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A CHARACTERIZATION AND SOLVABILITY OF QUASIHOMOGENEOUS SINGULARITIES

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Let $(V, 0)$ be an isolated hypersurface singularity defined by the holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. A local k-th $(0 \le k \le n+1)$ Hessian algebra $H_k(V)$ of isolated hypersurface singularity $(V, 0)$ is a finite-dimensional $\mathbb{C}$-algebra and it depends only on the isomorphism class of the germ $(V, 0)$. It is a natural question to ask for a necessary and sufficient condition for a complex analytic isolated hypersurface singularity to be quasihomogeneous in terms of its local k-th Hessian algebra $H_k(f)$. Xu and Yau proved that $(V, 0)$ admits a quasihomogeneous structure if and only if $H_0(f)$ is isomorphic to a finite-dimensional nonnegatively graded algebra in the early 1980s. In this paper, on the one hand, we generalize Xu and Yau's result to $H_{n+1}(f)$. On the other hand, a new series of finite-dimensional Lie algebras $L_k(V)$ (resp. $L^k(V)$) was defined to be the Lie algebra of derivations of the k-th $(0 \le k \le n+1)$ Hessian algebra $H_k(V)$ (resp. $A^k(V) := \mathcal{O}_{n+1}/(f, m^k J_f)$) and is finite-dimensional. We prove that $(V, 0)$ is quasihomogeneous singularity if $L_{n+1}(V)$ (resp. $L^k(V) := \mathrm{Der}(A^k(V))$) satisfies certain conditions. Moreover, we investigate whether the Lie algebras $L_k(V)$ (resp. $L^k(V)$) are solvable.

1. Introduction

A polynomial $f(z_0, \dots, z_n)$ is weighted homogeneous of type $(q_0, \dots, q_n; d)$, where $q_0, \dots, q_n$ and d are fixed positive integers, if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $q_0 i_0 + q_1 i_1 + \cdots + q_n i_n = d$. In this case, we say that z_i has weight q_i and f has weight d. Recall that an isolated hypersurface singularity $(V, 0) = \{(z_0, \dots, z_n) : f(z_0, \dots, z_n) = 0 \subset \mathbb{C}^{n+1}\}$ is quasihomogeneous if f is in the Jacobian ideal J_f, i.e., $f \in J_f = \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)$. By a theorem of Saito [1971], if f is quasihomogeneous with isolated singularity at 0, then after a biholomorphic change of coordinates, f becomes a weighted homogeneous polynomial.

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Let $(V, 0)$ be an isolated hypersurface singularity defined by the holomorphic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Let $\mathcal{O}_{n+1}$ denote the $\mathbb{C}$-algebra of germs of analytic functions defined at the origin of $\mathbb{C}^{n+1}$. Recall that the moduli algebra is $A(V) := \mathcal{O}_{n+1}/\big(f, \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\big)$. Mather and Yau [1982] proved that two germs of complex analytic hypersurfaces of the same dimension with isolated singularities are contact equivalent if and only if their moduli algebras are isomorphic. Therefore the moduli algebra $A(V)$ is important in the study of the complex structures of $(V, 0)$. In 1983,Yau introduced the Yau algebra $L(V)$ which was defined as the Lie algebra of derivations of the moduli algebra $A(V)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$ [Seeley and Yau 1990]. It plays an important role in singularity theory [Chen 1995]. In a beautiful paper, Elashvili and Khimshiashvili [2006] first used it to characterize ADE singularities. It is known that $L(V)$ is a finite-dimensional Lie algebra and its dimension $\lambda(V)$ is called Yau number [Khimshiashvili 2006; Yu 1996]. Yau, Zuo and their collaborators have been systematically studying various Lie algebras of isolated singularities [Benson and Yau 1990; Chen et al. 1995; 2019; 2020a; 2020b; Hussain et al. 2018; 2020; 2021b; Yau and Zuo 2016a; 2016b]. In this article, we study two kinds of new derivation Lie algebra arising from the isolated hypersurface singularity $(V, 0)$ as follows.

Hussain, Yau and Zuo [Hussain et al. 2020; 2021b], introduced the new series of k-th Yau algebras $L^k(V)$ which was defined to be the Lie algebra of derivations of the moduli algebra $A^k(V) = \mathcal{O}_{n+1}/(f, m^k J_f)$, $k \geq 0$, where m is the maximal ideal of $\mathcal{O}_n$, i.e., $L^k(V) := \mathrm{Der}(A^k(V), A^k(V))$. Its dimension was denoted as $\lambda^k(V)$. This series of integers $\lambda^k(V)$ are new numerical analytic invariants of singularities. It is natural to call it the k-th Yau number. In particular, when $k = 0$, these are exactly the previous Yau algebra and Yau number, i.e., $L(V) = L^0(V)$, $\lambda^0(V) = \lambda(V)$.

Let $\mathrm{Hess}(f)$ be the Hessian matrix (f_{ij}) of the second order partial derivatives of f, and $h(f)$ (the Hessian of f) be the determinant of $\mathrm{Hess}(f)$. More generally, for each k satisfying $0 \leq k \leq n+1$ we denote by $h_k(f)$ the ideal in $\mathcal{O}_{n+1}$ generated by all $k \times k$-minors in the matrix $\mathrm{Hess}(f)$. In particular, the ideal $h_{n+1}(f) = (h(f))$ is a principal ideal. For each k as above, consider the graded k-th Hessian algebra of the polynomial f defined by

$$H_k(f) = \mathcal{O}_{n+1}/((f) + J_f + h_k(f)).$$

In particular, $H_0(f)$ is exactly the well-known moduli algebra $A(V)$. It is easy to check that the isomorphism class of the local k-th Hessian algebra $H_k(f)$ is a contact invariant of f, i.e., $H_k(f)$ depends only on the isomorphism class of the germ $(V, 0)$ [Dimca and Sticlaru 2015].

Hussain, Yau and Zuo [Hussain et al. 2021a] defined a series of new derivation Lie algebras

$$L_k(V) := \mathrm{Der}(H_k(f), H_k(f)), \quad 0 \leq k \leq n+1.$$

Since $H_0(f) = A(V)$, so $L_k(V)$ is also a generalization of Yau algebra $L(V)$ and $L_0(V) = L(V)$. $L_k(V)$ is a finite-dimensional Lie algebra and the dimension of $L_k(V)$ is denoted by $\lambda_k(V)$ which is new numerical analytic invariant of isolated hypersurface singularities. It is natural to ask how to use $H_k(f)$ (resp. $L_{n+1}(V)$) to characterize the quasihomogeneity of an isolated hypersurface singularity. In this paper, we shall answer this question partially and prove that $(V, 0)$ admits a quasihomogeneous structure if and only if $H_{n+1}(f)$ (resp. $L_{n+1}(V)$) is isomorphic to a finite-dimensional nonnegatively graded algebra (resp. nonnegatively graded Lie algebra). We propose the following two conjectures.

Conjecture 1.1. *Let* $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be an isolated hypersurface singularity. Then the following are equivalent*:

(1) $(V, 0)$ *is quasihomogeneous.*

(2) *There exists a* k, $0 \le k \le n+1$, *such that the* k*-th Hessian algebra* $H_k(f)$ *is isomorphic to a finite-dimensional graded commutative local algebra* $\bigoplus_{i\ge 0} A_i$ *with* $A_0 = \mathbb{C}$.

(3) *For all* k, $0 \le k \le n+1$, *the* k*-th Hessian algebra* $H_k(f)$ *is isomorphic to a finite-dimensional graded commutative local algebra* $\bigoplus_{i\ge 0} A_i$ *with* $A_0 = \mathbb{C}$.

Conjecture 1.2. *Let* $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be an isolated hypersurface singularity with* $n \ge 1$. *Then* $(V, 0)$ *is a quasihomogeneous singularity if there exists* k, $0 \le k \le n+1$, *such that the following conditions are satisfied*:

(1) $L_k(V)$ *(resp.* $L^k(V)$*) is isomorphic to a nonnegatively graded Lie algebra* $\bigoplus_{i=0}^{\ell}(L_k(V))_i$ *without center.*

(2) *There exists* $E \in (L_k(V))_0$ *(resp.* $(L^k(V))_0$*) such that* $[E, D_i] = i(D_i)$ *for any* $D_i \in (L_k(V))_i$.

(3) *For any element* $\alpha \in m - m^2$, *where* m *is the maximal ideal of* $H_k(V)$ *(resp.* $A^k(V)$*),* αE *is not in* $(L_k(V))_0$ *(resp.* $(L^k(V))_0$*).*

Remark 1.1. For Conjecture 1.1, the implication (3) ⇒ (2) is obvious. Meanwhile, (1) ⇒ (2) and (1) ⇒ (3) are immediate corollaries of the well-known theorem of Saito [1971]. Thus the key point to prove Conjecture 1.1 is the implication (2) ⇒ (1) (see Theorem A). Conjectures 1.1 and 1.2 are verified in [Xu and Yau 1996] when $k = 0$. One of our main goals in this paper is to verify these two conjectures for the case of $k = n+1$. We obtain the following two main results.

Theorem A. *Let* $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ *be an isolated hypersurface singularity. Then* $(V, 0)$ *is quasihomogeneous if and only if its* $(n+1)$*-th Hessian algebra* $H_{n+1}(f)$ *is isomorphic to a finite-dimensional graded commutative local algebra* $\bigoplus_{i\ge 0} A_i$ *with* $A_0 = \mathbb{C}$.

