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The concept of *tame quasiconformal motions* was first introduced by Jiang et al. (2018). The concept of *monodromy* of holomorphic motions was first introduced by Beck et al. (2012). In this paper, we will show that the concept of *monodromy* of tame quasiconformal motions can be defined, whereas it cannot be defined for quasiconformal motions, in the sense of Sullivan and Thurston (1986). We also study some other properties of tame quasiconformal motions.

1. Introduction

The concept of quasiconformal motions was first introduced by Sullivan and Thurston [12]. Theorem 3 of [12] claimed that every quasiconformal motion of any set over an interval can be extended to the Riemann sphere. Jiang et al. [7] presented a counterexample to Theorem 3 of [12]. They [7] introduced a new concept, called *tame quasiconformal motions*, and showed that Theorem 3 of [12] holds for tame quasiconformal motions over any simply connected Hausdorff space. They also showed that this extension can be done in a conformally natural way, for tame quasiconformal motions. The crucial idea was to show that tame quasiconformal motions have a certain "universal property" that quasiconformal motions (in the sense of Sullivan and Thurston) do not have.

Beck et al. [2] introduced the concept of *monodromy* associated with a holomorphic motion of a closed subset of the Riemann sphere over a hyperbolic Riemann surface. Jiang and Mitra [6] proved that the triviality of the monodromy for this holomorphic motion is a necessary and sufficient condition for the given holomorphic motion to be extended to the whole Riemann sphere over the same hyperbolic Riemann surface. However, the concept of monodromy cannot be defined for a quasiconformal motion of a closed subset of the Riemann sphere over a hyperbolic Riemann surface, due to the counterexample in [7]. In the present paper, we show that the concept of monodromy can be defined for a tame quasiconformal motion

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of a closed subset of the Riemann sphere over any connected Hausdorff space. We prove that the triviality of the monodromy for a tame quasiconformal motion of a closed subset of the Riemann sphere over a path-connected Hausdorff space is a necessary and sufficient condition for this tame quasiconformal motion to be extended to a quasiconformal motion of the whole Riemann sphere over the same path-connected Hausdorff space. We also study some other properties of tame quasiconformal motions.

This paper is organized as follows. In Section 2, we give all basic definitions and note the various facts that are needed in this paper, and then state the two main theorems. In Section 3 we present three lemmas and in Sections 4 and 5, we prove the two main theorems.

2. Basic definitions and statements of the main theorems

Throughout this paper, $\mathbb{C}$ denotes the complex plane, $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere and $E \subset \hat{\mathbb{C}}$ is a closed subset such that $0, 1, \infty \in E$.

When we write $\widetilde{V}$ or $\widetilde{W}$ or $\widetilde{X}$ is "simply connected", we mean that it is a path-connected topological space and that its fundamental group is trivial.

We begin with some definitions.

Definition 1. Let $E \subset \hat{\mathbb{C}}$ and let X be a connected Hausdorff space with basepoint x_0. A *motion of E over X* is a map $\phi : X \times E \to \hat{\mathbb{C}}$ satisfying

(i) $\phi(x_0, z) = z$ for all $z \in E$, and

(ii) for all $x \in X$, the map $\phi(x, \cdot) : E \to \hat{\mathbb{C}}$ is injective.

We say that X is the *parameter space* of the motion ϕ. We will assume that 0, 1, and ∞ belong to E and that the motion ϕ is *normalized*, i.e., 0, 1, and ∞ are fixed points of the map $\phi(x, \cdot)$ for every x in X.

Let $E \subset \hat{E}$, $\phi : X \times E \to \hat{\mathbb{C}}$ and $\hat{\phi} : X \times \hat{E} \to \hat{\mathbb{C}}$ be two motions. We say that $\hat{\phi}$ *extends* ϕ if $\hat{\phi}(x, z) = \phi(x, z)$ for all $(x, z) \in X \times E$.

For any motion $\phi : X \times E \to \hat{\mathbb{C}}$, x in X, and any quadruplet of distinct points a, b, c, d of points in E, let $\phi_x(a, b, c, d)$ denote the cross-ratio of the values $\phi(x, a), \phi(x, b), \phi(x, c)$ and $\phi(x, d)$. We will often write $\phi(x, z)$ as $\phi_x(z)$ for x in X and z in E. So we have

$$\phi_x(a, b, c, d) = \frac{(\phi_x(a) - \phi_x(c))(\phi_x(b) - \phi_x(d))}{(\phi_x(a) - \phi_x(d))(\phi_x(b) - \phi_x(c))} \quad \text{for each } x \text{ in } X.$$

It is obvious that condition (ii) in Definition 1 holds if and only if $\phi_x(a, b, c, d)$ is a well-defined point in the thrice-punctured sphere $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ for all x in X and all quadruplets a, b, c, d of distinct points in E.

Let ρ be the Poincaré distance on $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Sullivan and Thurston [12] introduced the following definition.

Definition 2. A *quasiconformal motion* is a motion $\phi : X \times E \to \hat{\mathbb{C}}$ of E over X with the following additional property:

(iii) Given any x in X and any $\epsilon > 0$, there exists a neighborhood U_x of x such that for any quadruplet of distinct points a, b, c, d in E, we have

$$\rho\big(\phi_y(a, b, c, d), \phi_{y'}(a, b, c, d)\big) < \epsilon \quad \text{for all } y \text{ and } y' \text{ in } U_x.$$

Definition 3. A *continuous motion* of $\hat{\mathbb{C}}$ over X is a motion $\phi : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that the map ϕ is continuous.

