_r,\\
X &= X_1 + \cdots + X_r,\\
Y &= Y_1 + \cdots + Y_r,
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
H_j = {} & (l_j-1)x_{l_1+\cdots+l_{j-1}+1}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+1}}\\
&+ (l_j-3)x_{l_1+\cdots+l_{j-1}+2}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+2}} + \cdots\\
&+ (-(l_j-3))x_{l_1+\cdots+l_j-1}\frac{\partial}{\partial x_{l_1+\cdots+l_j-1}}\\
&+ (-(l_j-1))x_{l_1+\cdots+l_j}\frac{\partial}{\partial x_{l_1+\cdots+l_j}},
\end{aligned} \tag{11}$$

$$\begin{aligned}
X_j = {} & (l_j-1)x_{l_1+\cdots+l_{j-1}+1}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+2}} + \cdots\\
&+ i(l_j-i)x_{l_1+\cdots+l_{j-1}+i}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+i+1}} + \cdots\\
&+ (l_j-1)x_{l_1+\cdots+l_j-1}\frac{\partial}{\partial x_{l_1+\cdots+l_j}},
\end{aligned} \tag{12}$$

$$\begin{aligned}
Y_j = {} & x_{l_1+\cdots+l_{j-1}+2}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+1}} + \cdots\\
&+ x_{l_1+\cdots+l_{j-1}+i}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+i-1}} + \cdots\\
&+ x_{l_1+\cdots+l_j}\frac{\partial}{\partial x_{l_1+\cdots+l_j-1}}.
\end{aligned} \tag{13}$$

with $[X_j, Y_j] = H_j,\ [H_j, X_j] = 2X_j,\ [H_j, Y_j] = -2Y_j$.

Here we call r the irreducible representation number. A polynomial g is called of weight j if $H(g) = jg$ for some $j \in \mathbb{Z}$. Note that $l_i \geq 2$ for all $i = 1, \ldots, r$.

Theorem 4.2 [Yau 1991]. *Let $sl(2, \mathbb{C})$ act on M_n^d, the space of homogeneous polynomial of degree $d \geq 2$ as in Theorem 4.1 with $l_1 \geq l_2 \geq \cdots \geq l_r \geq 2$. Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$ where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2, \mathbb{C})$-submodule, then the singular set of f contains the x_1-axis and the x_{l_1}-axis.*

The solvability of $L^0(V)$ has been proved in [Yau 1991]. The solvability of $L^k(V)$ for $k \geq 2$ is proved below while $k = 1$ is much harder. We can only prove $A^1(V)$ does not admit some special $sl(2, \mathbb{C})$-action. (This is equivalent that $(f, mJ_f) = (mJ_f)$ does not admit certain special $sl(2, \mathbb{C})$-action, because a derivation D in $L^1(V)$ has the property $D(mJ_f) \subset (mJ_f)$.)

The key point of the proof for $k \geq 2$ is to show f is $sl(2, \mathbb{C})$-invariant, then Theorem 4.2 leads to contradiction.

Case 1: $k \geq 2$.

Proposition 4.1. *Let f be a homogeneous isolated singularity in n variables $x_1, \ldots, x_n$ of degree $d \geq 4$. Then $L^k(V)$ is solvable for $k \geq 2$.*

Proof. Let $D \in L^k(V)$ be a derivation, then $D(f, m^k J_f) \subset (f, m^k J_f)$. By Leibniz rule, we obtain that $D(m^k J_f) = D(m^k) J_f + m^k D(J_f)$. Moreover $D(m^k) J_f \subset m^k J_f$, hence $D(I) \subset I$ is equivalent to $m^k D(J_f) \subset (f, m^k J_f)$ and $D(f) \subset (f, m^k J_f)$. (Here $I = (f, m^k J_f)$.)

We obtain

$$D(f) = a^D \cdot f + \sum_i b_i^D \cdot \frac{\partial f}{\partial x_i}, \tag{14}$$

where $a^D \in \mathcal{O}_n$ and $b_i^D \in m^k$. Whenever $D = H$, X or Y, it preserves the degree of f, hence the left-hand side of equation (14) is of degree d. However, $\deg\big(b_i^D \cdot \frac{\partial f}{\partial x_i}\big) > \deg(f)$ when $k \geq 2$, thus the term $\big(\sum_i b_i^D \cdot \frac{\partial f}{\partial x_i}\big)$ is zero. Equation (14) becomes

$$D(f) = a^D \cdot f$$

for $D = H$, X or Y. This means that f is $sl(2, \mathbb{C})$-invariant. Therefore J_f is $sl(2, \mathbb{C})$-invariant. By Theorem 4.2, f is singular on x_1-axis, which is a contradiction. $\square$

Case 2: $k = 1$.

Now we consider the case of $k = 1$. The key point is as follows: If $L^1(V)$ is not solvable, then $(f, mJ_f) = (mJ_f)$ admits an action as in Theorem 4.1. Selecting a generator $g \in (mJ_f)$, we know that $H(g), X(g), Y(g) \in (mJ_f)$. Repeating this procedure, we can find that the number of generators is greater than n^2, which leads to a contradiction.

Case 2.1: $k = 1,\ n = 2$.

Proposition 4.2. *Let f be a homogeneous isolated singularity in 2 variables x_1, x_2 of degree $d \geq 4$. Then $L^1(V)$ is solvable.*

Proof. In the case $n = 2$, the action of $sl(2, \mathbb{C})$ is given by

$$X = x_1 \frac{\partial}{\partial x_2}, \quad Y = x_2 \frac{\partial}{\partial x_1}.$$

By Lemma 2.3, f is of one of the following two forms:

Form (1): $f = x_1^d + a_1 x_1^{d-1} x_2 + \cdots + a_d x_2^d$.

Form (2): $f = x_1^{d-1} x_2 + a_2 x_1^{d-2} x_2^2 + \cdots + a_d x_2^d$.

If f is of Form (1), then

$$x_1 \frac{\partial f}{\partial x_1} = d x_1^d + a_1(d-1) x_1^{d-1} x_2 + \cdots + a_{d-1} x_1 x_2^{d-1} \in (m J_f).$$

Hence,

$$X^d Y^d \left(x_1 \frac{\partial f}{\partial x_1} \right) = c \cdot x_1^d \in (m J_f)$$

where c is a constant. This implies that

$$x_1^d, Y(x_1^d) = x_1^{d-1} x_2, \ldots, Y^d(x_1^d) = x_2^d$$

are all in $(m J_f)$. These are $d + 1 > 4$ monomials. However $\dim_{\mathbb{C}}(m J_f \cap M_2^d) = 4$, which is a contradiction. (The basis of $m J_f \cap M_2^d$ are $x_i \frac{\partial f}{\partial x_j}$ with $i, j \in \{1, 2\}$.)

If f is of Form (2), then

$$x_1 \frac{\partial f}{\partial x_2} = d x_1^d + 2a_2(d-1) x_1^{d-1} x_2 + \cdots + d a_d x_1 x_2^{d-1} \in (m J_f).$$

By similar reasoning, we get a contradiction. □

Remark 4.3. The proof for $n = 2$ can be generalized to more variables. However, we must require the $sl(2, \mathbb{C})$-action to be irreducible. For general action it is still open.

Recall in Theorem 4.1, for $H = H_1 + \cdots + H_r$, we call r the irreducible component number.

Definition 4.1. The $sl(2, \mathbb{C})$-action is called irreducible if the irreducible component number $r = 1$ and $l_1 = n$.

Case 2.2: $k = 1, n \geq 2, r = 1$ and $l_1 = n$.

Theorem 4.4 (weak Theorem D). *Let f be a homogeneous isolated singularity in n variables $x_1, \ldots, x_n$ of degree $d \geq 4$. Then $(m J_f)$ does not admit irreducible $sl(2, \mathbb{C})$-action.*

Proof. By Theorem 4.1, we obtain that

$$\begin{aligned}
H &= (n-1)x_1\frac{\partial}{\partial x_1} + (n-3)x_2\frac{\partial}{\partial x_2} \\
&\quad + \cdots + (-(n-3))x_{n-1}\frac{\partial}{\partial x_{n-1}} + (-(n-1))x_n\frac{\partial}{\partial x_n} \\
X &= (n-1)x_1\frac{\partial}{\partial x_2} + 2(n-2)x_2\frac{\partial}{\partial x_3} + \cdots + i(n-i)x_i\frac{\partial}{\partial x_{i+1}} \\
&\quad + \cdots + (n-1)x_{n-1}\frac{\partial}{\partial x_n} \\
Y &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + \cdots + x_i\frac{\partial}{\partial x_{i-1}} + \cdots + x_n\frac{\partial}{\partial x_{n-1}}.
\end{aligned}$$

By Lemma 2.3, we obtain $f = x_1^d + a_1(x_2, \dots, x_n)x_1^{d-1} + \cdots + a_d(x_2, \dots, x_n)$ (Form (1)) or $f = x_1^{d-1}x_s + a_2(x_2, \dots, x_n)x_1^{d-2} + \cdots + a_d(x_2, \dots, x_n)$ (Form (2)), where $a_i(x_2, \dots, x_n)$ is a polynomial of degree i in variable $x_2, \dots, x_n$. (We omit the constant coefficient in later discussion for simplicity.)

If f is of Form (1), then

$$x_i\frac{\partial f}{\partial x_1} = x_i x_1^{d-1} + \text{lower weight terms.}$$

If f is of Form (2), then

$$x_i\frac{\partial f}{\partial x_s} = x_i x_1^{d-1} + \text{lower weight terms.}$$

The following lemma shows that $x_i x_1^{d-1} \in (mJ_f)$ whenever f is of Form (1) or (2).

Lemma 4.1. *Let $g = \sum g^j$ be a homogeneous polynomial in (mJ_f), where g^j is weight j component of g, then $g^j \in (mJ_f)$.*

By Lemma 4.1, we obtain that these polynomials are in mJ_f:

$$\begin{array}{llll}
x_1^d; & & & \\
x_1^{d-1}x_2; & & & \\
x_1^{d-1}x_3, & Y(x_1^{d-1}x_2); & & \\
x_1^{d-1}x_4, & Y(x_1^{d-1}x_3), & \dots, & Y^2(x_1^{d-1}x_2); \\
\vdots & & & \\
x_1^{d-1}x_n, & Y(x_1^{d-1}x_{n-1}), & \dots, & Y^{n-2}(x_1^{d-1}x_2); \\
Y^2(x_1^{d-1}x_{n-1}), & \dots, & Y^{n-1}(x_1^{d-1}x_2). &
\end{array}$$

Here on each row the polynomials are of same weight. We call these polynomials "Block 1".

All these polynomials are linear independent. Their weights are greater than or equal to $wt(x_1^{d-1}x_n) - 2 = (d-2)(n-1) - 2$. A similar discussion shows that

$$\begin{array}{llll}
x_n^d; & & & \\
x_n^{d-1}x_{n-1}; & & & \\
x_n^{d-1}x_{n-2}, & X(x_n^{d-1}x_{n-1}); & & \\
x_n^{d-1}x_{n-3}, & X(x_n^{d-1}x_{n-2}), & \ldots, & X^2(x_n^{d-1}x_{n-1}); \\
\vdots & & & \\
x_n^{d-1}x_1, & X(x_n^{d-1}x_2), & \ldots, & X^{n-2}(x_1^{d-1}x_{n-1}); \\
X^2(x_n^{d-1}x_2), & \ldots, & X^{n-1}(x_1^{d-1}x_{n-1}); &
\end{array}$$

are in mJ_f, with weight less than or equal to $-(d-2)(n-1)+2$. We call these polynomials "Block 2".

Since $d \geq 4$ and $n > 2$, $-(d-2)(n-1)+2 < (d-2)(n-1)-2$. Thus polynomials in Block 1 are of weights greater than those in Block 2, which implies the polynomials in Block 1 and Block 2 are linearly independent.

In Block 1 and Block 2, there are $2(1+1+2+\cdots+n-1+n-2) = n(n+1)-2$ linear independent polynomials of degree d, while $\dim_{\mathbb{C}}(mJ_f \cap M_n^d) = n^2$, which is a contradiction. $\square$

Observation: In the proof of $r = 1$, we construct two "blocks". The first one starts from $x_1^{d-1}x_i$, which is constructed by acting with Y. The second one starts from $x_n^{d-1}x_i$ and is constructed by acting with X.

Now for $r \neq 1$, firstly we assume $l_1 + \cdots + l_r = n$. We hope to construct blocks as above, then comparing the number of generators will lead to contradiction.

Case 3: $r > 1$, $l_1 + \cdots + l_r = n$.

We construct the following blocks (here $1 \leq i, j \leq r$.):

Block 1.1

$$\begin{array}{ll}
\frac{\partial f}{\partial x_1}x_1; & \\
\frac{\partial f}{\partial x_1}x_2; & \\
\frac{\partial f}{\partial x_1}x_3, & Y\left(\frac{\partial f}{\partial x_1}x_2\right);
\end{array}$$

$$\frac{\partial f}{\partial x_1}x_4, \qquad Y\left(\frac{\partial f}{\partial x_1}x_3\right), \quad Y^2\left(\frac{\partial f}{\partial x_1}x_2\right);$$

$$\vdots$$

$$\frac{\partial f}{\partial x_1}x_{l_1}, \qquad Y\left(\frac{\partial f}{\partial x_1}x_{l_1-1}\right), \qquad \dots, \qquad Y^{l_1-2}\left(\frac{\partial f}{\partial x_1}x_2\right);$$

$$Y^2\left(\frac{\partial f}{\partial x_1}x_{l_1-1}\right), \qquad \dots, \qquad Y^{l_1-1}\left(\frac{\partial f}{\partial x_1}x_2\right).$$

Block 1.2

$$\frac{\partial f}{\partial x_1}x_{l_1+1};$$

$$\frac{\partial f}{\partial x_1}x_{l_1+2};$$

$$\frac{\partial f}{\partial x_1}x_{l_1+3}, \qquad Y\left(\frac{\partial f}{\partial x_1}x_{l_1+2}\right);$$

$$\vdots$$

$$\frac{\partial f}{\partial x_1}x_{l_1+l_2}, \qquad Y\left(\frac{\partial f}{\partial x_1}x_{l_1+l_2-1}\right), \qquad \dots, \qquad Y^{l_2-2}\left(\frac{\partial f}{\partial x_1}x_{l_1+2}\right);$$

$$Y^2\left(\frac{\partial f}{\partial x_1}x_{l_1+l_2-1}\right), \qquad \dots, \qquad Y^{l_2-1}\left(\frac{\partial f}{\partial x_1}x_{l_1+2}\right).$$

Block 1.*r*

$$\frac{\partial f}{\partial x_1}x_{l_1+\dots+l_{r-1}+1};$$

$$\frac{\partial f}{\partial x_1}x_{l_1+\dots+l_{r-1}+2};$$

$$\frac{\partial f}{\partial x_1}x_{l_1+\dots+l_{r-1}+3}, \quad Y\left(\frac{\partial f}{\partial x_1}x_{l_1+\dots+l_{r-1}+2}\right);$$

$$\vdots$$

$$\frac{\partial f}{\partial x_1}x_{l_1+\dots+l_r}, \qquad Y\left(\frac{\partial f}{\partial x_1}x_{l_1+\dots+l_r-1}\right), \qquad \dots, \qquad Y^{l_r-2}\left(\frac{\partial f}{\partial x_1}x_{l_1+\dots+l_{r-1}+2}\right);$$

$$Y^2\left(\frac{\partial f}{\partial x_1}x_{l_1+\dots+l_r-1}\right), \qquad \dots, \qquad Y^{l_r-1}\left(\frac{\partial f}{\partial x_1}x_{l_1+\dots+l_{r-1}+2}\right).$$

$$\vdots$$

Block *i.j*

$$\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+1};$$

$$\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+2};$$

$$\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+3}, \quad Y\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+2}\right);$$

$$\vdots$$

$$\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_j}, \quad Y\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_j-1}\right),$$

$$\vdots \qquad \ldots, \qquad Y^{l_j-2}\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+2}\right);$$

$$Y^2\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_j-1}\right), \qquad \ldots, \qquad Y^{l_j-1}\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+2}\right).$$

$$\vdots$$

The number of linear independent polynomials in Block $i.j$ is

$$2(1+1+2+\cdots+l_j-1+l_j-2) = 2(l_j+1)l_j-2.$$

Similar to the construction of Block 1 and Block 2, we can construct another Block "dual" to Block $i.j$ with $2(l_j+1)l_j-2$ polynomials. If all above polynomials are linear independent, the whole number of linear independent polynomials is $4r(l_1(l_1+1)+\cdots+l_r(l_r+1)-2r)$. However

$$4r(l_1(l_1+1)+\cdots+l_r(l_r+1)-2r) > (l_1+\cdots+l_r)^2 = n^2.$$

This is a contradiction.

The problem arises on the linear independence of different blocks. To be more precise, there may exist variables in other blocks with same weight, so we cannot get linear independence by comparing weight. We use an example to explain this phenomenon.

Example 4.5. In the case $n=4$ and $l_1=l_2=2$,

$$H = H_1+H_2, \quad X = X_1+X_2, \quad Y = Y_1+Y_2.$$

x_1 and x_3 are of same weight. Let $f=(x_1+x_3)^4+x_2^4+x_4^4+x_1^3x_2+x_1^3x_4$ which defines an isolated singularity. The operation of taking highest weight is restricting polynomial to $x_2 = x_4 = 0$. For example,

$$\frac{\partial f}{\partial x_1} = 4(x_1 + x_3)^3 + 3x_1^2(x_2 + x_4).$$

The highest weight part of $\frac{\partial f}{\partial x_1}$ is $4(x_1 + x_3)^3$. However

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_2=x_4=0} = \frac{\partial f}{\partial x_3}.$$

Thus $\left.\frac{\partial f}{\partial x_1}\right|_{x_2=x_4=0}$, $\frac{\partial f}{\partial x_3}$ are linear dependent.

In this example, we only need to exchange $\frac{\partial f}{\partial x_3}$ to $\left.\frac{\partial f}{\partial x_2}\right|_{x_2=x_4=0}$. Then $\left.\frac{\partial f}{\partial x_1}\right|_{x_2=x_4=0}$, $\left.\frac{\partial f}{\partial x_2}\right|_{x_2=x_4=0}$ are linear independent. It reminds us that there exists a suitable way to select linear independent polynomials. This is illustrated in the following lemma:

Lemma 4.2. *If $r = 2$ and $l_1 = l_2$, then there exists g_1, g_2 of weight $(d-1)(l_1-1)$ in (mJ_f), such that the following four polynomials are linear independent*:

$$g_1 x_1, \quad g_1 x_{l_1+1}, \quad g_2 x_1, \quad g_2 x_{l_1+1}.$$

Proof. We first show how to construct g_1, g_2 from the derivatives of f. Then we prove the linear independence of above four polynomials. Let us consider the following polynomials:

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_2=\cdots=x_{l_1}=x_{l_1+2}=\cdots=x_n=0}, \qquad \left.\frac{\partial f}{\partial x_2}\right|_{x_2=\cdots=x_{l_1}=x_{l_1+2}=\cdots=x_n=0},$$

$$\ldots, \qquad \left.\frac{\partial f}{\partial x_n}\right|_{x_2=\cdots=x_{l_1}=x_{l_1+2}=\cdots=x_n=0}.$$

These are polynomials in x_1, x_{l_1+1} of degree $d-1$, for simplicity we write them as

$$h_1, \quad \ldots, \quad h_n.$$

Let the common factor of $h_1, \ldots, h_n$ be h. Define

$$Y := \{h = 0\} \cap \{x_2 = \cdots = x_{l_1} = x_{l_1+2} = \cdots = x_n = 0\}.$$

Here $h, x_2, \ldots, x_{l_1}, x_{l_1+2}, \ldots, x_n$ are $n-1$ functions, and thus $\dim Y \geq 1$. However, by the definition of Y, $f|_Y = h_i|_Y = 0$ for all $i = 1, \ldots, n$. This contradicts that f defines an isolated singularity. Thus the common factor of $h_1, \ldots, h_n$ is 1.

We claim there exists $a_1, \ldots, a_n \in \mathbb{C}$ and $j \in \{1, \ldots, n\}$, such that $a_1 h_1 + \cdots + a_n h_n$ and h_j do not have common factor. If the claim holds, then we denote $h_j = g_1$, $\sum_{i=1}^n a_i h_i = g_2$.

Now we prove the linear independence of $g_1x_1, g_1x_{l_1+1}, g_2x_1, g_2x_{l_1+1}$. Assume the contrary. Then there exists b_1, b_2, b_3, b_4 which are not all zero such that

$$g_1(b_1x_1 + b_3x_{l_1+1}) = g_2(b_2x_1 + b_4x_{l_1+1}).$$

Without loss of generality, we assume that $b_1, b_4 \neq 0$. $b_1x_1 + b_3x_{l_1+1}$ and $b_2x_1 + b_4x_{l_1+1}$ are coprime, otherwise g_1, g_2 have common factor. Thus the above equality implies

$$(b_1x_1 + b_3x_{l_1+1}) \mid g_2, \quad (b_2x_1 + b_4x_{l_1+1}) \mid g_1.$$

Observe that g_1, g_2 have degree $d - 1 \geq 3 > 1 = \deg(b_1x_1 + b_3x_{l_1+1})$; hence $\deg(g_2/(b_1x_1 + b_3x_{l_1+1})) \geq 2$. This means $g_2/(b_1x_1 + b_3x_{l_1+1})$ is a nontrivial polynomial, and is a factor of g_1, which contradicts that g_1, g_2 have no common factor.

