**LIMIT THEOREMS AND WRAPPING TRANSFORMS
IN BI-FREE PROBABILITY THEORY**

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We characterize idempotent distributions with respect to the bi-free multiplicative convolution on the bi-torus. The bi-free analogous Lévy triplet of an infinitely divisible distribution on the bi-torus without nontrivial idempotent factors is obtained. This triplet is unique and generates a homomorphism from the bi-free multiplicative semigroup of infinitely divisible distributions to the classical one. Also, the relevances of the limit theorems associated with four convolutions, classical and bi-free additive convolutions and classical and bi-free multiplicative convolutions, are analyzed. The analysis relies on the convergence criteria for limit theorems and the use of push-forward measures induced by the wrapping map from the plane to the bi-torus.

1. Introduction

The main aim of the present paper is to build the association among various limit theorems and their convergence criteria in classical and bi-free probability theories.

Bi-free probability theory, introduced by Voiculescu in [20], is an outspread research field of free probability theory, which grew out to intend to simultaneously study the left and right actions of algebras over reduced free product spaces. Since its creation, a great deal of research work has been conducted to better understand this theory and its connections to other parts of mathematics [17; 19; 21; 22]. Aside from the combinatorial means, the utilization of analytic functions as transformations and the bond to classical probability theory also play crucial roles in the study and comprehension of this theory [12; 13]. Especially, recent developments of bi-free harmonic analysis enable one to investigate bi-free limit theorems and other related topics from the probabilistic point of view [11].

To work in the probabilistic framework, we thereby consider the family $\mathscr{P}_X$ of Borel probability measures on a complete separable metric space X and endow this family with a commutative and associative binary operation $\Diamond$. Classical and bi-free *convolutions*, respectively denoted by $*$ and $\boxplus\boxplus$, are two examples of such operations performed on $\mathscr{P}_{\mathbb{R}^2}$. In probabilistic terms, $\mu_1 * \mu_2$ is the probability

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distribution of the sum of two independent bivariate random vectors respectively
having distributions μ_1 and μ_2. When restricted to compactly supported measures
in $\mathscr{P}_{\mathbb{R}^2}$, $\mu_1 \boxplus\boxplus \mu_2$ is the distribution of the sum of two bi-free bipartite self-adjoint
pairs with distributions μ_1 and μ_2, respectively [20]. This new notion of convolution
was later extended, without any limitation, to the whole class $\mathscr{P}_{\mathbb{R}^2}$ by the continuity
theorem of transforms [11]. The product of two independent random vectors
having distributions on the bi-torus $\mathbb{T}^2$ gives rise to the classical multiplicative
convolution $\circledast$, and the bi-free analog of multiplicative convolution $\boxtimes\boxtimes$ is defined
in a similar manner [22].

In (noncommutative) probability theory, the limit theorem and its related subject,
the notion of *infinite divisibility* of distributions, have attracted much attention.
By saying that a distribution in $(\mathscr{P}_X, \Diamond)$ is infinitely divisible we mean that it
can be expressed as the operation $\Diamond$ of an arbitrary number of copies of identical
distributions from $\mathscr{P}_X$. The collection of measures having this infinitely divisible
feature forms a semigroup and will be denoted by $\mathcal{ID}(X, \Diamond)$, or simply by $\mathcal{ID}(\Diamond)$
if the identification of the metric space is unnecessary. Any measure satisfying
$\mu = \mu \Diamond \mu$, known as *idempotent*, is an instance of infinitely divisible distributions. In
the case of $X = \mathbb{R}$, these topics have been thoroughly studied in classical probability
by the efforts of de Finetti, Kolmogorov, Lévy and Khintchine (see [16]), and the
same themes in the free contexts have also been deeply explored in the literature [5].

Bi-free probability, as expected, also parallels perfectly aspects of classical
and free probability theories [3]. For example, the theory of bi-freely infinitely
divisible distributions generalizes *bi-free central limit theorem* as they also serve
as the limit laws for sums of bi-freely independent and identically distributed
faces. Specifically, it was shown in [11] that for some infinitesimal triangular array
$\{\mu_{n,k}\}_{n\geq 1, 1\leq k\leq n_k} \subset \mathscr{P}_{\mathbb{R}^2}$ and sequence $\{v_n\} \subset \mathbb{R}^2$, the sequence

(1-1) $$\delta_{v_n} * \mu_{n1} * \cdots * \mu_{nk_n}$$

converges weakly if and only if so does the sequence

(1-2) $$\delta_{v_n} \boxplus\boxplus \mu_{n1} \boxplus\boxplus \cdots \boxplus\boxplus \mu_{nk_n}.$$

The limiting distributions in (1-1) and (1-2) respectively belong to the semigroups
$\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus\boxplus)$, and their classical and bi-free *Lévy triplets* agree. This con-
formity consequently brings out an isomorphism Λ between these two semigroups.

Same tasks are performed in the case of bi-free multiplicative convolution in
this paper. We determine $\boxtimes\boxtimes$-idempotent elements and identify measures in $\mathscr{P}_{\mathbb{T}^2}$
bearing no nontrivial $\boxtimes\boxtimes$-idempotent factors. Specifically, we demonstrate that
$\nu \in \mathcal{ID}(\boxtimes\boxtimes)$ has no nontrivial $\boxtimes\boxtimes$-idempotent factor if and only if it belongs
to $\mathscr{P}_{\mathbb{T}^2}^\times$, the subcollection of $\mathscr{P}_{\mathbb{T}^2}$ with the attributes

$$\int_{\mathbb{T}^2} s_j \, d\nu(s_1, s_2) \neq 0, \quad j = 1, 2.$$

Fix an infinitesimal triangular array $\{\nu_{nk}\}_{n\geq 1, 1\leq k\leq k_n} \subset \mathscr{P}_{\mathbb{T}^2}$ and a sequence $\{\boldsymbol{\xi}_n\} \subset \mathbb{T}^2$. We also manifest that the weak convergence of the sequence

$$(1\text{-}3) \qquad \delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes \nu_{n1} \boxtimes\boxtimes \cdots \boxtimes\boxtimes \nu_{nk_n}$$

to some element in $\mathscr{P}_{\mathbb{T}^2}^{\times}$ yields the same property of the sequence

$$(1\text{-}4) \qquad \delta_{\boldsymbol{\xi}_n} \circledast \nu_{n1} \circledast \cdots \circledast \nu_{nk_n},$$

and that their limiting distributions are both infinitely divisible. This is done by distinct types of equivalent convergence criteria offered in the present paper. As in the case of addition, there exists a triplet concurrently serving as the classical and bi-free multiplicative Lévy triplets of the limiting distributions in (1-3) and (1-4). The consistency of their Lévy triplets, together with the description of $\mathcal{ID}(\boxtimes\boxtimes)\backslash\mathscr{P}_{\mathbb{T}^2}^{\times}$, consequently produces a homomorphism Γ from $\mathcal{ID}(\boxtimes\boxtimes)$ to $\mathcal{ID}(\circledast)$.

Because of the nature of $\mathcal{ID}(\boxtimes\boxtimes)\backslash\mathscr{P}_{\mathbb{T}^2}^{\times}$ and that the limit in (1-4) may generally not have a unique Lévy measure, the homomorphism stated above is neither surjective nor injective. However, postulating the uniqueness of the Lévy measure, the weak convergence of (1-4) derives that of (1-3).

In addition to the previously mentioned conjunctions, what we would like to point out is that measures in $\mathscr{P}_{\mathbb{R}^2}$ and $\mathscr{P}_{\mathbb{T}^2}$ can be linked through the wrapping map $W : \mathbb{R}^2 \to \mathbb{T}^2$, $(x, y) \mapsto (e^{ix}, e^{iy})$. This wrapping map induces a map $W_* : \mathscr{P}_{\mathbb{R}^2} \to \mathscr{P}_{\mathbb{T}^2}$ so that the measure $\nu_{nk} = W_*(\mu_{nk}) = \mu_{nk}W^{-1}$ enjoys the property: the weak convergence of (1-1) or (1-2) yields the weak convergence of (1-3) and (1-4) with $\boldsymbol{\xi}_n = W(\boldsymbol{v}_n)$. Furthermore, the synchronous convergence allows one to construct a homomorphism $W_{\boxplus\boxplus} : \mathcal{ID}(\boxplus\boxplus) \to \mathcal{ID}(\boxtimes\boxtimes)$ making the following diagram commute:

$$(1\text{-}5) \qquad
\begin{array}{ccc}
\mathcal{ID}(*) & \xrightarrow{\;\;W_*\;\;} & \mathcal{ID}(\circledast) \\
\Big\downarrow{\scriptstyle\Lambda} & & \Big\uparrow{\scriptstyle\Gamma} \\
\mathcal{ID}(\boxplus\boxplus) & \xrightarrow{\;W_{\boxplus\boxplus}\;} & \mathcal{ID}(\boxtimes\boxtimes)
\end{array}$$

This diagram is a two-dimensional analog of [6, Theorem 1].

The rest of the paper is organized as follows. In Section 2 we provide the necessary background in classical and noncommutative probability theories. In Section 3 we characterize $\boxtimes\boxtimes$-idempotent distributions. In Section 4 we make comparisons of the convergence criteria of limit theorems, as well as those through wrapping transforms. Section 5 is devoted to offering bi-free multiplicative Lévy triplets of infinitely divisible distributions and investigating the relationships among limit theorems in additive and multiplicative cases. Section 6 provides the derivation of the diagram in (1-5).

2. Preliminary

2A. *Convergence of measures.* Let $\mathscr{B}_X$ be the collection of Borel sets on a complete separable metric space (X, d). A point is selected from X and fixed, named the *origin* and denoted by x_0 in the following. In the present paper, we will be mostly concerned with the abelian groups $X = \mathbb{R}^d$ and $X = \mathbb{T}^d$ endowed with the relative topology from $\mathbb{C}^d$, where the origin is chosen to be the unit. They are respectively the *d*-dimensional Euclidean metric space and the *d*-dimensional torus (or the *d-torus* for short). The 1-torus is just the unit circle $\mathbb{T}$ on the complex plane. A set contained in $\{x \in X : \mathrm{d}(x, x_0) \geq r\}$ for some $r > 0$ is colloquially said to be *bounded away from the origin*.

Next, let us introduce several types of measures on X that will be discussed later. The first one is the collection $\mathscr{M}_X$ of finite positive Borel measures on X. We shall also consider the set $\mathscr{M}_X^{x_0}$ of all positive Borel measures that when confined to any Borel set bounded away from the origin yield a finite measure. Clearly, we have $\mathscr{M}_X \subset \mathscr{M}_X^{x_0}$. Another assortment concerned herein is the collection $\mathscr{P}_X$ of elements in $\mathscr{M}_X$ having unit total mass.

The set $C_b(X)$ of bounded continuous functions on X induces the weak topology on $\mathscr{M}_X$. Likewise, $\mathscr{M}_X^{x_0}$ is equipped with the topology generated by $C_b^{x_0}(X)$, bounded continuous functions having support bounded away from the origin. Concretely, basic neighborhoods of a $\tau \in \mathscr{M}_X^{x_0}$ are of the form

$$\bigcap_{j=1,\ldots,n} \left\{ \tilde{\tau} \in \mathscr{M}_X^{x_0} : \left| \int f_j \, d\tilde{\tau} - \int f_j \, d\tau \right| < \epsilon \right\},$$

where $\epsilon > 0$ and each $f_j \in C_b^{x_0}(X)$. Putting it differently, a sequence $\{\tau_n\} \subset \mathscr{M}_X^{x_0}$ converges to some τ in $\mathscr{M}_X^{x_0}$, written as $\tau_n \Rightarrow_{x_0} \tau$, if and only if

$$\lim_{n \to \infty} \int f \, d\tau_n = \int f \, d\tau, \quad f \in C_b^{x_0}(X).$$

We remark that τ is not unique as it may assign arbitrary mass to the origin. Nevertheless, any weak limit in $\mathscr{M}_X^{x_0}$ that comes across in our discussions will serve as the so-called *Lévy measure*, which does not charge the origin.

Portmanteau theorem and *continuous mapping theorem* in the framework of $\mathscr{M}_X^{x_0}$ are presented below (see [1; 14]). The push-forward measure $\tau h^{-1} : \mathscr{B}_{X'} \to [0, +\infty]$ of $\tau \in \mathscr{M}_X^{x_0}$ provoked by a measurable mapping $h : (X, \mathscr{B}_X) \to (X', \mathscr{B}_{X'})$ is defined as

(2-1) $$(\tau h^{-1})(B') = \tau(\{x \in X : h(x) \in B'\}), \quad B' \in \mathscr{B}_{X'}.$$

Proposition 2.1. *The following statements* (1)–(3) *are equivalent for* $\{\tau_n\}$ *and* τ *in* $\mathscr{M}_X^{x_0}$:

(1) *We have $\tau_n \Rightarrow_{x_0} \tau$.*

(2) *For any $f \in C_b(X)$ and any $B \subset \mathscr{B}_X$, which is bounded away from the origin and satisfies $\tau(\partial B) = 0$, we have*

$$\lim_{n \to \infty} \int_B f \, d\tau_n = \int_B f \, d\tau.$$

(3) *For every closed set F and open set G of X that are both bounded away from x_0, we have*

$$\limsup_{n \to \infty} \tau_n(F) \leq \tau(F) \quad and \quad \liminf_{n \to \infty} \tau_n(G) \geq \tau(G).$$

If $h : (X, \mathrm{d}) \to (X', \mathrm{d}')$ is measurable so that h is continuous at x_0, $h(x_0) = x'_0$, and the set of discontinuities of h has τ-measure zero, then $\tau_n \Rightarrow_{x_0} \tau$ implies $\tau_n h^{-1} \Rightarrow_{x'_0} \tau h^{-1}$.

Finally, let us introduce the subset $\tilde{\mathscr{M}}_X^{x_0}$ consisting of measures in $\mathscr{M}_X^{x_0}$ that do not charge the origin x_0. This set is metrizable and becomes a separable complete metric space [14, Theorem 2.2]. In particular, the relative compactness of a subset Y of $\tilde{\mathscr{M}}_X^{x_0}$ is equivalent to that any sequence of Y has a subsequence convergent in $\tilde{\mathscr{M}}_X^{x_0}$. We refer the reader to [14, Theorem 2.7] for an analog of Prokhorov's theorem, which characterizes the relative compactness of subsets in $\tilde{\mathscr{M}}_X^{x_0}$.

2B. *Notations.* Below, we collect notations that will be commonly used in the sequel. The customary symbol $\arg s \in (-\pi, \pi]$ stands for the principal argument of a point $s \in \mathbb{T}$, while $\Re s$ and $\Im s$ respectively represent the real and imaginary parts of s. Here and elsewhere, points in a multidimensional space will be written in bold letters, for instance, $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{T}^d$ and $\boldsymbol{p} = (p_1, \ldots, p_d) \in \mathbb{Z}^d$ with each $s_j \in \mathbb{T}$ and $p_j \in \mathbb{Z}$. For any $\epsilon > 0$, we shall use $\mathscr{V}_\epsilon = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| < \epsilon\}$ and $\mathscr{U}_\epsilon = \{\boldsymbol{s} \in \mathbb{T}^d : \|\arg \boldsymbol{s}\| < \epsilon\}$ to respectively express open neighborhoods of origins $\boldsymbol{0} \in \mathbb{R}^d$ and $\boldsymbol{1} \in \mathbb{T}^d$, where $\arg \boldsymbol{s} = (\arg s_1, \ldots, \arg s_d) \in \mathbb{R}^d$. Analogous expressions also apply to vectors $\Re \boldsymbol{s} = (\Re s_1, \ldots, \Re s_d)$ and $\Im \boldsymbol{s} = (\Im s_1, \ldots, \Im s_d)$. Besides, we adopt the operational conventions in multidimensional spaces in the sequel, such as $\boldsymbol{s}^{\boldsymbol{p}} = s_1^{p_1} \cdots s_d^{p_d}$, $\boldsymbol{s}\boldsymbol{t} = (s_1 t_1, \ldots, s_d t_d)$, $\boldsymbol{s}^{-1} = (1/s_1, \ldots, 1/s_d)$, and $e^{i\boldsymbol{s}} = (e^{is_1}, \ldots, e^{is_d})$.

The push-forward probabilities $\mu^{(j)} = \mu \pi_j^{-1}$, $j = 1, \ldots, d$, on the real line induced by projections $\pi_j : \mathbb{R}^d \to \mathbb{R}$, $\boldsymbol{x} \mapsto x_j$, are called *marginals* of $\mu \in \mathscr{P}_{\mathbb{R}^d}$. Marginals of probability measures on $\mathbb{T}^d$ are defined and displayed in the same way. On $\mathbb{T}^2$, we shall also consider the (right) coordinate-flip transform $h_{\mathrm{op}} : \mathbb{T}^2 \to \mathbb{T}^2$ defined as $h_{\mathrm{op}}(\boldsymbol{s}) = (s_1, 1/s_2)$. Denote by $\boldsymbol{s}^\star = h_{\mathrm{op}}(\boldsymbol{s})$ and $B^\star = \{\boldsymbol{s}^\star : \boldsymbol{s} \in B\}$ if $\boldsymbol{s} \in \mathbb{T}^2$ and $B \subset \mathbb{T}^2$. By the (right) coordinate-flip measure of $\rho \in \mathscr{M}_{\mathbb{T}^2}^{\boldsymbol{1}}$, we mean the push-forward measure $\rho^\star = \rho h_{\mathrm{op}}^{-1}$, alternatively defined as $\rho^\star(B) = \rho(B^\star)$ for $B \in \mathscr{B}_{\mathbb{T}^2}$.

2C. *Free probability and bi-free probability.* Aside from the classical convolution on $\mathscr{P}_{\mathbb{R}^2}$, we shall also consider the bi-free convolution $\boxplus\boxplus$, where the *bi-free ϕ-transform* takes the place of Fourier transform [11]: for $\mu_1, \mu_2 \in \mathscr{P}_{\mathbb{R}^2}$, one has $\phi_{\mu_1 \boxplus\boxplus \mu_2} = \phi_{\mu_1} + \phi_{\mu_2}$. All information about marginals of the bi-free convolution is carried over to the free convolution: $(\mu_1 \boxplus\boxplus \mu_2)^{(j)} = \mu_1^{(j)} \boxplus \mu_2^{(j)}$ for $j = 1, 2$.

Now, we turn to probability measures on the *d*-torus. The sequence

$$m_{\boldsymbol{p}}(\nu) = \int_{\mathbb{T}^d} \boldsymbol{s}^{\boldsymbol{p}} \, d\nu(\boldsymbol{s}), \quad \boldsymbol{p} \in \mathbb{Z}^d,$$

is called the *d-moment sequence* of $\nu \in \mathscr{P}_{\mathbb{T}^d}$. In some circumstances, *characteristic function* and $\hat{\nu}(\boldsymbol{p})$ are the precise terminology and notation used for this sequence. Owing to Stone–Weierstrass theorem, we have $m_{\boldsymbol{p}}(\nu) \equiv m_{\boldsymbol{p}}(\nu')$ only when $\nu = \nu'$. The classical convolution $\circledast$ of distributions on $\mathbb{T}^d$ is characterized by $m_{\boldsymbol{p}}(\nu_1 \circledast \nu_2) = m_{\boldsymbol{p}}(\nu_1) \, m_{\boldsymbol{p}}(\nu_2)$ for $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^d}$.

The *bi-free multiplicative convolution* of $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}^{\times}$ is determined by its marginals $(\nu_1 \boxtimes\boxtimes \nu_2)^{(j)} = \nu_1^{(j)} \boxtimes \nu_2^{(j)}$ and the bi-free multiplicative formula

$$\Sigma_{\nu_1 \boxtimes\boxtimes \nu_2}(z, w) = \Sigma_{\nu_1}(z, w) \cdot \Sigma_{\nu_2}(z, w)$$

for points $(z, w) \in \mathbb{C}^2$ in a neighborhood of $(0, 0)$ and $(0, \infty)$. Here the free multiplicative convolution can be rephrased by means of the free Σ-transform $\Sigma_{\nu_1^{(j)} \boxtimes \nu_2^{(j)}} = \Sigma_{\nu_1^{(j)}} \cdot \Sigma_{\nu_2^{(j)}}$ valid in a neighborhood of the origin of the complex plane. The reader is referred to [4; 5; 12; 13; 17; 19; 21; 22] for more details along with properties of the transforms in (bi)-free probability theory. We remark that given a measure $\nu \in \mathscr{P}_{\mathbb{T}^2}^{\times}$, the transform Σ_{ν} is the identity map if and only if ν is a product measure, which leads to

(2-2) $\qquad (\nu_1^{(1)} \times \nu_1^{(2)}) \boxtimes\boxtimes (\nu_2^{(1)} \times \nu_2^{(2)}) = (\nu_1^{(1)} \boxtimes \nu_2^{(1)}) \times (\nu_1^{(2)} \boxtimes \nu_2^{(2)}),$

whenever $\nu_1^{(1)} \times \nu_1^{(2)}, \nu_2^{(1)} \times \nu_2^{(2)} \in \mathscr{P}_{\mathbb{T}^2}^{\times}$. In fact, (2-2) holds for any $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$ by continuity arguments together with the facts that $m_{p,q}(\nu_1 \boxtimes\boxtimes \nu_2)$ can be expressed as a polynomial of $m_{k,l}(\nu_i)$ for $i = 1, 2$, $|k| \le |p|$, $|l| \le |q|$ and that $\nu \in \mathscr{P}_{\mathbb{T}^2}$ is a product measure if and only if $m_{p,q}(\nu) = m_p(\nu^{(1)}) \, m_q(\nu^{(2)})$ for any $p, q \in \mathbb{Z}$.

