*Pacific Journal of Mathematics*

# COMBINATORICS OF THE TAUTOLOGICAL LAMINATION

DANNY CALEGARI

Volume 329 No. 1 March 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 329, No. 1, 2024
https://doi.org/10.2140/pjm.2024.329.39

# COMBINATORICS OF THE TAUTOLOGICAL LAMINATION

DANNY CALEGARI

**The *tautological lamination* arises in holomorphic dynamics as a combinatorial model for the geometry of 1-dimensional slices of the shift locus. In each degree $q$ the tautological lamination defines an iterated sequence of partitions of 1 (one for each integer $n$) into numbers of the form $2^m q^{-n}$. Denote by $N_q(n, m)$ the number of times $2^m q^{-n}$ arises in the $n$-th partition. We prove a recursion formula for $N_q(n, 0)$, and a gap theorem: $N_q(n, n) = 1$ and $N_q(n, m) = 0$ for $\lfloor n/2 \rfloor < m < n$.**



## 1. Introduction

The *tautological lamination*, introduced in [Calegari 2022], is a combinatorially defined object which gives a holomorphic model for certain 1 complex dimensional slices of the *shift locus*, a fundamental object in the theory of holomorphic dynamics. There is a shift locus $\mathcal{S}_q$ for each degree $q$; it is the space of depressed monic polynomials $z^q + a_2 z^{q-2} + a_3 z^{q-3} + \cdots + a_q$ in a complex variable $z$ (thought of as a subset of $\mathbb{C}^{q-1}$ with coordinates $a_j$) for which every critical point is in the attracting basin of infinity.

There is a tautological lamination $\Lambda_T(C)$ for each degree $q$ and for each choice of *critical data* $C$ (certain holomorphic parameters which determine the slice of $\mathcal{S}_q$). For the complex dynamics reader: the tautological lamination records the combinatorics of the 1 complex dimensional slices of the shift locus where $q-2$ critical Böttcher coordinates are fixed, and one critical point (with a smaller escape rate than any of the others) is allowed to vary.

*MSC2020:* 37F10, 68R15.
*Keywords:* shift locus, elaminations, tautological lamination, bordered words, tau sequence.

Each tautological lamination determines a sequence of operations, called pinching, which cut the unit circle $S^1$ up into pieces and reglue them into a collection of smaller circles, denoted $S^1 \bmod \Lambda_{T,n}(C)$. Subsequent operations refine the previous ones, so each component of $S^1 \bmod \Lambda_{T,n}(C)$ is cut up and reglued into a union of components of $S^1 \bmod \Lambda_{T,n+1}(C)$. The precise cut and paste operations depend on $C$, but the set of *lengths* of the components of $\Lambda_{T,n}(C)$ (counted with multiplicity) depends only on $n$ and the degree $q$. These lengths are all of the form $2^m q^{-n}$ for various nonnegative integers $m$, and we can define $N_q(n,m)$ to be the number of components of $S^1 \bmod \Lambda_{T,n}(C)$ of length $2^m q^{-n}$.

The *short components* of $S^1 \bmod \Lambda_{T,n}(C)$ are those with length $q^{-n}$. The number of short components is $N_q(n,0)$. Our first main result is an exact recursive formula for $N_q(n,0)$ (which can be solved in closed form):

**Theorem 3.10** (recursive formula). *$N_q(n,0)$ satisfies the recursion $N_q(0,0)=1$, $N_q(1,0)=(q-2)$ and*

$$N_q(2n,0)=qN_q(2n-1,0) \quad \text{and} \quad N_q(2n+1,0)=qN_q(2n,0)-2N_q(n,0)$$

*and has the generating function $(\beta(t)-1)/qt$, where a closed form for $\beta(t)$ is given in Proposition 2.2.*

At the other extreme, there is a unique largest component of $S^1 \bmod \Lambda_{T,n}(C)$ of length $2^n q^{-n}$. Our second main result is a "gap" theorem:

**Theorem 5.11** (gap theorem). *$N_q(n,m)=0$ for $\lfloor n/2\rfloor < m < n$.*

Both the recursive formula for $N_q(n,0)$ and the existence of a gap were observed experimentally. Our main motivation in writing this paper was to give a rigorous proof of these observations.

One of the striking things about the tautological lamination is the existence of a rather mysterious bijection between the components of $S^1 \bmod \Lambda_{T,n}(C)$ and some seemingly unrelated objects called *tree polynomials*, introduced in Section 4. This bijection is a corollary of one of the main theorems of [Calegari 2022], and the proof there is topological. We know of no direct combinatorial proof of this bijection, and raise the question of whether one can be found.

## 2. Unbordered words

Some words end like they begin, such as `abra`·`cad`·`abra` and `b`·`aoba`·`b`. Such words are said to be *bordered*. Others (most) are *unbordered*. A *border* is a *nonempty*, *proper* suffix of some word which is equal to a prefix.

If a word contains a border, then it contains one of at most half the length (for, a border of more than half the length will itself be bordered and now we can apply induction).

If $W$ is a word, let's denote its length by $|W|$. If $W$ is an unbordered word of even length, we can write it as $W = W_1 W_2$ where $|W_1| = |W_2|$, and then for every letter $\mathsf{c}$ the word $W_1 \mathsf{c} W_2$ is also unbordered. If $W$ is an unbordered word of odd length, we can write it as $W = W_1 W_2$ where $|W_1| + 1 = |W_2|$, and then for every letter $\mathsf{c}$ the word $W_1 \mathsf{c} W_2$ is unbordered *except* when $W_1 \mathsf{c} = W_2$. Thus: if $a_n$ denotes the number of unbordered words of length $n$ in a $q$-letter alphabet, then $a_0 = 1$ (there is one empty word) and

$$a_{2n+1} = q a_{2n} \quad \text{and} \quad a_{2n} = q a_{2n-1} - a_n.$$

Let's define a generating function $\alpha(t) := \sum_{n=0}^{\infty} a_n t^n$. Then the recurrence becomes the functional equation

$$\alpha(t) = \frac{2 - \alpha(t^2)}{1 - qt}.$$

Iteratively substituting $t^2$ for $t$ and being careful about convergence, one obtains the following formula:

$$\alpha(t) = 1 + q \sum_{j=0}^{\infty} (-1)^j t^{2^j} \prod_{i=0}^{j} \frac{1}{(1 - q \cdot t^{2^i})}.$$

These facts are not new. Unbordered words have been studied by many authors. They are also called *bifix-free*, and *primary* (neither of these terms seem very descriptive to us). As far as we know they were first considered by Silberger [1971]; see also, e.g., [Lothaire 1997, p. 153].

A minor variation on this idea is as follows. Let's take for our $q$-letter alphabet the elements of $\mathbb{Z}/q\mathbb{Z}$. If $W$ is a word in the alphabet, let $W'$ denote the result of adding 1 to the first letter (digit). Say a word is 1-*unbordered* if no suffix $S$ is equal to a prefix $P$ or to $P'$ (and say it is 1-bordered otherwise). Then reasoning as above gives:

**Proposition 2.1** (recursion). *Let $b_n$ denote the number of 1-unbordered words of length $n$ in a $q$-letter alphabet. Then $b_0 = 1$ and*

$$b_{2n+1} = q b_{2n} \quad \textit{and} \quad b_{2n} = q b_{2n-1} - 2b_n.$$

Define the generating function $\beta(t) := \sum_{n-0}^{\infty} b_n t^n$. Then

$$\beta(t) = \frac{3 - 2\beta(t^2)}{1 - qt}.$$

The following "closed form" for $\beta(t)$ (and the argument below) was kindly provided by Frank Calegari:

**Proposition 2.2** (closed form solution). *The generating function $\beta(t)$ converges for small $|t|$, and can be meromorphically continued throughout the unit disk with a simple pole at every $2^k$-th root of $1/q$.*

*Define a sequence of integers $h(n)$ by*

$$h(0) := 1 \quad \textit{and} \quad h(n) := (-q)^{s(n)}(1-(-2)^{k(n)}) \quad \textit{for } n > 0,$$

*where $2^{k(n)}$ is the biggest power of 2 dividing $n$, and $s(n)$ is the sum of the binary digits of $n$. Then throughout the unit disk,*

$$\beta(t) = \left(\sum_{n=0}^{\infty} h(n)t^n\right)\prod_{j=0}^{\infty}\frac{1}{(1-qt^{2^j})}.$$

*Proof.* From the growth rate of the coefficients it's clear that $\beta(t)$ has a pole at $q^{-1}$ and converges uniformly throughout the open disk of radius $q^{-1}$. It follows that $\beta(t^2)$ converges uniformly throughout the open disk of radius $q^{-1/2}$. Using the identity $(1-qt)\beta(t) = 3-2\beta(t^2)$ and induction, the first claim is proved.

