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ON THE COEFFICIENT INEQUALITIES FOR SOME CLASSES OF HOLOMORPHIC MAPPINGS IN COMPLEX BANACH SPACES

QINGHUA XU, XIAOHUA YANG AND TAISHUN LIU

Let $\mathcal{C}$ be the familiar class of normalized close-to-convex functions in the unit disk. Koepf (1987) proved that for a function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the class $\mathcal{C}$,

$$|a_3 - \lambda a_2^2| \le \begin{cases} 3 - 4\lambda, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{1}{3} + \frac{4}{9\lambda}, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \lambda \in \left[\frac{2}{3}, 1\right] \end{cases} \qquad \text{and} \qquad \big||a_3| - |a_2|\big| \le 1.$$

Recently, Xu et al. (2023) generalized the above results to a subclass of close-to-quasiconvex mappings of type B defined on the open unit polydisc in $\mathbb{C}^n$, and to a subclass of close-to-starlike mappings defined on the open unit ball of a complex Banach space, respectively. In the first part of this paper, by using different methods, we obtain the corresponding results of norm type and functional type on the open unit ball in a complex Banach space. We next give the coefficient inequalities for a subclass of g-starlike mappings of complex order λ on the open unit ball of a complex Banach space, which generalize many known results. Moreover, the proofs presented here are simpler than those given in the related papers.

1. Introduction

Let $\mathcal{S}$ be the class of functions of the form

$$f(\xi) = \xi + \sum_{m=2}^{\infty} a_m \xi^m, \tag{1-1}$$

which are univalent in the open unit disk

$$\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}.$$

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Let X be a complex Banach space with norm $\|\cdot\|$, $\mathbb{B}$ be the open unit ball of X.

Let $L(X, Y)$ denote the set of continuous linear operators from X into a complex Banach space Y. Let I be the identity in $L(X, X)$. For each $x \in X \setminus \{0\}$, we define

$$T(x) = \{T_x \in L(X, \mathbb{C}) : \|T_x\| = 1, T_x(x) = \|x\|\}.$$

According to the Hahn–Banach theorem, $T(x)$ is nonempty.

Let $H(\mathbb{B})$ denote the set of all holomorphic mappings from $\mathbb{B}$ into X. It is well known that if $f \in H(\mathbb{B})$, then

$$f(y) = \sum_{m=0}^{\infty} \frac{1}{m!} D^m f(x)((y-x)^m)$$

for all y in some neighborhood of $x \in \mathbb{B}$, where $D^m f(x)$ is the m-th Fréchet derivative of f at x, and for $m \geq 1$,

$$D^m f(x)((y-x)^m) = D^m f(x)(\underbrace{y-x, \ldots, y-x}_{m}).$$

Furthermore, $D^m f(x)$ is a bounded symmetric m-linear mapping from

$$X^m = \underbrace{X \times \cdots \times X}_{m} \quad \text{into } X.$$

A holomorphic mapping $f : \mathbb{B} \to X$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(\mathbb{B})$. A mapping $f \in H(\mathbb{B})$ is called locally biholomorphic if the Fréchet derivative $Df(x)$ has a bounded inverse for each $x \in \mathbb{B}$. If $f : \mathbb{B} \to X$ is a holomorphic mapping, then f is called normalized if $f(0) = 0$ and $Df(0) = I$, where I represents the identity operator from X into X. A mapping $f \in H(\mathbb{B})$ is called starlike if f is biholomorphic on $\mathbb{B}$ and $f(\mathbb{B})$ is a starlike domain. Let $\mathcal{S}^*(\mathbb{B})$ denote the class of normalized starlike mappings on $\mathbb{B}$, when $X = \mathbb{C}$, $\mathbb{B} = \mathbb{U}$, the class $\mathcal{S}^*(\mathbb{U})$ is denoted by $\mathcal{S}^*$. Suppose $f, g \in H(\mathbb{U})$. If there exists a Schwarz function φ (i.e., $\varphi \in H(\mathbb{U})$, $\varphi(0) = 0$, $\varphi(\mathbb{U}) \subseteq \mathbb{U}$) such that $f = g \circ \varphi$, then we say that f is subordinate to g(written $f \prec g$).

Now, we introduce the class of quasiconvex mappings of type B on $\mathbb{B}$ in X, which has been introduced by Roper and Suffridge [31] on the unit ball $\mathbb{B} \subset \mathbb{C}^n$.

