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KENSHIRO TASHIRO

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COLLAPSED LIMITS OF COMPACT HEISENBERG MANIFOLDS WITH SUB-RIEMANNIAN METRICS

KENSHIRO TASHIRO

We show that every collapsed Gromov–Hausdorff limit of compact Heisenberg manifolds endowed with left-invariant Riemannian/sub-Riemannian metrics is isometric to a flat torus. We say that a sequence of sub-Riemannian manifolds collapses if their total measure with respect to Popp's volume converges to zero.

1. Introduction

A sub-Riemannian manifold is a triple $(M, \mathcal{D}, g)$, where M is a smooth manifold, $\mathcal{D}$ is a subbundle of the tangent bundle, and g is a metric on $\mathcal{D}$. In the same way to Riemannian manifolds, we can put a length structure and the associated distance function on bracket generating sub-Riemannian manifolds (see Definition 2.2). Sub-Riemannian manifolds appear as Gromov–Hausdorff limits of sequences of Riemannian manifolds. In general their sectional, Ricci and scalar curvature diverge as they converge to (non-Riemannian) sub-Riemannian manifolds. However some sub-Riemannian manifolds have the measure contraction property which reflects the Ricci curvature lower bound in a sense [Juillet 2009; Rifford 2013; Rizzi 2016; Barilari and Rizzi 2018]. These results lead us to study sub-Riemannian manifolds as examples of the singular Gromov–Hausdorff limit spaces.

In [Tashiro 2020], the author began to study the topological type of the Gromov–Hausdorff limit space of a sequence of (sub-)Riemannian manifolds. Here we use the notation (sub-)Riemannian metrics to cover both Riemannian and (non-Riemannian) sub-Riemannian metrics. Let H_n be the n-Heisenberg Lie group, $\mathfrak{h}_n$ the associated Lie algebra, and Γ a lattice in H_n. A quotient space $\Gamma\backslash\mathrm{H}_n$ is called a *compact Heisenberg manifold*. Let $\mathfrak{v}$ be a subspace in $\mathfrak{h}_n$ and $\langle\cdot,\cdot\rangle$ a scalar product on $\mathfrak{v}$. It induces the left-invariant sub-Riemannian structure on H_n. Since the induced geodesic distance on H_n has the isometric action Γ from the left, we obtain a quotient distance on $\Gamma\backslash\mathrm{H}_n$. We also call such a quotient distance on $\Gamma\backslash\mathrm{H}_n$ *left-invariant*. The author studied noncollapsed limits of compact Heisenberg manifolds with left-invariant (sub-)Riemannian metrics. Here we say that a sequence is *noncollapsed* if

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the total measure with respect to Popp's volume have a strictly positive lower bound. Popp's volume is a generalization of a canonical volume form of a Riemannian manifold (see Section 2C). We showed that the noncollapsed limit of a sequence of compact Heisenberg manifolds with left-invariant (sub-)Riemannian metrics are again diffeomorphic to a compact Heisenberg manifold of the same dimension.

In this paper, we study collapsed Gromov–Hausdorff limits of compact Heisenberg manifolds with left-invariant (sub-)Riemannian metrics. We say that a sequence of (sub-)Riemannian manifolds *collapses* if the total measure with respect to Popp's volume converges to zero. It complements our previous result [Tashiro 2020].

Theorem 1.1 (Main result). *Let $\{(\Gamma_k\backslash \mathrm{H}_n, \overline{\mathrm{dist}}_k)\}_{k\in\mathbb{N}}$ be a sequence of compact Heisenberg manifolds endowed with left-invariant (sub-)Riemannian metrics. Assume that this sequence converges in the Gromov–Hausdorff topology with a diameter upper bound $D > 0$ and the total measure with respect to Popp's measure converges to zero. Then the limit space is isometric to a flat torus of lower dimension.*

The idea of the proof is the following. It is well known that a compact Heisenberg manifold has a circle bundle structure $S^1 \to \Gamma\backslash \mathrm{H}_n \to \mathbb{T}^{2n}$. We show that if a sequence collapses, then the circle fiber also collapses to a point. Once we show that the fibers collapse, then the Gromov–Hausdorff limit is isometric to the limit of the base tori with the quotient distances. It is also known that a Gromov–Hausdorff limit of tori with flat metrics is isometric to a flat torus [Bettiol et al. 2018, Proposition 3.1]. This concludes the theorem.

2. Preliminaries from sub-Riemannian Lie group

In this section we prepare notation on sub-Riemannian metrics on Lie groups.

2A. ***Sub-Riemannian structure.*** Let G be a connected Lie group, $\mathfrak{g}$ the associated Lie algebra, $\mathfrak{v} \subset \mathfrak{g}$ a subspace and $\langle\cdot,\cdot\rangle$ a scalar product on $\mathfrak{v}$. For $x \in G$, denote by $L_x : G \to G$ the left translation by x. Define a sub-Riemannian metric on G by

$$\mathcal{D}_x = L_{x*}\mathfrak{v}, \qquad g_x(u, v) = \langle L_{x*}^{-1}u, L_{x*}^{-1}v\rangle.$$

Such a sub-Riemannian metric $(\mathcal{D}, g)$ is called *left-invariant*. We sometimes write a left-invariant sub-Riemannian metric by $(\mathfrak{v}, \langle\cdot,\cdot\rangle)$. Moreover, if $\dim(\mathfrak{g}/\mathfrak{v}) = k$, we say that a sub-Riemannian metric $(\mathfrak{v}, \langle\cdot,\cdot\rangle)$ is corank k. Notice that if $\mathfrak{v} = \mathfrak{g}$, i.e., corank 0, then $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ is a Riemannian metric.

Remark 2.1. From now on we shall declare the corank of sub-Riemannian metrics. If we do not declare the corank, then the word "sub-Riemannian metric" cover sub-Riemannian metrics of all corank.

For simplicity, we shall consider a Lie group with a left-invariant sub-Riemannian metric $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$. The associated distance function is given as follows. We say that an absolutely continuous path $c : [0, 1] \to G$ is *admissible* if $\dot{c}(t) \in L_{c(t)*}\mathfrak{v}$ a.e. $t \in [0, 1]$. We define the length of an admissible path by

$$\text{length}(c) = \int_0^1 \sqrt{\langle \dot{c}(t), \dot{c}(t)\rangle}\, dt.$$

For $x, y \in G$, define the distance function by

$$\text{dist}(x, y) = \inf\{\text{length}(c) \mid c(0) = x,\ c(1) = y,\ c \text{ is admissible}\}.$$

In general not every pair of points in G is joined by an admissible path. This implies that the value of the function dist may be the infinity. The following *bracket generating* condition ensures that any two points are joined by an admissible path.