Let's define $H(t) := \sum_{n=0}^{\infty} h(n)t^n$ and $B(t) := H(t)\prod_{j=0}^{\infty}(1-qt^{2^j})^{-1}$. Then the proposition will follow if we can show that $B(t)$ satisfies $B(t)(1-qt) = 3-2B(t^2)$. First observe that $h(n) = 0$ if $n$ is odd; and furthermore,

$$\frac{2h(n)+h(2n)}{3} = \frac{(-q)^{s(n)}}{3}\left(3-2(-2)^{k(n)}+2(-2)^{k(n)}\right) = (-q)^{s(n)}.$$

The required identity is equivalent to

$$(1-qt)B(t)\prod_{k=1}^{\infty}(1-qt^{2^k}) = (3-2B(t^2))\prod_{k=1}^{\infty}(1-qt^{2^k})$$

or

$$B(t)\prod_{k=0}^{\infty}(1-qt^{2^k}) = (3-2B(t^2))\prod_{k=0}^{\infty}(1-q(t^2)^{2^k})$$

or

$$H(t)+2H(t^2) = 3\prod_{k=0}^{\infty}(1-q(t^2)^{2^k}).$$

Since $h(n) = 0$ for $n$ odd this is equivalent to

$$\sum_{n=0}^{\infty} h(2n)t^{2n} + \sum_{n=0}^{\infty} 2h(n)t^{2n} = 3\prod_{k=0}^{\infty}(1-q(t^2)^{2^k}).$$

Replacing $t^2$ by $t$ and using $h(2n)+2h(n) = 3(-q)^{s(n)}$ this is equivalent to

$$\sum_{n=0}^{\infty}(-q)^{s(n)}t^n = \prod_{k=0}^{\infty}(1-qt^{2^k})$$

which is clear. □

The definition of 1-unbordered words would seem utterly unmotivated — except that it just so happens that they arise naturally in an entirely different context which is the subject of the rest of the paper.

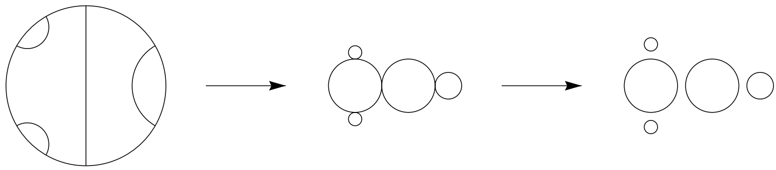

**Figure 1.** Pinching a circle along a finite lamination to obtain a collection of smaller circles.

## 3. Tautological laminations

**3A.** ***Laminations.*** A *leaf* is an unordered pair of distinct points in a circle $S$. Two leaves in $S$ are *linked* if they are disjoint (as subsets of $S$) and each separates the other in $S$. A *lamination of* $S$ is a set of leaves in $S$, no two of which are linked. A finite lamination is one with finitely many leaves.

If $\Lambda$ is a finite lamination of $S$ we may *pinch* $S$ along $\Lambda$. This means that we quotient each leaf to a point, so that $S$ collapses to a "tree" of smaller circles (sometimes called a *cactus*), and then split this tree apart into its constituent circles. We denote the result by $S \bmod \Lambda$. See Figure 1.

If there is a Riemannian metric on $S$ then we get a Riemannian metric on $S \bmod \Lambda$, so it makes sense to talk about the *length* of the components of $S \bmod \Lambda$, and observe that the sum of these lengths is equal to the length of $S$.

Now suppose $\Lambda$ is the increasing union of $\Lambda_n$ (for $n = 1$ to $\infty$) where each $\Lambda_n$ is finite. The *depth* $n$ *leaves* are those in $\Lambda_n - \Lambda_{n-1}$ and for each $n$ we can form $S \bmod \Lambda_n$ for each $n$ and obtain in this way a sequence of increasingly refined partitions of $|S|$.

**3B.** ***Tautological elaminations and complex dynamics.*** We are interested in some naturally occurring laminations called *tautological laminations*. These objects were introduced in [Calegari 2022] to study the geometry and topology of the *shift locus* — a certain parameter space that arises naturally in holomorphic dynamics. For example, in degree 2, the shift locus is the complement (in $\mathbb{C}$) of the Mandelbrot set.

The tautological laminations in [Calegari 2022] have some extra structure — they are actually "extended laminations" or *elaminations*. If we identify the circle $S^1$ with the boundary of the closed unit disk $\overline{\mathbb{D}} \subset \mathbb{C}$, leaves in a lamination $\Lambda$ corresponds to (infinite, unoriented) geodesics in $\mathbb{D}$ thought of as the hyperbolic plane in the Poincaré disk model. The unlinking property of leaves in a lamination corresponds to the condition that the geodesics in $\mathbb{D}$ they span are disjoint (except at their ideal "endpoints"). In an elamination these geodesics extend beyond $S^1$ to a pair of radial segments in $\mathbb{C} - \overline{\mathbb{D}}$. An elamination determines a lamination of $S^1$ (or equivalently, a geodesic lamination of $\mathbb{D}$) by forgetting these "extended" segments.

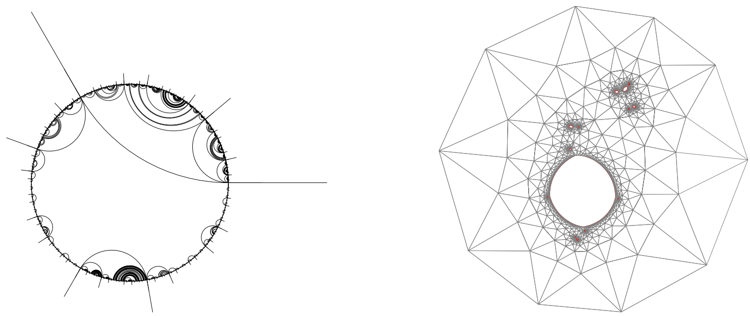

**Figure 2.** A finite elamination approximating the degree 3 tautological elamination for $z = 2$, and the result of pinching.

As mentioned in the introduction, the tautological lamination records the combinatorics of the 1 complex dimensional slices of the shift locus where $q-2$ critical Böttcher coordinates are fixed, and one critical point (with a smaller escape rate than any of the others) is allowed to vary. The extra structure of the tautological elamination records not only the combinatorics, but the holomorphic structure on these slices.

A finite elamination may be *pinched*, giving rise to a planar Riemann surface which may be (partially) compactified by a finite collection of circles, which are precisely the result of pinching the associated lamination of $S^1$. Figure 2 gives an example, approximating an infinite (tautological) elamination.

To orient the reader and to motivate the remainder of this paper, let us now describe the relationship between the tautological elamination and the holomorphic geometry of the shift locus, in the special case of degree 3. A depressed cubic polynomial $f(z) := z^3 + pz + q$ is in the shift locus $\mathcal{S}_3$ if the critical points $c_1, c_2$ (not necessarily distinct) are in the basin of attraction of infinity. These critical points have canonical *Böttcher coordinates* $C_1$, $C_2$, whose absolute value is well-defined and strictly greater than 1, and whose arguments are multivalued, where different values differ by multiples of $2\pi/3$. For $z \in \mathbb{C} - \overline{\mathbb{D}}$ let us define the *Böttcher slice* $B(z)$ of $\mathcal{S}_3$ to be the 1-complex dimensional subset where $C_1 = \{z, e^{2\pi i/3}z\}$ and $|C_1| > |C_2|$. The open dense subset of $\mathcal{S}_3$ for which the critical points are distinct and their Böttcher coordinates have distinct absolute values is foliated by such Böttcher slices, and in fact the Böttcher slices form the fibers of a (topological) fiber bundle over $\mathbb{C} - \overline{\mathbb{D}}$. Associated to each $z$ is a tautological elamination $\Lambda_T(z)$, and the Böttcher slice $B(z)$ is obtained from $\Lambda_T(z)$ by pinching.

Figure 2 depicts the tautological elamination $\Lambda_T(z)$ for $z = 2$ and the Riemann surface obtained from $\Lambda_T(z)$ by pinching. The laminations of $S^1$ associated to $\Lambda_T(z)$ are the main objects of interest throughout this section; they depend only on the argument of $z$. We shall give them a precise definition in Section 3D.