Definition 1.1. Let $h : \mathbb{B} \to X$ be a normalized locally biholomorphic mapping. If

$$\text{(1-2)} \quad \Re e\big\{T_x\big[(Dh(x))^{-1}\big(D^2h(x)(x^2) + Dh(x)\,x\big)\big]\big\} > 0, \quad x \in \mathbb{B}\setminus\{0\},\ T_x \in T(x),$$

then h is called a quasiconvex mapping of type B on $\mathbb{B}$.

Let $\mathcal{Q}_B(\mathbb{B})$ denote the class of quasiconvex mappings of type B on $\mathbb{B}$. When $X = \mathbb{C}$, $\mathbb{B} = \mathbb{U}$, we deduce easily that relation (1-2) is equivalent to

$$\Re e\left(1 + \frac{\xi h''(\xi)}{h'(\xi)}\right) > 0, \quad \xi \in \mathbb{U},$$

which is the well-known criterion of convex functions on $\mathbb{U}$. Let $\mathcal{K}$ denote the class of normalized convex functions on $\mathbb{U}$.

Xu et al. [35] introduced the following class of mappings on the open unit ball of a complex Banach space.

Definition 1.2 [35]. Suppose that $f : \mathbb{B} \to X$ is a normalized holomorphic mapping. If there exists a mapping $h \in \mathcal{Q}_B(\mathbb{B})$ such that

$$\text{(1-3)} \qquad \Re e\{T_x[(Dh(x))^{-1} Df(x)\, x]\} > 0, \quad x \in \mathbb{B} \setminus \{0\},\ T_x \in T(x),$$

then f is called a close-to-quasiconvex mapping of type B on $\mathbb{B}$.

If $X = \mathbb{C}^n$, $\mathbb{B} = \mathbb{U}^n$, then it is obvious that the relation (1-3) is equivalent to

$$\Re e \frac{p_j(z)}{z_j} > 0, \quad z \in \mathbb{U}^n \setminus \{0\},$$

where $p(z) = (p_1(z), \ldots, p_n(z))' = (Dh(z))^{-1} Df(z)\, z$ is a column vector in $\mathbb{C}^n$, and j satisfies $|z_j| = \|z\| = \max_{1 \le k \le n}\{|z_k|\}$.

The following definition has been introduced by Pfaltzgraff and Suffridge [29] on the unit ball with respect to an arbitrary norm in $\mathbb{C}^n$.

Definition 1.3. Suppose that $f : \mathbb{B} \to X$ is a normalized locally biholomorphic mapping. If there exists a mapping $h \in \mathcal{S}^*(\mathbb{B})$ such that

$$\text{(1-4)} \qquad \Re e\{T_x[(Df(x))^{-1} h(x)]\} > 0, \quad x \in \mathbb{B} \setminus \{0\},\ T_x \in T(x),$$

then f is called a close-to-starlike mapping on $\mathbb{B}$.

Remark 1.4. Clearly, if $X = \mathbb{C}$, $\mathbb{B} = \mathbb{U}$, then the relation (1-3) (respectively, the relation (1-4)) is equivalent to $\Re e \frac{f'(\xi)}{h'(\xi)} > 0$, $\xi \in \mathbb{U}$, here $h \in \mathcal{K}$ (respectively, $\Re e \frac{\xi f'(\xi)}{h(\xi)} > 0$, $\xi \in \mathbb{U}$, here $h \in \mathcal{S}^*$), which is the usual definition of close-to-convex functions on $\mathbb{U}$.

Koepf [23] obtained the following Fekete and Szegő inequality for the class $\mathcal{C}$.

Theorem 1.5 [23]. *Let the function* $f(\xi)$ *be defined by* (1-1). *If* $f \in \mathcal{C}$, *then*

$$|a_3 - \lambda a_2^2| \le \begin{cases} 3 - 4\lambda, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{1}{3} + \frac{4}{9\lambda}, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \lambda \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

As an interesting application of Theorem 1.5, it was proved that $\big||a_3| - |a_2|\big| \le 1$ for the class $\mathcal{C}$.

In recent years, the Fekete and Szegő inequality for subclass of biholomorphic mappings in several complex variables has been studied by some authors (see [3; 4; 5; 6; 7; 15; 20; 21; 32; 36; 38]).