At last we prove the claim. For j such that $h_j \neq 0$, we express h_j as product of irreducible polynomials:

$$h_j = s_1^{r_1} s_2^{r_2} \cdots s_l^{r_l}.$$

If h_{i_0} and h_j do not have common factor then we are done. So we assume each h_i and h_j have a common factor for any $i = 1, \ldots, n$. Since the common factor of $h_1, \ldots, h_n$ is 1, there exists two polynomials, say h_1, h_2, such that they have a different common factor with h_j. Without loss of generality, we assume $s_1 \mid h_1$, $s_2 \mid h_2$, $s_1 \nmid h_2$, $s_2 \nmid h_1$. Then $s_1, s_2 \nmid (h_1 + h_2)$. If $h_1 + h_2$ does not have common factor with h_j, then we are done. So we assume $s_3 \mid (h_1 + h_2)$. If $s_3 \mid h_1$, then $s_3 \mid (h_1 + h_2 - h_1)$, which contradicts that h_1, h_2 have a different common factor with h_j. Thus $s_3 \nmid h_1, h_2$. Then $s_1, s_2, s_3 \nmid ((h_1 + h_2) + h_1)$, by the same induction we know $s_4 \mid (2h_1 + h_2)$ or $2h_1 + h_2$ has no common factor with h_j. Since r_l is finite, this implies that the induction procedure must terminate, and so finally we can find a linear combination of h_1, h_2 such that it has no common factor with h_j. □

Case 3.1: $r = 2$.

The following proposition follows from Lemma 4.2 immediately.

Proposition 4.3. *Let f be homogeneous isolated singularity of degree d. Then (mJ_f) does not admit an $sl(2, \mathbb{C})$-action when $r = 2$, $l_1 + l_2 = n$.*

Proof. We divide it into two cases:

Case 1: $l_1 = l_2$.

Choose g_1, g_2 as in Lemma 4.2. Then we consider the following four blocks:

Block 1.1

g_1x_1;

g_1x_2;

g_1x_3, $\quad Y(g_1x_2)$;

$$
\begin{array}{lll}
g_1x_4, & Y(g_1x_3), & Y^2(g_1x_2); \\
\vdots & & \\
g_1x_{l_1}, & Y(g_1x_{l_1-1}), & Y^{l_1-2}(g_1x_2); \\
Y^2(g_1x_{l_1-1}), & Y^{l_1-1}(g_1x_2); &
\end{array}
$$

Block 1.2

$$
\begin{array}{lll}
g_2x_1; & & \\
g_2x_2; & & \\
g_2x_3, & Y(g_2x_2); & \\
g_2x_4, & Y(g_2x_3), & Y^2(g_2x_2); \\
\vdots & & \\
g_2x_{l_1}, & Y(g_2x_{l_1-1}), & Y^{l_1-2}(g_2x_2); \\
Y^2(g_2x_{l_1-1}), & Y^{l_1-1}(g_2x_2); &
\end{array}
$$

Block 2.1

$$
\begin{array}{lll}
g_1x_{l_1+1}; & & \\
g_1x_{l_1+2}; & & \\
g_1x_{l_1+3}, & Y(g_1x_{l_1+2}); & \\
g_1x_{l_1+4}, & Y(g_1x_{l_1+3}), & Y^2(g_1x_{l_1+2}); \\
\vdots & & \\
g_1x_{l_1+l_2}, & Y(g_1x_{l_1+l_2-1}), & Y^{l_2-2}(g_1x_{l_1+2}); \\
Y^2(g_1x_{l_1+l_2-1}), & \ldots, & Y^{l_2-1}(g_1x_{l_1+2});
\end{array}
$$

Block 2.2

$$
\begin{array}{lll}
g_2x_{l_1+1}; & & \\
g_2x_{l_1+2}; & & \\
g_2x_{l_1+3}, & Y(g_2x_{l_1+2}); & \\
g_2x_{l_1+4}, & Y(g_2x_{l_1+3}), & Y^2(g_2x_{l_1+2}); \\
\vdots & & \\
g_2x_{l_1+l_2}, & Y(g_2x_{l_1+l_2-1}), & Y^{l_2-2}(g_2x_{l_1+2}); \\
Y^2(g_2x_{l_1+l_2-1}), & \ldots, & Y^{l_2-1}(g_2x_{l_1+2}).
\end{array}
$$

The number of polynomials in all the blocks is $2(l_1(l_1+1)-2+l_2(l_2+1)-2)$. Replacing x_1, x_{l_1+1} by x_{l_1}, x_n and Y by X, we can get another $2(l_1(l_1+1)+l_2(l_2+1)-4)$ polynomials. However $4(l_1^2+l_2^2+l_1+l_2-4) > n^2$, which is a contradiction.

Case 2: $l_1 > l_2$.

In this case we can use same argument as in the irreducible case that

$$x_1^{d-1}x_i \in (mJ_f) \quad \text{for all } i.$$

And the block can be constructed as follows:

In Block 1.1, 2.1, we choose g_1 to be x_1^{d-1}. In Block 1.2, 2.2, we choose g_2 to be $x_{l_1+1}^{d-1}+g_3(x_1,\dots,x_{l_1})$, where g_3 is a polynomial of weight $(d-1)(l_2-1)$ such that $(x_{l_1+1}^{d-1}+g_3) \in J_f$. Then it leads to a contradiction similarly. □

Proof of Theorem D. When $k \geq 2$, the theorem follows immediately from Proposition 4.1. In the case of $n=4,\ k=1,\ r$ has to be 1 or 2. If $r=2$, we obtain that $l_1+l_2=4$ by Theorem 4.1. And the result follows from Proposition 4.3. If $r=1,\ l_1=4$, the result follows from Theorem 4.4. We only have to consider the cases $r=1,\ l_1=2$ or 3.

Case 1: $r=1,\ l_1=2$. The $sl(2,\mathbb{C})$-action is as follows:

$$H = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}, \quad X = x_1\frac{\partial}{\partial x_2}, \quad Y = x_2\frac{\partial}{\partial x_1}.$$

By Lemma 4.1, $x_1^{d-1}x_i \in mJ_f$. By the discussion in Proposition 4.2, x_1^{d-1}, x_3^{d-1}, x_4^{d-1} are in J_f. Thus

$$\begin{array}{llll}
x_1^d, & Y(x_1^d) = x_1^{d-1}x_2, & \dots, & Y^d(x_1^d) = x_2^d; \\
x_1^{d-1}x_3, & Y(x_1^{d-1}x_3), & \dots, & Y^{d-1}(x_1^{d-1}x_3) = x_2^{d-1}x_3; \\
x_1^{d-1}x_4, & Y(x_1^{d-1}x_4), & \dots, & Y^{d-1}(x_1^{d-1}x_4) = x_2^{d-1}x_4; \\
x_3^d; \quad x_4^d; & & & \\
x_3^{d-1}x_1, \quad x_3^{d-1}x_2; & x_4^{d-1}x_1, \quad x_4^{d-1}x_2; & &
\end{array}$$

are in mJ_f. The number of linear independent polynomials of degree d are $3d+6 > 16$, which is a contradiction.

Case 2: $r=1, l_1=3$. By the discussion in Theorem 4.4, we can find $3(3+1)-2=10$ linear independent polynomials in x_1, x_2, x_3. Thus we only need to find more than 6 polynomials. x_4^d, $x_4^{d-1}x_1$, $x_4^{d-1}x_2$, $x_4^{d-1}x_3$, $x_1^{d-1}x_4$, $x_2^{d-1}x_4$, $x_3^{d-1}x_4$ are satisfied. □

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STABLE VALUE OF DEPTH OF SYMBOLIC POWERS OF EDGE IDEALS OF GRAPHS

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Dedicated to Professor Ngo Viet Trung on the occasion of his 70th birthday

Let G be a simple graph on n vertices. We introduce the notion of bipartite connectivity of G, denoted by $\mathrm{bc}(G)$ and prove that

$$\lim_{s\to\infty} \operatorname{depth}(S/I(G)^{(s)}) \le \mathrm{bc}(G),$$

where $I(G)$ denotes the edge ideal of G and $S = \mathrm{k}[x_1, \dots, x_n]$ is a standard graded polynomial ring over a field k. We further compute the depth of symbolic powers of edge ideals of several classes of graphs, including odd cycles and whisker graphs of complete graphs to illustrate the cases where the above inequality becomes equality.

1. Introduction

Let I be a homogeneous ideal in a standard graded polynomial ring $S = \mathrm{k}[x_1, \dots, x_n]$ over a field k. While the depth function of powers of I is convergent by the result of Brodmann [1979], the depth function of symbolic powers of I is more exotic. Nguyen and N. V. Trung [2019] proved that for every positive eventually periodic function $f : \mathbb{N} \to \mathbb{N}$ there exists an ideal I such that $\operatorname{depth} S/I^{(s)} = f(s)$ for all $s \ge 1$, where $I^{(s)}$ denotes the s-th symbolic power of I. On the other hand, when I is a squarefree monomial ideal, by the result of Hoa et al. [2017] and Varbaro [2011],

$$\lim_{s\to\infty} \operatorname{depth} S/I^{(s)} = \min\{\operatorname{depth} S/I^{(s)} \mid s \ge 1\} = n - \ell_s(I),$$

where $\ell_s(I)$ is the symbolic analytic spread of I. Nonetheless, given a squarefree monomial ideal I, computing the stable value of depth of symbolic powers of I is a difficult problem even in the case of edge ideals of graphs.

Let us now recall the notion of the edge ideals of graphs. Let G be a simple graph with the vertex set $V(G) = \{1, \dots, n\}$ and edge set $E(G)$. The edge ideal of G, denoted by $I(G)$, is the squarefree monomial ideal generated by $x_i x_j$ where $\{i, j\}$ is an edge of G. Trung [2016] showed that $\lim_{s\to\infty} \operatorname{depth} S/I(G)^s$ equals the number

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of bipartite connected components of G, and that depth $S/I(G)^s$ stabilizes when it reaches the limit depth. By the results in [Nguyen and Vu 2019; Hà et al. 2020], we may assume that G is a connected graph when considering the depth of (symbolic) powers of the edge ideal of G. In this case, the result of Trung [2016] can be written as

$$\lim_{s\to\infty} \operatorname{depth} S/I(G)^s = \begin{cases} 1 & \text{if } G \text{ is bipartite,} \\ 0, & \text{otherwise,} \end{cases} \tag{1-1}$$

and the stabilization index of depth of powers of $I(G)$, denoted by $\operatorname{dstab}(I(G))$, is the smallest exponent s such that depth $S/I(G)^s$ equals the limit depth of powers. Since we expect that the depth functions of symbolic powers of edge ideals are nonincreasing, this property should hold for symbolic powers of $I(G)$ as well. Hien, Lam, and Trung [2024] characterized graphs for which $\lim_{s\to\infty} \operatorname{depth} S/I(G)^{(s)} = 1$ and proved that the stabilization index of depth of symbolic powers in this case is also the smallest exponent s such that depth $S/I(G)^{(s)} = 1$. For a general nonbipartite graph G, we do not know the value $\lim_{s\to\infty} \operatorname{depth} S/I(G)^{(s)}$.

In this paper, we introduce the notion of bipartite connectivity of G and show that this is tightly connected to the stable value of depth of symbolic powers of $I(G)$. Let $\mathcal{B}(G)$ denote the set of maximal induced bipartite subgraphs H of G, i.e., for any $v \in V(G) \setminus V(H)$, the induced subgraph of G on $V(H) \cup \{v\}$ is not bipartite. Note that H might contain isolated vertices. Since H is maximal, it contains at least one edge. Then we define $\operatorname{bc}(G) = \min\{c(H) \mid H \in \mathcal{B}(G)\}$ and call it the bipartite connectivity number of G, where $c(H)$ is the number of connected components of H. With this notation, the result of Hien et al. [2024] can be stated as $\lim_{s\to\infty} \operatorname{depth} S/I(G)^{(s)} = 1$ if and only if $\operatorname{bc}(G) = 1$, i.e., there exists an induced connected bipartite subgraph H of G such that H dominates G. In this paper, we generalize this result and prove:

Theorem 1.1. *Let G be a simple graph. Then*

$$\lim_{s\to\infty} \operatorname{depth} S/I(G)^{(s)} \le \operatorname{bc}(G).$$

In contrast to (1-1), we show that the limit depth of symbolic powers of $I(G)$ could be any positive number even when G is a connected graph.

Proposition 1.2. *Let $n \ge 2$ be a positive number and $W_n = W(K_n)$ be the whisker graph on the complete graph on n vertices. Then*, $\operatorname{bc}(W_n) = n - 1$ *and*

$$\operatorname{depth} S/I(W_n)^{(s)} = \begin{cases} n & \text{if } s = 1, \\ n-1 & \text{if } s \ge 2. \end{cases}$$

We also note that the inequality in Theorem 1.1 could be strict as given in the following example.

Example 1.3. Let W be the graph obtained by gluing two whiskers at the vertices of a 3-cycle. Then $\operatorname{bc}(W) = 3$ while

$$\operatorname{depth} S/I(W)^{(s)} = \begin{cases} 7 & \text{if } s = 1, \\ 4 & \text{if } s = 2, \\ 2 & \text{if } s \geq 3. \end{cases}$$

Nonetheless, if we cluster the isolated points in a maximal bipartite subgraph H of G by the bouquets in G then we obtain a finer invariant of G that gives the stable value of depth of symbolic powers. More precisely, assume that

$$H = H_1 \cup \cdots \cup H_c \cup \{p_1, \ldots, p_t\},$$

where H_i are connected components of H with at least one edge and $p_1, \ldots, p_t$ are isolated points in H. We say that $p_{i_1}, \ldots, p_{i_u}$ are clustered if there exists a $v \in V(G) \setminus V(H)$ such that the induced subgraph of G on $\{v, p_{i_1}, \ldots, p_{i_u}\}$ is a bouquet. Let $\operatorname{bou}_G(H)$ be the smallest number b such that the set $\{p_1, \ldots, p_t\}$ can be clustered into b bouquets in G. We call $c'(H) = c + \operatorname{bou}_G(H)$ the number of restricted connected components of H. We then define

$$\operatorname{bc}'(G) = \min\{c'(H) \mid H \in \mathcal{B}(G)\},$$

the restricted bipartite connectivity number of G. It is easy to see that for the graph W in Example 1.3, we have $\operatorname{bc}'(W) = 2$. We conjecture that:

Conjecture 1.4. Let G be a simple graph. Then

$$\lim_{s \to \infty} \operatorname{depth} S/I(G)^{(s)} = \operatorname{bc}'(G).$$

We verify this conjecture for whisker graphs of complete graphs.

Theorem 1.5. *Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $W_{\boldsymbol{a}}$ be the graph obtained by gluing a_i leaves to the vertex i of a complete graph K_n. Assume that $a_i \geq 1$ for all $i = 1, \ldots, n$. Then $\operatorname{bc}'(W_{\boldsymbol{a}}) = n - 1$ and*

$$\lim_{s \to \infty} \operatorname{depth} S/I(W_{\boldsymbol{a}})^{(s)} = n - 1.$$

Finally, we compute the depth of symbolic powers of edge ideals of odd cycles by extending our argument in [Minh et al. 2023]. This shows that the bound for the index of depth stability of symbolic powers of I given in [Hien et al. 2024] is sharp.

Theorem 1.6. *Let $I(C_n)$ be the edge ideal of a cycle of length $n = 2k + 1 \geq 5$. Then*

$$\operatorname{depth} S/I(C_n)^{(s)} = \begin{cases} \left\lceil \frac{n-1}{3} \right\rceil & \text{if } s = 1, \\ \max\left(1, \left\lceil \frac{n-s+1}{3} \right\rceil\right) & \text{if } s \geq 2. \end{cases}$$

In particular, $\operatorname{sdstab}(I(C_n)) = n - 2$, where $\operatorname{sdstab}(I)$ is the index of depth stability of symbolic powers of I.

We structure the paper as follows. In Section 2, we set up the notation and provide some background. In Section 3, we prove Theorem 1.1 and compute the depth of symbolic powers of edge ideals of whisker graphs of complete graphs. In Section 4, we prove Theorem 1.6.

2. Preliminaries

In this section, we recall some definitions and properties concerning depth, graphs and their edge ideals, and the symbolic powers of squarefree monomial ideals. The interested readers are referred to [Bruns and Herzog 1993] for more details.

Throughout the paper, we denote by $S = \mathrm{k}[x_1, \dots, x_n]$ a standard graded polynomial ring over a field k. Let $\mathfrak{m} = (x_1, \dots, x_n)$ be the maximal homogeneous ideal of S.

Depth. For a finitely generated graded S-module L, the depth of L is defined to be

$$\operatorname{depth}(L) = \min\{i \mid H^i_{\mathfrak{m}}(L) \neq 0\},$$

where $H^i_{\mathfrak{m}}(L)$ denotes the i-th local cohomology module of L with respect to $\mathfrak{m}$. We have the following estimates on depth along short exact sequences (see [Bruns and Herzog 1993, Proposition 1.2.9]).

Lemma 2.1. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated graded S-modules. Then*:

(1) $\operatorname{depth} M \geq \min\{\operatorname{depth} L, \operatorname{depth} N\}$.

(2) $\operatorname{depth} L \geq \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}$.

We make repeated use of the following two results in the sequence. The first one is [Rauf 2010, Corollary 1.3]. The second one is [Caviglia et al. 2019, Theorem 4.3].

Lemma 2.2. *Let I be a monomial ideal and f a monomial such that $f \notin I$. Then*

$$\operatorname{depth} S/I \leq \operatorname{depth} S/(I : f).$$

Lemma 2.3. *Let I be a monomial ideal and f a monomial. Then*

$$\operatorname{depth} S/I \in \{\operatorname{depth}(S/I : f), \operatorname{depth}(S/(I, f))\}.$$

Finally, we also use the following simple result.

Lemma 2.4. *Let $S = \mathrm{k}[x_1, \dots, x_n]$, $R_1 = \mathrm{k}[x_1, \dots, x_a]$, and $R_2 = \mathrm{k}[x_{a+1}, \dots, x_n]$ for some natural number a such that $1 \leq a < n$. Let I and J be homogeneous ideals of R_1 and R_2, respectively. Then*:

(1) $\operatorname{depth}(S/(I + J)) = \operatorname{depth}(R_1/I) + \operatorname{depth}(R_2/J)$.

(2) *Let $P = I + (x_{a+1}, \dots, x_b)$. Then* $\operatorname{depth}(S/P) = \operatorname{depth}(R_1/I) + (n - b)$.

Proof. (1) The proof is standard; see, e.g., [Nguyen and Vu 2019, Lemma 2.3].

(2) It follows from (1) and the fact that $\operatorname{depth}(R_2/(x_{a+1}, \dots, x_b)) = (n-b)$. □

Depth of Stanley–Reisner rings. Let Δ be a simplicial complex on the vertex set $V(\Delta) = [n] = \{1, \dots, n\}$. For a face $F \in \Delta$, the link of F in Δ is the subsimplicial complex of Δ defined by

$$\operatorname{lk}_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta,\ F \cap G = \varnothing\}.$$

For each subset F of $[n]$, let $x_F = \prod_{i \in F} x_i$ be a squarefree monomial in S. We now recall the Stanley–Reisner correspondence.

Definition 2.5. For a squarefree monomial ideal I, the Stanley–Reisner complex of I is defined by

$$\Delta(I) = \{F \subseteq [n] \mid x_F \notin I\}.$$

For a simplicial complex Δ, the Stanley–Reisner ideal of Δ is defined by

$$I_\Delta = (x_F \mid F \notin \Delta).$$

The Stanley–Reisner ring of Δ is $\mathrm{k}[\Delta] = S/I_\Delta$.

Definition 2.6. The q-th reduced homology group of Δ with coefficients over k, denoted $\widetilde{H}_q(\Delta; \mathrm{k})$ is defined to be the q-th homology group of the augmented oriented chain complex of Δ over k.

From the Hochster's formula, we deduce that:

Lemma 2.7. *Let Δ be a simplicial complex. Then*

$$\operatorname{depth}(\mathrm{k}[\Delta]) = \min\{|F| + i \mid \widetilde{H}_{i-1}(\operatorname{lk}_\Delta F; \mathrm{k}) \neq 0,\ F \in \Delta\}.$$

Proof. By definition, $\operatorname{depth}(k[\Delta]) = \min\{i \mid H^i_{\mathfrak{m}}(k[\Delta]) \neq 0\}$. By Hochster's formula [Bruns and Herzog 1993, Theorem 5.3.8], the conclusion follows. □

We will also use the following nerve theorem from [Borsuk 1948]. First, we recall the definition of the nerve complex. Assume that the set of maximal facets of Δ is $A = \{A_1, \dots, A_r\}$. The nerve complex of Δ, denoted by $N(\Delta)$ is the simplicial complex on the vertex set $[r] = \{1, \dots, r\}$ such that $F \subseteq [r]$ is a face of $N(\Delta)$ if and only if

$$\bigcap_{j \in F} A_j \neq \varnothing.$$

Theorem 2.8. *Let Δ be a simplicial complex. Then for all integer i, we have*

$$\widetilde{H}_i(N(\Delta); \mathrm{k}) \cong \widetilde{H}_i(\Delta; \mathrm{k}).$$

Graphs and their edge ideals. Let G denote a finite simple graph over the vertex set $V(G) = [n] = \{1, 2, \dots, n\}$ and the edge set $E(G)$. The edge ideal of G is defined to be

$$I(G) = (x_i x_j \mid \{i, j\} \in E(G)) \subseteq S.$$

For simplicity, we often write $i \in G$ (resp. $ij \in G$) instead of $i \in V(G)$ (resp. $\{i, j\} \in E(G)$). By abuse of notation, we also call x_i a vertex of G and $x_i x_j \in I(G)$ an edge of G.

A path P_n of length $n-1$ is the graph on $[n]$ whose edges are $\{i, i+1\}$ for $i = 1, \dots, n-1$. A cycle C_n of length $n \geq 3$ is the graph on $[n]$ whose edges are $\{i, i+1\}$ for $i = 1, \dots, n-1$ and $\{1, n\}$.

A clique in G is a complete subgraph of G of size at least 2.

A graph H on $[n]$ is called bipartite if there exists a partition $[n] = X \cup Y$, $X \cap Y = \varnothing$ such that $E(H) \subseteq X \times Y$. When $E(H) = X \times Y$, H is called a complete bipartite graph, denoted by $K_{X,Y}$. A bouquet is a complete bipartite graph with $|X| = 1$.