Remark. If ϕ is a continuous motion of $\hat{\mathbb{C}}$, then each ϕ_x, x in X, is a map from $\hat{\mathbb{C}}$ to itself that fixes 0, 1, and ∞. Since ϕ_x is injective and continuous, it is a homeomorphism of $\hat{\mathbb{C}}$ onto itself, by invariance of domain.

Recall that a homeomorphism of $\hat{\mathbb{C}}$ is called *normalized* if it fixes the points 0, 1, and ∞. We use $M(\mathbb{C})$ to denote the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\hat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

We will need the following properties that were proved in [11].

Proposition 4. *A motion $\phi : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is quasiconformal if and only if it satisfies*:

(i) *The map $\phi_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is quasiconformal for each x in X.*

(ii) *The map from X to $M(\mathbb{C})$ that sends x to the Beltrami coefficient of ϕ_x for each x in X is continuous.*

Part (ii) means that the map $x \mapsto \mu_x = (\phi_x)_{\bar{z}}/(\phi_x)_z,\ x \in X$, is continuous.

Proposition 5. *Every quasiconformal motion of $\hat{\mathbb{C}}$ is a continuous motion.*

Definition 6. Assume that W is a connected complex manifold with basepoint x_0. A *holomorphic motion* of E over W is a motion $\phi : W \times E \to \hat{\mathbb{C}}$ of E over W such that the map $\phi(\,\cdot\,, z) : W \to \hat{\mathbb{C}}$ is holomorphic for each z in E.

Definition 7. Let X be a connected Hausdorff space with a basepoint x_0, and E be a set in $\hat{\mathbb{C}}$ (containing the points 0, 1, and ∞). A *tame quasiconformal motion* is a motion $\phi : X \times E \to \hat{\mathbb{C}}$ of E over X with the following additional property:

(iii) Given any x in X, there exists a quasiconformal map $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, a neighborhood $N(x)$, with basepoint x, and a quasiconformal motion $\psi : N(x) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ over $N(x)$ such that $\phi(y, z) = \psi(y, w(z))$ for all $(y, z) \in N(x) \times E$.

The lemma below was proved in [7].

Lemma 8. *A motion $\phi : X \times E \to \hat{\mathbb{C}}$ is a tame quasiconformal motion if and only if given any $x \in X$, there exists a neighborhood $N(x)$, and a continuous map $g_x : N(x) \to M(\mathbb{C})$ such that $\phi(y, z) = w^{g_x(y)}(z)$ for all $(y, z) \in N(x) \times E$.*

Definition 9. Let X and Y be connected Hausdorff spaces with basepoint, and f be a continuous basepoint preserving map of X into Y. If ϕ is a motion of E over Y its *pullback* by f is the motion

$$f^*(\phi)(x,z) = \phi(f(x),z) \quad \text{for all } (x,z) \in X \times E$$

of E over X.

Remark. If the motion ϕ is continuous, or tame quasiconformal, $f^*(\phi)$ has the same property. If X and Y are complex manifolds, f holomorphic and ϕ is a holomorphic motion, then so is $f^*(\phi)$.

Proposition 10. *If $\phi : X \times E \to \hat{\mathbb{C}}$ is a holomorphic motion where X is a connected complex Banach manifold with a basepoint x_0. Then ϕ is a tame quasiconformal motion.*

See Proposition 6 in [7].

Remark. In [7] it was shown that holomorphic motions $\Rightarrow$ tame quasiconformal motions $\Rightarrow$ quasiconformal motions $\Rightarrow$ continuous motions.

Definition 11. Let $\phi : X \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion. Let G be a group of Möbius transformations, and suppose that E is invariant under G (which means, $g(E) = E$ for all g in G). We say that ϕ is *G-equivariant* if and only if for each g in G, and x in X, there is a Möbius transformation $\theta_x(g)$ such that

$$\phi(x, g(z)) = (\theta_x(g))(\phi(x,z)) \quad \text{for all } z \in E. \tag{2-1}$$

Definition 12. Let G be a subgroup of $\mathrm{PSL}(2,\mathbb{C})$, and suppose that E is invariant under G. An isomorphism $\eta : G \to \mathrm{PSL}(2,\mathbb{C})$ is said to be induced by an injection $f : E \to \hat{\mathbb{C}}$ if $f(g(z)) = \eta(g)(f(z))$ for all $g \in G$ and for $z \in E$. An isomorphism induced by a quasiconformal self-map of $\hat{\mathbb{C}}$ is called a *quasiconformal deformation* of G.

Definition 13. Let X be a connected Hausdorff space and let G be a subgroup of $\mathrm{PSL}(2,\mathbb{C})$. A *continuous family $\{\theta_x\}$ of isomorphisms of G* is such that:

(i) For each $x \in X$, $\theta_x : G \to \mathrm{PSL}(2,\mathbb{C})$ is an isomorphism.

(ii) The map $x \mapsto \theta_x(g)$ is continuous for each $g \in G$, and for each $x \in X$.

We will need the following result; see Corollaries 1 and 2 of [7].

Theorem 14. *Let $\widetilde{V}$ be a simply connected Hausdorff space with a basepoint, and let $\phi : \widetilde{V} \times E \to \hat{\mathbb{C}}$ be a G-equivariant tame quasiconformal motion. Then, there exists a G-equivariant quasiconformal motion $\tilde{\phi} : \widetilde{V} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends ϕ.*

This means the following:

(i) For each x in $\widetilde{V}$, the map $\tilde{\phi}_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map; let its Beltrami coefficient be μ_x.