Xu et al. [39] obtained the following Fekete and Szegő inequality for the subclass of close-to-quasiconvex mappings of type B on the open unit polydisk $\mathbb{U}^n$ in $\mathbb{C}^n$ with respect to $H \in \mathcal{Q}_B(\mathbb{U}^n)$, which could be regarded as a generalization of Theorem 1.5 to several complex variables.

Theorem 1.6 [39]. *Let $f : \mathbb{U}^n \to \mathbb{C}$, $h : \mathbb{U}^n \to \mathbb{C}$ be holomorphic functions, and $H(z) = zh(z) \in \mathcal{Q}_B(\mathbb{U}^n)$. Suppose that $F(z) = zf(z)$ is a close-to-quasiconvex mapping of type B with respect to $H(z)$. Then, for $\lambda \in [0, 1]$, $z \in \mathbb{U}^n$, we have*

$$\left\| \tfrac{1}{3!} D^3F(0)(z^3) - \lambda \tfrac{1}{2!} D^2F(0)\left(z, \frac{D^2F(0)(z^2)}{2!}\right)\right\| \le \begin{cases} (3-4\lambda)\|z\|^3, & \lambda \in \left[0, \frac{1}{3}\right], \\ \left(\frac{1}{3} + \frac{4}{9\lambda}\right)\|z\|^3, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ \|z\|^3, & \lambda \in \left[\frac{2}{3}, 1\right] \end{cases}$$

and

$$\left| \left\| \frac{D^3F(0)(z^3))}{3!} \right\| - \left\| \frac{D^2F(0)(z^2)\|z\|}{2!} \right\| \right| \le \|z\|^3.$$

The above estimates are sharp.

Hamada [15] generalized Theorem 1.6 to the open unit ball of a complex Banach space under weaker assumptions than in Theorem 1.6. Moreover, in the same paper, Hamada also obtained the Fekete and Szegő inequality of functional type for the subclasses of close-to-quasiconvex mappings of type B on the open unit ball $\mathbb{B}$ in a complex Banach space.

Theorem 1.7 [15]. *Let G be a quasiconvex mapping of type B on $\mathbb{B}$ such that*

$$\tfrac{1}{2!} D^2G(0)(x^2) = L_G(x)\,x, \quad x \in X,$$

where $L_G(\cdot) \in L(X, \mathbb{C})$. Let F be a close-to-quasiconvex mapping of type B on $\mathbb{B}$ with respect to G such that

$$\tfrac{1}{2!} D^2F(0)(x^2) = L_F(x)\,x, \quad x \in X,$$

where $L_F(\cdot) \in L(X, \mathbb{C})$ and

$$\tfrac{1}{3!} D^3F(0)(x^3) = Q_F(x)\,x, \quad x \in X,$$

where $Q_F(x)$ *is a homogeneous polynomial of degree* 2 *with values in* $\mathbb{C}$. *Let* $x_0 \in X$ *with* $\|x_0\| = 1$. *Then, for* $\lambda \in [0, 1]$, *it holds that*

$$\left\| \tfrac{1}{3!} D^3 F(0)(x_0^3) - \lambda \tfrac{1}{2!} D^2 F(0)\left(x_0, \frac{D^2 F(0)(x_0^2)}{2!} \right) \right\| \le \begin{cases} 3 - 4\lambda, & \lambda \in \left[0, \tfrac{1}{3}\right], \\ \tfrac{1}{3} + \tfrac{4}{9\lambda}, & \lambda \in \left[\tfrac{1}{3}, \tfrac{2}{3}\right], \\ 1, & \lambda \in \left[\tfrac{2}{3}, 1\right]. \end{cases}$$

The above estimates are sharp.

More recently, Xu et al. [37] gave another extension of Theorem 1.5 to higher dimensions, and established the following Fekete and Szegő inequality for the subclass of close-to-starlike mappings on the open unit ball $\mathbb{B}$ in a complex Banach space with respect to $H \in \mathcal{S}^*(\mathbb{B})$.

