Sp(1)-SYMMETRIC HYPERKÄHLER QUANTISATION

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We provide a new general scheme for the geometric quantisation of Sp(1)-symmetric hyperkähler manifolds, considering Hilbert spaces of holomorphic sections with respect to the complex structures in the hyperkähler 2-sphere. Under properness of an associated moment map, or other finiteness assumptions, we construct unitary (super) representations of groups acting by Riemannian isometries preserving the 2-sphere, and we study their decomposition in irreducible components. We apply this scheme to hyperkähler vector spaces, the Taub–NUT metric on $\mathbb{R}^4$, moduli spaces of framed SU(r)-instantons on $\mathbb{R}^4$, and in part to the Atiyah–Hitchin manifold of magnetic monopoles in $\mathbb{R}^3$.

1. Introduction

The constructions of geometric quantisation offer a recipe for addressing problems related to the quantum mechanics of an object moving in an arbitrary, possibly curved, phase space [42; 89]. The process, abstracting canonical quantisation, is fundamentally based on the structure of a symplectic manifold. Two of the main goals are to obtain operators subject to commutation relations prescribed by the Poisson bracket, and unitary representations of groups associated to Hamiltonian flows. However, there are strong limitations to the extent to which these can be achieved in general. One of the most typical problems is the need of a polarisation, whose existence is generally not guaranteed, nor is its uniqueness ever satisfied.

Furthermore, the choice of a particular polarisation poses serious constraints on which functions and Hamiltonian flows can be quantised.

A common approach to this issue consists in considering instead a *family* of polarisations, parametrised by some smooth manifold. One then attempts to assemble their corresponding quantum Hilbert spaces into a vector bundle and identify them via the holonomy of some appropriate connection. In this framework, the natural way to quantise Hamiltonian group actions is by automorphisms of the bundle as a whole rather than of the individual vector spaces. A group representation is then usually obtained by considering the space of (projectively) flat sections. The prototypical example of this is the Hitchin connection [16; 45], further discussed below. The latter also has a simple yet interesting adaptation to the case of a symplectic linear space, providing a quantisation of its full symplectic group in the form of a representation of a double cover of it, the metaplectic representation [89, Chapter 10].

Because polarisations on a symplectic manifold often arise as compatible complex structures, it is rather common in geometric quantisation to work with Kähler manifolds [30; 89]. This approach has been successfully applied to a number of moduli spaces arising from differential and algebraic geometry, representation theory, and mathematical physics. Notable examples include unitary flat connections [16; 45] and vector bundles on Riemann surfaces, compact coadjoint orbits [53], and polygons [29; 51]. Unitary flat connections in particular are a good example of the scheme sketched above, as the moduli space comes with a family of Kähler structures parametrised by the Teichmüller space. The construction of a projectively flat connection in that setting is due to Hitchin [45] and Axelrod, Della Pietra, and Witten [16] and was extended to a broader framework in later works [1; 3; 8]. The role of flat connections in Chern–Simons theory [38; 86] also motivated further study of the relation between geometric quantisation and other formulations of the theory, including deformation quantisation [2; 6; 24; 52; 63; 78; 79] and other approaches [9; 10; 11; 12; 59].

In many cases, spaces similar to those above, and related to interesting quantisation problems, come with natural *hyperkähler* structures rather than just Kähler. Some of these may be viewed as complexifications of those already mentioned, e.g., flat connections for complex groups [5; 7; 63; 73; 87], Higgs bundles [4; 31; 35; 44; 80], semisimple/nilpotent (co)adjoint orbits in (dual) complex Lie algebras [17; 55; 58], hyperpolygons, and Nakajima quiver varieties [41; 70]. Others, on the other hand, arise independently of an underlying "real" version, including for instance the Taub–NUT metrics on $\mathbb{R}^4$, moduli spaces of framed $\mathrm{SU}(r)$-instantons and magnetic monopoles, and the Nahm moduli spaces.

Crucially, complex structures on these spaces give rise to families of Kähler forms, whose parametrising spaces come with their own Kähler structures — isomorphic

to that of $\mathbb{C}P^1$. Unlike in the Kähler case, geometric quantisation does not directly apply in this situation because no preferred symplectic structure is given in general. What is more, many interesting symmetries of such spaces act by hyperkähler rotations, i.e., by permuting the sphere of Kähler structures rather than fixing each of them individually. If one of the symplectic forms is fixed by the action, one may focus on that particular structure and apply quantisation with respect to it, an approach that was carried out by Andersen, Gukov, and Pei [4] in the case of the Hitchin moduli space. Nonetheless, one may still wish to obtain a version of quantisation with respect to other Kähler forms, or to the hyperkähler structure as a whole. In addition, the induced action on the sphere is in many cases transitive, suggesting again that a more "global" approach should be taken in that situation.

The latter is precisely the setup that we are going to address in this work. Namely, we shall consider a hyperkähler manifold M acted on by a compact Lie group G by hyperkähler isometries and assume that the induced action on $\mathbb{C}P^1$ is transitive. We will also assume that a smooth family of prequantum line bundles on M is given, parametrised by $\mathbb{C}P^1$, together with a lifted equivariant G-action. Carrying out geometric quantisation for each individual symplectic form will give rise to a family of Hilbert spaces, typically of infinite dimension. We will then attempt to use representation theory to "break down" these spaces into finite-dimensional components and assemble each family into a vector bundle over $\mathbb{C}P^1$. We study these objects explicitly and show that their structure is determined by the combinatorics of irreducible subrepresentations in the Hilbert spaces. In particular, we construct natural connections on these bundles and explicitly characterise their curvatures. While the resulting connection on the overall family fails to be projectively flat, we notice that it defines a *holomorphic* structure on it. Based on this, we propose a definition of the overall quantum Hilbert space as the supercohomology of this object, thus obtaining a natural G-representation as a space of sections of a bundle over $\mathbb{C}P^1$.

1A. *Description of the main construction.*

Let us expand and detail the description sketched above. Suppose a hyperkähler manifold M is given and that G is a compact connected Lie group acting on it by hyperkähler rotations — by this we mean that G acts on M by isometries which permute the Kähler structures on M; we will additionally require that the induced action be transitive. Since Kähler forms are parametrised by $\mathbb{C}P^1$, this corresponds to a surjective group homomorphism $G \twoheadrightarrow \mathrm{SO}(3)$. As we shall see, this implies that G is covered by a product $\mathrm{Sp}(1) \times G_0$, with $\mathrm{Sp}(1)$ acting on $\mathbb{C}P^1$ in the usual way and G_0 fixing all Kähler structures.

Since no preferred symplectic form is given on M, it makes little sense to talk about a prequantum line bundle over the hyperkähler manifold. Instead, we will assume that M is equipped with a Hermitian line bundle (L, h) and a prequantum connection ∇_q for each symplectic form ω_q, depending smoothly on $q \in \mathbb{C}P^1$ in

an appropriate sense (see Section 2B). We will further assume that the G-action lifts to L in such a way as to permute the connections equivariantly.

If M_q denotes the Kähler manifold corresponding to $q \in \mathbb{C}P^1$, its geometric quantisation consists of the Hilbert space $\mathcal{H}_q$ of L^2 holomorphic sections of the corresponding prequantum line bundle. The embedding into $\mathbb{C}P^1 \times L^2(M, L)$ defines a Hermitian structure on this family, viewed informally as a vector bundle over $\mathbb{C}P^1$, together with a compatible connection and Sp(1)-equivariant G-action. We study this object by decomposing each fibre $\mathcal{H}_q$ into isotypical components as a representation of (appropriate subgroups of) $G_q := \text{Stab}_G(q)$. By transitivity of the Sp(1)-action, all these stabilisers are conjugated, and the respective isotypical components form constant-rank subfamilies of $\mathcal{H}$. The following is the central theorem of our work.

Theorem 1.1 (see Theorem 2.11). *Suppose that*:

- *M is a hyperkähler manifold.*
- *G is a connected compact Lie group acting on M by fixing the metric and permuting the symplectic forms transitively.*
- *$L \to M$ is a Hermitian line bundle with a family of prequantum connections as in Section 2B and a G-action covering that on M.*
- *ρ is an irreducible representation of $G_q := \text{Stab}_G(\omega_q)$ for ω_q one of the symplectic forms on M.*
- *ρ has finite multiplicity $m^{(\rho)}$ in the space $\mathcal{H}_q$ of L^2 holomorphic sections of $L \to M$ with respect to the structure associated to ω_q.*

For each other symplectic form $\omega_{q'}$, denote by $\mathcal{H}_{q'}^{(\rho)}$ the isotypical component in $\mathcal{H}_{q'}$ corresponding to ρ under the identification $G_q \simeq G_{q'}$ by conjugation in G. Then the collection of spaces $\mathcal{H}^{(\rho)}$ has a canonical structure of Hermitian vector bundle over $\mathbb{C}P^1$ with compatible connection. Moreover, for some integer $d = d_\rho$ there exists an isomorphism

$$(1) \qquad\qquad \mathcal{H}^{(\rho)} \simeq (\mathcal{L}^d \otimes \underline{V_\rho})^{\oplus m^\rho}$$

preserving the Hermitian structure and connection, where $\underline{V_\rho} = \mathbb{C}P^1 \times V_\rho$ carries the trivial connection and $\mathcal{L} \to \mathbb{C}P^1$ is the standard degree-1 Sp(1)-equivariant Hermitian line bundle with connection.

The result implies that, informally speaking, the family $\mathcal{H}$ decomposes as a sum of vector bundles with connections, as long as the appropriate multiplicities are finite. The holonomies of the various components may then be assembled to form parallel transport operators on $\mathcal{H}$. However, (1) also determines the curvature of the connection on each component, which is proportional to the degree d. Consequently,

the parallel transport operators on $\mathcal{H}$ depend essentially on the choice of paths on the base and fail to unambiguously identify the different Hilbert spaces — even projectively.

Nonetheless, the components $\mathcal{H}^{(\rho)}$ may also be regarded as G-equivariant *holomorphic* bundles over $\mathbb{C}P^1$. We then obtain G-representations not as spaces of flat sections as customary, but as the cohomology of $\mathcal{H}^{(\rho)}$ as a super vector space. The following then descends from Theorem 1.1.

Theorem 1.2 (see Section 2F). *In the setting of Theorem 1.1, define*

$$(2) \qquad\qquad H^{(\rho)} := H^*(\mathbb{C}P^1, \mathcal{H}^{(\rho)})$$

as a super vector space. Then $H^{(\rho)}$ comes with a Hermitian structure and compatible G-action. With d as in Theorem 1.1, it is a direct sum of $|d+1|m^{(\rho)}$ copies of V_ρ, all in even (resp. odd) degree if $d \geq 0$ (resp. $d < 0$). In particular, the completed orthogonal sum

$$(3) \qquad\qquad H := \overline{\bigoplus_\rho H^{(\rho)}},$$

with ρ ranging over all the isomorphism classes of irreducible G-representations, defines a Hilbert space G-representation, and (3) is the isotypical decomposition.

This viewpoint also lends itself to an approach in terms of rank-generating series and localisation formulæ, something which we address in Section 2H.

Again, the space H of (3) may informally be thought of as the cohomology of the sum of all the $\mathcal{H}^{(\rho)}$'s, regarded now as a *holomorphic* vector bundle over $\mathbb{C}P^1$. It is interesting to note how this is reminiscent of the description of M in terms of its twistor space, a holomorphic fibration $Z \to \mathbb{C}P^1$ (plus additional holomorphic data). It would be an interesting problem to investigate whether our setup can be obtained in terms of twistor data by purely holomorphic constructions, something which we would like to address in a separate work.

The most crucial assumptions in our construction, besides the surjectivity of $G \to SO(3)$, is the finite-dimensionality of the isotypical components in $\mathcal{H}_q$. For that reason, we also investigate sufficient conditions to ensure it. They can be summarised as follows.

Theorem 1.3 (see Theorems 2.14 and 2.17). *Suppose one of the Kähler forms ω_q on M is fixed, $S \subseteq \mathrm{Stab}_G(\omega_q)$ is a connected Lie subgroup, and ρ is an isomorphism class of S-representations. Then each of the following is a sufficient condition for the corresponding S-isotypical component in $\mathcal{H}_q$ to have finite dimension.*

- *The Kostant moment map for S is proper on M_q, and its action extends holomorphically to the complexification of S.*

- *S is a torus, M_q has the structure of an affine scheme or Stein space, and the Kostant moment map for S is proper.*

- *S is a torus, M_q has the structure of an affine scheme or Stein space, and $M /\!\!/_w S$, w the weight of ρ, admits a compactification with rational singularities and boundary of codimension greater than or equal to 2.*

A further way to ensure finite dimensionality can be found in the discussion of meromorphic torus actions in [90].