For a vertex $x \in V(G)$, let the neighborhood of x be the subset

$$N_G(x) = \{y \in V(G) \mid \{x, y\} \in E(G)\}$$

and set $N_G[x] = N_G(x) \cup \{x\}$. The degree of a vertex x, denoted by $\deg_G(x)$ is the number of neighbors of x. A leaf is a vertex of degree 1. The unique edge attached to a leaf is called a leaf edge. Denote $d_G(x)$ the number of nonleaf edges incident to x.

Projective dimension of edge ideals of weakly chordal graphs. A graph G is called weakly chordal if G and its complement do not contain an induced cycle of length at least 5. The projective dimension of edge ideals of weakly chordal graphs can be computed via the notion of strongly disjoint families of complete bipartite subgraphs, introduced in [Kimura 2016]. For a graph G, we consider all families of (noninduced) subgraphs $B_1, \dots, B_g$ of G such that:

(1) Each B_i is a complete bipartite graph for $1 \leq i \leq g$.

(2) The graphs $B_1, \dots, B_g$ have pairwise disjoint vertex sets.

(3) There exist an induced matching $e_1, \dots, e_g$ of G for each $e_i \in E(B_i)$ for $1 \leq i \leq g$.

Such a family is termed a strongly disjoint family of complete bipartite subgraphs. We define

$$d(G) = \max\left(\sum_1^g |V(B_i)| - g\right),$$

where the maximum is taken over all the strongly disjoint families of complete bipartite subgraphs $B_1, \dots, B_g$ of G. We have the following result of Nguyen and Vu [Nguyen and Vu 2016, Theorem 7.7].

Theorem 2.9. *Let G be a weakly chordal graph with at least one edge. Then*

$$\operatorname{pd}(S/I(G)) = d(G).$$

We now use it to compute the depth of the edge ideals of whisker graphs of complete graphs.

Lemma 2.10. *Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $W_{\boldsymbol{a}}$ be the graph obtained by gluing a_i leaves to the vertex i of a complete graph K_n. Assume that*

$$a_1 \geq \ldots \geq a_k > 0 = a_{k+1} = \cdots = a_n.$$

Then

$$\operatorname{depth}(S/I(W_{\boldsymbol{a}})) = 1 + a_2 + \cdots + a_k.$$

Proof. From the definition of $W_{\boldsymbol{a}}$, it is clear that $W_{\boldsymbol{a}}$ is a chordal graph. For simplicity of notation, we assume that

$$V(G) = \{x_1, \ldots, x_n\} \cup \{y_{i,j} \mid i = 1, \ldots, k,\ j = 1, \ldots, a_i\},$$
$$E(G) = \{\{x_i, x_j\} \mid i \neq j \in [n]\} \cup \{\{x_i, y_{i,j}\} \mid i = 1, \ldots, k,\ j = 1, \ldots, a_i\}.$$

For any edges e_1, e_2 of $W_{\boldsymbol{a}}$, we have $N_{W_{\boldsymbol{a}}}[e_1] \cap e_2 \neq \varnothing$. Hence, the induced matching number of $W_{\boldsymbol{a}}$ is 1. Now, let B be a complete bipartite subgraph of $W_{\boldsymbol{a}}$ with bipartition $V(B) = U_1 \cup U_2$. Let

$$X = \{x_1, \ldots, x_n\} \quad \text{and} \quad Y = \{y_{i,j} \mid i = 1, \ldots, k,\ j = 1, \ldots, a_i\}.$$

If $V(B) \cap Y = \varnothing$ then $|V(B)| \leq n$. Now, assume that $y_{i,j} \in U_1$ for some i, j. Then $x_i \in U_2$ and $y_{k,l} \notin V(B)$ for any $k \neq i$ since B is a complete bipartite graph. Hence, $|V(B)| \leq n + a_i$. Therefore, for any complete bipartite subgraph B of $W_{\boldsymbol{a}}$, we have

$$|V(B)| \leq n + \max\{a_i \mid i = 1, \ldots, n\} = n + a_1.$$

Furthermore, let $U_1 = \{x_1\}$, $U_2 = \{x_2, \ldots, x_n, y_{1,1}, \ldots, y_{1,a_1}\}$ and $B = K_{U_1,U_2}$ then B is a complete bipartite subgraph of $W_{\boldsymbol{a}}$ with $|V(B)| = n + a_1$. By Theorem 2.9, we deduce that

$$\operatorname{pd}(S/I(W_{\boldsymbol{a}})) = n + a_1 - 1.$$

The conclusion follows from the Auslander–Buchsbaum formula. □

Symbolic powers of edge ideals. Let I be a squarefree monomial ideal in S with the irreducible decomposition

$$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m.$$

The s-th symbolic power of I is defined by

$$I^{(s)} = \mathfrak{p}_1^s \cap \cdots \cap \mathfrak{p}_m^s.$$

By the proof of [Kimura et al. 2018, Theorem 5.2], we have:

Lemma 2.11. *Assume that e is a leaf edge of G. Then for all $s \geq 2$ we have $I(G)^{(s)} : e = I(G)^{(s-1)}$. In particular,* depth $S/I(G)^{(s)}$ *is a nonincreasing function.*

We also have the following simple result that will be used later.

Lemma 2.12. *Assume that $n \geq 2$ be an integer. Let K_n be the complete graph on n vertices. Then $I(K_n)^{(n)} : (x_1 \cdots x_n) = I(K_n)$.*

Proof. For each $i = 1, \dots, n$, let $\mathfrak{p}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Then, we have $I(K_n) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$. Since $x_i \notin \mathfrak{p}_i$, we deduce that $\mathfrak{p}_i^n : (x_1 \cdots x_n) = \mathfrak{p}_i$. Hence,

$$
\begin{aligned}
I(K_n)^{(n)} : (x_1 \cdots x_n) &= (\mathfrak{p}_1^n \cap \cdots \cap \mathfrak{p}_n^n) : (x_1 \cdots x_n) \\
&= (\mathfrak{p}_1^n : (x_1 \cdots x_n)) \cap \cdots \cap (\mathfrak{p}_n^n : (x_1 \cdots x_n)) \\
&= \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n = I(K_n).
\end{aligned}
$$

The conclusion follows. □

3. Stable value of depth of symbolic powers of edge ideals

In this section, we prove that the stable value of depth of symbolic powers of edge ideals is at most the bipartite connectivity number of G. We assume that $S = \mathrm{k}[x_1, \dots, x_n]$ and G is a simple graph on $V(G) = \{1, \dots, n\}$. For an exponent $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, we set $x^{\boldsymbol{a}} = x_1^{a_1} \cdots x_n^{a_n}$ and $|\boldsymbol{a}| = a_1 + \cdots + a_n$.

We first introduce some notation. Let H be a connected bipartite graph with the partition $V(H) = X \cup Y$. The bipartite completion of H, denoted by $\widetilde{H}$ is the complete bipartite graph $K_{X,Y}$. Now, assume that $H = H_1 \cup \cdots \cup H_c \cup \{p_1, \dots, p_t\}$ where $H_1, \dots, H_c$ are connected components of H with at least one edge, and $p_1, \dots, p_t$ are isolated points of H. Then the bipartite completion of H is defined by $\widetilde{H} = \widetilde{H}_1 \cup \cdots \cup \widetilde{H}_c \cup \{p_1, \dots, p_t\}$. We have:

Lemma 3.1. *Let H be a bipartite graph. Let $\boldsymbol{a} = \boldsymbol{d}(H) = (d_H(1), \dots, d_H(n)) \in \mathbb{N}^n$ and $s = \frac{|\boldsymbol{a}|}{2}$. Then*

$$
\sqrt{I(H)^{s+1} : x^{\boldsymbol{a}}} = I(\widetilde{H}),
$$

where $\widetilde{H}$ is the bipartite completion of H.

Proof. Since variables corresponding to isolated points do not appear in $I(H)$, we may assume that H does not have isolated points. Assume that $H = H_1 \cup \cdots \cup H_c$ where H_i are connected components of H with at least one edge. Let $\boldsymbol{a}_i = \boldsymbol{d}(H_i)$. Note that $x^{\boldsymbol{a}_i}$ is equal to the product of nonleaf edges of H_i, hence $|\boldsymbol{a}_i|$ is even for all i. Let $s_i = \frac{|\boldsymbol{a}_i|}{2}$. Now assume that $f \in \sqrt{I(H)^{s+1} : x^{\boldsymbol{a}}}$ with $f = f_1 \cdots f_c$ and $\operatorname{supp} f_i \subseteq V(H_i)$. Then we have $f^m x^{\boldsymbol{a}} \in I(H)^{s+1}$ for some $m > 0$. Thus, we must have $f_i^m x^{\boldsymbol{a}_i} \in I(H_i)^{s_i+1}$ for some i. Hence, we may assume that H is connected. The conclusion then follows from [Trung 2016, Lemma 3.1] and [Minh et al. 2022, Lemma 2.19]. □

Now, assume that H is a maximal induced bipartite subgraph of G, that is, for any $v \in V(G) \setminus V(H)$ the induced subgraph of G on $V(H) \cup \{v\}$ is not bipartite. In particular, H contains at least one edge. Let $H = H_1 \cup \cdots \cup H_c \cup \{p_1, \dots, p_t\}$ where H_i are connected components of H with at least one edge and $p_1, \dots, p_t$ are isolated points of H. Then $c(H) = c + t$ is the number of connected components of H. We have

Lemma 3.2. *Let H be a maximal induced bipartite subgraph of G. Then*

$$\operatorname{depth}(S/(I(G)^{(s)}) \le c(H)$$

for all $s \ge |E(H)| + 1$, where $c(H)$ is the number of connected components of H.

Proof. Assume that $H = H_1 \cup \cdots \cup H_c \cup \{p_1, \dots, p_t\}$ where $H_1, \dots, H_c$ are connected components of H with at least one edge and $p_1, \dots, p_t$ are isolated points of H. Let $\boldsymbol{b} = \boldsymbol{d}(H)$ and $x^{\boldsymbol{a}} = x^{\boldsymbol{b}} \cdot \prod(e \mid e \text{ is a leaf edge of } H)$. Then $x^{\boldsymbol{a}}$ is the product of edges of H. Let $s = \frac{|\boldsymbol{a}|}{2} = |E(H)|$. By [Minh et al. 2022, Corollary 2.7], $x^{\boldsymbol{a}} \notin I(G)^{(s+1)}$. We claim that

$$\sqrt{I(G)^{(s+1)} : x^{\boldsymbol{a}}} = I(\widetilde{H}) + (x_j \mid j \in V(G) \setminus V(H)). \tag{3-1}$$

By Lemma 3.1, it is sufficient if we prove that $x_j \in \sqrt{I(G)^{(s+1)} : x^{\boldsymbol{a}}}$ for all $j \in V(G) \setminus V(H)$. Since the induced subgraph of G on $\{j\} \cup H$ is not bipartite, there must exist a connected component, say H_1 of H such that the induced subgraph of G on $V(H_1) \cup \{j\}$ has an odd cycle. Let G_1 be the induced subgraph of G on $H_1 \cup \{j\}$. Let $j, 1, \dots, 2k$ be an induced odd cycle in G_1. Then $x_j x_1 \cdots x_{2k} \in I(G_1)^{(k+1)}$. Furthermore, $x_1 \cdots x_{2k} = \prod_{j=1}^k e_j$ is a product of k edges of H_1. By the definition of $\boldsymbol{a}$, we have $x^{\boldsymbol{a}_1}$ equals the products of all edges of H_1. In other words, we have $x^{\boldsymbol{a}_1} = x_1 \cdots x_{2k} \cdot h$ with $h \in I(H_1)^{|E(H_1)|-k}$. Hence, $x_j x^{\boldsymbol{a}_1} \in I(G_1)^{(s_1+1)}$ where $s_1 = |E(H_1)|$. Equation (3-1) follows.

By Lemma 2.2 and equation (3-1), we deduce that

$$\operatorname{depth} S/I(G)^{(s+1)} \le \operatorname{depth} S/(I(G)^{(s+1)} : x^{\boldsymbol{a}}) \le \operatorname{depth} S/\sqrt{I(G)^{(s+1)} : x^{\boldsymbol{a}}} = c(H).$$

For any $t \ge s+1$, let $x^{\boldsymbol{c}} = x^{\boldsymbol{a}} \cdot e^{t-s-1}$ where e is an arbitrary edge of H. Then we have $x^{\boldsymbol{c}} \notin I(G)^{(t)}$ and $\sqrt{I(G)^{(t)} : x^{\boldsymbol{c}}} \supseteq \sqrt{I(G)^{(s+1)} : x^{\boldsymbol{a}}}$. Hence, $\operatorname{depth} S/I(G)^{(t)} \le c(H)$ for all $t \ge s+1$. The conclusion follows. $\square$

Definition 3.3. Let G be a simple graph. Denote by $\mathcal{B}(G)$ the set of all maximal induced bipartite subgraphs of G. The bipartite connectivity number of G is defined by

$$\operatorname{bc}(G) = \min\{c(H) \mid H \in \mathcal{B}(G)\}.$$

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. The conclusion follows immediately from the definition and Lemma 3.2. $\square$

We now prove Proposition 1.2 giving an example of connected graphs for which the above inequality is equality and that the limit depth of symbolic powers of $I(G)$ could be any positive number.

Proof of Proposition 1.2. We may assume that

$$V(W_n) = \{x_1, \dots, x_n, y_1, \dots, y_n\} \text{ and}$$
$$E(W_n) = \{\{x_i, x_j\}, \{x_i, y_i\} \mid 1 \le i \ne j \le n\}.$$

Let H be a maximal bipartite subgraph of W_n. Then $y_1, \dots, y_n \in H$ and H contains at most two vertices in $\{x_1, \dots, x_n\}$. By the maximality of H, we deduce that H must be the induced subgraph of W_n on $\{y_1, \dots, y_n\} \cup \{x_i, x_j\}$ for some $i \ne j$. Hence, $c(H) = n - 1$. Thus, $\operatorname{bc}(W_n) = n - 1$.

By Lemma 2.11, depth $S/I(W_n)^{(s)}$ is nonincreasing. Furthermore, we have

$$I(W_n)^{(2)} : (x_1 x_2) = (x_1 y_1, x_2 y_2, x_1 x_2, y_1 y_2, x_3, \dots, x_n).$$

Hence, depth $S/I(W_n)^{(2)} \le n - 1$.

It remains to prove that depth $S/I(W_n)^{(s)} \ge n - 1$ for all $s \ge 2$. We prove by induction on n and s the following statement. Let $I_k = I(K_n) + (x_1 y_1, \dots, x_k y_k)$ and $S_k = \mathrm{k}[x_1, \dots, x_n, y_1, \dots, y_k]$. Then depth $S_k/I_k^{(s)} \ge k - 1$ for all $2 \le k \le n$ and all $s \ge 1$.

Note that $I_k = I(G_k)$ where $G_k = K_n \cup \{\{x_i, y_i\} \mid i = 1, \dots, k\}$. By Lemma 2.10, depth $S_k/I_k = k$.

Since $\mathfrak{m}_k$, the maximal homogeneous ideal of S_k, is not an associated prime of I_k, depth $S_k/I_k^{(s)} \ge 1$ for all k. Thus, we may assume that $s \ge 2$ and $n \ge k \ge 3$. By Lemma 2.3,

$$\operatorname{depth} S_k/I_k^{(s)} \in \{\operatorname{depth}(S_k/(I_k^{(s)}, x_k y_k)), \operatorname{depth}(S_k/I_k^{(s)} : x_k y_k)\}.$$

By Lemma 2.11, $I_k^{(s)} : x_k y_k = I_k^{(s-1)}$. Thus, by induction, it suffices to prove that

$$\operatorname{depth} S_k/(I_k^{(s)}, x_k y_k) \ge k - 1.$$

We have $J = (I_k^{(s)}, x_k y_k) = (J, x_k) \cap (J, y_k)$. The conclusion follows from induction on k and Lemma 2.1. □

The inequality in Theorem 1.1 might be strict. We will now define a finer invariant of G which we conjecture to be equal to the stable value of depth of symbolic powers of $I(G)$. Let $H = H_1 \cup \dots \cup H_c \cup \{p_1, \dots, p_t\}$ be a maximal induced bipartite subgraph of G where $H_1, \dots, H_c$ are connected components of H with at least one edge and $p_1, \dots, p_t$ are isolated points. We say that $\{p_{i_1}, \dots, p_{i_u}\}$ are clustered if there exists $v \in V(G) \setminus V(H)$ such that the induced subgraph of G on $\{v, p_{i_1}, \dots, p_{i_u}\}$ is a bouquet. Let $\operatorname{bou}_G(H)$ be the smallest number b

such that the set $\{p_1, \dots, p_t\}$ can be clustered into b bouquets in G. We call $c'(H) = c + \mathrm{bou}_G(H)$ the number of restricted connected components of H.

Definition 3.4. Let G be a simple graph. The restricted bipartite connectivity number of G is defined by

$$\mathrm{bc}'(G) = \min\{c'(H) \mid H \in \mathcal{B}(G)\}.$$

We need a preparation lemma to prove Theorem 1.5.

Lemma 3.5. *Let $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ be such that $a_i \geq 1$ for all $i = 1, \dots, n$. Let $W_{\boldsymbol{a}}$ be a graph whose vertex set and edge set are*

$$V(W_{\boldsymbol{a}}) = \{x_1, \dots, x_n, y_{1,1}, \dots, y_{1,a_1}, \dots, y_{n,1}, \dots, y_{n,a_n}\},$$
$$E(W_{\boldsymbol{a}}) = \{\{x_i, x_j\}, \{x_i, y_{i,\ell}\} \mid \textit{for all } i, j, \ell \textit{ such that } 1 \leq i \neq j \leq n,\ 1 \leq \ell \leq a_i\}.$$

Then

$$I(W_{\boldsymbol{a}})^{(n)} : (x_1 \cdots x_n) = I(W_{\boldsymbol{a}}) + (y_{1,1}, \dots, y_{1,a_1})(y_{2,1}, \dots, y_{2,a_2}) \cdots (y_{n,1}, \dots, y_{n,a_n}).$$

Proof. For simplicity of notation, we set

$$X = \{x_1, \dots, x_n\} \quad \text{and} \quad Y = \{y_{i,j} \mid i = 1, \dots, n,\ j = 1, \dots, a_i\}.$$

We also denote $I = I(W_{\boldsymbol{a}})$ and

$$J = I(W_{\boldsymbol{a}}) + (y_{1,1}, \dots, y_{1,a_1})(y_{2,1}, \dots, y_{2,a_2}) \cdots (y_{n,1}, \dots, y_{n,a_n}).$$

For each $C \subseteq V(W_{\boldsymbol{a}})$, let $m_C = \prod_{x \in C} x$ be a monomial in

$$S = \mathrm{k}[x_1, \dots, x_n, y_{1,1}, \dots, y_{1,a_1}, \dots, y_{n,1}, \dots, y_{n,a_n}].$$

Since $W_{\boldsymbol{a}}$ is a chordal graph, by [Sullivant 2008, Theorem 3.10], we have

$$I^{(n)} = \left(m_{C_1} \cdots m_{C_t} \mid C_1, \dots, C_t \text{ are cliques of } W_{\boldsymbol{a}} \text{ and } \sum_{i=1}^{t} (|C_i| - 1) = n\right). \tag{3-2}$$

The cliques $C_1, \dots, C_t$ are not necessarily distinct. In $W_{\boldsymbol{a}}$, $C \subseteq V(W_{\boldsymbol{a}})$ is a clique if and only if either $C = \{x_i, y_{i,j}\}$ for some $i = 1, \dots, t$ and $j = 1, \dots, a_i$ or $C \subseteq X$. In particular, $(x_1 \cdots x_n)e \in I^{(n)}$ for all edges e of $W_{\boldsymbol{a}}$ and $(x_1 y_{1,j_1}) \cdots (x_n y_{n,j_n}) \in I^{(n)}$ for all $j_1, \dots, j_n$ such that $1 \leq j_\ell \leq a_\ell$. Hence,

$$J \subseteq I(W_{\boldsymbol{a}})^{(n)} : (x_1 \cdots x_n). \tag{3-3}$$

We now prove by induction on n the reverse containment

$$I(W_{\boldsymbol{a}})^{(n)} : (x_1 \cdots x_n) \subseteq J. \tag{3-4}$$

The base case $n = 2$ is clear. Thus, assume that $n \geq 3$. Let $C_1, \ldots, C_t$ be cliques of $W_{\boldsymbol{a}}$ such that $\sum_{i=1}^{t}(|C_i| - 1) = n$. Let $M = m_{C_1} \cdots m_{C_t}$ and $f = x_1 \cdots x_n$. It suffices to prove that $M/\gcd(M, f) \in J$. Since $|C_i| \leq n$ for all $i = 1, \ldots, t$, we must have $t \geq 2$. We have two cases.

Case 1. $C_i \cap Y = \varnothing$ for all $i = 1, \ldots, t$. In this case, we have $M \in I(K_n)^{(n)}$. By Lemma 2.12, we deduce that $M/\gcd(M, f) \in I(K_n) \subseteq I(W_{\boldsymbol{a}})$.

Case 2. $C_i \cap Y \neq \varnothing$ for some $i \in \{1, \ldots, t\}$. Since Y is the set of leaves, we deduce that $|C_i| = 2$. For simplicity, we assume that $C_1 = \{x_1, y_{1,1}\}$. If there exists a clique C_i for some $i = 2, \ldots, t$ such that $C_1 \cap C_i \neq \varnothing$, then we must have $x_1 \in C_1 \cap C_i$. In particular, we deduce that $x_1 y_{1,1} \mid M/\gcd(M, f)$. Hence, $M/\gcd(M, f) \in J$. Thus, we may assume that $C_i \cap C_1 = \varnothing$ for all $i = 2, \ldots, t$. In other words, $C_i \subseteq X' \cup Y'$ where

$$X' = \{x_2, \ldots, x_n\} \quad \text{and} \quad Y' = \{y_{i,j} \mid i = 2, \ldots, n,\ j = 1, \ldots, a_i\}.$$

Furthermore, we have $\sum_{i=2}^{t}(|C_i| - 1) = n - 1$. By equation (3-2), we deduce that $M' = m_{C_2} \cdots m_{C_t} \in I(W_{\boldsymbol{a}'})^{(n-1)}$, where $\boldsymbol{a}' = (a_2, \ldots, a_n)$ and $W_{\boldsymbol{a}'}$ is the whisker graph obtained by gluing a_i leaves to the vertex i of the complete graph on $\{2, \ldots, n\}$. Since $y_{1,1}$ does not divide f, we deduce that $y_{1,1}(M'/\gcd(M', f')) \mid M/\gcd(M, f)$, where $f' = x_2 \cdots x_n$. By induction on n, the conclusion follows. □

Proof of Theorem 1.5. We may assume that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$. We keep the notations as in Lemma 3.5.