Fix $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$, and let $\nu = \nu_1 \boxtimes\boxtimes \nu_2$. In order to analyze ν, it will be convenient to treat it as the distribution of a certain bipartite pair $(u_1 u_2, v_1 v_2)$, where (u_1, v_1) and (u_2, v_2) are bi-free bipartite unitary pairs in some C^*-probability space having distributions ν_1 and ν_2, respectively. Below, we briefly introduce the construction of such pairs carrying the mentioned properties. For more information, we refer the reader to [13; 20; 22]. Associating each ν_j with the Hilbert space $\mathcal{H}_j = L^2(\nu_j)$ with specified unit vector ξ_j, the constant function one in $\mathcal{H}_j$, consider the Hilbert space free product $(\mathcal{H}, \xi) = *_{j=1,2}(\mathcal{H}_j, \xi_j)$. The left and right factorizations of $\mathcal{H}_j$ from $\mathcal{H}$ can be respectively done via natural isomorphisms $V_j : \mathcal{H}_j \otimes \mathcal{H}(\ell, j) \to \mathcal{H}$

and $W_j : \mathcal{H}(r, j) \otimes \mathcal{H}_j \to \mathcal{H}$. Then for any $T \in B(\mathcal{H}_j)$, these isomorphisms induce the so-called *left* and *right operators*

$$\lambda_j(T) = V_j(T \otimes I_{\mathcal{H}(\ell, j)})V_j^{-1} \quad \text{and} \quad \rho_j(T) = W_j(I_{\mathcal{H}(r, j)} \otimes T)W_j^{-1} \qquad \text{on } \mathcal{H}.$$

For any $S_j, T_j \in B(\mathcal{H}_j)$, pairs $(\lambda_1(S_1), \rho_1(T_1))$ and $(\lambda_2(S_2), \rho_2(T_2))$ are, by definition, bi-free in the C^*-probability space $(B(\mathcal{H}), \varphi_\xi)$, where $\varphi_\xi(\cdot) = \langle \cdot\, \xi, \xi \rangle$. Particularly, the multiplication operators $(S_j f)(s, t) = sf(s, t)$ and $(T_j f)(s, t) = tf(s, t)$ for $f \in \mathcal{H}_j$ furnish the desired pairs (u_1, v_1) and (u_2, v_2), where $u_j = \lambda_j(S_j)$ and $v_j = \rho_j(T_j)$.

Recall from [13] that one can perform the *opposite bi-free multiplicative convolution* of v_1 and v_2:

$$(2\text{-}3) \qquad\qquad v_1 \boxtimes\boxtimes {}^{\mathrm{op}} v_2 = (v_1^\star \boxtimes\boxtimes v_2^\star)^\star.$$

Then $v_1 \boxtimes\boxtimes {}^{\mathrm{op}} v_2$ is the distribution of $(u_1 u_2, v_2 v_1)$, the pair obtained by performing the opposite multiplication on the right face $(u_1, v_1) \cdot^{\mathrm{op}} (u_2, v_2) = (u_1 u_2, v_2 v_1)$. The coordinate-flip map h_{op} gives rise to a homeomorphism from the semigroup $(\mathscr{P}_{\mathbb{T}^2}, \boxtimes\boxtimes)$ to another $(\mathscr{P}_{\mathbb{T}^2}, \boxtimes\boxtimes^{\mathrm{op}})$ satisfying

$$(v_1 \boxtimes\boxtimes v_2)h_{\mathrm{op}}^{-1} = (v_1 h_{\mathrm{op}}^{-1}) \boxtimes\boxtimes {}^{\mathrm{op}} (v_2 h_{\mathrm{op}}^{-1}),$$

which is the distribution of

$$h_{\mathrm{op}}((u_1, v_1)(u_2, v_2)) = (u_1 u_2, v_2^{-1} v_1^{-1}) = h_{\mathrm{op}}((u_1, v_1)) \cdot^{\mathrm{op}} h_{\mathrm{op}}((u_2, v_2)).$$

Passing to the transform

$$\Sigma_v^{\mathrm{op}}(z, w) = \Sigma_{v^\star}(z, 1/w),$$

the equation (2-3) is translated into $\Sigma_{v_1 \boxtimes\boxtimes^{\mathrm{op}} v_2}^{\mathrm{op}}(z, w) = \Sigma_{v_1}^{\mathrm{op}}(z, w) \cdot \Sigma_{v_1}^{\mathrm{op}}(z, w)$.

2D. *Limit theorem.* Either in classical or in (bi-)free probability theory, one is concerned with the asymptotic behavior of the sequence

$$(2\text{-}4) \qquad\qquad \delta_{x_n} \Diamond \mu_{n1} \Diamond \cdots \Diamond \mu_{nk_n}, \quad n = 1, 2, \dots,$$

where δ_x is the Dirac measure concentrated at $x \in X$ and $\{\mu_{nk_n}\}_{n\geq 1, 1\leq k \leq k_n}$ is an infinitesimal triangular array in $\mathscr{P}_X$. The *infinitesimality* of $\{\mu_{nk}\}$, by definition, means that $k_1 < k_2 < \cdots$ and that for any $\epsilon > 0$, we have

$$\lim_{n \to \infty} \max_{1 \leq k \leq k_n} \mu_{nk}(\{x \in X : \mathrm{d}(x, x_0) \geq \epsilon\}) = 0.$$

One phenomenon related to equation (2-4) is the concept of infinite divisibility: $\mu \in (\mathscr{P}_X, \Diamond)$ is said to be *infinitely divisible* if for any $n \in \mathbb{N}$, it coincides with the n-fold $\Diamond$-operation $\mu_n^{\Diamond n}$ of some $\mu_n \in \mathscr{P}_X$.

Commutative and associative binary operations to be considered throughout the paper are classical convolutions $*$ and $\circledast$ on $\mathscr{P}_{\mathbb{R}^d}$ and $\mathscr{P}_{\mathbb{T}^d}$, respectively, and bi-free additive and multiplicative convolutions $\boxplus\boxplus$ and $\boxtimes\boxtimes$ on $\mathscr{P}_{\mathbb{R}^2}$ and $\mathscr{P}_{\mathbb{T}^2}$, respectively. The following convergence criteria play an essential role in the asymptotic analysis of limit theorems of $\mathscr{P}_{\mathbb{R}^d}$.

Condition 2.2. Let $\{\tau_n\}$ be a sequence in $\mathscr{M}^{\mathbf{0}}_{\mathbb{R}^d}$.

(I) For $j = 1, \ldots, d$, the sequence $\{\sigma_{nj}\}_{n\geq 1}$ defined by

$$d\sigma_{nj}(\boldsymbol{x}) = \frac{x_j^2}{1+x_j^2}\, d\tau_n(\boldsymbol{x})$$

belongs to $\mathscr{M}_{\mathbb{R}^d}$ and converges weakly to some $\sigma_j \in \mathscr{M}_{\mathbb{R}^d}$.

(II) For $j, \ell = 1, \ldots, d$, the following limit exists in $\mathbb{R}$:

$$L_{j\ell} = \lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{x_j\, x_\ell}{(1+x_j^2)(1+x_\ell^2)}\, d\tau_n(\boldsymbol{x}).$$

Condition 2.3. Let $\{\tau_n\}$ be a sequence in $\mathscr{M}^{\mathbf{0}}_{\mathbb{R}^d}$.

(III) There is some $\tau \in \mathscr{M}^{\mathbf{0}}_{\mathbb{R}^d}$ with $\tau(\{\mathbf{0}\}) = 0$ (that is $\tau \in \tilde{\mathscr{M}}^{\mathbf{0}}_{\mathbb{R}^d}$) so that $\tau_n \Rightarrow_{\mathbf{0}} \tau$.

(IV) For any vector $\boldsymbol{u} \in \mathbb{R}^d$, the following limits exist in $\mathbb{R}$:

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \int_{\mathscr{V}_\epsilon} \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2\, d\tau_n(\boldsymbol{x}) = Q(\boldsymbol{u}) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} \int_{\mathscr{V}_\epsilon} \langle \boldsymbol{u}, \boldsymbol{x}\rangle^2\, d\tau_n(\boldsymbol{x}).$$

Although we describe the conditions in a higher dimension setup, the reader can effortlessly mimic the proof in [11] to obtain the equivalence of Conditions 2.2 and 2.3, and draw the following consequences:

(1) The function $Q(\cdot) = \langle \boldsymbol{A}\cdot, \cdot\rangle$ in (IV) defines a nonnegative quadratic form on $\mathbb{R}^d$, where the matrix $\boldsymbol{A} = (a_{j\ell})$ is given by

$$a_{j\ell} = L_{j\ell} - \int_{\mathbb{R}^d} \frac{x_j\, x_\ell}{(1+x_j^2)(1+x_\ell^2)}\, d\tau(\boldsymbol{x}), \quad j, \ell = 1, \ldots, d.$$

In particular, $a_{jj} = \sigma_j(\{\mathbf{0}\})$ for $j = 1, \ldots, d$.

(2) Measures τ and $\sigma_1, \ldots, \sigma_d$ are uniquely determined by the relations

$$d\sigma_j(\boldsymbol{x}) = \frac{x_j^2}{1+x_j^2}\, d\tau(\boldsymbol{x}) + Q(\boldsymbol{e}_j)\, \delta_{\mathbf{0}}(d\boldsymbol{x}),$$

where $\{\boldsymbol{e}_j\}$ is the standard basis of $\mathbb{R}^d$.

(3) The function $\boldsymbol{x} \mapsto \min\{1, \|\boldsymbol{x}\|^2\}$ is τ-integrable.

Now, let us briefly introduce the limit theorems of (1-1) and (1-2). Throughout our discussions in the paper,

$$(2\text{-}5) \qquad\qquad \theta \in (0, 1)$$

is an arbitrary but fixed quantity. To meet the purpose, consider the shifted triangular array

$$\mathring{\mu}_{nk}(B) = \mu_{nk}(B + \boldsymbol{v}_{nk}), \quad B \in \mathscr{B}_{\mathbb{R}^d},$$

associated with an infinitesimal triangular array $\{\mu_{nk}\}_{n \geq 1, 1 \leq k \leq k_n} \subset \mathscr{P}_{\mathbb{R}^d}$ and the vector

$$(2\text{-}6) \qquad\qquad \boldsymbol{v}_{nk} = \int_{V_\theta} \boldsymbol{x} \, d\mu_{nk}(\boldsymbol{x}).$$

Due to $\lim_{n \to \infty} \max_{k \leq k_n} \|\boldsymbol{v}_{nk}\| = 0$, $\{\mathring{\mu}_{nk}\}$ so obtained is also infinitesimal. In conjunction with this centered triangular array, we focus on the positive measures

$$(2\text{-}7) \qquad\qquad \tau_n = \sum_{k=1}^{k_n} \mathring{\mu}_{nk}.$$

It turns out that the sequence in (1-1) converges weakly to a certain $\mu_* \in \mathscr{P}_{\mathbb{R}^d}$ if and only if τ_n defined in (2-7) meets Condition 2.3 (as well as Condition 2.2 since these two conditions are equivalent) and the limit

$$(2\text{-}8) \qquad\qquad \boldsymbol{v} = \lim_{n \to \infty} \left[\boldsymbol{v}_n + \sum_{k=1}^{k_n} \left(\boldsymbol{v}_{nk} + \int_{\mathbb{R}^d} \frac{\boldsymbol{x}}{1 + \|\boldsymbol{x}\|^2} \, d\mathring{\mu}_{nk}(\boldsymbol{x}) \right) \right]$$

exists in $\mathbb{R}^d$. Additionally, μ_* is $*$-infinitely divisible and possesses the characteristic function read as

$$(2\text{-}9) \quad \hat{\mu}_*(\boldsymbol{u}) = \exp \left[i \langle \boldsymbol{u}, \boldsymbol{v} \rangle - \tfrac{1}{2} \langle A\boldsymbol{u}, \boldsymbol{u} \rangle + \int_{\mathbb{R}^d} \left(e^{i \langle \boldsymbol{u}, \boldsymbol{x} \rangle} - 1 - \frac{i \langle \boldsymbol{u}, \boldsymbol{x} \rangle}{1 + \|\boldsymbol{x}\|^2} \right) d\tau(\boldsymbol{x}) \right],$$

which is known as the *Lévy–Khintchine representation*. The limiting distribution is uniquely determined by the formula (2-9) and denoted by $\mu_*^{(\boldsymbol{v}, A, \tau)}$, and $(\boldsymbol{v}, A, \tau)$ is referred to as its *Lévy triplet*. The set $\mathcal{ID}(*)$ is completely parameterized by the triplets $(\boldsymbol{v}, A, \tau)$, where

$$(2\text{-}10) \quad \boldsymbol{v} \in \mathbb{R}^d, \ A \in M_d(\mathbb{R}) \text{ is positive semidefinite, and } \tau \text{ is a positive measure}$$
$$\text{on } \mathbb{R}^d \text{ satisfying } \tau(\{\boldsymbol{0}\}) = 0 \text{ and } \min\{1, \|\boldsymbol{x}\|^2\} \in L^1(\tau).$$

As a matter of fact, when $d = 2$, the same convergence criteria are also necessary and sufficient to assure the weak convergence of (1-2). Paralleling to the classical case, the limiting distribution of (1-2) is ⊞⊞-infinitely divisible and owns the bi-free ϕ-transform, called *bi-free Lévy–Khintchine* representation, of the form

$$\phi(z, w) = \frac{v_1}{z} + \frac{v_2}{w} + \left(\frac{a_{11}}{z^2} + \frac{a_{12}}{zw} + \frac{a_{22}}{w^2} \right)$$
$$+ \int_{\mathbb{R}^2} \left[\frac{zw}{(z - x_1)(w - x_2)} - 1 - \frac{x_1 z^{-1} + x_2 w^{-1}}{1 + \|x\|^2} \right] d\tau(x).$$

Analogically, this limiting distribution is always expressed as $\mu_{\boxplus\boxplus}^{(v, A, \tau)}$ and said to own the *bi-free Lévy triplet* (v, A, τ). Those triplets (v, A, τ) satisfying (2-10) also give a complete parametrization of the set $\mathcal{ID}(\boxplus\boxplus)$, and therefore output a bijective homomorphism Λ from $\mathcal{ID}(*)$ onto $\mathcal{ID}(\boxplus\boxplus)$, sending an element $\mu_*^{(v, A, \tau)}$ in the first set to the distribution $\mu_{\boxplus\boxplus}^{(v, A, \tau)}$ in the second one. No matter in the classical or bi-free probability, $*$- and $\boxplus\boxplus$-infinitely divisible distributions both appear as limiting distributions in the limit theorem.

Next, we turn our attention to the limit theorem on the d-torus, on which the Borel probability measures of interest are sometimes imposed the nonvanishing mean conditions:

$$(2\text{-}11) \qquad\qquad \int_{\mathbb{T}^d} s_j \, dv(s) \neq 0, \quad j = 1, \ldots, d.$$

For convenience, we adopt the symbol $\mathscr{P}_{\mathbb{T}^d}^{\times}$ to signify the collection of probability measures carrying such features. As will be shown in Theorem 3.12, when $d = 2$, these conditions (2-11) turn out to be necessary and sufficient for a $\boxtimes\boxtimes$-infinitely divisible distribution to contain no nontrivial $\boxtimes\boxtimes$-idempotent factors. We would also like to remind the reader that the symbol $\mathscr{P}_{\mathbb{T}^2}^{\times}$ introduced here is distinct from that in [13] as Theorem 3.10 of the present paper designates that the requirement $m_{1,1}(v) \neq 0$ in the limit theorem is redundant.

Given an infinitesimal triangular array $\{v_{nk}\}_{n \geq 1, 1 \leq k \leq k_n}$ in $\mathscr{P}_{\mathbb{T}^d}$, one works with the rotated probability measures $d\mathring{v}_{nk}(s) = dv_{nk}(b_{nk}s)$, where

$$(2\text{-}12) \qquad\qquad b_{nk} = \exp\left[i \int_{\mathcal{U}_\theta} (\arg s) \, dv_{nk}(s) \right].$$

Once again, $\{\mathring{v}_{nk}\}$ is infinitesimal because of $\lim_n \max_k \|\arg b_{nk}\| = 0$. Given a sequence $\{\xi_n\} \subset \mathbb{T}^d$, further define vectors

$$(2\text{-}13) \qquad \gamma_n = \xi_n \exp\left[i \sum_{k=1}^{k_n} \left(\arg b_{nk} + \int_{\mathbb{T}^2} (\Im s) \, d\mathring{v}_{nk}(s) \right) \right] \in \mathbb{T}^d.$$

The bi-free multiplicative limit theorem on the bi-torus has been shown in [13, Theorem 3.4]:

Theorem 2.4. *The necessary and sufficient condition for the sequence* (1-3) *to converge weakly to a certain* $v_{\boxtimes\boxtimes} \in \mathscr{P}_{\mathbb{T}^2}^{\times}$ *is that the limit*

$$(2\text{-}14) \qquad\qquad \lim_{n \to \infty} \gamma_n = \gamma$$

exists and the positive measures

$$(2\text{-}15) \qquad \rho_n = \sum_{k=1}^{k_n} \mathring{\nu}_{nk}$$

satisfy Condition 2.5 *stated below with* $d = 2$.

Condition 2.5. Let $\{\rho_n\}$ be a sequence in $\mathcal{M}_{\mathbb{T}^d}^1$.

(i) For $j = 1, \dots, d$, the sequence $\{\lambda_{nj}\}_{n \geq 1}$ defined by

$$d\lambda_{nj}(s) = (1 - \Re s_j) \, d\rho_n(s)$$

belongs to $\mathcal{M}_{\mathbb{T}^d}$ and converges weakly to some $\lambda_j \in \mathcal{M}_{\mathbb{T}^d}$.

(ii) For $1 \leq j, \ell \leq d$, the following limit exists in $\mathbb{R}$:

$$L_{j\ell} = \lim_{n \to \infty} \int_{\mathbb{T}^d} (\Im s_j)(\Im s_\ell) \, d\rho_n(s).$$

The limiting distribution $\nu = \nu_{\boxtimes\boxtimes}$ in Theorem 2.4 is $\boxtimes\boxtimes$-infinitely divisible, as expected, and uniquely determined by the formulas [13]

$$(2\text{-}16) \qquad \Sigma_{\nu^{(j)}}(\xi) = \exp[u_j(\xi)] \quad \text{and} \quad \Sigma_\nu(z, w) = \exp[u(z, w)].$$

Here the functions u_j, $j = 1, 2$, are defined on $\mathbb{D}$ and given by

$$u_j(\xi) = -i \arg \gamma_j + \int_{\mathbb{T}^2} \frac{1 + \xi s_j}{1 - \xi s_j} \, d\lambda_j(s),$$

and for $(z, w) \in (\mathbb{C} \setminus \mathbb{D})^2$, the function u satisfies

$$\frac{(1-z)(1-w)}{1-zw} u(z, w) = \int_{\mathbb{T}^2} \frac{1 + zs_1}{1 - zs_1} \frac{1 + ws_2}{1 - ws_2} (1 - \Re s_2) \, d\lambda_1(s)$$
$$- i \int_{\mathbb{T}^2} \frac{1 + zs_1}{1 - zs_1} (\Im s_2) \, d\lambda_1(s)$$
$$- i \int_{\mathbb{T}^2} \frac{1 + ws_2}{1 - ws_2} (\Im s_1) \, d\lambda_2(s) - L_{12}.$$

In turn, any measure in $\mathcal{ID}(\boxtimes\boxtimes) \cap \mathcal{P}_{\mathbb{T}^2}^\times$ truly arises as a weak-limit point of (1-3).

Remark 2.6. Suppose that $\nu \in \mathcal{ID}(\boxtimes\boxtimes) \setminus \mathcal{P}_{\mathbb{T}^2}^\times$, and let $m_j = \int s_j \, d\nu^{(j)}$ for $j = 1, 2$. Then $\Sigma_{\nu^{(j)}}(0) = 1/m_j$, $\arg \gamma_j = \arg m_j$, and $\lambda_j(\mathbb{T}^2) = -\log|m_j| \in [0, \infty)$. We remind the reader that the parameter γ_j in $u_j(\xi)$ and that appearing in [13] are conjugate complex numbers. With the help of the equation

$$(2\text{-}17) \qquad \frac{1 + \xi s}{1 - \xi s}(1 - \Re s) = i \Im s + \frac{(1 - \xi)(1 - s)}{1 - \xi s}, \qquad (\xi, s) \in \mathbb{D} \times \mathbb{T},$$

one can see that

$$u_j(\xi) = -i \arg \gamma_j + \lim_{n \to \infty} \int_{\mathbb{T}^2} \frac{1 + \xi s_j}{1 - \xi s_j} (1 - \Re s_j) \, d\rho_n(s)$$

and

$$u(z, w) = \lim_{n \to \infty} \int_{\mathbb{T}^2} \frac{(1 - zw)(1 - s_1)(1 - s_2)}{(1 - zs_1)(1 - ws_2)} \, d\rho_n(s)$$

for some sequence $\{\rho_n\} \subset \mathscr{M}^1_{\mathbb{T}^2}$ satisfying Condition 2.5.

3. ⊠⊠-Idempotent distributions

Let $\mu \in \mathscr{P}_X$. A measure $\mu' \in \mathscr{P}_X$ is called a ◊-factor of μ if $\mu = \mu' ◊ \mu''$ for some $\mu'' \in \mathscr{P}_X$. Particularly, μ is said to be ◊-idempotent when $\mu' = \mu = \mu''$. Idempotent distributions and other related subjects in classical probability have been extensively studied in [16]. It is to questions of these sorts in the bi-free probability theory that the present section is devoted.

The normalized Lebesgue measure $\mathrm{m} = d\theta/(2\pi)$ on $\mathbb{T}$ is the only ⊠-idempotent element except for the trivial one, the Dirac measure at 1. On $\mathbb{T}^2$, the probability measure

$$P(B) = \mathrm{m}(\{s \in \mathbb{T} : (s, \bar{s}) \in B\}), \quad B \in \mathscr{B}_{\mathbb{T}^2},$$

is ⊛-idempotent because $m_{p,q}(P) = 1$ for $p = q \in \mathbb{Z}$ and zero otherwise. As a matter of fact, this singularly continuous measure is also ⊠⊠-idempotent proved below.

The following result is a direct consequence of Voiculescu's two-bands moment formula in [21, Lemma 2.1] and we provide its proof and notations for the later use.