Theorem B. *Let $(V, 0) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(z_0, \dots, z_n) = 0\}$ be an isolated hypersurface singularity with $n \geq 1$. Then $(V, 0)$ is a quasihomogeneous singularity if the following conditions are satisfied*:

(1) *$L_{n+1}(V)$ is isomorphic to a nonnegatively graded Lie algebra $\bigoplus_{i=0}^{k} L_i$ without center.*

(2) *There exists $E \in L_0$ such that $[E, D_i] = i D_i$ for any $D_i \in L_i$.*

(3) *For any element $\alpha \in m - m^2$ where m is the maximal ideal of $H_{n+1}(f)$, αE is not in L_0 (For brevity, we use L_i to denote $(L_{n+1}(V))_i$).*

Remark 1.2. We can only prove Conjectures 1.1 and 1.2 for $k = n+1$. The reason is that the proof of Theorem B depends on Theorem A. In our proof of Theorem A, we use a beautiful result of Saito [1974, Corollary 3.8], which cannot be generalized to general k. As for $L^k(V)$, we can only verify the conjectures when k is sufficiently large (see Theorem C), $k = 1$ is still a open problem.

Theorem C. *Let $(V, 0)$ be an isolated hypersurface singularity defined by f with multiplicity of at least three. Then $(V, 0)$ is quasihomogeneous if there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$*:

(1) *$L^k(V) \cong \bigoplus_{i=0}^{j} L_i$ which is nonnegatively graded and without center.*

(2) *There exists $E \in L_0$ such that $[E, D_i] = i D_i$ for all $D_i \in L_i$.*

(3) *For any element $\alpha \in m - m^2$ where m is the maximal ideal of $A(V)$, αE is not in L_0.*

In [Yau 1991], the Lie algebra $L_0(V) = L^0(V)$ was shown to be solvable. Thus a necessary condition for a commutative local Artinian algebra to be a moduli algebra is that its algebra of derivations is a solvable Lie algebra. Naturally one expects that $L_k(V)$ and $L^k(V)$ are also solvable. We prove that $L^k(V)$ ($k \geq 2$) is indeed solvable for any dimension n, and $k = 1$ is solvable for some special cases. For the sake of convenience to the readers, we abuse the notations of x and z. The subscript of x we shall use in the following theorem begins with 1 instead of 0 which is slightly different with the above two main theorems. We do this in order to be consistent with the symbols in [Yau 1983; 1986; 1991], so that the reader can easily refer to them.

Theorem D. *Let f be a homogeneous isolated singularity in n variables $x_1, \dots, x_n$ of degree $d \geq 4$. Then $L^k(V)$ is solvable for $k \geq 2$ or $k = 1, n = 4$.*

Remark 1.3. In Theorem D, the condition $d \geq 4$ cannot be omitted. In fact, there is a counterexample when $d = 3$.

Let $f = x^2y + xy^2$, then the $A^1(V)$ is $\mathcal{O}_2$ module the following relations:

$$x^2y + xy^2 = 0, \quad 2x^2y + xy^2 = 0, \quad 2xy^2 + y^3 = 0,$$
$$2xy^2 + x^2y = 0, \quad 2x^2y + x^3 = 0.$$

The monomial basis for $A^1(V)$ is

$$1, \quad x, \quad y, \quad x^2, \quad xy, \quad y^2.$$

It is easy to check that $x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\in L^1(V)$, Hence $L^1(V)$ is not solvable.

2. The derivation Lie algebra of a graded commutative Artinian algebra

We first state some elementary properties of the derivation Lie algebra of a graded commutative Artinian local algebra.

Lemma 2.1. *Let $A=\bigoplus_{i=0}^{t} A_i$ be a graded commutative Artinian local algebra. Then the derivation algebra of A denoted by $L(A)$ is a graded Artinian Lie algebra. (Here Artinian Lie algebra means $L(A)$ is finite dimension as $\mathbb{C}$-vector space.)*

Proof. See Lemma 2.1 in [Xu and Yau 1996]. □

Definition 2.1. The socle of a local Artinian algebra A with maximal ideal m is the complex vector subspace $\operatorname{Soc} A=\{a\in A : a\cdot m=0\}$ in A. The type of A is the complex dimension of $\operatorname{Soc} A$ as a vector space. The algebra A is Gorenstein when its type is one.

Lemma 2.2. *Let A be a commutative Artinian local algebra. Let $D\in L(A)$ be any derivation of A. Then D preserves the m-adic filtration of A, i.e., $D(m)\subset m$, where m is the maximal ideal of A.*

Proof. See Lemma 2.5 in [Xu and Yau 1996]. □

Proposition 2.1. *Let $A=\bigoplus_{i=0}^{k} A_i$ be a graded commutative Artinian local algebra with $A_0=\mathbb{C}$. Suppose the maximal ideal of A is generated by A_j for some $j>0$. Then $L(A)$ is a graded Lie algebra without negative weight.*

Proof. See Proposition 2.6 in [Xu and Yau 1996]. □

Lemma 2.3. *Let f be a weighted homogeneous polynomial with isolated singularity in $z_0,\dots,z_n$ variables of type $(\alpha_0,\dots,\alpha_n;d)$. Assume $\mathrm{wt}(z_0)=\alpha_0\ge \mathrm{wt}(z_1)=\alpha_1\ge\cdots\ge \mathrm{wt}(z_n)=\alpha_n$. Then f must be of either the form*

$$f=z_0^m+a_1(z_1,\dots,z_n)z_0^{m-1}+\cdots+a_{m-1}(z_1,\dots,z_n)z_0+a_m(z_1,\dots,z_n),$$

or

$$f=z_0^m z_i+a(z_1,\dots,z_n)z_0^{m-1}+\cdots+a_{m-1}(z_1,\dots,z_n)z_0+a_m(z_1,\dots,z_n).$$

Proof. See Lemma 2.1 in [Chen et al. 1995]. □

3. Proof of Theorems A and B

We first recall the following useful lemma.

Lemma 3.1 (Rossi). *Let* $(V, 0) = \{(z_0, \dots, z_n) : f(z_0, \dots, z_n) = 0\} \subset \mathbb{C}^{n+1}$ *be an isolated hypersurface singularity. Let* $\theta = \sum_{i=0}^n a_i(z) \frac{\partial}{\partial z_i}$ *be a holomorphic vector field of* $(V, 0)$. *Then* $a_i(0) = 0$ *for* $0 \le i \le n$.

Proof. See [Rossi 1963]. □

Proof of Theorem A. If $(V, 0)$ is a quasihomogeneous singularity, then by the theorem of Saito, we can assume that f is a weight homogeneous polynomial after a biholomorphic change if necessary. So the moduli ideal $(f) + J_f + h_{n+1}(f) = J_f + h(f)$ is a graded ideal and $H_{n+1}(f) = \mathcal{O}_{n+1}/((f) + J_f + h_{n+1}(f)) = \bigoplus_{i \ge 0} A_i$ with $A_0 = \mathbb{C}$.

On the other side, we assume that $H_{n+1}(f) = \bigoplus_{i \ge 0} A_i$ with $A_0 = \mathbb{C}$. Let $m = \bigoplus_{i \ge 1} A_i$ be the maximal ideal of $H_{n+1}(f)$. It is not difficult to find a $\mathbb{C}$-basis of m/m^2, denoted by $\{x_0, \dots, x_n\}$, with $x_i \in A_{q_i}$ for $0 \le i \le n$. Let $E : H_{n+1}(f) \to H_{n+1}(f)$ be the linear map such that the restriction of E on A_i is just multiplication by i. Then it is easy to see E satisfies Leibniz rule on $H_{n+1}(f)$, i.e., E is a derivation of $H_{n+1}(f)$. E can be viewed as a derivation of $\mathbb{C}[x_0, \dots, x_n]$ which leaves the moduli ideal $(f) + J_f + h_{n+1}(f)$ in $\mathcal{O}_{n+1}$ invariant. E is of the form $\sum_{i=0}^n q_i x_i \frac{\partial}{\partial x_i}$. If we let the degree of x_i be q_i for $0 \le i \le n$, then $\mathbb{C}[x_0, \dots, x_n]$ is graded and the natural map $\mathbb{C}[x_0, \dots, x_n] \to H_{n+1}(f)$ is a graded homomorphism of degree 0. Let $\bigoplus_{r>0} J_r$ be the grading of the moduli ideal $(f) + J_f + h(f)$. As E is a graded derivation of degree 0, E leaves J_r invariant for all $r > 0$. Since $\ker(E|_{J_r}) = 0$ and $\dim_{\mathbb{C}} J_r < \infty$, we obtain that $E|_{J_r}$ is surjective for all $r > 0$. Hence $E : (f) + J_f + h_{n+1}(f) \to (f) + J_f + h_{n+1}(f)$ is bijective. Let b_i, r_i and $a_{i0}, a_{i1}, \dots, a_{in}$ be such that

$$E\left(\frac{\partial f}{\partial x_i}\right) = b_i f + \sum_{j=0}^n a_{ij} \frac{\partial f}{\partial x_j} + r_i h(f)$$

for all $0 \le i \le n$. Let e, h and p_j be such that

$$E(h(f)) = ef + \sum_{j=0}^n p_j \frac{\partial f}{\partial x_j} + h \cdot h(f).$$