1B. *Applications and further directions.* In Section 3 we showcase applications of the main construction. The first one is a hyperkähler vector space V of real dimension $4n$, with $n \in \mathbb{Z}_{\geq 1}$. In this case

$$\mathrm{Hk}(V) \simeq \mathrm{Sp}(n) \cdot \mathrm{Sp}(1),$$

identifying $V \simeq \mathbb{H}^n$ (see Remark 3.2). Indeed, under this isomorphism, $\mathrm{Sp}(1)$ acts on V via right multiplication of unit-norm imaginary quaternions, and commutes with the natural $\mathrm{Sp}(n)$-action. Furthermore the norm associated to the hyperkähler metric provides a hyperkähler potential and we can apply the abstract construction (see Theorems 3.5 and 3.6).

Importantly, there are many more examples of (nonflat) $\mathrm{Sp}(1)$-symmetric hyperkähler manifolds. These include moduli spaces of magnetic monopoles on $\mathbb{R}^3$ by the work of Atiyah and Hitchin [14] or equivalently, by the work of Donaldson [34], the moduli spaces of based rational maps from $\mathbb{C}P^1$ to itself; moduli spaces of framed $\mathrm{SU}(r)$-instantons on $\mathbb{R}^4$, by the work of Maciocia [62]; the hyperkähler structure on nilpotent orbits, by the work of Kronheimer [57], and more generally the hyperkähler Swann bundle over any quaternionic Kähler manifold [82]. In four dimensions a complete classification of $\mathrm{Sp}(1)$-symmetric hyperkähler manifolds is given (up to finite covers) by the work of Gibbons and Pope [40] and by Atiyah and Hitchin [14]. The three examples are the flat metric on $\mathbb{H}$, the Taub–NUT metric, and the hyperkähler metric on the moduli space of charge-2 monopoles, i.e., the Atiyah–Hitchin manifold.

We establish in Sections 3B and 3C that the Theorems 1.1, 1.2 and 2.17 (or slight modifications thereof) apply to some of these examples, producing a quantisation and corresponding irreducible unitary (super)representations of distinguished groups of hyperkähler isometries.

2. Abstract Sp(1)-symmetric hyperkähler quantisation

2A. *Hyperkähler manifolds and their symmetry groups.* Let n be a positive integer and M a smooth manifold of dimension $4n$.

Definition 2.1. A hyperkähler structure on M consists of a Riemannian metric g and an ordered triple (I, J, K) of covariant constant orthogonal automorphism of TM satisfying the quaternionic identities $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{TM}$.

It follows that I, J, K are g-skew-symmetric global sections of $\operatorname{End}(TM) \to M$, and we denote by $\mathfrak{su}(2)_M$ the three-dimensional real Lie algebra they span.

The hyperkähler 2-sphere of complex structures of (M, g, I, J, K) is

(4) $\quad \mathbb{S}_{IJK} := \{I_q = aI + bJ + cK \mid q = (a, b, c) \in \mathbb{R}^3, \ a^2 + b^2 + c^2 = 1\} \subseteq \mathfrak{su}(2)_M$.

As customary, the structure on M identifies (4) with the 2-sphere of unit-norm imaginary quaternions, i.e., with $\mathbb{C}P^1$ as a Kähler manifold. In particular for $q \in \mathbb{C}P^1$ there is a (real) symplectic form on M defined by

$$\omega_q(v, w) := g(I_q v, w) \quad \text{for } v, w \in TM.$$

The triple $M_q := (M, I_q, \omega_q)$ is a Kähler manifold, and for further use we denote by $\mu_q = \operatorname{d vol} \in \Omega^{\text{top}}(M)$ the Liouville volume form — independent of $q \in \mathbb{C}P^1$ as it agrees with the Riemannian volume form of (M, g).

Remark 2.2. The above data can be encoded in a fibration $\pi_{\mathbb{C}P^1} : Z \to \mathbb{C}P^1$ of Kähler manifolds over the Riemann sphere, the *twistor space* of (M, g, I, J, K). Clearly this family comes with a natural global trivialisation $Z \simeq M \times \mathbb{C}P^1$ as a smooth fibre bundle, but not as fibre bundle with symplectic or complex fibres. Nonetheless the natural complex structure on Z makes $Z \to \mathbb{C}P^1$ into a *holomorphic* fibre bundle [46, pp. 141-142].

Now consider the group $\operatorname{Sp}(M) = \operatorname{Sp}(M, g, I, J, K) \subseteq \operatorname{Iso}(M, g)$ of Riemannian isometries of (M, g) preserving the Kähler forms ω_q (or equivalently the complex structures I_q) simultaneously for all $q \in \mathbb{C}P^1$. This group is sometimes referred to as the *hyperunitary* group. Denoting $\operatorname{Aut}_0(Z)$ the group of holomorphic automorphisms of $Z \to \mathbb{C}P^1$ over the identity, there is a natural group homomorphism

(5) $$\operatorname{Sp}(M) \to \operatorname{Aut}_0(Z),$$

given by the fibrewise action of $\operatorname{Sp}(M)$.

We shall consider a group of isometries that preserve the hyperkähler structure in a looser sense, relaxing the condition that differentials should commute with I, J, and K individually.

Definition 2.3. Let $\operatorname{Hk}(M) \subseteq \operatorname{Iso}(M, g)$ be the subgroup stabilising the Lie algebra $\mathfrak{su}(2)_M$:

$$\operatorname{Hk}(M) = \operatorname{Hk}(M, g, I, J, K) := \{\varphi \in \operatorname{Iso}(M, g) \mid \operatorname{Ad}_{d\varphi}(\mathfrak{su}(2)_M) = \mathfrak{su}(2)_M\}.$$

Hence Hk(M) acts on $\mathfrak{su}(2)_M$, and Sp(M) $\subseteq$ Hk(M) is the kernel of this action. Moreover the adjoint action Ad on $\mathfrak{su}(2)_M \simeq \mathbb{R}^3$ is by positive isometries for the standard Euclidean structure, resulting in a group morphism

$$\mathrm{Ad} : \mathrm{Hk}(M) \to \mathrm{SO}(3), \quad \mathrm{Ad} : \varphi \mapsto \mathrm{Ad}_{\mathrm{d}\varphi},$$

and an action on the hyperkähler 2-sphere (4)— simply denoted by $q \mapsto \varphi.q$. The combination of the actions of $\mathrm{Hk}(M)$ on $\mathbb{C}P^1$ and M itself naturally extends (5) to a map

$$(6) \qquad\qquad\qquad \mathrm{Hk}(M) \to \mathrm{Aut}(Z),$$

where $\mathrm{Aut}(Z)$ denotes the full group of biholomorphisms of Z compatible with the fibration map and covering arbitrary Kähler automorphisms of $\mathbb{C}P^1$.

Suppose now given a connected compact Lie group G, and a G-action

$$\rho : G \to \mathrm{Hk}(M)$$

on M by transformations in $\mathrm{Hk}(M)$. We will sometimes denote by

$$\rho^Z : G \to \mathrm{Aut}(Z)$$

the composition of ρ with (6); where unambiguous, we will often denote the G-action simply by $(\rho(g))(p) = gp$, and similarly for ρ^Z. As in the introduction, we require that the induced G-action on $\mathbb{C}P^1$ be transitive, or equivalently that the corresponding map $G \to \mathrm{SO}(3)$ be surjective. The kernel G_0 of this action is then also a compact Lie group, and by construction it acts on M by transformations in Sp(M).

Lemma 2.4. *The induced G-action on $\mathbb{C}P^1$ factors through a morphism*

$$(7) \qquad\qquad\qquad \sigma : \mathrm{Sp}(1) \to G$$

from the universal cover $\mathrm{Sp}(1)$ *of* $\mathrm{SO}(3)$. *This, moreover, arises from a covering map* $G_0 \times \mathrm{Sp}(1) \to G$.

Proof. By compactness, the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ admits a nondegenerate invariant pairing. Once such a pairing is fixed, the orthogonal complement of $\mathfrak{g}_0 := \mathrm{Lie}(G_0)$ is a Lie subalgebra which maps isomorphically to $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$. This induces a section $\mathfrak{su}(2) \to \mathfrak{g}$ which integrates to the desired map σ. In fact, since $\mathfrak{g}_0$ and $\mathfrak{g}_0^\perp$ commute with each other, the splitting $\mathfrak{g} \simeq \mathfrak{g}_0 \oplus \mathfrak{g}_0^\perp$ is an isomorphism of Lie algebras. In particular, every element of G_0 commutes with $\sigma(\mathrm{Sp}(1))$, resulting in a map $G_0 \times \mathrm{Sp}(1) \to G$. $\qquad\qquad\square$

In other words, for G as above, a G-action by transitive hyperkähler rotations always comes from an $\mathrm{Sp}(1)$-action, and up to covers it splits as the product with an action by Sp(M). Henceforth we shall fix a group homomorphism σ as in (7).

2B. *Prequantum data*. As already noted, a notion of prequantum line bundle on M is ill-posed, since a hyperkähler manifold comes with a continuous family of incompatible prequantum conditions. Instead, we will assume given a Hermitian line bundle (L, h) on M together with a smooth family of compatible connections ∇_q, $q \in \mathbb{C}P^1$, each with curvature $F_q = -i\omega_q$. The smoothness in q may be expressed by the condition that, if a section ψ of $\pi_M^* L \to Z$ is smooth, then so is the family $\nabla_q \psi|_q$, as a section of $\pi_M^*(L \otimes T^*M)$. Equivalently, for every local trivialisation of L, the induced connection potentials should depend smoothly on $q \in \mathbb{C}P^1$. Together with the trivial derivative along the directions of $\mathbb{C}P^1$ in $Z \simeq M \times \mathbb{C}P^1$, these ∇_q's assemble to form a connection on $\pi^* L \to Z$. Additionally, we will require that L be equipped with a Hermitian G-action

$$\rho^L : G \to \mathrm{Aut}(L, h),$$

which lifts the one on M and permutes the connections equivariantly.

In practice, the G-action may not come with preferred prequantum data as above. Now we investigate criteria to determine whether such data exist for a given action.

A necessary condition for the existence of a prequantum line bundle on a symplectic manifold is that the symplectic form represent an integral class in cohomology. Conversely, in that case a prequantum line bundle can be constructed by a diagram chasing procedure on the Čech–de Rham complex [89].

In our situation, we will require that

$$[\omega_q] \in H^2(M, \mathbb{Z}) \quad \text{for all } q \in \mathbb{C}P^1.$$

In fact, if the condition holds for at least one q, then by the Sp(1)-action it does for all q, and it then follows by continuity that $[\omega_q]$ is independent of q. The diagram chasing procedure mentioned above may then be carried out with differential forms on M depending smoothly on q. Hence a family of prequantum line bundles exists if and only if $[\omega_q]$ is integral for some q.

Suppose such a family is fixed, with underlying Hermitian line bundle (L, h), and let $L_q := (L, h, \nabla_q)$ for each q. For every $g \in G$, the structure of $g^* L_{g.q} \otimes L_q^{-1}$ defines a family of *flat* Hermitian connections. Since such objects are classified up to isomorphism by $\Gamma := H^1(M, \mathrm{U}(1))$, this defines a map

$$(8) \qquad u : G \to C^\infty(\mathbb{C}P^1, \Gamma), \quad u : g \mapsto (q \mapsto [g^* L_{g.q} \otimes L_q^{-1}]).$$

Viewing the abelian group $\Gamma' := C^\infty(\mathbb{C}P^1, \Gamma)$ as a G-module under the pull-back action, u defines a cocycle in $C^1(G, \Gamma')$.

Lemma 2.5. *Suppose (L, h) is a Hermitian line bundle over M with a family of prequantum connections ∇_q smoothly parametrised by $\mathbb{C}P^1$. The cohomology class of the coycle u from (8) vanishes in $H^1(G, \Gamma')$ if and only if there exist*:

- *a Hermitian line bundle B and*
- *a family of Hermitian flat connections ∇_q^B smoothly parametrised by $\mathbb{C}P^1$*

such that, for all $q \in \mathbb{C}P^1$ and $g \in G$, we have

$$(9) \qquad g^*(L_{g.q} \otimes B_{g.q}^{-1}) \simeq L_q \otimes B_q^{-1}$$

as Hermitian line bundles with connection, where $B_q := (B, \nabla_q^B)$.