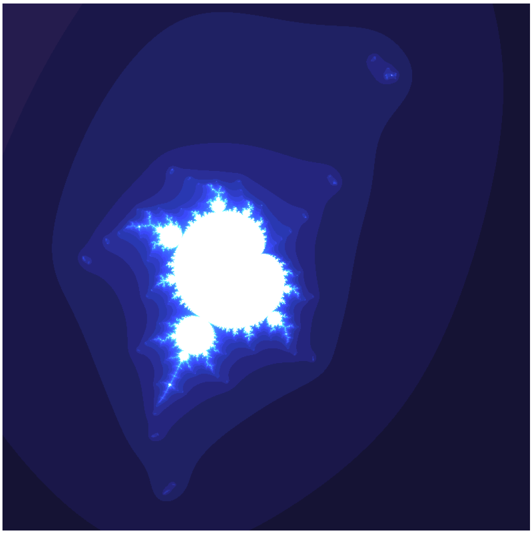

**Figure 3.** Part of the degree 3 shift locus (in blue) in a coordinate slice $f(z) = z^3 + pz + 1$.

It is computationally difficult to transform from Böttcher coordinates to polynomial coordinates. Fortunately, because the shift locus is more or less foliated by Böttcher slices, we may obtain a qualitatively reasonable picture of a Böttcher slice by instead giving (part of) a "coordinate slice" of $\mathcal{S}_3$ consisting of polynomials $z^3 + pz + q$ where the linear term $q$ is fixed, at least in a region where such a coordinate slice lies close to a Böttcher slice. Figure 3 is (part of) a "coordinate slice" of $\mathcal{S}_3$ parametrizing shift polynomials of the form $z^3 + pz + 1$. There is one "large" continent, which resembles a lopsided Mandelbrot, surrounded by a few visible small islands; and there is a little archipelago to the northeast; compare with Figure 2.

There is a refinement of the tautological elamination (called the *completed* tautological elamination) which (conjecturally) parametrizes the cut points on the components of the complement of $\mathcal{S}_q$ in a Böttcher slice. When $q = 2$ this recovers Thurston's combinatorial model for the cut points in the Mandelbrot set. For a definition see [Calegari 2022, §8.7].

**3C.** ***Degree 3, a worked example.*** Tautological laminations are laminations of the unit circle, which we normalize as $S^1 = \mathbb{R}/\mathbb{Z}$ so that it has length 1. Tautological laminations depend on a degree $q \geq 2$ and a continuous parameter $C$ (morally, a vector of $q - 2$ arguments of Böttcher coordinates) which, for $q = 3$, is encoded by a single angle $\theta \in \mathbb{R}/\mathbb{Z}$.

Although the laminations depend on the parameter, the result of pinching the circle to any finite depth does not. Thus, for each degree $q$ and each depth $n$ we obtain a *partition* of 1 into a vector of lengths of the components of $S^1 \bmod \Lambda_n$.

These lengths are all integer multiples of $q^{-n}$ (in fact, they are of the form $2^m q^{-n}$ for various $m$). Our goal is to count the number of components of length exactly $q^{-n}$ (the *short components*), and the main result of this section (Theorem 3.10) gives the generating function for the number of short components in degree $n$, and shows that it is related in a rather simple way to the function $\beta(t)$ from Proposition 2.2.

We shall first give an ad hoc (though precise) definition in the special case $q = 3$ and work out a few examples by hand. Multiplication by 3 gives a map from $S^1$ to itself. If $\lambda$ is the leaf $\{p, q\}$ for distinct points $p, q \in S^1$ then $3\lambda$ is the leaf $\{3p, 3q\}$. For any $\theta \in S^1$ let $L(\theta)$ denote the leaf $\{\theta, \theta + 1/3\}$. By abuse of notation we let $L^-(\theta)$ be the limit of leaves $L(\theta - \epsilon)$ as $\epsilon \to 0$ from above, and we say that a leaf $\lambda$ *links* $L^-(\theta)$ if it links $L(\theta - \epsilon)$ for all sufficiently small positive $\epsilon$. Likewise, we let $L^+(\theta)$ be the limit of leaves $L(\theta + \epsilon)$ as $\epsilon \to 0$ from above.

**Definition 3.1** (ad hoc definition, degree 3). There will be one depth $n$ leaf of the tautological lamination $\Lambda_T$ for each $x \in [1/3, 2/3)$ for which $3^n x = 0$. We claim there is a unique $y \in S^1$ such that

(1) $3^n y = 1/3$; and
(2) if $\lambda$ denotes the leaf $\{x, y\}$ then $3^m \lambda$ does not link $L^-(0)$ or $L^+(x)$ for $m = 0, 1, 2, \ldots, n-1$.

Then $\{3x, 3y\}$ is a depth $n$ leaf of $\Lambda_T$, and all depth $n$ leaves arise this way.

**Example 3.2** (depth 1). The only $x \in [1/3, 2/3)$ with $3x = 0$ is $x = 1/3$. The only $y$ with $3y = 1/3$ and for which $\{1/3, y\}$ does not link $L^-(0)$ or $L^+(1/3)$ is $y = 7/9$. Thus $\{x, y\} = \{1/3, 7/9\}$ and the leaf $3\{x, y\} = \{0, 1/3\}$ is the unique depth 1 leaf of $\Lambda_T$. Thus $S^1$ mod $\Lambda_T$ to depth 1 has two components of length 1/3 and 2/3 respectively.

**Example 3.3** (depth 2). For $x \in [1/3, 2/3)$ with $9x = 0$ we must have one of $x = 1/3, 4/9, 5/9$. For $x = 1/3$ we may check for $\lambda = \{1/3, 19/27\}$ that $\lambda$ and $3\lambda$ do not link $L^-(0)$ or $L^+(1/3)$. Likewise for $\lambda = \{4/9, 22/27\}$ that $\lambda$ and $3\lambda$ do not link $L^-(0)$ or $L^+(4/9)$, and for $\lambda = \{5/9, 25/27\}$ that $\lambda$ and $3\lambda$ do not link $L^-(0)$ or $L^+(5/9)$. Thus the unique depth 2 leaves of $\Lambda_T$ are $\{0, 1/9\}$, $\{1/3, 4/9\}$ and $\{2/3, 7/9\}$. Thus $S^1$ mod $\Lambda_T$ to depth 2 has 3 components of length 1/9, 1 component of length 2/9, and 1 component of length 4/9.

**Example 3.4** (depth 3). For $x \in [1/3, 2/3)$ with $27x = 0$ we must have one of $x = 1/3, 10/27, 11/27, \ldots, 17/27$. Let's do one example. For $x = 11/27$ we want $y$ with $27y = 1/3$ so for $\lambda = \{11/27, y\}$ that $\lambda, 3\lambda, 9\lambda$ do not link $L^-(0)$ or $L^+(11/27)$. One might naively guess (based on the examples in depth 1 and depth 2 in which $y = x + 3^{-1} + 3^{-n-1}$ is always the correct choice) that $y = 61/81$ would work, but $9\{11/27, 61/81\} = \{2/3, 7/9\}$ which links $L(11/27) = \{11/27, 20/27\}$. In fact $y = 58/81$, and $3\{11/27, 58/81\} = \{2/9, 4/27\}$ is a depth 3 leaf of $\Lambda_T$.

One may check that $S^1$ mod $\Lambda_T$ to depth 3 has 7 components of length $1/27$, 6 components of length $2/27$, and 1 component of length $8/27$.

The set of leaves to depth 3 is

(1) $\{0, 1/3\}$;

(2) $\{0, 1/9\}$, $\{1/3, 4/9\}$, $\{2/3, 7/9\}$;

(3) $\{0, 1/27\}$, $\{1/9, 7/27\}$, $\{2/9, 4/27\}$, $\{1/3, 10/27\}$, $\{4/9, 13/27\}$, $\{5/9, 16/27\}$, $\{2/3, 19/27\}$, $\{7/9, 22/27\}$, $\{8/9, 25/27\}$.

See the left side of Figure 2.

Continuing out to greater depth, the number of components of $S^1$ mod $\Lambda_T$ to depth $n$ of length $3^{-n}$ is $1, 3, 7, 21, 57, 171, 499$ and so on.

**3D. *Tautological laminations.*** Let us now give a more precise definition. Fix a degree $q$ which is an integer $\geq 2$. Multiplication by $q$ defines a degree $q$ map from $S^1$ to itself; if $\lambda$ is a leaf whose points do not differ by a multiple of $1/q$ then it makes sense to define the leaf $q\lambda$. Let $C$ be a finite lamination $C$ consisting of $q-2$ leaves $C_1, \ldots, C_{q-2}$ such that for each $j$ the points of $C_j$ differ by $1/q$. Thus we can write $C_j = \{\theta_j, \theta_j + 1/q\}$ for some $\theta_j \in S^1$. For simplicity we assume that $C$ is *generic*, meaning that the $\theta_j$ are irrational and irrationally related.