Definition 2.2 (bracket generating distribution). For a sub-Riemannian Lie group $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$ and an integer $i \in \mathbb{N}$, let $\mathfrak{v}^i$ be the subspace in $\mathfrak{g}$ inductively defined by

$$\mathfrak{v}^1 = \mathfrak{v}, \quad \mathfrak{v}^{i+1} = \mathfrak{v} + [\mathfrak{v}, \mathfrak{v}^i].$$

We say that a subspace $\mathfrak{v}$ is bracket generating if there is $r \in \mathbb{N}$ such that $\mathfrak{v}^r = \mathfrak{g}$. We say $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$ is r-step if $\mathfrak{v}^{r-1} \subsetneq \mathfrak{v}^r = \mathfrak{g}$.

Theorem 2.3 (See, e.g., Theorem 3.31 in [Agrachev et al. 2020]). *Let* $(G, \mathfrak{v}, \langle\cdot,\cdot\rangle)$ *be a sub-Riemannian Lie group with a bracket generating distribution. Then the following two assertions hold*:

(1) (G, dist) *is a metric space.*

(2) *The topology induced by* dist *is equivalent to the manifold topology.*

In particular, $\text{dist} : G \times G \to \mathbb{R}$ *is continuous.*

Remark 2.4. Since the sub-Riemannian structure is left-invariant, the distance function is also left-invariant, that is, $\text{dist}(hx, hy) = \text{dist}(x, y)$ for all $h, x, y \in G$.

2B. ***Length minimizer.*** In sub-Riemannian geometry, there are two types of length minimizers; normal geodesics and abnormal geodesics. Normal geodesics are characterized as solutions to a specific differential equation, called the Hamiltonian equation. On the other hand, abnormal geodesics are not solutions to that equation. It sometimes appear in sub-Riemannian geometry, however, it is known that there is no nontrivial (i.e., nonconstant) abnormal geodesic if $\mathfrak{v}$ is *fat* (see [Montgomery 2002]). Here we say that a bracket generating subspace $\mathfrak{v} \subset \mathfrak{h}_n$ is fat if for all $U \in \mathfrak{v} \setminus \{0\}$, we have $\mathfrak{v} + [U, \mathfrak{v}] = \mathfrak{g}$. In the next section, we shall check that if G is the Heisenberg group, then every bracket generating subspace is fat. Therefore we omit the explanation of abnormal geodesics.

We say that a basis $\{U_1, \dots, U_n\}$ of $\mathfrak{g}$ is adapted if $\{U_1, \dots, U_m\}$ is an orthonormal basis of a sub-Riemannian metric $(\mathfrak{v}, \langle \cdot, \cdot \rangle)$. Let $H : T^*G \to \mathbb{R}$ be the function defined by

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^{m} p(L_{x*}U_i)^2 \quad (\lambda = (x, p) \in T^*G).$$

This function is called the *sub-Riemannian Hamiltonian*.

We say that a Lipschitz curve $\lambda = (x, p) : [0, T] \to T^*G$ is a solution to the Hamiltonian equation if it satisfies

$$\dot{x}(t) = \frac{\partial H}{\partial p}, \quad \dot{p}(t) = -\frac{\partial H}{\partial x}. \tag{1}$$

Such a curve $\lambda(t)$ is called a *normal extremal*, and its projection $x(t)$ is called a *normal geodesic*. It is known that every minimizer in sub-Riemannian manifold is either normal or abnormal geodesic. In particular, if a subspace $\mathfrak{v}$ is fat, then any length minimizer is a normal geodesic.

2C. *Popp's volume.* On a Riemannian Lie group (G, g), one has a canonical volume form defined by

$$\mathrm{dvol}_R = \nu_1 \wedge \cdots \wedge \nu_n,$$

where $\{\nu_1, \cdots \nu_n\}$ is a dual coframe of an orthonormal basis. The induced measure $m(\Omega) := \left| \int_\Omega \mathrm{dvol}_R \right|$ $(\Omega \subset G)$ is called the *volume measure*.

In sub-Riemannian geometry, we also have a canonical volume form, called *Popp's volume* introduced in [Montgomery 2002]. For simplicity, we only consider the 2-step case.

We do not introduce the original definition of Popp's volume, however, we define it with local coordinates given in [Barilari and Rizzi 2013]. Let $U_1, \dots, U_n$ be an adapted frame. Define the constant c_{ij}^l by

$$[U_i, U_j] = \sum_{l=1}^{n} c_{ij}^l U_l.$$

We call them the *structure constants*. We define the $(n-m)$ square matrix B by

$$B_{hl} = \sum_{i,j=1}^{m} c_{ij}^h c_{ij}^l.$$

Theorem 2.5 [Barilari and Rizzi 2013, Theorem 1]. *Let $U_1, \dots, U_n$ be a local adapted frame, and $\nu^1, \dots, \nu^n$ the dual coframe. Then Popp's volume* dvol_{sR} *is locally written by*

$$\mathrm{dvol}_{sR} = (\det B)^{-\frac{1}{2}} \nu^1 \wedge \cdots \wedge \nu^n.$$

The induced measure $m(\Omega) := \left| \int_\Omega \mathrm{dvol}_{sR} \right|$ $(\Omega \subset G)$ is called Popp's measure.

Remark 2.6. If a sub-Riemannian metric is corank 0, i.e., 1-step, then Popp's volume coincides with the canonical volume form. Indeed, an adapted frame of corank 0 sub-Riemannian metric is an orthonormal basis.

3. Compact Heisenberg manifolds

In this section, we recall fundamental properties on compact Heisenberg manifolds.

3A. ***Heisenberg groups.*** For $n \in \mathbb{N}$, the n-Heisenberg group H_n is the $(2n+1)$-dimensional Lie group diffeomorphic to $\mathbb{C}^n \times \mathbb{R}$ with the group product law

$$(w, z)(w', z') = \big(w + w', z + z' + \tfrac{1}{2}\mathfrak{Im}(w \cdot w')\big),$$

where $w \cdot w'$ is the Hermitian product on $\mathbb{C}^n$ and $\mathfrak{Im}$ denotes the imaginary part. We shall denote the associated Lie algebra by $\mathfrak{h}_n$.