Proof. Suppose such a family exists. Then (9) is equivalent to

$$g^* L_{g.q} \otimes L_q^{-1} \simeq g^* B_{g.q} \otimes B_q^{-1},$$

i.e., $u = \delta\Lambda$ for $\Lambda(q) := [B_q] \in \Gamma$, and therefore $[u] = 0$.

Conversely, suppose that $u = \delta\Lambda$ for some $\Lambda \in \Gamma'$. It follows from the exact sequence

$$H^1(M, \mathbb{R}) \to H^1(M, \mathrm{U}(1)) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$$

that the components of $\Gamma = H^1(M, \mathrm{U}(1))$ are labelled by the torsion of $H^2(M, \mathbb{Z})$, while the identity component is covered by $H^1(M, \mathbb{R})$. Since $\Lambda : \mathbb{C}P^1 \to \Gamma$ is a continuous map, we may fix some $\Lambda_0 \in \Gamma$ so that $\Lambda - \Lambda_0$ takes values in the identity component. Since $\mathbb{C}P^1$ is simply connected, this lifts to a map $\tilde{\Lambda} : \mathbb{C}P^1 \to H^1(M, \mathbb{R})$. Choose a collection of 1-forms $\alpha_1, \ldots, \alpha_n$ on M whose de Rham cohomology classes form a basis of $H^1(M, \mathbb{R})$. Expressing $\tilde{\Lambda}$ as

$$\tilde{\Lambda}(q) = \sum_{i=1}^n c_i(q)[\alpha_i],$$

we see that each c_i is a smooth function of q and therefore $\alpha = \sum_{i=1}^n c_i \alpha_i$ is a smooth family of 1-forms on M parametrised by $\mathbb{C}P^1$. Choosing a representative (B, ∇_0^B) of $\Lambda_0 \in H^1(M, \mathrm{U}(1))$ and setting $\nabla_q^B := \nabla_0^B + \alpha(q)$, it follows by construction that

$$\Lambda(q) = [(B, \nabla_q^B)].$$

Expanding and manipulating the condition $u = \delta\Lambda$ leads to (9). $\square$

Lemma 2.5 shows that, if $[u] = 0$, then L may be replaced by a new family of prequantum line bundles on which the action of every element of G admits an equivariant lift. In that case, the action of the group

$$G' := \{\varphi \in \mathrm{Aut}(L, h) \mid \varphi \text{ covers some } g \in G\}$$

covers that of G on M surjectively while permuting the connections equivariantly. Notice moreover that G' is also compact and connected, being a central extension of the image of G in $\mathrm{Hk}(M)$, so the discussion from the previous section also applies to it. In particular, by Lemma 2.4, there exists a map $\sigma^L : \mathrm{Sp}(1) \to G'$ lifting the $\mathrm{Sp}(1)$-action on M. Even though there may not be a lifted G-action on L, we obtain

one by replacing the group with G', which does not essentially change the action on M.

The simplest vanishing $[u] = 0$ is obtained if Γ is trivial (which we will see in some examples) or by the existence of a hyperkähler potential (which we discuss in Section 2I).

Up to the necessary replacements, in what follows we thus assume to have fixed a family of prequantum connections with an equivariant action $\rho^L : G \to L$.

2C. *Geometric quantisation.* Following the prescription of geometric quantisation, for $q \in \mathbb{C}P^1$ consider the separable Hilbert space

$$(10) \qquad \mathcal{H}_q := \left\{ \psi \in H^0(M_q, L_q) \,\Big|\, \int_M h(\psi, \psi) \, \mathrm{d\,vol} < \infty \right\} \subseteq L^2(M, L),$$

using the holomorphic structure $\bar{\partial}_q = \nabla_q^{0,1}$ and the standard L^2 Hermitian product:

$$(11) \qquad \langle \psi \mid \psi' \rangle := \int_M h(\psi, \psi') \, \mathrm{d\,vol}, \quad \psi, \psi' \in \mathcal{H}_q.$$

Let us denote by $\mathcal{H}$ the family of Hilbert spaces thus defined over $\mathbb{C}P^1$.

By construction there are unitary isomorphisms

$$\rho_g^{\mathcal{H}} : \mathcal{H}_q \to \mathcal{H}_{g.q}, \quad q \in \mathbb{C}P^1, \, g \in G,$$

explicitly given by

$$(12) \qquad (\rho_g^{\mathcal{H}} \psi)(m) := \rho_g^L(\psi(\rho_{g^{-1}}^Z m)), \quad m \in M.$$

2D. *Decomposition of $\mathcal{H}_q$.* We will now consider the decompositions of the spaces (10) induced by viewing them as representations under the action (12). For a given $q \in \mathbb{C}P^1$, restricting $\rho^{\mathcal{H}}$ to

$$G_q := \mathrm{Stab}_G(q),$$

defines a group action on $\mathcal{H}_q$ by unitary operators, i.e., a Hilbert space representation. By the Peter–Weyl theorem [54, Theorem 1.12], $\mathcal{H}_q$ decomposes a completed orthogonal sum of irreducible components. Similarly, denoting by

$$T_q := \mathrm{Stab}_{\mathrm{Sp}(1)}(q)$$

the maximal torus in $\mathrm{Sp}(1)$ fixing q, its action on $\mathcal{H}_q$ gives a decomposition

$$\mathcal{H}_q = \overline{\bigoplus_{d \in \mathbb{Z}} \mathcal{H}_q^{(d)}},$$

where $\mathcal{H}_q^{(d)} \subseteq \mathcal{H}_q$ is the isotypical component corresponding to the character

$$T_q \simeq \mathrm{U}(1) \to \mathbb{C}^\times, \quad z \mapsto z^d,$$

under the natural identification with the standard torus $U(1) \subseteq \mathbb{C}^\times$. Since T_q commutes with $G_0 \subseteq G_q$, each component $\mathcal{H}_q^{(d)}$ is a representation of G_0. Therefore, we obtain a refinement of the decomposition above as

$$(13) \qquad \qquad \mathcal{H}_q^{(d)} = \overline{\bigoplus_{\lambda \in \Lambda} \mathcal{H}_{q,\lambda}^{(d)}},$$

where Λ denotes the set of (analytically) integral weights of G_0 and $\mathcal{H}_{q,\lambda}^{(d)}$ is the isotypical component in $\mathcal{H}_q^{(d)}$ of maximal weight λ. In what follows we shall often denote by $\Lambda^{(d)} \subseteq \Lambda$ the subset of "active" representations.[1]

Remark 2.6. It is not difficult to see, given that G_q is covered by $T_q \times G_0$, that every irreducible representation of G_q has a single weight for T_q and is also irreducible for G_0. Therefore, every irreducible G_q-representation induces a pair (d, λ) of weights for T_q and G_0, by which the representation itself is unambiguously determined. In particular, the decomposition (13) is equivalent to the one into isotypical components under G_q.

In a similar way we may also consider a maximal torus $T \subseteq G_0$ and find

$$\mathcal{H}_q^{(d)} = \overline{\bigoplus_{a \in T^\vee} \mathcal{H}_{a,q}^{(d)}},$$

where $\mathcal{H}_{a,q}^{(d)} \subseteq \mathcal{H}_q^{(d)}$ is the isotypical component of the character $a : T \to \mathbb{C}^\times$. Again, the decomposition above is equivalent to the one we would obtain by considering the action of the maximal torus $T'_q := T_q \cdot T \subseteq G_q$ on $\mathcal{H}_q$.

We denote $\mathcal{H}^{(d)}$, $\mathcal{H}_a^{(d)}$ and $\mathcal{H}_\lambda^{(d)}$ the families of Hilbert spaces thus defined over $\mathbb{C}P^1$, so that we have L^2-completed orthogonal direct sums

$$\mathcal{H} = \overline{\bigoplus_{d \in \mathbb{Z}} \mathcal{H}^{(d)}}, \quad \overline{\bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda^{(d)}} = \mathcal{H}^{(d)} = \overline{\bigoplus_{a \in T^\vee} \mathcal{H}_a^{(d)}}.$$

2E. *Structure of $\mathcal{H}_\lambda^{(d)}$.* Our main assumption, unless otherwise stated, is as follows:

The spaces $\mathcal{H}_{q,\lambda}^{(d)}$ are finite-dimensional.

The goal of this section is to make each family $\mathcal{H}_\lambda^{(d)}$, under the finite-dimensionality condition above, into a vector bundle, and equip it with a connection. The simplest way to define a smooth structure on $\mathcal{H}_\lambda^{(d)}$ is to assume it is a Banach sumbanifold of the product $\mathbb{C}P^1 \times L^2(M, L)$, with G acting smoothly on it. We shall now investigate the structure induced by this assumption, and later show that the same data can be obtained canonically from the group structure.

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[1]The subset $\Lambda^{(d)}$ is independent of $q \in \mathbb{C}P^1$ since the $Sp_0{'}(M)$-modules $\mathcal{H}_q^{(d)}$ are isomorphic under the $Sp(1)$-action.

Now let (temporarily, see Remark 2.12 below) the family of Hilbert spaces $\mathcal{H}_\lambda^{(d)}$ forms a *smooth* Banach submanifold of the trivial Hilbert bundle $\mathrm{L}^2(M, L) \to \mathbb{C}P^1$. We can then differentiate smooth local sections ψ of $\mathcal{H}_\lambda^{(d)} \to \mathbb{C}P^1$, viewed as maps $\mathbb{C}P^1 \to \mathrm{L}^2(M, L)$, along tangent vectors on $\mathbb{C}P^1$. Then, since $\mathcal{H}_{q,\lambda}^{(d)} \subseteq \mathrm{L}^2(M, L)$ is a closed subspace, there are orthogonal projections

$$\pi_{q,\lambda}^{(d)} : \mathrm{L}^2(M, L) \to \mathcal{H}_{q,\lambda}^{(d)}.$$

Definition 2.7. For any tangent vector $X \in T_q\mathbb{C}P^1$ set

$$\nabla_X^{\mathcal{H}_\lambda^{(d)}} \psi := \pi_{q,\lambda}^{(d)}(X[\psi]) \in \mathcal{H}_{q,\lambda}^{(d)}.$$

Remark 2.8. The same definition (of the standard L^2-connection) can be given verbatim in the case where the families $\mathcal{H}_a^{(d)} \subseteq \mathcal{H}^{(d)}$ also constitute smooth submanifolds.

Remark 2.9. This covariant derivative is characterised by the property that

$$\langle \nabla_X^{\mathcal{H}_\lambda^d} \psi | \psi' \rangle = \langle X[\psi] | \psi' \rangle$$

for all X, ψ, ψ' as appropriate.

Proposition 2.10. *The covariant-derivative operators of Definition 2.7 are compatible with the action $\rho^{\mathcal{H}}$ of (12) and with the Hermitian structure of $\mathcal{H}_\lambda^{(d)} \to \mathbb{C}P^1$.*

Proof. The operators $\nabla_X^{\mathcal{H}_\lambda^{(d)}}$ satisfy Leibniz and preserve the Hermitian pairing by construction. We need only show that they are $\rho^{\mathcal{H}}$-equivariant. Given $g \in G$, $q \in \mathbb{C}P^1$, a section ψ of $\mathcal{H}_\lambda^{(d)} \to \mathbb{C}P^1$, and a tangent vector $X \in T_q\mathbb{C}P^1$, we have

$$X[\rho_g \psi] = \rho_g((g_*^{-1}X)[\psi]),$$

where the superscripts in the actions were removed for convenience. Combining the above with a change of variables in (11), one sees that

$$\langle X[\rho_g \psi] | \psi' \rangle = \langle (g_*^{-1}X)[\psi] | \rho_{g^{-1}} \psi' \rangle$$

for all $\psi' \in \mathcal{H}_{g.q,\lambda}^{(d)}$. By Remark 2.9, this shows that $\nabla_X(\rho_g \psi) = \rho_g(\nabla_{g_*^{-1}X}[\psi])$. $\square$

Recall now that for every integer d there exists an Sp(1)-equivariant Hermitian line bundle of degree d with compatible connection over $\mathbb{C}P^1$, unique up to isomorphism. This can be characterised as the holomorphic line bundle $\mathcal{O}(d) \to \mathbb{C}P^1$ together with the standard Hermitian metric and its corresponding Chern connection. Alternatively, it can also be described as the quotient of an appropriate line bundle over Sp(1) under the identification $\mathbb{C}P^1 \simeq \mathrm{Sp}(1)/\mathrm{U}(1)$. More precisely, consider the d-th character $\chi^{(d)} : \mathfrak{u}(1) \to \mathbb{R}$ and its unique $\mathrm{Ad}_{\mathrm{U}(1)}$-invariant extension to $\mathfrak{sp}(1)$.