Proposition 3.1. *A ⊠⊠-idempotent distribution in $\mathscr{P}_{\mathbb{T}^2}$ is one of five types $\delta_{(1,1)}$, $\mathrm{m} \times \delta_1$, $\delta_1 \times \mathrm{m}$, $\mathrm{m} \times \mathrm{m}$, and P. A measure in $\mathscr{P}_{\mathbb{T}^2}$ is ⊠⊠$^{\mathrm{op}}$-idempotent if and only if it is $\delta_{(1,1)}$, $\mathrm{m} \times \delta_1$, $\delta_1 \times \mathrm{m}$, $\mathrm{m} \times \mathrm{m}$, or $P^\star$.*

Proof. Let ν be ⊠⊠-idempotent. Since each marginal satisfies $\nu^{(j)} = \nu^{(j)} \boxtimes \nu^{(j)}$, it follows that $\nu^{(j)}$ is ⊠-infinitely divisible. If $\nu^{(j)}$ has nonzero mean, then $\Sigma_{\nu^{(j)}}(0) = 1$, yielding $\nu^{(j)} = \delta_1$ by [4, Lemma 2.7]. Otherwise, we can infer from [4, Lemma 6.1] that $\nu^{(j)} = \mathrm{m}$. Thus, consideration given to the case $\nu^{(1)} = \mathrm{m} = \nu^{(2)}$ is sufficient to complete the proof. To continue the proof, we realize $\nu = \nu_1 \boxtimes\boxtimes \nu_2$ as the distribution of $(u, v) = (u_1 u_2, v_1 v_2)$, where $(u_j, v_j) = (\lambda_j(S_j), \rho_j(T_j))$, $j = 1, 2$, are bi-free unitary faces respectively following $\nu_j = \nu$ in the C^*-probability space $(B(\mathcal{H}), \varphi_\xi)$, as constructed in Section 2C.

From $\varphi_\xi(u_j) = 0$ for $j = 1, 2$, it follows that $S_j^{\pm 1} \xi_j \in \mathring{\mathcal{H}}_j = \mathcal{H}_j \ominus \mathbb{C}\xi_j$, which supplies a simplistic representation for $u^p \xi$ for any $p \in \mathbb{N}$, namely,

$$(3\text{-}1) \qquad u^p \xi = ((S_1 \xi_1) \otimes (S_2 \xi_2))^{\otimes p} \quad \text{and} \quad u^{-p} \xi = ((S_2^{-1} \xi_2) \otimes (S_1^{-1} \xi_1))^{\otimes p}$$

lying in spaces $(\mathring{\mathcal{H}}_1 \otimes \mathring{\mathcal{H}}_2)^{\otimes p}$ and $(\mathring{\mathcal{H}}_2 \otimes \mathring{\mathcal{H}}_1)^{\otimes p}$. Similarly, $\varphi_\xi(v_1) = 0 = \varphi_\xi(v_2)$ implies that

$$(3\text{-}2) \qquad v^q \xi = ((T_2 \xi_2) \otimes (T_1 \xi_1))^{\otimes q} \in (\mathring{\mathcal{H}}_2 \otimes \mathring{\mathcal{H}}_1)^{\otimes q}, \quad q \in \mathbb{N}.$$

We consequently arrive at that for $(p, q) \in (\mathbb{Z} \setminus \{0\}) \times (\mathbb{N} \cup \{0\})$,

$$m_{p,q}(v) = \varphi_\xi(u^p v^q) = \langle v^q \xi, u^{-p} \xi \rangle = \delta_{p,q} [\varphi_\xi(u_1 v_1)\, \varphi_\xi(u_2 v_2)]^p = \delta_{p,q}\, m_{1,1}(v)^{2p}$$

and that $m_{0,q}(v) = \varphi_\xi(v^q) = \delta_{0,q}$ for $q \in \mathbb{N} \cup \{0\}$. If $m_{1,1}(v) = 0$, then $m_{p,q}(v) = 0$ for any $(p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, which occurs only when $v = m \times m$. If $m_{1,1}(v) \neq 0$, then the equation $m_{1,1}(v) = m_{1,1}(v)^2$ results in $m_{1,1}(v) = 1$, yielding $v = P$ as they have a common 2-moment sequence.

The $\boxtimes\boxtimes^{\mathrm{op}}$-idempotent elements can be easily ascertained by formula (2-3) and established results. This finishes the proof. $\qquad\square$

It is known that for any $v_1, v_2 \in \mathscr{P}_{\mathbb{T}^2}$, $m_{p,q}(v_1 \boxtimes\boxtimes v_2) = m_{p,q}(v_1)\, m_{p,q}(v_2)$ holds when $(p, q) = (0, 1), (1, 0)$.

Lemma 3.2. *Identities*

$$m_{1,1}(v_1 \boxtimes\boxtimes v_2) = m_{1,1}(v_1)\, m_{1,1}(v_2)$$

and

$$m_{1,-1}(v_1 \boxtimes\boxtimes {}^{\mathrm{op}} v_2) = m_{1,-1}(v_1)\, m_{1,-1}(v_2)$$

hold for any $v_1, v_2 \in \mathscr{P}_{\mathbb{T}^2}$.

Proof. Following the notations in Section 2C, let $\alpha_j = \langle S_j^{-1} \xi_j, \xi_j \rangle$, $\beta_j = \langle T_j \xi_j, \xi_j \rangle$, $h_j = S_j^{-1} \xi_j - \alpha_j \xi_j$, and $k_j = T_j \xi_j - \beta_j \xi_j$ for $j = 1, 2$. Then

$$m_{1,1}(v_j) = \langle T_j \xi_j, S_j^{-1} \xi_j \rangle = \overline{\alpha_j} \beta_j + \langle k_j, h_j \rangle.$$

On the other hand, we have $u_2^{-1} u_1^{-1} \xi = \alpha_1 \alpha_2 \xi + \alpha_2 h_1 + \alpha_1 h_2 + h_2 \otimes h_1$ and $v_1 v_2 \xi = \beta_1 \beta_2 \xi + \beta_2 k_1 + \beta_1 k_2 + k_2 \otimes k_1$. Thus, the first desired result follows from the representation of $m_{1,1}(v_j)$ given above and the computations

$$\begin{aligned}
m_{1,1}(v_1 \boxtimes\boxtimes v_2) &= \langle v_1 v_2 \xi, u_2^{-1} u_1^{-1} \xi \rangle \\
&= \overline{\alpha_1 \alpha_2}\, \beta_1 \beta_2 + \overline{\alpha_2}\, \beta_2 \langle k_1, h_1 \rangle + \overline{\alpha_1}\, \beta_1 \langle k_2, h_2 \rangle + \langle k_1, h_1 \rangle \langle k_2, h_2 \rangle.
\end{aligned}$$

Thanks to (2-3) and the first result, we obtain

$$m_{1,-1}(v_1 \boxtimes\boxtimes^{\mathrm{op}} v_2) = m_{1,1}(v_1^\star \boxtimes\boxtimes v_2^\star) = m_{1,-1}(v_1)\, m_{1,-1}(v_2). \qquad\square$$

Remark 3.3. Results in Lemma 3.2 can also be easily derived by the moment-cumulant formula and vanishing of bi-free mixed cumulants [8].

In the sequel, except for $\delta_{(1,1)}$, the other four $\boxtimes\boxtimes$-idempotent distributions are called nontrivial. The abusing notation $0^0 = 1$ is used in the following proposition and elsewhere.

Proposition 3.4. *Let $\nu \in \mathscr{P}_{\mathbb{T}^2}$.*

(1) *ν has the $\boxtimes\boxtimes$-factor $\mathrm{m} \times \delta_1$ if and only if $\nu = \mathrm{m} \times \nu^{(2)}$.*

(2) *ν has the $\boxtimes\boxtimes$-factor $\delta_1 \times \mathrm{m}$ if and only if $\nu = \nu^{(1)} \times \mathrm{m}$.*

(3) *ν has the $\boxtimes\boxtimes$-factor $\mathrm{m} \times \mathrm{m}$ if and only if $\nu = \mathrm{m} \times \mathrm{m}$.*

(4) *P is a $\boxtimes\boxtimes$-factor of ν if and only if*

$$(3\text{-}3) \qquad m_{p,q}(\nu) = \delta_{p,q}\, m_{1,1}(\nu)^p, \quad (p,q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\}),$$

where $\delta_{p,q}$ is the Kronecker function of p and q.

Statements (1)–(3) *remain true if the convolution $\boxtimes\boxtimes$ is replaced with $\boxtimes\boxtimes^{\mathrm{op}}$. Moreover, $P^\star$ is a $\boxtimes\boxtimes^{\mathrm{op}}$-factor of ν if and only if*

$$(3\text{-}4) \qquad m_{p,q}(\nu) = \delta_{p,-q} m_{1,-1}(\nu)^p, \quad (p,q) \in \mathbb{Z} \times (-\mathbb{N} \cup \{0\}).$$

Remark 3.5. For negative integers q, by taking complex conjugate, formula (3-3) becomes $m_{p,q}(\nu) = \delta_{p,q}\, m_{-1,-1}(\nu)^{-p}$.

Proof. Write $\nu = \nu_1 \boxtimes\boxtimes \nu_2$, where neither ν_1 nor ν_2 is $\delta_{(1,1)}$. We shall stay employing the notations for ν_1, ν_2 introduced in Section 2C to accomplish the proof.

First, let $\nu_2 = \mathrm{m} \times \delta_1$. In order to obtain $\nu = \mathrm{m} \times \nu^{(2)}$ as desired in (1), it amounts to proving that $m_{p,q}(\nu) = 0$ for any $p \in \mathbb{Z}^\times = \mathbb{Z}\setminus\{0\}$ and $q \in \mathbb{Z}$ because a probability measure on the bi-torus is uniquely determined by its moments. To this end, we take operator models (u_1, v_1) and (u_2, v_2) as in the proof of Proposition 3.1. A consequence of [20, Lemma 5.3] is that $m_{p,q}(\nu) = \varphi_\xi((u_1 u_2)^p (v_1 v_2)^q)$ can be expressed as a sum of products of quantities from the set $\{\varphi_\xi(u_i^{m_i} v_i^{n_i}) : m_1, m_2, n_1, n_2 \in \mathbb{Z}\}$. Moreover, since $p \neq 0$, each product in the sum contains at least one factor $\varphi_\xi(u_2^{m_2} v_2^{n_2})$ with $m_2 \neq 0$, which vanishes because (u_2, v_2) follows $\mathrm{m} \times \delta_1$. This verifies the "only if" part of (1). The "if" part of (1) is a direct consequence of (2-2). Alternatively, one can obtain the result by considering the measure $\tilde{\nu} = \nu\boxtimes\boxtimes(\mathrm{m}\times\delta_1)$. Indeed, $\tilde{\nu}$ has the $\boxtimes\boxtimes$-factor $\mathrm{m} \times \delta_1$, and so $\tilde{\nu} = \mathrm{m} \times \tilde{\nu}^{(2)}$ by the result proved above. Since $\tilde{\nu}^{(2)} = \nu^{(2)} \boxtimes \delta_1 = \nu^{(2)}$, it follows that

$$m_{p,q}(\nu) = m_p(\mathrm{m})\, m_q(\nu^{(2)}) = m_p(\mathrm{m})\, m_q(\tilde{\nu}^{(2)}) = m_{p,q}(\tilde{\nu})$$

for any $p, q \in \mathbb{Z}$. Hence we have $\nu = \tilde{\nu}$, which proves the "if" part.

By similar reasonings, (2) holds. If $\mathrm{m} \times \mathrm{m}$ is a $\boxtimes\boxtimes$-factor of ν, then so are distributions $\mathrm{m} \times \delta_1$ and $\delta_1 \times \mathrm{m}$, from which we see that (3) holds by (1) and (2).

Finally, we suppose $\nu_2 = P$ and justify (4). In view of P being $\boxtimes\boxtimes$-idempotent, $\nu_1 \boxtimes\boxtimes P$ may take the place of ν_1, and we do assume so below, without affecting the

convolution $\nu = \nu_1 \boxtimes\boxtimes P$. Since $m_{p,q}(\nu_1) = 0 = m_{p,q}(\nu_2)$ for $(p, q) = (0, 1)$, $(1, 0)$, formulas (3-1) and (3-2), together with Lemma 3.2, allow one to see that

$$m_{p,q}(\nu) = \langle v^q \xi, u^{-p} \xi \rangle = \delta_{p,q} \langle S_1 T_1 \xi_1, \xi_1 \rangle^p \langle S_2 T_2 \xi_2, \xi_2 \rangle^p = \delta_{p,q} m_{1,1}(\nu)^p$$

for $(p, q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$. This furnishes all mixed moments (3-3) of ν.

That $\nu \boxtimes\boxtimes P$ has the $\boxtimes\boxtimes$-factor P and the established result implies that for any $(p, q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$, $m_{p,q}(\nu \boxtimes\boxtimes P) = \delta_{p,q} m_{1,1}(\nu \boxtimes\boxtimes P)^p = \delta_{p,q} m_{1,1}(\nu)^p$ by Lemma 3.2. Thus $m_{p,q}(\nu \boxtimes\boxtimes P) = m_{p,q}(\nu)$ or, equivalently, $\nu \boxtimes\boxtimes P = \nu$ if (3-3) holds, proving the converse of (4).

All assertions regarding $\boxtimes\boxtimes^{\mathrm{op}}$-idempotent factors are direct consequences of statements (1)–(4), equation (2-3), and the formula $m_{p,q}(\nu^\star) = m_{p,-q}(\nu)$. $\quad\square$

Remark 3.6. From $m_{1,1}(\mathrm{m} \times \mathrm{m}) = 0$, assertion (4) of Proposition 3.4 can be strengthened as that P is the only nontrivial $\boxtimes\boxtimes$-idempotent factor of $\nu \in \mathscr{P}_{\mathbb{T}^2}$ if and only if $m_{1,1}(\nu) \neq 0$ and (3-3) holds.

Remark 3.7. The notions of bi-R-diagonality and Haar bi-unitary elements were first introduced in [18, Example 4.7] and [7, Definition 10.1.2], respectively. A Haar bi-unitary element is a bipartite pair having distribution $P^\star$ [15, Definition 2.15]. The opposite multiplication plays a key role when characterizing bi-R-diagonal pairs in terms of Haar bi-unitary elements [15, Theorem 4.4]. Moreover, measures $\nu \in \mathscr{P}_{\mathbb{T}^2}$ satisfying (3-4) are bi-R-diagonal because of $\nu = \nu \boxtimes\boxtimes^{\mathrm{op}} P^\star$ according to Proposition 3.4 and because of [15, Theorem 4.4].

For any $c \in \mathbb{D}$, define

$$d\kappa_c(s) = \frac{1 - |c|^2}{|1 - \bar{c}s|^2} \, d\mathrm{m}(s),$$

which is the probability measure on $\mathbb{T}$ induced by the Poisson kernel. It is the normalized Haar measure on $\mathbb{T}$ in case $c = 0$. By taking the weak limit we define $\kappa_c = \delta_c$ for $c \in \mathbb{T}$. Alternatively, κ_c with $c \in \mathbb{D} \cup \mathbb{T}$ is the unique probability measure on $\mathbb{T}$ determined by the requirement $m_p(\kappa_c) = c^p$ for $p \in \mathbb{N}$. Also, we have $m_p(\kappa_c) = \bar{c}^{|p|}$ for $p \in -\mathbb{N}$.

Observe that for any $c, d \in \mathbb{D} \cup \mathbb{T}$, we have

(3-5) $\qquad\qquad \nu \boxtimes\boxtimes (\kappa_c \times \kappa_d) = \nu \circledast (\kappa_c \times \kappa_d), \quad \nu \in \mathscr{P}_{\mathbb{T}^2}.$

To see this, consider ν and $\kappa_c \times \kappa_d$ as the distributions of two bi-free commuting unitary faces (u_1, v_1) and (u_2, v_2), respectively, in some C^*-probability space $(B(\mathcal{H}), \varphi_\xi)$. Observe that both pairs of faces (u_2, v_2) and $(cI_{B(\mathcal{H})}, dI_{B(\mathcal{H})})$ are commuting, bi-free from (u_1, v_1), and have the same (p, q)-moments $c^p d^q$ for $(p, q) \in (\mathbb{N} \cup \{0\})^2$. In view of the universal calculation formula for mixed moments

[20, Lemma 5.2], we may replace (u_2, v_2) with $(c I_{B(\mathcal{H})}, d I_{B(\mathcal{H})})$. This entails

$$\varphi_\xi((u_1 u_2)^p (v_1 v_2)^q) = c^p d^q \varphi_\xi(u_1^p v_1^q),$$

and hence $m_{p,q}(\nu \boxtimes\boxtimes (\kappa_c \times \kappa_d)) = m_{p,q}(\nu \circledast (\kappa_c \times \kappa_d))$ for $(p, q) \in (\mathbb{N} \cup \{0\})^2$. Similarly, one can obtain the same identity for $(p, q) \in (\mathbb{N} \cup \{0\}) \times (-\mathbb{N} \cup \{0\})$ by using that (u_2, v_2) and $(c I_{B(\mathcal{H})}, (1/\bar{d}) I_{B(\mathcal{H})})$ have the same (p, q)-moments $c^p \bar{d}^{|q|}$. Therefore, we justify (3-5).

A special case of (3-5) is the validity of

$$(\kappa_{c_1} \times \kappa_{d_1}) \boxtimes\boxtimes (\kappa_{c_2} \times \kappa_{d_2}) = \kappa_{c_1 c_2} \times \kappa_{d_1 d_2} = (\kappa_{c_1} \times \kappa_{d_1}) \circledast (\kappa_{c_2} \times \kappa_{d_2})$$

for any $c_1, c_2, d_1, d_2 \in \mathbb{D} \cup \mathbb{T}$, yielding the following results.

Proposition 3.8. *The measure $\kappa_c \times \kappa_d$ is both $\circledast$- and $\boxtimes\boxtimes$-infinitely divisible for any $c, d \in \mathbb{D} \cup \mathbb{T}$.*

Proposition 3.9. *Any $\nu \in \mathscr{P}_{\mathbb{T}^2}$ with moments satisfying (3-3) can be expressed as $P \circledast (\kappa_c \times \delta_1)$, where $c = m_{1,1}(\nu)$. In particular, ν is both $\circledast$- and $\boxtimes\boxtimes$-infinitely divisible.*

Proof. Clearly, we have $m_{p,q}(P \circledast (\kappa_c \times \delta_1)) = \delta_{p,q} c^p$ for $(p, q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$, and hence $\nu = P \circledast (\kappa_c \times \delta_1)$. The $\circledast$-infinitely divisibility of P and Proposition 3.8 yield that ν is $\circledast$-infinitely divisible. Also, the identity $\nu = P \boxtimes\boxtimes (\kappa_c \times \delta_1)$ obtained by (3-5) proves the $\boxtimes\boxtimes$-infinite divisibility of ν. $\square$

A consequence of (3-5) and Proposition 3.9 is that the following identity holds for every $\nu \in \mathscr{P}_{\mathbb{T}^2}$:

$$(3\text{-}6) \qquad P \boxtimes\boxtimes (\kappa_{m_{1,1}(\nu)} \times \delta_1) = P \boxtimes\boxtimes \nu = P \circledast (\kappa_{m_{1,1}(\nu)} \times \delta_1).$$

The following is a bi-free multiplicative analog of the classical multiplicative limit theorem.

Theorem 3.10. *Let $\{\nu_{nk_n}\}_{n \geq 1, 1 \leq k \leq k_n}$ be an infinitesimal triangular array in $\mathscr{P}_{\mathbb{T}^2}$ and $\{\boldsymbol{\xi}_n\}$ be a sequence in $\mathbb{T}^2$. If the sequence in (1-3) has a weak limit ν, then ν is $\boxtimes\boxtimes$-infinitely divisible. If $m_{1,0}(\nu) \neq 0 \neq m_{0,1}(\nu)$, then $m_{1,1}(\nu) \neq 0$. Moreover, if $m_{1,0}(\nu) = 0$, then $\nu = \mathrm{m} \times \nu^{(2)}$ and if $m_{0,1}(\nu) = 0$, then $\nu = \nu^{(1)} \times \mathrm{m}$.*

Proof. We separately consider three possible statuses (i) $m_{1,0}(\nu) \neq 0 \neq m_{0,1}(\nu)$, (ii) $m_{1,0}(\nu) = 0 \neq m_{0,1}(\nu)$ (the case $m_{1,0}(\nu) \neq 0 = m_{0,1}(\nu)$ is treated similarly to (ii)), and (iii) $m_{1,0}(\nu) = 0 = m_{0,1}(\nu)$.

(i) Once we can prove that $m_{1,1}(\nu) \neq 0$, then the $\boxtimes\boxtimes$-infinite divisibility of ν will follow from [13, Theorem 4.2]. Assume to the contrary that $m_{1,1}(\nu) = 0$, which together with Lemma 3.2 implies that as $n \to \infty$,

$$m_{1,1}(\delta_{\boldsymbol{\xi}_n}) m_{1,1}(\nu_{n1}) \cdots m_{1,1}(\nu_{nk_n}) = m_{1,1}(\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes \nu_{n1} \boxtimes\boxtimes \cdots \boxtimes\boxtimes \nu_{nk_n}) \to 0.$$

Then there exists a sequence $\{\ell_n\} \subset \mathbb{N}$ so that as $n \to \infty$, we have

$$m_{1,1}(\delta_{\xi_n})\, m_{1,1}(\nu_{n1}) \cdots m_{1,1}(\nu_{n\ell_n}) \to 0 \quad \text{and} \quad m_{1,1}(\nu_{n,\ell_n+1}) \cdots m_{1,1}(\nu_{nk_n}) \to 0,$$

namely, one sees from Lemma 3.2 that

$$m_{1,1}(\delta_{\xi_n} \boxtimes\!\boxtimes \nu_{n1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{n\ell_n}) \to 0 \quad \text{and} \quad m_{1,1}(\nu_{n,\ell_n+1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{nk_n}) \to 0$$

as $n \to \infty$. To obtain such a sequence $\{\ell_n\}$, one can select, for example,

$$\ell_n = \min\Big\{k : |m_{1,1}(\delta_{\xi_n} \boxtimes\!\boxtimes \nu_{n1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{nk})| \\ \leq |m_{1,1}(\delta_{\xi_n} \boxtimes\!\boxtimes \nu_{n1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{nk_n})|^{1/2}\Big\}.$$

One may assume, by passing to a subsequence if needed, that

$$\delta_{\xi_n} \boxtimes\!\boxtimes \nu_{n1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{n\ell_n} \Rightarrow \nu_1' \in \mathscr{P}_{\mathbb{T}^2}, \quad \nu_{n,\ell_n+1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{nk_n} \Rightarrow \nu_1'' \in \mathscr{P}_{\mathbb{T}^2}.$$

Then we have $\nu = \nu_1' \boxtimes\!\boxtimes \nu_1''$ and $m_{1,1}(\nu_1') = 0 = m_{1,1}(\nu_1'')$. Also, the formula

$$m_{1,0}(\nu) = m_{1,0}(\nu_1')\, m_{1,0}(\nu_1'')$$

indicates that either $|m_{1,0}(\nu_1')| \geq |m_{1,0}(\nu)|^{1/2}$ or $|m_{1,0}(\nu_1'')| \geq |m_{1,0}(\nu)|^{1/2}$ must occur; assume, without loss of generality, that the first inequality is valid. Carrying out the same arguments on ν_1' allows us to obtain $\nu_2', \nu_2'' \in \mathscr{P}_{\mathbb{T}^2}$ fulfilling requirements $\nu_1' = \nu_2' \boxtimes\!\boxtimes \nu_2''$, $m_{1,1}(\nu_2') = 0 = m_{1,1}(\nu_2'')$, and $|m_{1,0}(\nu_2')| \geq |m_{1,0}(\nu_1')|^{1/2}$. Continuing this process then results in the existence of sequences $\{\nu_n'\}, \{\nu_n''\} \subset \mathscr{P}_{\mathbb{T}^2}$ for which $\nu_n' = \nu_{n+1}' \boxtimes\!\boxtimes \nu_{n+1}''$, $m_{1,1}(\nu_n') = 0 = m_{1,1}(\nu_n'')$, and $|m_{1,0}(\nu_{n+1}')| \geq |m_{1,0}(\nu_n')|^{1/2}$ hold.