By the surjectivity of $E : (f) + J_f + h_{n+1}(f) \to (f) + J_f + h_{n+1}(f)$, there exist c_i, s_i and $d_{i0}, d_{i1}, \dots, d_{in}$ such that

$$(1) \quad \frac{\partial f}{\partial x_i} = E\left(c_i f + \sum_{j=0}^n d_{ij} \frac{\partial f}{\partial x_j} + s_i h(f)\right)$$

$$
\begin{aligned}
&= E(c_i)f + c_i\sum_{j=0}^{n} q_j x_j \frac{\partial f}{\partial x_j} + \sum_{j=0}^{n} E(d_{ij})\frac{\partial f}{\partial x_j} \\
&\quad + \sum_{j=0}^{n} d_{ij}\left(b_j f + \sum_{l=0}^{n} a_{jl}\frac{\partial f}{\partial x_l} + r_j h(f)\right) \\
&\quad + E(s_i)h(f) + s_i\left(ef + \sum_{j=0}^{n} p_j \frac{\partial f}{\partial x_j} + h\cdot h(f)\right) \\
&= \left(E(c_i) + \sum_{j=0}^{n} d_{ij}b_j + s_i e\right) f + c_i \sum_{j=0}^{n} q_j x_j \frac{\partial f}{\partial x_j} + \sum_{j=0}^{n} E(d_{ij})\frac{\partial f}{\partial x_j} \\
&\quad + \sum_{j=0}^{n} E(d_{ij})\frac{\partial f}{\partial x_j} + \sum_{j=0}^{n} d_{ij}\sum_{l=0}^{n} a_{jl}\frac{\partial f}{\partial x_l} + s_i \sum_{j=0}^{n} p_j \frac{\partial f}{\partial x_j} \\
&\quad + (E(s_i) + s_i h)h(f)
\end{aligned}
$$

$$
\begin{aligned}
(2) \qquad &= \left(E(c_i) + \sum_{j=0}^{n} d_{ij}b_j + s_i e\right) f \\
&\quad + \sum_{j=0}^{n}\left[c_i q_j x_j + E(d_{ij}) + \sum_{l=0}^{n} d_{il}a_{lj} + s_i p_j\right]\frac{\partial f}{\partial x_j} + (E(s_i) + s_i h)h(f).
\end{aligned}
$$

Now we assume that f is not quasihomogeneous. Recall the beautiful result of Saito [1974, Corollary 3.8]: Let $f \in \mathcal{O}_{n+1}$ be a germ of a holomorphic function which defines a hypersurface with an isolated singularity at 0, then f is not quasihomogeneous, precisely when

$$
h(f) = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{0\le i,j\le n} \in (f) + J_f.
$$

Without loss of generality, we assume that $r_i = 0$, $s_i = 0$ for $0 \le i \le n$ and $h = 0$. Thus

$$
\begin{aligned}
(3) \qquad \frac{\partial f}{\partial x_i} &= E\left(c_i f + \sum_{j=0}^{n} d_{ij}\frac{\partial f}{\partial x_j}\right) \\
&= \left(E(c_i) + \sum_{j=0}^{n} d_{ij}b_j\right) f + \sum_{j=0}^{n}\left[c_i q_j x_j + E(d_{ij}) + \sum_{l=0}^{n} d_{il}a_{lj}\right]\frac{\partial f}{\partial x_j}.
\end{aligned}
$$

Let

$$
\theta_i = \sum_{j=0}^{n}\left[c_i q_j x_j + E(d_{ij}) + \sum_{l=0}^{n} d_{il}a_{lj} - \delta_{ij}\right]\frac{\partial}{\partial x_j}.
$$

Then $\theta_i(f) = \big(E(c_i) + \sum_{j=0}^n d_{ij}b_j\big)f$. So θ_i is a holomorphic vector field of $\{f(x_0, \ldots, x_n) = 0\}$. By Lemma 3.1, $\theta_{ij}(0) = 0$ for all $0 \le j \le n$, where we write $\theta_i = \sum_{j=0}^n \theta_{ij} \frac{\partial}{\partial x_j}$. Observe that for any $g \in \mathbb{C}[x_0, \ldots, x_n]$, $E(g)$ vanishes at 0. Therefore we conclude that

$$\left(\sum_{l=0}^n d_{il}a_{lj} - \delta_{ij}\right)(0) = 0$$

for all $0 \le i \le n$. This means that

$$\begin{bmatrix} d_{00}(0) & d_{01}(0) & \ldots & d_{0n}(0) \\ d_{10}(0) & d_{11}(0) & \ldots & d_{1n}(0) \\ \cdots & \cdots & \cdots & \cdots \\ d_{n0}(0) & d_{n1}(0) & \ldots & d_{nn}(0) \end{bmatrix} \cdot \begin{bmatrix} a_{00}(0) & a_{01}(0) & \ldots & a_{0n}(0) \\ a_{10}(0) & a_{11}(0) & \ldots & a_{1n}(0) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n0}(0) & a_{n1}(0) & \ldots & a_{nn}(0) \end{bmatrix} = I,$$

where I is the identity matrix. On the other hand, by the surjectivity of

$$E : (f) + J_f + h(f) \to (f) + J_f + h(f),$$

there exist c and $d_0, \ldots, d_n$ such that

$$\begin{aligned} (4) \quad f &= E\left(cf + \sum_{i=0}^n d_i \frac{\partial f}{\partial x_i}\right) \\ &= E(c)f + c\sum_{j=0}^n q_j x_j \frac{\partial f}{\partial x_j} + \sum_{i=0}^n E(d_i)\frac{\partial f}{\partial x_i} + \sum_{i=0}^n d_i\left(b_i f + \sum_{j=0}^n a_{ij}\frac{\partial f}{\partial x_j}\right) \\ &= \left(E(c) + \sum_{i=0}^n b_i d_i\right)f + \sum_{j=0}^n\left(cq_j x_j + E(d_j) + \sum_{i=0}^n d_i a_{ij}\right)\frac{\partial f}{\partial x_j}. \end{aligned}$$

Let

$$H = \sum_{j=0}^n \left(cq_j x_j + E(d_j) + \sum_{i=0}^n d_i a_{ij}\right)\frac{\partial}{\partial x_j}.$$

Then $H(f) = [1 - E(c) - b_0d_0 - b_1d_1 - \cdots - b_nd_n]f$. So H is a vector field of $\{f(x_0, \ldots, x_n) = 0\}$. By Lemma 3.1, $H_i(0) = 0$ for $0 \le i \le n$, where $H = \sum_{i=0}^n H_i \frac{\partial}{\partial x_i}$. Since $E(d_i)$ vanishes at the origin for $i = 0, 1, \ldots, n$, we conclude that

$$\left(\sum_{i=0}^n d_i a_{ij}\right)(0) = 0$$

for all $0 \le j \le n$, i.e.,

$$\begin{bmatrix} d_0(0) & d_1(0) & \cdots & d_n(0) \end{bmatrix} \cdot \begin{bmatrix} a_{00}(0) & a_{01}(0) & \cdots & a_{0n}(0) \\ a_{10}(0) & a_{11}(0) & \cdots & a_{1n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0}(0) & a_{n1}(0) & \cdots & a_{nn}(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Since the matrix

$$\begin{bmatrix} a_{00}(0) & a_{01}(0) & \dots & a_{0n}(0) \\ a_{10}(0) & a_{11}(0) & \dots & a_{1n}(0) \\ \dots & \dots & \dots & \dots \\ a_{n0}(0) & a_{n1}(0) & \dots & a_{nn}(0) \end{bmatrix}$$

is nonsingular, we deduce that $\begin{bmatrix} d_0(0) & d_1(0) & \cdots & d_n(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$. It follows that $1 - E(c) - b_0 d_0 - b_1 d_1 - \cdots - b_n d_n$ is a unit in $\mathcal{O}_{n+1} = \mathbb{C}\{x_0, \dots, x_n\}$ since $E(c)$ vanishes at the origin. Because $(1 - E(c) - b_0 d_0 - b_1 d_1 - \cdots - b_n d_n) f = H(f)$, we conclude that $f \in \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)\mathcal{O}_{n+1}$. By definition of quasihomogeneity, $(V, 0)$ is quasihomogeneous which is contradict to our assumption. Hence f is quasihomogeneous, i.e., $(V, 0)$ is quasihomogeneous. □

Theorem 3.1. *Let $(V, 0)$ be a hypersurface singularity defined by a weighted homogeneous polynomial $f(z_0, \dots, z_n)$ which has an isolated singularity at the origin with multiplicity at least three. Suppose that $n \ge 1$. When the multiplicity is equal to three, we also need to suppose that $n > 1$. Then the Lie algebra $L_{n+1}(V)$ is graded and without center.*

Proof. Since f is a weighted homogeneous polynomial, the moduli ideal

$$(f) + J_f + h_{n+1}(f) = J_f + h_{n+1}(f)$$

is graded and hence

$$H_{n+1}(f) := \mathbb{C}[z_0, \dots, z_n]/((f) + J_f + h_{n+1}(f))$$

is graded. By Lemma 2.1, $L_{n+1}(V)$ is graded. Let D be an element in the center of $L_{n+1}(V)$. Write $D = \sum_i D_i$ where D_i is a derivation with weight i. Let

$$E = \sum_{i=0}^{n} q_i z_i \frac{\partial}{\partial z_i}$$

be the Euler derivation where $q_i = wt(z_i)$. Then

$$0 = [E, D] = \left[E, \sum_i D_i\right] = \sum_i i D_i$$

which implies $D_i = 0$ for $i \ne 0$. Hence D is a homogeneous element of weight 0.