For ease of reading, we divide the proof into several steps.

Step 1. $\operatorname{bc}'(W_{\boldsymbol{a}}) = n - 1$. As in the proof of Proposition 1.2, we deduce that a maximal induced bipartite subgraph H of $W_{\boldsymbol{a}}$ is an induced subgraph of $W_{\boldsymbol{a}}$ on $Y \cup \{x_i, x_j\}$ for some $i \neq j$. For such H, we have $c(H) = |\boldsymbol{a}| - (a_i + a_j) + 1$ but $c'(H) = n - 1$ as $\{y_{\ell,1}, \ldots, y_{\ell,a_\ell}\}$ can be clustered into a bouquet in G for all $\ell = 1, \ldots, n$. Thus, $\operatorname{bc}(W_{\boldsymbol{a}}) = a_3 + \cdots + a_n + 1$ and $\operatorname{bc}'(W_{\boldsymbol{a}}) = n - 1$.

Step 2. $\operatorname{depth} S/I(W_{\boldsymbol{a}})^{(s)} \geq n - 1$ for all $s \geq 1$ and all $\boldsymbol{a}$ such that $a_i \geq 1$ for $i = 1, \ldots, n$.

First, assume that $s = 1$. By Lemma 2.10, $\operatorname{depth} S/I(W_{\boldsymbol{a}}) = a_2 + \cdots + a_n + 1$. When $a_1 = \cdots = a_n = 1$, the conclusion follows from Proposition 1.2. Thus, we may assume that $s \geq 2$ and $a_1 \geq 2$. By induction, Lemmas 2.3 and 2.11, it suffices to prove that

$$\operatorname{depth} S/(I(W_{\boldsymbol{a}})^{(s)}, x_1 y_{1,a_1}) \geq n - 1.$$

Let $J = I(W_{\boldsymbol{a}})^{(s)}$. Then $(J, x_1 y_{1,a_1}) = (J, x_1) \cap (J, y_{1,a_1})$. Let $\boldsymbol{a}' = (a_2, \ldots, a_n)$ and $W_{\boldsymbol{a}'}$ the whisker graph obtained by gluing a_i leaves to the vertex i of the complete graph on $\{2, \ldots, n\}$. We have

$$(J, x_1) = (I(W_{\boldsymbol{a}'})^{(s)}, x_1) \quad \text{and} \quad (J, x_1, y_{1,a_1}) = (I(W_{\boldsymbol{a}'})^{(s)}, x_1, y_{1,a_1}).$$

By Lemma 2.4,

$$\operatorname{depth} S/(J, x_1) = a_1 + \operatorname{depth} R/I(W_{\boldsymbol{a}'})^{(s)},$$
$$\operatorname{depth} S/(J, x_1, y_{1,a_1}) = a_1 - 1 + \operatorname{depth} R/I(W_{\boldsymbol{a}'})^{(s)},$$

where $R = \mathrm{k}[x_2, \dots, x_n, y_{2,1}, \dots, y_{2,a_2}, \dots, y_{n,1}, \dots, y_{n,a_n}]$. By induction, both terms are at least $n-1$. Finally, we have

$$(J, y_{1,a_1}) = (I(W_{\boldsymbol{a}''})^{(s)}, y_{1,a_1}),$$

where $\boldsymbol{a}'' = (a_1 - 1, a_2, \dots, a_n)$. Hence,

$$\operatorname{depth} S/(J, y_{1,a_1}) = \operatorname{depth} T/I(W_{\boldsymbol{a}''})^{(s)},$$

where $T = \mathrm{k}[x_1, \dots, x_n, y_{1,1}, \dots, y_{1,a_1-1}, \dots, y_{n,1}, \dots, y_{n,a_n}]$. Thus, the conclusion of Step 2 follows from induction and Lemma 2.1.

Step 3. $\operatorname{depth} S/I(W_{\boldsymbol{a}})^{(s)} \le n-1$ for all $s \ge n$.

By Lemmas 2.2 and 2.11, it suffices to prove that

$$\operatorname{depth} S/I(W_{\boldsymbol{a}})^{(n)} : (x_1 \cdots x_n) \le n-1.$$

Let $J = I(W_{\boldsymbol{a}})^{(n)} : (x_1 \cdots x_n)$. By Lemma 3.5, we have that

$$J = I(W_{\boldsymbol{a}}) + (y_{1,1}, \dots, y_{1,a_1})(y_{2,1}, \dots, y_{2,a_2}) \cdots (y_{n,1}, \dots, y_{n,a_n}).$$

Therefore, the Stanley–Reisner complex $\Delta(J)$ of J has exactly n facets

$$F_i = \{x_i\} \cup \{y_{j,\ell} \mid j \ne i,\ \ell = 1, \dots, a_j\}.$$

Hence, $F_1 \cap \cdots \cap F_n = \varnothing$ and for any j, we have

$$F_1 \cap \cdots \cap F_{j-1} \cap F_{j+1} \cap \cdots \cap F_n = \{y_{j,1}, \dots, y_{j,a_j}\}.$$

Therefore, the nerve complex of $\Delta(J)$ is isomorphic to the $n-2$-sphere. By Theorem 2.8, $\widetilde{H}_{n-2}(\Delta(J); \mathrm{k}) \ne 0$. By Lemma 2.7, the conclusion follows. □

Remark 3.6. (1) The notion of maximal bipartite subgraphs of a graph has been studied by many researchers as early as in [Erdős 1965; Malle 1982]. They are interested in finding the maximum number of edges of a maximal bipartite subgraph of G.

(2) In general, the problem of finding a maximum induced bipartite subgraph of a graph is NP-complete [Lewis and Yannakakis 1980]. Nonetheless, we do not know if the problem of computing the bipartite connectivity number or restricted bipartite connectivity number is NP-complete.

Remark 3.7. (1) The Cohen–Macaulay property, or depth of the edge ideal of a graph might depend on the characteristic of the base field. For example, consider the following ideal in [Villarreal 2015, Exercise 5.3.31]:

$$\begin{aligned}I = (&x_1x_3, x_1x_4, x_1x_7, x_1x_{10}, x_1x_{11}, x_2x_4, x_2x_5, x_2x_8, x_2x_{10}, x_2x_{11},\\ &x_3x_5, x_3x_6, x_3x_8, x_3x_{11}, x_4x_6, x_4x_9, x_4x_{11},\\ &x_5x_7, x_5x_9, x_5x_{11}, x_6x_8, x_6x_9, x_7x_9, x_7x_{10}, x_8x_{10}).\end{aligned}$$

Then

$$\operatorname{depth} S/I = \begin{cases} 2 & \text{if } \operatorname{char} \mathrm{k} = 2,\\ 3, & \text{otherwise.}\end{cases}$$

But depth $S/I^{(s)} = 1$ for all $s \geq 2$, regardless of the characteristic of the base field k.

(2) By the result of Trung [2016], the stable value of depth of powers of edge ideals of graphs does not depend on the characteristic of the base field k. If Conjecture 1.4 holds, the stable value of depth of symbolic powers of edge ideals also does not depend on the characteristic of the base field k. This is in contrast to the asymptotic behavior of the regularity of (symbolic) powers of edge ideals as [Minh and Vu 2022, Corollary 5.3] shows that the linearity constant of the regularity function of (symbolic) powers of edge ideals of graphs might depend on the characteristic of the base field k.

4. Depth of symbolic powers of edge ideals of cycles

In this section, we compute the depth of symbolic powers of edge ideals of cycles. The purpose of this is twofold. First, together with Proposition 1.2, this gives the first classes of nonbipartite graphs where one computes explicitly the depth of symbolic powers of their edge ideals. Second, this shows that the stabilization index of depth of symbolic powers of G is tightly connected to the stabilization index of depth of powers of maximal induced bipartite subgraphs of G.

We fix the following notation. Let $S = \mathrm{k}[x_1, \dots, x_n]$ and C_n be a cycle of length n. For each $i = 1, \dots, n-1$, we denote $e_i = x_ix_{i+1}$. Let

$$\varphi(n,t) = \left\lceil \frac{n-t+1}{3} \right\rceil.$$

We recall the following results (Lemmas 3.4, 3.10, 3.11, and Theorem 1.1) from [Minh et al. 2023].

Lemma 4.1. *Let H be any subgraph of P_n. Then, for any positive integer t with $t < n$, we have that*

$$\operatorname{depth}\big(S/(I(P_n)^t + I(H))\big) \geq \varphi(n,t).$$

Lemma 4.2. *Let H be a nonempty subgraph of C_n. Then for $t \geq 2$, we have that*

$$\operatorname{depth}\big(S/(I(C_n)^t + I(H))\big) \geq \varphi(n,t).$$

Lemma 4.3. *Assume that $I = I(C_n)$ and $t \le n-2$. Then*

$$\operatorname{depth}\big(S/(I^t : (e_2 \cdots e_t))\big) \le \varphi(n,t).$$

Theorem 4.4. *Let $I(C_n)$ be the edge ideal of a cycle of length $n \ge 5$. Then*

$$\operatorname{depth}(S/I(C_n)^t) = \begin{cases} \lceil \frac{n-1}{3} \rceil & \text{if } t = 1, \\ \lceil \frac{n-t+1}{3} \rceil & \text{if } 2 \le t < \lceil \frac{n+1}{2} \rceil, \\ 1 & \text{if } n \text{ is even and } t \ge \frac{n}{2}+1, \\ 0 & \text{if } n \text{ is odd and } t \ge \frac{n+1}{2}. \end{cases}$$

Now, assume that $n = 2k+1$ where $k \ge 2$ is a positive integer. For a positive integer $s \in \mathbb{N}$, we write $s = a(k+1)+b$ for some $a, b \in \mathbb{N}$ and $0 \le b \le k$. Let $f = x_1 \cdots x_n$. By [Gu et al. 2020, Theorem 3.4], we have

$$I(C_n)^{(s)} = \sum_{j=0}^{a} I(C_n)^{s-j(k+1)} f^j. \tag{4-1}$$

We now establish some preparation results.

Lemma 4.5. *Assume that $I = I(C_n)$, $e_i = x_i x_{i+1}$ for all $i = 1, \dots, n-1$. Then for all $s \le n-2$, we have*

$$\operatorname{depth} S/I^{(s)} \le \operatorname{depth} S/(I^{(s)} : e_2 \cdots e_{s-1}) \le \varphi(n,s).$$

Proof. Let $f = x_1 \cdots x_n$. By (4-1), we have that $I^{(s)} = I^s$ when $s \le k$. Now, assume that $k+1 \le s \le n-2 = 2k-1$. By (4-1), we have that

$$I^{(s)} = I^s + f I^{s-k-1}.$$

Since $f/\gcd(f, e_2 \cdots e_{s-1}) \in I \subseteq I^s : (e_2 \cdots e_{s-1})$, we deduce that

$$I^{(s)} : (e_2 \cdots e_{s-1}) = I^s : (e_2 \cdots e_{s-1}).$$

The conclusion follows from Lemma 4.3. □

Lemma 4.6. *Let $f = x_1 \cdots x_n$. Then for all integer s such that $k+1 \le s \le n-2$,*

$$I^{(s)} : f = I^{s-k-1}.$$

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the associated primes of I. Then

$$I^{(s)} = \mathfrak{p}_1^s \cap \cdots \cap \mathfrak{p}_t^s.$$

Since $\mathfrak{p}_i$ is generated by $k+1$ variables for all $i = 1, \dots, t$, we have $\mathfrak{p}_i^s : f = \mathfrak{p}_i^{s-k-1}$. Hence, $I^{(s)} : f = I^{(s-k-1)} = I^{s-k-1}$ since $s \le 2k-1$. □

We are now ready for the proof of Theorem 1.6.

Proof of Theorem 1.6. By (4-1) and Theorem 4.4, it remains to consider the cases where $k+1 \le s \le 2k-1$. Let $f = x_1 \cdots x_n$. By Lemmas 2.3, 4.5, 4.6, and Theorem 4.4, it suffices to prove that

$$\operatorname{depth}(S/(I^{(s)} + f)) \ge \varphi(n, s). \tag{4-2}$$

Write $f = e_1 f_1$ where $f_1 = x_3 \cdots x_n$. We have $I^{(s)} + f = (I^{(s)}, e_1) \cap (I^{(s)}, f_1)$. For each $i = 1, \dots, k-1$, we can write $f_i = e_{2i+1} f_{i+1}$. By repeated use of Lemma 2.1 and the fact that for any subgraph H of C_n we have

$$I^{(s)} + I(H) + f_i = (I^{(s)} + I(H) + (e_{2i+1})) \cap (I^{(s)} + I(H) + (f_{i+1})).$$

It suffices to prove the following two claims.

Claim 1. *For any nonempty subgraph H of C_n, we have*

$$\operatorname{depth} S/(I^{(s)} + I(H)) \ge \varphi(n, s).$$

Claim 2. *For any (possibly empty) subgraph H of C_n, we have*

$$\operatorname{depth}\big(S/(I^{(s)} + I(H) + (x_{n-2}x_{n-1}x_n))\big) \ge \varphi(n, s).$$

Proof of Claim 1. Since $k+1 \le s \le 2k-1$, by (4-1), we have that

$$I^{(s)} = I^s + f I^{s-k-1}.$$

For any nonempty subgraph H of C_n, we have $f \in I(H)$. Therefore, we have $I^{(s)} + I(H) = I^s + I(H)$. The conclusion follows from Lemma 4.2.

Proof of Claim 2. Let $J = I^{(s)} + I(H) + (x_{n-2}x_{n-1}x_n)$ and $e = x_{n-2}x_{n-1}$. Note that $J + (e)$ can be expressed as $I^{(s)} + I(H_1)$ for some subgraph H_1 of C_n and $J : e = I(P_{n-1})^{s-1} + I(H') + (x_n)$ where H' is a subgraph of P_{n-1}. The claim follows from Lemma 2.3, Claim 1, and Lemma 4.1. The conclusion follows. $\square$

Remark 4.7. For cycles C_{2k} of even length, by the result of Simis, Vasconcelos, and Villarreal [1994], $I(C_{2k})^{(s)} = I(C_{2k})^s$ for all $s \ge 1$. The depth of powers of the edge ideal of C_{2k} has been computed in [Minh et al. 2023, Theorem 1.1].

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COLLAPSED LIMITS OF COMPACT HEISENBERG MANIFOLDS WITH SUB-RIEMANNIAN METRICS

KENSHIRO TASHIRO

We show that every collapsed Gromov–Hausdorff limit of compact Heisenberg manifolds endowed with left-invariant Riemannian/sub-Riemannian metrics is isometric to a flat torus. We say that a sequence of sub-Riemannian manifolds collapses if their total measure with respect to Popp's volume converges to zero.

1. Introduction

A sub-Riemannian manifold is a triple $(M, \mathcal{D}, g)$, where M is a smooth manifold, $\mathcal{D}$ is a subbundle of the tangent bundle, and g is a metric on $\mathcal{D}$. In the same way to Riemannian manifolds, we can put a length structure and the associated distance function on bracket generating sub-Riemannian manifolds (see Definition 2.2). Sub-Riemannian manifolds appear as Gromov–Hausdorff limits of sequences of Riemannian manifolds. In general their sectional, Ricci and scalar curvature diverge as they converge to (non-Riemannian) sub-Riemannian manifolds. However some sub-Riemannian manifolds have the measure contraction property which reflects the Ricci curvature lower bound in a sense [Juillet 2009; Rifford 2013; Rizzi 2016; Barilari and Rizzi 2018]. These results lead us to study sub-Riemannian manifolds as examples of the singular Gromov–Hausdorff limit spaces.

In [Tashiro 2020], the author began to study the topological type of the Gromov–Hausdorff limit space of a sequence of (sub-)Riemannian manifolds. Here we use the notation (sub-)Riemannian metrics to cover both Riemannian and (non-Riemannian) sub-Riemannian metrics. Let H_n be the n-Heisenberg Lie group, $\mathfrak{h}_n$ the associated Lie algebra, and Γ a lattice in H_n. A quotient space $\Gamma\backslash\mathrm{H}_n$ is called a *compact Heisenberg manifold*. Let $\mathfrak{v}$ be a subspace in $\mathfrak{h}_n$ and $\langle\cdot,\cdot\rangle$ a scalar product on $\mathfrak{v}$. It induces the left-invariant sub-Riemannian structure on H_n. Since the induced geodesic distance on H_n has the isometric action Γ from the left, we obtain a quotient distance on $\Gamma\backslash\mathrm{H}_n$. We also call such a quotient distance on $\Gamma\backslash\mathrm{H}_n$ *left-invariant*. The author studied noncollapsed limits of compact Heisenberg manifolds with left-invariant (sub-)Riemannian metrics. Here we say that a sequence is *noncollapsed* if

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the total measure with respect to Popp's volume have a strictly positive lower bound. Popp's volume is a generalization of a canonical volume form of a Riemannian manifold (see Section 2C). We showed that the noncollapsed limit of a sequence of compact Heisenberg manifolds with left-invariant (sub-)Riemannian metrics are again diffeomorphic to a compact Heisenberg manifold of the same dimension.

In this paper, we study collapsed Gromov–Hausdorff limits of compact Heisenberg manifolds with left-invariant (sub-)Riemannian metrics. We say that a sequence of (sub-)Riemannian manifolds *collapses* if the total measure with respect to Popp's volume converges to zero. It complements our previous result [Tashiro 2020].

Theorem 1.1 (Main result). *Let $\{(\Gamma_k\backslash \mathrm{H}_n, \overline{\mathrm{dist}_k})\}_{k\in\mathbb{N}}$ be a sequence of compact Heisenberg manifolds endowed with left-invariant (sub-)Riemannian metrics. Assume that this sequence converges in the Gromov–Hausdorff topology with a diameter upper bound $D > 0$ and the total measure with respect to Popp's measure converges to zero. Then the limit space is isometric to a flat torus of lower dimension.*

The idea of the proof is the following. It is well known that a compact Heisenberg manifold has a circle bundle structure $S^1 \to \Gamma\backslash \mathrm{H}_n \to \mathbb{T}^{2n}$. We show that if a sequence collapses, then the circle fiber also collapses to a point. Once we show that the fibers collapse, then the Gromov–Hausdorff limit is isometric to the limit of the base tori with the quotient distances. It is also known that a Gromov–Hausdorff limit of tori with flat metrics is isometric to a flat torus [Bettiol et al. 2018, Proposition 3.1]. This concludes the theorem.

2. Preliminaries from sub-Riemannian Lie group

In this section we prepare notation on sub-Riemannian metrics on Lie groups.

2A. ***Sub-Riemannian structure.*** Let G be a connected Lie group, $\mathfrak{g}$ the associated Lie algebra, $\mathfrak{v} \subset \mathfrak{g}$ a subspace and $\langle\cdot,\cdot\rangle$ a scalar product on $\mathfrak{v}$. For $x \in G$, denote by $L_x : G \to G$ the left translation by x. Define a sub-Riemannian metric on G by

$$\mathcal{D}_x = L_{x*}\mathfrak{v}, \quad g_x(u, v) = \langle L_{x*}^{-1}u, L_{x*}^{-1}v\rangle.$$

Such a sub-Riemannian metric $(\mathcal{D}, g)$ is called *left-invariant*. We sometimes write a left-invariant sub-Riemannian metric by $(\mathfrak{v}, \langle\cdot,\cdot\rangle)$. Moreover, if $\dim(\mathfrak{g}/\mathfrak{v}) = k$, we say that a sub-Riemannian metric $(\mathfrak{v}, \langle\cdot,\cdot\rangle)$ is corank k. Notice that if $\mathfrak{v} = \mathfrak{g}$, i.e., corank 0, then $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is a Riemannian metric.

Remark 2.1. From now on we shall declare the corank of sub-Riemannian metrics. If we do not declare the corank, then the word "sub-Riemannian metric" cover sub-Riemannian metrics of all corank.

For simplicity, we shall consider a Lie group with a left-invariant sub-Riemannian metric $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$. The associated distance function is given as follows. We say that an absolutely continuous path $c : [0, 1] \to G$ is *admissible* if $\dot{c}(t) \in L_{c(t)*}\mathfrak{v}$ a.e. $t \in [0, 1]$. We define the length of an admissible path by

$$\text{length}(c) = \int_0^1 \sqrt{\langle \dot{c}(t), \dot{c}(t)\rangle}\, dt.$$

For $x, y \in G$, define the distance function by

$$\text{dist}(x, y) = \inf\{\text{length}(c) \mid c(0) = x,\ c(1) = y,\ c \text{ is admissible}\}.$$

In general not every pair of points in G is joined by an admissible path. This implies that the value of the function dist may be the infinity. The following *bracket generating* condition ensures that any two points are joined by an admissible path.

Definition 2.2 (bracket generating distribution). For a sub-Riemannian Lie group $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$ and an integer $i \in \mathbb{N}$, let $\mathfrak{v}^i$ be the subspace in $\mathfrak{g}$ inductively defined by

$$\mathfrak{v}^1 = \mathfrak{v}, \quad \mathfrak{v}^{i+1} = \mathfrak{v} + [\mathfrak{v}, \mathfrak{v}^i].$$

We say that a subspace $\mathfrak{v}$ is bracket generating if there is $r \in \mathbb{N}$ such that $\mathfrak{v}^r = \mathfrak{g}$. We say $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$ is r-step if $\mathfrak{v}^{r-1} \subsetneq \mathfrak{v}^r = \mathfrak{g}$.