We fix the coordinates of $\mathrm{H}_n \simeq \mathbb{C}^n \times \mathbb{R}$ by

$$(w, z) = (x_1, \dots, x_n, y_1, \dots, y_n, z),$$

where $w = \vec{x} + \vec{y}\sqrt{-1}$. We also fix the basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$ of the Lie algebra $\mathfrak{h}_n$ by

$$X_i = \partial_{x_i} - \tfrac{1}{2} y_i \partial_z, \quad Y_i = \partial_{y_i} + \tfrac{1}{2} x_i \partial_z, \quad Z = \partial_z.$$

A straightforward computation shows that $[X_i, Y_i] = Z$ for all $i = 1, \dots, n$ and the other brackets are zero.

For $U \in \mathfrak{h}_n$, let $\phi_U^t : \mathrm{H}_n \to \mathrm{H}_n$ be the flow of the vector field U at time t. The exponential map $\exp : \mathfrak{h}_n \to \mathrm{H}_n$ is defined by $\exp(U) := \phi_U^1(e)$, where e is the identity element. It is well defined since a left-invariant vector field is complete. It is well known that the exponential map on the Heisenberg group is a diffeomorphism. This fact allows us to identify the Heisenberg group H_n to its Lie algebra $\mathfrak{h}_n$ by

$$\exp : \mathfrak{h}_n \ni \sum_{i=1}^{n} (x_i X_i + y_i Y_i) + zZ \overset{\sim}{\longmapsto} (x_1, \dots, x_n, y_1, \dots, y_n, z) \in \mathrm{H}_n.$$

Let $(\mathfrak{v}, \langle \cdot, \cdot \rangle)$ be a left-invariant sub-Riemannian metric on H_n. A subspace $\mathfrak{v}$ is bracket generating if and only if

$$\mathfrak{v} + \mathrm{Span}(Z) = \mathfrak{h}_n. \tag{2}$$

In particular, the corank of a bracket generating subspace is 0 or 1. From now on we always assume the bracket generating condition (2). Moreover, by (2), we can easily check that if a subspace $\mathfrak{v} \subset \mathfrak{h}_n$ satisfies bracket generating condition, then it is fat. Therefore the sub-Riemannian Heisenberg group does not have nontrivial abnormal minimizers.

Let $\Gamma < \mathrm{H}_n$ be a lattice in H_n, that is, a discrete cocompact subgroup. Since a sub-Riemannian metric $(\mathfrak{v}, \langle\cdot,\cdot\rangle)$ is left-invariant, the left multiplication by Γ induces an isometric action on H_n. Therefore we can define the sub-Riemannian metric on $\Gamma\backslash\mathrm{H}_n$ via the quotient map. We shall denote such a quotient sub-Riemannian metric on $\Gamma\backslash\mathrm{H}_n$ by $\overline{\mathrm{dist}}$.

3B. ***Isometry classes of compact Heisenberg manifolds.*** In this section, we consider isometry classes of left-invariant sub-Riemannian metrics on a compact Heisenberg manifold $\Gamma\backslash\mathrm{H}_n$. The detail is in [Tashiro 2020].

First of all, we recall the isomorphism classes of compact Heisenberg manifolds. Let D_n be the set of n-tuples of integers $\boldsymbol{r} = (r_1, \dots, r_n)$ such that r_i divides r_{i+1} for all $i = 1, \dots, n$. For $\boldsymbol{r} \in D_n$, let $\Gamma_{\boldsymbol{r}} < \mathrm{H}_n$ be the discrete subgroup defined by

$$\Gamma_{\boldsymbol{r}} = \langle r_1 X_1, \dots, r_n X_n, Y_1, \dots, Y_n, Z\rangle.$$

This gives a classification of lattices in the Heisenberg Lie group.

Theorem 3.1 [Gordon and Wilson 1986, Theorem 2.4]. *For any uniform lattice $\Gamma < \mathrm{H}_n$, there is an automorphism of H_n which sends Γ onto $\Gamma_{\boldsymbol{r}}$ for some $\boldsymbol{r} \in D_n$. Moreover, $\Gamma_{\boldsymbol{r}}$ is isomorphic to $\Gamma_{\boldsymbol{s}}$ if and only if $\boldsymbol{r} = \boldsymbol{s}$.*

Next we consider isometry classes of $\Gamma_{\boldsymbol{r}}\backslash\mathrm{H}_n$ for a fixed lattice $\Gamma_{\boldsymbol{r}}$. Fix a scalar product $\langle\cdot,\cdot\rangle_0$ on $\mathfrak{h}_n$ such that its orthonormal basis is $\{X_1, \dots, Y_n, Z\}$. Let A be a matrix of the form

$$A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \rho_A \end{pmatrix},$$

where $\tilde{A} \in \mathrm{GL}_{2n}(\mathbb{R})$ and $\rho_A \in \mathbb{R}$. Moreover let $J_n \in \mathrm{Skew}_{2n}(\mathbb{R})$ be a skew-symmetric matrix given by

$$J_n = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix},$$

where I_n is the identity matrix of size n. We say that a matrix A is of *canonical form* if

$$ {}^t\tilde{A} J_n \tilde{A} = \begin{pmatrix} O & \mathrm{diag}(d_1(A), \dots, d_n(A)) \\ -\mathrm{diag}(d_1(A), \dots, d_n(A)) & O \end{pmatrix},$$

where $d_1(A), \dots, d_n(A)$ are nondecreasing positive numbers such that the imaginary numbers $\pm\sqrt{-1}d_1, \dots, \pm\sqrt{-1}d_n$ are the eigenvalues of ${}^t\tilde{A} J_n \tilde{A}$.

For a matrix A of canonical form, define the scalar product $\langle\cdot,\cdot\rangle_A$ on $\mathrm{Im}(A)$ by the norm

$$\|u\|_A := \min\{\|w\|_0 \mid u = Aw\}.$$

It is equivalent to the following definition: $\langle\cdot,\cdot\rangle_A$ is the scalar product which has an orthonormal basis $\{\tilde{A}X_1, \dots, \tilde{A}Y_n, \rho_A Z\}$. A pair $(\mathrm{Im}(A), \langle\cdot,\cdot\rangle_A)$ gives a sub-Riemannian metric on H_n of corank 0 (resp. corank 1) if $\rho_A \neq 0$ (resp. $\rho_A = 0$). If $\rho_A = 0$, then the subspace $\mathrm{Im}(A)$ is $\mathfrak{v}_0$, where

$$\mathfrak{v}_0 := \mathrm{Span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}.$$

These types of metrics cover all isometry classes of left invariant (sub-)Riemannian metrics on compact Heisenberg manifolds.