Denoting by $\alpha^{(d)}$ the corresponding left-invariant 1-form on $\mathrm{Sp}(1)$, the connection $\mathrm{d} + 2\pi i \alpha^{(d)}$ on $\mathrm{Sp}(1) \times \mathbb{C}$ is then invariant under the actions

$$A \cdot (x, z) := (Ax, z) \quad \text{and} \quad (x, z) \cdot h := (xh, h^{-d}z)$$

for $A \in \mathrm{Sp}(1)$ and $h \in \mathrm{U}(1)$. Furthermore, the right $\mathrm{U}(1)$-action is by construction horizontal for this connection. Therefore, the latter descends to a metric and $\mathrm{Sp}(1)$-equivariant connection on $(\mathrm{Sp}(1) \times \mathbb{C})/\mathrm{U}(1) \to \mathrm{Sp}(1)/\mathrm{U}(1) \simeq \mathbb{C}P^1$.

   Uniqueness can be established by noticing that the difference of two such line bundles comes with a connection whose curvature is $\mathrm{Sp}(1)$-invariant and vanishes in cohomology, and is therefore zero. The space of flat sections is then a 1-dimensional $\mathrm{Sp}(1)$-representation, so that choosing one unit element in this space gives an isomorphism of the line bundles intertwining the Hermitian structures and connections.

   We will refer to this object as $\mathcal{L}^d$.

   For each $d \in \mathbb{Z}$ and $\lambda \in \Lambda^{(d)}$, denote by $m_\lambda^{(d)}$ the multiplicity of V_λ in $\mathcal{H}_q^{(d)}$.[2] Finally, call $\underline{V_\lambda} \to \mathbb{C}P^1$ the trivial Hermitian bundle with fibre V_λ with ∇^{Tr} the trivial connection and $\mathrm{Sp}(1)$ acting on it trivially on the fibres.

Theorem 2.11. *Fix an integer d and a dominant weight λ of G_0. Suppose, for some $q \in \mathbb{C}P^1$, that the multiplicity $m_\lambda^{(d)}$ of the corresponding isotypical component in the Hilbert space $\mathcal{H}_q$ is finite. Consider the collection $\mathcal{H}_\lambda^{(d)}$ of corresponding isotypical components, and suppose it forms a Banach submanifold of $\mathbb{C}P^1 \times \mathrm{L}^2(M, L)$ acted on smoothly by G. Then $\mathcal{H}_\lambda^{(d)}$ is a Hermitian vector bundle over $\mathbb{C}P^1$ and there exists a G-invariant isomorphism*

$$\mathcal{H}_\lambda^{(d)} \simeq (\mathcal{L}^d \otimes \underline{V_\lambda})^{\oplus m_\lambda^{(d)}}$$

of Hermitian vector bundles which intertwines the covariant derivative operators $\nabla^{\mathcal{H}_\lambda^{(d)}}$ of Definition 2.7 with the natural connection on the right-hand side.

Proof. Introducing for simplicity the notation $m := m_\lambda^{(d)}$, fix $q \in \mathbb{C}P^1$, and identify T_q with $\mathrm{U}(1)$ by the orientation defined by q. Consider on $\mathrm{Sp}(1)$ the trivial vector bundle $\mathrm{Sp}(1) \times V_\lambda^{\oplus m}$ with the left and right actions

$$(A, g) \cdot (x, v) := (Ax, gv) \quad \text{and} \quad (x, v) \cdot h = (xh, h^{-d}v)$$

of $\mathrm{Sp}(1) \times G_0$ and $\mathrm{U}(1)$, respectively. Choose an isomorphism $\varphi : V_\lambda^{(\oplus m)} \to \mathcal{H}_{q,\lambda}^{(d)}$ as G_0-modules and define

$$\Phi : \mathrm{Sp}(1) \times V_\lambda^{\oplus m} \to \mathcal{H}_\lambda^{(d)}, \quad \Phi : (x, v) \mapsto \rho^{\mathcal{H}}(x)(\varphi(v)).$$

By construction, Φ is invariant under the right $\mathrm{U}(1)$-action and intertwines the $\mathrm{Sp}(1) \times G_0$-actions. It is also a surjective smooth map covering the projection

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[2]The integer $m^{(d)}$ is independent of $q \in \mathbb{C}P^1$ (see the previous footnote).

$\pi : \mathrm{Sp}(1) \to \mathbb{C}P^1$, $\pi(x) := xq$ and restricts fibrewise to unitary isomorphisms. It follows that Φ is a submersion, and therefore the induced bijection

$$(14) \qquad (\mathrm{Sp}(1) \times V_\lambda^{\oplus m})/\mathrm{U}(1) \to \mathcal{H}_\lambda^{(d)}$$

is a diffeomorphism, thus showing that $\mathcal{H}_\lambda^{(d)}$ is a vector bundle as claimed. It then follows from Proposition 2.10 that $\nabla^{\mathcal{H}_\lambda^{(d)}}$ is a Hermitian G-invariant connection.

The map Φ may also be regarded as a unitary isomorphism

$$(15) \qquad \mathrm{Sp}(1) \times V_\lambda^{\oplus m} \simeq \pi^* \mathcal{H}_\lambda^{(d)}.$$

Both sides come with $\mathrm{Sp}(1) \times G_0$- and $\mathrm{U}(1)$-invariant Hermitian connections, both making the right $\mathrm{U}(1)$-action horizontal. Such a connection, however, is uniquely characterised by these properties. Indeed, left $\mathrm{Sp}(1)$-invariance implies that such a connection is determined by the potential over any element of $\mathrm{Sp}(1)$. On the other hand, combining the left and right $\mathrm{U}(1)$-invariance shows that the operation of lifting elements of $\mathrm{T}_{\mathrm{Id}} \mathrm{Sp}(1) \simeq \mathfrak{sp}(1)$ horizontally is $\mathrm{Ad}_{\mathrm{U}(1)}$-equivariant. The condition that the right $\mathrm{U}(1)$-action be horizontal, moreover, determines the lifts of vectors in $\mathfrak{u}(1)$, and therefore of those in $\mathfrak{sp}(1)$ by $\mathrm{Ad}_U(1)$-invariance. We conclude that the isomorphism (15) also identifies the connections on the two bundles, which is to say that the isomorphism (14) is also horizontal. The left-hand side of (14), however, is clearly isomorphic to $\mathcal{L}^d \otimes \underline{V_\lambda}^{\oplus m}$. Finally, since the kernel of the covering map $\mathrm{Sp}(1) \times G_0 \to G$ acts trivially on the right-hand side, it follows that the group action on the left-hand side descends to G. $\qquad \square$

Remark 2.12. Theorem 2.11 yields an alternative definition of the bundles of isotypical components, without smoothness assumptions. Indeed, a map Φ constructed as above uniquely defines a smooth structure on $\mathcal{H}_\lambda^{(d)}$ making it a vector bundle which comes with an isomorphism with $\mathcal{L}^d \otimes \underline{V_\lambda}^m$, and therefore inducing also a connection with the desired properties. Given that the only ambiguity in the construction of Φ lies in the choice of φ, any two such maps are related by precomposition with a G_0-invariant automorphism of $V_\lambda^{\oplus m}$. Since this operation preserves the structure on $\mathrm{Sp}(1) \times V_\lambda^{\oplus m}$, the two choices induce the same data on $\mathcal{H}_\lambda^{(d)}$.

This yields finite-rank smooth G-equivariant Hermitian vector bundles over the Riemann sphere, equipped with Hermitian connections, defined from the combinatorial data of the multiplicities of $\mathcal{H}_q$ as a representation, as long as the main assumption that the $\mathcal{H}_{q,\lambda}^{(d)}$'s be finite-dimensional is verified.

Together with Remark 2.6, the content of this section proves Theorem 1.1.

2F. *Quantum super Hilbert spaces and unitary representations.* We now denote by $H_\lambda^{(d)}$ the super vector space obtained by taking the holomorphic cohomology of

the bundles of isotypical components:

$$H_\lambda^{(d)} := H^*(\mathbb{C}P^1, \mathcal{H}_\lambda^d).$$

By Remark 2.6, the above is equivalent to the space $H^{(\rho)}$ of (2). Since $\mathcal{H}_\lambda^{(d)}$ is Hermitian and $\mathbb{C}P^1$ is Kähler, the L^2-pairing on harmonic representatives gives each of the above a natural Hermitian structure.

If $W^{(d)} = W_+^{(d)} \oplus W_-^{(d)}$ is the unitary super $\mathrm{Sp}(1)$-representation defined by

$$W_+^{(d)} := H^0(\mathbb{C}P^1, \mathcal{L}^d), \quad W_-^{(d)} := H^1(\mathbb{C}P^1, \mathcal{L}^d),$$

then $H_\lambda^{(d)} \simeq W^{(d)} \otimes V_\lambda^{\oplus m_\lambda^{(d)}}$ as super G-representations, where V_λ is endowed with the trivial $\mathbb{Z}_2$-grading. Moreover, $\dim W_+^{(d)}$ is equal to $d+1$ if $d \geq 0$ and 0 otherwise, while similarly $\dim W_-^{(d)}$ vanishes for $d \geq 0$ and is equal to $-d-1$ otherwise.

Finally consider the nested L^2-completed orthogonal direct sums

$$H := \overline{\bigoplus_{d \in \mathbb{Z}} H^{(d)}}, \quad H^{(d)} := \overline{\bigoplus_{\lambda \in \Lambda^{(d)}} H_\lambda^{(d)}}.$$

This provides a G-representation quantising the G-action on (M, g, I, J, K), thus proving Theorem 1.2 from the introduction.

2G. *Finite-rank conditions.* We shall now consider conditions which entail finite-dimensionality for the isotypical components of Section 2D.

In general, if K is a compact Lie group with Lie algebra $\mathfrak{k} = \mathrm{Lie}(K)$, acting on a Kähler manifold X with a lifted K-action on a prequantum line bundle (L, ∇), there is a natural moment map $\mu : X \to \mathfrak{k}^\vee$ defined by Kostant's formula

$$(16) \qquad\qquad 2\pi i \langle \mu, \xi \rangle \frac{\partial}{\partial \theta} = \xi_X^H - \xi_L$$

for every $\xi \in \mathfrak{k}$, where ξ_L is the vector field corresponding to ξ on L, ξ_X^H the one on X lifted horizontally, and $\partial/\partial\theta$ is the fibrewise "angular" vector field. In this setup, we will make use of the following version of the general principle that "quantisation commutes with reduction".

Theorem 2.13 [43; 81]. *In the setup above, if the K-action extends holomorphically to the complexified group $K^\mathbb{C}$, and if the moment map* (16) *is proper, then for every dominant weight γ of K there is an identification*

$$\mathrm{Hom}_K(V_\gamma, H^0(X, L)) \simeq H^0(X_\gamma, L_\gamma),$$

where V_γ denotes a simple K-module of highest weight γ, $X_\gamma = X /\!\!/_\gamma K$ is the symplectic reduction of X at level γ, and L_γ is the induced (V-)bundle on X_γ.

This result was first established by Guillemin and Sternberg [43] in the case X is compact, with additional regularity conditions, and then extended by Sjamaar [81]. The statement has been subsequently generalised in various works including those

of Meinrenken [67; 68], Meinrenken and Sjamaar [69], Vergne [84; 85], Ma [60], Ma and Zhang [61], and Hochs and Song [48].

We emphasise that this formulation of "quantisation commutes with reduction" requires no assumptions on γ being a regular value or the K-action being free on $\mu^{-1}(\gamma)$. In the statement of Theorem 2.13, X_γ and L_γ are regarded as a complex analytic space and a coherent sheaf, respectively. See [81] for further detail.

Returning to our setting, for any fixed q and Lie subgroup $S \subset G_q$ we have a moment map

$$\mu_S : M \to \mathrm{Lie}(S)^\vee,$$

given by Kostant's formula. Then Theorem 2.13 yields the following.

Theorem 2.14. *Fix $q \in \mathbb{C}P^1$ and a (connected) Lie subgroup $S \subseteq G_q$, and denote by μ_S the Kostant moment map of the S-action on M_q. Assume that μ_S is proper, and suppose that the S-action has a holomorphic extension to the complexified group $S^\mathbb{C}$ on M_q. Then every S-isotypical component in $\mathcal{H}_q$ has finite multiplicity.*

Proof. Properness of the moment map implies that, for any dominant weight γ of S, the symplectic reduction $M /\!\!/_\gamma S$ is a *compact* complex analytic space. On the other hand, L_γ is a coherent sheaf on it by [81, Section 2.2], and by compactness the space of sections is finite-dimensional [28].