Theorem 1.8 [37]. *Let* $f : \mathbb{B} \to \mathbb{C}$, $h : \mathbb{B} \to \mathbb{C}$ *be holomorphic functions, and* $H(x) = xh(x) \in \mathcal{S}^*(\mathbb{B})$. *Suppose that* $F(x) = xf(x)$ *is a close-to-starlike mapping with respect to* $H(x)$. *Then, for* $x \in \mathbb{B} \setminus \{0\}$, $T_x \in T(x)$, $\lambda \in [0, 1]$, *we have*

$$\left| \frac{T_x(D^3 F(0)(x^3))}{3!\,\|x\|^3} - \lambda \left(\frac{T_x(D^2 F(0)(x^2))}{2!\,\|x\|^2} \right)^2 \right| \le \begin{cases} 3 - 4\lambda, & \lambda \in \left[0, \tfrac{1}{3}\right], \\ \tfrac{1}{3} + \tfrac{4}{9\lambda}, & \lambda \in \left[\tfrac{1}{3}, \tfrac{2}{3}\right], \\ 1, & \lambda \in \left[\tfrac{2}{3}, 1\right] \end{cases}$$

and

$$\left| \left| \frac{T_x(D^3 F(0)(x^3))}{3!\,\|x\|^3} \right| - \left| \frac{T_x(D^2 F(0)(x^2))}{2!\,\|x\|^2} \right| \right| \le 1.$$

The above estimates are sharp.

Zhang et al. [40] introduced the following class of g-starlike mapping of complex order λ on $\mathbb{B}$ in X, which has been introduced by Hu et al. [22] on $\mathbb{B}^n$.

Definition 1.9 [40]. Let $g : \mathbb{U} \to \mathbb{C}$ be a biholomorphic function such that $g(0) = 1$, $\Re e g(\xi) > 0$ on $\mathbb{U}$. Let $\lambda \in \mathbb{C}$ with $\Re e \lambda \le 0$ and let $f : \mathbb{B} \to X$ be a normalized locally biholomorphic mapping. If

$$(1 - \lambda) \frac{\|x\|}{T_x((Df(x))^{-1} f(x))} + \lambda \in g(\mathbb{U}), \quad x \in \mathbb{B} \setminus \{0\},\ T_x \in T(x),$$

then f is called a g-starlike mapping of complex order λ.

Let $\mathcal{S}^*_{g,\lambda}(\mathbb{B})$ denote the class of g-starlike mapping of complex order λ on $\mathbb{B}$. In particular, when $X = \mathbb{C}$, $\mathbb{B} = \mathbb{U}$, the above relation implies that

$$f \in \mathcal{S}^*_{g,\lambda}(\mathbb{U}) \qquad \text{if and only if } (1 - \lambda) \frac{\xi f'(\xi)}{f(\xi)} + \lambda \prec g, \quad \xi \in \mathbb{U}.$$

In view of Remark 2.4 of [40], we know that some important subclasses of $\mathcal{S}(\mathbb{B})$ coincide with the classes of $\mathcal{S}^*_{g,\lambda}(\mathbb{B})$ for certain choices of g and λ.

Since the authors [37; 39] used Definitions 1.2 and 1.3 directly, the proofs of Theorems 1.6 and 1.8 are long and rather complicated. In Section 3, under the same conditions as in Theorems 1.6 and 1.8, we will establish the corresponding inequalities of norm type and functional type for the subclasses of close-to-quasiconvex mappings of type B and close-to-starlike mappings on the open unit ball of a Banach space, respectively. Next, In Section 4, we obtain the coefficient inequalities for a subclass of g-starlike mappings of complex order λ on the open unit ball of a complex Banach space. The various results of this paper would generalize many known results. Moreover, the proof methods presented here simplify those appeared in some earlier papers [25; 26; 28; 33; 36; 37; 39].

Some investigations concerning the coefficient estimates for subclasses of holomorphic mappings in several variables have been obtained by Bracci et al. [1; 2], Graham et al. [8; 9; 10; 11; 12; 13; 14], Hamada et al. [16; 18; 19], Kohr [24], Liu and Wu [27], Liu et al. [28], and Xu et al. [34; 35].

2. Some lemmas

In order to prove the desired results, we need to provide the following lemmas.

Lemma 2.1 [17]. *Let $h : \mathbb{B} \to X$ be a normalized locally biholomorphic mapping. Then h is a starlike mapping on $\mathbb{B}$ if and only if*

$$\Re e\big(T_x(Dh(x)^{-1}h(x))\big) > 0, \quad x \in \mathbb{B} \setminus \{0\},\ T_x \in T(x).$$

Comparing Lemma 2.1 with Definition 1.3, we remark that any normalized starlike mapping on $\mathbb{B}$ is close-to-starlike (with respect to itself).