One has $\nu = \nu_n' \boxtimes\!\boxtimes \nu_n'''$ for some $\nu_n''' \in \mathscr{P}_{\mathbb{T}^2}$ and $|m_{1,0}(\nu_n')| \geq |m_{1,0}(\nu)|^{1/2^n}$. Passing to subsequences if needed again, let $\nu_n' \Rightarrow \nu_1 \in \mathscr{P}_{\mathbb{T}^2}$ and $\nu_n''' \Rightarrow \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$, and so $\nu = \nu_1 \boxtimes\!\boxtimes \nu_2$, $m_{1,1}(\nu_1) = 0$, and $|m_{1,0}(\nu_1)| = 1$. The last identity reveals that $\nu_1 = \delta_\alpha \times \nu_1^{(2)}$, $\alpha = m_{1,0}(\nu_1) \in \mathbb{T}$. Also, using $0 \neq m_{0,1}(\nu) = m_{0,1}(\nu_1)\, m_{0,1}(\nu_2)$ we get $m_{0,1}(\nu_1) \neq 0$. However, these discussions would lead to $m_{1,1}(\nu_1) = \alpha m_{0,1}(\nu_1) \neq 0$, a contradiction. Hence we must have $m_{1,1}(\nu) \neq 0$, as desired.

(ii) Note that the marginal $\nu^{(2)}$ is $\boxtimes$-infinitely divisible by [2, Theorem 2.1]. The $\boxtimes\!\boxtimes$-infinite divisibility of ν will follow immediately if one can argue that $\nu = \mathrm{m} \times \nu^{(2)}$. The proof, presented below, is basically similar to that of (1).

First, applying the strategy employed in the first paragraph of (1) to $m_{1,0}(\nu) = 0$ indicates the presence of $\ell_n \in \mathbb{N}$ satisfying $m_{1,0}(\delta_{\xi_n} \boxtimes\!\boxtimes \nu_{n1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{n\ell_n}) \to 0$ and $m_{1,0}(\nu_{n,\ell_n+1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{nk_n}) \to 0$ as $n \to \infty$. Assume, dropping a subsequence if necessary, that

$$\delta_{\xi_n} \boxtimes\!\boxtimes \nu_{n1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{n\ell_n} \Rightarrow \nu_1' \in \mathscr{P}_{\mathbb{T}^2}, \quad \nu_{n,\ell_n+1} \boxtimes\!\boxtimes \cdots \boxtimes\!\boxtimes \nu_{nk_n} \Rightarrow \nu_1'' \in \mathscr{P}_{\mathbb{T}^2}.$$

Thus we have $\nu = \nu'_1 \boxtimes\boxtimes \nu''_1$ and $m_{1,0}(\nu'_1) = 0 = m_{1,0}(\nu''_1)$. We may further assume $|m_{0,1}(\nu'_1)| \geq |m_{0,1}(\nu)|^{1/2}$. Mimicking the arguments in (1) constructs sequences $\{\nu'_n\}, \{\nu''_n\} \subset \mathscr{P}_{\mathbb{T}^2}$ meeting conditions $\nu = \nu'_n \boxtimes\boxtimes \nu''_n$, $m_{1,0}(\nu'_n) = 0 = m_{1,0}(\nu''_n)$, and $|m_{0,1}(\nu'_n)| \geq |m_{0,1}(\nu)|^{1/2^n}$. Passing to subsequences if needed again, assume that $\nu'_n \Rightarrow \nu_1 \in \mathscr{P}_{\mathbb{T}^2}$ and $\nu''_n \Rightarrow \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$. Then we come to that $\nu = \nu_1 \boxtimes\boxtimes \nu_2$, $m_{1,0}(\nu_1) = 0 = m_{1,0}(\nu_2)$, and $\nu_1^{(2)} = \delta_\alpha$ for some $\alpha \in \mathbb{T}$. To proceed the proof, we shall use notations introduced in Section 2C. Since $\nu_1 = \alpha I_{B(\mathcal{H})}$, it follows that $v^q \xi = \alpha^q v_2^q \xi \in \mathbb{C}\xi \oplus \mathring{\mathcal{H}}_2$ for any $q \in \mathbb{Z}$. Thus, equation (3-1) implies that $m_{p,q}(\nu) = \langle v^q \xi, u^{-p}\xi \rangle = 0$ for any $(p,q) \in \mathbb{N} \times \mathbb{Z}$, proving $\nu = \mathrm{m} \times \nu^{(2)}$.

(iii) In this case, we have $\nu^{(1)} = \mathrm{m} = \nu^{(2)}$ by [2, Theorem 2.1] and [4, Lemma 6.1]. Further, one can employ the proof in (ii) to show that there are $\nu_1, \nu_2 \in \mathscr{P}_{\mathbb{T}^2}$ so that $\nu = \nu_1 \boxtimes\boxtimes \nu_2$ and $m_{1,0}(\nu_1) = 0 = m_{1,0}(\nu_2)$. Then $m_{0,1}(\nu_1) m_{0,1}(\nu_2) = m_{0,1}(\nu) = 0$. If $m_{0,1}(\nu_1) = 0 = m_{0,1}(\nu_2)$, then (3-1) and (3-2) yield that $m_{p,q}(\nu) = \delta_{p,q} m_{1,1}(\nu)^p$ for $(p,q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$, whence ν is $\boxtimes\boxtimes$-infinitely divisible by Proposition 3.9. For the other case, say $m_{0,1}(\nu_1) \neq 0$, the established conclusion in (ii) then shows that $\nu_1 = \mathrm{m} \times \nu_1^{(2)}$. In such a situation, the measure ν_1, as well as ν, has the $\boxtimes\boxtimes$-factor $\mathrm{m} \times \delta_1$. Thus, Proposition 3.4 says that $\nu = \mathrm{m} \times \mathrm{m}$, which is clearly $\boxtimes\boxtimes$-infinitely divisible.   □

Corollary 3.11. *The set $\mathcal{ID}(\boxtimes\boxtimes)$ is weakly closed.*

We are now in a position to characterize distributions in $\mathcal{ID}(\boxtimes\boxtimes)$ carrying no nontrivial $\boxtimes\boxtimes$-idempotent factors.

Theorem 3.12. *In order that a measure $\nu \in \mathcal{ID}(\boxtimes\boxtimes)$ contains no nontrivial $\boxtimes\boxtimes$-idempotent factor, it is necessary and sufficient that $m_{1,0}(\nu) \neq 0 \neq m_{0,1}(\nu)$, in which case $m_{1,1}(\nu) \neq 0$.*

Proof. According to Proposition 3.4, only the necessity requires a proof. We merely prove that ν has a nontrivial $\boxtimes\boxtimes$-idempotent factor when $m_{1,0}(\nu) = 0$, because the case $m_{0,1}(\nu) = 0$ can be handled in the same way. To do so, let $m_{1,0}(\nu) = 0$, and consider two possible cases (i) $m_{0,1}(\nu) = 0$ and (ii) $m_{0,1}(\nu) \neq 0$, which are discussed separately below. Note that $m_{p,0}(\nu) = 0$ for all $p \in \mathbb{N}$ since $\nu^{(1)} = m$.

Case (i): Since $\nu^{(j)} = \mathrm{m}$ for $j = 1, 2$, one can mimic the proof of Proposition 3.1, especially employ equations (3-1) and (3-2), to obtain $m_{p,q}(\nu) = \delta_{p,q} m_{1,1}(\nu)^p$ for $(p,q) \in \mathbb{Z} \times (\mathbb{N} \cup \{0\})$. Hence P is a $\boxtimes\boxtimes$-factor of ν by Proposition 3.4.

Case (ii): To treat this case, let $\nu'_n \in \mathscr{P}_{\mathbb{T}^2}$ be an n-th $\boxtimes\boxtimes$-convolution root of ν for any $n \in \mathbb{N}$, i.e., $(\nu'_n)^{\boxtimes\boxtimes n} = \nu$. Then we have $\nu = \nu'_n \boxtimes\boxtimes \nu''_n$, where $\nu''_n = (\nu'_n)^{\boxtimes\boxtimes(n-1)}$, $m_{1,0}(\nu'_n) = 0 = m_{1,0}(\nu''_n)$ and $|m_{0,1}(\nu'_n)| = |m_{0,1}(\nu)|^{1/n}$. If ν' and ν'' are any weak limits of $\{\nu'_n\}$ and $\{\nu''_n\}$, respectively, then we further obtain $\nu = \nu' \boxtimes\boxtimes \nu''$, $m_{1,0}(\nu') = 0 = m_{1,0}(\nu'')$, and $|m_{0,1}(\nu')| = 1$. This leads to $(\nu')^{(2)} = \delta_\alpha$ for some $\alpha \in \mathbb{T}$, which is exactly the situation dealt in the last part of the proof (ii) of

Theorem 3.10. Thus we conclude that $\nu = m \times \nu^{(2)}$, which has the $\boxtimes\boxtimes$-idempotent factor $m \times \delta_1$ by Proposition 3.4.

Lastly, we turn to argue that $m_{1,1}(\nu) \neq 0$ if $m_{1,0}(\nu) \neq 0 \neq m_{0,1}(\nu)$. Any sequence $\{\nu_n\}$ satisfying $\nu = \nu_n^{\boxtimes\boxtimes 2^n}$ has a subsequence $\{\nu_{n_j}\}$ converging weakly to $\delta_{\boldsymbol{\xi}}$ for some $\boldsymbol{\xi} \in \mathbb{T}^2$ (see (i) of Theorem 3.10). Then Lemma 3.2 implies that $|m_{1,1}(\nu)|^{2^{-n_j}} = |m_{1,1}(\nu_{n_j})| \to |m_{1,1}(\delta_{\boldsymbol{\xi}})| = 1$, leading to the desired result. $\qquad\square$

Propositions 3.4 and 3.9, and Theorem 3.12 readily imply the following.

Corollary 3.13. *Any measure ν in $\mathcal{ID}(\boxtimes\boxtimes) \setminus \mathscr{P}_{\mathbb{T}^2}^\times$ is either $\nu^{(1)} \times m$, $m \times \nu^{(2)}$, $m \times m$ or $P \circledast (\kappa_c \times \delta_1)$, where $\nu^{(1)}$ and $\nu^{(2)}$ are in $\mathcal{ID}(\boxtimes)$ with nonzero mean and $c \in (\mathbb{D} \cup \mathbb{T}) \setminus \{0\}$.*

4. Equivalent conditions on limit theorems

This section is devoted to exploring the associations among the conditions introduced in Section 2D and the following one.

Condition 4.1. Let $\{\rho_n\}$ be a sequence in $\mathscr{M}_{\mathbb{T}^d}^{\mathbf{1}}$.

(iii) There exists some $\rho \in \mathscr{M}_{\mathbb{T}^d}^{\mathbf{1}}$ with $\rho(\{\mathbf{1}\}) = 0$ (i.e., $\rho \in \tilde{\mathscr{M}}_{\mathbb{T}^d}^{\mathbf{1}}$) so that $\rho_n \Rightarrow_{\mathbf{1}} \rho$.

(iv) The following limits exist in $\mathbb{R}$ for any $\boldsymbol{p} \in \mathbb{Z}^d$:

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{\mathscr{U}_\epsilon} \langle \boldsymbol{p}, \Im \boldsymbol{s} \rangle^2 \, d\rho_n(\boldsymbol{s}) = Q(\boldsymbol{p}) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \int_{\mathscr{U}_\epsilon} \langle \boldsymbol{p}, \Im \boldsymbol{s} \rangle^2 \, d\rho_n(\boldsymbol{s}).$$

Condition 2.5 with $d = 2$ was used in [13, Theorem 3.4] to prove the limit theorem for the bi-free multiplicative convolution, while Condition 4.1 is beneficial for the corresponding classical limit theorem [10]. More properties regarding these two conditions are presented below.

Proposition 4.2. Condition 2.5 *is equivalent to* Condition 4.1, *in which*

$$(4\text{-}1) \qquad d\lambda_j(\boldsymbol{s}) = (1 - \Re s_j) \, d\rho(\boldsymbol{s}) + \frac{Q(\boldsymbol{e}_j)}{2} \delta_{\mathbf{1}}(d\boldsymbol{s}), \quad j = 1, \ldots, d,$$

$$(4\text{-}2) \qquad \int_{\mathbb{T}^d} \|\mathbf{1} - \Re \boldsymbol{s}\| \, d\rho(\boldsymbol{s}) < \infty,$$

and the quadratic form $Q(\cdot) = \langle \boldsymbol{A} \cdot, \cdot \rangle$ on $\mathbb{Z}^d$ is determined by the positive semidefinitive matrix $\boldsymbol{A} = (a_{j\ell})$ whose entries are

$$(4\text{-}3) \qquad a_{j\ell} = L_{j\ell} - \int_{\mathbb{T}^d} (\Im s_j)(\Im s_\ell) \, d\rho(\boldsymbol{s}) \in \mathbb{R}, \quad j, \ell = 1, \ldots, d.$$

Moreover, $a_{jj} = 2\lambda_j(\{\mathbf{1}\})$ for $j = 1, \ldots, d$.

Proof. Suppose first that Condition 2.5 is satisfied. Then the relation

$$(1 - \Re s_j) \, d\lambda_\ell = (1 - \Re s_\ell) \, d\lambda_j, \quad j, \ell = 1, \dots, d,$$

guaranteed by item (i) of Condition 2.5 ensures that the measure

$$(4\text{-}4) \qquad\qquad d\rho(\boldsymbol{s}) = \frac{1_{\{s_j \neq 1\}}(\boldsymbol{s})}{1 - \Re s_j} \, d\lambda_j(\boldsymbol{s})$$

is unambiguous and does not depend on j. In addition, it satisfies requirements $\rho(\mathbb{T}^d \setminus \mathscr{U}_\epsilon) < \infty$ for any $\epsilon > 0$ and (4-2). Hence the measure ρ that we just constructed belongs to $\mathscr{M}^1_{\mathbb{T}^d}$.

To see $\rho_n \Rightarrow_1 \rho$, pick a continuous function f on $\mathbb{T}^d$ with support contained within $\mathbb{T}^d \setminus \mathscr{U}_\delta$ for some $\delta > 0$. Then this f produces d continuous functions on $\mathbb{T}^d$, which are

$$f_j(\boldsymbol{s}) = \frac{\text{dist}(U_j, \boldsymbol{s})}{\text{dist}(U_1, \boldsymbol{s}) + \cdots + \text{dist}(U_d, \boldsymbol{s})} f(\boldsymbol{s}),$$

where $U_j = \{\boldsymbol{u} \in \mathbb{T}^d : |\arg u_j| < \delta / \sqrt{2d}\}$ and $\text{dist}(U_j, \boldsymbol{s}) = \inf\{\|\arg \boldsymbol{s} - \arg \boldsymbol{u}\| : \boldsymbol{u} \in U_j\}$ for $j = 1, \dots, d$. Obviously, the relation $f = f_1 + \cdots + f_d$ holds and each $f_j / (1 - \Re s_j)$ is continuous on $\mathbb{T}^d$. These observations and the weak convergence $\lambda_{nj} \Rightarrow \lambda_j$ then yield that

$$\int_{\mathbb{T}^d} f(\boldsymbol{s}) \, d\rho_n(\boldsymbol{s}) = \sum_{j=1}^d \int_{\mathbb{T}^d} \frac{f_j(\boldsymbol{s})}{1 - \Re s_j} \, d\lambda_{nj}(\boldsymbol{s}) \xrightarrow[n \to \infty]{} \sum_{j=1}^d \int_{\mathbb{T}^d} \frac{f_j(\boldsymbol{s})}{1 - \Re s_j} \, d\lambda_j(\boldsymbol{s})$$

$$= \int_{\mathbb{T}^d} f(\boldsymbol{s}) \, d\rho(\boldsymbol{s}).$$

Therefore, we have completed the verification of item (iii) of Condition 4.1.

We next demonstrate the validity of the following identities for $1 \leq j, \ell \leq d$,

$$(4\text{-}5) \qquad \lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{\mathscr{U}_\epsilon} (\Im s_j)(\Im s_\ell) \, d\rho_n = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \int_{\mathscr{U}_\epsilon} (\Im s_j)(\Im s_\ell) \, d\rho_n,$$

which confirms that of Condition 4.1(iv). To continue, observe that the mapping $s \mapsto (\Im s)^2 / (1 - \Re s)$ is continuous on $\mathbb{T}$ and at the origin, it takes value

$$(4\text{-}6) \qquad\qquad \lim_{\arg s \to 0} \frac{(\Im s)^2}{1 - \Re s} = 2.$$

Then (4-2), (4-6), and the Hölder inequality imply that $(\Im s_j)(\Im s_\ell) \in L^1(\rho)$ for $j, \ell = 1, \dots, d$. In order to get results (4-3) and (4-5), we examine the following differences which are related to them:

$$D_n(\epsilon) = \int_{\mathbb{T}^d} (\Im s_j)(\Im s_\ell) \, d\rho_n - \int_{\mathbb{T}^d} (\Im s_j)(\Im s_\ell) \, d\rho - \int_{\mathscr{U}_\epsilon} (\Im s_j)(\Im s_\ell) \, d\rho_n,$$

which further splits into the sum of

$$I_{n1}(\epsilon) = \int_{\mathbb{T}^d \setminus \mathscr{U}_\epsilon} (\Im s_j)(\Im s_\ell) \, d\rho_n - \int_{\mathbb{T}^d \setminus \mathscr{U}_\epsilon} (\Im s_j)(\Im s_\ell) \, d\rho$$

and

$$I_2(\epsilon) = - \int_{\mathscr{U}_\epsilon} (\Im s_j)(\Im s_\ell) \, d\rho.$$

Apparently, we have $\lim_{\epsilon \to 0} I_2(\epsilon) = 0$ owing to $(\Im s_j)(\Im s_\ell) \in L^1(\rho)$. Next, take an $\epsilon' \in (\epsilon, 2\epsilon)$ and an $\epsilon'' \in \left(\frac{\epsilon}{2}, \epsilon\right)$ with the attributes that $\rho(\partial \mathscr{U}_{\epsilon'}) = 0$ and $\rho(\partial \mathscr{U}_{\epsilon''}) = 0$, the presence of which are insured by the finiteness of the measure $1_{\mathbb{T}^d \setminus \mathscr{U}_{\epsilon/2}} \, \rho$ on $\mathbb{T}^d$. Then applying Proposition 2.1 to the established result $\rho_n \Rightarrow_1 \rho$ results in

$$\lim_{n \to \infty} \int_{\mathbb{T}^d \setminus \mathscr{U}_{\epsilon'}} (\Im s_j)(\Im s_\ell) \, d\rho_n = \int_{\mathbb{T}^d \setminus \mathscr{U}_{\epsilon'}} (\Im s_j)(\Im s_\ell) \, d\rho.$$

On the other hand, working with the closed subset $F_\epsilon = \{s \in \mathbb{T}^d : \epsilon'' \leq \|\arg s\| \leq \epsilon'\}$ and employing Proposition 2.1, we come to

(4-7) $$\left(\limsup_{n \to \infty} \int_{F_\epsilon} |\Im s_j| \, |\Im s_\ell| \, d\rho_n\right)^2 \leq \limsup_{n \to \infty} \int_{F_\epsilon} (\Im s_j)^2 \, d\rho_n \cdot \int_{F_\epsilon} (\Im s_\ell)^2 \, d\rho_n$$

$$\leq \int_{F_\epsilon} (\Im s_j)^2 \, d\rho \cdot \int_{F_\epsilon} (\Im s_\ell)^2 \, d\rho \to 0$$

as $\epsilon \to 0$. With the help of the facts $\mathbb{T}^d \setminus \mathscr{U}_{\epsilon''} = (\mathbb{T}^d \setminus \mathscr{U}_\epsilon) \cup (\mathscr{U}_\epsilon \setminus \mathscr{U}_{\epsilon''})$ and $\mathscr{U}_\epsilon \setminus \mathscr{U}_{\epsilon''} \subset F_\epsilon$, we are able to conclude that $\lim_{\epsilon \to 0} \limsup_{n \to \infty} |I_{n1}(\epsilon)| = 0$. Consequently, we have shown $\lim_{\epsilon \to 0} \limsup_{n \to \infty} |D_n(\epsilon)| = 0$, which together with Condition 2.5(ii) accounts for (4-3) and (4-5) with any indices j and ℓ.

If ϵ' is also chosen so that $\lambda_j(\partial \mathscr{U}_{\epsilon'}) = 0$, then we draw once again from (4-6) that $a_{jj} = 2\lambda_j(\{\mathbf{1}\})$ because

$$\limsup_{n \to \infty} \int_{\mathscr{U}_\epsilon} |2(1 - \Re s_j) - (\Im s_j)^2| \, d\rho_n \leq \limsup_{n \to \infty} \int_{\mathscr{U}_{\epsilon'}} \left|2 - \frac{(\Im s)^2}{1 - \Re s}\right| \, d\lambda_{nj}$$

$$= \int_{\mathscr{U}_{\epsilon'}} \left|2 - \frac{(\Im s_j)^2}{1 - \Re s_j}\right| \, d\lambda_j \xrightarrow[\epsilon \to \infty]{}.$$

This conclusion and (4-4) give (4-1). It is easy to see that the limits in (iv) of Condition 4.1 are equal to $\langle A\mathbf{p}, \mathbf{p}\rangle$ for any $\mathbf{p} \in \mathbb{Z}^d$ (in fact, for any $\mathbf{p} \in \mathbb{R}^d$ as well) with $A = (a_{j\ell})$ and $a_{j\ell}$ the value of the limit given in (4-5). Also, it is clear that the quadratic form Q extends to $\mathbb{R}^d$ and is positive therein. Then the positivity of $A \geq 0$ can be gained by that of Q on $\mathbb{R}^d$.