(ii) The map $x \mapsto \mu_x$ is continuous for x in $\widetilde{V}$.

(iii) $\tilde{\phi}(x, z) = \phi(x, z)$ for all $(x, z) \in \widetilde{V} \times E$.

(iv) $\tilde{\phi}_x \circ g \circ \tilde{\phi}_x^{-1} = \theta_x(g)$ for each g in G.

We also need the following result; see Remark 4 in [7].

Lemma 15. *Assume that $\phi : X \times E \to \hat{\mathbb{C}}$ is a tame quasiconformal motion where X is a connected Hausdorff space with a basepoint x_0. For each z in E, the map $\phi(\,\cdot\,, z) : X \to \hat{\mathbb{C}}$ is continuous.*

2A. ***Teichmüller space of a closed set E.*** Two normalized quasiconformal self-mappings f and g of $\hat{\mathbb{C}}$ are said to be E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The *Teichmüller space* $T(E)$ is the set of all E-equivalence classes of normalized quasiconformal self-mappings of $\hat{\mathbb{C}}$. The basepoint of $T(E)$ is the E-equivalence class of the identity map.

Recall that $M(\mathbb{C})$ denotes the open unit ball of the complex Banach space $L^\infty(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is the Beltrami coefficient of a unique normalized quasiconformal homeomorphism w^μ of $\hat{\mathbb{C}}$ onto itself. The basepoint of $M(\mathbb{C})$ is the zero function.

We can define the quotient map $P_E : M(\mathbb{C}) \to T(E)$ by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ, written as $[w^\mu]_E$. Clearly, P_E maps the basepoint of $M(\mathbb{C})$ to the basepoint of $T(E)$.

G. Lieb [8] proved that $T(E)$ is a complex Banach manifold such that the projection map P_E from $M(\mathbb{C})$ to $T(E)$ is a holomorphic split submersion. (The result was also proved in [3].)

2B. ***Changing the basepoint.*** Let w be a normalized quasiconformal self-mapping of $\hat{\mathbb{C}}$, and let $\hat{E} = w(E)$. By definition, the *allowable map* g from $T(\hat{E})$ to $T(E)$ maps the $\hat{E}$-equivalence class of f (written as $[f]_{\hat{E}}$) to the E-equivalence class of $f \circ w$ (written as $[f \circ w]_E$) for every normalized quasiconformal self-mapping f of $\hat{\mathbb{C}}$.

Proposition 16. *The allowable map $g : T(\hat{E}) \to T(E)$ is biholomorphic. If μ is the Beltrami coefficient of w, then g maps the basepoint of $T(\hat{E})$ to the point $P_E(\mu)$ in $T(E)$.*

See Proposition 7.20 in [3] or Proposition 6.7 in [9].

2C. ***Universal holomorphic motion of E.*** The *universal holomorphic motion* Ψ_E of E over $T(E)$ is defined as

$$\Psi_E(P_E(\mu), z) = w^\mu(z) \quad \text{for } \mu \in M(\mathbb{C}) \text{ and } z \in E.$$

The definition of P_E in Section 2A guarantees that Ψ_E is well defined. It is a holomorphic motion since P_E is a holomorphic split submersion and $\mu \mapsto w^\mu(z)$ is a holomorphic map from $M(\mathbb{C})$ to $\hat{\mathbb{C}}$ for every fixed z in $\hat{\mathbb{C}}$ (by Theorem 11 in [1]). This holomorphic section is "universal" in the following sense.

Theorem 17. *Let $\phi : \widetilde{W} \times E \to \hat{\mathbb{C}}$ be a holomorphic motion where $\widetilde{W}$ is a simply connected complex Banach manifold with a basepoint x_0, there exists a unique basepoint preserving holomorphic map $f : \widetilde{W} \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

For a proof, see Section 14 in [9].

By Proposition 10, every holomorphic motion is also a tame quasiconformal motion. Hence, $\Psi_E : T(E) \times E \to \hat{\mathbb{C}}$ is also a tame quasiconformal motion. In [7], it was proved that this is the universal tame quasiconformal motion of the closed set E over a simply connected Hausdorff space. Here is the precise statement:

Theorem 18. *Let $\phi : \widetilde{X} \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion where $\widetilde{X}$ is a simply connected Hausdorff space with a basepoint x_0. There exists a unique basepoint preserving continuous map $f : \widetilde{X} \to T(E)$ such that $f^*(\Psi_E) = \phi$.*

See Theorem II in [7].

2D. ***Douady–Earle section.*** Below we present some important facts.

Proposition 19. *There is a continuous basepoint preserving map s from $T(E)$ to $M(\mathbb{C})$ such that $P_E \circ s$ is the identity map on $T(E)$.*

See [3] or [5] for a proof. It immediately implies that:

Corollary 20. *The Teichmüller space $T(E)$ is contractible.*

Definition 21. The map s from $T(E)$ to $M(\mathbb{C})$ is called the *Douady–Earle section of P_E* for the Teichmüller space $T(E)$.

2E. ***Monodromy associated with a tame quasiconformal motion.*** We now discuss the concept of *monodromy* of a tame quasiconformal motion. Let $\phi : X \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion, where X is a connected Hausdorff space with a basepoint x_0. Let $\pi : \widetilde{X} \to X$ be a universal covering, with the group of deck transformations Γ. We choose a point $\tilde{x}_0$ in $\widetilde{X}$ such that $\pi(\tilde{x}_0) = x_0$. Let $\pi_1(X, x_0)$ denote the fundamental group of X with basepoint x_0.