Lemma 2.2. *Let $g : \mathbb{U} \to \mathbb{C}$ satisfy the conditions of* Definition 1.9, *$f \in H(\mathbb{B}, C)$, $f(0)=1$, $F(x)=xf(x)$. Fix $x \in \mathbb{B}\setminus\{0\}$ and denote $x_0 = \frac{x}{\|x\|}$. Let $l(\xi)=T_x(F(\xi x_0))$, $\xi \in \mathbb{U}$. Then*

$$l \in \mathcal{S}^*_{g,\lambda}(\mathbb{U}) \Leftrightarrow F \in \mathcal{S}^*_{g,\lambda}(\mathbb{B}).$$

Proof. Since $F \in \mathcal{S}^*_{g,\lambda}(\mathbb{B})$, we deduce from Definition 1.9 that

$$(1-\lambda)\frac{\|x\|}{T_x((DF(x))^{-1}F(x))} + \lambda \in g(\mathbb{U}), \quad x \in \mathbb{B} \setminus \{0\},\ T_x \in T(x).$$

It follows that F is locally biholomorphic on $\mathbb{B}$, and thus $f(x) \neq 0$, $x \in \mathbb{B}$. Using a similar method to that in [8] (also see [9, Theorem 7.1.14]), we have

$$[DF(x)]^{-1} = \frac{1}{f(x)}\left(I - \frac{\frac{xDf(x)}{f(x)}}{1+\frac{Df(x)x}{f(x)}}\right). \tag{2-1}$$

Hence,

$$(DF(x))^{-1}F(x) = x\left(\frac{1}{1+\frac{Df(x)x}{f(x)}}\right) = \frac{xf(x)}{f(x)+Df(x)x}, \quad x \in \mathbb{B}.$$

We find from the above equality that

$$(2\text{-}2)\quad (1-\lambda)\frac{\|x\|}{T_x((DF(x))^{-1}F(x))}+\lambda=(1-\lambda)\frac{f(x)+Df(x)\,x}{f(x)}+\lambda\in g(\mathbb{U}).$$

Since $l(\xi)=T_x(F(\xi x_0))=T_x(\xi x_0\, f(\xi x_0))=\xi f(\xi x_0)$, we have

$$l'(\xi)=f(\xi x_0)+Df(\xi x_0)\,\xi x_0$$

and

$$(2\text{-}3)\qquad (1-\lambda)\frac{\xi l'(\xi)}{l(\xi)}+\lambda=(1-\lambda)\frac{f(\xi x_0)+Df(\xi x_0)\,\xi x_0}{f(\xi x_0)}+\lambda\in g(\mathbb{U}),$$

which implies that $l\in\mathcal{S}^*_{g,\lambda}(\mathbb{U})$.

Conversely, we assume $l\in\mathcal{S}^*_{g,\lambda}(\mathbb{U})$. Then it is clear that $\frac{\xi l'(\xi)}{l(\xi)}\neq 0,\ \xi\in\mathbb{U}$. Hence we have

$$1+\frac{Df(x)\,x}{f(x)}\neq 0,\qquad x\in\mathbb{B}.$$

It is not hard to deduce from this and (2-1) that F is locally biholomorphic on $\mathbb{B}$.

On the other hand, in view of (2-2) and (2-3), we can conclude that

$$(1-\lambda)\frac{\xi}{T_{\xi x_0}((DF(\xi x_0))^{-1}F(\xi x_0))}+\lambda=(1-\lambda)\frac{\xi l'(\xi)}{l(\xi)}+\lambda\in g(\mathbb{U}).$$

Taking $\xi=\|x\|$ in the above relation, we obtain

$$(1-\lambda)\frac{\|x\|}{T_x((DF(x))^{-1}F(x))}+\lambda\in g(\mathbb{U}),$$

as desired. □

Lemma 2.3 [20]. *Let $g(\xi)=1+g'(0)\,\xi+\frac{g''(0)}{2}\xi^2+\dots$ be a holomorphic function on $\mathbb{U}$ such that $g'(0)\neq 0$. Let $s(\xi)=1+s'(0)\,\xi+\frac{s''(0)}{2}\xi^2+\dots$ be a holomorphic function on $\mathbb{U}$ such that $s\prec g$. Then for every $\mu\in\mathbb{C}$, it holds that*

$$\left|\frac{s''(0)}{2}-\mu(s'(0))^2\right|\le\max\left\{|g'(0)|,\left|\frac{g''(0)}{2}-\mu(g'(0))^2\right|\right\}.$$

This estimate is sharp.