Theorem 3.2 [Tashiro 2020, Theorem 3.4]. *For any compact Heisenberg manifold with a bracket generating left-invariant sub-Riemannian metric* $(\Gamma\backslash\mathrm{H}_n, \mathfrak{v}, \langle\cdot,\cdot\rangle)$, *there exists an n-tuple* $\boldsymbol{r} \in D_n$ *and a matrix A of canonical form such that* $(\Gamma_{\boldsymbol{r}}\backslash\mathrm{H}_n, \mathrm{Im}(A), \langle\cdot,\cdot\rangle_A)$ *is isometric to* $(\Gamma\backslash\mathrm{H}_n, \mathfrak{v}, \langle\cdot,\cdot\rangle)$.

We shall denote the induced left-invariant distance function on H_n by dist_A and a quotient distance on $\Gamma\backslash\mathrm{H}_n$ by $\overline{\mathrm{dist}}_A$.

3C. *j-operator.* We recall the j-operator which plays an important role in the study of nilpotent Lie groups. Let $Z^* \in \mathfrak{h}_n^*$ be the dual covector of the vector $Z \in [\mathfrak{h}_n, \mathfrak{h}_n] \subset \mathfrak{h}_n$. For a matrix A of canonical form, define a skew symmetric operator $j(A) : \mathfrak{v}_0 \to \mathfrak{v}_0$ by

$$\langle j(A)(X), Y\rangle_A = Z^*([X, Y]).$$

Lemma 3.3 [Tashiro 2020, Lemma 4.1]. *The operator* $j(A) : \mathfrak{v}_0 \to \mathfrak{v}_0$ *has a matrix representation* ${}^t\tilde{A} J_n \tilde{A}$ *in the basis* $\{AX_1, \dots, AX_n, AY_1, \dots, AY_n\}$.

The positive number d_n can be regarded as the ℓ^∞-norm of the matrix ${}^t\tilde{A} J_n \tilde{A}$ as an element in the Euclidean space $\mathbb{R}^{4n^2}$. We also mention its ℓ^2-norm, the Hilbert–Schmidt norm of matrices.

Definition 3.4. For a matrix A of canonical form, we define $\delta(A) = \|{}^t\tilde{A} J_n \tilde{A}\|_{HS}$.

The following lemma is useful for later calculations.

Lemma 3.5 [Tashiro 2020, Lemma 5.1]. *For a matrix A of canonical form, we have*

(1) $\delta(A) = \sqrt{2\sum_{i=1}^n d_i(A)^2}$,

(2) $|\det(\tilde{A})| = \prod_{i=1}^n d_i(A)$.

3D. *Geodesics on Heisenberg groups.* Let A be a matrix of canonical form. For $i = 1, \dots, n$, define the functions $h_{x_i}, h_{y_i}, h_z : T^*\mathrm{H}_n \to \mathbb{R}$ by

$$h_{x_i}(\lambda) = p(L_{g*}AX_i), \quad h_{y_i}(\lambda) = p(L_{g*}AY_i), \quad h_z(p) = p(L_{g*}Z)$$

for $\lambda = (g, p) \in T^*\mathrm{H}_n$. Suppose that an admissible path

$$\gamma(t) = \sum_{i=1}^{n} x_i(t)AX_i + y_i(t)AY_i + z(t)Z$$

is length minimizing. By the Hamiltonian equation (1) with a linear modification, there is a lift $\lambda(t)$ of $\gamma(t)$ such that

$$\begin{cases} \dot{h}_{x_i} = d_i(A)h_z h_{y_i} & (i = 1, \dots, n), \\ \dot{h}_{y_i} = -d_i(A)h_z h_{x_i} & (i = 1, \dots, n), \\ \dot{h}_z = 0, & \\ \dot{x}_i = h_{x_i} & (i = 1, \dots, n), \\ \dot{y}_i = h_{y_i} & (i = 1, \dots, n), \\ \dot{z} = \frac{1}{2}\sum_{i=1}^{n} d_i(A)(x_i h_{y_i} - y_i h_{x_i}) + \rho_A^2 p_z, & \end{cases}$$

where we write

$$h_{x_i}(t) = h_{x_i} \circ \lambda(t), \quad h_{y_i}(t) = h_{y_i} \circ \lambda(t), \quad h_z(t) = h_z \circ \lambda(t).$$

By proving this equation, we obtain the following parametrization of length minimizers.

Lemma 3.6 ([Eberlein 1994, Proposition 3.5] for corank 0 and [Rizzi 2016, Lemma 14] for corank 1 cases). *Let A be a matrix of canonical form and $\lambda : [0, T] \to T^*\mathrm{H}_n$ be the normal extremal with the initial data*

$$(h_{x_1}(0), \dots, h_{y_n}(0), h_z(0)) = (p_{x_1}, \dots, p_{y_n}, p_z) \in T_e^*\mathrm{H}_n = \mathfrak{h}_n^*.$$

Then the associated normal geodesic γ is given as follows.

If $p_z \neq 0$, then

$$\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} = \frac{1}{p_z d_i(A)} \begin{pmatrix} \sin(p_z d_i(A)t) & \cos(p_z d_i(A)t) - 1 \\ -\cos(p_z d_i(A)t) + 1 & \sin(p_z d_i(A)t) \end{pmatrix} \begin{pmatrix} p_{x_i} \\ p_{y_i} \end{pmatrix},$$

$$z(t) = \rho_A^2 p_z t + \frac{1}{2p_z} \sum_{i=1}^{n} \left(t - \frac{1}{p_z d_i(A)} \sin(p_z d_i(A)t) \right) (p_{x_i}^2 + p_{y_i}^2).$$

Moreover, the normal geodesic fails to be length minimizing over the time

$$T = \frac{2\pi}{|p_z| d_{i_m}(A)},$$

where $i_m \in \{1, \dots, n\}$ is the minimum integer such that $(p_{x_i}, p_{y_i}) \neq (0, 0)$.

If $p_z = 0$, then

$$\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} = \begin{pmatrix} p_{x_i} \\ p_{y_i} \end{pmatrix} t, \quad z(t) \equiv 0.$$

Remark 3.7. The initial data $(p_{x_1}, \dots, p_{y_n})$ is identified with the projection of initial vector into $\mathfrak{v}_0 \subset \mathfrak{h}_n = T_e \mathrm{H}_n$ via the identification

$$\mathbb{R}^{2n} \ni (p_{x_1}, \dots, p_{y_n}) \simeq \sum_{i=1}^{n} (p_{x_i} A X_i + p_{y_i} A Y_i) \in \mathfrak{v}_0.$$

For later arguments, we give an explicit distance from the identity to points in the horizontal direction and the vertical direction.