If we write

$$D(f_i) = \sum c_j f_j + c \cdot h(f),$$

then by comparing weight of both sides, we get $c = 0$. This shows $D(f_i) \in J_f$. From now on we consider D as a derivation on $\mathcal{O}_{n+1}/J_f$. Let $D = \sum_{i=0}^{n} b_i \frac{\partial}{\partial z_i}$. Then

$$0 = [z_i E, D] = z_i[E, D] + [z_i, D]E = -b_i E.$$

This implies that $b_i \in$ Socle of $\mathcal{O}_{n+1}/J_f$ for all $0 \le i \le n$, i.e., $z_j \cdot b_i \in J_f$ for any $0 \le j \le n$. By local duality, we know that the socle of $\mathcal{O}_{n+1}/J_f$ is the highest degree nonzero subspace of $\mathcal{O}_{n+1}/J_f$. We shall assume without loss of generality that $d \ge 2q_0 \ge 2q_1 \ge \cdots \ge 2q_n$. By Lemma 2.1 in [Chen et al. 1995], we obtain that f must satisfy one of the following two cases:

$$f = \begin{cases} z_0^m + a_1(z_1, \ldots, z_n) z_0^{m-1} + \cdots + a_m(z_1, \ldots, z_n), & \text{Case (1)}, \\ z_0^{m-1} z_i + a_1(z_1, \ldots, z_n) z_0^{m-2} + \cdots + a_m(z_1, \ldots, z_n). & \text{Case (2)}. \end{cases}$$

Hence

$$\begin{aligned} wt\ h(f) &= (d - 2q_0) + (d - 2q_1) + \cdots + (d - 2q_n) \\ &= \begin{cases} m(n+1)q_0 - 2\sum_{j=0}^{n} q_j, & \text{Case (1)}, \\ (m-1)(n+1)q_0 + (n+1)q_i - 2\sum_{j=0}^{n} q_j, & \text{Case (2)}. \end{cases} \end{aligned}$$

If the multiplicity of f is at least four, we have $wt\ h(f) > 2q_0$ and $wt\big(\frac{\partial f}{\partial z_n}\big) \ge \cdots \ge wt\big(\frac{\partial f}{\partial z_0}\big) > 2q_0$. The fact that D is a homogeneous element of weight 0 implies that $wt(b_i) = wt(z_i) = q_i$ for all $0 \le i \le n$. Hence $wt(z_j \cdot b_i) \le 2q_0$. This would lead to a contradiction unless $b_i = 0$ for all $0 \le i \le n$. Hence $D = 0$.

Now we consider the case of $\operatorname{mult}(f) = 3$.

Case (1) $\quad f = z_0^3 + a_1(z_1, \ldots, z_n) z_0^2 + a_2(z_1, \ldots, z_n) z_0 + a_3(z_1, \ldots, z_n).$

In this case $wt\ h(f) = 3(n+1)q_0 - 2\sum_{i=0}^{n} q_i$ which implies that

$$wt\ h(f) > 3q_0 - q_n = wt\left(\frac{\partial f}{\partial z_n}\right) \ge \cdots \ge wt\left(\frac{\partial f}{\partial z_0}\right)$$

for all n. Since D is a homogeneous element of weight 0, we obtain that $D\big(\frac{\partial f}{\partial z_j}\big) \in J_f$ for all $0 \le j \le n$, i.e., D is a derivation of the algebra $\mathbb{C}[z_0, \ldots, z_n]/((f) + J_f)$. By Proposition 3.1 in [Xu and Yau 1996], we obtain that $D = 0$.

Case (2) $\quad f = z_0^2 z_i + a_1(z_1, \ldots, z_n) z_0 + a_2(z_1, \ldots, z_n).$

In this case $wt\ h(f) = 2(n+1)q_0 + nq_i - 2\sum_{j=0}^{n} q_j$, which implies that

$$wt\ h(f) > 2q_0 - q_n = wt\left(\frac{\partial f}{\partial z_n}\right) \ge \cdots \ge wt\left(\frac{\partial f}{\partial z_0}\right)$$

when $n \geq 2$. Since D is a homogeneous element of weight 0, we obtain that $D\big(\frac{\partial f}{\partial z_j}\big) \in J_f$ for all $0 \leq j \leq n$, i.e., D is a derivation of the algebra $\mathbb{C}[z_0, \dots, z_n]/(f) + J_f$. By Proposition 3.1 in [Xu and Yau 1996] we obtain that $D = 0$. □

Notice that L_0 has no center for $\mathrm{mult}(f) \geq 3$ and $n \geq 1$ [Xu and Yau 1996]. However, for L_{n+1}, some interesting new phenomena have been discovered, e.g., the following remark.

Remark 3.2. A counterexample when $\mathrm{mult}(f) = 3$ and $n = 1$ is as follows:

$$f = z_0^2 z_1 + a_1(z_1) z_0 + a_2(z_1).$$

Let $q_0 = sq_1$, then $a_1(z_1) = a z_1^{s+1}$, $a_2(z_1) = b z_1^{2s+1}$.

If $b = 0$ and $s = 1$, then $f = z_0^2 z_1 + a z_0 z_1^2$. Hence $\frac{\partial f}{\partial z_0} = 2z_0 z_1 + a z_1^2$, $\frac{\partial f}{\partial z_1} = z_0^2 + 2a z_0 z_1$ and $h(f) = -4(z_0^2 + a z_0 z_1 + a^2 z_1^2)$. It is obvious that D is a linear combination of $z_0 \frac{\partial}{\partial z_0}$, $z_0 \frac{\partial}{\partial z_1}$, $z_1 \frac{\partial}{\partial z_0}$ and $z_1 \frac{\partial}{\partial z_1}$.

It is easy to verify that $\big(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}, h(f)\big) = (z_0^2, z_1^2, z_0 z_1)$. Hence for any derivation $D' = (a_0 z_0 + a_1 z_1)\frac{\partial}{\partial z_0} + (b_0 z_0 + b_1 z_1)\frac{\partial}{\partial z_1}$, we obtain that

$$\begin{aligned}
\Big[z_0 \frac{\partial}{\partial z_0}, D'\Big] &= b_0 z_0 \frac{\partial}{\partial z_1} - a_1 z_1 \frac{\partial}{\partial z_0};\\
\Big[z_1 \frac{\partial}{\partial z_0}, D'\Big] &= a_0 z_1 \frac{\partial}{\partial z_0} + b_0 z_1 \frac{\partial}{\partial z_1} - b_0 z_0 \frac{\partial}{\partial z_0} - b_1 z_1 \frac{\partial}{\partial z_0};\\
\Big[z_0 \frac{\partial}{\partial z_1}, D'\Big] &= a_1 z_0 \frac{\partial}{\partial z_0} + b_1 z_0 \frac{\partial}{\partial z_1} - a_0 z_0 \frac{\partial}{\partial z_1} - a_1 z_1 \frac{\partial}{\partial z_1};\\
\Big[z_1 \frac{\partial}{\partial z_1}, D'\Big] &= a_1 z_1 \frac{\partial}{\partial z_0} - b_0 z_0 \frac{\partial}{\partial z_1}.
\end{aligned} \tag{5}$$

Let $D = z_0 \frac{\partial}{\partial z_0} + z_1 \frac{\partial}{\partial z_1}$, then $[D, D'] = 0$ for all derivations D', i.e., D is in the center.

Proof of Theorem B. By conditions (1) and (2), the adjoint representation of $L_{n+1}(V)$ is faithful and $ad\ E$ is semisimple. Take the Jordan decomposition of $E = S + N$, where S is semisimple and N is nilpotent. In view of the theorem on page 99 of [Humphreys 1975], we know that $N = 0$. Therefore, there exists a coordinate $x_0, \dots, x_n$ such that

$$E = \alpha_0 x_0 \frac{\partial}{\partial x_0} + \alpha_1 x_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_n x_n \frac{\partial}{\partial x_n}.$$

Observe that

$$[E, x_i E] = -x_i [E, E] + [E, x_i] E = \alpha_i x_i E. \tag{6}$$

Write $x_i E = D_0 + D_1 + \cdots + D_k$ where $D_i \in L_i$ for all $0 \le i \le k$. Then

$$[E, x_i E] = \sum_{j=0}^{k} [E, D_j] = \sum_{j=0}^{k} j D_j. \tag{7}$$

On the other hand, equation (6) says that

$$[E, x_i E] = \alpha_i \sum_{j=0}^{k} D_j. \tag{8}$$

If $\alpha_i = 0$, equations (7) and (8) imply $D_j = 0$ for all $1 \le j \le k$, i.e., $x_i E \in L_0$. This contradicts hypothesis (3) of the Theorem A. Therefore, $\alpha_i = j$ for some positive integer j between 1 and k in the view of equations (7) and (8). Since E acts on $H_{n+1}(f)$, $H_{n+1}(f)$ is graded according to the eigenspace of E. $H_{n+1}(f)$ is nonnegatively graded because all the α_i's are positive integers. Notice that the kernel of E on $H_{n+1}(f)$ is precisely $\mathbb{C}$. Hence we can apply Theorem A to conclude that $(V, 0)$ is a quasihomogeneous singularity. □

For the proof of Theorem C, it is much simpler:

Proof of Theorem C. By the proof of Theorem B, we know there is an Euler derivation in $L^k(V)$, written as $E = \sum_i \alpha_i x_i \frac{\partial}{\partial x_i}$. Notice that

$$E(f) = \sum_i \alpha_i x_i \frac{\partial f}{\partial x_i} \in (f, m^k J_f).$$

Take $k_0 \in \mathbb{N}$ such that $\mathrm{mult}(f) - 1 + k_0 > \deg(f)$. For $k \ge k_0$,

$$\deg(E(f)) = \deg\left(\sum_i \alpha_i x_i \frac{\partial f}{\partial x_i}\right) < \mathrm{mult}(m^k J_f).$$

(Here, $\mathrm{mult}(m^k J_f) := \min\{\mathrm{mult}(g) \mid g \in (m^k J_f) \text{ and } g \ne 0\}$, $\deg(f)$ means the degree of the highest degree monomial in f.) This means $E(f)$ can only be some multiple of f:

$$E(f) = \sum_i \alpha_i x_i \frac{\partial f}{\partial x_i} = af.$$

Comparing degrees of both sides shows that a is a nonzero constant. This tells us that $f \in (J_f)$, thus f is quasihomogeneous. □

The following theorem tells us that the condition "without center" is necessary:

Theorem 3.3. *Let f be weight homogeneous of multiplicity at least three, with weights given in Theorem C, then $L^k(V)$ is without center.*

Proof. Let D be in the center of $L^k(V)$ written as $D = \sum_i D_i$, where D_i is a derivation of weight i. Let

$$E = \sum \alpha_i x_i \frac{\partial}{\partial x_i}$$

be the Euler derivation. Then

$$0 = [E, D] = \left[E, \sum_i D_i\right] = \sum_i i D_i$$

which implies only $D_0 \neq 0$. Hence, D is homogeneous of weight 0. If we write

$$D = \sum a_i \frac{\partial}{\partial x_i}.$$

Then

$$0 = [x_i E, D] = x_i[E, D] + [x_i, D]E = -a_i E.$$

This means if we regard $a_i E$ as a derivation of $\mathbb{C}\{x_0, \ldots, x_n\}$, then for all $g \in \mathbb{C}\{x_0, \ldots, x_n\}$,

$$a_i E(g) \in (m^k J_f, f).$$

Since $(m^l J_f, f) \supset (m^k J_f, f)$ for all $l \leq k$, we know $a_i E$ maps any $g \in \mathbb{C}\{x_0, \ldots, x_n\}$ into $(m^l J_f, f)$. Let $l = 0$, therefore $a_i E$ can be regarded as a zero derivation of $A^0(V)$. This leads to a_i is in the socle of $A^0(V)$. By Lemma 2.3, we obtain that

$$d \geq wt(x_n) + 2wt(x_0) = \alpha_n + 2\alpha_0.$$

Since the socle of $A^0(V)$ is generated by Hess(f), we have

$$wt(a_i) = (d - 2\alpha_0) + \cdots + (d - 2\alpha_n) > \alpha_0.$$

However, D with weight 0 means $wt(a_i) = wt(x_i) \leq \alpha_0$, which is a contradiction. Hence, D must be zero as a derivation of $A^0(V)$, which implies that $a_i \in J_f$. Again, since f is of multiplicity at least three, $a_i \in J_f$ implies that $wt(a_i) \geq wt(f) - wt(x_0) \geq \alpha_0 + \alpha_n > wt(x_i)$. This is a contradiction. Therefore, $a_i = 0$. □

4. Solvability of $L^k(V)$

Firstly, we recall two classical results given in [Yau 1986; 1991].

Theorem 4.1 [Yau 1991]. *Let $sl(2, \mathbb{C})$ act on the formal power series ring $\mathbb{C}[[x_1, \ldots, x_n]]$, preserving the m-adic filtration where m is the maximal ideal* *in* $\mathbb{C}[\![x_1, \dots, x_n]\!]$. *Then there exists a coordinate system*

$$\begin{array}{l} x_1, \quad x_2, \quad \dots, \quad x_{l_1}; \\ x_{l_1+1}, \quad x_{l_1+2}, \quad \dots, \quad x_{l_1+l_2}; \\ \qquad\qquad \vdots \\ x_{l_1+l_2+\cdots+l_{r-1}+1}, \quad \dots, \quad x_{l_1+l_2+\cdots+l_r}; \\ x_{l_1+l_2+\cdots+l_r+1}, \quad \dots, \quad x_n \end{array} \tag{9}$$

via

$$\begin{aligned} H &= H_1 + \cdots + H_r, \\ X &= X_1 + \cdots + X_r, \\ Y &= Y_1 + \cdots + Y_r, \end{aligned} \tag{10}$$

where

$$\begin{aligned} H_j = {} & (l_j - 1)x_{l_1+\cdots+l_{j-1}+1}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+1}} \\ & + (l_j - 3)x_{l_1+\cdots+l_{j-1}+2}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+2}} + \cdots \\ & + (-(l_j - 3))x_{l_1+\cdots+l_j-1}\frac{\partial}{\partial x_{l_1+\cdots+l_j-1}} \\ & + (-(l_j - 1))x_{l_1+\cdots+l_j}\frac{\partial}{\partial x_{l_1+\cdots+l_j}}, \end{aligned} \tag{11}$$

$$\begin{aligned} X_j = {} & (l_j - 1)x_{l_1+\cdots+l_{j-1}+1}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+2}} + \cdots \\ & + i(l_j - i)x_{l_1+\cdots+l_{j-1}+i}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+i+1}} + \cdots \\ & + (l_j - 1)x_{l_1+\cdots+l_j-1}\frac{\partial}{\partial x_{l_1+\cdots+l_j}}, \end{aligned} \tag{12}$$

$$\begin{aligned} Y_j = {} & x_{l_1+\cdots+l_{j-1}+2}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+1}} + \cdots \\ & + x_{l_1+\cdots+l_{j-1}+i}\frac{\partial}{\partial x_{l_1+\cdots+l_{j-1}+i-1}} + \cdots \\ & + x_{l_1+\cdots+l_j}\frac{\partial}{\partial x_{l_1+\cdots+l_j-1}}. \end{aligned} \tag{13}$$

with $[X_j, Y_j] = H_j,\ [H_j, X_j] = 2X_j,\ [H_j, Y_j] = -2Y_j$.

Here we call r the irreducible representation number. A polynomial g is called of weight j if $H(g) = jg$ for some $j \in \mathbb{Z}$. Note that $l_i \geq 2$ for all $i = 1, \dots, r$.

Theorem 4.2 [Yau 1991]. *Let $sl(2, \mathbb{C})$ act on M_n^d, the space of homogeneous polynomial of degree $d \geq 2$ as in Theorem 4.1 with $l_1 \geq l_2 \geq \cdots \geq l_r \geq 2$. Let I be the complex vector subspace spanned by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$ where f is a homogeneous polynomial of degree $k+1$. If I is a $sl(2, \mathbb{C})$-submodule, then the singular set of f contains the x_1-axis and the x_{l_1}-axis.*

The solvability of $L^0(V)$ has been proved in [Yau 1991]. The solvability of $L^k(V)$ for $k \geq 2$ is proved below while $k = 1$ is much harder. We can only prove $A^1(V)$ does not admit some special $sl(2, \mathbb{C})$-action. (This is equivalent that $(f, mJ_f) = (mJ_f)$ does not admit certain special $sl(2, \mathbb{C})$-action, because a derivation D in $L^1(V)$ has the property $D(mJ_f) \subset (mJ_f)$.)

The key point of the proof for $k \geq 2$ is to show f is $sl(2, \mathbb{C})$-invariant, then Theorem 4.2 leads to contradiction.

Case 1: $k \geq 2$.

Proposition 4.1. *Let f be a homogeneous isolated singularity in n variables $x_1, \dots, x_n$ of degree $d \geq 4$. Then $L^k(V)$ is solvable for $k \geq 2$.*

Proof. Let $D \in L^k(V)$ be a derivation, then $D(f, m^k J_f) \subset (f, m^k J_f)$. By Leibniz rule, we obtain that $D(m^k J_f) = D(m^k) J_f + m^k D(J_f)$. Moreover $D(m^k) J_f \subset m^k J_f$, hence $D(I) \subset I$ is equivalent to $m^k D(J_f) \subset (f, m^k J_f)$ and $D(f) \subset (f, m^k J_f)$. (Here $I = (f, m^k J_f)$.)

We obtain

$$D(f) = a^D \cdot f + \sum_i b_i^D \cdot \frac{\partial f}{\partial x_i}, \tag{14}$$

where $a^D \in \mathcal{O}_n$ and $b_i^D \in m^k$. Whenever $D = H$, X or Y, it preserves the degree of f, hence the left-hand side of equation (14) is of degree d. However, $\deg\bigl(b_i^D \cdot \frac{\partial f}{\partial x_i}\bigr) > \deg(f)$ when $k \geq 2$, thus the term $\bigl(\sum_i b_i^D \cdot \frac{\partial f}{\partial x_i}\bigr)$ is zero. Equation (14) becomes

$$D(f) = a^D \cdot f$$

for $D = H$, X or Y. This means that f is $sl(2, \mathbb{C})$-invariant. Therefore J_f is $sl(2, \mathbb{C})$-invariant. By Theorem 4.2, f is singular on x_1-axis, which is a contradiction. □

Case 2: $k = 1$.

Now we consider the case of $k = 1$. The key point is as follows: If $L^1(V)$ is not solvable, then $(f, mJ_f) = (mJ_f)$ admits an action as in Theorem 4.1. Selecting a generator $g \in (mJ_f)$, we know that $H(g), X(g), Y(g) \in (mJ_f)$. Repeating this procedure, we can find that the number of generators is greater than n^2, which leads to a contradiction.

Case 2.1: $k = 1,\ n = 2$.

Proposition 4.2. *Let f be a homogeneous isolated singularity in 2 variables x_1, x_2 of degree $d \geq 4$. Then $L^1(V)$ is solvable.*

Proof. In the case $n = 2$, the action of $sl(2, \mathbb{C})$ is given by

$$X = x_1 \frac{\partial}{\partial x_2}, \quad Y = x_2 \frac{\partial}{\partial x_1}.$$

By Lemma 2.3, f is of one of the following two forms:

Form (1): $f = x_1^d + a_1 x_1^{d-1} x_2 + \cdots + a_d x_2^d$.

Form (2): $f = x_1^{d-1} x_2 + a_2 x_1^{d-2} x_2^2 + \cdots + a_d x_2^d$.