Let $\Phi = \pi^*(\phi)$. Then, $\Phi : \widetilde{X} \times E \to \hat{\mathbb{C}}$ is a tame quasiconformal motion of E over $\widetilde{X}$ with $\tilde{x}_0$ as the basepoint. By Theorem 18, there exists a unique basepoint preserving continuous map $f : \widetilde{X} \to T(E)$ such that $f^*(\Psi_E) = \phi$. Then by Proposition 19, there is a continuous basepoint preserving map $\tilde{f} = s \circ f$ from $\widetilde{X} \to M(\mathbb{C})$ such that

$$\Phi(x, z) = w^{\tilde{f}(x)}(z) \quad \text{for each } x \in \widetilde{X} \text{ and each } z \in E.$$

For each $z \in E$ and each $\gamma \in \Gamma$, we have

$$w^{\tilde{f} \circ \gamma(\tilde{x}_0)}(z) = \Phi(\gamma(\tilde{x}_0), z) = \phi(\pi \circ \gamma(\tilde{x}_0), z) = \phi(x_0, z) = z.$$

Therefore, $w^{\tilde{f}\circ\gamma(\tilde{x}_0)}$ keeps every point of E fixed. Since s may not be unique, $\tilde{f}$ is not necessarily unique. So we need the next lemma.

Lemma 22. *The homotopy class of $w^{\tilde{f}\circ\gamma(\tilde{x}_0)}$ relative to E does not depend on the choice of the continuous map $\tilde{f}$.*

Proof. Let $\tilde{f}_1, \tilde{f}_2 : \widetilde{X} \to M(\mathbb{C})$ be basepoint preserving continuous maps which are obtained from the given tame quasiconformal motion $\phi : X \times E \to \hat{\mathbb{C}}$. For each $\gamma \in \Gamma$, take a path $c_\gamma : [0, 1] \to \widetilde{X}$ which connects $\tilde{x}_0$ and $\gamma(\tilde{x}_0)$ and write

$$H(z, t) := w^{\tilde{f}_1\circ\gamma(\tilde{x}_0)} \circ \{w^{\tilde{f}_1\circ c_\gamma(t)}\}^{-1} \circ w^{\tilde{f}_2\circ c_\gamma(t)}(z)$$

for $(z, t) \in \hat{\mathbb{C}} \times [0, 1]$. Then, we see that $H(\cdot, \cdot)$ gives a homotopy from $w^{\tilde{f}_1\circ\gamma(\tilde{x}_0)}$ to $w^{\tilde{f}_2\circ\gamma(\tilde{x}_0)}$ relative to E. Hence, we conclude that $w^{\tilde{f}_1\circ\gamma(\tilde{x}_0)}$ and $w^{\tilde{f}_2\circ\gamma(\tilde{x}_0)}$ belong to the same homotopy class relative to E, as claimed. □

We now assume that E' is a finite set containing n points where $n \geq 4$; as usual, 0, 1, and ∞ are in E'. Let $\phi : X \times E' \to \hat{\mathbb{C}}$ be a tame quasiconformal motion. The map $w^{\tilde{f}\circ\gamma(\tilde{x}_0)}$ is quasiconformal self-map of the hyperbolic Riemann surface $X'_E := \hat{\mathbb{C}} \setminus E'$. Therefore, it represents a mapping class of X'_E, and by Lemma 22, we have a homomorphism $\rho_\phi : \pi_1(X, x_0) \to \mathrm{Mod}(0, n)$ given by

$$\rho_\phi(c) = [w^{\tilde{f}\circ\gamma(\tilde{x}_0)}],$$

where $\mathrm{Mod}(0, n)$ is the mapping class group of the n-times punctured sphere, $\gamma \in \Gamma$ is the element corresponding to $c \in \pi_1(X, x_0)$, and $[w]$ denotes the mapping class group of X'_E for w.

Definition 23. Suppose $\phi : X \times E \to \hat{\mathbb{C}}$ is a tame quasiconformal motion where X is a connected Hausdorff space. We say ϕ has trivial monodromy if for every finite subset $\{0, 1, \infty\} \subset E' \subseteq E$, the homomorphism ρ_ϕ for the tame quasiconformal motion $\phi : X \times E' \to \hat{\mathbb{C}}$ is *trivial*, that is, it maps every element of $\pi_1(X, x_0)$ to the identity of $\mathrm{Mod}(0, n)$.

We now state the two main theorems of this paper.

Theorem A. *Let $\phi : V \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion where V is a path-connected Hausdorff space. Then the following are equivalent.*

(i) *There exists a continuous motion $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends ϕ.*

(ii) *There exists a quasiconformal motion $\hat{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\hat{\phi}$ extends ϕ.*

(iii) *There exists a unique basepoint preserving continuous map $F : V \to T(E)$ such that $F^*(\Psi_E) = \phi$.*

(iv) *The monodromy of ϕ is trivial.*

Remark. For (ii) ⇔ (iii), X does not have to be path-connected; a connected Hausdorff space with basepoint is sufficient.

Theorem B. *Let G be a subgroup of* $\mathrm{PSL}(2, \mathbb{C})$, *and suppose E is a closed set in $\hat{\mathbb{C}}$ which is invariant under G. Let $\phi : X \times E \to \hat{\mathbb{C}}$ be a G-equivariant tame quasiconformal motion where X is a connected Hausdorff space. Then*

(i) *$\{\theta_x\}$ is a continuous family of isomorphisms of G, and*

(ii) *θ_x is a quasiconformal deformation of G for every x in X.*

3. Three lemmas

In what follows, V is a path-connected Hausdorff space with a basepoint x_0. Let $\mathcal{H}(\hat{\mathbb{C}})$ be the group of homeomorphisms of $\hat{\mathbb{C}}$ onto itself, with the topology of uniform convergence in the spherical metric.