Lemma 3.8 ([Eberlein 1994, Proposition 3.11] for corank 0 and [Tashiro 2020, Lemma 5.2] for corank 1 cases). *For $U \in \mathfrak{v}_0$ and $V \in [\mathfrak{h}_n, \mathfrak{h}_n]$, we have*

$$\mathrm{dist}_A(e, U + V) \geq \|U\|_A.$$

Moreover, the equality holds if and only if $V = 0$.

Lemma 3.9. *For $z_0 \in \mathbb{R}$, the distance from e to $z_0 Z = (0, \dots, 0, z_0) \in \mathrm{H}_n$ is given by*

$$\mathrm{dist}_A(e, z_0 Z) = \min\left\{ \left| \frac{z}{\rho_A} \right|, \frac{2}{d_n(A)} \sqrt{|z_0| \pi d_n(A) - \pi^2 \rho_A^2} \right\},$$

with the convention

$$\begin{cases} \left| \dfrac{z_0}{\rho_A} \right| = +\infty & \text{if } \rho_A = 0, \\ \dfrac{2}{d_n(A)} \sqrt{|z_0| \pi d_n(A) - \pi^2 \rho_A^2} = +\infty & \text{if } |z_0| \pi d_n(A) - \pi^2 \rho_A^2 < 0. \end{cases}$$

Proof. For the simplicity we assume $z_0 > 0$. First let us consider a unit speed geodesic of the initial data $(0, \dots, 0, p_z)$. Then a unit speed normal geodesic $\gamma : [0, T] \to \mathrm{H}_n$ with $\gamma(T) = z_0 Z$ needs to satisfy

$$\begin{cases} z(T) = \rho_A^2 p_z T = z_0, \\ |p_z \rho_A| = 1 \end{cases}$$

Then the length is equal to the time $T = |z_0 / \rho_A|$ with the convention $|z_0/\rho_A| = +\infty$ if $\rho_A = 0$, i.e., sub-Riemannian metric of corank 1.

Next we consider geodesics of the initial data $(p_{x_1}, \dots, p_{y_n}) \neq 0$. Since the endpoint $z_0 Z$ is in the center $[\mathfrak{h}_n, \mathfrak{h}_n]$, the initial data $(p_{x_1}, \dots, p_{y_n}) \in \mathbb{R}^{2q} \simeq \mathfrak{v}_0$ need to be inside the eigenspace of $\pm\sqrt{d_i(A)}$ of j_A (with the multiplicity), where we use the identification

$$\mathbb{R}^{2n} \ni (p_{x_1}, \dots, p_{y_n}) \simeq \sum_{i=1}^{n} (p_{x_i} A X_i + p_{y_i} A Y_i) \in \mathfrak{v}_0.$$

Indeed, by the parametrization of $x_i(t)$, $y_i(t)$ in Lemma 3.6, the geodesic ends at $[\mathfrak{h}_n, \mathfrak{h}_n]$ only if the frequency of the trigonometric function is the same. Moreover, its length $T = 2\pi / (|p_z| d_i(A))$ is independent of the choice of the initial data

$(p_{x_1}, \dots, p_{y_n})$ as long as it belongs to the same eigenspace. Therefore we can assume that the initial covector is $(0, \dots, 0, p_{y_i}, 0, \dots, 0, p_z)$. If such a normal geodesic is unit speed and its endpoint is $z_0 Z$, then it needs to satisfy

$$\begin{cases} z(T) = \dfrac{2\pi\rho_A^2}{d_i(A)} + \dfrac{\pi}{p_z^2 d_i(A)} p_{y_i}^2 = z_0, \\ p_{y_i}^2 + (\rho_A p_z)^2 = 1. \end{cases}$$

This equation has a solution only if $z_0 \geq 2\pi\rho_A^2/d_i(A)$ with

$$p_{y_i} = \sqrt{\frac{z_0 d_i(A) - 2\pi\rho_A^2}{z_0 d_i(A) - \pi\rho_A^2}}, \quad p_z = \sqrt{\frac{\pi}{z_0 d_i(A) - \pi\rho_A^2}}.$$

Its length is

$$\frac{2\pi}{p_z d_i(A)} = \frac{2\sqrt{\pi}}{d_i(A)}\sqrt{z_0 d_i(A) - \pi\rho_A^2}.$$

Therefore the distance from e to $z_0 Z = (0, \dots, 0, z_0)$ is the minimum of two values

$$\min\left\{\left|\frac{z_0}{\rho_A}\right|, \frac{2\sqrt{\pi}}{d_1(A)}\sqrt{z_0 d_1(A) - \pi\rho_A^2}, \dots, \frac{2\sqrt{\pi}}{d_n(A)}\sqrt{z_0 d_n(A) - \pi\rho_A^2}\right\} \\ = \min\left\{\left|\frac{z_0}{\rho_A}\right|, \frac{2\sqrt{\pi}}{d_n(A)}\sqrt{z_0 d_n(A) - \pi\rho_A^2}\right\},$$

where we use $d_1(A) \leq \cdots \leq d_n(A)$. □

3E. ***Popp's volume form on Heisenberg group.*** In this section, we discuss Popp's volume form on the Heisenberg Lie group.

For a matrix A of canonical form with $\rho_A \neq 0$, denote by $\mathrm{dvol}_R(A)$ the canonical Riemannian volume form. Since it is the wedge of the dual coframe of an orthonormal frame, we have

$$\mathrm{dvol}_R(A) = \rho_A^{-1}(\det \tilde{A})^{-1} X_1^* \wedge \cdots \wedge Y_n^* \wedge Z^*.$$

In particular, the total measure of a Riemannian compact Heisenberg manifold $(\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n, \langle\cdot,\cdot\rangle_A)$ is

$$\text{(3)} \qquad \mathrm{meas}(\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n, \langle\cdot,\cdot\rangle_A) := \left|\int_{\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n} \mathrm{dvol}_R(A)\right| = \prod_{i=1}^n r_i |\rho_A^{-1}(\det \tilde{A})^{-1}|.$$

Next let A be a matrix of canonical form with $\rho_A = 0$. Denote by $\mathrm{dvol}_{sR}(A)$ Popp's volume associated to the sub-Riemannian metric $(\mathfrak{v}_0, \langle\cdot,\cdot\rangle_A)$.