Lemma 2.4. *Let $g:\mathbb{U}\to\mathbb{C}$ satisfy the conditions of* Definition 1.9 *and*

$$l(\xi)=\xi+\sum_{m=2}^{\infty}l_m\,\xi^m\in\mathcal{S}^*_{g,\lambda}(\mathbb{U}).$$

Then

$$|l_3-\nu l_2^2|\le\frac{|g'(0)|}{2|1-\lambda|}\max\left\{1,\left|\frac{1}{2}\frac{g''(0)}{g'(0)}+\frac{1-2\nu}{1-\lambda}g'(0)\right|\right\},\qquad \nu\in\mathbb{C}.$$

The above estimate is sharp for the function

$$l(\xi)=\xi\exp\frac{1}{1-\lambda}\int_0^{\xi}(g(t)-1)\frac{1}{t}dt$$

if $\left|\frac{1}{2}\frac{g''(0)}{g'(0)}+\frac{1-2\nu}{1-\lambda}g'(0)\right|\geq 1$, *and for*

$$l(\xi)=\xi\exp\frac{1}{1-\lambda}\int_0^{\xi}(g(t^2)-1)\frac{1}{t}dt$$

if $\left|\frac{1}{2}\frac{g''(0)}{g'(0)}+\frac{1-2\nu}{1-\lambda}g'(0)\right|\leq 1$.

Proof. Since $l\in\mathcal{S}^*_{g,\lambda}(\mathbb{U})$, we have

$$b(\xi)=(1-\lambda)\frac{\xi l'(\xi)}{l(\xi)}+\lambda\in g(\mathbb{U}),\quad \xi\in\mathbb{U},\ b\prec g.$$

A computation shows that

$$b'(0)=(1-\lambda)\,l_2,\quad \frac{b''(0)}{2}=2(1-\lambda)\,l_3-(1-\lambda)\,l_2^2.$$

By using Lemma 2.3, we have

$$\left|\frac{b''(0)}{2}-\mu(b'(0))^2\right|\leq\max\left\{|g'(0)|,\left|\frac{g''(0)}{2}-\mu(g'(0))^2\right|\right\},\quad \mu\in\mathbb{C}.$$

From the above relations, we obtain that

$$|l_3-\nu l_2^2|\leq\frac{|g'(0)|}{2|1-\lambda|}\max\left\{1,\left|\frac{1}{2}\frac{g''(0)}{g'(0)}+\frac{1-2\nu}{1-\lambda}g'(0)\right|\right\},\quad \nu\in\mathbb{C}.\qquad\square$$

Remark 2.5. Lemma 2.4 generalizes Theorem 3.1 of [36], when $\lambda=0$, Lemma 2.4 obtained by Xu et al. [36]. Moreover, the proof presented here is simpler than that in [36, Theorem 3.1].

Lemma 2.6. *Let* $g:\mathbb{U}\to\mathbb{C}$ *be a convex function which satisfies the conditions of* Definition 1.9 *and*

$$l(\xi)=\xi+\sum_{m=2}^{\infty}l_m\,\xi^m\in\mathcal{S}^*_{g,\lambda}(\mathbb{U}).$$

Then

$$|l_m|\leq\frac{1}{(m-1)!}\prod_{k=2}^{m}\left((k-2)+\frac{|g'(0)|}{|1-\lambda|}\right),\quad m=2,3,4,\ldots.$$

Proof. Since $l\in\mathcal{S}^*_{g,\lambda}(\mathbb{U})$, the function p is defined by

$$p(\xi)=(1-\lambda)\frac{\xi l'(\xi)}{l(\xi)}+\lambda,\quad \xi\in\mathbb{U},$$

in view of Definition 1.9, we have $p\prec g$. If

$$p(\xi)=1+p_1\,\xi+p_2\,\xi^2+\cdots+p_m\,\xi^m+\ldots,\quad \xi\in\mathbb{U},$$

then from Rogosinski's theorem [30], we obtain that $|p_m| \le |g'(0)|$, $m = 1, 2, 3, \ldots$. Comparing the coefficients in the power series of $l(\xi)p(\xi) = (1-\lambda)\,\xi l'(\xi) + \lambda l(\xi)$, we deduce that $(1-\lambda)\,l_2 = p_1$ and

$$(1-\lambda)(m-1)\,l_m = p_{m-1} + l_2\,p_{m-2} + l_3\,p_{m-3} + \cdots + l_{m-1}\,p_1, \quad m = 2, 3, 4, \ldots.$$