Lemma 24. *Let $h : V \to \mathcal{H}(\hat{\mathbb{C}})$ be a continuous map such that $h(x)(z) = z$ for all x in V and for all z in E. If $h(x_0)$ is isotopic to the identity* rel E *for some fixed x_0 in V, then $h(x)$ is isotopic to the identity* rel E *for all x in V.*

See Lemma 12.1 in [9].

Lemma 25. *Let $s : T(E) \to M(\mathbb{C})$ be the Douady–Earle section, and let $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be any homeomorphism. There is at most one point t in $T(E)$ such that ψ is isotopic to $w^{s(t)}$* rel E.

Proof. If $w^{s(t)}$ and $w^{s(t')}$ are both isotopic to ψ rel E, then they are E-equivalent, and hence $t = P_E(s(t)) = P_E(s(t')) = t'$. □

Lemma 26. *If the continuous maps f and g from V into $T(E)$ satisfy*

(1) *$\Psi_E(f(x), z) = \Psi_E(g(x), z)$ for all x in V, and for all z in E, and*

(2) *$f(p) = g(p)$ for some p in V,*

then $f(x) = g(x)$ for all x in V.

See Lemma 12.2 in [9].

4. Proof of Theorem A

We first prove the following theorem. The proof is similar to that given in [10] (which was for holomorphic motions). We include the details for the reader's convenience, and also to make our paper self-contained.

Theorem 27. *Let V be a path-connected Hausdorff space with a basepoint x_0, and let $\phi : V \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion. If there exists a continuous motion $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\phi}$ extends ϕ, then there exists a unique basepoint preserving continuous map $F : V \to T(E)$ such that $F^*(\Psi_E) = \phi$.*

Proof. Let S be the set of points x in V with the following property: there exists a neighborhood N of x and a continuous map $h : N \to T(E)$ such that $w^{s(h(x'))}$ is isotopic to $\tilde{\phi}_{x'}$ rel E for all x' in N. We claim that $S = V$.

It is clear that S is an open set. We first show that S is nonempty; in fact, $x_0 \in S$. Choose a simply connected neighborhood N of x_0 in V, and give N the basepoint x_0. By Theorem 18, there exists a basepoint preserving continuous map $h : N \to T(E)$ such that $h^*(\Psi_E) = \phi$ on $N \times E$. Define

$$H(x) = (w^{s(h(x))})^{-1} \circ \tilde{\phi}_x \quad \text{for each } x \text{ in } N.$$

Clearly, $H(x_0)$ is the identity. Also, for all x in N, and for all z in E, we have

$$\tilde{\phi}_x(z) = \tilde{\phi}(x, z) = \phi(x, z) = \Psi_E(h(x), z) = w^{s(h(x))}(z).$$

Hence, for all z in E, $H(x)(z) = z$. Since $H(x)$ is continuous in x, it follows from Lemma 24 that $H(x)$ is isotopic to the identity rel E. Hence, for each x in N, $w^{s(h(x))}$ is isotopic to $\tilde{\phi}_x$ rel E. This shows that x_0 belongs to S.

Now we shall prove that S is closed. Let y be a limit point of S; choose a simply connected neighborhood B of y. Since y is a limit point of S, B contains a point p in S. Choose p to be the basepoint of B. Let

$$\hat{E} = \phi_p(E) = \{\phi(p, z) : z \in E\}$$

and define $\hat{\phi} : B \times \hat{E} \to \hat{\mathbb{C}}$ as

$$\hat{\phi}(x, \phi_p(z)) = \phi(x, z), \quad (x, z) \in B \times E.$$

It is easy to see that $\hat{\phi} : B \times \hat{E} \to \hat{\mathbb{C}}$ is a tame quasiconformal motion of $\hat{E}$ over B with basepoint p. By Theorem 18, there exists a basepoint preserving continuous map $f : B \to T(\hat{E})$ such that $f^*(\Psi_{\hat{E}}) = \hat{\phi}$ on $B \times \hat{E}$ (where $\Psi_{\hat{E}} : T(\hat{E}) \times \hat{E} \to \hat{\mathbb{C}}$ is the universal tame quasiconformal motion of $\hat{E}$).

This means

$$\Psi_{\hat{E}}(f(x), \phi_p(z)) = \hat{\phi}(x, \phi_p(z)) \tag{4-1}$$

for all x in B and for all z in E.

Since $p \in S$, there is a point t in $T(E)$ such that $\tilde{\phi}_p$ is isotopic to $w^{s(t)}$ rel E. Thus, $w^{s(t)}$ maps E onto $\hat{E}$; so it induces a biholomorphic map $g : T(\hat{E}) \to T(E)$ as in Section 2B. Define $\hat{h} : B \to T(E)$ by $\hat{h} = g \circ f$. We will show that $w^{s(\hat{h}(x))}$ is isotopic to $\tilde{\phi}_x$ rel E for all x in B.