If f is of Form (1), then

$$x_1 \frac{\partial f}{\partial x_1} = d x_1^d + a_1 (d-1) x_1^{d-1} x_2 + \cdots + a_{d-1} x_1 x_2^{d-1} \in (m J_f).$$

Hence,

$$X^d Y^d \left(x_1 \frac{\partial f}{\partial x_1} \right) = c \cdot x_1^d \in (m J_f)$$

where c is a constant. This implies that

$$x_1^d, Y(x_1^d) = x_1^{d-1} x_2, \ldots, Y^d(x_1^d) = x_2^d$$

are all in $(m J_f)$. These are $d + 1 > 4$ monomials. However $\dim_{\mathbb{C}}(m J_f \cap M_2^d) = 4$, which is a contradiction. (The basis of $m J_f \cap M_2^d$ are $x_i \frac{\partial f}{\partial x_j}$ with $i, j \in \{1, 2\}$.)

If f is of Form (2), then

$$x_1 \frac{\partial f}{\partial x_2} = d x_1^d + 2 a_2 (d-1) x_1^{d-1} x_2 + \cdots + d a_d x_1 x_2^{d-1} \in (m J_f).$$

By similar reasoning, we get a contradiction. □

Remark 4.3. The proof for $n = 2$ can be generalized to more variables. However, we must require the $sl(2, \mathbb{C})$-action to be irreducible. For general action it is still open.

Recall in Theorem 4.1, for $H = H_1 + \cdots + H_r$, we call r the irreducible component number.

Definition 4.1. The $sl(2, \mathbb{C})$-action is called irreducible if the irreducible component number $r = 1$ and $l_1 = n$.

Case 2.2: $k = 1, n \geq 2, r = 1$ and $l_1 = n$.

Theorem 4.4 (weak Theorem D). *Let f be a homogeneous isolated singularity in n variables $x_1, \ldots, x_n$ of degree $d \geq 4$. Then $(m J_f)$ does not admit irreducible $sl(2, \mathbb{C})$-action.*

Proof. By Theorem 4.1, we obtain that

$$\begin{aligned} H &= (n-1)x_1\frac{\partial}{\partial x_1} + (n-3)x_2\frac{\partial}{\partial x_2} \\ &\quad + \cdots + (-(n-3))x_{n-1}\frac{\partial}{\partial x_{n-1}} + (-(n-1))x_n\frac{\partial}{\partial x_n} \\ X &= (n-1)x_1\frac{\partial}{\partial x_2} + 2(n-2)x_2\frac{\partial}{\partial x_3} + \cdots + i(n-i)x_i\frac{\partial}{\partial x_{i+1}} \\ &\quad + \cdots + (n-1)x_{n-1}\frac{\partial}{\partial x_n} \\ Y &= x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + \cdots + x_i\frac{\partial}{\partial x_{i-1}} + \cdots + x_n\frac{\partial}{\partial x_{n-1}}. \end{aligned}$$

By Lemma 2.3, we obtain $f = x_1^d + a_1(x_2, \ldots, x_n)x_1^{d-1} + \cdots + a_d(x_2, \ldots, x_n)$ (Form (1)) or $f = x_1^{d-1}x_s + a_2(x_2, \ldots, x_n)x_1^{d-2} + \cdots + a_d(x_2, \ldots, x_n)$ (Form (2)), where $a_i(x_2, \ldots, x_n)$ is a polynomial of degree i in variable $x_2, \ldots, x_n$. (We omit the constant coefficient in later discussion for simplicity.)

If f is of Form (1), then

$$x_i\frac{\partial f}{\partial x_1} = x_i x_1^{d-1} + \text{lower weight terms.}$$

If f is of Form (2), then

$$x_i\frac{\partial f}{\partial x_s} = x_i x_1^{d-1} + \text{lower weight terms.}$$

The following lemma shows that $x_i x_1^{d-1} \in (mJ_f)$ whenever f is of Form (1) or (2).

Lemma 4.1. *Let $g = \sum g^j$ be a homogeneous polynomial in (mJ_f), where g^j is weight j component of g, then $g^j \in (mJ_f)$.*

By Lemma 4.1, we obtain that these polynomials are in mJ_f:

$$\begin{array}{llll} x_1^d; & & & \\ x_1^{d-1}x_2; & & & \\ x_1^{d-1}x_3, & Y(x_1^{d-1}x_2); & & \\ x_1^{d-1}x_4, & Y(x_1^{d-1}x_3), & \ldots, & Y^2(x_1^{d-1}x_2); \\ \vdots & & & \\ x_1^{d-1}x_n, & Y(x_1^{d-1}x_{n-1}), & \ldots, & Y^{n-2}(x_1^{d-1}x_2); \\ Y^2(x_1^{d-1}x_{n-1}), & \ldots, & Y^{n-1}(x_1^{d-1}x_2). & \end{array}$$

Here on each row the polynomials are of same weight. We call these polynomials "Block 1".

All these polynomials are linear independent. Their weights are greater than or equal to $wt(x_1^{d-1}x_n) - 2 = (d-2)(n-1) - 2$. A similar discussion shows that

$$\begin{array}{llll} x_n^d; & & & \\ x_n^{d-1}x_{n-1}; & & & \\ x_n^{d-1}x_{n-2}, & X(x_n^{d-1}x_{n-1}); & & \\ x_n^{d-1}x_{n-3}, & X(x_n^{d-1}x_{n-2}), & \dots, & X^2(x_n^{d-1}x_{n-1}); \\ \vdots & & & \\ x_n^{d-1}x_1, & X(x_n^{d-1}x_2), & \dots, & X^{n-2}(x_1^{d-1}x_{n-1}); \\ X^2(x_n^{d-1}x_2), & \dots, & X^{n-1}(x_1^{d-1}x_{n-1}); & \end{array}$$

are in mJ_f, with weight less than or equal to $-(d-2)(n-1)+2$. We call these polynomials "Block 2".

Since $d \geq 4$ and $n > 2$, $-(d-2)(n-1)+2 < (d-2)(n-1)-2$. Thus polynomials in Block 1 are of weights greater than those in Block 2, which implies the polynomials in Block 1 and Block 2 are linearly independent.

In Block 1 and Block 2, there are $2(1+1+2+\cdots+n-1+n-2) = n(n+1)-2$ linear independent polynomials of degree d, while $\dim_{\mathbb{C}}(mJ_f \cap M_n^d) = n^2$, which is a contradiction. $\square$

Observation: In the proof of $r=1$, we construct two "blocks". The first one starts from $x_1^{d-1}x_i$, which is constructed by acting with Y. The second one starts from $x_n^{d-1}x_i$ and is constructed by acting with X.

Now for $r \neq 1$, firstly we assume $l_1 + \cdots + l_r = n$. We hope to construct blocks as above, then comparing the number of generators will lead to contradiction.

Case 3: $r > 1$, $l_1 + \cdots + l_r = n$.

We construct the following blocks (here $1 \leq i, j \leq r$.):

Block 1.1

$$\begin{array}{ll} \frac{\partial f}{\partial x_1}x_1; & \\ \frac{\partial f}{\partial x_1}x_2; & \\ \frac{\partial f}{\partial x_1}x_3, & Y\left(\frac{\partial f}{\partial x_1}x_2\right); \end{array}$$

$$\frac{\partial f}{\partial x_1}x_4, \qquad Y\left(\frac{\partial f}{\partial x_1}x_3\right), \quad Y^2\left(\frac{\partial f}{\partial x_1}x_2\right);$$

$$\vdots$$

$$\frac{\partial f}{\partial x_1}x_{l_1}, \quad Y\left(\frac{\partial f}{\partial x_1}x_{l_1-1}\right), \qquad \ldots, \qquad Y^{l_1-2}\left(\frac{\partial f}{\partial x_1}x_2\right);$$

$$Y^2\left(\frac{\partial f}{\partial x_1}x_{l_1-1}\right), \qquad \ldots, \qquad Y^{l_1-1}\left(\frac{\partial f}{\partial x_1}x_2\right).$$

Block 1.2

$$\frac{\partial f}{\partial x_1}x_{l_1+1};$$

$$\frac{\partial f}{\partial x_1}x_{l_1+2};$$

$$\frac{\partial f}{\partial x_1}x_{l_1+3}, \qquad Y\left(\frac{\partial f}{\partial x_1}x_{l_1+2}\right);$$

$$\vdots$$

$$\frac{\partial f}{\partial x_1}x_{l_1+l_2}, \quad Y\left(\frac{\partial f}{\partial x_1}x_{l_1+l_2-1}\right), \qquad \ldots, \qquad Y^{l_2-2}\left(\frac{\partial f}{\partial x_1}x_{l_1+2}\right);$$

$$Y^2\left(\frac{\partial f}{\partial x_1}x_{l_1+l_2-1}\right), \qquad \ldots, \qquad Y^{l_2-1}\left(\frac{\partial f}{\partial x_1}x_{l_1+2}\right).$$

Block 1.*r*

$$\frac{\partial f}{\partial x_1}x_{l_1+\cdots+l_{r-1}+1};$$

$$\frac{\partial f}{\partial x_1}x_{l_1+\cdots+l_{r-1}+2};$$

$$\frac{\partial f}{\partial x_1}x_{l_1+\cdots+l_{r-1}+3}, \; Y\left(\frac{\partial f}{\partial x_1}x_{l_1+\cdots+l_{r-1}+2}\right);$$

$$\vdots$$

$$\frac{\partial f}{\partial x_1}x_{l_1+\cdots+l_r}, \quad Y\left(\frac{\partial f}{\partial x_1}x_{l_1+\cdots+l_r-1}\right), \qquad \ldots, \qquad Y^{l_r-2}\left(\frac{\partial f}{\partial x_1}x_{l_1+\cdots+l_{r-1}+2}\right);$$

$$Y^2\left(\frac{\partial f}{\partial x_1}x_{l_1+\cdots+l_r-1}\right), \qquad \ldots, \qquad Y^{l_r-1}\left(\frac{\partial f}{\partial x_1}x_{l_1+\cdots+l_{r-1}+2}\right).$$

$$\vdots$$

Block *i.j*

$$\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+1};$$
$$\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+2};$$
$$\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+3}, \quad Y\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+2}\right);$$
$$\vdots$$
$$\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_j}, \qquad Y\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_j-1}\right),$$
$$\vdots \qquad \ldots, \qquad Y^{l_j-2}\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+2}\right);$$
$$Y^2\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_j-1}\right), \qquad \ldots, \qquad Y^{l_j-1}\left(\frac{\partial f}{\partial x_{l_1+\cdots+l_i}} x_{l_1+\cdots+l_{j-1}+2}\right).$$
$$\vdots$$

The number of linear independent polynomials in Block $i.j$ is

$$2(1+1+2+\cdots+l_j-1+l_j-2) = 2(l_j+1)l_j-2.$$

Similar to the construction of Block 1 and Block 2, we can construct another Block "dual" to Block $i.j$ with $2(l_j+1)l_j-2$ polynomials. If all above polynomials are linear independent, the whole number of linear independent polynomials is $4r(l_1(l_1+1)+\cdots+l_r(l_r+1)-2r)$. However

$$4r(l_1(l_1+1)+\cdots+l_r(l_r+1)-2r) > (l_1+\cdots+l_r)^2 = n^2.$$

This is a contradiction.