It follows from Theorem 2.13 that the irreducible representation of S of highest weight γ has finite multiplicity inside $H^0(M_q, L_q)$, so *a fortiori* inside $\mathcal{H}_q$. □

Remark 2.15. Another way to ensure finite-dimensionality is to assume there are compactifications of the symplectic reductions, with rational singularities and boundary of (complex) codimension at least two; then Hartogs's theorem applies on the reduction (see, e.g., [83] for such generalisations, and see Theorem 2.17).

As briefly noted in the introduction, another approach to controlling the dimension of the isotypical components is offered by the results of [90]. Indeed, the cited work introduces a notion of meromorphicity for certain group actions which, under appropriate conditions (see Assumption 2.14 of the same work), ensures finite-dimensionality.

2H. *Rank-generating series and localisation formulæ.* If either $\mathcal{H}^{(d)}$, $\mathcal{H}_a^{(d)}$ or $\mathcal{H}_\lambda^{(d)}$ have finite rank, we consider the (formal) generating series:

$$(17) \qquad\qquad H(t) = \sum_d \mathrm{rk}(\mathcal{H}^{(d)}) \cdot t^d,$$

and

$$(18) \qquad\qquad H'(t, \tilde{t}) = \sum_{d,a} \mathrm{rk}(\mathcal{H}_a^{(d)}) \cdot t^d \, \tilde{t}^a,$$

as well as

$$(19) \qquad G(t, \tilde{t}) = \sum_{d \in \mathbb{Z}} \sum_{\lambda \in \Lambda^{(d)}} m_\lambda^{(d)} \cdot t^d \, \tilde{t}^\lambda.$$

Note that if $\mathcal{H}_a^{(d)}$ and $\mathcal{H}_\lambda^{(d)}$ are both finite-rank then (18) can be obtained from (19) via the substitution $\tilde{t}^\lambda \mapsto \chi_\lambda(\tilde{t})$, where

$$\chi_\lambda(\tilde{t}) = \sum_{a \in E_\lambda} n_a^{(\lambda)} \cdot \tilde{t}^a,$$

and where E_λ is the set of weights of V_λ with multiplicities $n_a^{(\lambda)} \in \mathbb{Z}_{\geq 0}$.

If in particular G_0 is semisimple then the Weyl character formula yields

$$\chi_\lambda(\tilde{t}) = \frac{\sum_{w \in W} \epsilon(w) \, \tilde{t}^{w(\lambda + \rho)}}{\sum_{w \in W} \epsilon(w) \, \tilde{t}^{w(\rho)}},$$

where $W = N(T)/T$ is the Weyl group and $\rho \in \mathfrak{t}^\vee$ the half-sum of positive roots.

Conversely (19) can be recovered from (18) (when both are defined) as follows. Fix $d \in \mathbb{Z}$ and let $H_d(\tilde{t})$ be the coefficient of t^d in (18). Let λ be maximal among the weights such that $\tilde{t}^\lambda$ appears in $H_d(\tilde{t})$. In particular, the weight $\lambda_d^{(0)}$ can only appear in an irreducible component of $\mathcal{H}_q^{(d)}$ (as a G_0-module) if it is the highest. Therefore, the coefficient of $\tilde{t}^\lambda$ in $H_d(\tilde{t})$ is equal to $m_\lambda^{(d)}$. One may now consider $H_d(\tilde{t}) - m_\lambda^{(d)} \chi_\lambda(\tilde{t})$ and repeat the procedure inductively. Since each step strictly decreases one of the maximal weights the process terminates — exactly when the polynomial vanishes. This results in a decomposition

$$H_d(\tilde{t}) = \sum_{\lambda \in \Lambda^{(d)}} m_\lambda^{(d)} \chi_\lambda(\tilde{t}),$$

recovering all multiplicities and ultimately (19).

Furthermore the generating series (17), (18), and (19) can sometimes be computed by localisation formulæ. We refer to [49] for general results, and we review here the simpler versions used in what follows.

Suppose the action of T_q on M_q has a finite number of fixed points $|M_q| \subseteq M_q$, and let $\bar{R}(T_q)$ be the formal completion of the character ring $R(T_q)$ of T_q.

Since the fixed points $p \in |M_q|$ are isolated we see that $\Lambda_{-1}(\mathrm{T}_p M_q) \in \bar{R}(T_q)$ is invertible. Suppose now we have a decomposition

$$H^i(M_q, L_q) = \overline{\bigoplus_{d \in \mathbb{Z}} H^i(M_q, L_q)^{(d)}},$$

such that T_q acts on $H^i(M_q, L_q)^{(d)}$ via the d-th power of the standard representation, and such that the spaces $H^i(M_q, L_q)^{(d)}$ are finite-dimensional.

Proposition 2.16 [25; 49]. *The following formula holds*:

$$\sum_{i=0}^{2n}(-1)^i \dim H^i(M_q, L_q)^{(d)} t^d = \sum_{p\in|M_q|} \frac{L_{q,p}}{\Lambda_{-1}(T_p M_q)}.$$

Hence if $H^i(M_q, L_q) = (0)$ for $i > 0$ then simply

(20) $$H(t) = \sum_{p\in|M_q|} \frac{L_{q,p}}{\Lambda_{-1}(T_p M_q)}.$$

Considering the action of $T'_q = T_q \cdot T$ on M_q we get an analogous result, provided T'_q has finitely many fixed points and all spaces $H^i(M_q, L_q)^{(d)}_{a,q}$ are finite-dimensional, and interpreting the right-hand side as an element of $\bar{R}(T) \simeq \mathbb{Z}[\![t^{\pm1}, \tilde{t}_i^{\pm1}]\!]$. In particular,

(21) $$H'(t, \tilde{t}) = \sum_{p\in|M_q|} \frac{L_{q,p}}{\Lambda_{-1}(T_p M_q)}.$$

Now recall that if M_q is a Stein space, or has the structure of an affine scheme, then Cartan's theorem yields the vanishing of higher cohomology groups [27]. Thus putting together the previous results we have established the following.

Theorem 2.17. *Suppose there exists $q \in \mathbb{C}P^1$ such that M_q is a Stein space, or has the structure of an affine scheme, and that the T_q-action (resp. T'_q-action) has finitely many fixed points. Assume further that one of the following holds*:

- *There is a proper moment map for the T_q-action (resp. T'_q-action).*

- *There exists a compactification of the symplectic reductions with rational singularities, with boundary of codimension at least two (see Remark 2.15).*

Then the family $\mathcal{H}^{(d)}$ (resp. $\mathcal{H}^{(d)}_a$) has finite rank, and the associated localisation formula (20) (resp. (21)) holds for the rank-generating series (17) (resp. (18)).

Remark 2.18. If the higher cohomology groups do not vanish one could replace (10) by the super space

$$\widetilde{\mathcal{H}}_q = H^{\mathrm{even}}(M_q, L) \oplus H^{\mathrm{odd}}(M_q, L),$$

in which case formulæ (20) and (21) hold for the super representations $\widetilde{\mathcal{H}}_q$ of T_q and T'_q. (In this setup one need not assume that M_q be a Stein space or an affine scheme.)

Remark 2.19. Alternatively, in the setting of [90], Wu's localisation results (Theorem 3.14 of the same work) yield the generating series (17) and (18) by an index computation of the fixed-point locus for the T_q- and T'_q-action, respectively.

2I. Sp(1)-*symmetric hyperkähler potentials.*

Definition 2.20. A hyperkähler potential on the hyperkähler manifold (M, g, I, J, K) is a smooth map $\mu : M \to \mathbb{R}$ such that $\omega_q = i \partial_q \bar{\partial}_q \mu$ for every $q \in \mathbb{C}P^1$.

One can also use such potentials to obtain equivariant prequantum data, as discussed below. Assume further that μ is Sp(1)-invariant and that it generates the T_q-actions, i.e., $i\mu : M_q \to i\mathbb{R} \simeq \mathfrak{t}_q^\vee$ is a moment map.

In this case we consider the trivial Hermitian line bundle, and lift the G-action by the identity on each fibre. Natural symplectic potentials are given by

$$\theta_q = \tfrac{1}{2}(\bar{\partial}_q \mu - \partial_q \mu) \in \Omega^1(M),$$

hence $\nabla_q = \mathrm{d} + (\theta_q/\hbar)$ is a prequantum connection for all $q \in \mathbb{C}P^1$, and the resulting prequantum data are G-equivariant since μ is Sp(1)-invariant.

Now if $\mathrm{grad}(\mu)$ is complete then each T_q-action extends holomorphically to $\mathbb{C}^\times$, and if in addition μ is proper then the subspaces $\mathcal{H}_q^{(d)}$ are finite-dimensional by Theorem 2.14.

Proposition 2.21. *Suppose that M admits a G-invariant hyperkähler potential μ which, for every $q \in \mathbb{C}P^1$, is also an ω_q-moment map for the T_q-action. Assume moreover that μ is bounded below and that it has finitely many critical values. Then for every $q \in \mathbb{C}P^1$ the function $\psi_0 := e^{-\mu/2\hbar}$ is square-integrable, and it is a holomorphic frame for the prequantum line bundle constructed above.*

Proof. Nonvanishing and holomorphicity are a straightforward consequence of the definition.

On the other hand, the L^2-square-norm of ψ_0 can be expressed as

$$(22) \qquad \|\psi_0\|_{L^2}^2 = \int_M e^{-\mu/\hbar}\, \mathrm{d\,vol} = \int_B^\infty e^{-\xi/\hbar} \mu_*(\mathrm{d\,vol}),$$

where $B \in \mathbb{R}$ is a lower bound for μ and $\mu_*(\mathrm{d\,vol})$ the push-forward of the Liouville measure. By the Duistermaat–Heckman theorem [36] the push-forward admits a density which restricts to a polynomial on every interval $I \subset \mathbb{R}$ not containing critical values for μ. Since there are finitely many such values, (22) splits as a finite sum of converging integrals. □

By construction, the compact torus $T_q \simeq \mathrm{U}(1)$ acts on the complex vector space of holomorphic functions on M_q — by (inverse) pullback. By definition, such a function is *d-homogeneous* if it transforms (under the T_q-action) in the irreducible representation corresponding to the character $z \mapsto z^d \in \mathrm{U}(1)$, where $d \in \mathbb{Z}$. Under the assumptions of Proposition 2.21 we thus get an isomorphism

$$\Psi : \mathrm{L}^2\, H^0(M_q, \mathcal{O}, e^{-\mu/\hbar}\, \mathrm{d\,vol})^{(d)} \to \mathcal{H}_q^{(d)},$$

given by $\Psi(f) = f\psi_0$, where the left-hand side denotes the space of d-homogeneous holomorphic functions with finite L^2-norm with respect to $e^{-\mu/\hbar}\,$d vol.

3. Examples of applications

3A. *Hyperkähler vector spaces.* Let $n > 0$ be integer and V a real vector space of dimension $4n$.

Definition 3.1. A linear hyperkähler structure on V is a scalar product g and an ordered triple (I, J, K) of orthogonal automorphisms of V satisfying the quaternionic identities $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_V$.

Equivalently, a linear hyperkähler structure on V may be regarded as a Hermitian representation of the quaternion algebra

$$\mathbb{H} = \{q = d + ai + bj + ck \mid a, b, c, d \in \mathbb{R}\},$$

on V, where the quaternionic Hermitian form is

$$\boldsymbol{h} := g - i\omega_I - j\omega_J - k\omega_K, \qquad \text{with} \quad \omega_\bullet(v, w) := g(\bullet v, w) \quad \text{for } \bullet \in \{I, J, K\}.$$

It follows that I, J, K are g-skew-symmetric, and hence they span a real Lie subalgebra $\mathfrak{su}(2)_V \subseteq \mathfrak{o}(V, g)$.

Attached to the hyperkähler vector space is the group $\operatorname{Sp}(V, \boldsymbol{h}) \subseteq O(V, g)$ of $\mathbb{R}$-linear endomorphisms of V preserving $\boldsymbol{h}$ — and hence g and each of the forms $\omega_I, \omega_J, \omega_K$. As above we are interested in transformations that preserve the hyperkähler structure in a looser sense, but here we restrict to *linear* ones:

$$\operatorname{Hk}(V) = \operatorname{Hk}(V, g, I, J, K) := \{A \in O(V, g) \mid \operatorname{Ad}_A(\mathfrak{su}(2)_V) = \mathfrak{su}(2)_V\}.$$

As a subgroup of $O(V, g)$, the above is compact.