The quotient $S^1$ mod $C$ is a union of $q-1$ circles, $q-2$ of them of length $1/q$ and one of length $2/q$; we refer to this as the *big circle* and denote it $B'$. The preimage of $B'$ in $S^1$ is a finite collection of arcs of $S^1$ bounded by points in leaves of $C$; denote this $B$. Note that the projection from $B$ to $B'$ is one-to-one away from points in leaves of $C$.

We shall now define the depth $n$ leaves of the tautological lamination $\Lambda_T(C)$. Let $x \in B$ be a point for which $q^n x = \theta_j$; this $x$ will be a point of a depth $n$ leaf of *type* $j$. Let $z \in B$ be the unique point for which the projections $x', z' \in B'$ are antipodal (i.e., they are distance $1/q$ apart). Define a finite lamination $C(x)$ to be the union of $C$ together with the leaf $\{x, z\}$.

Now, $S^1$ mod $C(x)$ is a union of $q$ circles, all of length $1/q$. Furthermore if we denote by $\pi : S^1 \to S^1$ mod $C(x)$ the projection (which is well-defined away from the points of $C(x)$) the map $z \to q\pi^{-1}z$ extends from its domain of definition over the missing points to a homeomorphism from each component of $S^1$ mod $C(x)$ to $S^1$. By abuse of notation, we denote this map by $q : S^1$ mod $C(x) \to S^1$ and think of it as a $q$–1 map.

**Lemma 3.5** (division by $q$). *If $\lambda := \{a, b\}$ is a generic leaf unlinked with $C(x)$, and $a' \in S^1$ satisfies $qa' = a$ then there is a unique $b'$ with $qb' = b$ so that $\lambda' := \{a', b'\}$ is unlinked with $C(x)$.*

*Proof.* "Generic" is just to rule out boundary cases where, e.g., $a'$ or $b'$ is equal to a point in $C(x)$. In particular, if $a'$ maps to a component $S_i$ of $S^1$ mod $C(x)$ then we can pull back $\lambda$ under the map $q : S_i \to S^1$ and then take its preimage in $S^1$ to obtain $\lambda'$. □

Given $x$, we consider the sequence of points $x_i := q^i x$ for $0 \le i \le n$. By definition, $x_n = \theta_j$. Define $y_n = \theta_j + 1/q$ and $\lambda_n := \{x_n, y_n\}$ so that $\lambda_n = C_j$, and then inductively let $\lambda_i$ be obtained from $\lambda_{i+1}$ as in Lemma 3.5 so that $\lambda_i := \{x_i, y_i\}$ where $qy_i = y_{i+1}$ and $\lambda_i$ is unlinked from $C(x)$. Finally we obtain the leaf $\lambda_0$ which, because it depends on $x$, we should really denote $\lambda_0(x)$.

**Definition 3.6.** With notation as above, the depth $n$ leaves of $\Lambda_T(C)$ are the leaves $q\lambda_0(x)$ of $S^1$ as $x$ ranges over the points in $B$ with $q^n x = \theta_j$ and $j$ ranges over $1, \dots, q-2$.

Notice that in this definition every leaf is enumerated exactly *twice*; if $x$ and $z$ in $B$ have antipodal image in $B'$ then the images of $\lambda_0(x)$ and $\lambda_0(z)$ are antipodal in $B'$ so that $q\lambda_0(x) = q\lambda_0(z)$. So we only need to find a subset $A \subset B$ projecting to half of $B'$ and add leaves $q\lambda_0(x)$ for $x \in A$ with $q^n x = \theta_j$. Thus the number of leaves of depth $n$ is equal to $q^{n-1}(q-2)$. In particular $\Lambda_T(C)$ is empty if $q = 2$.

**Proposition 3.7** (lamination). *The leaves of $\Lambda_T(C)$ are pairwise unlinked; thus $\Lambda_T(C)$ really is a lamination. Furthermore, if $\Lambda_{T,n}(C)$ denotes the leaves of $\Lambda_T(C)$ of depth at most n, the set of lengths of components of $S^1$* mod $\Lambda_{T,n}(C)$ (*counted with multiplicity*) *is independent of C and depends only on q.*

For a proof, see [Calegari 2022, §7]. Tautological laminations for $q = 3, 4, 5, 6$ (for a rather symmetric choice of $C$) are displayed in Figure 4.

Since the set of lengths of $S^1$ mod $\Lambda_{T,n}(C)$ (with multiplicity) is independent of $C$ we can fix a normalization $\theta_j = (j-1)/q$ and suppress $C$ in our notation in the sequel. This set of values is not generic; so we interpret the values of $\theta_j$ as limits as we approach $(j-1)/q$ from below. So we should interpret $C_j$ as a "leaf" whose endpoints span the interval $[(j-1)/q, j/q)$, for the purposes of determining when leaves and their preimages are linked.

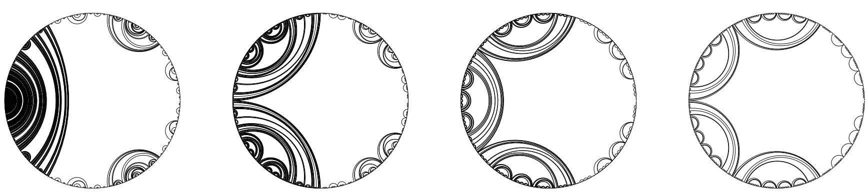

**Figure 4.** Tautological laminations for $q = 3, 4, 5, 6$.

Then every depth $n$ leaf is of the form $q\lambda$, where $\lambda = \{x, y\}$, $q^n x = (j-1)/q$ and $q^n y = j/q$. It follows that every depth $n$ leaf of $\Lambda_T$ consists of a pair of points which are integer multiples of $q^{-n}$, and therefore every component of $S^1$ mod $\Lambda_{T,n}$ has length which is an integer multiple of $q^{-n}$. What is not obvious, but is nevertheless true, is that these integer multiples are all *powers of* 2 (we shall deduce this in the sequel). Write the length of a component as $\ell \cdot q^{-n}$ where $\ell$ is a power of 2, and define $N_q(n, m)$ to be the number of components of $S^1$ mod $\Lambda_{T,n}$ with $\ell = 2^m$.

Let's spell out Definition 3.6 in this normalization. We can take $B$ and $A$ to be the half-open intervals

$$B = [(q-2)/q, 1) \quad \text{and} \quad A = [(q-2)/q, (q-1)/q).$$

The base $q$ expansion of $x \in A$ with $q^n x = (j-1)/q$ is a word of length $n+1$ in the alphabet $\{0, 1, \dots, (q-1)\}$ starting with the digit $(q-2)$ and ending with the digit $(j-1)$. If we denote the digits of $x$ as $x_0 \cdots x_n$ then

$$x := \cdot(q-2)x_1x_2\cdots x_{n-1}(j-1) \quad \text{and} \quad z := \cdot(q-1)x_1x_2\cdots x_{n-1}(j-1).$$

Likewise, we denote the digits of $y$ as $y_0 \cdots y_n$. Then

(1) $y_n = j$; and recursively,

(2) if $x_i \neq q-2$ or $q-1$ then $y_i = x_i$; and

(3) if $x_i = q-2$ or $q-1$ then $y_i$ is the unique one of $q-1$ or $q-2$ so that $\cdot x_i \cdots x_n$ and $\cdot y_i \cdots y_n$ do not link $x$ and $z$.

Although we are not able to give a simple formula for $N_q(n, m)$, it turns out there is a relatively simple formula for $N_q(n, 0)$ — i.e., the number of components of $S^1$ mod $\Lambda_{T,n}$ of length $q^{-n}$. These are the *short components*.

**3E. *Short components.*** One of the nice things about our normalization $C$ is that there is a simple relationship between short components of $S^1$ mod $\Lambda_{T,n}$ and certain depth $n$ leaves of $\Lambda_{T,n}$, a relationship which is substantially more complicated for longer components. Say a *short leaf* is a depth $n$ leaf of $\Lambda_{T,n}$ whose points differ by exactly $q^{-n}$ (this is the least it can be). Then:

**Lemma 3.8** (short leaf). *There is a bijection between short components and short leaves of any fixed depth.*

*Proof.* Let $S$ be a short component at depth $n$, and consider the preimage $X$ in $S^1$. Then $X$ is a union of finitely many disjoint arcs and isolated points bounded by leaves of depth $\leq n$. The total length of $X$ is $q^{-n}$ by the definition of short component. But leaves of depth $k$ consist of points which are integer multiples of $q^{-k}$ so the only possibility is that $X$ consists of a single arc $Y$ of $S^1$ together with finitely many (possibly zero) isolated points joined to the endpoints of $Y$ by a chain of leaves $\gamma_0, \dots, \gamma_n$, each sharing one endpoint with the next.