The problem arises on the linear independence of different blocks. To be more precise, there may exist variables in other blocks with same weight, so we cannot get linear independence by comparing weight. We use an example to explain this phenomenon.

Example 4.5. In the case $n=4$ and $l_1=l_2=2$,

$$H = H_1+H_2, \quad X = X_1+X_2, \quad Y = Y_1+Y_2.$$

x_1 and x_3 are of same weight. Let $f=(x_1+x_3)^4+x_2^4+x_4^4+x_1^3x_2+x_1^3x_4$ which defines an isolated singularity. The operation of taking highest weight is restricting polynomial to $x_2 = x_4 = 0$. For example,

$$\frac{\partial f}{\partial x_1} = 4(x_1 + x_3)^3 + 3x_1^2(x_2 + x_4).$$

The highest weight part of $\frac{\partial f}{\partial x_1}$ is $4(x_1 + x_3)^3$. However

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_2=x_4=0} = \frac{\partial f}{\partial x_3}.$$

Thus $\left.\frac{\partial f}{\partial x_1}\right|_{x_2=x_4=0}$, $\frac{\partial f}{\partial x_3}$ are linear dependent.

In this example, we only need to exchange $\frac{\partial f}{\partial x_3}$ to $\left.\frac{\partial f}{\partial x_2}\right|_{x_2=x_4=0}$. Then $\left.\frac{\partial f}{\partial x_1}\right|_{x_2=x_4=0}$, $\left.\frac{\partial f}{\partial x_2}\right|_{x_2=x_4=0}$ are linear independent. It reminds us that there exists a suitable way to select linear independent polynomials. This is illustrated in the following lemma:

Lemma 4.2. *If $r = 2$ and $l_1 = l_2$, then there exists g_1, g_2 of weight $(d-1)(l_1-1)$ in (mJ_f), such that the following four polynomials are linear independent*:

$$g_1x_1, \quad g_1x_{l_1+1}, \quad g_2x_1, \quad g_2x_{l_1+1}.$$

Proof. We first show how to construct g_1, g_2 from the derivatives of f. Then we prove the linear independence of above four polynomials. Let us consider the following polynomials:

$$\left.\frac{\partial f}{\partial x_1}\right|_{x_2=\cdots=x_{l_1}=x_{l_1+2}=\cdots=x_n=0}, \quad \left.\frac{\partial f}{\partial x_2}\right|_{x_2=\cdots=x_{l_1}=x_{l_1+2}=\cdots=x_n=0},$$

$$\ldots, \quad \left.\frac{\partial f}{\partial x_n}\right|_{x_2=\cdots=x_{l_1}=x_{l_1+2}=\cdots=x_n=0}.$$

These are polynomials in x_1, x_{l_1+1} of degree $d-1$, for simplicity we write them as

$$h_1, \quad \ldots, \quad h_n.$$

Let the common factor of $h_1, \ldots, h_n$ be h. Define

$$Y := \{h = 0\} \cap \{x_2 = \cdots = x_{l_1} = x_{l_1+2} = \cdots = x_n = 0\}.$$

Here $h, x_2, \ldots, x_{l_1}, x_{l_1+2}, \ldots, x_n$ are $n-1$ functions, and thus $\dim Y \geq 1$. However, by the definition of Y, $f|_Y = h_i|_Y = 0$ for all $i = 1, \ldots, n$. This contradicts that f defines an isolated singularity. Thus the common factor of $h_1, \ldots, h_n$ is 1.

We claim there exists $a_1, \ldots, a_n \in \mathbb{C}$ and $j \in \{1, \ldots, n\}$, such that $a_1h_1 + \cdots + a_nh_n$ and h_j do not have common factor. If the claim holds, then we denote $h_j = g_1$, $\sum_{i=1}^n a_ih_i = g_2$.

Now we prove the linear independence of $g_1x_1, g_1x_{l_1+1}, g_2x_1, g_2x_{l_1+1}$. Assume the contrary. Then there exists b_1, b_2, b_3, b_4 which are not all zero such that

$$g_1(b_1x_1 + b_3x_{l_1+1}) = g_2(b_2x_1 + b_4x_{l_1+1}).$$

Without loss of generality, we assume that $b_1, b_4 \neq 0$. $b_1x_1 + b_3x_{l_1+1}$ and $b_2x_1 + b_4x_{l_1+1}$ are coprime, otherwise g_1, g_2 have common factor. Thus the above equality implies

$$(b_1x_1 + b_3x_{l_1+1}) \mid g_2, \quad (b_2x_1 + b_4x_{l_1+1}) \mid g_1.$$

Observe that g_1, g_2 have degree $d - 1 \geq 3 > 1 = \deg(b_1x_1 + b_3x_{l_1+1})$; hence $\deg(g_2/(b_1x_1 + b_3x_{l_1+1})) \geq 2$. This means $g_2/(b_1x_1 + b_3x_{l_1+1})$ is a nontrivial polynomial, and is a factor of g_1, which contradicts that g_1, g_2 have no common factor.

At last we prove the claim. For j such that $h_j \neq 0$, we express h_j as product of irreducible polynomials:

$$h_j = s_1^{r_1} s_2^{r_2} \cdots s_l^{r_l}.$$

If h_{i_0} and h_j do not have common factor then we are done. So we assume each h_i and h_j have a common factor for any $i = 1, \ldots, n$. Since the common factor of $h_1, \ldots, h_n$ is 1, there exists two polynomials, say h_1, h_2, such that they have a different common factor with h_j. Without loss of generality, we assume $s_1 \mid h_1$, $s_2 \mid h_2$, $s_1 \nmid h_2$, $s_2 \nmid h_1$. Then $s_1, s_2 \nmid (h_1 + h_2)$. If $h_1 + h_2$ does not have common factor with h_j, then we are done. So we assume $s_3 \mid (h_1 + h_2)$. If $s_3 \mid h_1$, then $s_3 \mid (h_1 + h_2 - h_1)$, which contradicts that h_1, h_2 have a different common factor with h_j. Thus $s_3 \nmid h_1, h_2$. Then $s_1, s_2, s_3 \nmid ((h_1 + h_2) + h_1)$, by the same induction we know $s_4 \mid (2h_1 + h_2)$ or $2h_1 + h_2$ has no common factor with h_j. Since r_l is finite, this implies that the induction procedure must terminate, and so finally we can find a linear combination of h_1, h_2 such that it has no common factor with h_j. □

Case 3.1: $r = 2$.

The following proposition follows from Lemma 4.2 immediately.

Proposition 4.3. *Let f be homogeneous isolated singularity of degree d. Then (mJ_f) does not admit an $sl(2, \mathbb{C})$-action when $r = 2$, $l_1 + l_2 = n$.*

Proof. We divide it into two cases:

Case 1: $l_1 = l_2$.

Choose g_1, g_2 as in Lemma 4.2. Then we consider the following four blocks:

Block 1.1

g_1x_1;

g_1x_2;

g_1x_3, $\qquad Y(g_1x_2)$;

$$g_1x_4, \quad Y(g_1x_3), \quad Y^2(g_1x_2);$$
$$\vdots$$
$$g_1x_{l_1}, \quad Y(g_1x_{l_1-1}), \quad Y^{l_1-2}(g_1x_2);$$
$$Y^2(g_1x_{l_1-1}), \quad Y^{l_1-1}(g_1x_2);$$

Block 1.2

$$g_2x_1;$$
$$g_2x_2;$$
$$g_2x_3, \quad Y(g_2x_2);$$
$$g_2x_4, \quad Y(g_2x_3), \quad Y^2(g_2x_2);$$
$$\vdots$$
$$g_2x_{l_1}, \quad Y(g_2x_{l_1-1}), \quad Y^{l_1-2}(g_2x_2);$$
$$Y^2(g_2x_{l_1-1}), \quad Y^{l_1-1}(g_2x_2);$$

Block 2.1

$$g_1x_{l_1+1};$$
$$g_1x_{l_1+2};$$
$$g_1x_{l_1+3}, \quad Y(g_1x_{l_1+2});$$
$$g_1x_{l_1+4}, \quad Y(g_1x_{l_1+3}), \quad Y^2(g_1x_{l_1+2});$$
$$\vdots$$
$$g_1x_{l_1+l_2}, \quad Y(g_1x_{l_1+l_2-1}), \quad Y^{l_2-2}(g_1x_{l_1+2});$$
$$Y^2(g_1x_{l_1+l_2-1}), \quad \dots, \quad Y^{l_2-1}(g_1x_{l_1+2});$$

Block 2.2

$$g_2x_{l_1+1};$$
$$g_2x_{l_1+2};$$
$$g_2x_{l_1+3}, \quad Y(g_2x_{l_1+2});$$
$$g_2x_{l_1+4}, \quad Y(g_2x_{l_1+3}), \quad Y^2(g_2x_{l_1+2});$$
$$\vdots$$
$$g_2x_{l_1+l_2}, \quad Y(g_2x_{l_1+l_2-1}), \quad Y^{l_2-2}(g_2x_{l_1+2});$$
$$Y^2(g_2x_{l_1+l_2-1}), \quad \dots, \quad Y^{l_2-1}(g_2x_{l_1+2}).$$

The number of polynomials in all the blocks is $2(l_1(l_1+1)-2+l_2(l_2+1)-2)$. Replacing x_1, x_{l_1+1} by x_{l_1}, x_n and Y by X, we can get another $2(l_1(l_1+1)+l_2(l_2+1)-4)$ polynomials. However $4(l_1^2+l_2^2+l_1+l_2-4) > n^2$, which is a contradiction.