By the definition, the (i, j)-th entry of the matrix ${}^t\tilde{A} J_n \tilde{A}$ is the structure constant c_{ij} of the basis $\{AX_1, \ldots, AY_n\}$, that is

$$\begin{pmatrix} Z^*([AX_i, AX_j]) & Z^*([AX_i, AY_{j-n}]) \\ Z^*([AY_{i-n}, AX_j]) & Z^*([AY_{i-n}, AY_{j-n}]) \end{pmatrix} = (c_{ij}) = {}^t\tilde{A} J_n \tilde{A} = \begin{pmatrix} O & \operatorname{diag}(d_1(A), \ldots, d_n(A)) \\ -\operatorname{diag}(d_1(A), \ldots, d_n(A)) & O \end{pmatrix}.$$

By Theorem 2.5, Popp's volume $\mathrm{dvol}_{sR}(A)$ is written by

$$\mathrm{dvol}_{sR}(A) = \delta(A)^{-1} (\det \tilde{A})^{-1} X_1^* \wedge \cdots \wedge Y_n^* \wedge Z^*.$$

In particular, the total measure of a sub-Riemannian compact Heisenberg manifold $(\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n, \mathfrak{v}_0, \langle \cdot, \cdot \rangle_A)$ is

$$\text{(4)} \quad \operatorname{meas}(\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n, \mathfrak{v}_0, \langle \cdot, \cdot \rangle_A) := \left| \int_{\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n} \mathrm{dvol}_{sR}(A) \right| = \prod_{i=1}^n r_i |\delta(A)^{-1} (\det \tilde{A})^{-1}|.$$

3F. ***The circle bundle structure.*** Fix a n-tuple of numbers $\boldsymbol{r} \in D_n$. We recall a circle bundle structure of a compact Heisenberg manifold $\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n$. Let $P : \mathrm{H}_n \to \mathfrak{h}_n \to \mathfrak{v}_0$ be the composition of the logarithm map and the projection. Denote the image of the lattice $\Gamma_{\boldsymbol{r}}$ by $\mathfrak{z}_{\boldsymbol{r}}$, which is again a lattice in $\mathfrak{v}_0$ isomorphic to $\mathbb{Z}^{2n}$. Then one obtains a surjective map $\bar{P} : \Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n \to \mathfrak{z}_{\boldsymbol{r}} \backslash \mathfrak{v}_0$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{H}_n & \xrightarrow{P} & \mathfrak{v}_0 \\ \downarrow{\scriptstyle P_{\Gamma_{\boldsymbol{r}}}} & & \downarrow{\scriptstyle P_{\mathfrak{z}_{\boldsymbol{r}}}} \\ \Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n & \xrightarrow{\bar{P}} & \mathfrak{z}_{\boldsymbol{r}} \backslash \mathfrak{v}_0 \end{array}$$

Here the vertical arrows are the quotient map. The compact Heisenberg manifold $\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n$ has a circle bundle structure by this map $\bar{P}$. For each $b \in \mathfrak{z}_{\boldsymbol{r}} \backslash \mathfrak{v}_0$, we denote by F_b the fiber over b.

Remark 3.10. Since a sub-Riemannian metric $\langle \cdot, \cdot \rangle_A$ is left-invariant, the diameter of a fiber is independent of the choice of a base point b. We shall denote the diameter of a fiber by $\operatorname{diam}(F_A)$.

The quotient metric on $\mathfrak{v}_0$ has an orthonormal basis $\{\tilde{A}X_1, \ldots, \tilde{A}X_{2n}\}$. Therefore we shall denote the induced distance on $\mathfrak{v}_0$ by $\mathrm{dist}_{\tilde{A}}$, and the quotient distance on $\mathfrak{z}_{\boldsymbol{r}} \backslash \mathfrak{v}_0$ by $\overline{\mathrm{dist}}_{\tilde{A}}$. In the next section, we use the circle bundle structure to show that the Gromov–Hausdorff limit of compact Heisenberg manifolds is isometric to that of the base flat tori.

4. Gromov–Hausdorff limits of compact Heisenberg manifolds

4A. ***Collapse of the circle fiber.*** We say that a sequence of compact metric spaces $\{M_k\}$ converges to a compact metric space N if the Gromov–Hausdorff distance $d_{GH}(M_k, N)$ converges to 0. We do not use the original definition of d_{GH} since it is complicated. Instead of the original definition, we use the ϵ-approximation map which is easier to compute.

Definition 4.1 [Fukaya 1990, Definition 1.1]. Let (M, d_M), (N, d_N) be a compact path metric spaces. For $\epsilon > 0$, we say a map $\phi : M \to N$ is an ϵ-Hausdorff approximation if it satisfies the following:

(i) The ϵ neighborhood of $\phi(M)$ in N is N.

(ii) For $u, v \in M$. we have

$$|d_M(u, v) - d_N(\phi(u), \phi(v))| < \epsilon.$$

It is known that if there is ϵ-Hausdorff approximation map between M, N, then $d_{GH}(M, N) < 2\epsilon$. Therefore if a metric space M_k has an ϵ_k-approximation to N such that $\epsilon_k \to 0$, then the sequence $\{M_k\}$ converges to N in the Gromov–Hausdorff topology.

We can check that the quotient map $\bar{P}$ is an ϵ-approximation with ϵ equal to the diameter of the fiber.

Lemma 4.2. *The quotient map* $\bar{P} : (\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n, \overline{\mathrm{dist}}_A) \to \mathfrak{z}_{\boldsymbol{r}} \backslash (\mathfrak{v}_0, \overline{\mathrm{dist}}_{\tilde{A}})$ *is a* $2\,\mathrm{diam}(F_A)$-*approximation map.*

Proof. Since the map $\bar{P}$ is surjective, we only need to check the almost isometric embeddability (Definition 4.1(ii)).