Next, we elaborate that Condition 4.1 implies Condition 2.5. Define λ_j's as in (4-1). These measures thus obtained are all in $\mathscr{M}_{\mathbb{T}^d}$, and the arguments for this go as follows. Select a sequence $\epsilon_m \downarrow 0$ as $m \to \infty$ and $\rho(\{\|\arg s\| = \epsilon_m\}) = 0$

for each m. Then (iv), along with Proposition 2.1, indicates that for any numbers $m < m'$ both large enough, one has

$$\int_{\{\epsilon_{m'} < \|\arg s\| < \epsilon_m\}} (\Im s_j)^2 \, d\rho(s) \leq 1 + Q(e_j).$$

Thanks to monotone convergence theorem, (4-6), and the assumption $\rho(\{1\}) = 0$, one further gets that for m large enough, $(1 - \Re s_j) 1_{\mathscr{U}_{\epsilon_m}} \in L^1(\rho)$ for any j. This proves that $\lambda_j(\mathbb{T}^d) < \infty$ and $(\Im s_j)^2 \in L^1(\rho)$ for any j.

After the previous preparations, we are in a position to justify the weak convergence $\lambda_{nj} \Rightarrow \lambda_j$. Given a continuous function f on $\mathbb{T}^d$, the difference

$$\left| \int_{\mathbb{T}^d} f \, d\lambda_{nj} - \int_{\mathbb{T}^d} f \, d\lambda_j \right|$$

is dominated by the sum of the following four terms:

$$D_{n1}(m) = \int_{\mathscr{U}_{\epsilon_m}} |f(s) - f(1)| \, d\lambda_{nj}(s),$$

$$D_{n2}(m) = |f(1)| \left| \lambda_{nj}(\mathscr{U}_{\epsilon_m}) - \tfrac{1}{2} Q(e_j) \right|,$$

$$D_3(m) = \int_{\mathscr{U}_{\epsilon_m} \setminus \{1\}} |f| \, d\lambda_j(s),$$

$$D_{n4}(m) = \left| \int_{\mathbb{T}^d \setminus \mathscr{U}_{\epsilon_m}} f \, d\lambda_{nj}(s) - \int_{\mathbb{T}^d \setminus \mathscr{U}_{\epsilon_m}} f \, d\lambda_j(s) \right|.$$

First, one can show that $\lim_{m \to \infty} \limsup_{n \to \infty} |D_{n2}(m)| = 0$ by applying (4-6) and item (iv) to

$$2\lambda_{nj}(\mathscr{U}_{\epsilon_m}) - Q(e_j) = \int_{\mathscr{U}_{\epsilon_m}} [2(1 - \Re s_j) - (\Im s_j)^2] \, d\rho_n(s) + \int_{\mathscr{U}_{\epsilon_m}} (\Im s_j)^2 \, d\rho_n(s) - Q(e_j).$$

Similarly, one can show

$$\lim_{m \to \infty} \limsup_{n \to \infty} |D_{n1}(m)| \leq \tfrac{1}{2} Q(e_j) \cdot \lim_{m \to \infty} \sup_{s \in \mathscr{U}_{\epsilon_m}} |f(s) - f(1)| = 0.$$

On the other hand, the finiteness of $\lambda_j(\mathbb{T}^d)$ leads to

$$\lim_{m \to \infty} D_3(m) \leq \|f\|_\infty \lim_{m \to \infty} \lambda_j(\mathscr{U}_{\epsilon_m} \setminus \{1\}) = 0.$$

That we have $\lim_{n \to \infty} D_{n4}(m) = 0$ for all m evidently follows from Condition 4.1(iii) and Proposition 2.1. Putting all these observations together illustrates $\lambda_{nj} \Rightarrow \lambda_j$.

It remains to deal with (ii) of Condition 2.5, in which the integral is rewritten as

$$\int_{\mathscr{U}_{\epsilon_m}} (\Im s_j)(\Im s_\ell) \, d\rho_n + \int_{\mathbb{T}^d \setminus \mathscr{U}_{\epsilon_m}} (\Im s_j)(\Im s_\ell) \, d\rho_n.$$

For any j, ℓ, taking the operations $\lim_{m\to\infty} \limsup_{n\to\infty}$ and $\lim_{m\to\infty} \liminf_{n\to\infty}$ of the first integral gives the same value $\frac{1}{2}[Q(e_j + e_\ell) - Q(e_j) - Q(e_\ell)]$, while doing the same thing to the second integral yields the value $\int_{\mathbb{T}^d} (\Im s_j)(\Im s_\ell) \, d\rho$ because of $\rho_n \Rightarrow_1 \rho$ and $(\Im s_j)^2 + (\Im s_\ell)^2 \in L^1(\rho)$. This finishes the proof of the proposition. $\square$

An intuitive thought is that measures on $\mathbb{T}^d$ obtained by rotating measures within controllable angles maintain the same structural properties, such as Condition 4.1, as the original ones. The statement and its rigorous proof are given below.

Proposition 4.3. *Suppose that $\{\nu_{nk}\} \subset \mathscr{P}_{\mathbb{T}^d}$ is a triangular array for which the measure $\rho_n = \sum_{k=1}^{k_n} \nu_{nk}$ satisfies* Condition 4.1. *If an array $\{\boldsymbol{\theta}_{nk}\} \subset (-\pi, \pi]^d$ fulfills the condition*

$$(4\text{-}8) \qquad \lim_{n\to\infty} \sum_{k=1}^{k_n} (\mathbf{1} - \cos \boldsymbol{\theta}_{nk}) = \mathbf{0},$$

then Condition 4.1 *is still applicable to measures $\tilde{\rho}_n(\cdot) = \sum_{k=1}^{k_n} \nu_{nk}(\cdot e^{i\boldsymbol{\theta}_{nk}})$, in which $\tilde{\rho}_n \Rightarrow_1 \rho$ and ρ_n and $\tilde{\rho}_n$ define the same quadratic form in* Condition 4.1(iv).

Proof. First of all, (4-8) reveals that $\lim_n \max_k \|\boldsymbol{\theta}_{nk}\| = 0$. We now argue that $\tilde{\rho}_n \Rightarrow_1 \rho$ as well by using Proposition 2.1. To do so, pick a closed subset $F \subset \mathbb{T}^d \setminus \mathscr{U}_r$ for some $r > 0$. Since $\rho(F) < \infty$, it follows that given any $\delta > 0$, there exists a closed set $F' \subset \mathbb{T}^d \setminus \mathscr{U}_{r/2}$ such that $e^{i\boldsymbol{\theta}_{nk}} F \subset F'$ for all sufficiently large n and for all $1 \le k \le k_n$, and $\rho(F' \setminus F) < \delta$. Then

$$\tilde{\rho}_n(F) = \sum_k \nu_{nk}(e^{i\boldsymbol{\theta}_{nk}} F) \le \sum_k \nu_{nk}(F') = \rho_n(F')$$

implies that $\limsup_{n\to\infty} \tilde{\rho}_n(F) \le \limsup_{n\to\infty} \rho_n(F') \le \rho(F') \le \rho(F) + \delta$. Consequently, we arrive at the inequality $\limsup_{n\to\infty} \tilde{\rho}_n(F) \le \rho(F)$. In the same vein, one can show that $\liminf_{n\to\infty} \tilde{\rho}_n(G) \ge \rho(G)$ for any set G which is open and bounded away from $\mathbf{1}$. Hence $\tilde{\rho}_n \Rightarrow_1 \rho$ by Proposition 2.1.

Next, we turn to demonstrate that both ρ_n and $\tilde{\rho}_n$ bring out the tantamount quantities in (4-5), which asserts that the quadratic form in (iv) output by them is unchanged on $\mathbb{Z}^d$. Any index n considered below is always sufficiently large. In the case $j = \ell$, we have the estimate

$$\int_{\mathscr{U}_\epsilon} (\Im s_j)^2 \, d\tilde{\rho}_n(s) = \sum_{k=1}^{k_n} \int_{e^{i\boldsymbol{\theta}_{nk}} \mathscr{U}_\epsilon} (\Im(e^{-i\theta_{nkj}} s_j))^2 \, d\nu_{nk}(s)$$

$$\le \sum_{k=1}^{k_n} \int_{\mathscr{U}_{2\epsilon}} (\Im(e^{-i\theta_{nkj}} s_j))^2 \, d\nu_{nk}(s),$$

where we express $\boldsymbol{\theta}_{nk} = (\theta_{nk1}, \ldots, \theta_{nkd})$. The inequality

$$(4\text{-}9) \qquad (\Im(e^{-i\theta_{nkj}} s_j))^2 \le (\Im s_j)^2 + 2|\sin \theta_{nkj}| \, |\Im s_j| + \sin^2(\theta_{nkj})$$

will help us to continue with the arguments. Consideration given to the first term
on the right-hand side of (4-9) gives

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \sum_{k=1}^{k_n} \int_{\mathcal{U}_{2\epsilon}} (\Im s_j)^2 \, dv_{nk}(s) = a_{jj}$$

by the hypothesis, while analyzing the second term results in

$$\sum_{k=1}^{k_n} |\sin \theta_{nkj}| \cdot \int_{\mathcal{U}_{2\epsilon}} |\Im s_j| \, dv_{nk}(s) \leq \left(\sum_{k=1}^{k_n} \sin^2 \theta_{nkj} \right)^{1/2} \left(\int_{\mathcal{U}_{2\epsilon}} (\Im s_j)^2 \, d\rho_n(s) \right)^{1/2}$$

by the Cauchy–Schwarz inequality. The simple fact $\sin^2 x \leq 2(1 - \cos x)$ for $x \in \mathbb{R}$
and the assumption (4-8) immediately yield that

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \sum_{k=1}^{k_n} \int_{\mathcal{U}_{2\epsilon}} |\sin \theta_{nkj}| \, |\Im s_j| \, dv_{nk}(s) = 0$$

and

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \sum_{k=1}^{k_n} \int_{\mathcal{U}_{2\epsilon}} \sin^2 \theta_{nkj} \, dv_{nk}(s) = 0.$$

These estimates then lead to $\lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{\mathcal{U}_{\epsilon}} (\Im s_j)^2 \, d\tilde{\rho}_n(s) \leq a_{jj}$. Employ-
ing the opposite inclusion $\mathcal{U}_{\epsilon/2} \subset e^{-i\theta_{nk}} \mathcal{U}_{\epsilon}$ and inequality

$$(\Im(e^{-i\theta_{nkj}} s_j))^2 \geq (\Im s_j)^2 - 2(1 - \cos \theta_{nkj}) - 2|\sin \theta_{nkj}| \, |\Im s_j| - \sin^2 \theta_{nkj}$$

allows us to obtain $\lim_{\epsilon \to 0} \liminf_{n \to \infty} \int_{\mathcal{U}_{\epsilon}} (\Im s_j)^2 \, d\rho_n(s) \geq a_{jj}$.

Now we deal with the situation $j \neq \ell$ in (4-5). After careful consideration of all
available information, the focus is only needed on the summand

$$\sum_{k=1}^{k_n} \int_{e^{i\theta_{nk}} \mathcal{U}_{\epsilon}} (\Im s_j)(\Im s_\ell) \, dv_{nk}(s)$$

and justifying that

(4-10) $$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \sum_{k=1}^{k_n} \int_{(e^{i\theta_{nk}} \mathcal{U}_{\epsilon}) \triangle \mathcal{U}_{\epsilon}} |\Im s_j| \, |\Im s_\ell| \, dv_{nk}(s) = 0,$$

where $\triangle$ denotes the operation of symmetric difference on sets. Using the fact
$(e^{i\theta_{nk}} \mathcal{U}_{\epsilon}) \triangle \mathcal{U}_{\epsilon} \subset \left\{ \frac{\epsilon}{2} \leq \|\arg s\| \leq 2\epsilon \right\}$ and mimicking the proof of (4-7) allow us to
get (4-10) done.   □

Recall from (2-1) that the push-forward measure $\tau W^{-1} \in \mathcal{M}_{\mathbb{T}^d}^1$ of a given $\tau \in \mathcal{M}_{\mathbb{R}^d}^0$
via the wrapping map $W(\boldsymbol{x}) = e^{i\boldsymbol{x}}$ from $\mathbb{R}^d$ to $\mathbb{T}^d$ is defined as

(4-11) $$(\tau W^{-1})(B) = \tau(\{\boldsymbol{x} \in \mathbb{R}^d : e^{i\boldsymbol{x}} \in B\}), \quad B \in \mathcal{B}_{\mathbb{T}^d}.$$

A useful and frequently used result regarding W is the change-of-variables formula stating that a Borel function f on $\mathbb{T}^d$ belongs to $L^1(\tau W^{-1})$ if and only if the function $\boldsymbol{x} \mapsto f(e^{i\boldsymbol{x}})$ lies in $L^1(\tau)$, and the equation

$$(4\text{-}12) \qquad \int_{\mathbb{T}^d} f(\boldsymbol{s}) \, d(\tau W^{-1})(\boldsymbol{s}) = \int_{\mathbb{R}^d} f(e^{i\boldsymbol{x}}) \, d\tau(\boldsymbol{x})$$

holds in either case. In the following, we will translate conditions introduced in Section 2D accordingly via the wrapping map W.

Proposition 4.4. *Assume that $\{\tau_n\}$ and τ are in $\mathcal{M}_{\mathbb{R}^d}^{\mathbf{0}}$ satisfying Condition 2.3 (or Condition 2.2). Then Condition 4.1, as well as Condition 2.5, applies to $\rho_n = \tau_n W^{-1}$ and $\rho = 1_{\mathbb{T}^d \setminus \{1\}} \tau W^{-1}$. Moreover, τ_n and ρ_n determine the same quadratic form on $\mathbb{Z}^d$, in particular, the same matrix in (IV) and (iv), respectively.*

Proof. Suppose that Condition 2.3 holds for τ_n and τ, and let $\boldsymbol{A} = (a_{j\ell})$ represent the matrix produced by these measures in (IV). According to Proposition 4.2, we shall only elaborate that Condition 4.1 is applicable to ρ_n and ρ.

That $\rho_n \Rightarrow_1 \rho$ is clearly valid according to the continuous mapping theorem, Proposition 2.1. It remains to argue that in Condition 4.1(iv), ρ_n also outputs $\boldsymbol{A}$. The simple observation that $e^{i\boldsymbol{x}} \in \mathcal{U}_\epsilon$ if and only if $\boldsymbol{x}$ belongs to the set

$$(4\text{-}13) \qquad \widetilde{\mathcal{V}_\epsilon} = \bigcup_{\boldsymbol{p} \in \mathbb{Z}^d} \{\boldsymbol{x} + 2\pi \boldsymbol{p} : \boldsymbol{x} \in \mathcal{V}_\epsilon\}$$

and formula (4-12) help us to establish that for $j, \ell = 1, \dots, d$,

$$\int_{\mathcal{U}_\epsilon} (\Im s_j)(\Im s_\ell) \, d\rho_n(\boldsymbol{s}) = \int_{\mathbb{T}^d} 1_{\mathcal{U}_\epsilon}(\boldsymbol{s})(\Im s_j)(\Im s_\ell) \, d\rho_n(\boldsymbol{s})$$

$$= \int_{\mathbb{R}^d} 1_{\mathcal{U}_\epsilon}(e^{i\boldsymbol{x}})(\Im e^{ix_j})(\Im e^{ix_\ell}) \, d\tau_n(\boldsymbol{x})$$

$$= \int_{\widetilde{\mathcal{V}_\epsilon}} \sin(x_j) \sin(x_\ell) \, d\tau_n(\boldsymbol{x}).$$

Observe next that we have $\widetilde{\mathcal{V}_\epsilon} \setminus \mathcal{V}_\epsilon = \bigcup_{m=1}^d \mathcal{D}_{\epsilon m}$, where $\mathcal{D}_{\epsilon m} = \widetilde{\mathcal{V}_\epsilon} \cap \{|x_m| \geq \pi\}$, provided that $\epsilon < \pi$. If we temporarily impose the requirement $\sigma_m(\partial \mathcal{D}_{\epsilon m}) = 0$ for some $m \in \{1, \dots, d\}$, then the weak convergence $\sigma_{nm} \Rightarrow \sigma_m$ implies that

$$\limsup_{n \to \infty} \int_{\mathcal{D}_{\epsilon m}} |\sin(x_j) \sin(x_\ell)| \, d\tau_n = \limsup_{n \to \infty} \int_{\mathcal{D}_{\epsilon m}} |\sin(x_j) \sin(x_\ell)| \cdot \frac{1 + x_m^2}{x_m^2} \, d\sigma_{nm}$$

$$= \int_{\mathcal{D}_{\epsilon m}} |\sin(x_j) \sin(x_\ell)| \cdot \frac{1 + x_m^2}{x_m^2} \, d\sigma_m \xrightarrow[\epsilon \to \infty]{} 0.$$

This, along with facts $x - \sin x = o(|x|^2)$ as $|x| \to 0$ and $x_j^2 \in L^1(\sigma_j)$, leads to

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{\mathcal{U}_\epsilon} (\Im s_j)(\Im s_\ell) \, d\rho_n(s) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{\mathcal{V}_\epsilon} \sin(x_j) \sin(x_\ell) \, d\tau_n(x)$$

$$= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{\mathcal{V}_\epsilon} x_j \, x_\ell \, d\tau_n(x).$$

The same arguments also elaborate the identity

$$\lim_{\epsilon \to 0} \liminf_{n \to \infty} \int_{\mathcal{U}_\epsilon} (\Im s_j)(\Im s_\ell) \, d\rho_n(s) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \int_{\mathcal{V}_\epsilon} x_j \, x_\ell \, d\tau_n(x).$$

Apparently, the selection of ϵ does not vary the validity of these identities, and so we have established that ρ_n generates the matrix A in (iv) as well. $\qquad\square$

Measures in $\mathcal{M}_{\mathbb{R}^d}^{\mathbf{0}}$ can be wrapped either clockwise or counterclockwise (see equation (4-11)) in all variables, and consequences, such as Proposition 4.4, are not affected at all by this slight change. As a matter of fact, it is also the case when one wraps some variables counterclockwise and others clockwise. Without loss of generality, we shall use the simplest circumstance, the 2-dimensional opposite wrapping map $W_2^\star : \mathbb{R}^2 \to \mathbb{T}^2$, $(x_1, x_2) \mapsto (e^{ix_1}, e^{-ix_2})$, to illustrate these features. The following result is merely an easy consequence of the continuous mapping theorem, the relations $(\tau(W_2^\star)^{-1})(B) = (\tau W_2^{-1})(B^\star) = (\tau W_2^{-1})^\star(B)$ for any $B \in \mathscr{B}_{\mathbb{T}^2}$, and Proposition 4.4.

Proposition 4.5. *If $\{\rho_n\}$ and ρ in $\mathcal{M}_{\mathbb{T}^2}^{\mathbf{1}}$ fulfill* Condition 4.1, *then*

(1) $\rho_n^\star \Rightarrow_1 \rho^\star$, *and*

(2) *for any $\boldsymbol{p} = (p_1, p_2) \in \mathbb{Z}^2$, denoting by $\boldsymbol{p}^\star = (p_1, -p_2)$, we have*

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \int_{\mathcal{U}_\epsilon} \langle \boldsymbol{p}, \Im s \rangle^2 \, d\rho_n^\star(s) = Q(\boldsymbol{p}^\star) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \int_{\mathcal{U}_\epsilon} \langle \boldsymbol{p}, \Im s \rangle^2 \, d\rho_n^\star(s).$$

Particularly, if $\{\tau_n\}$ and τ in $\mathcal{M}_{\mathbb{R}^2}^{\mathbf{0}}$ satisfy Condition 2.3 (*or* Condition 2.2), *then statements* (1) *and* (2) *above apply to* $\rho_n^\star = \tau_n(W_2^\star)^{-1}$ *and* $\rho^\star = \tau(W_2^\star)^{-1}$.

We add one remark on item (2) of the preceding proposition: if $Q(\boldsymbol{p}) = \langle A\boldsymbol{p}, \boldsymbol{p} \rangle$, then $Q(\boldsymbol{p}^\star) = \langle A^{\mathrm{op}} \boldsymbol{p}, \boldsymbol{p} \rangle$, where the (i, j)-entry of A^{op} is $(-1)^{i+j} A_{ij}$.

5. Limit theorems and bi-free multiplicative Lévy triplet

5A. *Bi-free multiplicative Lévy–Khintchine representation.* Thanks to Proposition 4.2, one can correlate the quantity L_{12} and measures λ_j given in the formulas (2-16) with the matrix A and measure $\rho \in \mathcal{M}_{\mathbb{T}^2}^{\mathbf{1}}$ determined by (4-1), (4-2), and (4-3). Therefore, instead of working with the parametrization $(\boldsymbol{\gamma}, \lambda_1, \lambda_2, L_{12})$

for measures in $\mathcal{ID}(\boxtimes\boxtimes) \cap \mathscr{P}^{\times}_{\mathbb{T}^2}$, one may take another parametrization $(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)$ (with the same $\boldsymbol{\gamma}$) having the following properties with $d = 2$:

(5-1) $\boldsymbol{\gamma} \in \mathbb{T}^d$, $\boldsymbol{A}$ is a positive semidefinite $d \times d$ symmetric matrix, and
     ρ is a positive measure on $\mathbb{T}^d$ so that $\rho(\{\mathbf{1}\}) = 0$ and $\|\mathbf{1} - \Re \boldsymbol{s}\| \in L^1(\rho)$.