Theorem 2.3 (See, e.g., Theorem 3.31 in [Agrachev et al. 2020]). *Let* $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$ *be a sub-Riemannian Lie group with a bracket generating distribution. Then the following two assertions hold*:

(1) (G, dist) *is a metric space.*

(2) *The topology induced by* dist *is equivalent to the manifold topology.*

In particular, $\text{dist} : G \times G \to \mathbb{R}$ *is continuous.*

Remark 2.4. Since the sub-Riemannian structure is left-invariant, the distance function is also left-invariant, that is, $\text{dist}(hx, hy) = \text{dist}(x, y)$ for all $h, x, y \in G$.

2B. ***Length minimizer.*** In sub-Riemannian geometry, there are two types of length minimizers; normal geodesics and abnormal geodesics. Normal geodesics are characterized as solutions to a specific differential equation, called the Hamiltonian equation. On the other hand, abnormal geodesics are not solutions to that equation. It sometimes appear in sub-Riemannian geometry, however, it is known that there is no nontrivial (i.e., nonconstant) abnormal geodesic if $\mathfrak{v}$ is *fat* (see [Montgomery 2002]). Here we say that a bracket generating subspace $\mathfrak{v} \subset \mathfrak{h}_n$ is fat if for all $U \in \mathfrak{v} \setminus \{0\}$, we have $\mathfrak{v} + [U, \mathfrak{v}] = \mathfrak{g}$. In the next section, we shall check that if G is the Heisenberg group, then every bracket generating subspace is fat. Therefore we omit the explanation of abnormal geodesics.

We say that a basis $\{U_1, \dots, U_n\}$ of $\mathfrak{g}$ is adapted if $\{U_1, \dots, U_m\}$ is an orthonormal basis of a sub-Riemannian metric $(\mathfrak{v}, \langle \cdot, \cdot \rangle)$. Let $H : T^*G \to \mathbb{R}$ be the function defined by

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^{m} p(L_{x*}U_i)^2 \quad (\lambda = (x, p) \in T^*G).$$

This function is called the *sub-Riemannian Hamiltonian.*

We say that a Lipschitz curve $\lambda = (x, p) : [0, T] \to T^*G$ is a solution to the Hamiltonian equation if it satisfies

$$\dot{x}(t) = \frac{\partial H}{\partial p}, \quad \dot{p}(t) = -\frac{\partial H}{\partial x}. \tag{1}$$

Such a curve $\lambda(t)$ is called a *normal extremal*, and its projection $x(t)$ is called a *normal geodesic*. It is known that every minimizer in sub-Riemannian manifold is either normal or abnormal geodesic. In particular, if a subspace $\mathfrak{v}$ is fat, then any length minimizer is a normal geodesic.

2C. *Popp's volume.* On a Riemannian Lie group (G, g), one has a canonical volume form defined by

$$\mathrm{dvol}_R = \nu_1 \wedge \cdots \wedge \nu_n,$$

where $\{\nu_1, \cdots \nu_n\}$ is a dual coframe of an orthonormal basis. The induced measure $m(\Omega) := \left|\int_\Omega \mathrm{dvol}_R\right|$ $(\Omega \subset G)$ is called the *volume measure*.

In sub-Riemannian geometry, we also have a canonical volume form, called *Popp's volume* introduced in [Montgomery 2002]. For simplicity, we only consider the 2-step case.

We do not introduce the original definition of Popp's volume, however, we define it with local coordinates given in [Barilari and Rizzi 2013]. Let $U_1, \dots, U_n$ be an adapted frame. Define the constant c_{ij}^l by

$$[U_i, U_j] = \sum_{l=1}^{n} c_{ij}^l U_l.$$

We call them the *structure constants*. We define the $(n-m)$ square matrix B by

$$B_{hl} = \sum_{i,j=1}^{m} c_{ij}^h c_{ij}^l.$$

Theorem 2.5 [Barilari and Rizzi 2013, Theorem 1]. *Let $U_1, \dots, U_n$ be a local adapted frame, and $\nu^1, \dots, \nu^n$ the dual coframe. Then Popp's volume* dvol_{sR} *is locally written by*

$$\mathrm{dvol}_{sR} = (\det B)^{-\frac{1}{2}} \nu^1 \wedge \cdots \wedge \nu^n.$$

The induced measure $m(\Omega) := \left|\int_\Omega \mathrm{dvol}_{sR}\right|$ $(\Omega \subset G)$ is called Popp's measure.

Remark 2.6. If a sub-Riemannian metric is corank 0, i.e., 1-step, then Popp's volume coincides with the canonical volume form. Indeed, an adapted frame of corank 0 sub-Riemannian metric is an orthonormal basis.

3. Compact Heisenberg manifolds

In this section, we recall fundamental properties on compact Heisenberg manifolds.

3A. ***Heisenberg groups.*** For $n \in \mathbb{N}$, the n-Heisenberg group H_n is the $(2n+1)$-dimensional Lie group diffeomorphic to $\mathbb{C}^n \times \mathbb{R}$ with the group product law

$$(w, z)(w', z') = \big(w + w', z + z' + \tfrac{1}{2}\mathfrak{Im}(w \cdot w')\big),$$

where $w \cdot w'$ is the Hermitian product on $\mathbb{C}^n$ and $\mathfrak{Im}$ denotes the imaginary part. We shall denote the associated Lie algebra by $\mathfrak{h}_n$.

We fix the coordinates of $\mathrm{H}_n \simeq \mathbb{C}^n \times \mathbb{R}$ by

$$(w, z) = (x_1, \dots, x_n, y_1, \dots, y_n, z),$$

where $w = \vec{x} + \vec{y}\sqrt{-1}$. We also fix the basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$ of the Lie algebra $\mathfrak{h}_n$ by

$$X_i = \partial_{x_i} - \tfrac{1}{2} y_i \partial_z, \quad Y_i = \partial_{y_i} + \tfrac{1}{2} x_i \partial_z, \quad Z = \partial_z.$$

A straightforward computation shows that $[X_i, Y_i] = Z$ for all $i = 1, \dots, n$ and the other brackets are zero.

For $U \in \mathfrak{h}_n$, let $\phi_U^t : \mathrm{H}_n \to \mathrm{H}_n$ be the flow of the vector field U at time t. The exponential map $\exp : \mathfrak{h}_n \to \mathrm{H}_n$ is defined by $\exp(U) := \phi_U^1(e)$, where e is the identity element. It is well defined since a left-invariant vector field is complete. It is well known that the exponential map on the Heisenberg group is a diffeomorphism. This fact allows us to identify the Heisenberg group H_n to its Lie algebra $\mathfrak{h}_n$ by

$$\exp : \mathfrak{h}_n \ni \sum_{i=1}^{n} (x_i X_i + y_i Y_i) + zZ \overset{\sim}{\longmapsto} (x_1, \dots, x_n, y_1, \dots, y_n, z) \in \mathrm{H}_n.$$

Let $(\mathfrak{v}, \langle \cdot, \cdot \rangle)$ be a left-invariant sub-Riemannian metric on H_n. A subspace $\mathfrak{v}$ is bracket generating if and only if

$$\mathfrak{v} + \mathrm{Span}(Z) = \mathfrak{h}_n. \tag{2}$$

In particular, the corank of a bracket generating subspace is 0 or 1. From now on we always assume the bracket generating condition (2). Moreover, by (2), we can easily check that if a subspace $\mathfrak{v} \subset \mathfrak{h}_n$ satisfies bracket generating condition, then it is fat. Therefore the sub-Riemannian Heisenberg group does not have nontrivial abnormal minimizers.

Let $\Gamma < \mathrm{H}_n$ be a lattice in H_n, that is, a discrete cocompact subgroup. Since a sub-Riemannian metric $(\mathfrak{v}, \langle\cdot,\cdot\rangle)$ is left-invariant, the left multiplication by Γ induces an isometric action on H_n. Therefore we can define the sub-Riemannian metric on $\Gamma\backslash\mathrm{H}_n$ via the quotient map. We shall denote such a quotient sub-Riemannian metric on $\Gamma\backslash\mathrm{H}_n$ by $\overline{\mathrm{dist}}$.

3B. ***Isometry classes of compact Heisenberg manifolds.*** In this section, we consider isometry classes of left-invariant sub-Riemannian metrics on a compact Heisenberg manifold $\Gamma\backslash\mathrm{H}_n$. The detail is in [Tashiro 2020].

First of all, we recall the isomorphism classes of compact Heisenberg manifolds. Let D_n be the set of n-tuples of integers $\boldsymbol{r} = (r_1, \dots, r_n)$ such that r_i divides r_{i+1} for all $i = 1, \dots, n$. For $\boldsymbol{r} \in D_n$, let $\Gamma_{\boldsymbol{r}} < \mathrm{H}_n$ be the discrete subgroup defined by

$$\Gamma_{\boldsymbol{r}} = \langle r_1 X_1, \dots, r_n X_n, Y_1, \dots, Y_n, Z\rangle.$$

This gives a classification of lattices in the Heisenberg Lie group.

Theorem 3.1 [Gordon and Wilson 1986, Theorem 2.4]. *For any uniform lattice $\Gamma < \mathrm{H}_n$, there is an automorphism of H_n which sends Γ onto $\Gamma_{\boldsymbol{r}}$ for some $\boldsymbol{r} \in D_n$. Moreover, $\Gamma_{\boldsymbol{r}}$ is isomorphic to $\Gamma_{\boldsymbol{s}}$ if and only if $\boldsymbol{r} = \boldsymbol{s}$.*

Next we consider isometry classes of $\Gamma_{\boldsymbol{r}}\backslash\mathrm{H}_n$ for a fixed lattice $\Gamma_{\boldsymbol{r}}$. Fix a scalar product $\langle\cdot,\cdot\rangle_0$ on $\mathfrak{h}_n$ such that its orthonormal basis is $\{X_1, \dots, Y_n, Z\}$. Let A be a matrix of the form

$$A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \rho_A \end{pmatrix},$$

where $\tilde{A} \in \mathrm{GL}_{2n}(\mathbb{R})$ and $\rho_A \in \mathbb{R}$. Moreover let $J_n \in \mathrm{Skew}_{2n}(\mathbb{R})$ be a skew-symmetric matrix given by

$$J_n = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix},$$

where I_n is the identity matrix of size n. We say that a matrix A is of *canonical form* if

$$^t\tilde{A} J_n \tilde{A} = \begin{pmatrix} O & \mathrm{diag}(d_1(A), \dots, d_n(A)) \\ -\mathrm{diag}(d_1(A), \dots, d_n(A)) & O \end{pmatrix},$$

where $d_1(A), \dots, d_n(A)$ are nondecreasing positive numbers such that the imaginary numbers $\pm\sqrt{-1}d_1, \dots, \pm\sqrt{-1}d_n$ are the eigenvalues of $^t\tilde{A} J_n \tilde{A}$.

For a matrix A of canonical form, define the scalar product $\langle\cdot,\cdot\rangle_A$ on $\mathrm{Im}(A)$ by the norm

$$\|u\|_A := \min\{\|w\|_0 \mid u = Aw\}.$$

It is equivalent to the following definition: $\langle\cdot,\cdot\rangle_A$ is the scalar product which has an orthonormal basis $\{\tilde{A}X_1, \dots, \tilde{A}Y_n, \rho_A Z\}$. A pair $(\mathrm{Im}(A), \langle\cdot,\cdot\rangle_A)$ gives a sub-Riemannian metric on H_n of corank 0 (resp. corank 1) if $\rho_A \neq 0$ (resp. $\rho_A = 0$). If $\rho_A = 0$, then the subspace $\mathrm{Im}(A)$ is $\mathfrak{v}_0$, where

$$\mathfrak{v}_0 := \mathrm{Span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}.$$

These types of metrics cover all isometry classes of left invariant (sub-)Riemannian metrics on compact Heisenberg manifolds.

Theorem 3.2 [Tashiro 2020, Theorem 3.4]. *For any compact Heisenberg manifold with a bracket generating left-invariant sub-Riemannian metric* $(\Gamma\backslash\mathrm{H}_n, \mathfrak{v}, \langle\cdot,\cdot\rangle)$, *there exists an n-tuple $\boldsymbol{r} \in D_n$ and a matrix A of canonical form such that* $(\Gamma_{\boldsymbol{r}}\backslash\mathrm{H}_n, \mathrm{Im}(A), \langle\cdot,\cdot\rangle_A)$ *is isometric to* $(\Gamma\backslash\mathrm{H}_n, \mathfrak{v}, \langle\cdot,\cdot\rangle)$.

We shall denote the induced left-invariant distance function on H_n by dist_A and a quotient distance on $\Gamma\backslash\mathrm{H}_n$ by $\overline{\mathrm{dist}}_A$.

3C. *j-operator.* We recall the j-operator which plays an important role in the study of nilpotent Lie groups. Let $Z^* \in \mathfrak{h}_n^*$ be the dual covector of the vector $Z \in [\mathfrak{h}_n, \mathfrak{h}_n] \subset \mathfrak{h}_n$. For a matrix A of canonical form, define a skew symmetric operator $j(A) : \mathfrak{v}_0 \to \mathfrak{v}_0$ by

$$\langle j(A)(X), Y\rangle_A = Z^*([X, Y]).$$

Lemma 3.3 [Tashiro 2020, Lemma 4.1]. *The operator $j(A) : \mathfrak{v}_0 \to \mathfrak{v}_0$ has a matrix representation ${}^t\tilde{A}J_n\tilde{A}$ in the basis $\{AX_1, \dots, AX_n, AY_1, \dots, AY_n\}$.*

The positive number d_n can be regarded as the ℓ^∞-norm of the matrix ${}^t\tilde{A}J_n\tilde{A}$ as an element in the Euclidean space $\mathbb{R}^{4n^2}$. We also mention its ℓ^2-norm, the Hilbert–Schmidt norm of matrices.

Definition 3.4. For a matrix A of canonical form, we define $\delta(A) = \|{}^t\tilde{A}J_n\tilde{A}\|_{HS}$.

The following lemma is useful for later calculations.

Lemma 3.5 [Tashiro 2020, Lemma 5.1]. *For a matrix A of canonical form, we have*

(1) $\delta(A) = \sqrt{2\sum_{i=1}^n d_i(A)^2}$,

(2) $|\det(\tilde{A})| = \prod_{i=1}^n d_i(A)$.

3D. *Geodesics on Heisenberg groups.* Let A be a matrix of canonical form. For $i = 1, \dots, n$, define the functions $h_{x_i}, h_{y_i}, h_z : T^*\mathrm{H}_n \to \mathbb{R}$ by

$$h_{x_i}(\lambda) = p(L_{g*}AX_i), \quad h_{y_i}(\lambda) = p(L_{g*}AY_i), \quad h_z(p) = p(L_{g*}Z)$$

for $\lambda = (g, p) \in T^*\mathrm{H}_n$. Suppose that an admissible path

$$\gamma(t) = \sum_{i=1}^{n} x_i(t) A X_i + y_i(t) A Y_i + z(t) Z$$

is length minimizing. By the Hamiltonian equation (1) with a linear modification, there is a lift $\lambda(t)$ of $\gamma(t)$ such that

$$\begin{cases} \dot{h}_{x_i} = d_i(A) h_z h_{y_i} & (i = 1, \dots, n), \\ \dot{h}_{y_i} = -d_i(A) h_z h_{x_i} & (i = 1, \dots, n), \\ \dot{h}_z = 0, & \\ \dot{x}_i = h_{x_i} & (i = 1, \dots, n), \\ \dot{y}_i = h_{y_i} & (i = 1, \dots, n), \\ \dot{z} = \frac{1}{2} \sum_{i=1}^{n} d_i(A)(x_i h_{y_i} - y_i h_{x_i}) + \rho_A^2 p_z, & \end{cases}$$

where we write

$$h_{x_i}(t) = h_{x_i} \circ \lambda(t), \quad h_{y_i}(t) = h_{y_i} \circ \lambda(t), \quad h_z(t) = h_z \circ \lambda(t).$$

By proving this equation, we obtain the following parametrization of length minimizers.

Lemma 3.6 ([Eberlein 1994, Proposition 3.5] for corank 0 and [Rizzi 2016, Lemma 14] for corank 1 cases). *Let A be a matrix of canonical form and $\lambda : [0, T] \to T^*\mathrm{H}_n$ be the normal extremal with the initial data*

$$(h_{x_1}(0), \dots, h_{y_n}(0), h_z(0)) = (p_{x_1}, \dots, p_{y_n}, p_z) \in T_e^*\mathrm{H}_n = \mathfrak{h}_n^*.$$

Then the associated normal geodesic γ is given as follows.

If $p_z \neq 0$, then

$$\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} = \frac{1}{p_z d_i(A)} \begin{pmatrix} \sin(p_z d_i(A) t) & \cos(p_z d_i(A) t) - 1 \\ -\cos(p_z d_i(A) t) + 1 & \sin(p_z d_i(A) t) \end{pmatrix} \begin{pmatrix} p_{x_i} \\ p_{y_i} \end{pmatrix},$$

$$z(t) = \rho_A^2 p_z t + \frac{1}{2 p_z} \sum_{i=1}^{n} \left(t - \frac{1}{p_z d_i(A)} \sin(p_z d_i(A) t) \right) (p_{x_i}^2 + p_{y_i}^2).$$

Moreover, the normal geodesic fails to be length minimizing over the time

$$T = \frac{2\pi}{|p_z| d_{i_m}(A)},$$

where $i_m \in \{1, \dots, n\}$ is the minimum integer such that $(p_{x_i}, p_{y_i}) \neq (0, 0)$.

If $p_z = 0$, then

$$\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} = \begin{pmatrix} p_{x_i} \\ p_{y_i} \end{pmatrix} t, \quad z(t) \equiv 0.$$

Remark 3.7. The initial data $(p_{x_1}, \dots, p_{y_n})$ is identified with the projection of initial vector into $\mathfrak{v}_0 \subset \mathfrak{h}_n = T_e \mathrm{H}_n$ via the identification

$$\mathbb{R}^{2n} \ni (p_{x_1}, \dots, p_{y_n}) \simeq \sum_{i=1}^{n} (p_{x_i} AX_i + p_{y_i} AY_i) \in \mathfrak{v}_0.$$

For later arguments, we give an explicit distance from the identity to points in the horizontal direction and the vertical direction.

Lemma 3.8 ([Eberlein 1994, Proposition 3.11] for corank 0 and [Tashiro 2020, Lemma 5.2] for corank 1 cases). *For $U \in \mathfrak{v}_0$ and $V \in [\mathfrak{h}_n, \mathfrak{h}_n]$, we have*

$$\mathrm{dist}_A(e, U + V) \geq \|U\|_A.$$

Moreover, the equality holds if and only if $V = 0$.

Lemma 3.9. *For $z_0 \in \mathbb{R}$, the distance from e to $z_0 Z = (0, \dots, 0, z_0) \in \mathrm{H}_n$ is given by*

$$\mathrm{dist}_A(e, z_0 Z) = \min\left\{ \left| \frac{z}{\rho_A} \right|, \frac{2}{d_n(A)} \sqrt{|z_0| \pi d_n(A) - \pi^2 \rho_A^2} \right\},$$

with the convention

$$\begin{cases} \left| \dfrac{z_0}{\rho_A} \right| = +\infty & \text{if } \rho_A = 0, \\ \dfrac{2}{d_n(A)} \sqrt{|z_0| \pi d_n(A) - \pi^2 \rho_A^2} = +\infty & \text{if } |z_0| \pi d_n(A) - \pi^2 \rho_A^2 < 0. \end{cases}$$

Proof. For the simplicity we assume $z_0 > 0$. First let us consider a unit speed geodesic of the initial data $(0, \dots, 0, p_z)$. Then a unit speed normal geodesic $\gamma : [0, T] \to \mathrm{H}_n$ with $\gamma(T) = z_0 Z$ needs to satisfy

$$\begin{cases} z(T) = \rho_A^2 p_z T = z_0, \\ |p_z \rho_A| = 1 \end{cases}$$

Then the length is equal to the time $T = |z_0/\rho_A|$ with the convention $|z_0/\rho_A| = +\infty$ if $\rho_A = 0$, i.e., sub-Riemannian metric of corank 1.

Next we consider geodesics of the initial data $(p_{x_1}, \dots, p_{y_n}) \neq 0$. Since the endpoint $z_0 Z$ is in the center $[\mathfrak{h}_n, \mathfrak{h}_n]$, the initial data $(p_{x_1}, \dots, p_{y_n}) \in \mathbb{R}^{2q} \simeq \mathfrak{v}_0$ need to be inside the eigenspace of $\pm\sqrt{d_i(A)}$ of j_A (with the multiplicity), where we use the identification

$$\mathbb{R}^{2n} \ni (p_{x_1}, \dots, p_{y_n}) \simeq \sum_{i=1}^{n} (p_{x_i} AX_i + p_{y_i} AY_i) \in \mathfrak{v}_0.$$

Indeed, by the parametrization of $x_i(t)$, $y_i(t)$ in Lemma 3.6, the geodesic ends at $[\mathfrak{h}_n, \mathfrak{h}_n]$ only if the frequency of the trigonometric function is the same. Moreover, its length $T = 2\pi/(|p_z| d_i(A))$ is independent of the choice of the initial data

$(p_{x_1}, \dots, p_{y_n})$ as long as it belongs to the same eigenspace. Therefore we can assume that the initial covector is $(0, \dots, 0, p_{y_i}, 0, \dots, 0, p_z)$. If such a normal geodesic is unit speed and its endpoint is $z_0 Z$, then it needs to satisfy

$$\begin{cases} z(T) = \dfrac{2\pi\rho_A^2}{d_i(A)} + \dfrac{\pi}{p_z^2 d_i(A)} p_{y_i}^2 = z_0, \\ p_{y_i}^2 + (\rho_A p_z)^2 = 1. \end{cases}$$

This equation has a solution only if $z_0 \geq 2\pi\rho_A^2/d_i(A)$ with

$$p_{y_i} = \sqrt{\frac{z_0 d_i(A) - 2\pi\rho_A^2}{z_0 d_i(A) - \pi\rho_A^2}}, \quad p_z = \sqrt{\frac{\pi}{z_0 d_i(A) - \pi\rho_A^2}}.$$

Its length is

$$\frac{2\pi}{p_z d_i(A)} = \frac{2\sqrt{\pi}}{d_i(A)} \sqrt{z_0 d_i(A) - \pi\rho_A^2}.$$

Therefore the distance from e to $z_0 Z = (0, \dots, 0, z_0)$ is the minimum of two values

$$\min\left\{ \left| \frac{z_0}{\rho_A} \right|, \frac{2\sqrt{\pi}}{d_1(A)} \sqrt{z_0 d_1(A) - \pi\rho_A^2}, \dots, \frac{2\sqrt{\pi}}{d_n(A)} \sqrt{z_0 d_n(A) - \pi\rho_A^2} \right\}$$
$$= \min\left\{ \left| \frac{z_0}{\rho_A} \right|, \frac{2\sqrt{\pi}}{d_n(A)} \sqrt{z_0 d_n(A) - \pi\rho_A^2} \right\},$$

where we use $d_1(A) \leq \cdots \leq d_n(A)$. □

3E. ***Popp's volume form on Heisenberg group.*** In this section, we discuss Popp's volume form on the Heisenberg Lie group.