Let $u_1, u_2 \in \Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n$ be points in the compact Heisenberg manifold. By definition of the distance on the quotient space, there are $v_1, v_2 \in \Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n$ such that

$$\overline{\mathrm{dist}}_A(v_1, v_2) = \overline{\mathrm{dist}}_{\tilde{A}}(\bar{P}(u_1), \bar{P}(u_2)).$$

By the triangle inequality, we have

$$\begin{aligned} |\overline{\mathrm{dist}}_A(u_1, u_2) - \overline{\mathrm{dist}}_{\tilde{A}}(\bar{P}(u_1), \bar{P}(u_2))| &\le \overline{\mathrm{dist}}_A(u_1, v_1) + \overline{\mathrm{dist}}_A(u_2, v_2) \\ &\le 2\,\mathrm{diam}(F). \end{aligned}$$

This proves almost isometric embeddability. □

Proposition 4.3. *Let* $(\Gamma_{\boldsymbol{r}(k)} \backslash \mathrm{H}_n, \overline{\mathrm{dist}}_{A_k})$ *be a sequence of compact Heisenberg manifolds which has a uniform upper bound of the diameter. Assume that the diameter of the fibers* $\mathrm{diam}(F_{A_k})$ *converges to* 0. *Then its Gromov–Hausdorff limit is isometric to that of base flat tori* $(\mathfrak{z}_{\boldsymbol{r}} \backslash \mathfrak{v}_0, \overline{\mathrm{dist}}_{\tilde{A}_k})$.

In particular, the limit is isometric to a flat torus of lower dimension.

Proof. Since the diameter of the base flat tori is also uniformly bounded, we can assume that the sequence of base flat tori subconverges to a flat torus (N, d) (possibly a point). It is a consequence of [Bettiol et al. 2018, Proposition 3.1]. Then there are ϵ_k-approximation map $\varphi_k : (\mathfrak{z}_{\boldsymbol{r}} \backslash \mathfrak{v}_0, \overline{\mathrm{dist}}_{\tilde{A}_k}) \to (N, d)$. By Lemma 4.2, the composition $\varphi_k \circ \overline{P}$ is $(2\,\mathrm{diam}(F_{A_k}) + \epsilon_k)$-approximation. By the assumption, the Gromov–Hausdorff limit of $(\Gamma_{\boldsymbol{r}(k)} \backslash \mathrm{H}_n, \overline{\mathrm{dist}}_{A_k})$ is isometric to (N, d). □

4B. ***Collapse of the fiber.*** Let $\{(\Gamma_{\boldsymbol{r}(k)} \backslash \mathrm{H}_n, \overline{\mathrm{dist}}_{A_k})\}$ be a sequence of compact sub-Riemannian Heisenberg manifolds with the diameter upper bound by $D > 0$. In this section, we show that if the sequence collapses, then the diameter of the circle fibers converge to zero.

The fiber over $b \in \mathfrak{z}_{\boldsymbol{r}} \backslash \mathfrak{v}_0$ is written by $F_b = \{\Gamma_{\boldsymbol{r}(k)} z_0 Z \cdot h_b \mid z_0 \in \mathbb{R}\}$, where we fix $h_b \in P^{-1}_{\Gamma_{\boldsymbol{r}(k)}}(\overline{P}^{-1}(b)) \subset \mathrm{H}_n$. In particular, the subset $\{z_0 Z \cdot h_b \mid z_0 \in [0, 1)\} \subset \mathrm{H}_n$ is a representative of F_b. By the left-invariance of the restricted distance on F_b, its diameter is the distance from h_b to $\frac{1}{2} Z \cdot h_b$ and is independent of the choice of h_b.

The above argument shows the following lemma.

Lemma 4.4. *The diameter of the fibers* $\mathrm{diam}(F_{A_k})$ *is given by*

$$\mathrm{diam}(F_{A_k}) = \mathrm{dist}_{A_k}\big(e, \tfrac{1}{2} Z\big).$$

Let us pass to the estimate of the diameter. First we consider a sequence $\{\Gamma_{\boldsymbol{r}(k)}\}_{k \in \mathbb{N}}$ such that $\boldsymbol{r}(k_1) \neq \boldsymbol{r}(k_2)$ for any $k_1 \neq k_2$. This implies that the sequence of numbers $\{r_n(k)\}$ diverges.

Proposition 4.5. *Assume that* $\mathrm{diam}(\Gamma_{\boldsymbol{r}(k)} \backslash \mathrm{H}_n, \langle \cdot, \cdot \rangle_{A_k}) \leq D$ *and* $r_n(k)$ *diverge to the infinity. Then the diameter of the fibers* $\mathrm{diam}(F_{A_k})$ *converge to zero.*

Proof. Let $\gamma_{n,k} = \frac{1}{2} r_n(k) X_n \in \mathrm{H}_n$. Since $\gamma_{n,k}$ is on the plane $\mathfrak{v}_0$, by Lemma 3.8, a length minimizer from e to $\gamma_{n,k}$ in H_n is the straight segment $\ell(t) := t X_n$, $t \in \big[0, \frac{1}{2} r_n(k)\big]$. Moreover its projection by $P_{\Gamma_{\boldsymbol{r}(k)}}$ is a length minimizer from $\Gamma_{\boldsymbol{r}(k)} e$ to $\Gamma_{\boldsymbol{r}(k)} \gamma_{n,k}$. Indeed, any element in $\Gamma_{\boldsymbol{r}(k)} \gamma_{n,k}$ is written by

$$r_n(k)\big(m + \tfrac{1}{2}\big) X_n + E,$$

where $m \in \mathbb{Z}$ and E is an element in $\mathfrak{h}_n \simeq \mathrm{H}_n$ transverse to X_n. Clearly a length minimizer from $\Gamma_{\boldsymbol{r}(k)} e$ to $\Gamma_{\boldsymbol{r}(k)} \gamma_{n,k}$ is realized when

$$m = 0, -1 \quad \text{and} \quad E = 0.$$

This shows that the projection of the straight segment $\ell(t)$ is length minimizing in $\Gamma_{\boldsymbol{r}} \backslash \mathrm{H}_n$.

Since the length of the straight segment $\ell(t)$ is $\big\| \frac{1}{2} r_n(k) X_n \big\|_{A_k}$, we obtain

$$\begin{aligned} \big\| \tfrac{1}{2} r_n(k) X_n \big\|_{A_k} &= \mathrm{dist}_{A_k}(e, \gamma_{n,k}) = \overline{\mathrm{dist}}_{A_k}(\Gamma_{\boldsymbol{r}(k)} e, \Gamma_{\boldsymbol{r}(k)} \gamma_{n,k}) \\ &\leq \mathrm{diam}(\Gamma_{\boldsymbol{r}(k)} \backslash \mathrm{H}_n, \overline{\mathrm{dist}}_{A_k}) \leq D. \end{aligned} \tag{5}$$