We shall refer to $(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)$ as the *bi-free multiplicative Lévy triplet* of the measure in $\mathcal{ID}(\boxtimes\boxtimes) \cap \mathscr{P}^{\times}_{\mathbb{T}^2}$ having (bi-)free Σ-transforms presented in (2-16), and signify this measure by $\nu^{(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)}_{\boxtimes\boxtimes}$ to comply with the correspondence. This triplet plays the role of the classical multiplicative Lévy triplet. We will clarify this in more details in Corollary 5.3, where limit theorems between classical and bi-free multiplicative convolutions are examined and in Section 6, where the commutativity of diagram (1-5) is verified.

A measure ν belongs to $\mathcal{ID}(\boxtimes\boxtimes^{\mathrm{op}}) \cap \mathscr{P}^{\times}_{\mathbb{T}^2}$ if and only if $\nu^{\star} \in \mathcal{ID}(\boxtimes\boxtimes) \cap \mathscr{P}^{\times}_{\mathbb{T}^2}$ by (2-3) and Theorem 3.12. Thus, we shall denote by $\nu^{(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)}_{\boxtimes\boxtimes^{\mathrm{op}}}$ the measure ν satisfying $\nu^{\star} = \nu^{(\boldsymbol{\gamma}^{\star}, \boldsymbol{A}^{\mathrm{op}}, \rho^{\star})}_{\boxtimes\boxtimes}$ and refer to $(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)$ as its *opposite bi-free multiplicative Lévy triplet*. Passing to analytic transforms, we have

$$\Sigma^{\mathrm{op}}_{\nu}(z, w) = \Sigma_{\nu^{(\boldsymbol{\gamma}^{\star}, \boldsymbol{A}^{\mathrm{op}}, \rho^{\star})}_{\boxtimes\boxtimes}}(z, 1/w) \quad \text{for } (z, w) \in \mathbb{D} \times (\mathbb{T} \cup \{0\})^c.$$

In terms of notations introduced above, we reformulate the basic limit theorem [13, Theorem 3.4] on the bi-free multiplicative convolution, including statements for $\boxtimes\boxtimes^{\mathrm{op}}$.

Theorem 5.1. *Given an infinitesimal array $\{\nu_{nk}\} \subset \mathscr{P}^{\times}_{\mathbb{T}^2}$ and a sequence $\{\boldsymbol{\xi}_n\} \subset \mathbb{T}^2$, define $\boldsymbol{\gamma}_n$ as in* (2-13). *The following are equivalent.*

(1) *The sequence*

(5-2) $\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes \nu_{n1} \boxtimes\boxtimes \cdots \boxtimes\boxtimes \nu_{nk}$

converges weakly to some $\nu_{\boxtimes\boxtimes} \in \mathscr{P}^{\times}_{\mathbb{T}^2}$.

(2) *The sequence*

(5-3) $\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes \,^{\mathrm{op}} \nu_{n1} \boxtimes\boxtimes \,^{\mathrm{op}} \cdots \boxtimes\boxtimes \,^{\mathrm{op}} \nu_{nk}$

converges weakly to some $\nu_{\boxtimes\boxtimes^{\mathrm{op}}} \in \mathscr{P}^{\times}_{\mathbb{T}^2}$.

(3) *The measure $\rho_n = \sum_{k=1}^{k_n} \mathring{\nu}_{nk}$ satisfies Condition 4.1 (or Condition 2.5) with $d = 2$ and $\lim_n \boldsymbol{\gamma}_n = \boldsymbol{\gamma}$ exists.*

If (1)–(3) *hold, then $\nu_{\boxtimes\boxtimes} = \nu^{(\boldsymbol{\gamma}, \boldsymbol{A}, \rho)}_{\boxtimes\boxtimes}$ and $(\nu_{\boxtimes\boxtimes^{\mathrm{op}}})^{\star} = \nu^{(\boldsymbol{\gamma}^{\star}, \boldsymbol{A}^{\mathrm{op}}, \rho^{\star})}_{\boxtimes\boxtimes}$, where ρ and $\boldsymbol{A}$ are as in* Condition 4.1 *and* Proposition 4.2, *respectively.*

Proof. We only prove (2)⇔(3). With $\{\boldsymbol{b}_{nk}\}$ defined in (2-12), the equality

$$\exp\left[i \int_{\mathcal{U}_{\theta}} (\arg \boldsymbol{s}) \, d\nu^{\star}_{nk}(\boldsymbol{s})\right] = \boldsymbol{b}^{\star}_{nk}$$

shows that $(\nu_{nk}^\star)^\circ(B) = \nu_{nk}^\star(\boldsymbol{b}_{nk}^\star B) = \nu_{nk}(\boldsymbol{b}_{nk}B^\star) = \mathring{\nu}_{nk}(B^\star) = (\mathring{\nu}_{nk})^\star(B)$ for any Borel set B on $\mathbb{T}^2$. Since the operations $\star$ and $\circ$ acting on ν_{nk} are interchangeable in order, we adopt the notation $\mathring{\nu}_{nk}^\star$ instead of $(\nu_{nk}^\star)^\circ = (\mathring{\nu}_{nk})^\star$ if no confusions arise.

Item (2) holds if and only if

$$\delta_{\boldsymbol{\xi}_n^\star} \boxtimes\boxtimes \nu_{n1}^\star \boxtimes\boxtimes \cdots \boxtimes\boxtimes \nu_{nk}^\star = (\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes {}^{\mathrm{op}} \nu_{n1} \boxtimes\boxtimes {}^{\mathrm{op}} \cdots \boxtimes\boxtimes {}^{\mathrm{op}} \nu_{nk})^\star \Rightarrow (\nu_{\boxtimes\boxtimes^{\mathrm{op}}})^\star$$

according to (2-3). This happens if and only if Condition 4.1 applies to the measure $\sum_{k=1}^n \mathring{\nu}_{nk}^\star = \left(\sum_{k=1}^n \mathring{\nu}_{nk}\right)^\star$ and the vector

$$\boldsymbol{\gamma}_n^\star = \boldsymbol{\xi}_n^\star \exp\left[i \sum_{k=1}^{k_n}\left(\arg \boldsymbol{b}_{nk}^\star + \int_{\mathbb{T}^2}(\Im s) \, d\mathring{\nu}_{nk}^\star(s)\right)\right]$$

has a limit by Theorem 2.4. Then Proposition 4.5 proves the equivalence (2)⇔(3) and the last assertion. □

Recall from [16] that a measure ν in $\mathcal{ID}(\circledast)$ has no nontrivial $\circledast$-idempotent factor if and only if its characteristic function takes the form

$$\hat{\nu}(\boldsymbol{p}) = \boldsymbol{\gamma}^{\boldsymbol{p}} \exp\left(-\tfrac{1}{2}\langle A\boldsymbol{p}, \, \boldsymbol{p}\rangle + \int_{\mathbb{T}^d}(s^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im s\rangle) \, d\rho(s)\right), \quad \boldsymbol{p} \in \mathbb{Z}^d$$

for certain triplet $(\boldsymbol{\gamma}, A, \rho)$ fulfilling the conditions in (5-1). We shall write $\nu_\circledast^{(\boldsymbol{\gamma}, A, \rho)}$ for this measure, and refer to ρ and $(\boldsymbol{\gamma}, A, \rho)$ as its multiplicative Lévy measure and multiplicative Lévy triplet, respectively. A known phenomenon is that a $\circledast$-infinitely divisible distribution on $\mathbb{T}^d$ has unique $\boldsymbol{\gamma}$ and A, but may have various Lévy measures. For example, it was pointed out in [6] that when $d = 1$, one has $\nu_\circledast^{(1,0,\pi\delta_i)} = \nu_\circledast^{(1,0,\pi\delta_{-i})}$. The uniqueness of multiplicative Lévy measures will be more systematically studied in [10]. This observation leads to the following definition.

Definition 5.2. Let ρ be a multiplicative Lévy measure on $\mathbb{T}^d$. The symbol $\mathcal{L}(\rho)$ stands for the collection of those measures serving as multiplicative Lévy measures for $\nu_\circledast^{(\boldsymbol{1},\boldsymbol{0},\rho)}$.

The following corollary, derived from Theorem 2.4 and [10], supplies the link between classical and bi-free limit theorems on the bi-torus. The attentive reader can also notice that the hypothesis $\mathcal{L}(\rho) = \{\rho\}$ is redundant in the implication (2) ⇒ (1).

Corollary 5.3. *Let $\{\nu_{nk}\} \subset \mathscr{P}_{\mathbb{T}^2}$ be infinitesimal, $\{\boldsymbol{\xi}_n\} \subset \mathbb{T}^2$, and $(\boldsymbol{\gamma}, A, \rho)$ be a multiplicative Lévy triplet such that $\mathcal{L}(\rho) = \{\rho\}$. With the notations in (2-13) and (2-15) for $d = 2$, the following statements are equivalent:*

(1) $\delta_{\boldsymbol{\xi}_n} \circledast \nu_{n1} \circledast \cdots \circledast \nu_{nk_n} \Rightarrow \nu_\circledast^{(\boldsymbol{\gamma}, A, \rho)}$.

(2) $\delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes \nu_{n1} \boxtimes\boxtimes \cdots \boxtimes\boxtimes \nu_{nk_n} \Rightarrow \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}, A, \rho)}$.

(3) $\lim_{n\to\infty} \boldsymbol{\gamma}_n = \boldsymbol{\gamma}$, $\rho_n \Rightarrow_1 \rho$, and

$$\lim_{\epsilon\to 0}\ \limsup_{n\to\infty} \int_{\mathscr{U}_\epsilon} \langle \boldsymbol{p}, \Im s\rangle^2\, d\rho_n(s) = \langle A\boldsymbol{p}, \boldsymbol{p}\rangle$$
$$= \lim_{\epsilon\to 0}\ \liminf_{n\to\infty} \int_{\mathscr{U}_\epsilon} \langle \boldsymbol{p}, \Im s\rangle^2\, d\rho_n(s), \quad \boldsymbol{p}\in\mathbb{Z}^2.$$

The one-dimensional multiplicative limit theorem, which was pointed out in the remark to [23, Corollary 4.2], is a consequence of Corollary 5.3, e.g., by considering product measures.

Corollary 5.4. *Let $\{v_{nk}\}\subset \mathscr{P}_\mathbb{T}$ be infinitesimal, $\{\xi_n\}\subset\mathbb{T}$, and (γ, a, ρ) be a multiplicative Lévy triplet such that $\mathcal{L}(\rho) = \{\rho\}$. With the notations in (2-13) and (2-15) for $d = 1$, the following statements are equivalent*:

(1) $\delta_{\xi_n}\circledast v_{n1}\circledast\cdots\circledast v_{nk_n} \Rightarrow v_{\circledast}^{(\gamma,a,\rho)}$.

(2) $\delta_{\xi_n}\boxtimes v_{n1}\boxtimes\cdots\boxtimes v_{nk_n} \Rightarrow v_{\boxtimes}^{(\gamma,a,\rho)}$.

(3) $\lim_{n\to\infty}\gamma_n = \gamma$, $\rho_n\Rightarrow_1\rho$, and

$$\lim_{\epsilon\to 0}\ \limsup_{n\to\infty} \int_{\mathscr{U}_\epsilon} (\Im s)^2 d\rho_n(s) = a = \lim_{\epsilon\to 0}\ \liminf_{n\to\infty}\int_{\mathscr{U}_\epsilon} (\Im s)^2 d\rho_n(s).$$

Apparently, the nonuniqueness of Lévy measures is the exclusive obstruction for reaching the equivalence of limit theorems, thus complementing the work of Chistyakov and Götze [9, Theorems 2.3 and 2.4].

The goal of this section is to provide an alternative description for the Σ-transform of a measure in $\mathcal{ID}(\boxtimes\boxtimes)\cap\mathscr{P}_{\mathbb{T}^2}^\times$ in terms of its bi-free multiplicative Lévy triplets. To achieve this, we need some basics. For any $p\in\mathbb{N}$, the function

$$\mathcal{K}_p(s) = \frac{s^p - 1 - ip\Im s}{1 - \Re s}$$

is continuous on $\mathbb{T}$ and equal to $-p^2$ at $s = 1$.

Lemma 5.5. *For any $p\in\mathbb{N}$, we have $\|\Im\mathcal{K}_p\|_\infty \le p^3$ and $\int_{-\pi}^{\pi}\mathcal{K}_p(e^{i\theta})\,d\theta = -2p\pi$.*

Proof. In the following arguments, we shall make use of the basic formula:

$$(5\text{-}4) \qquad\qquad \frac{1 - \cos(p\theta)}{1 - \cos\theta} = e^{i(1-p)\theta}\sum_{j,k=0}^{p-1} e^{i(j+k)\theta}.$$

Clearly, we have $\Im\mathcal{K}_1\equiv 0$. If $\|\Im\mathcal{K}_p\|_\infty\le p^3$ for some $p\ge 2$, then for $s\ne 1$, the inequality $|(1-\Re s^p)/(1-\Re s)|\le p^2$ following from (5-4) implies that

$$|\Im\mathcal{K}_{p+1}(s)| = \left|\Im s^p - \Im\mathcal{K}_p(s) + \frac{1-\Re s^p}{1-\Re s}\cdot\Im s\right| \le 1 + p^3 + p^2 \le (p+1)^3.$$

By induction, this finishes the proof of the first assertion. To prove the second assertion, it suffices to show $\int_{-\pi}^{\pi} (1 - \cos(p\theta))/(1 - \cos\theta)\, d\theta = 2p\pi$, which can be easily obtained by using (5-4) again. $\qquad\square$

Fix a measure $\nu \in \mathscr{P}_{\mathbb{T}^2}^{\times} \cap \mathcal{ID}(\boxtimes\boxtimes)$, and suppose that its (bi-)free Σ-transforms are given as in (2-16). Due to the integral representations, both u_1 and u_2 are analytic in $\Omega = (\mathbb{C}\backslash\mathbb{T}) \cup \{\infty\}$ and u is analytic in Ω^2. Hence the function

$$U_\nu(z, w) = \frac{zw}{1 - zw} u(z, w) - \frac{z}{1 - z} u_1(z) - \frac{w}{1 - w} u_2(w)$$

is analytic in Ω^2. If $\nu \in \mathcal{ID}(\boxtimes\boxtimes^{\mathrm{op}}) \cap \mathscr{P}_{\mathbb{T}^2}^{\times}$, then we define

$$U_\nu^{\mathrm{op}}(z, w) = U_{\nu^\star}(z, 1/w),$$

which is also an analytic function in Ω^2.

When $\nu \in \mathcal{ID}(\boxtimes\boxtimes^{\mathrm{op}}) \cap \mathscr{P}_{\mathbb{T}^2}^{\times}$, one can obtain an equivalent formula for U_ν in terms of the bi-free multiplicative Lévy triplet, which we call the *bi-free multiplicative Lévy–Khintchine representation*. Note that we acquire the following proof with the help of limit theorems, in spite of the algebraic nature of the statement. Also, it is simpler even though there exists an algebraic proof.

Theorem 5.6. *Letting $\nu = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}, A, \rho)}$, we have*

(5-5) $\qquad U_\nu(z, w) = \dfrac{iz}{1 - z} \arg\gamma_1 + \dfrac{iw}{1 - w} \arg\gamma_2 - N_\nu(z, w) + P_\nu(z, w),$

where

$$N_\nu(z, w) = \frac{a_{11}}{2} \cdot \frac{z(1 + z)}{(1 - z)^2} + \frac{a_{12}\, zw}{(1 - z)(1 - w)} + \frac{a_{22}}{2} \cdot \frac{w(1 + w)}{(1 - w)^2}$$

and

$$P_\nu(z, w) = (1 - z)(1 - w) \sum_{p=0}^{\infty} \left[\int_{\mathbb{T}^2} (s^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im s \rangle)\, d\rho(s) \right] z^{p_1} w^{p_2}.$$

Further, letting $\tilde{\nu} = \nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(\boldsymbol{\gamma}, A, \rho)}$, we have $U_{\tilde{\nu}}^{\mathrm{op}}(z, w) = U_{\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma^\star}, A^{\mathrm{op}}, \rho^\star)}}(z, 1/w).$

Proof. First of all, using Remark 2.6 and the function

$$f(z, w, \boldsymbol{s}) = \frac{zw(1 - s_1)(1 - s_2)}{(1 - zs_1)(1 - ws_2)} - \frac{z(1 + zs_1)(1 - \Re s_1)}{(1 - z)(1 - zs_1)} - \frac{w(1 + ws_2)(1 - \Re s_2)}{(1 - w)(1 - ws_2)},$$

one can rewrite U_ν as

$$U_\nu(z, w) = \frac{iz}{1 - z} \arg\gamma_1 + \frac{iw}{1 - w} \arg\gamma_2 + \lim_{n\to\infty} \int_{\mathbb{T}^2} f(z, w, \boldsymbol{s})\, d\rho_n(\boldsymbol{s}).$$

Below, $r > 0$ is taken so that $\rho(\partial \mathscr{U}_r) = 0$. The continuity of $s \mapsto f(z, w, s)$ on $\mathbb{T}^2$ for any fixed $(z, w) \in \mathbb{D}^2$ and Proposition 2.1 imply that

$$\lim_{n \to \infty} \int_{\mathbb{T}^2 \setminus \mathscr{U}_r} f(z, w, s) \, d\rho_n = \int_{\mathbb{T}^2 \setminus \mathscr{U}_r} f(z, w, s) \, d\rho.$$

Using dominated convergence theorem, we arrive at

$$\lim_{r \to 0} \lim_{n \to \infty} \int_{\mathbb{T}^2 \setminus \mathscr{U}_r} f(z, w, s) \, d\rho_n = \int_{\mathbb{T}^2} f(z, w, s) \, d\rho.$$

On the other hand, thanks to weak convergence $\lambda_{nj} = (1 - \Re s_j)\rho_n \Rightarrow \lambda_j, \; j = 1, 2$, we see that for $\xi \in \mathbb{D}$,

$$\limsup_{n \to \infty} \left| \int_{\mathscr{U}_r} \frac{1 + \xi s_j}{1 - \xi s_j} (1 - \Re s_j) \, d\rho_n - \frac{a_{jj}}{2} \cdot \frac{1 + \xi}{1 - \xi} \right|$$

$$\leq \limsup_{n \to \infty} \left(\left| \lambda_{nj}(\mathscr{U}_r) - \tfrac{1}{2} a_{jj} \right| \left| \frac{1 + \xi}{1 - \xi} \right| + \int_{\mathscr{U}_r} \left| \frac{1 + \xi s_j}{1 - \xi s_j} - \frac{1 + \xi}{1 - \xi} \right| d\lambda_{nj} \right)$$

$$\leq \frac{2}{(1 - |\xi|)^2} \limsup_{n \to \infty} \left(\left| \lambda_{nj}(\mathscr{U}_r) - \tfrac{1}{2} a_{jj} \right| + \int_{\mathscr{U}_r} |1 - s_j| \, d\lambda_{nj} \right) \xrightarrow[r \to \infty]{} 0.$$

Similarly, one can show that

$$\lim_{r \to 0} \lim_{n \to \infty} \int_{\mathbb{T}^2 \setminus \mathscr{U}_r} \frac{(1 - s_1)(1 - s_2)}{(1 - z s_1)(1 - w s_2)} \, d\rho_n = \int_{\mathbb{T}^2} \frac{(1 - s_1)(1 - s_2)}{(1 - z s_1)(1 - w s_2)} \, d\rho$$

and

$$\lim_{r \to 0} \limsup_{n \to \infty} \left| \int_{\mathscr{U}_r} \frac{(1 - s_1)(1 - s_2)}{(1 - z s_1)(1 - w s_2)} \, d\rho_n + \frac{a_{12}}{(1 - z)(1 - w)} \right| = 0.$$

Next, we shall make use of the equation (2-17). After some algebraic manipulations, we come to the result

$$\lim_{n \to \infty} \int_{\mathbb{T}^2} f(z, w, s) \, d\rho_n(s) = -N(z, w) + (1 - z)(1 - w) \int_{\mathbb{T}^2} \tilde{f}(z, w, s) \, d\rho(s),$$

where

$$\tilde{f}(z, w, s)$$
$$= \frac{1}{(1 - z s_1)(1 - w s_2)} - \frac{1}{(1 - z)(1 - w)} - \frac{i z \Im s_1}{(1 - z)^2(1 - w)} - \frac{i w \Im s_2}{(1 - z)(1 - w)^2}.$$

Lastly, the use of the power series expansion

$$\xi_j (1 - \xi_1)^{-2} (1 - \xi_2)^{-1} = \sum_{p \geq 0} p_j \, \xi_1^{p_1} \xi_2^{p_2} \quad \text{for } \xi_1, \xi_2 \in \mathbb{D},$$

allows us to get

$$\int_{\mathbb{T}^2} \tilde{f}(z, w, s) \, d\rho(s) = \int_{\mathbb{T}^2} \sum_{p \geq 0} (s^p - 1 - i \langle p, \Im s \rangle) \, z^{p_1} w^{p_2} \, d\rho(s).$$

The operations of integration and summation performed above are interchangeable due to Lemma 5.5. Indeed, one can utilize the uniform convergence of the summands to obtain

$$\int \sum_{p \geq 0} (s_j^{p_j} - 1 - i p_j \Im s_j) \, z^{p_1} w^{p_2} \, d\rho = \sum_{p \geq 0} \int \mathcal{K}_{p_j}(s) \, d\lambda_j \, z^{p_1} w^{p_2}$$

$$= \sum_{p \geq 0} \int (s_j^{p_j} - 1 - i p_j \Im s_j) \, d\rho \, z^{p_1} w^{p_2}$$

and similarly

$$\int \sum_{p \geq 0} (s_1^{p_1} - 1)(s_2^{p_2} - 1) \, z^{p_1} w^{p_2} \, d\rho = \sum_{p \geq 0} \int (s_1^{p_1} - 1)(s_2^{p_2} - 1) \, d\rho \, z^{p_1} w^{p_2}.$$

Putting all these findings together yields the desired result.