Thus by the mathematical induction, we obtain

$$|l_m| \le \frac{1}{(m-1)!}\prod_{k=2}^{m}\left((k-2) + \frac{|g'(0)|}{|1-\lambda|}\right), \quad m = 2, 3, 4, \ldots,$$

as desired. □

3. Simplified proofs of Fekete–Szegő inequalities for close-to-quasiconvex mappings of type B and close-to-starlike mappings

In this section, by using the proof methods different from those appeared in [39] and [37], we obtain the corresponding results of norm type and functional type for subclasses of close-to-quasiconvex mappings of type B and close-to-starlike mappings defined on the open unit ball in a complex Banach space (see Theorem 1.7).

Theorem 3.1. *Let $f : \mathbb{B} \to \mathbb{C}$, $h : \mathbb{B} \to \mathbb{C}$ be holomorphic functions, and let $H(x) = xh(x) \in \mathcal{Q}_B(\mathbb{B})$. Suppose that $F(x) = xf(x)$ is a close-to-quasiconvex mapping of type B with respect to $H(x)$. Then, for $x \in X \setminus \{0\}$, $T_x \in T(x)$ and* $\lambda \in [0, 1]$, *we have*

$$\left\| \frac{D^3F(0)(x^3)}{3!\,\|x\|^3} - \lambda\frac{1}{2\|x\|^3}D^2F(0)\left(x, \frac{D^2F(0)(x^2)}{2!}\right)\right\| \le \begin{cases} 3-4\lambda, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{1}{3} + \frac{4}{9\lambda}, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \lambda \in \left[\frac{2}{3}, 1\right], \end{cases}$$

$$\left| \frac{T_x(D^3F(0)(x^3))}{3!\,\|x\|^3} - \lambda\left(\frac{T_x(D^2F(0)(x^2))}{2!\,\|x\|^2}\right)^2\right| \le \begin{cases} 3-4\lambda, & \lambda \in \left[0, \frac{1}{3}\right], \\ \frac{1}{3} + \frac{4}{9\lambda}, & \lambda \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1, & \lambda \in \left[\frac{2}{3}, 1\right], \end{cases}$$

$$\left| \left|\frac{T_x(D^3F(0)(x^3))}{3!\,\|x\|^3}\right| - \left|\frac{T_x(D^2F(0)(x^2))}{2!\,\|x\|^2}\right| \right| \le 1$$

and

$$\left| \left\|\frac{D^3F(0)(x^3))}{3!\,\|x\|^3}\right\| - \left\|\frac{D^2F(0)(x^2)}{2!\,\|x\|^2}\right\| \right| \le 1.$$

The above estimates are sharp.

Proof. Fix $x \in X \setminus \{0\}$ and denote $x_0 = \frac{x}{\|x\|}$. Let $p : \mathbb{U} \to \mathbb{C}$ be given by

$$p(\xi) = \begin{cases} \frac{T_x((DH(\xi x_0))^{-1}(D^2H(\xi x_0)(\xi x_0)^2 + DH(\xi x_0)(\xi x_0))}{\xi}, & \xi \ne 0, \\ 1, & \xi = 0. \end{cases}$$

Then $p \in H(\mathbb{U})$, $p(0) = 1$, and

$$p(\xi) = \frac{T_x((DH(\xi x_0))^{-1}(D^2H(\xi x_0)(\xi x_0)^2 + DH(\xi x_0)(\xi x_0))}{\xi}$$
$$= \frac{T_{\xi x_0}((DH(\xi x_0))^{-1}(D^2H(\xi x_0)(\xi x_0)^2 + DH(\xi x_0)(\xi x_0))}{\|\xi x_0\|}, \quad \xi \in \mathbb{U} \setminus \{0\}.$$

Since $H \in \mathcal{Q}_B(\mathbb{B})$, using Definition 1.1, we obtain

$$\Re e(p(\xi)) > 0, \quad \xi \in \mathbb{U}. \tag{3-1}$$

On the other hand, by using an elementary computation, we have

$$(DH(x))^{-1} = \frac{1}{h(x)}\left(I - \frac{\frac{xDh(x)}{h(x)}}{1 + \frac{Dh(x)x}{h(x)}}\right)$$

and

$$(DH(x))^{-1}(D^2H(x)(x^2) + DH(x)x) = \frac{D^2h(x)(x^2) + 3Dh(x)x + h(x)}{h(x) + Dh(x)x}x.$$