Note that f maps p to the basepoint of $T(\hat{E})$ and by Proposition 16, g maps $f(p)$ to the point $P_E(s(t))$ in $T(E)$. So, $\hat{h}(p) = P_E(s(t))$ and since $\hat{h}(p) = P_E\big(s(\hat{h}(p))\big)$, we have $P_E(s(t)) = P_E\big(s(\hat{h}(p))\big)$. That means, $w^{s(t)}$ is isotopic to $w^{s(\hat{h}(p))}$ rel E;

so $\tilde{\phi}_p$ is isotopic to $w^{s(\hat{h}(p))}$ rel E. Let

$$\hat{H}(x) = (w^{s(\hat{h}(x))})^{-1} \circ \tilde{\phi}_x \quad \text{for all } x \text{ in } B. \tag{4-2}$$

By the above discussion, $\hat{H}(p)$ is isotopic to the identity rel E.

We have the standard projection map

$$P_{\hat{E}} : M(\mathbb{C}) \to T(\hat{E}),$$

and $\hat{s} : T(\hat{E}) \to M(\mathbb{C})$ is a continuous basepoint preserving map such that $P_{\hat{E}} \circ \hat{s}$ is the identity map on $T(\hat{E})$. Since $\tilde{\phi}_p$ is isotopic to $w^{s(t)}$ rel E, and $\tilde{\phi}_p(z) = \phi_p(z)$ for all z in E, it follows that

$$\phi_p(z) = w^{s(t)}(z) \tag{4-3}$$

for all z in E. Furthermore, for all $x \in B$, and $z \in E$, we have

$$\tilde{\phi}_x(z) = \phi_x(z) = \hat{\phi}_x(\phi_p(z)) = \Psi_{\hat{E}}(f(x), \phi_p(z))$$

by (4-1). Also,

$$\Psi_{\hat{E}}(f(x), \phi_p(z)) = w^{\hat{s}(f(x))}(\phi_p(z)) = w^{\hat{s}(f(x))}(w^{s(t)}(z))$$

by (4-3). We conclude that

$$\tilde{\phi}_x(z) = w^{\hat{s}(f(x))}(w^{s(t)}(z)) \tag{4-4}$$

for all x in B, and for all z in E.

For all x in B, we have $\hat{h}(x) = g(f(x))$. Also, $f(x) = P_{\hat{E}}\big(\hat{s}(f(x))\big) = [w^{\hat{s}(f(x))}]_{\hat{E}}$ and by Section 2B, we have

$$g : [w^{\hat{s}(f(x))}]_{\hat{E}} \mapsto [w^{\hat{s}(f(x))} \circ w^{s(t)}]_E.$$

Therefore,

$$\hat{h}(x) = [w^{\hat{s}(f(x))} \circ w^{s(t)}]_E.$$

We also have $\hat{h}(x) = P_E\big(s(\hat{h}(x))\big) = [w^{s(\hat{h}(x))}]_E$ for all x in B. Hence, for all x in B, and for all z in E, we have

$$w^{\hat{s}(f(x))}(w^{s(t)}(z)) = w^{s(\hat{h}(x))}(z). \tag{4-5}$$

Therefore, by (4-4) and (4-5), we get $\tilde{\phi}_x(z) = w^{s(\hat{h}(x))}(z)$ for all x in B and for all z in E. Hence, by (4-2), $\hat{H}(x)(z) = z$ for all x in B, and for all z in E. Since $\hat{H}$ is continuous in x, it follows from Lemma 24 that $\hat{H}(x)$ is isotopic to the identity rel E for all x in B. Therefore, $w^{s(\hat{h}(x))}$ is isotopic to $\tilde{\phi}_x$ rel E for all x in B. Hence B is contained in S. In particular, $y \in S$, so S is closed. As S is also open and nonempty, $S = V$.

We now define a continuous map $F : V \to T(E)$ as follows: Given any x in V, choose a neighborhood N of x and a continuous map $h : N \to T(E)$ such that

$w^{s(h(x'))}$ is isotopic to $\tilde{\phi}_{x'}$ rel E for all x' in N. Set $F = h$ in N. By Lemma 25, F is well defined on all of V. It is obviously continuous, and $w^{s(F(x))}$ is isotopic to $\hat{\phi}_x$ rel E for all x in V.

Finally, for all x in V, and for all z in E, we have

$$F^*(\Psi_E)(x,z) = \Psi_E(F(x),z) = \Psi_E\big(P_E\big(s(F(x))\big),z\big) = w^{s(F(x))}(z)$$

and $\phi(x,z) = \tilde{\phi}(x,z) = \tilde{\phi}_x(z) = w^{s(F(x))}(z)$ (since $w^{s(F(x))}$ is isotopic to $\tilde{\phi}_x$ rel E for all x in V). Therefore, $F^*(\Psi_E)(x,z) = \phi(x,z)$ for all x in V and for all z in E.

The uniqueness of F follows from Lemma 26. This completes the proof. □

Proof of Theorem A. Theorem 27 proved the direction (i) ⇒ (iii).

For (iii) ⇒ (ii), define $\widetilde{F} : V \to M(\mathbb{C})$ by $\widetilde{F} = s \circ F$. Then, $\widetilde{F} : V \to M(\mathbb{C})$ is a basepoint preserving continuous map. Define $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$\tilde{\phi}(x,z) = w^{\widetilde{F}(x)}(z) \quad \text{for all } x \text{ in } V \text{ and for all } z \text{ in } \hat{\mathbb{C}}.$$

By Proposition 4, $\tilde{\phi} : V \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal motion, and for all z in E,

$$\begin{aligned}\phi(x,z) = F^*(\Psi_E)(x,z) = \Psi_E(F(x),z) &= \Psi_E\big(P_E\big(s(F(x))\big),z\big)\\ &= w^{s(F(x))}(z) = w^{\widetilde{F}(x)}(z) = \tilde{\phi}(x,z).\end{aligned}$$

Hence $\tilde{\phi}$ extends ϕ.