For a matrix A of canonical form with $\rho_A \neq 0$, denote by $\mathrm{dvol}_R(A)$ the canonical Riemannian volume form. Since it is the wedge of the dual coframe of an orthonormal frame, we have

$$\mathrm{dvol}_R(A) = \rho_A^{-1} (\det \tilde{A})^{-1} X_1^* \wedge \cdots \wedge Y_n^* \wedge Z^*.$$

In particular, the total measure of a Riemannian compact Heisenberg manifold $(\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n, \langle \cdot, \cdot \rangle_A)$ is

$$\text{(3)} \qquad \mathrm{meas}(\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n, \langle \cdot, \cdot \rangle_A) := \left| \int_{\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n} \mathrm{dvol}_R(A) \right| = \prod_{i=1}^n r_i |\rho_A^{-1} (\det \tilde{A})^{-1}|.$$

Next let A be a matrix of canonical form with $\rho_A = 0$. Denote by $\mathrm{dvol}_{sR}(A)$ Popp's volume associated to the sub-Riemannian metric $(\mathfrak{v}_0, \langle \cdot, \cdot \rangle_A)$.

By the definition, the (i, j)-th entry of the matrix ${}^t\tilde{A}J_n\tilde{A}$ is the structure constant c_{ij} of the basis $\{AX_1, \dots, AY_n\}$, that is

$$\begin{pmatrix} Z^*([AX_i, AX_j]) & Z^*([AX_i, AY_{j-n}]) \\ Z^*([AY_{i-n}, AX_j]) & Z^*([AY_{i-n}, AY_{j-n}]) \end{pmatrix}$$
$$= (c_{ij}) = {}^t\tilde{A}J_n\tilde{A} = \begin{pmatrix} O & \operatorname{diag}(d_1(A), \dots, d_n(A)) \\ -\operatorname{diag}(d_1(A), \dots, d_n(A)) & O \end{pmatrix}.$$

By Theorem 2.5, Popp's volume $\mathrm{dvol}_{sR}(A)$ is written by

$$\mathrm{dvol}_{sR}(A) = \delta(A)^{-1}(\det\tilde{A})^{-1}X_1^* \wedge \cdots \wedge Y_n^* \wedge Z^*.$$

In particular, the total measure of a sub-Riemannian compact Heisenberg manifold $(\Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n, \mathfrak{v}_0, \langle\cdot,\cdot\rangle_A)$ is

$$\text{(4)}\quad \mathrm{meas}(\Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n, \mathfrak{v}_0, \langle\cdot,\cdot\rangle_A) := \left|\int_{\Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n} \mathrm{dvol}_{sR}(A)\right| = \prod_{i=1}^n r_i|\delta(A)^{-1}(\det\tilde{A})^{-1}|.$$

3F. ***The circle bundle structure.*** Fix a n-tuple of numbers $\boldsymbol{r} \in D_n$. We recall a circle bundle structure of a compact Heisenberg manifold $\Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n$. Let $P : \mathrm{H}_n \to \mathfrak{h}_n \to \mathfrak{v}_0$ be the composition of the logarithm map and the projection. Denote the image of the lattice $\Gamma_{\boldsymbol{r}}$ by $\mathfrak{z}_{\boldsymbol{r}}$, which is again a lattice in $\mathfrak{v}_0$ isomorphic to $\mathbb{Z}^{2n}$. Then one obtains a surjective map $\bar{P} : \Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n \to \mathfrak{z}_{\boldsymbol{r}}\backslash \mathfrak{v}_0$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathrm{H}_n \arrow[r, "P"] \arrow[d, "P_{\Gamma_{\boldsymbol{r}}}"'] & \mathfrak{v}_0 \arrow[d, "P_{\mathfrak{z}_{\boldsymbol{r}}}"] \\ \Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n \arrow[r, "\bar{P}"] & \mathfrak{z}_{\boldsymbol{r}}\backslash \mathfrak{v}_0 \end{tikzcd}$$

Here the vertical arrows are the quotient map. The compact Heisenberg manifold $\Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n$ has a circle bundle structure by this map $\bar{P}$. For each $b \in \mathfrak{z}_{\boldsymbol{r}}\backslash \mathfrak{v}_0$, we denote by F_b the fiber over b.

Remark 3.10. Since a sub-Riemannian metric $\langle\cdot,\cdot\rangle_A$ is left-invariant, the diameter of a fiber is independent of the choice of a base point b. We shall denote the diameter of a fiber by $\mathrm{diam}(F_A)$.

The quotient metric on $\mathfrak{v}_0$ has an orthonormal basis $\{\tilde{A}X_1, \dots, \tilde{A}X_{2n}\}$. Therefore we shall denote the induced distance on $\mathfrak{v}_0$ by $\mathrm{dist}_{\tilde{A}}$, and the quotient distance on $\mathfrak{z}_{\boldsymbol{r}}\backslash \mathfrak{v}_0$ by $\overline{\mathrm{dist}}_{\tilde{A}}$. In the next section, we use the circle bundle structure to show that the Gromov–Hausdorff limit of compact Heisenberg manifolds is isometric to that of the base flat tori.

4. Gromov–Hausdorff limits of compact Heisenberg manifolds

4A. ***Collapse of the circle fiber.*** We say that a sequence of compact metric spaces $\{M_k\}$ converges to a compact metric space N if the Gromov–Hausdorff distance $d_{GH}(M_k, N)$ converges to 0. We do not use the original definition of d_{GH} since it is complicated. Instead of the original definition, we use the ϵ-approximation map which is easier to compute.

Definition 4.1 [Fukaya 1990, Definition 1.1]. Let (M, d_M), (N, d_N) be a compact path metric spaces. For $\epsilon > 0$, we say a map $\phi : M \to N$ is an ϵ-Hausdorff approximation if it satisfies the following:

(i) The ϵ neighborhood of $\phi(M)$ in N is N.

(ii) For $u, v \in M$. we have

$$|d_M(u, v) - d_N(\phi(u), \phi(v))| < \epsilon.$$

It is known that if there is ϵ-Hausdorff approximation map between M, N, then $d_{GH}(M, N) < 2\epsilon$. Therefore if a metric space M_k has an ϵ_k-approximation to N such that $\epsilon_k \to 0$, then the sequence $\{M_k\}$ converges to N in the Gromov–Hausdorff topology.

We can check that the quotient map $\overline{P}$ is an ϵ-approximation with ϵ equal to the diameter of the fiber.

Lemma 4.2. *The quotient map* $\overline{P} : (\Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n, \overline{\mathrm{dist}}_A) \to \mathfrak{z}_{\boldsymbol{r}}\backslash(\mathfrak{v}_0, \overline{\mathrm{dist}}_{\tilde{A}})$ *is a* $2\,\mathrm{diam}(F_A)$-*approximation map.*

Proof. Since the map $\overline{P}$ is surjective, we only need to check the almost isometric embeddability (Definition 4.1(ii)).

Let $u_1, u_2 \in \Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n$ be points in the compact Heisenberg manifold. By definition of the distance on the quotient space, there are $v_1, v_2 \in \Gamma_{\boldsymbol{r}}\backslash \mathrm{H}_n$ such that

$$\overline{\mathrm{dist}}_A(v_1, v_2) = \overline{\mathrm{dist}}_{\tilde{A}}(\overline{P}(u_1), \overline{P}(u_2)).$$

By the triangle inequality, we have

$$\begin{aligned} |\overline{\mathrm{dist}}_A(u_1, u_2) - \overline{\mathrm{dist}}_{\tilde{A}}(\overline{P}(u_1), \overline{P}(u_2))| &\le \overline{\mathrm{dist}}_A(u_1, v_1) + \overline{\mathrm{dist}}_A(u_2, v_2) \\ &\le 2\,\mathrm{diam}(F). \end{aligned}$$

This proves almost isometric embeddability. □

Proposition 4.3. *Let* $(\Gamma_{\boldsymbol{r}(k)}\backslash \mathrm{H}_n, \overline{\mathrm{dist}}_{A_k})$ *be a sequence of compact Heisenberg manifolds which has a uniform upper bound of the diameter. Assume that the diameter of the fibers* $\mathrm{diam}(F_{A_k})$ *converges to* 0. *Then its Gromov–Hausdorff limit is isometric to that of base flat tori* $(\mathfrak{z}_{\boldsymbol{r}}\backslash\mathfrak{v}_0, \overline{\mathrm{dist}}_{\tilde{A}_k})$.

In particular, *the limit is isometric to a flat torus of lower dimension.*

Proof. Since the diameter of the base flat tori is also uniformly bounded, we can assume that the sequence of base flat tori subconverges to a flat torus (N, d) (possibly a point). It is a consequence of [Bettiol et al. 2018, Proposition 3.1]. Then there are ϵ_k-approximation map $\varphi_k : (\mathfrak{z}_{\boldsymbol{r}} \backslash \mathfrak{v}_0, \overline{\mathrm{dist}}_{\tilde{A}_k}) \to (N, d)$. By Lemma 4.2, the composition $\varphi_k \circ \overline{P}$ is $(2\,\mathrm{diam}(F_{A_k}) + \epsilon_k)$-approximation. By the assumption, the Gromov–Hausdorff limit of $(\Gamma_{\boldsymbol{r}(k)} \backslash \mathrm{H}_n, \overline{\mathrm{dist}}_{A_k})$ is isometric to (N, d). $\square$

4B. ***Collapse of the fiber.*** Let $\{(\Gamma_{\boldsymbol{r}(k)} \backslash \mathrm{H}_n, \overline{\mathrm{dist}}_{A_k})\}$ be a sequence of compact sub-Riemannian Heisenberg manifolds with the diameter upper bound by $D > 0$. In this section, we show that if the sequence collapses, then the diameter of the circle fibers converge to zero.

The fiber over $b \in \mathfrak{z}_{\boldsymbol{r}} \backslash \mathfrak{v}_0$ is written by $F_b = \{\Gamma_{\boldsymbol{r}(k)} z_0 Z \cdot h_b \mid z_0 \in \mathbb{R}\}$, where we fix $h_b \in P^{-1}_{\Gamma_{\boldsymbol{r}(k)}}(\overline{P}^{-1}(b)) \subset \mathrm{H}_n$. In particular, the subset $\{z_0 Z \cdot h_b \mid z_0 \in [0, 1)\} \subset \mathrm{H}_n$ is a representative of F_b. By the left-invariance of the restricted distance on F_b, its diameter is the distance from h_b to $\frac{1}{2} Z \cdot h_b$ and is independent of the choice of h_b.

The above argument shows the following lemma.

Lemma 4.4. *The diameter of the fibers* $\mathrm{diam}(F_{A_k})$ *is given by*

$$\mathrm{diam}(F_{A_k}) = \mathrm{dist}_{A_k}\big(e, \tfrac{1}{2} Z\big).$$

Let us pass to the estimate of the diameter. First we consider a sequence $\{\Gamma_{\boldsymbol{r}(k)}\}_{k \in \mathbb{N}}$ such that $\boldsymbol{r}(k_1) \neq \boldsymbol{r}(k_2)$ for any $k_1 \neq k_2$. This implies that the sequence of numbers $\{r_n(k)\}$ diverges.

Proposition 4.5. *Assume that* $\mathrm{diam}(\Gamma_{\boldsymbol{r}(k)} \backslash \mathrm{H}_n, \langle \cdot, \cdot \rangle_{A_k}) \leq D$ *and* $r_n(k)$ *diverge to the infinity. Then the diameter of the fibers* $\mathrm{diam}(F_{A_k})$ *converge to zero.*

Proof. Let $\gamma_{n,k} = \frac{1}{2} r_n(k) X_n \in \mathrm{H}_n$. Since $\gamma_{n,k}$ is on the plane $\mathfrak{v}_0$, by Lemma 3.8, a length minimizer from e to $\gamma_{n,k}$ in H_n is the straight segment $\ell(t) := t X_n$, $t \in \big[0, \frac{1}{2} r_n(k)\big]$. Moreover its projection by $P_{\Gamma_{\boldsymbol{r}(k)}}$ is a length minimizer from $\Gamma_{\boldsymbol{r}(k)} e$ to $\Gamma_{\boldsymbol{r}(k)} \gamma_{n,k}$. Indeed, any element in $\Gamma_{\boldsymbol{r}(k)} \gamma_{n,k}$ is written by

$$r_n(k)\big(m + \tfrac{1}{2}\big) X_n + E,$$

where $m \in \mathbb{Z}$ and E is an element in $\mathfrak{h}_n \simeq \mathrm{H}_n$ transverse to X_n. Clearly a length minimizer from $\Gamma_{\boldsymbol{r}(k)} e$ to $\Gamma_{\boldsymbol{r}(k)} \gamma_{n,k}$ is realized when

$$m = 0, -1 \quad \text{and} \quad E = 0.$$

This shows that the projection of the straight segment $\ell(t)$ is length minimizing in $\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n$.

Since the length of the straight segment $\ell(t)$ is $\big\| \frac{1}{2} r_n(k) X_n \big\|_{A_k}$, we obtain

$$\begin{aligned} \big\| \tfrac{1}{2} r_n(k) X_n \big\|_{A_k} &= \mathrm{dist}_{A_k}(e, \gamma_{n,k}) = \overline{\mathrm{dist}}_{A_k}(\Gamma_{\boldsymbol{r}(k)} e, \Gamma_{\boldsymbol{r}(k)} \gamma_{n,k}) \\ &\leq \mathrm{diam}(\Gamma_{\boldsymbol{r}(k)} \backslash \mathrm{H}_n, \overline{\mathrm{dist}}_{A_k}) \leq D. \end{aligned} \tag{5}$$

By the same argument we also show that

$$\left\|\tfrac{1}{2}Y_n\right\|_{A_k} \leq D. \tag{6}$$

On the other hand, let $c : [0, 4] \to \mathrm{H}_n$ be a path inductively defined by

$$c(t) = \begin{cases} -t\sqrt{\dfrac{r_n(k)}{2}}X_n & \text{for } t \in [0, 1], \\ c(1) \cdot \left(-(t-1)\dfrac{1}{\sqrt{2r_n(k)}}Y_n\right) & \text{for } t \in [1, 2], \\ c(2) \cdot \left((t-2)\sqrt{\dfrac{r_n(k)}{2}}X_n\right) & \text{for } t \in [2, 3], \\ c(3) \cdot \left((t-3)\dfrac{1}{\sqrt{2r_n(k)}}Y_n\right) & \text{for } t \in [3, 4]. \end{cases}$$

The endpoint of c is $c(4) = \frac{1}{2}Z$, and the length is

$$\begin{aligned} \mathrm{length}(c) &= \|\sqrt{2r_n(k)}X_n\|_{A_k} + \left\|\sqrt{\frac{2}{r_n(k)}}Y_n\right\|_{A_k} \\ &= \sqrt{2r_n(k)}\|X_n\|_{A_k} + \sqrt{\frac{2}{r_n(k)}}\|Y_n\|_{A_k} \\ &\leq \sqrt{2r_n(k)}\frac{2D}{r_n(k)} + \sqrt{\frac{2}{r_n(k)}}\,2D \\ &= \frac{4\sqrt{2}D}{\sqrt{r_n(k)}}. \end{aligned}$$

Here the third inequality follows from (5) and (6). Hence we obtain

$$\mathrm{diam}(F_{A_k}) = \mathrm{dist}_{A_k}\left(e, \tfrac{1}{2}Z\right) \leq \mathrm{length}(c) \leq \frac{4\sqrt{2}D}{\sqrt{r_n(k)}}.$$

Since $r_n(k)$ diverges to the infinity, the diameters of the fibers converge to zero. □

Next we consider a sequence consisting of a fixed isomorphism type $\Gamma_{\boldsymbol{r}}\backslash\mathrm{H}_n$. We start from Riemannian case.

Proposition 4.6. *Let* $\{\Gamma_{\boldsymbol{r}}\backslash\mathrm{H}_n, \overline{\mathrm{dist}}_{A_k}\}$ *be a sequence of compact Heisenberg manifolds with left-invariant Riemannian metrics with the diameter upper bound. If the total measure in the canonical Riemannian volume converges to zero, then the diameter of the fibers* $\mathrm{diam}(F_{A_k})$ *converges to zero.*

Proof. By (3), if the total measure converges to zero, then either/both of the following two cases holds:

(a) $|\rho_{A_k}|^{-1} \to 0$, or

(b) $|\det(\tilde{A}_k)|^{-1} \to 0$.

In the case (a), by using Lemmas 3.9 and 4.4, we have

$$\operatorname{diam}(F_{A_k}) = \operatorname{dist}_{A_k}\big(e, \tfrac{1}{2}Z\big) = \min\left\{ \left| \frac{1}{2\rho_{A_k}} \right|, \frac{2}{d_n(A_k)} \sqrt{\frac{\pi d_n(A_k)}{2} - \pi^2 \rho_{A_k}^2} \right\}$$
$$\leq \left| \frac{1}{2\rho_{A_k}} \right| \to 0 \quad (k \to \infty).$$

In the case (b), by using Lemmas 3.5, 3.9 and 4.4, we have

$$\operatorname{diam}(F_{A_k}) = \min\left\{ \left| \frac{1}{2\rho_{A_k}} \right|, \frac{2}{d_n(A_k)} \sqrt{\frac{\pi d_n(A_k)}{2} - \pi^2 \rho_{A_k}^2} \right\} \leq \sqrt{\frac{2\pi}{d_n(A_k)}}$$
$$\leq \sqrt{\frac{2\pi}{\sqrt[n]{|\det(\tilde{A}_k)|}}} \to 0 \quad (k \to \infty).$$

In both cases, the diameter of the fiber $\operatorname{diam}(F_{A_k})$ converges to zero. This concludes the proposition. □

A similar argument follows also for sub-Riemannian metrics of corank 1.

Proposition 4.7. *Let* $\{\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n, \overline{\operatorname{dist}}_{A_k}\}$ *be a sequence of compact Heisenberg manifolds with left-invariant sub-Riemannian metrics of corank* 1. *If the total measure in Popp's volume converges to zero, then the diameter of the fibers converges to zero.*

Proof. By (4), if the total measure converges to zero, then either/both of the following two cases holds:

(a) $\delta(A_k)^{-1} \to 0$, or

(b) $|\det(\tilde{A}_k)|^{-1} \to 0$.

In the case (a), by using Lemmas 3.5, 3.9 and 4.4, we have

$$\operatorname{diam}(F_{A_k}) = \operatorname{dist}_{A_k}\big(e, \tfrac{1}{2}Z\big) = \min\left\{ +\infty, \frac{2}{d_n(A_k)} \sqrt{\frac{\pi d_n(A_k)}{2}} \right\}$$
$$= \sqrt{\frac{2\pi}{d_n(A_k)}} \leq \frac{2\sqrt{n\pi}}{\delta(A_k)} \to 0 \quad (k \to \infty).$$

In the case (b), again by using Lemmas 3.5, 3.9 and 4.4, we have

$$\operatorname{diam}(F_{A_k}) = \sqrt{\frac{2\pi}{d_n(A_k)}} \leq \sqrt{\frac{2\pi}{\sqrt[n]{|\det(\tilde{A}_k)|}}} \to 0 \quad (k \to \infty).$$

In both cases, the diameter of the fiber $\operatorname{diam}(F_{A_k})$ converges to zero. This concludes the proposition. □

Now we are prepared to prove the main theorem.

Proof of Theorem 1.1. Suppose that there are infinitely many isomorphic classes of lattices $\Gamma_{\boldsymbol{r}_k}$ in the sequence. Then by Proposition 4.5, the diameter of the fibers converges to 0, and by Proposition 4.3, the Gromov–Hausdorff limit is isometric to a flat torus of lower dimension.

Assume there are finitely many isomorphic classes of lattices in the sequence. By taking a subsequence, we can assume that the lattices are isomorphism to $\Gamma_{\boldsymbol{r}}$ for a fixed $\boldsymbol{r}$ in D_n. By Propositions 4.6 and 4.7, if the total measure converges to 0, then the diameter of the fiber converges to 0. Again by Proposition 4.3, the Gromov–Hausdorff limit is isometric to a flat torus of lower dimension. □

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ON THE COEFFICIENT INEQUALITIES FOR SOME CLASSES OF HOLOMORPHIC MAPPINGS IN COMPLEX BANACH SPACES

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Let $\mathcal{C}$ be the familiar class of normalized close-to-convex functions in the unit disk. Koepf (1987) proved that for a function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the class $\mathcal{C}$,

$$|a_3 - \lambda a_2^2| \le \begin{cases} 3-4\lambda, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{1}{3} + \frac{4}{9\lambda}, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \lambda \in \left[\frac{2}{3}, 1\right] \end{cases} \qquad \text{and} \qquad \big||a_3| - |a_2|\big| \le 1.$$

Recently, Xu et al. (2023) generalized the above results to a subclass of close-to-quasiconvex mappings of type B defined on the open unit polydisc in $\mathbb{C}^n$, and to a subclass of close-to-starlike mappings defined on the open unit ball of a complex Banach space, respectively. In the first part of this paper, by using different methods, we obtain the corresponding results of norm type and functional type on the open unit ball in a complex Banach space. We next give the coefficient inequalities for a subclass of g-starlike mappings of complex order λ on the open unit ball of a complex Banach space, which generalize many known results. Moreover, the proofs presented here are simpler than those given in the related papers.