We claim that in fact there are no isolated points, so that $X = Y$ is a single arc of $S^1$ cut off by a single (necessarily) short leaf. To see this, let's enlarge the circle by a factor of $q^n$ so that depth $k$ leaves with $k < n$ consist of points which are divisible by $q$, and each depth $n$ leaf of type $j$ joins a point congruent to $(j-1) \bmod q$ to a point congruent to $j \bmod q$. By the nature of their construction distinct depth $n$ leaves of type $j$ cannot share an endpoint, so a depth $n$ leaf of type $j$ must be followed by a depth $n$ leaf of type $j+1$, and only a type 1 leaf of depth $n$ can follow a depth $< n$ leaf and only in the positive (i.e., anticlockwise) direction around $S^1$ (remember our understanding of $\theta_j$ as the limit of a sequence approaching $(j-1)/q$ from below). It follows that if there is some intermediate point, the endpoints of $Y$ differ by at least 2 mod $q$ so that $S$ is not short after all. This proves the claim. □

Note that this lemma is *false* for generic $C$.

Let $\lambda'$ be a short leaf of $\Lambda_T$ of depth $n$ of the form $q\lambda_0(x)$, where $x \in A$ and $q^n x = \theta_j$. For this normalization, $z = x + 1/q$ and $y = x + q^{-1-n}$, where $\lambda_0(x) = \{x, y\}$. The defining property of being a depth $n$ leaf means that $\lambda_k(x) = q^k \lambda_0(x)$ does not link $C(x) = C \cup \{x, z\}$ for any $0 \le k \le n$. Actually, for *any* integer $m \bmod q^{n+1}$, setting $x = mq^{-1-n}$ and $y = (m+1)q^{-1-n}$, the leaf $\lambda_k(x)$ does not link any $C_i$ for $0 \le i \le (q-2)$. So the short leaves are just the $x$ for which $\lambda_k(x)$ does not link $\{x, z\}$ for $0 \le k \le n$.

Remember that the base $q$ expansion of $x$ is a word of length $n+1$ in the alphabet $\{0, 1, \dots, (q-1)\}$ starting with the digit $(q-2)$ and ending with the digit $(j-1)$. The base $q$ expansion of $y$ is the same as that of $x$ with the last digit replaced by $j$. Similarly, the base $q$ expansion of $z$ is the same as that of $x$ with the first digit replaced by $(q-1)$. We deduce:

**Lemma 3.9** (short is 1-unbordered). *A word in the alphabet* $\{0, 1, \dots, (q-1)\}$ *of length* $(n+1)$ *starting with* $(q-2)$ *and ending with* $(j-1)$ *corresponds to a short leaf if and only if it is* 1*-unbordered.*

*Proof.* The leaf $\lambda_k(x) = \{q^k x, q^k y\}$, and the base $q$ expansions of $q^k x$ and $q^k y$ are obtained from the base $q$ expansions of $x$ and $y$ by the $k$-fold left shift. This leaf links $\{x, z\}$ if and only if the length $k$-suffix of $x$ is either equal to a prefix of $x$, or to a prefix of $z$. But this is the definition of a 1-unbordered word. □

Since $(j-1)$ is allowed to vary from 0 to $(q-3)$, and since a word that starts with $(q-2)$ and ends with $(q-2)$ or $(q-1)$ is already 1-bordered, it follows that $N_q(n, 0)$ is equal to the number of 1-unbordered words of length $(n+1)$ starting with $(q-2)$, which is just $q^{-1}$ times the number of 1-unbordered words of length $(n+1)$. In other words:

**Theorem 3.10** (recursive formula). *$N_q(n,0)$ satisfies the recursion $N_q(0,0)=1$, $N_q(1,0)=(q-2)$ and*

$$N_q(2n,0)=qN_q(2n-1,0) \quad \textit{and} \quad N_q(2n+1,0)=qN_q(2n,0)-2N_q(n,0)$$

*and has the generating function $(\beta(t)-1)/qt$ where a closed form for $\beta(t)$ is given in Proposition 2.2.*

## 4. Tree polynomials

We now discuss a rather different class of objects that turn out to be naturally in bijection with the components of $S^1$ mod $\Lambda_{T,n}$. These objects are called *tree polynomials*.

We give our definition in terms of rooted trees (with some auxiliary planar structure) and adopt the standard terminology of parents, children, siblings etc. Thus for every (nonroot) vertex there is a unique embedded path from that vertex to the root, and the *parent* of $v$ is the unique vertex $w$ on that path connected to $v$ by an edge, and conversely $v$ is the *child* of $w$; vertices are *siblings* if they share a common parent, and so on.

**Definition 4.1.** A *tree polynomial* is a finite rooted tree $T$ together with the following data:

(1) **depth:** all leaves have a common depth $n$; we call this the *depth of $T$*;

(2) **critical:** all vertices are *critical* or *ordinary*;

   (a) the root is critical;
   (b) every nonleaf critical vertex has exactly one critical child;
   (c) every ordinary vertex has no critical children;

(3) **order:** the children of every vertex are *ordered*, and the critical child of the root is first among its siblings;

(4) **self-map:** there is a simplicial self-map $f : T \to T$ such that

   (a) $f(\text{root}) = \text{root}$;
   (b) $f(v) = \text{root}$ for all children $v$ of the root; and
   (c) for all $v$ with nonroot parent $w$, the image $f(v)$ is a child of $f(w)$;
   (d) if $v$ is ordinary and not a leaf, then $f$ maps the children of $v$ bijectively and in an order-preserving way to the children of $f(v)$;
   (e) if $v$ is critical and not a leaf or the root, then $f$ maps the children of $v$ in an order nondecreasing way to the children of $f(v)$; this map is onto and two-to-one except for the critical child of $v$ which is the unique child mapping to its image;

(5) **length:** there is a *length* function $\ell$ from the vertices to $\mathbb{N}$;
   (a) $\ell(\text{root}) = 1$;
   (b) if $v$ is ordinary, $\ell(v) = \ell(f(v))$;
   (c) if $v$ is critical, $\ell(v) = 2\ell(f(v))$.

Another way of talking about the order structure on the children of each vertex is to say that $T$ is a *planar* tree, and the map $f$ is compatible with the planar structure.

### 4A. *Basic properties.*

**Definition 4.2** (degree). Let $T$ be a tree polynomial. The root has one critical child with $\ell = 2$ and some nonnegative number of ordinary children with $\ell = 1$. All children map to the root under $f$. Thus tree polynomials of depth 1 are classified by the number of children. The *degree* of a tree polynomial, denoted $q(T)$, is equal to the number of children of the root, plus one.

**Example 4.3** (degree 2). There is a unique tree polynomial of degree 2 of any positive depth, since every vertex is critical and all but the leaf have a unique child.

**Definition 4.4** (postcritical length). Let $T$ be a tree polynomial and let $c$ be the unique critical leaf. The *postcritical length* of $T$, denoted $\ell(T)$, is equal to $\ell(f(c)) = \ell(c)/2$.

By induction, $\ell(T)$ is always a power of 2.

The next proposition explains why we have introduced tree polynomials:

**Proposition 4.5** (bijection). *There is a natural bijection between the set of degree $q$ tree polynomials $T$ of depth $(n+1)$ with $\ell(T) = \ell$ and the set of components of $S^1 \bmod \Lambda_{n,T}$ of length $\ell \cdot q^{-n}$ where $\Lambda_{n,T}$ are the leaves of depth $\leq n$ in the tautological lamination from Section 3.*

*Proof.* This is a corollary of [Calegari 2022, Theorems 9.20 and 9.21]. The tree polynomials are combinatorial abstractions of the sausage polynomials defined in [Calegari 2022, Definition 9.4]. A sausage polynomial is a certain kind of infinite nodal genus 0 Riemann surface $\Sigma$ together with a holomorphic self-map of degree $q$ satisfying a number of properties. A tree polynomial records only the underlying combinatorics of $\Sigma$, which is enough to recover $\ell$. $\square$

It follows that for each $n$ and each $m$, the number of degree $q$ tree polynomials $T$ of depth $(n+1)$ with $\ell(T) = 2^m$ is $N_q(n, m)$.