The direction (ii) ⇒ (i) is obvious by Proposition 5.

Finally, we prove (i) ⇔ (iv).

Let $\pi : \widetilde{V} \to V$ be a universal covering with the group Γ of deck transformations, so that $V = \widetilde{V}/\Gamma$ and $\pi(\tilde{x}_0) = x_0$.

Suppose ϕ can be extended to a continuous motion $\tilde{\phi}$ of $\hat{\mathbb{C}}$ over V. Then, by Theorem 27, there exists a continuous map $f : V \to M(\mathbb{C})$ such that

$$\tilde{\phi}(x,z) = w^{f(x)}(z) \quad \text{for all } (x,z) \in V \times \hat{\mathbb{C}}.$$

Let $\tilde{f} = f \circ \pi$. Then, for any $c \in \pi_1(X, x_0)$ with corresponding $\gamma \in \Gamma$, we have

$$\rho_\phi(c) = [w^{\tilde{f}\circ\gamma(\tilde{x}_0)}] = [w^{f\circ\pi\circ\gamma(\tilde{x}_0)}] = [w^{f(x_0)}] = [Id].$$

This shows that the monodromy ρ_ϕ is trivial.

Let $\phi : V \times E \to \hat{\mathbb{C}}$ be a tame quasiconformal motion with trivial monodromy. Let $\Phi = \pi^*(\phi)$ be the tame quasiconformal motion of E over $\widetilde{V}$. By Theorem 18, there exists a unique basepoint preserving continuous map $\tilde{f} : \widetilde{V} \to T(E)$ such that $\tilde{f}^*(\Psi_E) = \Phi$. For any element $\gamma \in \Gamma$, we also have $\tilde{f} \circ \gamma : \widetilde{V} \to T(E)$. Note that

$$\begin{aligned}\Psi_E(\tilde{f}\circ\gamma(x),z) &= (\tilde{f}\circ\gamma)^*(\Psi_E)(x,z)\\ &= \Phi(\gamma(x),z)\\ &= \phi(\pi\circ\gamma(x),z)\\ &= \phi(\pi(x),z) = \Phi(x,z) = (\tilde{f})^*(\Psi_E)(x,z) = \Psi_E(\tilde{f}(x),z).\end{aligned}$$

By the triviality of the monodromy, we have $\tilde{f}\circ\gamma(x_0)=\tilde{f}(x_0)=[Id]$ for all $\gamma\in\Gamma$. Lemma 26 implies that $\tilde{f}\circ\gamma=\tilde{f}$ for all $\gamma\in\Gamma$. Thus, $\tilde{f}$ defines a unique basepoint preserving continuous map $f:V\to T(E)$ such that $\phi=f^*(\Psi_E)$. Thus there exists a continuous motion of $\hat{\mathbb{C}}$ over V that extends ϕ. □

5. Proof of Theorem B

Part (i). The proof is similar to the one given in [4]. We include the arguments for reader's convenience. Since E has at least three points, for each x in X and g in G, the Möbius transformation $\theta_x(g)$ is completely determined by (2-1). It easily follows that θ_x is a homomorphism for each x in X. Also, for x in X, θ_x is injective. For $\theta_x(g_1)=\theta_x(g_2)$, we have $\phi_x(g_1(z))=\phi_x(g_2(z))$ which implies $g_1(z)=g_2(z)$ for all z in E (by injectivity). We conclude that $g_1=g_2$. Hence, for each x in X, the map θ_x is an isomorphism.

Choose three distinct points z_1, z_2, z_3 in E. For x in X, let h_x be the unique Möbius transformation such that

$$h_x(z_i)=\phi_x(z_i) \qquad \text{for all } i=1,2,3,$$

$$\theta_x(g)(h_x(z_i))=\phi_x(g(z_i)) \quad \text{for all } i=1,2,3.$$

By Lemma 15, for each i, the right-hand sides of the above equations depend continuously on x. Therefore, $x\mapsto h_x$ and $x\mapsto\theta_x(g)\circ h_x$ are continuous maps. Hence so is $x\mapsto\theta_x(g)$ for each g in G.

Part (ii). Let Ω be the set of all x in X with the following property: for each x in Ω, there exists a neighborhood $N(x)$ such that θ_t is a quasiconformal deformation of G for every t in $N(x)$.

Clearly, Ω is open. Also, Ω is nonempty, for the basepoint x_0 is in Ω. To see this, choose a simply connected neighborhood V of x_0 and use Theorem 14.

We will show that Ω is closed. Let k be a limit point of Ω. Choose a simply connected neighborhood B of k. Then, B contains a point p in Ω. So, θ_p is a quasiconformal deformation of G. Choose p to be the basepoint of B. Let $\theta_p(G)=\hat{G}$ and $\phi_p(E)=\hat{E}$.

Define $\hat{\phi}:B\times\hat{E}\to\hat{\mathbb{C}}$ as

$$\hat{\phi}_x(\phi_p(z))=\phi_x(z) \quad \text{for } x\in B \text{ and } z\in E.$$

Since $\phi:X\times E\to\hat{\mathbb{C}}$ is a tame quasiconformal motion, for p in B, there exists a neighborhood $N(p)$ and a continuous map $f_p:N(p)\to M(\mathbb{C})$ such that $\phi_x(z)=w^{f_p(x)}(z)$ for x in $N(p)$ and z in E (see Lemma 8). Set $w=w^{f_p(p)}$. Then, $w:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ is a quasiconformal map and $\phi_p(z)=w(z)$ for all z in E.