Remark 3.2. We are thus slightly abusing the notation from Section 2. Indeed if V is regarded as a smooth hyperkähler manifold then the group of *all* transformations preserving g and $\mathfrak{su}(2)_V$ also contains the translations, and it is in fact generated by these two kinds of transformations. We shall still denote this subgroup $\operatorname{Hk}(V)$ in the linear case to simplify the notation.

Remark 3.3. In this case the twistor space is a rank-$2n$ holomorphic vector bundle $\pi_{\mathbb{C}P^1} : Z \to \mathbb{C}P^1$ isomorphic to $\underline{\mathbb{C}^{2n}} \otimes \mathcal{O}(1)$ (in the straightforward generalisation of the case $n = 1$ from [46, Example 2.4, p. 143]).

Lemma 3.4. *There is an exact sequence of Lie groups*:

$$1 \to \operatorname{Sp}(V, \boldsymbol{h}) \to \operatorname{Hk}(V) \xrightarrow{\operatorname{Ad}} SO(3) \to 1,$$

and an embedding $\sigma : \operatorname{Sp}(1) \to \operatorname{Hk}(V)$ *such that* $\operatorname{Ad} \circ \sigma : \operatorname{Sp}(1) \to SO(3)$ *is the natural surjection.*

Proof. The natural $\mathrm{Sp}(1)$-action on $\mathbb{H}$ by multiplication on the right induces the standard $\mathrm{Sp}(1)$-action on the unit sphere of complex structures $\mathbb{S}_{IJK}$. The conclusion follows from a choice of identification $V \simeq \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}^n$ as $\mathbb{H}$-module. $\square$

Hence a choice of orthonormal basis for $(V, \boldsymbol{h})$ (as a left $\mathbb{H}$-module) yields an identification

$$\mathrm{Hk}(V) = \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \simeq (\mathrm{Sp}(n) \times \mathrm{Sp}(1))/\mathbb{Z}_2.$$

Choosing $G = \mathrm{Hk}(V)$, we see that in the notation of the introduction we have

$$G_0 = \mathrm{Sp}(V, \boldsymbol{h}) \subseteq \mathrm{Hk}(V).$$

Geometric quantisation. Geometric quantisation on a Kähler vector space is straightforward and essentially unique up to the choice of a symplectic potential, which corresponds to a gauge choice on the prequantum line bundle. For $\hbar \in \mathbb{R}_{>0}$ one considers the triple (L, h, ∇_q), consisting of the trivial complex line bundle $L := V \times \mathbb{C} \to V$ with the tautological Hermitian metric h, and the connection $\nabla_q := d - \frac{i}{\hbar}\theta_q$ defined by the invariant symplectic potential

$$\theta_q(v)(X) = \tfrac{1}{2}\omega_q(v, X),$$

for $v \in V$ a point and X a tangent vector there. The above yields prequantum data for (V, ω_q) at level $\hbar^{-1}$. We may denote $L_q \to V$ the line bundle to emphasise the structure we are prequantising on V.

The bundle L_q comes endowed with a natural holomorphic frame

$$\psi_0(q, v) := \exp\left(-\frac{1}{4\hbar}g(v, v)\right),$$

which is manifestly independent of $q \in \mathbb{C}P^1$. For each q, the resulting quantum Hilbert space consists of sections $\psi = f\psi_0$, with $f : V \to \mathbb{C}$ an I_q-holomorphic *function* with finite L^2-norm with respect to the Gaussian measure. This space is well known to be densely generated by the polynomial functions, which induces a grading on each $\mathcal{H}_q$ — the Fock grading.

This setting is a particular case of the one discussed in Section 2I. Indeed, on a Kähler vector space, the function $\mu(v) = \frac{1}{2}\|v\|^2$ is a moment map for the $\mathrm{U}(1)$-action by scalar multiplication and a Kähler potential, and moreover

$$-\frac{i}{2}(\partial - \bar{\partial})\mu = \theta$$

is the invariant symplectic potential. Additionally, for each $q \in \mathbb{C}P^1$ the action of T_q is the standard one.

Furthermore d-homogeneous holomorphic functions on a complex vector space are d-homogeneous polynomials, whence the decomposition of $\mathcal{H}_q$ into isotypical components as a T_q-module reduces to the well-known Fock grading. By the

identification of the space of such homogeneous polynomials with $\operatorname{Sym}^d V_q^\vee$, the finite-dimensional spaces $\mathcal{H}_q^{(d)}$ assemble into finite-rank Hermitian subbundles $\mathcal{H}^{(d)} \to \mathbb{C}P^1$ of the trivial $\mathrm{L}^2(V, L)$-bundle, with a natural isomorphism

$$\operatorname{Sym}^d Z^\vee \to \mathcal{H}^{(d)}$$

of vector bundles over the Riemann sphere.

Group action on quantum spaces. The action $\rho^Z : \mathrm{Hk}(V) \to \operatorname{Aut}(Z)$ has a natural lift to $L = Z \times \mathbb{C}$ as $\rho^Z \times \mathrm{Id}$. Since $A^*\theta_q = \theta_{A.q}$ for $A \in \mathrm{Hk}(V)$ and $q \in \mathbb{C}P^1$, it follows that this action preserves the structure of L as a family of prequantum line bundles. This defines an action $\rho^{\mathcal{H}}$ on sections of $\mathcal{H}^{(d)}$ by pull-back, as in (12), and it is easy to check this is a graded fibrewise unitary $\mathrm{Hk}(V)$-action — covering that on the hyperkähler 2-sphere.

Theorem 3.5. *For $q \in \mathbb{C}P^1$ there is a canonical isomorphism $\mathcal{H}_q^{(d)} \simeq \operatorname{Sym}^d(V)$ of simple $\mathrm{Sp}(V, \boldsymbol{h})$-modules, and the bundle with connection $(\mathcal{H}^{(d)}, \nabla^{\mathcal{H}^{(d)}})$ is $\mathrm{Hk}(V)$-equivariantly isomorphic to $\mathcal{L}^d \otimes \underline{\operatorname{Sym}^d(V)} \to \mathbb{C}P^1$.*

Proof. This follows directly from the above discussion and from Theorem 2.11: The metric g, and hence the section ψ_0, are fixed by $\mathrm{Sp}(V, \boldsymbol{h})$. It is known the natural action on $\operatorname{Sym}^d V_q^\vee$ is irreducible [76]. $\qquad\square$

Altogether the statements of this section establish the assumptions needed to apply Theorem 1.2, which in this particular case yields the following.

Theorem 3.6 (see Theorem 1.2). *The $\mathrm{Sp}(1)$-symmetric geometric quantisation of the hyperkähler vector space V yields the super Hilbert space*

$$H = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^{(d)}},$$

in analogy with Section 2F. This carries a unitary $\mathrm{Hk}(V)$-representation preserving the splitting, and there is an isomorphism $H^{(d)} \simeq W^{(d)} \otimes \operatorname{Sym}^d(V)$ of simple $\mathrm{Hk}(V)$-modules.

For every $d \geq 0$ we thus have

$$\dim(W_+^{(d)}) = (d+1), \quad \dim(W_-^{(d)}) = 0, \quad \dim(H^{(d)}) = (d+1)\binom{2n+d}{d}.$$

The generating series (17) and (18) are obtained explicitly from the above:

$$H(t) = \frac{1}{(1-t)^{2n}}, \quad H'(t, \tilde{t}) = \frac{1}{\prod_{i=1}^n (1 - t\tilde{t}_i)(1 - t\tilde{t}_i^{-1})}.$$

On the other hand, since $V_q \simeq \mathbb{C}^{2n}$ is a Stein space, and since the actions of T_q and T_q' only fix the origin, Theorem 2.17 also applies, and the result from (20)

and (21) yields the same formulæ. Now by Theorem 3.6 we see that $m^{(d)}_{\mathrm{Sym}^d(V)} = 1$ for $d \in \mathbb{Z}_{\geq 0}$, whence

$$G(t, \tilde{t}) = \sum_{d=0}^{\infty} t^d \, \tilde{t}^{\lambda_{\mathrm{Sym}^d(V)}}.$$

3B. *Four-dimensional examples.* As mentioned in the introduction, in dimension 4 there is a complete classification of Sp(1)-symmetric hyperkähler manifolds up to finite quotients. Besides $\mathbb{H}$ with its flat metric there are the Taub–NUT metrics on $\mathbb{R}^4$, and the hyperkähler metric on the moduli space of charge-2 monopoles, i.e., the Atiyah–Hitchin manifold M_{AH}.

Taub–NUT metrics. Consider the case of $M = \mathbb{R}^4$ with the Taub–NUT metric g^a corresponding to a positive real parameter a — the case $a = 0$ corresponds to the standard flat metric on $\mathbb{H}$, which we already discussed. We will denote ω_q^a the corresponding symplectic structures. It is well known (e.g., [39, Remark 1]) that

$$\mathrm{Hk}(M) \simeq (\mathrm{Sp}(1) \times \mathrm{U}(1))/\mathbb{Z}_2 \simeq \mathrm{U}(2).$$

In particular there is a faithful Sp(1)-action rotating the sphere of hyperkähler structures, while $\mathrm{Sp}(M) = \mathrm{U}(1)$ is compact and commutes with Sp(1). Furthermore there exists, unique up to isomorphism, a family of prequantum line bundles for M, since $H^2(M, \mathbb{Z}) = 0 = H^1(M, \mathrm{U}(1))$.

The action of $T'_q = (\mathrm{U}(1) \times \mathrm{U}(1))/\mathbb{Z}_2 \subseteq \mathrm{Hk}(M)$ is studied explicitly by Gauduchon in [39, Section 3.2] for the complex structure J_+ corresponding to $q = i$. The subgroup is identified in that context with $\mathrm{U}(1) \times \mathrm{U}(1)$ via the isomorphism $(t, s) \mapsto (ts, ts^{-1})$. From equations (3.10) and (3.19) of the same work one concludes that the action of $T_q = \mathrm{U}(1) \times \{1\}$ on M_q is Hamiltonian with moment map $\mu_q = \mu_1^+ + \mu_2^+$ (borrowing Gauduchon's notation), which is easily seen to be proper from the definitions. Finally [39, Proposition 1] provides a biholomorphism $\Phi_+^a = \Phi : (M, J_+) \to \mathbb{C}^2$, and by a straightforward check this map intertwines the T_q-action on M_q with the standard U(1)-action on $\mathbb{C}^2$. In particular the T_q-action extends holomorphically to $\mathbb{C}^*$, and the hypotheses of Theorem 2.14 are verified.

Thus decomposing $\mathcal{H}_q$ with respect to the T_q-action yields

$$\mathcal{H}_q = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathcal{H}_q^{(d)}},$$

where the subspaces $\mathcal{H}_q^{(d)} \subseteq \mathcal{H}_q$ are finite-dimensional. Then we consider the action of the commuting compact group $\mathrm{Sp}_0(M) = \{1\} \times \mathrm{U}(1)$ on $\mathcal{H}_q^{(d)}$ to refine:

$$\mathcal{H}_q^{(d)} = \bigoplus_{d' \in \Lambda^{(d)}} \mathcal{H}_{d',q}^{(d)},$$

where $\Lambda^{(d)} \subseteq \mathbb{Z}_{\geq 0}$ is finite. In addition, we also have the following statement.

Proposition 3.7. *For $q = i$, the prequantum line bundle L_q admits a T_q-invariant holomorphic frame ψ_q such that $\Phi^*(f) \cdot \psi_q$ is L^2 for every polynomial function f on $\mathbb{C}^2$.*

Proof. Recall that, again in the notations of [39], x_1, x_2, and x_3 are three real-valued functions on M whose span is preserved by Sp(1), which acts on them by rotations in the standard way. Furthermore, all three functions are fixed by the action of U(1) = Sp(M). Writing $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, it follows from (3.19), (2.11), and (2.12) of [39] that the aforementioned moment map μ can be expressed as

$$\mu = r + a^2(x_2^2 + x_3^2).$$

Since $\mu_i := \mu$ is a moment map for T_i with respect to ω_i^a, it follows that for every $g \in$ Sp(1) the function $(g^{-1})^*\mu$ generates the $T_{g.i}$-action with respect to $\omega_{g.i}^a$. In particular, if $g.i = j$, then the flow associated to

$$\mu_j := (g^{-1})^*\mu = r + a^2(x_1^2 + x_3^2)$$

with respect to ω_j^a rotates the circle spanned by ω_3^a and ω_1^a. It is therefore a Kähler potential for ω_i^a [47], and repeating the argument when $g.i = k$ so is $\mu_k := r + a^2(x_1^2 + x_2^2)$. Therefore the T_i-invariant function

$$\varphi = \varphi_i := \frac{\mu_j + \mu_k}{2} = r + \frac{a^2}{2}(r^2 + x_1^2)$$

is also a Kähler potential. It follows that for every $g \in$ Sp(1) the function $(g^{-1})^*\varphi_i$ is completely determined by $q = g.i$, so that

$$\varphi_q := (g^{-1})^*\varphi_i$$

is well defined, and a potential for ω_q^a. From this we obtain an explicit realisation of the family of prequantum line bundles, for which the functions $\psi_{0,q} = e^{-\frac{1}{2\hbar}\varphi_q}$ define holomorphic frames.