**Lemma 4.6** (extension). *Let $T$ be a tree polynomial of depth $n$ and let $c$ be the unique critical leaf. Then tree polynomials $T'$ of depth $(n+1)$ that extend $T$ are in bijection with the children of $f(c)$.*

*Proof.* To extend $T$ to $T'$ we just add children to each of the leaves of $T$. For each ordinary leaf $v$ we add a copy of the children of $f(v)$. For the unique critical leaf $c$ we must choose a child $e$ of $f(c)$ and then add as children of $c$ one copy of $e$, and two copies of every other child of $f(c)$. The copy of $e$ becomes the unique critical child of $c$ in $T'$. The functions $f$ and $\ell$ extend to these new leaves uniquely. $\square$

The next two lemmas give direct proofs in the language of tree polynomials of the identities $\sum_\ell N_q(n,m)\cdot 2^m = q^n$ and $\sum_\ell N_q(n,m) = 1+(q-2)(q^n-1)/(q-1)$. Both identities follow immediately from Proposition 4.5 since the first just says that the sum of the lengths of the components of $S^1$ mod $\Lambda_{T,n}$ is equal to 1, and the second just says that $\Lambda_T$ has $(q-2)q^{n-1}$ leaves of depth $n$, both of which follow immediately from the definitions.

**Lemma 4.7** (multiplication by $q$). *Let $T$ have degree $q$. For each nonleaf vertex $v$ with children $w_i$ we have $q\cdot\ell(v) = \sum_i \ell(w_i)$. Consequently $\sum_\ell N_q(n,m)\cdot 2^m = q^n$.*

*Proof.* There is a unique tree polynomial of depth 1 and degree $d$. The root has $\ell(\text{root}) = 1$, and it has $q-1$ children with $\ell = 2, 1, \dots, 1$. Thus $q\cdot\ell(v) = \sum_i \ell(w_i)$ is true for the root vertex, and by induction on depth, it is true for each ordinary or critical nonleaf vertex.

For any $T$, the extensions $T'$ of $T$ are in bijection with the children of the postcritical vertex, and the formula we just proved shows $\sum \ell(T') = q\ell(T)$. $\square$

**Lemma 4.8** (number of children). *Let $T$ have degree $d$. For each nonleaf vertex $v$ the number of children of $v$ is $(q-2)\ell(v)+1$. Consequently*

$$\sum_m N_q(n,m) = 1+(q-2)(q^n-1)/(q-1).$$

*Proof.* First we prove the formula relating $\ell(v)$ to the number of children of $v$. The formula is true for the root vertex. If $v$ is ordinary then $\ell(v) = \ell(f(v))$ and $v$ has the same number of children as $f(v)$, so if the formula is true for $f(v)$ it is true for $v$. If $v$ is critical then $\ell(v) = 2\ell(f(v))$ and if $f(v)$ has $(q-2)\ell(f(v))+1$ children then $v$ has $2(q-2)\ell(f(v))+1$ children. So the formula is true by induction.

Since there are $N_q(n-1,m)$ depth $n$ degree $q$ tree polynomials of length $\ell = 2^m$, and since by Lemma 4.8 each has $(q-2)\ell(T)+1$ children, we obtain a recursion

$$\sum_m N_q(n,m) = \sum_m N_q(n-1,m)((q-2)2^m+1) = (q-2)q^{n-1} + \sum_m N_q(n-1,m)$$

Since $\sum_m N_q(0,m) = 1$ the lemma follows. $\square$

**Lemma 4.9** (length subdoubles). *Every child $w$ of $v$ has $\ell(w) \le 2\ell(v)$ with equality if and only if every sibling of $w$ has $\ell = 1$.*

*Proof.* By Lemmas 4.7 and 4.8 $w$ has $(q-2)\ell(v)+1$ children, whose lengths sum to $q\ell(v)$. $\square$

**Remark 4.10.** It is worth pointing out a close relationship between tree polynomials (as defined above) and the *polynomial-like tree maps* of [DeMarco and McMullen 2008]. The main difference seems to be that the latter objects forget the planar structure (i.e., the data of the ordering on each set of siblings). One should also mention that there is a close relationship between the *dynamical elaminations* (see [Calegari 2022, §4.4]) and the *pictographs* of [DeMarco and Pilgrim 2017] which are in turn closely related to the pattern and tableau of [Branner and Hubbard 1992]. The tautological elaminations we discuss in this article are (roughly speaking) related to dynamical elaminations as the shift locus is related to individual shift polynomials.

## 5. $F$-sequences

**Definition 5.1** (critical vein). Let $T$ be a tree polynomial of depth $n$. The *critical vein* is the segment of $T$ containing all the critical vertices. We denote it by $\gamma$ and label the critical points on $\gamma$ as $c_i$, where $c_0$ is the root and $c_n$ is the critical leaf.

For each vertex $w$ of $T$, define $F(w)$ to be equal to $f^k(w)$ for the least positive $k$ so that $f^k(w)$ is critical. Thus we can think of $F$ as a map from the critical vein to itself, and by abuse of notation, for integers $i$, $j$ we write $F(i) = j$ if $F(c_i) = c_j$ so that we can and do think of $F$ as a function from $\{0, \dots, n\}$ to itself. We also write $\ell(i)$ for $\ell(c_i)$.

**Lemma 5.2** (properties of $F$). *$F(0) = 0$ and for every positive $i$, $F(i) < i$ and $F(i+1) \le F(i) + 1$. Furthermore, $\ell(i) = 2\ell(F(i))$ for $i > 0$ so that $\ell(i) = 2^k$ where $k$ is the least integer so that $F^k(i) = 0$*

*Proof.* Since $f(w)$ has smaller depth than $w$ unless $w$ is the root, $F(i) < i$. Moreover, $F(c_{i+1})$ is equal to $F(w)$ for some child $w$ of $F(c_i)$, so $F(i+1) \le F(i) + 1$.

Finally, $\ell(w) = \ell(f(w))$ when $w$ is ordinary or the root, and $\ell(w) = 2\ell(f(w))$ when $w$ is critical and not the root. □

Let $\gamma^+$ denote the union of $\gamma$ together with the siblings of every critical vertex. We may think of it as a digraph (i.e., a directed graph), where every edge points away from the root. Define $\Gamma$ to be the quotient of $\gamma^+$ obtained by identifying every sibling $w$ of a critical vertex with its image $F(w)$. Note that $\Gamma$ is a digraph.

The next proposition gives a characterization of the functions $F$ that can arise from tree polynomials.

**Proposition 5.3** ($F$-sequence). *A function $F$ from $\{0, 1, \dots, n\}$ to itself arises from some tree polynomial $T$ of depth $n$ if and only if it satisfies the following properties*:

(1) $F(0) = 0$ *and* $F(1) = 0$;

(2) *each $j$ has a finite set of* ***options*** *which are the admissible values of $F(i+1)$ when $F(i)=j$;*
   (a) *the options of* $0$ *are* $0$ *and* $1$;
   (b) *if $F(i)=0$ then the options of $i$ are $i+1$ and whichever option of* $0$ *is not equal to $F(i+1)$;*
   (c) *if $F(i)\neq 0$ and $F(i+1)$ is not equal to $F(i)+1$ then the options of $i$ are $i+1$ together with the options of $F(i)$;*
   (d) *if $F(i)\neq 0$ and $F(i+1)=F(i)+1$ then the options of $i$ are $i+1$ together with all the options of $F(i)$ except $F(i)+1$.*

*A function $F$ is an* ***$F$-sequence*** *if it satisfies these properties.*

*Proof.* Imagine growing a tree $T$ by iterated extensions from a tree of depth 1. The extensions at each stage are the children of the critical image $f(v)$, which in turn may be identified with the children of $F(v)$.

When we grow $T$ of depth $i$ to $T'$ we grow $\Gamma$ to $\Gamma'$ by adding a new edge from $i$ to a new vertex $i+1$, and adding two new edges from $i$ to $j$ for each edge from $F(i)$ to $j$ *except* for the edge from $F(i)$ to $F(i+1)$ (outgoing edges at $F(i)$ may be identified with the *options* of $F(i)$ as above). The root 0 is joined by a single edge both to itself and to 1, but every other vertex $i$ is joined by a single edge to $i+1$ and by an *even* number of edges to each of its options (this can be seen by induction). The proposition follows. □

**Remark 5.4.** The referee has pointed out that the $F$-function carries essentially the same information as the Yoccoz' $\tau$-function derived from the tableau of Branner–Hubbard (see, e.g., [Branner and Hubbard 1992, §4.2; DeMarco and McMullen 2008, §11]). Proposition 5.3 is essentially equivalent to [DeMarco and Schiff 2010, Proposition 2.1].

The map from tree polynomials to $F$-sequences is many to one, since for every vertex $i$ of $\Gamma$ except the root, if there is an edge from $i$ to $j\neq i+1$ then there are at least two such edges. Nevertheless, if $F$ is an $F$-sequence corresponding to $T$ of depth $n$, the extended sequence defined by $F(n+1)=F(n)+1$ corresponds to the unique extension of $T$ for which $F(c_{n+1})$ is the critical child of $F(c_n)$.