Case 2: $l_1 > l_2$.

In this case we can use same argument as in the irreducible case that

$$x_1^{d-1}x_i \in (mJ_f) \quad \text{for all } i.$$

And the block can be constructed as follows:

In Block 1.1, 2.1, we choose g_1 to be x_1^{d-1}. In Block 1.2, 2.2, we choose g_2 to be $x_{l_1+1}^{d-1}+g_3(x_1,\dots,x_{l_1})$, where g_3 is a polynomial of weight $(d-1)(l_2-1)$ such that $(x_{l_1+1}^{d-1}+g_3) \in J_f$. Then it leads to a contradiction similarly. □

Proof of Theorem D. When $k \geq 2$, the theorem follows immediately from Proposition 4.1. In the case of $n=4,\ k=1,\ r$ has to be 1 or 2. If $r=2$, we obtain that $l_1+l_2=4$ by Theorem 4.1. And the result follows from Proposition 4.3. If $r=1,\ l_1=4$, the result follows from Theorem 4.4. We only have to consider the cases $r=1,\ l_1=2$ or 3.

Case 1: $r=1,\ l_1=2$. The $sl(2,\mathbb{C})$-action is as follows:

$$H = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}, \quad X = x_1\frac{\partial}{\partial x_2}, \quad Y = x_2\frac{\partial}{\partial x_1}.$$

By Lemma 4.1, $x_1^{d-1}x_i \in mJ_f$. By the discussion in Proposition 4.2, x_1^{d-1}, x_3^{d-1}, x_4^{d-1} are in J_f. Thus

$$\begin{array}{llll}
x_1^d, & Y(x_1^d)=x_1^{d-1}x_2, & \dots, & Y^d(x_1^d)=x_2^d;\\
x_1^{d-1}x_3, & Y(x_1^{d-1}x_3), & \dots, & Y^{d-1}(x_1^{d-1}x_3)=x_2^{d-1}x_3;\\
x_1^{d-1}x_4, & Y(x_1^{d-1}x_4), & \dots, & Y^{d-1}(x_1^{d-1}x_4)=x_2^{d-1}x_4;\\
x_3^d; \quad x_4^d; & & & \\
x_3^{d-1}x_1, & x_3^{d-1}x_2; & x_4^{d-1}x_1, & x_4^{d-1}x_2;
\end{array}$$

are in mJ_f. The number of linear independent polynomials of degree d are $3d+6 > 16$, which is a contradiction.

Case 2: $r=1, l_1=3$. By the discussion in Theorem 4.4, we can find $3(3+1)-2=10$ linear independent polynomials in x_1, x_2, x_3. Thus we only need to find more than 6 polynomials. x_4^d, $x_4^{d-1}x_1$, $x_4^{d-1}x_2$, $x_4^{d-1}x_3$, $x_1^{d-1}x_4$, $x_2^{d-1}x_4$, $x_3^{d-1}x_4$ are satisfied. □

References

[Benson and Yau 1990] M. Benson and S. S.-T. Yau, "Equivalences between isolated hypersurface singularities", *Math. Ann.* **287**:1 (1990), 107–134. MR Zbl

[Chen 1995] H. Chen, "On negative weight derivations of the moduli algebras of weighted homogeneous hypersurface singularities", *Math. Ann.* **303**:1 (1995), 95–107. MR Zbl

[Chen et al. 1995] H. Chen, Y.-J. Xu, and S. S.-T. Yau, "Nonexistence of negative weight derivation of moduli algebras of weighted homogeneous singularities", *J. Algebra* **172**:2 (1995), 243–254. MR Zbl

[Chen et al. 2019] H. Chen, S. S.-T. Yau, and H. Zuo, "Non-existence of negative weight derivations on positively graded Artinian algebras", *Trans. Amer. Math. Soc.* **372**:4 (2019), 2493–2535. MR Zbl

[Chen et al. 2020a] B. Chen, H. Chen, S. S.-T. Yau, and H. Zuo, "The nonexistence of negative weight derivations on positive dimensional isolated singularities: generalized Wahl conjecture", *J. Differential Geom.* **115**:2 (2020), 195–224. MR Zbl

[Chen et al. 2020b] B. Chen, N. Hussain, S. S.-T. Yau, and H. Zuo, "Variation of complex structures and variation of Lie algebras, II: New Lie algebras arising from singularities", *J. Differential Geom.* **115**:3 (2020), 437–473. MR Zbl

[Dimca and Sticlaru 2015] A. Dimca and G. Sticlaru, "Hessian ideals of a homogeneous polynomial and generalized Tjurina algebras", *Doc. Math.* **20** (2015), 689–705. MR Zbl

[Elashvili and Khimshiashvili 2006] A. Elashvili and G. Khimshiashvili, "Lie algebras of simple hypersurface singularities", *J. Lie Theory* **16**:4 (2006), 621–649. MR Zbl

[Humphreys 1975] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics **21**, Springer, 1975. MR Zbl

[Hussain et al. 2018] N. Hussain, S. S.-T. Yau, and H. Zuo, "On the derivation Lie algebras of fewnomial singularities", *Bull. Aust. Math. Soc.* **98**:1 (2018), 77–88. MR Zbl

[Hussain et al. 2020] N. Hussain, S. S.-T. Yau, and H. Zuo, "On the new k-th Yau algebras of isolated hypersurface singularities", *Math. Z.* **294**:1-2 (2020), 331–358. MR Zbl

[Hussain et al. 2021a] N. Hussain, S. S.-T. Yau, and H. Zuo, "Inequality conjectures on derivations of local k-th Hessain algebras associated to isolated hypersurface singularities", *Math. Z.* **298**:3-4 (2021), 1813–1829. MR Zbl

[Hussain et al. 2021b] N. Hussain, S. S.-T. Yau, and H. Zuo, "k-th Yau number of isolated hypersurface singularities and an inequality conjecture", *J. Aust. Math. Soc.* **110**:1 (2021), 94–118. MR Zbl

[Khimshiashvili 2006] G. Khimshiashvili, "Yau algebras of fewnomial singularities", preprint, Universiteit Utrecht, 2006, available at http://www.math.uu.nl/publications/preprints/1352.pdf.

[Mather and Yau 1982] J. N. Mather and S. S.-T. Yau, "Classification of isolated hypersurface singularities by their moduli algebras", *Invent. Math.* **69**:2 (1982), 243–251. MR Zbl

[Rossi 1963] H. Rossi, "Vector fields on analytic spaces", *Ann. of Math.* (2) **78** (1963), 455–467. MR Zbl

[Saito 1971] K. Saito, "Quasihomogene isolierte Singularitäten von Hyperflächen", *Invent. Math.* **14** (1971), 123–142. MR Zbl

[Saito 1974] K. Saito, "Einfach-elliptische Singularitäten", *Invent. Math.* **23** (1974), 289–325. MR Zbl

[Seeley and Yau 1990] C. Seeley and S. S.-T. Yau, "Variation of complex structures and variation of Lie algebras", *Invent. Math.* **99**:3 (1990), 545–565. MR Zbl

[Xu and Yau 1996] Y.-J. Xu and S. S.-T. Yau, "Micro-local characterization of quasi-homogeneous singularities", *Amer. J. Math.* **118**:2 (1996), 389–399. MR Zbl

[Yau 1983] S. S. T. Yau, “Continuous family of finite-dimensional representations of a solvable Lie algebra arising from singularities”, *Proc. Nat. Acad. Sci. U.S.A.* **80**:24 (1983), 7694–7696. MR Zbl

[Yau 1986] S. S.-T. Yau, “Solvable Lie algebras and generalized Cartan matrices arising from isolated singularities”, *Math. Z.* **191**:4 (1986), 489–506. MR Zbl

[Yau 1991] S. S.-T. Yau, “Solvability of Lie algebras arising from isolated singularities and nonisolatedness of singularities defined by sl(2, **C**) invariant polynomials”, *Amer. J. Math.* **113**:5 (1991), 773–778. MR Zbl

[Yau and Zuo 2016a] S. S.-T. Yau and H. Zuo, “Derivations of the moduli algebras of weighted homogeneous hypersurface singularities”, *J. Algebra* **457** (2016), 18–25. MR Zbl

[Yau and Zuo 2016b] S. S.-T. Yau and H. Q. Zuo, “A sharp upper estimate conjecture for the Yau number of a weighted homogeneous isolated hypersurface singularity”, *Pure Appl. Math. Q.* **12**:1 (2016), 165–181. MR Zbl

[Yu 1996] Y. Yu, “On Jacobian ideals invariant by a reducible sl(2, **C**) action”, *Trans. Amer. Math. Soc.* **348**:7 (1996), 2759–2791. MR Zbl

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