1. Introduction

Let $\mathcal{S}$ be the class of functions of the form

$$f(\xi) = \xi + \sum_{m=2}^{\infty} a_m \xi^m, \tag{1-1}$$

which are univalent in the open unit disk

$$\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}.$$

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Let X be a complex Banach space with norm $\|\cdot\|$, $\mathbb{B}$ be the open unit ball of X.

Let $L(X, Y)$ denote the set of continuous linear operators from X into a complex Banach space Y. Let I be the identity in $L(X, X)$. For each $x \in X \setminus \{0\}$, we define

$$T(x) = \{T_x \in L(X, \mathbb{C}) : \|T_x\| = 1,\, T_x(x) = \|x\|\}.$$

According to the Hahn–Banach theorem, $T(x)$ is nonempty.

Let $H(\mathbb{B})$ denote the set of all holomorphic mappings from $\mathbb{B}$ into X. It is well known that if $f \in H(\mathbb{B})$, then

$$f(y) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m f(x)((y-x)^m)$$

for all y in some neighborhood of $x \in \mathbb{B}$, where $D^m f(x)$ is the m-th Fréchet derivative of f at x, and for $m \geq 1$,

$$D^m f(x)((y-x)^m) = D^m f(x)(\underbrace{y-x, \dots, y-x}_{m}).$$

Furthermore, $D^m f(x)$ is a bounded symmetric m-linear mapping from

$$X^m = \underbrace{X \times \cdots \times X}_{m} \quad \text{into } X.$$

A holomorphic mapping $f : \mathbb{B} \to X$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(\mathbb{B})$. A mapping $f \in H(\mathbb{B})$ is called locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in \mathbb{B}$. If $f : \mathbb{B} \to X$ is a holomorphic mapping, then f is called normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X. A mapping $f \in H(\mathbb{B})$ is called starlike if f is biholomorphic on $\mathbb{B}$ and $f(\mathbb{B})$ is a starlike domain. Let $\mathcal{S}^*(\mathbb{B})$ denote the class of normalized starlike mappings on $\mathbb{B}$, when $X = \mathbb{C}$, $\mathbb{B} = \mathbb{U}$, the class $\mathcal{S}^*(\mathbb{U})$ is denoted by $\mathcal{S}^*$. Suppose $f, g \in H(\mathbb{U})$. If there exists a Schwarz function φ (i.e., $\varphi \in H(\mathbb{U})$, $\varphi(0) = 0$, $\varphi(\mathbb{U}) \subseteq \mathbb{U}$) such that $f = g \circ \varphi$, then we say that f is subordinate to g(written $f \prec g$).

Now, we introduce the class of quasiconvex mappings of type B on $\mathbb{B}$ in X, which has been introduced by Roper and Suffridge [31] on the unit ball $\mathbb{B} \subset \mathbb{C}^n$.

Definition 1.1. Let $h : \mathbb{B} \to X$ be a normalized locally biholomorphic mapping. If

$$\text{(1-2)} \quad \Re e\big\{T_x\big[(Dh(x))^{-1}\big(D^2h(x)(x^2)+Dh(x)\,x\big)\big]\big\} > 0, \quad x \in \mathbb{B}\backslash\{0\},\ T_x \in T(x),$$

then h is called a quasiconvex mapping of type B on $\mathbb{B}$.

Let $\mathcal{Q}_B(\mathbb{B})$ denote the class of quasiconvex mappings of type B on $\mathbb{B}$. When $X=\mathbb{C}$, $\mathbb{B}=\mathbb{U}$, we deduce easily that relation (1-2) is equivalent to

$$\Re e\left(1+\frac{\xi h''(\xi)}{h'(\xi)}\right)>0, \quad \xi\in\mathbb{U},$$

which is the well-known criterion of convex functions on $\mathbb{U}$. Let $\mathcal{K}$ denote the class of normalized convex functions on $\mathbb{U}$.

Xu et al. [35] introduced the following class of mappings on the open unit ball of a complex Banach space.

Definition 1.2 [35]. Suppose that $f:\mathbb{B}\to X$ is a normalized holomorphic mapping. If there exists a mapping $h\in\mathcal{Q}_B(\mathbb{B})$ such that

$$\Re e\{T_x[(Dh(x))^{-1}Df(x)\,x]\}>0, \quad x\in\mathbb{B}\setminus\{0\},\ T_x\in T(x), \tag{1-3}$$

then f is called a close-to-quasiconvex mapping of type B on $\mathbb{B}$.

If $X=\mathbb{C}^n$, $\mathbb{B}=\mathbb{U}^n$, then it is obvious that the relation (1-3) is equivalent to

$$\Re e\frac{p_j(z)}{z_j}>0, \quad z\in\mathbb{U}^n\setminus\{0\},$$

where $p(z)=(p_1(z),\dots,p_n(z))'=(Dh(z))^{-1}Df(z)\,z$ is a column vector in $\mathbb{C}^n$, and j satisfies $|z_j|=\|z\|=\max_{1\le k\le n}\{|z_k|\}$.

The following definition has been introduced by Pfaltzgraff and Suffridge [29] on the unit ball with respect to an arbitrary norm in $\mathbb{C}^n$.

Definition 1.3. Suppose that $f:\mathbb{B}\to X$ is a normalized locally biholomorphic mapping. If there exists a mapping $h\in\mathcal{S}^*(\mathbb{B})$ such that

$$\Re e\{T_x[(Df(x))^{-1}h(x)]\}>0, \quad x\in\mathbb{B}\setminus\{0\},\ T_x\in T(x), \tag{1-4}$$

then f is called a close-to-starlike mapping on $\mathbb{B}$.

Remark 1.4. Clearly, if $X=\mathbb{C}$, $\mathbb{B}=\mathbb{U}$, then the relation (1-3) (respectively, the relation (1-4)) is equivalent to $\Re e\frac{f'(\xi)}{h'(\xi)}>0$, $\xi\in\mathbb{U}$, here $h\in\mathcal{K}$ (respectively, $\Re e\frac{\xi f'(\xi)}{h(\xi)}>0$, $\xi\in\mathbb{U}$, here $h\in\mathcal{S}^*$), which is the usual definition of close-to-convex functions on $\mathbb{U}$.

Koepf [23] obtained the following Fekete and Szegő inequality for the class $\mathcal{C}$.

Theorem 1.5 [23]. *Let the function $f(\xi)$ be defined by* (1-1). *If $f\in\mathcal{C}$, then*

$$|a_3-\lambda a_2^2|\le\begin{cases}3-4\lambda, & \lambda\in\left[0,\frac{1}{3}\right],\\ \frac{1}{3}+\frac{4}{9\lambda}, & \lambda\in\left[\frac{1}{3},\frac{2}{3}\right],\\ 1, & \lambda\in\left[\frac{2}{3},1\right].\end{cases}$$

As an interesting application of Theorem 1.5, it was proved that $\big||a_3| - |a_2|\big| \le 1$ for the class $\mathcal{C}$.

In recent years, the Fekete and Szegő inequality for subclass of biholomorphic mappings in several complex variables has been studied by some authors (see [3; 4; 5; 6; 7; 15; 20; 21; 32; 36; 38]).

Xu et al. [39] obtained the following Fekete and Szegő inequality for the subclass of close-to-quasiconvex mappings of type B on the open unit polydisk $\mathbb{U}^n$ in $\mathbb{C}^n$ with respect to $H \in \mathcal{Q}_B(\mathbb{U}^n)$, which could be regarded as a generalization of Theorem 1.5 to several complex variables.

Theorem 1.6 [39]. *Let $f : \mathbb{U}^n \to \mathbb{C}$, $h : \mathbb{U}^n \to \mathbb{C}$ be holomorphic functions, and $H(z) = zh(z) \in \mathcal{Q}_B(\mathbb{U}^n)$. Suppose that $F(z) = zf(z)$ is a close-to-quasiconvex mapping of type B with respect to $H(z)$. Then, for $\lambda \in [0, 1]$, $z \in \mathbb{U}^n$, we have*

$$\left\| \tfrac{1}{3!} D^3F(0)(z^3) - \lambda \tfrac{1}{2!} D^2F(0)\left(z, \frac{D^2F(0)(z^2)}{2!}\right) \right\| \le \begin{cases} (3-4\lambda)\|z\|^3, & \lambda \in \left[0, \tfrac{1}{3}\right], \\ \left(\tfrac{1}{3} + \tfrac{4}{9\lambda}\right)\|z\|^3, & \lambda \in \left[\tfrac{1}{3}, \tfrac{2}{3}\right], \\ \|z\|^3, & \lambda \in \left[\tfrac{2}{3}, 1\right] \end{cases}$$

and

$$\left| \left\| \frac{D^3F(0)(z^3))}{3!} \right\| - \left\| \frac{D^2F(0)(z^2)\|z\|}{2!} \right\| \right| \le \|z\|^3.$$

The above estimates are sharp.

Hamada [15] generalized Theorem 1.6 to the open unit ball of a complex Banach space under weaker assumptions than in Theorem 1.6. Moreover, in the same paper, Hamada also obtained the Fekete and Szegő inequality of functional type for the subclasses of close-to-quasiconvex mappings of type B on the open unit ball $\mathbb{B}$ in a complex Banach space.

Theorem 1.7 [15]. *Let G be a quasiconvex mapping of type B on $\mathbb{B}$ such that*

$$\tfrac{1}{2!} D^2G(0)(x^2) = L_G(x)\,x, \quad x \in X,$$

where $L_G(\,\cdot\,) \in L(X, \mathbb{C})$. Let F be a close-to-quasiconvex mapping of type B on $\mathbb{B}$ with respect to G such that

$$\tfrac{1}{2!} D^2F(0)(x^2) = L_F(x)\,x, \quad x \in X,$$

where $L_F(\,\cdot\,) \in L(X, \mathbb{C})$ and

$$\tfrac{1}{3!} D^3F(0)(x^3) = Q_F(x)\,x, \quad x \in X,$$

where $Q_F(x)$ is a homogeneous polynomial of degree 2 *with values in* $\mathbb{C}$. *Let* $x_0 \in X$ *with* $\|x_0\| = 1$. *Then, for* $\lambda \in [0, 1]$, *it holds that*

$$\left\| \tfrac{1}{3!} D^3 F(0)(x_0^3) - \lambda \tfrac{1}{2!} D^2 F(0)\left(x_0, \frac{D^2 F(0)(x_0^2)}{2!} \right) \right\| \leq \begin{cases} 3 - 4\lambda, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{1}{3} + \frac{4}{9\lambda}, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \lambda \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

The above estimates are sharp.

More recently, Xu et al. [37] gave another extension of Theorem 1.5 to higher dimensions, and established the following Fekete and Szegő inequality for the subclass of close-to-starlike mappings on the open unit ball $\mathbb{B}$ in a complex Banach space with respect to $H \in \mathcal{S}^*(\mathbb{B})$.

Theorem 1.8 [37]. *Let* $f : \mathbb{B} \to \mathbb{C}$, $h : \mathbb{B} \to \mathbb{C}$ *be holomorphic functions*, *and* $H(x) = xh(x) \in \mathcal{S}^*(\mathbb{B})$. *Suppose that* $F(x) = xf(x)$ *is a close-to-starlike mapping with respect to* $H(x)$. *Then*, *for* $x \in \mathbb{B} \setminus \{0\}$, $T_x \in T(x)$, $\lambda \in [0, 1]$, *we have*

$$\left| \frac{T_x(D^3 F(0)(x^3))}{3!\,\|x\|^3} - \lambda \left(\frac{T_x(D^2 F(0)(x^2))}{2!\,\|x\|^2} \right)^2 \right| \leq \begin{cases} 3 - 4\lambda, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{1}{3} + \frac{4}{9\lambda}, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \lambda \in \left[\frac{2}{3}, 1\right] \end{cases}$$

and

$$\left| \left| \frac{T_x(D^3 F(0)(x^3))}{3!\,\|x\|^3} \right| - \left| \frac{T_x(D^2 F(0)(x^2))}{2!\,\|x\|^2} \right| \right| \leq 1.$$

The above estimates are sharp.

Zhang et al. [40] introduced the following class of g-starlike mapping of complex order λ on $\mathbb{B}$ in X, which has been introduced by Hu et al. [22] on $\mathbb{B}^n$.

Definition 1.9 [40]. Let $g : \mathbb{U} \to \mathbb{C}$ be a biholomorphic function such that $g(0) = 1$, $\Re e g(\xi) > 0$ on $\mathbb{U}$. Let $\lambda \in \mathbb{C}$ with $\Re e \lambda \leq 0$ and let $f : \mathbb{B} \to X$ be a normalized locally biholomorphic mapping. If

$$(1 - \lambda) \frac{\|x\|}{T_x(Df(x))^{-1} f(x))} + \lambda \in g(\mathbb{U}), \quad x \in \mathbb{B} \setminus \{0\}, T_x \in T(x),$$

then f is called a g-starlike mapping of complex order λ.

Let $\mathcal{S}^*_{g,\lambda}(\mathbb{B})$ denote the class of g-starlike mapping of complex order λ on $\mathbb{B}$. In particular, when $X = \mathbb{C}$, $\mathbb{B} = \mathbb{U}$, the above relation implies that

$$f \in \mathcal{S}^*_{g,\lambda}(\mathbb{U}) \qquad \text{if and only if } (1 - \lambda) \frac{\xi f'(\xi)}{f(\xi)} + \lambda \prec g, \quad \xi \in \mathbb{U}.$$

In view of Remark 2.4 of [40], we know that some important subclasses of $\mathcal{S}(\mathbb{B})$ coincide with the classes of $\mathcal{S}^*_{g,\lambda}(\mathbb{B})$ for certain choices of g and λ.

Since the authors [37; 39] used Definitions 1.2 and 1.3 directly, the proofs of Theorems 1.6 and 1.8 are long and rather complicated. In Section 3, under the same conditions as in Theorems 1.6 and 1.8, we will establish the corresponding inequalities of norm type and functional type for the subclasses of close-to-quasiconvex mappings of type B and close-to-starlike mappings on the open unit ball of a Banach space, respectively. Next, In Section 4, we obtain the coefficient inequalities for a subclass of g-starlike mappings of complex order λ on the open unit ball of a complex Banach space. The various results of this paper would generalize many known results. Moreover, the proof methods presented here simplify those appeared in some earlier papers [25; 26; 28; 33; 36; 37; 39].

Some investigations concerning the coefficient estimates for subclasses of holomorphic mappings in several variables have been obtained by Bracci et al. [1; 2], Graham et al. [8; 9; 10; 11; 12; 13; 14], Hamada et al. [16; 18; 19], Kohr [24], Liu and Wu [27], Liu et al. [28], and Xu et al. [34; 35].

2. Some lemmas

In order to prove the desired results, we need to provide the following lemmas.

Lemma 2.1 [17]. *Let $h : \mathbb{B} \to X$ be a normalized locally biholomorphic mapping. Then h is a starlike mapping on $\mathbb{B}$ if and only if*

$$\Re e\big(T_x(Dh(x)^{-1}h(x))\big) > 0, \quad x \in \mathbb{B} \setminus \{0\},\ T_x \in T(x).$$

Comparing Lemma 2.1 with Definition 1.3, we remark that any normalized starlike mapping on $\mathbb{B}$ is close-to-starlike (with respect to itself).

Lemma 2.2. *Let $g : \mathbb{U} \to \mathbb{C}$ satisfy the conditions of* Definition 1.9, $f \in H(\mathbb{B}, C)$, $f(0)=1$, $F(x)=xf(x)$. *Fix $x \in \mathbb{B}\setminus\{0\}$ and denote $x_0 = \frac{x}{\|x\|}$. Let $l(\xi) = T_x(F(\xi x_0))$, $\xi \in \mathbb{U}$. Then*

$$l \in \mathcal{S}^*_{g,\lambda}(\mathbb{U}) \Leftrightarrow F \in \mathcal{S}^*_{g,\lambda}(\mathbb{B}).$$

Proof. Since $F \in \mathcal{S}^*_{g,\lambda}(\mathbb{B})$, we deduce from Definition 1.9 that

$$(1-\lambda)\frac{\|x\|}{T_x((DF(x))^{-1}F(x))} + \lambda \in g(\mathbb{U}), \quad x \in \mathbb{B} \setminus \{0\},\ T_x \in T(x).$$

It follows that F is locally biholomorphic on $\mathbb{B}$, and thus $f(x) \neq 0$, $x \in \mathbb{B}$. Using a similar method to that in [8] (also see [9, Theorem 7.1.14]), we have

$$[DF(x)]^{-1} = \frac{1}{f(x)}\left(I - \frac{\frac{xDf(x)}{f(x)}}{1 + \frac{Df(x)x}{f(x)}}\right). \tag{2-1}$$

Hence,

$$(DF(x))^{-1}F(x) = x\left(\frac{1}{1+\frac{Df(x)x}{f(x)}}\right) = \frac{xf(x)}{f(x)+Df(x)x}, \quad x \in \mathbb{B}.$$

We find from the above equality that

$$(2\text{-}2)\quad (1-\lambda)\frac{\|x\|}{T_x((DF(x))^{-1}F(x))}+\lambda=(1-\lambda)\frac{f(x)+Df(x)\,x}{f(x)}+\lambda\in g(\mathbb{U}).$$

Since $l(\xi)=T_x(F(\xi x_0))=T_x(\xi x_0\, f(\xi x_0))=\xi f(\xi x_0)$, we have

$$l'(\xi)=f(\xi x_0)+Df(\xi x_0)\,\xi x_0$$

and

$$(2\text{-}3)\qquad (1-\lambda)\frac{\xi l'(\xi)}{l(\xi)}+\lambda=(1-\lambda)\frac{f(\xi x_0)+Df(\xi x_0)\,\xi x_0}{f(\xi x_0)}+\lambda\in g(\mathbb{U}),$$

which implies that $l\in\mathcal{S}^*_{g,\lambda}(\mathbb{U})$.

Conversely, we assume $l\in\mathcal{S}^*_{g,\lambda}(\mathbb{U})$. Then it is clear that $\frac{\xi l'(\xi)}{l(\xi)}\neq 0,\ \xi\in\mathbb{U}$. Hence we have

$$1+\frac{Df(x)\,x}{f(x)}\neq 0,\quad x\in\mathbb{B}.$$

It is not hard to deduce from this and (2-1) that F is locally biholomorphic on $\mathbb{B}$.

On the other hand, in view of (2-2) and (2-3), we can conclude that

$$(1-\lambda)\frac{\xi}{T_{\xi x_0}((DF(\xi x_0))^{-1}F(\xi x_0))}+\lambda=(1-\lambda)\frac{\xi l'(\xi)}{l(\xi)}+\lambda\in g(\mathbb{U}).$$

Taking $\xi=\|x\|$ in the above relation, we obtain

$$(1-\lambda)\frac{\|x\|}{T_x((DF(x))^{-1}F(x))}+\lambda\in g(\mathbb{U}),$$

as desired. □

Lemma 2.3 [20]. *Let $g(\xi)=1+g'(0)\,\xi+\frac{g''(0)}{2}\xi^2+\dots$ be a holomorphic function on $\mathbb{U}$ such that $g'(0)\neq 0$. Let $s(\xi)=1+s'(0)\,\xi+\frac{s''(0)}{2}\xi^2+\dots$ be a holomorphic function on $\mathbb{U}$ such that $s\prec g$. Then for every $\mu\in\mathbb{C}$, it holds that*

$$\left|\frac{s''(0)}{2}-\mu(s'(0))^2\right|\le\max\left\{|g'(0)|,\ \left|\frac{g''(0)}{2}-\mu(g'(0))^2\right|\right\}.$$

This estimate is sharp.

Lemma 2.4. *Let $g:\mathbb{U}\to\mathbb{C}$ satisfy the conditions of* Definition 1.9 *and*

$$l(\xi)=\xi+\sum_{m=2}^{\infty}l_m\,\xi^m\in\mathcal{S}^*_{g,\lambda}(\mathbb{U}).$$

Then

$$|l_3-\nu l_2^2|\le\frac{|g'(0)|}{2|1-\lambda|}\max\left\{1,\ \left|\frac{1}{2}\frac{g''(0)}{g'(0)}+\frac{1-2\nu}{1-\lambda}g'(0)\right|\right\},\quad \nu\in\mathbb{C}.$$

The above estimate is sharp for the function

$$l(\xi)=\xi\exp\frac{1}{1-\lambda}\int_0^{\xi}(g(t)-1)\frac{1}{t}dt$$

if $\left|\frac{1}{2}\frac{g''(0)}{g'(0)}+\frac{1-2\nu}{1-\lambda}g'(0)\right|\geq 1$, *and for*

$$l(\xi)=\xi\exp\frac{1}{1-\lambda}\int_0^{\xi}(g(t^2)-1)\frac{1}{t}dt$$

if $\left|\frac{1}{2}\frac{g''(0)}{g'(0)}+\frac{1-2\nu}{1-\lambda}g'(0)\right|\leq 1$.