Now note the function μ is bounded below, and its only critical point is the origin — the only fixed point of the induced action. We may then apply the Duistermaat–Heckman theorem [36] as in Proposition 2.21 to conclude that $e^{-\alpha\mu}$ is integrable with respect to the Taub–NUT volume $d\,vol^a$ for every parameter $\alpha \in \mathbb{R}_{>0}$. The same clearly applies to μ_j and μ_k, and from this it is easily deduced that

$$e^{-\alpha\varphi_q} \in L^2(M, d\,vol^a)$$

for every $q \in \mathbb{C}P^1$ and $\alpha > 0$. In particular, the holomorphic frames constructed above are L^2.

To conclude we recall that the two components of Φ are defined as

$$w_1 = e^{a^2 x_1} z_1, \quad w_2 = e^{-a^2 x_2} z_2,$$

where z_1 and z_2 are the standard i-holomorphic coordinates on $M = \mathbb{H}$ with respect to the usual flat metric (see [39, (3.4)]). We need to show that, for every $n, m \in \mathbb{Z}$, the function $w_1^n w_2^m \psi_0$ is also L^2. Expanding the definition of φ yields

$$\frac{1}{\hbar}\varphi - 2a^2(n-m)\,x_1 = \frac{r}{\hbar} + \frac{a^2}{2\hbar}(r^2 + x_1^2) - 2a^2(n-m)\,x_1$$
$$\geq \frac{r}{2\hbar} + \frac{a^2}{4\hbar}(r^2 + x_1^2) - C = \frac{1}{2\hbar}\varphi - C \geq \frac{1}{4\hbar}\mu_j - C,$$

provided $C \in \mathbb{R}_{>0}$ is large enough. Furthermore, it is a simple consequence of the definitions in [39] that $|z_1|^2 + |z_2|^2 = 2r$, whence

$$|z_1^n z_2^m|^2 \leq (2r)^{2(n+m)} \leq \mu_j^{2(n+m)}.$$

Collecting the estimates and using again the Duistermaat–Heckman theorem we conclude

$$\int_M |w_1^n w_2^m|^2\, \psi_0^2 \,\mathrm{d}\,\mathrm{vol}^a \leq e^C \int_M \mu_j^{2(m+n)} e^{-\frac{1}{4\hbar}\mu_j} \,\mathrm{d}\,\mathrm{vol}^a < \infty. \qquad \square$$

As a consequence of this result we have $\dim \mathcal{H}_{d'}^{(d)} = 1$ for all $d \in \mathbb{Z}_{\geq 0}$ and $d' = d - 2j$ with $j \in \{0, \ldots, d\}$, and we conclude that $\mathcal{H}_{d'}^{(d)} \simeq \mathcal{L}^d$ for such values of d and d'.

Theorem 3.8. *The generating series* (18) *is the same as for the flat metric*, *namely*

$$H'(t, \tilde{t}) = \frac{1}{(1 - t\tilde{t})(1 - t\tilde{t}^{-1})}.$$

We thus have

$$H = \overline{\bigoplus_{d \in \mathbb{Z}_{\geq 0}} H^{(d)}}, \quad H^{(d)} = \bigoplus_{d' \in \Lambda^{(d)}} H_{d'}^{(d)} = H^0(\mathbb{C}P^1, \mathcal{L}^d)^{\oplus(d+1)}.$$

The Atiyah–Hitchin manifold. Let us consider the Atiyah–Hitchin manifold M_{AH}, the last four-dimensional case. We shall discuss the extent to which our methods apply here.

The Atiyah–Hitchin manifold can be realised as the moduli space of charge-2 centred magnetic monopoles in $\mathbb{R}^3$, and it comes with a natural Riemannian metric preserved by the SO(3)-action induced by rotating monopoles. The quaternionic nature of the Bogomolny equation, of which the monopoles represented by M_{AH} are a particular class of solutions, induces a family of almost complex structures, which can be better understood via Donaldson's description in terms of rational maps [34]. More precisely, the choice of an oriented line through the origin in $\mathbb{R}^3$ induces an identification

$$\widetilde{M}_{\mathrm{AH}} = \left\{ S(z) = \frac{uz + v}{z^2 - w} \in \mathbb{C}(z) \,\Big|\, v^2 - wu^2 = 1 \right\} =: R_2^0,$$

where the left-hand side denotes the (two-fold) universal cover of M_{AH}. The Atiyah–Hitchin manifold is recovered from the monodromy action, generated by $(u, v, w) \mapsto (-u, -v, w)$. The resulting map is a biholomorphism with respect to one of the aforementioned almost complex structures, establishing that the latter is integrable and the former is Kähler. Rotations around the preferred direction induce a U(1)-action of R_2^0 by

$$(23) \qquad\qquad t.(u, v, w) = (tu, v, t^{-2}w).$$

As the preferred direction changes across all possible choices, this results in a family of Kähler structures parametrised by $\mathbb{C}P^1$, which is clearly rotated by the SO(3)-action (see [15, Chapter 2]).

The above identification is not isometric with respect to the Riemannian embedding $R_2^0 \subseteq \mathbb{C}^3$; nonetheless, the Riemannian structure on M_{AH} can be described by studying the SO(3)-orbits [15, Chapters 8–11]. The generic stabiliser of a monopole is the Klein four-group K_4, while orbits are parametrised by $k = \sin(\alpha)$ for an angle $\alpha \in \left[0, \frac{\pi}{2}\right]$, resulting in a description of an open dense of M_{AH} as the product $(0, 1) \times \mathrm{SO}(3)/K_4$; furthermore, as $k \to 0$ the orbit degenerates to a diffeomorphic copy of $\mathbb{R}P^2$, onto which M_{AH} deformation-retracts.

According to Swann's work [82, Section 6, Four-manifolds], M_{AH} does not admit a hyperkähler potential. Furthermore, one sees from (23) that the stabiliser of each Kähler structure has exactly one fixed point, and since the manifold has the homotopy type of $\mathbb{R}P^2$ there can be no proper moment map. Nonetheless the above homotopy equivalence yields

$$H^1(M_{\mathrm{AH}}, \mathrm{U}(1)) \simeq H^2(M_{\mathrm{AH}}, \mathbb{Z}) \simeq \mathbb{Z}_2.$$

Hence by Section 2B there are exactly two inequivalent SO(3)-equivariant families of prequantum line bundles. They differ by a twist by a family of flat connections on the nontrivial complex line bundle on M_{AH}.

The family supported on the trivial bundle can be constructed by means of the Kähler potentials of Olivier [72]. Namely the metric on the Atiyah–Hitchin manifold is the completion of

$$(24) \qquad \mathrm{d}s^2 = \frac{\beta^2 \gamma^2 \delta^2}{(4k^2(1 - k^2)\, K^2)^2} \mathrm{d}m^2 + \beta^2 \sigma_x^2 + \gamma^2 \sigma_y^2 + \delta^2 \sigma_z^2,$$

defined on $\left(0, \frac{\pi}{2}\right) \times \mathrm{SO}(3)/K_4$. We follow the conventions of [72]. Namely, $m = k^2$ is used as a coordinate in place of k, while $(\sigma_x, \sigma_y, \sigma_z)$ is an orthonormal frame of $\mathrm{T}^*\mathrm{SO}(3) \to \mathrm{SO}(3)$ and the coefficients β, γ, δ are functions of k determined by

$$\beta\gamma = -EK, \quad \gamma\delta = -EK + K^2, \quad \beta\delta = -EK + (1 - k^2)\, K^2,$$

where

$$K := K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad E := E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi}\, \mathrm{d}\phi$$

are the complete elliptic integrals of the first and second kind, respectively.

Oliver [72] then uses the Euler angles (φ, θ, ψ) as coordinates on SO(3) to give an explicit Kähler potential Ω for one of the complex structures, say I_3, preserved by rotations in the angle φ. This is given in of [72, (55)] and can be written explicitly using equations (6), (24), (25) and (36) therein, getting the formula

$$\Omega = \frac{\beta\gamma + \gamma\delta + \delta\beta}{8} + \tfrac{1}{8}(\gamma\delta \sin^2 \theta \cos^2 \psi + \delta\beta \sin^2 \theta \sin^2 \psi + \gamma\beta \cos^2 \theta).$$

Note for $k \in (0, 1)$ this function extends continuously to the whole of SO(3), and the trigonometric functions of (θ, ψ) descend to the projective space at $k = 0$; hence the potential extends to the completion M_{AH}. Finally, we emphasise that this potential is independent of the variable φ, which is to say that it is invariant under the action of the I_3-stabiliser. It follows that Ω defines an equivariant family of potentials under the SO(3)-action, whence an equivariant family of prequantum line bundles by the usual construction, together with a holomorphic frame $\psi_0 = e^{-\frac{1}{2\hbar}\Omega}$ for I_3.

Proposition 3.9. *The function $e^{-\alpha\Omega}$ is integrable on M_{AH} for $\alpha \in \mathbb{R}_{>0}$.*

Proof. From (24) we obtain the following expression for the volume form on (the complement of a negligible set in) M_{AH}:

$$\mathrm{d}\,\mathrm{vol} = \frac{\beta^2\gamma^2\delta^2}{4k^2(1 - k^2)\,K^2}\, \mathrm{d}m\sigma_x\sigma_y\sigma_z.$$

We need to show that

$$\int_{(0,1)\times\mathrm{SO}(3)} e^{-\alpha\Omega} \frac{\beta^2\gamma^2\delta^2}{4k^2(1 - k^2)\,K^2}\, \mathrm{d}m\sigma_x\sigma_y\sigma_z < \infty.$$

Note that $\beta\gamma \le 0$, $\gamma\delta \ge 0$, and $\beta\delta \le 0$ yield

$$\Omega \ge \frac{\gamma\delta}{8}.$$

We may then use these bounds and the Fubini–Tonelli theorem to reduce the statement to

$$\int_0^1 e^{-\frac{\alpha}{8}\gamma\delta} \frac{\beta^2\gamma^2\delta^2}{k^2(1 - k^2)\,K^2}\, \mathrm{d}m < \infty.$$

We will proceed by studying the asymptotic behaviour of the integrand in the limit $k \to 1$; the integral is necessarily regular for $k \to 0$. It is well known that

$$K \sim \tfrac{1}{2}\log(1 - k^2),$$

and since $E(1) = 1$ we find that

$$\beta\gamma \sim -\tfrac{1}{2}\log^2(1 - k^2), \quad \gamma\delta \sim \tfrac{1}{4}\log^2(1 - k^2), \quad \beta\delta \sim -\tfrac{1}{2}\log^2(1 - k^2),$$

and hence the integral converges by comparison with

$$\int_0^1 \exp\!\left(-\frac{\alpha}{32}\log^2(1 - k^2)\right)\frac{\log^2(1 - k^2)}{(1 - k^2)}\,\mathrm{d}m = \int_0^\infty e^{-\frac{\alpha}{32}x^2}x^2\,\mathrm{d}x < \infty,$$

which concludes the proof. $\square$

For $\alpha = 1/\hbar$ this implies the holomorphic frame ψ_0 is L^2, and hence an element of $\mathcal{H}_{I_3}^{(0)}$; in principle more L^2 holomorphic sections may be found considering functions of the holomorphic coordinates (u, v, w) on R_0^2. If all the monomials that descend to M_{AH} are L^2, then one concludes that $\mathcal{H}_q^{(d)}$ has infinite rank for every integer d, since $u^a v^b w^c$ is $(a - 2c)$-homogeneous and well defined on M_{AH} if $a + b$ is even. We obtain a partial result in this direction, showing that all powers of w are L^2.