**Example 5.5.** $\ell(T)=1$ if and only if $F(n)=0$, where $n$ is the depth of the tree. The number of depth $n$ tree polynomials with this property is $N_q(n-1,0)$ by definition.

**Example 5.6** (maximal type). There is a unique depth $n$ tree polynomial $T$ of any degree with $2\ell(T)=\ell(n)=2^n$ namely the tree polynomial for which $F(i)=i-1$ for all positive $i$. Thus $N_q(n,n)=1$. We call these trees of *maximal type*.

The components of $S^1 \bmod \Lambda_{T,n}$ corresponding to the trees of maximal type are clearly evident in Figure 4 (they correspond to the large components of “white space”).

**Example 5.7** (degree 3 maximal component). Consider the tree polynomial sequence of maximal type of degree 3. Let's work, as in Section 3D in the normalization $C = \{0, 1/3\}$. For each $n$ the result of pinching $S^1$ mod $\Lambda_{T,n}(C)$ has a unique component of length $2^n/3^n$, and this sequence of components corresponds precisely to the (degree 3) tree polynomial sequence of maximal type; we call this component, for each $n$, the *maximal component*.

For each $n$ we can let $K_n(C)$ be the preimage of the maximal component in $S^1$. As a subset of $S^1$ this depends on $C$, but for the specific normalization $C = \{0, 1/3\}$ it does not, and we abbreviate $K_n(C) = K_n$. It turns out that there is a very explicit description of $K_n$: it consists of numbers in $[0, 1)$ whose base 3 expansion is of two types:

(1) the first $n$ digits contain no 0; or
(2) numbers of the form $\cdot x\dot{0}$ where $x$ is a string of $< n$ digits containing no 0.

In other words:

$$K_1 = 0 \cup [1/3, 1), \quad K_2 = 0 \cup 1/3 \cup [4/9, 2/3] \cup [7/9, 1), \quad \text{etc.}$$

If we denote $K := \cap_n K_n$, then $K$ is the set of numbers in $[0, 1]$ whose base three expansion either contains no 0s, or is of the form $\cdot x\dot{0}$ for some finite string $x$ containing no 0s. See Figure 5. Compare with the left side of Figure 2. This component corresponds to the "lopsided Mandelbrot" in Figure 3.

The "boundary" of the region $K$ consists of a union of $2^{n-1}$ short components of each positive depth $n$, as follows from Lemma 3.9, and no other components (because otherwise the length of the maximal component would be strictly less than $2^n/3^{-n}$ for some $n$).

For each positive $k$ let $i_k$ denote the least index (if any) for which $\ell(i_k) = 2^k$. Note that $i_0 = 0$.

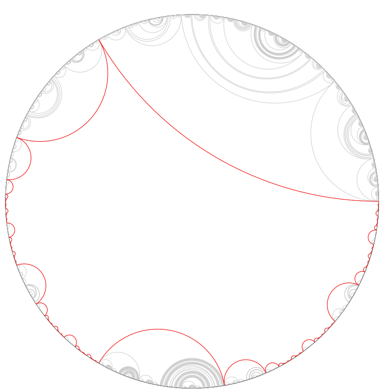

**Figure 5.** The maximal component $K$ of $S^1$ mod $\Lambda_T$.

**Lemma 5.8** (increments grow). *$F(i_k) = i_{k-1}$ for $k > 0$. Consequently $|i_{k+1} - i_k| \geq |i_k - i_{k-1}|$.*

*Proof.* By definition $F(i_k)$ is some value of $j$ with $\ell(j) = 2^{k-1}$. But if $j > i_{k-1}$ there was some $i' < i_k$ with $F(i') = i_{k-1}$, contrary to the definition of $i_k$.

The inequality follows from $F(i+1) \leq F(i) + 1$. □

**Definition 5.9** ($S$ and $B$). Let $F$ be an $F$-sequence. Let $S$ be the set of indices $i$ such that $F(i+1) = F(i) + 1$ and let $B$ be the rest. Note that 0 is in $B$.

The *prior options* of $i$ are the options other than $i+1$. We denote these by $P(i)$. Thus, if $i \in S$ then $P(i) = P(F(i))$.

**Lemma 5.10** (backslide). *Let $F$ be an $F$-sequence and let $i \in B$. Then $F(i+1) < F(i)$ and $F(i+1) \in P(b)$, where $b = F^k(i)$ for some $k$ and $b \in B$.*

*Proof.* Since $i \in B$ we must have $F(i+1) \in P(F(i))$ so that necessarily $F(i+1) < F(i)$. Furthermore, if $F(i) \in S$, then $P(F(i)) = P(F^2(i))$ and so on by induction until the first $k$ so that $F^k(i) \in B$. □

Using $F$-sequences we may deduce the following "gap" theorem, that was observed experimentally.

**Theorem 5.11** (gap). *$N_q(n, m) = 0$ for $\lfloor n/2 \rfloor < m < n$.*

*Proof.* Let $T$ be a tree polynomial of depth $n+1$. If $m < n$ then $T$ is not of maximal type, so there is some first positive index $k \in B$. Note that $i_k = k$ and in fact $i_j = j$ for all $j \leq k$. Since $k \in B$, by Lemma 5.10, $F(k+1) \in P(b)$, where $b \in B$ is $< k$. But then $b = 0$ so $F(k+1) = 0$. It follows that $i_{k+1} \geq 2k+1$ and, successively, $i_{k+j} \geq k + j(k+1)$. From this the desired inequality follows. □

## 6. Tautological tree

Degree $q$ tree polynomials of various depth can themselves be identified with the vertices of a (rooted, planar) *tautological tree* $\mathbb{T}_q$, whose vertices at depth $n$ are the tree polynomials of degree $q$ and depth $n$, and for each vertex $T$ of $\mathbb{T}_q$, the children of $T$ are the extensions of $T$.

Note that for each vertex $T$ of $\mathbb{T}_q$ we can recover $\ell(T)$ from the number of children of $T$ in $\mathbb{T}_q$, since this number is $(q-2)\ell(T) + 1$. So all the data of $N_q(n, m)$ can be read off from the abstract underlying tree of $\mathbb{T}_q$ (in fact, even the root can be recovered from the fact that it is the unique vertex of valence $q - 1$).

The tree $\mathbb{T}_3$ up to depth 4 (with vertices labeled by $\ell$ value, from which one could easily extend it another row as an unlabeled tree) is depicted in Figure 6

Every vertex labeled 1 has two children labeled 2 and 1. Every vertex labeled 2 has three children, but these might be labeled 4, 1, 1 or 2, 2, 2. Components of the complement of the shift locus $\mathcal{S}_q$ in a 1-dimensional slice are in bijection with the

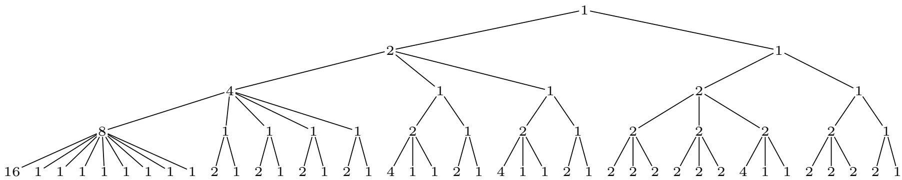


**Figure 6.** $\mathbb{T}_3$ up to depth 4.

ends of $\mathbb{T}_q$. Each such end gives rise to a sequence $\ell(n)$ of $\ell$-values, and when $\sum 1/\ell(n)$ diverges, the corresponding component consists of a single point. Such ends are called *small*; those with $\sum 1/\ell(n) < \infty$ are *big*. Big ends are *dense* in the space of ends of $\mathbb{T}_q$:

**Proposition 6.1** (big ends dense). *Big ends are dense. In other words, every finite rooted path in $\mathbb{T}_q$ can be extended to an infinite path converging to a big end.*

*Proof.* Let $T$ be a tree polynomial of some finite depth $n$ and let $F$ be the associated $F$-sequence. Suppose $F(n) = i$. There is a unique infinite sequence of extensions of $T$ defined recursively by $F(m+1) = F(m) + 1$ for all $m \geq n$. Then $F(m) = m + i - n$ for all sufficiently large $m$, so that $\ell(m) = 2\ell(m + i - n)$ and $\sum 1/\ell(m) < \infty$. $\square$

Let's call an end *type $S$* if the associated $F$-sequence satisfies $F(m+1) = F(m)+1$ for all sufficiently large $m$ (i.e., if it is of the sort constructed in Proposition 6.1). For example, the sequence of maximal type is of type $S$.