According to the definition of $\tilde{\nu}$, which is characterized by $(\tilde{\nu})^\star = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}^\star, A^{\mathrm{op}}, \rho^\star)}$, the last assertion follows from the definition of U_ν^{op}. $\qquad\square$

Performing the power series expansion to $N_\nu(z, w)$ in Theorem 5.6 further yields that

$$\frac{U_\nu(z, w)}{(1 - z)(1 - w)}$$
$$= \sum_{p=0}^{\infty} \left[i \langle \boldsymbol{p}, \arg \boldsymbol{\gamma} \rangle - \tfrac{1}{2} \langle A\boldsymbol{p}, \boldsymbol{p} \rangle + \int_{\mathbb{T}^2} (s^{\boldsymbol{p}} - 1 - i \langle \boldsymbol{p}, \Im s \rangle) \, d\rho(s) \right] z^{p_1} w^{p_2},$$

which offers the generating series for the exponent of the characteristic function

$$(5\text{-}6) \qquad \hat{\nu}(\boldsymbol{p}) = \boldsymbol{\gamma}^{\boldsymbol{p}} \exp\left[-\tfrac{1}{2} \langle A\boldsymbol{p}, \boldsymbol{p} \rangle + \int_{\mathbb{T}^2} (s^{\boldsymbol{p}} - 1 - i \langle \boldsymbol{p}, \Im s \rangle) \, d\rho(s) \right], \qquad \boldsymbol{p} \in \mathbb{Z}^2$$

of a measure in $\mathcal{ID}(\mathbb{T}^2, \circledast) \cap \mathscr{P}_{\mathbb{T}^2}^\times$ (cf. Corollary 5.3.)

5B. *Limit theorems via wrapping transformations.* We next present the limit theorems through the wrapping transformations.

Theorem 5.7. *Let $(\boldsymbol{v}, A, \tau)$ be a triplet satisfying (2-10) with $d = 2$, and let $\{\mu_{nk}\} \subset \mathscr{P}_{\mathbb{R}^2}$ be an infinitesimal triangular array and $\{\boldsymbol{v}_n\}$ a sequence of vectors in $\mathbb{R}^2$. If the sequence in (1-2) converges weakly to $\mu_{\boxplus\boxplus}^{(\boldsymbol{v}, A, \tau)}$, then the sequences in (5-2) and (5-3) generated by $\nu_{nk} = \mu_{nk} W^{-1}$ and $\boldsymbol{\xi}_n = e^{i\boldsymbol{v}_n}$ converge weakly to $\nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}, A, \rho)}$ and $\nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(\boldsymbol{\gamma}, A, \rho)}$, respectively, where*

$$(5\text{-}7) \qquad\qquad\qquad \rho = 1_{\mathbb{T}^2 \setminus \{\mathbf{1}\}} (\tau W^{-1})$$

and

$$(5\text{-}8) \qquad \gamma = \exp\left[i\,\boldsymbol{v} + i\int_{\mathbb{R}^2}\left(\sin(\boldsymbol{x}) - \frac{\boldsymbol{x}}{1 + \|\boldsymbol{x}\|^2}\right)d\tau(\boldsymbol{x})\right].$$

Proof. Before carrying out the main proof, let us record some properties instantly inferred from the hypotheses for the later utilization. Because the index n goes to infinity ultimately, it is always big enough whenever mentioned in the proof.

Firstly, observe that v_{nk} belongs to $\mathscr{P}_{\mathbb{T}^2}^{\times}$ and the vector

$$(5\text{-}9) \qquad \boldsymbol{\theta}_{nk} = \sum_{\boldsymbol{p}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \int_{\mathscr{V}_\theta} \boldsymbol{x}\, d\mu_{nk}(\boldsymbol{x} + 2\pi\,\boldsymbol{p})$$

satisfies $\lim_{n\to\infty} \max_k \|\boldsymbol{\theta}_{nk}\| = 0$ by the infinitesimality of $\{\mu_{nk}\}$. Secondly, following the notations in (2-6) and (4-13), an application of (4-12) gives

$$\boldsymbol{v}_{nk} + \boldsymbol{\theta}_{nk} = \int_{\mathbb{R}^d} 1_{\{e^{i\boldsymbol{x}} : \boldsymbol{x}\in\widetilde{\mathscr{V}_\theta}\}}(e^{i\boldsymbol{x}})\arg(e^{i\boldsymbol{x}})\,d\mu_{nk}(\boldsymbol{x})$$

$$= \int_{\mathbb{T}^d} 1_{\{\boldsymbol{s}:\|\arg \boldsymbol{s}\|<\theta\}}(\boldsymbol{s})\arg(\boldsymbol{s})\,d\nu_{nk}(\boldsymbol{s}) = \int_{\mathscr{U}_\theta}\arg(\boldsymbol{s})\,d\nu_{nk}(\boldsymbol{s}).$$

This and equation (2-12) provide us with the relations $\arg \boldsymbol{b}_{nk} = \boldsymbol{v}_{nk} + \boldsymbol{\theta}_{nk}$ and $d\mathring{\nu}_{nk}(\boldsymbol{s}) = d(\mathring{\mu}_{nk}W^{-1})(e^{i\boldsymbol{\theta}_{nk}}\boldsymbol{s})$ as for any $B \in \mathscr{B}_{\mathbb{T}^2}$, we have

$$(\mathring{\mu}_{nk}W^{-1})(B) = \mu_{nk}(\{e^{i\boldsymbol{x}} \in e^{i\boldsymbol{v}_{nk}}B\}) = \nu_{nk}(e^{i\boldsymbol{v}_{nk}}B) = \mathring{\nu}_{nk}(e^{-\boldsymbol{\theta}_{nk}}B).$$

Except for the beforehand mentioned results, the array $\{\boldsymbol{\theta}_{nk}\}$ in (5-9) also fulfills the condition in (4-8), which will play a dominant role in our arguments. Its proof, provided below, is based on the convergence $\tau_n = \sum_k \mathring{\mu}_{nk} \Rightarrow_{\mathbf{0}} \tau$ and some estimates. For convenience, denote $\boldsymbol{\theta}_{nk} = (\theta_{nk1}, \theta_{nk2})$ and $\boldsymbol{v}_{nk} = (v_{nk1}, v_{nk2})$, and consider the positive Borel measure $\varrho_{nk}(\cdot) = \sum_{\boldsymbol{p}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \mathring{\mu}_{nk}(\cdot + 2\pi\,\boldsymbol{p})1_{\overline{\mathscr{V}_{2\theta}}}$ on the closure of $\mathscr{V}_{2\theta}$. The infinitesimality of $\{\mathring{\mu}_{nk}\}$ indicates that $\lim_{n\to\infty} \max_{1\le k\le k_n} \varrho_{nk}(\mathscr{V}_{2\theta}) = 0$ and the assumption $\theta \in (0, 1)$ in (2-5) shows that

$$\varrho_{nk}(\mathscr{V}_\theta - \boldsymbol{v}_{nk}) \le \sum_{\boldsymbol{p}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \mathring{\mu}_{nk}(\mathscr{V}_{2\theta} + 2\pi\,\boldsymbol{p}) = \mathring{\mu}_{nk}(\widetilde{\mathscr{V}_{2\theta}}\setminus\mathscr{V}_{2\theta}).$$

This, together with Cauchy–Schwarz inequality, enables us to obtain

$$\sum_{k=1}^{k_n} \theta_{nkj}^2 = \sum_{k=1}^{k_n}\left(\int_{\mathscr{V}_\theta - \boldsymbol{v}_{nk}}(x_j + v_{nkj})\,d\varrho_{nk}(\boldsymbol{x})\right)^2$$

$$\le \sum_{k=1}^{k_n}\varrho_{nk}(\mathscr{V}_\theta - \boldsymbol{v}_{nk})\int_{\mathscr{V}_\theta - \boldsymbol{v}_{nk}}(x_j + v_{nkj})^2\,d\varrho_{nk}(\boldsymbol{x})$$

$$\le \theta^2\tau_n(\widetilde{\mathscr{V}_{2\theta}}\setminus\mathscr{V}_{2\theta}) \max_{1\le k\le k_n} \varrho_{nk}(\mathscr{V}_{2\theta}).$$

Since $\widetilde{\mathscr{V}_{2\theta}} \backslash \mathscr{V}_{2\theta}$ is bounded away from $\mathbf{1} \in \mathbb{T}^2$, the relation $\tau_n \Rightarrow_{\mathbf{0}} \tau$ leads us to $\limsup_n \tau_n(\widetilde{\mathscr{V}_{2\theta}} \backslash \mathscr{V}_{2\theta}) < \infty$. Thus, we are able to conclude that $\sum_{k=1}^{k_n} \theta_{nkj}^2 \to 0$ as $n \to \infty$, yielding (4-8) by the inequality $1 - \cos x \leq \frac{x^2}{2}$ on $\mathbb{R}$.

After these preparations, we are ready to present the proof of the theorem. Since (1-2) converges weakly, τ_n meets Condition 2.3, and thus $\rho_n = \tau_n W^{-1}$ satisfies Condition 2.5 according to Proposition 4.4. Then Proposition 4.3 consequently yields that Condition 2.5 also applies to $\tilde{\rho}_n = \sum_{k=1}^{k_n} \mathring{\nu}_{nk}$.

To finish the proof, we just need to verify (2-14) due to Theorem 5.1. The existence of the limit in (2-8) implies that the vector

$$\boldsymbol{E}_n = i \left[\boldsymbol{v}_n + \sum_{k=1}^{k_n} (\boldsymbol{v}_{nk} + \int \sin(\boldsymbol{x}) \, d\mathring{\mu}_{nk}) \right]$$

also has a limit when $n \to \infty$. Indeed, the limit $-i \lim_{n \to \infty} \boldsymbol{E}_n$ disintegrates into the sum of that in (2-8) and

$$\lim_{n \to \infty} \sum_{k=1}^{k_n} \int_{\mathbb{R}^2} \left(\sin(\boldsymbol{x}) - \frac{\boldsymbol{x}}{1 + \|\boldsymbol{x}\|^2} \right) d\mathring{\mu}_{nk}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \left(\sin(\boldsymbol{x}) - \frac{\boldsymbol{x}}{1 + \|\boldsymbol{x}\|^2} \right) d\tau(\boldsymbol{x}).$$

The validity of the equality displayed above is just because of that the integrand is $O(\|\boldsymbol{x}\|^3)$ as $\|\boldsymbol{x}\| \to 0$ and the function $\min\{1, \|\boldsymbol{x}\|^2\}$ is τ-integrable.

In order to go further, we analyze the difference

$$\left(\arg \boldsymbol{b}_{nk} + \int_{\mathbb{T}^2} \Im \boldsymbol{s} \, d\mathring{\nu}_{nk}(\boldsymbol{s}) \right) - \left(\boldsymbol{v}_{nk} + \int_{\mathbb{R}^2} \sin(\boldsymbol{x}) \, d\mathring{\mu}_{nk}(\boldsymbol{x}) \right),$$

which, along with the help of equation $\int \sin(\boldsymbol{x}) \, d\mathring{\mu}_{nk} = \int \Im(e^{i\theta_{nk}} \boldsymbol{s}) \, d\mathring{\nu}_{nk}$, becomes

$$(5\text{-}10) \quad (\boldsymbol{\theta}_{nk} - \sin \boldsymbol{\theta}_{nk}) + \sin(\boldsymbol{\theta}_{nk}) \int_{\mathbb{T}^2} (1 - \Re \boldsymbol{s}) \, d\mathring{\nu}_{nk}(\boldsymbol{s}) + (1 - \cos \boldsymbol{\theta}_{nk}) \int_{\mathbb{T}^2} \Im \boldsymbol{s} \, d\mathring{\nu}_{nk}(\boldsymbol{s}).$$

Using the elementary inequality

$$(5\text{-}11) \qquad\qquad\qquad |x - \sin x| \leq 1 - \cos x, \quad |x| \leq \frac{\pi}{4},$$

we see from the established result that

$$\sum_{k=1}^{k_n} |\theta_{nkj} - \sin \theta_{nkj}| \leq \sum_{k=1}^{k_n} (1 - \cos \theta_{nkj}) \to 0 \quad \text{as } n \to \infty.$$

For the second term in (5-10), $\lambda_{nj} = (1 - \Re s_j) \tilde{\rho}_n \Rightarrow \lambda_j \in \mathscr{M}_{\mathbb{T}^2}$ yields that

$$\sum_{k=1}^{k_n} \left| \sin(\theta_{nkj}) \int_{\mathbb{T}^2} (1 - \Re s_j) \, d\mathring{\nu}_{nk}(\boldsymbol{s}) \right| \leq \left(\max_{1 \leq k \leq k_n} |\sin \theta_{nkj}| \right) \lambda_{nj}(\mathbb{T}^2) \xrightarrow[n \to \infty]{} 0.$$

As for the last term, we then have

$$\sum_{k=1}^{k_n}(1-\cos\theta_{nkj})\left|\int_{\mathbb{T}^2}(\Im s_j)\,d\mathring{v}_{nk}(s)\right|\le\sum_{k=1}^{k_n}(1-\cos\theta_{nkj})\xrightarrow[n\to\infty]{}0.$$

Consequently, we have arrived at that the limit in (2-14) exists and equals the vector in (5-8). $\qquad\square$

The employment of the wrapping limit theorem with $\boldsymbol{v}_n=\boldsymbol{0}$ gives the following identically distributed limit theorem, which is the bi-free version of [6, Theorem 3.9].

Corollary 5.8. *Let $(\boldsymbol{v},\boldsymbol{A},\tau)$ be a triplet satisfying* (2-10) *with $d=2$, $\{\mu_n\}$ a sequence in $\mathscr{P}_{\mathbb{R}^2}$, and $\{k_n\}$ a strictly increasing sequence in $\mathbb{N}$. If $\mu_n^{\boxplus\boxplus k_n}\Rightarrow\mu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{A},\tau)}$, then $(\mu_n W^{-1})^{\boxtimes\boxtimes k_n}\Rightarrow v_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$ and $(\mu_n W^{-1})^{\boxtimes\boxtimes^{op}k_n}\Rightarrow v_{\boxtimes\boxtimes^{op}}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$, where $\boldsymbol{\gamma}$ and ρ are as in* Theorem 5.7.

Example 5.9. Given a 2×2 real matrix $\boldsymbol{A}=(a_{ij})\ge0$ with $a_{11}\ge a_{22}>0$, consider planar probability measures $\mu_n=\frac{1}{4}(\delta_{\boldsymbol{\alpha}_n}+\delta_{-\boldsymbol{\alpha}_n}+\delta_{\boldsymbol{\beta}_n}+\delta_{-\boldsymbol{\beta}_n})$, where $\boldsymbol{\alpha}_n=(\sqrt{2\det\boldsymbol{A}},0)/\sqrt{na_{22}}$ and $\boldsymbol{\beta}_n=(\sqrt{2}a_{12},\sqrt{2}a_{22})/\sqrt{na_{22}}$. Clearly, $\mathring{\mu}_n=\mu_n$ for all n and $\tau_n:=n\mu_n\Rightarrow_0 0$ as $n\to\infty$. Furthermore, for any $\theta>0$, if n is large enough, then $\int_{\mathscr{V}_\theta}x_j^2\,d\tau_n=a_{jj}$ and $\int_{\mathscr{V}_\theta}x_1x_2\,d\tau_n=a_{12}$. Hence the identically distributed limit theorem introduced in Section 2D indicates that $\mu_n^{\boxplus\boxplus n}$ converges weakly to $\mu_{\boxplus\boxplus}^{(\boldsymbol{0},\boldsymbol{A},0)}$, which is known as the bi-free Gaussian distribution with bi-free Lévy triplet $(\boldsymbol{0},\boldsymbol{A},0)$. For the measures

$$v_n=\mu_n W^{-1}=\tfrac{1}{4}(\delta_{e^{i\boldsymbol{\alpha}_n}}+\delta_{e^{-i\boldsymbol{\alpha}_n}}+\delta_{e^{i\boldsymbol{\beta}_n}}+\delta_{e^{-i\boldsymbol{\beta}_n}})\in\mathscr{P}_{\mathbb{T}^2},$$

a direct verification or an application of Corollary 5.8 shows that $v_n^{\boxtimes\boxtimes n}\Rightarrow v_{\boxtimes\boxtimes}^{(\boldsymbol{1},\boldsymbol{A},0)}$ and $v_n^{\boxtimes\boxtimes^{op}n}\Rightarrow v_{\boxtimes\boxtimes^{op}}^{(\boldsymbol{1},\boldsymbol{A},0)}$. Analogically, $v_{\boxtimes\boxtimes}=v_{\boxtimes\boxtimes}^{(\boldsymbol{1},\boldsymbol{A},0)}$ is called the *bi-free multiplicative Gaussian distribution* with Lévy triplet $(\boldsymbol{1},\boldsymbol{A},0)$. Note that the component $P_{v_{\boxtimes\boxtimes}}$ in the representation (5-5), called the bi-free multiplicative compound Poisson part (see Example 5.10), vanishes.

Example 5.10. Given any $r>0$ and $\mu\in\mathscr{P}_{\mathbb{R}^2}$, let $\mu_n=(1-r/n)\,\delta_{\boldsymbol{0}}+r/n\mu$, $\tau_n=n\mu_n$, and $\tau=r1_{\mathbb{R}^2\setminus\{\boldsymbol{0}\}}\mu$. A straightforward verification reveals that Condition 2.3 applies to τ_n, τ, and $Q\equiv0$. Hence [11, Theorem 5.6] shows that $\mu_n^{\boxplus\boxplus n}$ converges weakly to the so-called bi-free compound Poisson distribution $\mu_{\boxplus\boxplus}^{(\boldsymbol{v},\boldsymbol{0},\tau)}$ with rate r and jump distribution μ, where $\boldsymbol{v}=r\int\boldsymbol{x}(1+\|\boldsymbol{x}\|^2)^{-1}d\mu$. Applying Corollary 5.8 shows that

$$\left((1-r/n)\,\delta_{\boldsymbol{1}}+r/n(\mu W^{-1})\right)^{\boxtimes\boxtimes n}\Rightarrow v_{\boxtimes\boxtimes}^{(e^{i\boldsymbol{u}},\boldsymbol{0},\rho)},$$

as well as $(\mu_n W^{-1})^{\boxtimes\boxtimes^{op}n}\Rightarrow v_{\boxtimes\boxtimes^{op}}^{(e^{i\boldsymbol{u}},\boldsymbol{0},\rho)}$, where

$$\rho=r1_{\mathbb{T}^2\setminus\{\boldsymbol{1}\}}(\mu W^{-1})\quad\text{and}\quad\boldsymbol{u}=r\int\sin\boldsymbol{x}\,d\mu.$$

Analogous to the planar case, we refer to measures of the form $\nu_{\boxtimes\boxtimes} = \nu_{\boxtimes\boxtimes}^{(e^{iu},\mathbf{0},rv)}$, where $r > 0$, $v \in \mathscr{P}_{\mathbb{T}^2}$ with $v(\{\mathbf{1}\}) = 0$, and $\mathbf{u} = r\int\Im s\,dv$ as the *bi-free multiplicative compound Poisson distribution* with rate r and jump distribution v. In (5-5), we have the bi-free Gaussian component $N_{\nu_{\boxtimes\boxtimes}} \equiv 0$.

5C. *Limit theorems for identically distributed case.* The following is a special case of the limit theorem in the context of identically distributed random vectors on the bi-torus.

Proposition 5.11. *Let $\rho_n = k_n v_n$, where $\{v_n\} \subset \mathscr{P}_{\mathbb{T}^2}^{\times}$ and $\{k_n\} \subset \mathbb{N}$ with $k_1 < k_2 < \dots$. If ρ_n satisfies* Condition 4.1 *(or* Condition 2.5*) and the limit*

$$\mathbf{v} = \lim_{n\to\infty}\int_{\mathbb{T}^2}\Im\boldsymbol{\xi}\,d\rho_n(\boldsymbol{\xi})$$

exists, then $v_n^{\boxtimes\boxtimes k_n} \Rightarrow \nu_{\boxtimes\boxtimes}^{(e^{i\mathbf{v}},\mathbf{A},\rho)}$ and $v_n^{\boxtimes\boxtimes^{\mathrm{op}} k_n} \Rightarrow \nu_{\boxtimes\boxtimes^{\mathrm{op}}}^{(e^{i\mathbf{v}},\mathbf{A},\rho)}$, where ρ and $\mathbf{A}$ are as in Condition 4.1 *and* Proposition 4.2, *respectively.*

Proof. Let $h : \mathbb{T}^2 \to (-\pi,\pi]^2$ be the inverse of the wrapping map $W(\mathbf{x}) = e^{i\mathbf{x}}$ restricted to $(-\pi,\pi]^2$, namely, $h(\boldsymbol{\xi}) = \arg\boldsymbol{\xi}$. Further let $\mu_n = v_n h^{-1} \in \mathscr{P}_{\mathbb{R}^2}$ and $\tau = \rho h^{-1} \in \mathscr{M}_{\mathbb{R}^2}^{\mathbf{0}}$, whose supports are all contained in $[-\pi,\pi]^2$. Then $v_n = \mu_n W^{-1}$, and $\tau_n = \rho_n h^{-1} \Rightarrow_{\mathbf{0}} \tau$ by the continuous mapping theorem. Also, (4-2) and (4-12) show that $\min\{1, \|\mathbf{x}\|^2\} \in L^1(\tau)$. One can utilize (5-11) to justify

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\int_{\mathscr{U}_\epsilon}(\arg s_j)(\arg s_\ell)\,d\rho_n(\mathbf{s}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{\mathscr{U}_\epsilon}(\arg s_j)(\arg s_\ell)\,d\rho_n(\mathbf{s}).$$

On the other hand, one has the equation $\int_{\mathscr{V}_\epsilon}x_j\,x_\ell\,d\tau_n = \int_{\mathscr{U}_\epsilon}(\arg s_j)(\arg s_\ell)\,d\rho_n$ by the change-of-variables formula (4-12), which implies that τ_n satisfies (IV) of Condition 2.3. Ultimately, observe that

$$\int_{\mathbb{R}^2}\frac{\mathbf{x}}{1 + \|\mathbf{x}\|^2}\,d\tau_n(\mathbf{x}) = \int_{\mathbb{T}^2}\Im\mathbf{s}\,d\rho_n(\mathbf{s}) + \int_{\mathbb{R}^2}\left(\frac{\mathbf{x}}{1 + \|\mathbf{x}\|^2} - \sin(\mathbf{x})\right)d\tau_n(\mathbf{x})$$

has a limit when $n \to \infty$ owing to $\mathbf{x}/(1 + \|\mathbf{x}\|^2) - \sin(\mathbf{x}) = O(\|\mathbf{x}\|^3)$ as $\|\mathbf{x}\| \to 0$ and $\min\{1, \|\mathbf{x}\|^2\} \in L^1(\tau)$. Thus, $\mu_n^{\boxplus\boxplus k_n} \Rightarrow \mu_{\boxplus\boxplus}^{(\mathbf{v},\mathbf{A},\tau)}$ by [11, Theorem 5.6], and so we accomplish the proof by Corollary 5.8. $\qquad\square$

We shall also consider the rotated probabilities

$$d\tilde{v}_n(\mathbf{s}) = dv_n(\boldsymbol{\omega}_n\,\mathbf{s})$$

associated with a sequence $\{v_n\} \subset \mathscr{P}_{\mathbb{T}^2}^{\times}$, where $\boldsymbol{\omega}_n = (\omega_{n1}, \omega_{n2}) \in \mathbb{T}^2$ has components

$$\omega_{nj} = \int_{\mathbb{T}^2}s_j\,dv_n(\mathbf{s})\bigg/\left|\int_{\mathbb{T}^2}s_j\,dv_n(\mathbf{s})\right|.$$

Through this sort of rotated distributions, we next present the bi-freely identically distributed limit theorem, which is the bi-free analog of [6, Proposition 3.6].