Now, assume $t\in B$. There exists a neighborhood $N(t)$ and a continuous map $f_t:N(t)\to M(\mathbb{C})$ such that $\phi_x(z)=w^{f_t(x)}(z)$ for x in $N(t)$ and z in E. This means there exists a quasiconformal motion $w^{f_t} : N(x) \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ over $N(x)$ such that

$$\hat{\phi}_x(\phi_p(z)) = \phi_x(z) = w^{f_t(x)}(z)$$

for all x in $N(t)$ and z in E.

Let $\phi_p(z) = \hat{z} \in \hat{E}$. Then, we have

$$\hat{\phi}_x(\hat{z}) = w^{f_t(x)}(w^{-1}(\hat{z})) \quad \text{for all } x \in N(t).$$

It follows that $\hat{\phi} : B \times \hat{E} \to \hat{\mathbb{C}}$ is a tame quasiconformal motion with basepoint p.

Next, note that $\hat{E}$ is $\hat{G}$-invariant. In fact, for $\hat{g}$ in $\hat{G}$, we have

$$\hat{g}(\hat{E}) = \theta_p(g)(\phi_p(E)) = \phi_p(g(E)) = \phi_p(E) = \hat{E}.$$

Recall that $\phi_p(z) = \hat{z}$ and $\theta_p(g) = \hat{g}$. By (2-1), we have

$$\begin{aligned}
\hat{\phi}_x\big(\theta_p(g)(\phi_p(z))\big) = \hat{\phi}_x\big(\phi_p(g(z))\big) = \phi_x(g(z)) &= \theta_x(g)(\phi_x(z)) \\
&= \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)(\phi_x(z)) \\
&= \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)\big(\hat{\phi}_x(\phi_p(z))\big) \\
&= \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)(\hat{\phi}_x(\hat{z})).
\end{aligned}$$

It follows that $\hat{\phi} : B \times \hat{E} \to \hat{\mathbb{C}}$ is a tame quasiconformal motion with the property

$$\hat{\phi}_x(\hat{g}(\hat{z})) = \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)(\hat{\phi}(x, \hat{z})) \quad \text{for all } x \text{ in } B \text{ and } \hat{z} \text{ in } \hat{E}.$$

Therefore, by Theorem 14, there exists a quasiconformal motion $\tilde{\hat{\phi}} : B \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\hat{\phi}}$ extends $\hat{\phi}$, and for each x in B, $\tilde{\hat{\phi}}_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map. We also have

$$\tilde{\hat{\phi}}(\hat{g}(z)) = \big(\theta_x(\theta_p^{-1}(\hat{g}))\big)(\tilde{\hat{\phi}}(z)) \quad \text{for all } x \text{ in } B \text{ and for all } z \text{ in } \hat{\mathbb{C}}.$$

This implies that

$$\tilde{\hat{\phi}} \circ \hat{g} \circ \tilde{\hat{\phi}}^{-1} = \theta_x \circ \theta_p^{-1}(\hat{g}).$$

Using $\theta_p(g) = \hat{g}$, it follows that

$$\tilde{\hat{\phi}} \circ \theta_p(g) \circ \tilde{\hat{\phi}}^{-1} = \theta_x(g).$$

Recall that θ_p is a quasiconformal deformation of G. Hence, there exists a quasiconformal map $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $w \circ g \circ w^{-1} = \theta_p(g)$. By the above equation, we get

$$\tilde{\hat{\phi}} \circ w \circ g \circ w^{-1} \circ \tilde{\hat{\phi}}^{-1} = \theta_x(g) \quad \text{for } x \in B.$$

Let $\tilde{f}_x = \tilde{\hat{\phi}} \circ w$; so, for each x in B, $\tilde{f}_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal map and

$$\tilde{f}_x \circ g \circ \tilde{f}_x^{-1} = \theta_x(g) \quad \text{for each } x \in B.$$

Hence, k is in Ω, and therefore, Ω is closed. Since X is connected, it follows that $\Omega = X$. □

Below is a direct and short proof of part (ii) of Theorem B. We thank one of the referees for bringing this to our attention.

An alternative proof of Theorem B(ii). Let $\tilde{\phi}:\widetilde{X}\times E\to\hat{\mathbb{C}}$ be the lift of $\phi: X\times E\to\hat{\mathbb{C}}$ to the universal covering $\widetilde{X}$ of X. Namely, $\tilde{\phi}$ is defined as

$$\tilde{\phi}(\tilde{x}, z) = \phi(\pi(\tilde{x}), z) \quad (\tilde{x}, z) \in \widetilde{X} \times E,$$

where $\pi : \widetilde{X} \to X$ is the canonical projection. It is a tame quasiconformal motion of E over $\widetilde{X}$ because tame quasiconformal motion is a local property. Moreover, we have

$$\tilde{\phi}(\tilde{x}, g(z)) = \phi(\pi(\tilde{x}), g(z)) = \theta_{\pi(\tilde{x})}(g)(\phi(\pi(\tilde{x}), z)) = \theta_{\pi(\tilde{x})}(g)(\tilde{\phi}(\tilde{x}, z)) \quad \text{for } g \in G.$$

Hence, it is G-equivariant with isomorphisms $\theta_{\pi(\tilde{x})} : G \to \mathrm{PSL}(2, \mathbb{C})$ where $\tilde{x} \in \widetilde{X}$. Since $\widetilde{X}$ is simply connected, it follows from Theorem 14 that $\theta_{\pi(\tilde{x})}(G)$ is a quasiconformal deformation of G and so is $\theta_x(G)$. □

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