**Example 6.2** (littlebrot). The right side of Figure 7 depicts the second biggest complementary component in a Böttcher's slice (this is a speck in the northeast corner in Figure 3). It corresponds to an end of type $S$ with $\ell(n) = 2^{\lfloor n/2 \rfloor}$.

In the normalization $C = \{0, 1/3\}$ the base 3 decimal expansions of the points in the subset of $S^1$ associated to this component is a regular language in the alphabet $\{0, 1, 2\}$ (whose precise description is somewhat complicated and not very enlightening). Compare with Example 5.7.

Theorem 9.1 of [Branner and Hubbard 1992] implies (in degree 3, but the same result should hold in every degree) that every big end is of type $S$ and a component of the complement of $\mathcal{S}_q$ in a slice has positive diameter if and only if it corresponds to a big end of $\mathbb{T}_q$.

**Conjecture 6.3.** *In the normalization $C = \{0, 1/3\}$, every big end corresponds to a subset of $S^1$ whose base 3 decimal expansion is a regular language in the alphabet $\{0, 1, 2\}$.*

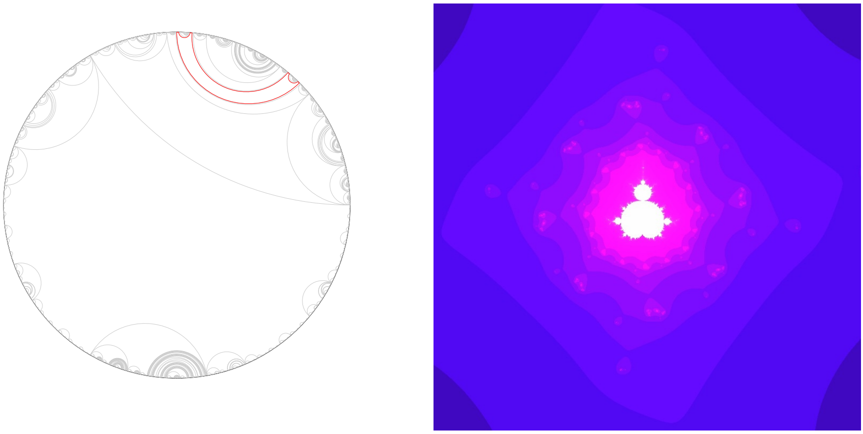

**Figure 7.** A slice through $z^3 + pz + 2$ of width 0.0003 centered at $1.72572 + 3.09778i$ and the corresponding component of the tautological lamination.

## 7. Tables of values

Values of $N_3(n, m)$ for $0 \leq n, m \leq 12$ are contained in Table 1. Values of $N_q(n, m)$ for $0 \leq n, m \leq 11$ and $q = 4, 5$ are in Tables 2 and 3. These tables were computed with the aid of the program `taut`, written by Alden Walker.

| $n$ | $m =$ 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | | |
| 1 | 1 | 1 | | | | | | | | | | | |
| 2 | 3 | 1 | 1 | | | | | | | | | | |
| 3 | 7 | 6 | 0 | 1 | | | | | | | | | |
| 4 | 21 | 16 | 3 | 0 | 1 | | | | | | | | |
| 5 | 57 | 51 | 13 | 0 | 0 | 1 | | | | | | | |
| 6 | 171 | 149 | 39 | 5 | 0 | 0 | 1 | | | | | | |
| 7 | 499 | 454 | 117 | 23 | 0 | 0 | 0 | 1 | | | | | |
| 8 | 1497 | 1348 | 360 | 66 | 9 | 0 | 0 | 0 | 1 | | | | |
| 9 | 4449 | 4083 | 1061 | 207 | 41 | 0 | 0 | 0 | 0 | 1 | | | |
| 10 | 13347 | 12191 | 3252 | 591 | 126 | 17 | 0 | 0 | 0 | 0 | 1 | | |
| 11 | 39927 | 36658 | 9738 | 1799 | 370 | 81 | 0 | 0 | 0 | 0 | 0 | 1 | |
| 12 | 119781 | 109898 | 29292 | 5351 | 1125 | 240 | 33 | 0 | 0 | 0 | 0 | 0 | 1 |

**Table 1.** Values of $N_3(n, m)$.

| $n$ | $m=$ 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | |
| 1 | 2 | 1 | | | | | | | | | | |
| 2 | 8 | 2 | 1 | | | | | | | | | |
| 3 | 28 | 14 | 0 | 1 | | | | | | | | |
| 4 | 112 | 52 | 6 | 0 | 1 | | | | | | | |
| 5 | 432 | 220 | 30 | 0 | 0 | 1 | | | | | | |
| 6 | 1728 | 872 | 120 | 10 | 0 | 0 | 1 | | | | | |
| 7 | 6856 | 3540 | 472 | 54 | 0 | 0 | 0 | 1 | | | | |
| 8 | 27424 | 14120 | 1924 | 204 | 18 | 0 | 0 | 0 | 1 | | | |
| 9 | 109472 | 56744 | 7620 | 828 | 98 | 0 | 0 | 0 | 0 | 1 | | |
| 10 | 437888 | 226768 | 30752 | 3212 | 396 | 34 | 0 | 0 | 0 | 0 | 1 | |
| 11 | 1750688 | 908040 | 122852 | 12872 | 1556 | 194 | 0 | 0 | 0 | 0 | 0 | 1 |

**Table 2.** Values of $N_4(n,m)$.

| $n$ | $m=$ 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | | |
| 1 | 3 | 1 | | | | | | | | | | |
| 2 | 15 | 3 | 1 | | | | | | | | | |
| 3 | 69 | 24 | 0 | 1 | | | | | | | | |
| 4 | 345 | 114 | 9 | 0 | 1 | | | | | | | |
| 5 | 1695 | 597 | 51 | 0 | 0 | 1 | | | | | | |
| 6 | 8475 | 2973 | 255 | 15 | 0 | 0 | 1 | | | | | |
| 7 | 42237 | 15018 | 1245 | 93 | 0 | 0 | 0 | 1 | | | | |
| 8 | 211185 | 75012 | 6306 | 438 | 27 | 0 | 0 | 0 | 1 | | | |
| 9 | 1055235 | 375951 | 31287 | 2199 | 171 | 0 | 0 | 0 | 0 | 1 | | |
| 10 | 5276175 | 1879269 | 157098 | 10767 | 858 | 51 | 0 | 0 | 0 | 0 | 1 | |
| 11 | 26377485 | 9400644 | 784596 | 53799 | 4230 | 339 | 0 | 0 | 0 | 0 | 0 | 1 |

**Table 3.** Values of $N_5(n,m)$.

## Acknowledgements

I would like to thank Laurent Bartholdi, Frank Calegari, Eric Rains and Alden Walker; furthermore I would like to especially thank the referee for extensive and detailed feedback on several previous iterations of this article.

## References

[Branner and Hubbard 1992] B. Branner and J. H. Hubbard, "The iteration of cubic polynomials, II: Patterns and parapatterns", *Acta Math.* **169**:3-4 (1992), 229–325. MR Zbl

[Calegari 2022] D. Calegari, "Sausages and butcher paper", pp. 155–200 in *Recent progress in mathematics*, edited by N.-G. Kang et al., KIAS Springer Ser. Math. **1**, Springer, 2022. MR Zbl

[DeMarco and McMullen 2008] L. G. DeMarco and C. T. McMullen, "Trees and the dynamics of polynomials", *Ann. Sci. Éc. Norm. Supér.* (4) **41**:3 (2008), 337–382. MR Zbl

[DeMarco and Pilgrim 2017] L. DeMarco and K. Pilgrim, "The classification of polynomial basins of infinity", *Ann. Sci. Éc. Norm. Supér.* (4) **50**:4 (2017), 799–877. MR Zbl

[DeMarco and Schiff 2010] L. DeMarco and A. Schiff, "Enumerating the basins of infinity of cubic polynomials", *J. Difference Equ. Appl.* **16**:5-6 (2010), 451–461. MR Zbl

[Lothaire 1997] M. Lothaire, *Combinatorics on words*, Cambridge University Press, 1997. MR Zbl

[Silberger 1971] D. M. Silberger, "Borders and roots of a word", *Portugal. Math.* **30** (1971), 191–199. MR Zbl

Received November 4, 2021. Revised March 27, 2024.

DANNY CALEGARI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
CHICAGO, IL
UNITED STATES

dannyc@math.uchicago.edu

# PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm





See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2024 is US $645/year for the electronic version, and $875/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

# PACIFIC JOURNAL OF MATHEMATICS

Volume 329 No. 1 March 2024

---