The problem of describing u, v, and w in terms of the setup above is addressed in [15, Chapter 6-7], by making use of the twistor description and spectral curves [50]. Introducing parameters

$$k_1 = \frac{\sqrt{k\sqrt{1 - k^2}}\,K}{2}, \quad k_2 = \frac{1 - 2k^2}{3k\sqrt{1 - k^2}},$$

consider the elliptic curve

$$y^2 = 4k_1^2(x^3 - 3k_2 x^2 - x)$$

and let $\wp$, ζ be its corresponding Weierstrass functions, η the real period of ζ. Suppose that $a, b \in \mathbb{C}$ are the entries of a matrix in SU(2), thought of as a parametrisation of SO(3)/K_4, and let $\xi \in \mathbb{C}$ be such that

(25) $$\wp(\xi) = \frac{b}{a} - k_2.$$

Then the corresponding point in M_{AH} has holomorphic coordinates

$$u = \frac{\sinh\!\big(2k_1\zeta(\xi) - \frac{\eta\xi}{2} + k_1\bar{a}\bar{b}\wp'(\xi)\big)}{k_1\bar{a}^2\wp'(\xi)},$$

$$v = \cosh\!\left(2k_1\zeta(\xi) - \frac{\eta\xi}{2} + k_1\bar{a}\bar{b}\wp'(\xi)\right),$$

$$w = k_1^2\bar{a}^4\wp'(\xi)^2,$$

up to the sign ambiguity resulting from the monodromy. Substituting (25) in the differential equation for $\wp$, and using g_2 and g_3 as given in [50], we obtain

$$w = k_1^2\bar{a}(-12\bar{a}b^2 k_2 + 4b^3 - 4\bar{a}^2 b).$$

Now since $|a|^2 + |b|^2 = 1$ a straightforward check shows that

$$|w|^2 \leq 16k_1^4(9k_2^2 + 2) \sim 4K^2 \sim \log^2(1 - k^2)$$

for $k \to 1$. Adapting the proof of Proposition 3.9 and using (23) we obtain:

Proposition 3.10. *For every integer $n \geq 0$ the holomorphic section $w^n \psi_0$ is L^2 and therefore an element of $\mathcal{H}_{I_3}^{(-2n)}$.*

The analysis is more delicate for the functions u and v. Using (25) one can express $\bar{a}\bar{b}$ in terms of ξ and write the argument of the hyperbolic functions as

$$\Phi(\xi) = 2k_1\zeta(\xi) - \frac{\eta\xi}{2} + k_1\wp'(\xi)\frac{k_2 + \overline{\wp(\xi)}}{1 + |k_2 + \wp(\xi)|^2}.$$

It follows from the definitions and the Legendre relation that this function is periodic for the real period of $\wp$ and quasiperiodic for the imaginary period, with step πi, whence the sign ambiguity of u and v. Moreover one can show the poles of the summands cancel out, leaving a nonholomorphic analytic function — hallmark of the fact that the SO(3)-action does not preserve the complex structure. In particular its real part is bounded for fixed "k".

3C. *Moduli spaces of framed $\mathrm{SU}(r)$-instantons.* Let $r \geq 2$ and $k \geq 0$ be integers, and consider the moduli space $M_{k,r}$ of charge-k framed $\mathrm{SU}(r)$-instantons on $\mathbb{R}^4$, which is a hyperkähler manifold [13; 33]. Each of its complex structures can be described in terms of the ADHM construction as follows, after fixing an identification $\mathbb{R}^4 \simeq \mathbb{C}^2$. Consider the product

$$\mathbb{M} := \mathrm{End}(\mathbb{C}^k)^2 \times \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^r) \times \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k),$$

with $\mathrm{GL}(\mathbb{C}^k)$-action given by

$$g.(\alpha_0, \alpha_1, a, b) = (g\alpha_0 g^{-1}, g\alpha_1 g^{-1}, ga, bg^{-1}).$$

Remark 3.11. $\mathbb{M}$ is a space of representations of a quiver on two nodes and that the action naturally extends to $\mathrm{GL}(\mathbb{C}^k) \times \mathrm{GL}(\mathbb{C}^r)$ (which controls isomorphisms of representations).

Let $\mathbb{M}_0$ denote the set of elements of $\mathbb{M}$ satisfying the additional conditions:

(i) $[\alpha_0, \alpha_1] + ab = 0$.

(ii) For all $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{pmatrix} \alpha_0 + \lambda \\ \alpha_1 + \mu \\ a \end{pmatrix}$$

injective and $\begin{pmatrix} \lambda - \alpha_0 & \alpha_1 - \mu & b \end{pmatrix}$ surjective.

Then the restricted $U(r)$-action is Hamiltonian with moment map

$$\mu(\alpha_0, \alpha_1, a, b) := [\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^*a,$$

and there is an identification

(26) $$M_{k,r} \simeq \mathbb{M}_0 /\!\!/_\mu U(k).$$

The rotation group $SO(4)$ acts on $M_{k,r}$, and in particular the subgroup $Sp(1)$, in the identification $\mathbb{R}^4 \simeq \mathbb{H}$, transitively permutes the complex structures. Furthermore, Maciocia [62] shows that for each $q \in \mathbb{C}P^1$ the T_q-action has moment map

$$m_2(A) = \frac{1}{16\pi^2} \int_{\mathbb{R}^4} \|x\|^2 \operatorname{tr} F_A^2.$$

This function is clearly $Sp(1)$-invariant, and therefore a hyperkähler potential, so one can construct an $Sp(1)$-invariant family of prequantum line bundles endowed with holomorphic frames as in Section 2I.

The function m_2 is not, however, a proper map. By [62], under the identification (26) it corresponds to the norm-squared function $f : \mathbb{M}_0 \to \mathbb{R}$, which is $U(k)$-invariant but not proper, on account of the open condition (ii). However, Donaldson [33] identifies the symplectic reduction (26) with the GIT quotient of $\mathbb{M}_0$ by $GL(k, \mathbb{C})$, whereupon (ii) translates into a stability condition. One may then include the semistable points to obtain a partial compactification

$$\overline{M}_{k,r} := \mathbb{M} /\!\!/_{\mathrm{GIT}} GL(k, \mathbb{C}),$$

which is smooth by the work of Nakajima and Yoshioka [71, Corollary 2.2]. The map f descends then to a proper one on $\overline{M}_{k,r}$; it is also clear that its gradient is complete on the quotient, showing that geometric quantisation on this space yields finite-rank isotypical components by Theorem 2.14. On the other hand, the codimension of the boundary $\overline{M}_{k,r} \setminus M_{k,r}$ is greater than 2, so that Hartogs's theorem allows for the extension of holomorphic functions on $M_{k,r}$, which yields the finite-dimensionality of the isotypic components over this latter space.

4. Outlook and further perspectives

There are more spaces that fit some of the requirements for our quantisation scheme.

By the work of Kronheimer [57] the nilpotent (co)adjoint orbits of complex semisimple (1-connected) Lie groups are hyperkähler manifolds with transitively permuting $SO(3)$-actions, and by Swann's work [82] they admit hyperkähler potentials. Indeed, Proposition 5.5 of the same work states that such a potential exists on a hyperkähler manifold if it admits an $Sp(1)$-action permuting the complex structure and such that, denoting by X_q the vector field generating the T_q-action for each $q \in \mathbb{C}P^1$, the vector field $I_q X_q$ is independent of q. After [82, Proposition 6.5],

Swann goes on to check that this condition is verified for Kronheimer's space, thus establishing the existence of a hyperkähler potential. This is a particular instance of hyperkähler moduli spaces of solutions of Nahm's equations, specifically on a half-line with nilpotent boundary conditions.[3] Since Nahm's equations come naturally with a quaternionic structure and Sp(1)-action, the resulting manifolds have symmetries of the kind considered in this paper, and different choices of domain and boundary conditions give rise to different hyperkähler structures. For instance, semisimple boundary conditions on a half-line result in orbits of semisimple elements [56], while the study of Nahm's equations on a compact interval leads to the cotangent bundle T^*G [32; 58]. By the works of Mayrand [64; 65; 66], the latter comes with natural Sp(1)-equivariant families of Kähler potentials and moment maps for the stabilizers T_q, rather than a hyperkähler one, and they enjoy interesting properties that might lead to a variation of our main construction.

Also, as mentioned in the introduction, many new interesting hyperkähler metrics can be defined on moduli spaces of irregular singular connections/Higgs bundles over (wild generalisations of) Riemann surfaces [18; 77; 88], with simple examples reviewed in [23]: the "multiplicative" versions of the Eguchi–Hanson space and Calabi's examples (whose standard "additive" versions are quiver varieties on two nodes). This fits into a more general (new) multiplicative theory of quiver varieties [22], involving a "fission" operation generalising the construction of moduli spaces of flat connections *à la* TFT [19; 21]; note that conjecturally this produces a lot more new hyperkähler manifolds [20], beyond (wild) nonabelian Hodge spaces. See [26; 37; 74; 75] about quantum moduli spaces of meromorphic connections.

Finally the example of Section 3C, i.e., the moduli spaces of framed $\mathrm{SU}(r)$-instantons, opens the way for further discussion on the relation between the generating series produced by this new quantisation scheme and the well-known Nekrasov partition functions.

Appendix: Comparison with the standard approach

In this section we shall correct the family of quantum Hilbert spaces $\mathcal{H}_q$ to obtain finite-rank flat vector bundles of isotypical components (under the main assumption), as well as unitary equivalences between the quantisation of M with respect to the given Kähler polarisations.

Based on Theorem 2.11, we do this by a correcting twist of the finite-rank bundles $\mathcal{H}_\lambda^{(d)} \to \mathbb{C}P^1$; namely consider the tensor product

$$\widetilde{\mathcal{H}}_\lambda^{(d)} := \mathcal{H}_\lambda^{(d)} \otimes \mathcal{L}^{-d}, \quad d \in \mathbb{Z}, \ \lambda \in \Lambda^{(d)}.$$

---

[3]The hyperkähler metric on general orbits was constructed in [17; 55].

This new vector bundle comes with a Hk$_0$'(M)-action, and we denote $\nabla^{\widetilde{\mathcal{H}}_\lambda^{(d)}}$ the resulting Hk$_0$'(M)-invariant flat connection.

Since $\mathbb{C}P^1$ is simply connected the parallel transport defines canonical unitary isomorphisms

$$\widetilde{\mathcal{H}}_{q,\lambda}^{(d)} \to \widetilde{\mathcal{H}}_{q',\lambda}^{(d)}, \quad q,q' \in \mathbb{C}P^1, \tag{27}$$

satisfying 1-cocycle identities. In analogy with the above we then define

$$\overline{\bigoplus_{\lambda \in \Lambda^{(d)}} \widetilde{\mathcal{H}}_{q,\lambda}^{(d)}} =: \widetilde{\mathcal{H}}_q^{(d)} \subseteq \widetilde{\mathcal{H}}_q := \overline{\bigoplus_{d \in \mathbb{Z}} \widetilde{\mathcal{H}}_q^{(d)}},$$

and these families of Hilbert spaces carry a 1-cocycle of unitary isomorphisms induced from (27): This is the usual geometric quantisation construction.

Now we can introduce super Hilbert spaces $\widetilde{H}_{\lambda,j}^{(d)}$ in analogy with Section 2F, taking the holomorphic cohomology of the twisted vector bundles $\widetilde{\mathcal{H}}_\lambda^{(d)} \to \mathbb{C}P^1$.

Theorem A.1 (see Theorem 1.2). *There is a unitary action* Hk$_0$'(M) $\to$ U($\widetilde{H}$) *preserving the nested splittings*

$$\widetilde{H} := \overline{\bigoplus_{d \in \mathbb{Z}} \widetilde{H}^{(d)}}, \quad \widetilde{H}^{(d)} := \overline{\bigoplus_{\lambda \in \Lambda^{(d)}} \widetilde{H}_\lambda^{(d)}}, \quad \widetilde{H}_\lambda^{(d)} := \bigoplus_{j=1}^{m_\lambda^{(d)}} \widetilde{H}_{\lambda,j}^{(d)}.$$

Finally we can compare this representation with the one constructed in Section 2F, finding that twisting trivializes part of the action. Namely, the present super Hilbert space $\widetilde{H}_{\lambda,j}^{(d)} \simeq V_\lambda \otimes W^{(0)}$ replaces the original $H_{\lambda,j}^{(d)} \simeq V_\lambda \otimes W^{(d)}$ as a Hk$_0$'(M)-module, recalling that $W^{(0)}$ is the trivial one-dimensional Sp(1)-module. This should be compared with the (more) interesting irreducible representations of Hk$_0$'(M) obtained from Theorem 1.2.

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