By the same argument we also show that

$$\left\|\tfrac{1}{2}Y_n\right\|_{A_k} \le D. \tag{6}$$

On the other hand, let $c : [0, 4] \to \mathrm{H}_n$ be a path inductively defined by

$$c(t) = \begin{cases} -t\sqrt{\dfrac{r_n(k)}{2}}X_n & \text{for } t \in [0, 1], \\ c(1)\cdot\left(-(t-1)\dfrac{1}{\sqrt{2r_n(k)}}Y_n\right) & \text{for } t \in [1, 2], \\ c(2)\cdot\left((t-2)\sqrt{\dfrac{r_n(k)}{2}}X_n\right) & \text{for } t \in [2, 3], \\ c(3)\cdot\left((t-3)\dfrac{1}{\sqrt{2r_n(k)}}Y_n\right) & \text{for } t \in [3, 4]. \end{cases}$$

The endpoint of c is $c(4) = \frac{1}{2}Z$, and the length is

$$\begin{aligned} \mathrm{length}(c) &= \|\sqrt{2r_n(k)}X_n\|_{A_k} + \left\|\sqrt{\frac{2}{r_n(k)}}Y_n\right\|_{A_k} \\ &= \sqrt{2r_n(k)}\|X_n\|_{A_k} + \sqrt{\frac{2}{r_n(k)}}\|Y_n\|_{A_k} \\ &\le \sqrt{2r_n(k)}\frac{2D}{r_n(k)} + \sqrt{\frac{2}{r_n(k)}}2D \\ &= \frac{4\sqrt{2}D}{\sqrt{r_n(k)}}. \end{aligned}$$

Here the third inequality follows from (5) and (6). Hence we obtain

$$\mathrm{diam}(F_{A_k}) = \mathrm{dist}_{A_k}\left(e, \tfrac{1}{2}Z\right) \le \mathrm{length}(c) \le \frac{4\sqrt{2}D}{\sqrt{r_n(k)}}.$$

Since $r_n(k)$ diverges to the infinity, the diameters of the fibers converge to zero. □

Next we consider a sequence consisting of a fixed isomorphism type $\Gamma_{\boldsymbol{r}}\backslash\mathrm{H}_n$. We start from Riemannian case.

Proposition 4.6. *Let* $\{\Gamma_{\boldsymbol{r}}\backslash\mathrm{H}_n, \overline{\mathrm{dist}}_{A_k}\}$ *be a sequence of compact Heisenberg manifolds with left-invariant Riemannian metrics with the diameter upper bound. If the total measure in the canonical Riemannian volume converges to zero, then the diameter of the fibers* $\mathrm{diam}(F_{A_k})$ *converges to zero.*

Proof. By (3), if the total measure converges to zero, then either/both of the following two cases holds:

(a) $|\rho_{A_k}|^{-1} \to 0$, or
(b) $|\det(\tilde{A}_k)|^{-1} \to 0$.

In the case (a), by using Lemmas 3.9 and 4.4, we have

$$\operatorname{diam}(F_{A_k}) = \operatorname{dist}_{A_k}\big(e, \tfrac{1}{2}Z\big) = \min\left\{\left|\frac{1}{2\rho_{A_k}}\right|, \frac{2}{d_n(A_k)}\sqrt{\frac{\pi d_n(A_k)}{2} - \pi^2\rho_{A_k}^2}\right\}$$
$$\leq \left|\frac{1}{2\rho_{A_k}}\right| \to 0 \quad (k \to \infty).$$

In the case (b), by using Lemmas 3.5, 3.9 and 4.4, we have

$$\operatorname{diam}(F_{A_k}) = \min\left\{\left|\frac{1}{2\rho_{A_k}}\right|, \frac{2}{d_n(A_k)}\sqrt{\frac{\pi d_n(A_k)}{2} - \pi^2\rho_{A_k}^2}\right\} \leq \sqrt{\frac{2\pi}{d_n(A_k)}}$$
$$\leq \sqrt{\frac{2\pi}{\sqrt[n]{|\det(\tilde{A}_k)|}}} \to 0 \quad (k \to \infty).$$

In both cases, the diameter of the fiber $\operatorname{diam}(F_{A_k})$ converges to zero. This concludes the proposition. □

A similar argument follows also for sub-Riemannian metrics of corank 1.

Proposition 4.7. *Let* $\{\Gamma_r\backslash \mathrm{H}_n, \overline{\operatorname{dist}}_{A_k}\}$ *be a sequence of compact Heisenberg manifolds with left-invariant sub-Riemannian metrics of corank* 1. *If the total measure in Popp's volume converges to zero, then the diameter of the fibers converges to zero.*

Proof. By (4), if the total measure converges to zero, then either/both of the following two cases holds:

(a) $\delta(A_k)^{-1} \to 0$, or

(b) $|\det(\tilde{A}_k)|^{-1} \to 0$.

In the case (a), by using Lemmas 3.5, 3.9 and 4.4, we have

$$\operatorname{diam}(F_{A_k}) = \operatorname{dist}_{A_k}\big(e, \tfrac{1}{2}Z\big) = \min\left\{+\infty, \frac{2}{d_n(A_k)}\sqrt{\frac{\pi d_n(A_k)}{2}}\right\}$$
$$= \sqrt{\frac{2\pi}{d_n(A_k)}} \leq \frac{2\sqrt{n\pi}}{\delta(A_k)} \to 0 \quad (k \to \infty).$$

In the case (b), again by using Lemmas 3.5, 3.9 and 4.4, we have

$$\operatorname{diam}(F_{A_k}) = \sqrt{\frac{2\pi}{d_n(A_k)}} \leq \sqrt{\frac{2\pi}{\sqrt[n]{|\det(\tilde{A}_k)|}}} \to 0 \quad (k \to \infty).$$

In both cases, the diameter of the fiber $\operatorname{diam}(F_{A_k})$ converges to zero. This concludes the proposition. □

Now we are prepared to prove the main theorem.

Proof of Theorem 1.1. Suppose that there are infinitely many isomorphic classes of lattices $\Gamma_{\boldsymbol{r}_k}$ in the sequence. Then by Proposition 4.5, the diameter of the fibers converges to 0, and by Proposition 4.3, the Gromov–Hausdorff limit is isometric to a flat torus of lower dimension.

Assume there are finitely many isomorphic classes of lattices in the sequence. By taking a subsequence, we can assume that the lattices are isomorphism to $\Gamma_{\boldsymbol{r}}$ for a fixed $\boldsymbol{r}$ in D_n. By Propositions 4.6 and 4.7, if the total measure converges to 0, then the diameter of the fiber converges to 0. Again by Proposition 4.3, the Gromov–Hausdorff limit is isometric to a flat torus of lower dimension. □

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KENSHIRO TASHIRO
MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI
JAPAN
kenshiro.tashiro.b2@tohoku.ac.jp

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