Theorem 5.12. *The following are equivalent for a sequence $\{\nu_n\}$ in $\mathscr{P}_{\mathbb{T}^2}^\times$ and a strictly increasing sequence $\{k_n\}$ in $\mathbb{N}$.*

(1) *The sequence $\nu_n^{\boxtimes\boxtimes k_n}$ converges weakly to some $\nu_{\boxtimes\boxtimes} \in \mathscr{P}_{\mathbb{T}^2}^\times$.*

(2) *The sequence $\nu_n^{\boxtimes\boxtimes^{\mathrm{op}} k_n}$ converges weakly to some $\nu_{\boxtimes\boxtimes^{\mathrm{op}}} \in \mathscr{P}_{\mathbb{T}^2}^\times$.*

(3) *Condition 4.1 holds for $\rho_n = k_n\tilde{\nu}_n$ and the limit $\boldsymbol{\gamma} = \lim_{n\to\infty}(\omega_{n1}^{k_n}, \omega_{n2}^{k_n})$ exists in $\mathbb{T}^2$.*

If (1)–(3) hold, then $\nu_{\boxtimes\boxtimes} = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},\boldsymbol{A},\rho)}$ and $(\nu_{\boxtimes\boxtimes^{\mathrm{op}}})^\star = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma}^\star,\boldsymbol{A}^{\mathrm{op}},\rho^\star)}$, where ρ and $\boldsymbol{A}$ are respectively as in Condition 4.1 *and* Proposition 4.2.

Proof. Only the equivalence (1) $\Leftrightarrow$ (3) needs a proof, which relies on Proposition 4.3. First of all, the weak convergence of $\nu_n^{\boxtimes\boxtimes k_n}$ to $\nu \in \mathscr{P}_{\mathbb{T}^2}^\times$ yields that $\tilde{\nu}_n \Rightarrow \delta_{(1,1)}$. Indeed, $m_{1,0}(\nu_n)^{k_n} = \Sigma_{\nu_n^{(1)}}(0)^{-k_n} \to \Sigma_{\nu^{(1)}}(0)^{-1} = m_{1,0}(\nu)$ shows that

$$\omega_{n1}^{k_n} \to m_{1,0}(\nu)/|m_{1,0}(\nu)| := \omega_1.$$

Since $\Sigma_{\tilde{\nu}_n^{(1)}}(z)^{k_n} = \omega_{n1}^{k_n}\Sigma_{\nu_n^{(1)}}(z)^{k_n} \to \omega_1\Sigma_{\nu^{(1)}}(z) = \Sigma_{\tilde{\nu}^{(1)}}(z)$ uniformly for z in a neighborhood of zero as $n \to \infty$ by [4, Proposition 2.9], it follows from [4, Lemma 2.7] that $\tilde{\nu}_n^{(1)} \Rightarrow \delta_1$. In the same vein, one can obtain $\tilde{\nu}_n^{(2)} \Rightarrow \delta_1$, giving the desired weak convergence. On other hand, the $\mathscr{M}_{\mathbb{T}^2}^1$-weak convergence of $\rho_n = k_n\tilde{\nu}_n$ also implies $\tilde{\nu}_n \Rightarrow \delta_{\mathbf{1}}$. In other words, $\tilde{\nu}_n$ is infinitesimal if assertion (1) or (3) holds.

Write $\nu_n^{\boxtimes\boxtimes k_n} = \delta_{\boldsymbol{\xi}_n} \boxtimes\boxtimes \tilde{\nu}_n^{\boxtimes\boxtimes k_n}$ and consider measures $d\mathring{\tilde{\nu}}_n(s) = d\tilde{\nu}_n(\tilde{\boldsymbol{b}}_n s)$, where $\boldsymbol{\xi}_n = \boldsymbol{\omega}_n^{k_n}$ and $\tilde{\boldsymbol{b}}_n = \exp[i\int_{\mathscr{U}_\theta}(\arg s)\, d\tilde{\nu}_n]$. Then as indicated in Theorem 5.1, assertion (1) holds if and only if $\rho_n' = k_n\mathring{\tilde{\nu}}_n$ satisfies Condition 2.5 and $\boldsymbol{\gamma}_n = \boldsymbol{\xi}_n\exp(i\boldsymbol{E}_n)$ has a finite limit, where $\boldsymbol{E}_n = k_n[\arg\tilde{\boldsymbol{b}}_n + \int(\Im s)\, d\mathring{\tilde{\nu}}_n]$. The infinitesimality of $\tilde{\nu}_n$ reveals that $\boldsymbol{\theta}_n = (\theta_{n1}, \theta_{n2}) \to \mathbf{0}$ as $n \to \infty$, where

$$(5\text{-}12) \qquad \theta_{nj} = \arg\tilde{b}_{nj} = \int_{\mathscr{U}_\theta}\arg s_j\, d\tilde{\nu}_n(s).$$

This simple fact will be often utilized in the following proof, and all the indices n considered below are sufficiently large. The equivalence of Condition 2.5 and Condition 4.1 is employed below as well. With a view toward applying Proposition 4.3 to ρ_n and ρ_n', we shall prove that $\lim_{n\to\infty} k_n\|\boldsymbol{\theta}_n\|^2 = 0$.

Now, we argue that $\rho_n'(\cdot) = k_n\mathring{\tilde{\nu}}_n(\cdot) = \rho_n(e^{i\boldsymbol{\theta}_n}\cdot)$ satisfies Condition 2.5 if the same condition applies to $\rho_n = k_n\tilde{\nu}_n$. Let $\lambda_{nj} = (1 - \Re s_j)\rho_n$. Using the fact

$$(5\text{-}13) \qquad \int_{\mathbb{T}^2}\Im s_j\, d\tilde{\nu}_n(s) = 0, \quad j = 1, 2,$$

we have

$$k_n\theta_{nj} = \int_{\mathcal{U}_\theta} \arg s_j \, d\rho_n(s) - \int_{\mathbb{T}^2} \Im s_j \, d\rho_n(s)$$

$$= \int_{\mathbb{T}^2} \frac{\arg s_j - \Im s_j}{1 - \Re s_j} \, d\lambda_{nj}(s) - \int_{\mathbb{T}^2 \setminus \mathcal{U}_\theta} \arg s_j \, d\rho_n(s).$$

Then the continuity of $s \mapsto (\arg s - \Im s)/(1 - \Re s)$ on $\mathbb{T}$ implies that

$$\limsup_{n \to \infty} k_n |\theta_{nj}| < \infty,$$

and so $\lim_{n \to \infty} k_n \|\boldsymbol{\theta}_n\|^2 = 0$. Thus, ρ_n' meets Condition 2.5 by Proposition 4.3.

Conversely, suppose that ρ_n' satisfies Condition 2.5. We first rewrite (5-12) as $\arg \tilde{b}_{nj} = \int_{\tilde{\boldsymbol{b}}_n^{-1}\mathcal{U}_\theta} (\arg s_j + \arg \tilde{b}_{nj}) \, d\mathring{\tilde{v}}_n$. On the other hand, the integral in (5-13) can be decomposed into the sum

$$\Im\tilde{b}_{nj} - (\Im\tilde{b}_{nj}) \int_{\mathbb{T}^2} (1 - \Re s_j) \, d\mathring{\tilde{v}}_n(s) - (1 - \Re\tilde{b}_{nj}) \int_{\mathbb{T}^2} (\Im s_j) \, d\mathring{\tilde{v}}_n(s) + \int_{\mathbb{T}^2} (\Im s_j) \, d\mathring{\tilde{v}}_n(s).$$

Since $\tilde{b}_{nj} = \cos\theta_{nj} + i \sin\theta_{nj}$, some simple calculations allow us to obtain

$$\theta_{nj} = \arg\tilde{b}_{nj} - \int_{\mathbb{T}^2} (\Im s_j) \, d\tilde{v}_n(s) = B_{nj} + R_{nj},$$

where

$$R_{nj} = (\theta_{nj} - \sin\theta_{nj}) + (1 - \cos\theta_{nj}) \int (\Im s_j) \, d\mathring{\tilde{v}}_n$$

and

$$B_{nj} = -\theta_{nj}\mathring{\tilde{v}}_n(\mathbb{T}^2 \setminus \tilde{\boldsymbol{b}}_n^{-1}\mathcal{U}_\theta) - \int_{\mathbb{T}^2 \setminus \tilde{\boldsymbol{b}}_n^{-1}\mathcal{U}_\theta} (\arg s_j) \, d\mathring{\tilde{v}}_n(s)$$

$$+ \sin(\theta_{nj}) \int_{\mathbb{T}^2} (1 - \Re s_j) \, d\mathring{\tilde{v}}_n(s) + \int_{\mathbb{T}^2} \frac{\arg s_j - \Im s_j}{1 - \Re s_j} \, d(1 - \Re s_j)\mathring{\tilde{v}}_n(s).$$

Note that sets $\mathbb{T}^2 \setminus \tilde{\boldsymbol{b}}_n^{-1}\mathcal{U}_\theta$ are uniformly bounded away from $\mathbf{1}$, whence we see that $\limsup_{n \to \infty} k_n |B_{nj}| < \infty$ by the $\mathcal{M}_{\mathbb{T}^2}^1$-convergence assumption of ρ_n'. Then $|R_{nj}| \leq |\theta_{nj}|^3 + |\theta_{nj}|^2$ leads to

$$\limsup_{n \to \infty} k_n |\theta_{nj}| [1 - |\theta_{nj}| - |\theta_{nj}|^2] \leq \limsup_{n \to \infty} k_n |B_{nj}| < \infty.$$

We thus obtain $\limsup_n k_n |\theta_{nj}| < \infty$, and so $\lim_n k_n \|\boldsymbol{\theta}_n\|^2 = 0$. Consequently, ρ_n satisfies Condition 2.5 by Proposition 4.3 again.

Finally, by using (5-13), one can express components of $\boldsymbol{E}_n = (E_{n1}, E_{n2})$ as

$$E_{nj} = k_n\theta_{nj} + k_n(\Im\tilde{b}_{nj}^{-1}) \int_{\mathbb{T}^2} (\Re s_j) \, d\tilde{v}_n(s)$$

$$= k_n(\theta_{nj} - \sin\theta_{nj}) + \sin(\theta_{nj}) \int_{\mathbb{T}^2} (1 - \Re s_j) \, d\rho_n(s).$$

As noted above that ρ_n meets Condition 2.5 if and only if so does ρ_n' and that $\lim_{n\to\infty} k_n|\theta_{nj}|^2 = 0$ in either case. Consequently, we have shown $\lim_{n\to\infty} E_{nj} = 0$ for $j = 1, 2$ and arrived at $\boldsymbol{\gamma} = \lim_{n\to\infty} \boldsymbol{\gamma}_n$ if (1) or (3) holds.   □

Remark 5.13. In spite of $\delta_{-1}^{\boxtimes\boxtimes 2n} = \delta_1$, $2n\delta_{-1}$ fails to converge in $\mathscr{M}_{\mathbb{T}^2}^1$. This example demonstrates that in Theorem 5.12, the rotated probabilities $\tilde{\nu}_n$ are a necessary medium in the convergence criteria of the bi-free multiplicative limit theorem. For the same inference, the converse statement of Proposition 5.11 does not hold, yet it does in the additive setting [11, Theorem 5.6].

6. Homomorphisms between infinitely divisible distributions

This section will provide explanations for the diagram (1-5). The bijection

$$\Lambda : \mathcal{ID}(*) \to \mathcal{ID}(\boxplus\boxplus)$$

was already defined in [11], specifically,

$$\Lambda(\mu_*^{(\boldsymbol{v},A,\tau)}) = \mu_{\boxplus\boxplus}^{(\boldsymbol{v},A,\tau)}.$$

If $\nu = \mu_*^{(\boldsymbol{v},A,\tau)} W^{-1}$, then (2-9) and (4-12) show that

$$\hat{\nu}(\boldsymbol{p}) = \exp\left[i\langle \boldsymbol{p}, \boldsymbol{v}\rangle - \tfrac{1}{2}\langle A\boldsymbol{p}, \boldsymbol{p}\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle \boldsymbol{p},\boldsymbol{x}\rangle} - 1 - \frac{i\langle \boldsymbol{p}, \boldsymbol{x}\rangle}{1 + \|\boldsymbol{x}\|^2}\right)d\tau(\boldsymbol{x})\right]$$

$$= \boldsymbol{\gamma}^{\boldsymbol{p}}\exp\left[-\tfrac{1}{2}\langle A\boldsymbol{p}, \boldsymbol{p}\rangle + \int_{\mathbb{T}^d}(s^{\boldsymbol{p}} - 1 - i\langle \boldsymbol{p}, \Im s\rangle)\,d\rho(s)\right],$$

where ρ and $\boldsymbol{\gamma}$ are respectively given in (5-7) and (5-8). Putting it differently, the wrapping map induces a homomorphism $W_* : \mathcal{ID}(*) \to \mathcal{ID}(\circledast)$ satisfying

(6-1) $$W_*(\mu_*^{(\boldsymbol{v},A,\tau)}) = \nu_{\circledast}^{(\boldsymbol{\gamma},A,\rho)}.$$

Motivated by (6-1), we analogously define $W_{\boxplus\boxplus} : \mathcal{ID}(\boxplus\boxplus) \to \mathcal{ID}(\boxtimes\boxtimes)$ as

$$W_{\boxplus\boxplus}(\nu_{\boxplus\boxplus}^{(\boldsymbol{v},A,\tau)}) = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},A,\rho)},$$

where $\boldsymbol{\gamma}$ and ρ are given as before. It was shown in Theorem 5.7 that the weak convergence of (1-2) to some $\nu_{\boxplus\boxplus}^{(\boldsymbol{v},A,\tau)}$ implies that equation (1-3) converges weakly to $W_{\boxplus\boxplus}(\nu_{\boxplus\boxplus}^{(\boldsymbol{v},A,\tau)})$.

For the last ingredient $\Gamma : \mathcal{ID}(\boxtimes\boxtimes) \to \mathcal{ID}(\circledast)$, recall from Proposition 3.9 that $\boxtimes\boxtimes$-idempotent elements also belong to $\mathcal{ID}(\circledast)$. Also, [6, Definition 3.3] introduced a homomorphism $\Gamma_1 : \mathcal{ID}(\mathbb{T}, \boxtimes) \to \mathcal{ID}(\mathbb{T}, \circledast)$ (which was denoted by Γ therein), which leads to the following definition.

Definition 6.1. Let $\nu \in \mathcal{ID}(\boxtimes\boxtimes)$. Define $\Gamma(\nu) = \nu_{\circledast}^{(\boldsymbol{\gamma},A,\rho)}$ if $\nu = \nu_{\boxtimes\boxtimes}^{(\boldsymbol{\gamma},A,\rho)}$. For $\nu \in \mathscr{P}_{\mathbb{T}^2}\backslash\mathscr{P}_{\mathbb{T}^2}^{\times}$, define $\Gamma(\nu) = \nu$ if $\nu = P\boxtimes\boxtimes(\kappa_c \times \delta_1)$, and let $\Gamma(\nu) = \mathrm{m} \times \Gamma_1(\nu^{(2)})$ if $\nu = \mathrm{m} \times \nu^{(2)}$ and $\Gamma(\nu) = \Gamma_1(\nu^{(1)}) \times \mathrm{m}$ if $\nu = \nu^{(1)} \times \mathrm{m}$.

One can check that $\Gamma : \mathcal{ID}(\boxtimes\boxtimes) \to \mathcal{ID}(\circledast)$ is a homomorphism and that the diagram (1-5) commutes. The latter result comes from the definition, while the former one requires convolution identities in Section 3. For example, if we write $\mu = P\boxtimes\boxtimes(\kappa_c \times \delta_1)$ and $\nu = \mathrm{m} \times \nu^{(2)}$ with $\nu^{(2)} \in \mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$, then we have $\mu \boxtimes\boxtimes \nu = P\boxtimes\boxtimes (\mathrm{m} \times \nu^{(2)}) = \mathrm{m} \times \mathrm{m}$, where the last equality can be confirmed by the use of (3-6) and computing moments. On the other hand,

$$\Gamma(\mu) \circledast \Gamma(\nu) = \mu \circledast (\mathrm{m} \times \Gamma_1(\nu^{(2)})) = P \circledast (\mathrm{m} \times \Gamma_1(\nu^{(2)})) = \mathrm{m} \times \mathrm{m},$$

where the last equality is again obtained by computing moments. Consequently, we arrive at $\Gamma(\mu \boxtimes\boxtimes \nu) = \Gamma(\mu) \circledast \Gamma(\nu)$.

This map Γ is neither injective nor surjective as we have

$$\nu_{\circledast}^{((1,0),\mathbf{0},\pi\delta_{(i,0)})} = \nu_{\circledast}^{((1,0),\mathbf{0},\pi\delta_{(-i,0)})}$$

and $P \circledast (\mu \times \delta_1)$ lies in $\mathcal{ID}(\circledast)\backslash\Gamma(\mathcal{ID}(\boxtimes\boxtimes))$ for any $\mu \in \mathcal{ID}(\mathbb{T}, \circledast)\backslash\{\kappa_c : c \in \mathbb{D}\cup\mathbb{T}\}$. Further, Γ is not weakly continuous. More strongly, we prove the following.

Proposition 6.2. (1) *The restriction of Γ_1 to the set $\mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$ has no weakly continuous extension to $\mathcal{ID}(\boxtimes)$.*

(2) *The restriction of Γ to the set $\mathcal{ID}(\boxtimes\boxtimes) \cap \mathscr{P}_{\mathbb{T}^2}^{\times}$ has no weakly continuous extension to $\mathcal{ID}(\boxtimes\boxtimes)$.*

Proof. Since $\Gamma(\mu^{(1)} \times \mu^{(2)}) = \Gamma_1(\mu^{(1)}) \times \Gamma_1(\mu^{(2)})$ for $\mu^{(1)}, \mu^{(2)} \in \mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$, assertion (2) follows immediately from (1).

Suppose that $\Gamma_1^0 := \Gamma_1|_{\mathcal{ID}(\boxtimes)\cap\mathscr{P}_{\mathbb{T}}^{\times}}$ has a weakly continuous extension $\tilde{\Gamma}_1$ to $\mathcal{ID}(\boxtimes)$. Observe that $\kappa_c \in \mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$ and $\Gamma_1^0(\kappa_c) = \kappa_c$ for any $c \in (\mathbb{D}\cup\mathbb{T})\backslash\{0\}$. The latter identity is shown below. From the moments $m_p(\kappa_c) = c^p$ for $p \in \mathbb{N}$, the formula

$$\Sigma_{\kappa_c}(z) = \frac{1}{c} = \frac{1}{c/|c|}\exp\left[(-\log|c|)\int_{\mathbb{T}^{\times}}\frac{1+sz}{1-sz}(1-\Re s)\frac{d\mathrm{m}(s)}{1-\Re s}\right]$$

yields that κ_c has $(c/|c|, 0, \rho)$, where $\rho(ds) = [-\log|c|/(1-\Re s)]\,\mathrm{m}(ds)$ on $\mathbb{T}^{\times}$, as its free multiplicative Lévy triplet (also known as $\boxtimes$-characteristic triplet in [6, p. 2437]). On the other hand, Lemma 5.5 says that the same triplet $(c/|c|, 0, \rho)$ also serves as the classical multiplicative Lévy triplet of κ_c. Thus we have shown that $\Gamma_1^0(\kappa_c) = \kappa_c$. That $\kappa_c \Rightarrow \mathrm{m}$ as $c \to 0$ allows us to further obtain $\tilde{\Gamma}_1(\mathrm{m}) = \mathrm{m}$.

Next, denote by ν_n the probability distribution in $\mathcal{ID}(\boxtimes) \cap \mathscr{P}_{\mathbb{T}}^{\times}$ having the free multiplicative Lévy triplet $(1, 0, n\delta_{-1})$, and let $\mu_n = \Gamma_1^0(\nu_n)$. Then (5-6) shows that for any $p \in \mathbb{Z}$,

$$\hat{\mu}_n(p) = \exp[n((-1)^p - 1)] = \begin{cases} 1, & p \text{ is even,} \\ e^{-2n}, & p \text{ is odd,} \end{cases}$$

which readily implies that $\mu_n \Rightarrow \frac{1}{2}(\delta_{-1} + \delta_1)$. However, we will explain in the next paragraph that $\nu_n \Rightarrow \mathrm{m}$, which apparently leads to a contradiction.

To see why $\nu_n \Rightarrow \mathrm{m}$, select a weakly convergent subsequence of $\{\nu_n\}$ (still denoted by $\{\nu_n\}$ in the remaining arguments) and denote the weak limit by ν. Let ν'_n be the probability measure having the free multiplicative Lévy triplet $\left(1, 0, \left(\frac{n}{2}\right)\delta_{-1}\right)$. Passing to a further subsequence we may assume that ν'_n weakly converge to ν'. Then letting $n \to \infty$ in the identity $\nu_n = \nu'_n \boxtimes \nu'_n$ gives $\nu = \nu' \boxtimes \nu'$. On the other hand, we see from (2-16) or from [6, Section 2.5] that $\Sigma_{\nu'_n}(0) = e^n$, i.e., $m_1(\nu'_n) = e^{-n} \to 0$ as $n \to \infty$ by Remark 2.6, whence $m_1(\nu') = 0$. By the definition of freeness, we can further conclude that $m_p(\nu) = 0$ for all $p \in \mathbb{Z}\backslash\{0\}$ or, equivalently, $\nu = \mathrm{m}$. $\square$

Acknowledgements

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