Proof. Since $l\in\mathcal{S}^*_{g,\lambda}(\mathbb{U})$, we have

$$b(\xi)=(1-\lambda)\frac{\xi l'(\xi)}{l(\xi)}+\lambda\in g(\mathbb{U}),\quad \xi\in\mathbb{U},\ b\prec g.$$

A computation shows that

$$b'(0)=(1-\lambda)\,l_2,\qquad \frac{b''(0)}{2}=2(1-\lambda)\,l_3-(1-\lambda)\,l_2^2.$$

By using Lemma 2.3, we have

$$\left|\frac{b''(0)}{2}-\mu(b'(0))^2\right|\leq\max\left\{|g'(0)|,\left|\frac{g''(0)}{2}-\mu(g'(0))^2\right|\right\},\quad \mu\in\mathbb{C}.$$

From the above relations, we obtain that

$$|l_3-\nu l_2^2|\leq\frac{|g'(0)|}{2|1-\lambda|}\max\left\{1,\left|\frac{1}{2}\frac{g''(0)}{g'(0)}+\frac{1-2\nu}{1-\lambda}g'(0)\right|\right\},\quad \nu\in\mathbb{C}.\qquad\square$$

Remark 2.5. Lemma 2.4 generalizes Theorem 3.1 of [36], when $\lambda=0$, Lemma 2.4 obtained by Xu et al. [36]. Moreover, the proof presented here is simpler than that in [36, Theorem 3.1].

Lemma 2.6. *Let* $g:\mathbb{U}\to\mathbb{C}$ *be a convex function which satisfies the conditions of* Definition 1.9 *and*

$$l(\xi)=\xi+\sum_{m=2}^{\infty}l_m\,\xi^m\in\mathcal{S}^*_{g,\lambda}(\mathbb{U}).$$

Then

$$|l_m|\leq\frac{1}{(m-1)!}\prod_{k=2}^{m}\left((k-2)+\frac{|g'(0)|}{|1-\lambda|}\right),\quad m=2,3,4,\ldots.$$

Proof. Since $l\in\mathcal{S}^*_{g,\lambda}(\mathbb{U})$, the function p is defined by

$$p(\xi)=(1-\lambda)\frac{\xi l'(\xi)}{l(\xi)}+\lambda,\quad \xi\in\mathbb{U},$$

in view of Definition 1.9, we have $p\prec g$. If

$$p(\xi)=1+p_1\,\xi+p_2\,\xi^2+\cdots+p_m\,\xi^m+\ldots,\quad \xi\in\mathbb{U},$$

then from Rogosinski's theorem [30], we obtain that $|p_m| \le |g'(0)|,\ m = 1, 2, 3, \dots$. Comparing the coefficients in the power series of $l(\xi)p(\xi) = (1-\lambda)\,\xi l'(\xi) + \lambda l(\xi)$, we deduce that $(1-\lambda)\, l_2 = p_1$ and

$$(1-\lambda)(m-1)\, l_m = p_{m-1} + l_2\, p_{m-2} + l_3\, p_{m-3} + \cdots + l_{m-1}\, p_1, \quad m = 2, 3, 4, \dots.$$

Thus by the mathematical induction, we obtain

$$|l_m| \le \frac{1}{(m-1)!} \prod_{k=2}^{m} \left((k-2) + \frac{|g'(0)|}{|1-\lambda|} \right), \quad m = 2, 3, 4, \dots,$$

as desired. □

3. Simplified proofs of Fekete–Szegő inequalities for close-to-quasiconvex mappings of type *B* and close-to-starlike mappings

In this section, by using the proof methods different from those appeared in [39] and [37], we obtain the corresponding results of norm type and functional type for subclasses of close-to-quasiconvex mappings of type B and close-to-starlike mappings defined on the open unit ball in a complex Banach space (see Theorem 1.7).

Theorem 3.1. *Let $f : \mathbb{B} \to \mathbb{C}$, $h : \mathbb{B} \to \mathbb{C}$ be holomorphic functions, and let $H(x) = xh(x) \in \mathcal{Q}_B(\mathbb{B})$. Suppose that $F(x) = xf(x)$ is a close-to-quasiconvex mapping of type B with respect to $H(x)$. Then, for $x \in X \setminus \{0\}$, $T_x \in T(x)$ and $\lambda \in [0, 1]$, we have*

$$\left\| \frac{D^3F(0)(x^3)}{3!\,\|x\|^3} - \lambda \frac{1}{2\|x\|^3} D^2F(0)\left(x, \frac{D^2F(0)(x^2)}{2!} \right) \right\| \le \begin{cases} 3 - 4\lambda, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{1}{3} + \frac{4}{9\lambda}, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \lambda \in \left[\frac{2}{3}, 1\right], \end{cases}$$

$$\left| \frac{T_x(D^3F(0)(x^3))}{3!\,\|x\|^3} - \lambda \left(\frac{T_x(D^2F(0)(x^2))}{2!\,\|x\|^2} \right)^2 \right| \le \begin{cases} 3 - 4\lambda, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{1}{3} + \frac{4}{9\lambda}, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \lambda \in \left[\frac{2}{3}, 1\right], \end{cases}$$

$$\left| \left| \frac{T_x(D^3F(0)(x^3))}{3!\,\|x\|^3} \right| - \left| \frac{T_x(D^2F(0)(x^2))}{2!\,\|x\|^2} \right| \right| \le 1$$

and

$$\left| \left\| \frac{D^3F(0)(x^3))}{3!\,\|x\|^3} \right\| - \left\| \frac{D^2F(0)(x^2)}{2!\,\|x\|^2} \right\| \right| \le 1.$$

The above estimates are sharp.

Proof. Fix $x \in X \setminus \{0\}$ and denote $x_0 = \frac{x}{\|x\|}$. Let $p : \mathbb{U} \to \mathbb{C}$ be given by

$$p(\xi) = \begin{cases} \frac{T_x((DH(\xi x_0))^{-1}(D^2H(\xi x_0)(\xi x_0)^2 + DH(\xi x_0)(\xi x_0))}{\xi}, & \xi \ne 0, \\ 1, & \xi = 0. \end{cases}$$

Then $p \in H(\mathbb{U})$, $p(0) = 1$, and

$$p(\xi) = \frac{T_x((DH(\xi x_0))^{-1}(D^2H(\xi x_0)(\xi x_0)^2 + DH(\xi x_0)(\xi x_0))}{\xi}$$
$$= \frac{T_{\xi x_0}((DH(\xi x_0))^{-1}(D^2H(\xi x_0)(\xi x_0)^2 + DH(\xi x_0)(\xi x_0))}{\|\xi x_0\|}, \quad \xi \in \mathbb{U} \setminus \{0\}.$$

Since $H \in \mathcal{Q}_B(\mathbb{B})$, using Definition 1.1, we obtain

$$\Re e(p(\xi)) > 0, \quad \xi \in \mathbb{U}. \tag{3-1}$$

On the other hand, by using an elementary computation, we have

$$(DH(x))^{-1} = \frac{1}{h(x)}\left(I - \frac{\frac{x\,Dh(x)}{h(x)}}{1 + \frac{Dh(x)\,x}{h(x)}}\right)$$

and

$$(DH(x))^{-1}(D^2H(x)(x^2) + DH(x)\,x) = \frac{D^2h(x)(x^2) + 3Dh(x)\,x + h(x)}{h(x) + Dh(x)\,x}x.$$

This allows us to rewrite p in the form

$$p(\xi) = \frac{D^2h(\xi x_0)((\xi x_0)^2) + 3Dh(\xi x_0)(\xi x_0) + h(\xi x_0)}{h(\xi x_0) + Dh(\xi x_0)(\xi x_0)}, \quad \xi \in \mathbb{U}. \tag{3-2}$$

Let $k(\xi) = T_x(H(\xi x_0)) = \xi h(\xi x_0)$ for $\xi \in \mathbb{U}$. Elementary computations using this inequality yield that

$$k'(\xi) = h(\xi x_0) + Dh(\xi x_0)(\xi x_0) \tag{3-3}$$

and

$$1 + \frac{\xi k''(\xi)}{k'(\xi)} = \frac{D^2h(\xi x_0)((\xi x_0)^2) + 3Dh(\xi x_0)(\xi x_0) + h(\xi x_0)}{h(\xi x_0) + Dh(\xi x_0)(\xi x_0)}. \tag{3-4}$$

Using (3-1), (3-2) and (3-4), we obtain $k \in \mathcal{K}$.

Let $s(x) = (DH(x))^{-1}DF(x)\,x$, and let

$$r(\xi) = \begin{cases} \frac{T_x(s(\xi x_0))}{\xi}, & \xi \in \mathbb{U} \setminus \{0\}, \\ 1, & \xi = 0. \end{cases}$$

Then r is holomorphic on $\mathbb{U}$, $r(0) = 1$ and

$$r(\xi) = \frac{T_x(s(\xi x_0))}{\xi} = \frac{T_{x_0}(s(\xi x_0))}{\xi} = \frac{T_{\xi x_0}(s(\xi x_0))}{\|\xi x_0\|}, \quad \xi \in \mathbb{U} \setminus \{0\}.$$

Since $F(x)$ is a close-to-quasiconvex mapping of type B with respect to $H(x)$, from Definition 1.2, we obtain

$$\Re e(r(\xi)) > 0, \quad \xi \in \mathbb{U}. \tag{3-5}$$

A simple computation shows that

$$r(\xi)=\frac{T_x(s(\xi x_0))}{\xi}=\frac{T_x((DH(\xi x_0))^{-1}DF(\xi x_0)\,\xi x_0)}{\xi}=\frac{f(\xi x_0)+Df(\xi x_0)\,\xi x_0}{h(\xi x_0)+Dh(\xi x_0)\,\xi x_0},\quad \xi\in\mathbb{U}\setminus\{0\}. \tag{3-6}$$

Letting $l(\xi)=T_x(F(\xi x_0))$, $\xi\in\mathbb{U}$, we have

$$l'(\xi)=T_x(DF(\xi x_0)\,x_0)=f(\xi x_0)+Df(\xi x_0)\,\xi x_0. \tag{3-7}$$

Hence from (3-3), (3-5), (3-6) and (3-7), we obtain

$$\Re e\left(\frac{l'(\xi)}{k'(\xi)}\right)=\Re e\left(\frac{f(\xi x_0)+Df(\xi x_0)\,\xi x_0}{h(\xi x_0)+Dh(\xi x_0)\,\xi x_0}\right)>0,$$

which means that $l\in\mathcal{C}$. Thus from Theorem 1.5, we have

$$\left|\frac{l'''(0)}{3!}-\lambda\left(\frac{l''(0)}{2!}\right)^2\right|\le\begin{cases}3-4\lambda, & \lambda\in\left[0,\frac{1}{3}\right],\\ \frac{1}{3}+\frac{4}{9\lambda}, & \lambda\in\left[\frac{1}{3},\frac{2}{3}\right],\\ 1, & \lambda\in\left[\frac{2}{3},1\right]\end{cases} \tag{3-8}$$

and

$$\left|\left|\frac{l'''(0)}{3!}\right|-\left|\frac{l''(0)}{2!}\right|\right|\le 1. \tag{3-9}$$

Furthermore, since $l(\xi)=T_x(F(\xi x_0))=T_x(\xi x_0 f(\xi x_0))=\xi f(\xi x_0)$ for $\xi\in\mathbb{U}$, a simple computation yields that

$$\frac{l'''(0)}{3!}=\frac{T_x(D^3F(0)(x_0^3))}{3!}=\frac{D^2f(0)(x_0^2)}{2},$$

$$\left(\frac{l''(0)}{2!}\right)^2=\left(\frac{T_x(D^2F(0)(x_0^2))}{2!}\right)^2=(Df(0)(x_0))^2,$$

and

$$\frac{D^3F(0)(x_0^3)}{3!}=\frac{D^2f(0)(x_0^2)}{2!}x_0,\quad \tfrac{1}{2}D^2F(0)\left(x_0,\frac{D^2F(0)(x_0^2)}{2!}\right)=(Df(0)(x_0))^2x_0.$$

Using the above equalities, (3-8) and (3-9), we obtain all of the desired conclusions about Theorems 3.1. The example which shows the sharpness of Theorem 3.1 is the same as the mapping defined in [37]. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $f:\mathbb{B}\to\mathbb{C}$, $h:\mathbb{B}\to\mathbb{C}$ be holomorphic functions, and let $H(x)=xh(x)\in\mathcal{S}^*(\mathbb{B})$. Suppose that $F(x)=xf(x)$ is a close-to-starlike mapping with respect to $H(x)$. Then, for $x\in X\setminus\{0\}$, $T_x\in T(x)$ and $\lambda\in[0,1]$, we have the same conclusions as in* Theorem 3.1.

Proof. Fix $x \in X\backslash\{0\}$ and denote $x_0 = \frac{x}{\|x\|}$. Let $p : \mathbb{U} \to \mathbb{C}$ be given by

$$p(\xi) = \begin{cases} \frac{T_x((DH(\xi x_0))^{-1}H(\xi x_0))}{\xi}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Then $p \in H(\mathbb{U})$, $p(0) = 1$, and

$$p(\xi) = \frac{T_x((DH(\xi x_0))^{-1}H(\xi x_0))}{\xi} = \frac{T_{\xi x_0}((DH(\xi x_0))^{-1}H(\xi x_0))}{\|\xi x_0\|}, \quad \xi \in \mathbb{U} \setminus \{0\}.$$

Since $H \in \mathcal{S}^*(\mathbb{B})$, by Lemma 2.1, we have

$$\Re e(p(\xi)) > 0, \quad \xi \in \mathbb{U}. \tag{3-10}$$

At the same time, a short computation yields the relation

$$(DH(x))^{-1}H(x) = \frac{h(x)}{h(x) + Dh(x)\,x}x.$$

Hence the above relations imply that

$$p(\xi) = \frac{h(\xi x_0)}{h(\xi x_0) + Dh(\xi x_0)(\xi x_0)}, \quad \xi \in \mathbb{U}. \tag{3-11}$$

Let

$$k(\xi) = T_x(H(\xi x_0)) = \xi h(\xi x_0), \quad \xi \in \mathbb{U}.$$

Then, we have

$$k'(\xi) = h(\xi x_0) + Dh(\xi x_0)(\xi x_0)$$

and

$$\frac{\xi k'(\xi)}{k(\xi)} = \frac{h(\xi x_0) + Dh(\xi x_0)(\xi x_0)}{h(\xi x_0)}. \tag{3-12}$$

By using (3-10), (3-11) and (3-12), we obtain $k \in \mathcal{S}^*$.

Let $s(x) = (DF(x))^{-1}H(x)$, and let

$$r(\xi) = \begin{cases} \frac{T_x(s(\xi x_0))}{\xi}, & \xi \in \mathbb{U} \setminus \{0\}, \\ 1, & \xi = 0. \end{cases}$$

Then r is holomorphic on $\mathbb{U}$, $r(0) = 1$ and

$$r(\xi) = \frac{T_x(s(\xi x_0))}{\xi} = \frac{T_{x_0}(s(\xi x_0))}{\xi} = \frac{T_{\xi x_0}(s(\xi x_0))}{\|\xi x_0\|}, \quad \xi \in \mathbb{U} \setminus \{0\}.$$

Since $F(x)$ is a close-to-starlike mapping with respect to $H(x)$, from Definition 1.3, we obtain

$$\Re e(r(\xi)) > 0, \quad \xi \in \mathbb{U}.$$

A simple computation shows that

$$r(\xi)=\frac{T_x(s(\xi x_0))}{\xi}=\frac{T_x((DF(\xi x_0))^{-1}H(\xi x_0))}{\xi}$$
$$=\frac{h(\xi x_0)}{f(\xi x_0)+Df(\xi x_0)\,\xi x_0},\qquad \xi\in\mathbb{U}\setminus\{0\}.$$

Letting $l(\xi)=T_x(F(\xi x_0)),\ \xi\in\mathbb{U}$ again, we obtain

$$l'(\xi)=T_x(DF(\xi x_0)\,x_0)=f(\xi x_0)+Df(\xi x_0)\,\xi x_0.$$

Consequently, combining this observation with the preceding relations, we have

$$\Re e\left(\frac{\xi l'(\xi)}{k(\xi)}\right)=\Re e\left(\frac{f(\xi x_0)+Df(\xi x_0)\,\xi x_0}{h(\xi x_0)}\right)>0,$$

which implies that $l\in\mathcal{C}$. The remaining part of the proof of Theorem 3.2 is similar to that in the proof of Theorem 3.1, so we omit the details. □

4. Simplified proofs of coefficient inequalities for a subclass of g-starlike mappings of complex order λ

By using Lemmas 2.2, 2.4 and 2.6, we establish bounds of all terms of homogeneous expansions and the Fekete–Szegő inequality for a subclass of g-starlike mappings of complex order λ on the open unit ball of a complex Banach space, which generalize the corresponding results appeared in [25; 26; 28; 33; 36].

Theorem 4.1. *Let $g:\mathbb{U}\to\mathbb{C}$ be a convex function which satisfies the conditions of* Definition 1.9, *and $f\in H(\mathbb{B},C)$, $f(0)=1$. Suppose that $F(x)=xf(x)\in\mathcal{S}^*_{g,\lambda}(\mathbb{B})$. Then for $x\in\mathbb{B}$, we have*

$$\frac{\|D^mF(0)(x^m)\|}{m!}\le\frac{\prod_{r=2}^m\bigl[r-2+\frac{1}{1-\lambda}|g'(0)|\bigr]}{(m-1)!}\|x\|^m,\quad m=2,3,4,\ldots.$$

Proof. Fix $x\in\mathbb{B}\setminus\{0\}$ and denote $x_0=\frac{x}{\|x\|}$. Let $l(\xi)=T_x(F(\xi x_0)),\ \xi\in\mathbb{U}$. In view of Lemma 2.2, we have $l\in\mathcal{S}^*_{g,\lambda}(\mathbb{U})$. Since $l(\xi)=T_x(F(\xi x_0))=\xi f(\xi x_0)$, we obtain

$$\frac{D^mF(0)(x_0^m)}{m!}=x_0\frac{D^{m-1}f(0)(x_0^{m-1})}{(m-1)!}=\frac{l^{(m)}(0)}{m!}.$$

By using Lemma 2.6, we have

$$\frac{\|D^mF(0)(x^m)\|}{m!}\le\frac{\prod_{r=2}^m\bigl[r-2+\frac{1}{1-\lambda}|g'(0)|\bigr]}{(m-1)!}\|x\|^m,\quad m=2,3,4,\ldots,$$

as desired. □

Remark 4.2. Theorem 4.1 generalizes many known results. In Theorem 4.3 if we set $\lambda = 0$ and $g(\xi) = \frac{1+\xi}{1-\xi}$, $\xi \in \mathbb{U}$, $\lambda = 0$ and $\lambda = -\frac{\alpha}{1-\alpha}$, $0 \le \alpha < 1$, we can readily deduce the corresponding results of [28], [26] and [33], respectively. Moreover, the proofs presented here are simpler than those given in [28], [26] and [33].

Theorem 4.3. *Let* $g : \mathbb{U} \to \mathbb{C}$ *satisfy the conditions of* Definition 1.9, *and let* $f \in H(\mathbb{B}, C)$, $f(0) = 1$. *Suppose that* $F(x) = xf(x) \in \mathcal{S}^*_{g,\lambda}(\mathbb{B})$. *Then for* $\nu \in \mathbb{C}$, $x \in \mathbb{B} \setminus \{0\}$, *we have*

$$\left\| \frac{D^3F(0)(x^3)}{3!\,\|x\|^3} - \nu \frac{1}{2\|x\|^3} D^2F(0)\left(x, \frac{D^2F(0)(x^2)}{2!}\right) \right\| \le \frac{|g'(0)|}{2|1-\lambda|} \max\left\{1, \left|\frac{1}{2}\frac{g''(0)}{g'(0)} + \frac{1-2\nu}{1-\lambda} g'(0)\right|\right\}$$

and

$$\left| \frac{T_x(D^3F(0)(x^3))}{3!\,\|x\|^3} - \nu \left(\frac{T_x(D^2F(0)(x^2))}{2!\,\|x\|^2}\right)^2 \right| \le \frac{|g'(0)|}{2|1-\lambda|} \max\left\{1, \left|\frac{1}{2}\frac{g''(0)}{g'(0)} + \frac{1-2\nu}{1-\lambda} g'(0)\right|\right\}.$$

The above estimates are sharp.

Proof. Fix $x \in \mathbb{B} \setminus \{0\}$ and denote $x_0 = \frac{x}{\|x\|}$. Let $l(\xi) = T_x(F(\xi x_0))$, $\xi \in \mathbb{U}$. From Lemma 2.2, we have $l \in \mathcal{S}^*_{g,\lambda}(\mathbb{U})$. Since $l(\xi) = T_x(F(\xi x_0)) = \xi f(\xi x_0)$, we have

$$\frac{l'''(0)}{3!} = \frac{T_x(D^3F(0)(x_0^3))}{3!} = \frac{D^2 f(0)(x_0^2)}{2},$$
$$\left(\frac{l''(0)}{2!}\right)^2 = \left(\frac{T_x(D^2F(0)(x_0^2))}{2!}\right)^2 = (Df(0)(x_0))^2,$$

and

$$\frac{D^3F(0)(x_0^3)}{3!} = \frac{D^2 f(0)(x_0^2)}{2!} x_0, \quad \frac{1}{2} D^2F(0)\left(x_0, \frac{D^2F(0)(x_0^2)}{2!}\right) = (Df(0)(x_0))^2 x_0.$$

Using the above equalities and Lemma 2.4, we obtain the desired conclusion.

In order to prove that the estimates of Theorem 4.3 are sharp, it suffices to consider the following examples.

If $\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + \frac{1-2\nu}{1-\lambda} g'(0)\right| \ge 1$, we consider the example

$$F(x) = x \exp \frac{1}{1-\lambda} \int_0^{T_u(x)} (g(t) - 1)\frac{dt}{t}, \quad x \in \mathbb{B},\ \|u\| = 1.$$

If $\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + \frac{1-2\nu}{1-\lambda} g'(0)\right| \le 1$, we consider the example

$$F(x) = x \exp \frac{1}{1-\lambda} \int_0^{T_u(x)} (g(t^2) - 1)\frac{dt}{t}, \quad x \in \mathbb{B},\ \|u\| = 1. \qquad \square$$

Remark 4.4. If we set $\lambda = -i \tan \beta$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ and $\lambda = 0$ in Theorem 4.3, we obtain the corresponding results of [25] and [36], respectively. Moreover, the proofs presented here are simpler than those given in [25] and [36].

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