This allows us to rewrite p in the form

$$p(\xi) = \frac{D^2h(\xi x_0)((\xi x_0)^2) + 3Dh(\xi x_0)(\xi x_0) + h(\xi x_0)}{h(\xi x_0) + Dh(\xi x_0)(\xi x_0)}, \quad \xi \in \mathbb{U}. \tag{3-2}$$

Let $k(\xi) = T_x(H(\xi x_0)) = \xi h(\xi x_0)$ for $\xi \in \mathbb{U}$. Elementary computations using this inequality yield that

$$k'(\xi) = h(\xi x_0) + Dh(\xi x_0)(\xi x_0) \tag{3-3}$$

and

$$1 + \frac{\xi k''(\xi)}{k'(\xi)} = \frac{D^2h(\xi x_0)((\xi x_0)^2) + 3Dh(\xi x_0)(\xi x_0) + h(\xi x_0)}{h(\xi x_0) + Dh(\xi x_0)(\xi x_0)}. \tag{3-4}$$

Using (3-1), (3-2) and (3-4), we obtain $k \in \mathcal{K}$.

Let $s(x) = (DH(x))^{-1}DF(x)x$, and let

$$r(\xi) = \begin{cases} \frac{T_x(s(\xi x_0))}{\xi}, & \xi \in \mathbb{U} \setminus \{0\}, \\ 1, & \xi = 0. \end{cases}$$

Then r is holomorphic on $\mathbb{U}$, $r(0) = 1$ and

$$r(\xi) = \frac{T_x(s(\xi x_0))}{\xi} = \frac{T_{x_0}(s(\xi x_0))}{\xi} = \frac{T_{\xi x_0}(s(\xi x_0))}{\|\xi x_0\|}, \quad \xi \in \mathbb{U} \setminus \{0\}.$$

Since $F(x)$ is a close-to-quasiconvex mapping of type B with respect to $H(x)$, from Definition 1.2, we obtain

$$\Re e(r(\xi)) > 0, \quad \xi \in \mathbb{U}. \tag{3-5}$$

A simple computation shows that

$$r(\xi)=\frac{T_x(s(\xi x_0))}{\xi}=\frac{T_x((DH(\xi x_0))^{-1}DF(\xi x_0)\,\xi x_0)}{\xi}=\frac{f(\xi x_0)+Df(\xi x_0)\,\xi x_0}{h(\xi x_0)+Dh(\xi x_0)\,\xi x_0},\quad \xi\in\mathbb{U}\setminus\{0\}. \tag{3-6}$$

Letting $l(\xi)=T_x(F(\xi x_0))$, $\xi\in\mathbb{U}$, we have

$$l'(\xi)=T_x(DF(\xi x_0)\,x_0)=f(\xi x_0)+Df(\xi x_0)\,\xi x_0. \tag{3-7}$$

Hence from (3-3), (3-5), (3-6) and (3-7), we obtain

$$\Re e\left(\frac{l'(\xi)}{k'(\xi)}\right)=\Re e\left(\frac{f(\xi x_0)+Df(\xi x_0)\,\xi x_0}{h(\xi x_0)+Dh(\xi x_0)\,\xi x_0}\right)>0,$$

which means that $l\in\mathcal{C}$. Thus from Theorem 1.5, we have

$$\left|\frac{l'''(0)}{3!}-\lambda\left(\frac{l''(0)}{2!}\right)^2\right|\le\begin{cases}3-4\lambda, & \lambda\in\left[0,\frac{1}{3}\right],\\ \frac{1}{3}+\frac{4}{9\lambda}, & \lambda\in\left[\frac{1}{3},\frac{2}{3}\right],\\ 1, & \lambda\in\left[\frac{2}{3},1\right]\end{cases} \tag{3-8}$$

and

$$\left|\left|\frac{l'''(0)}{3!}\right|-\left|\frac{l''(0)}{2!}\right|\right|\le 1. \tag{3-9}$$

Furthermore, since $l(\xi)=T_x(F(\xi x_0))=T_x(\xi x_0 f(\xi x_0))=\xi f(\xi x_0)$ for $\xi\in\mathbb{U}$, a simple computation yields that

$$\frac{l'''(0)}{3!}=\frac{T_x(D^3F(0)(x_0^3))}{3!}=\frac{D^2f(0)(x_0^2)}{2},$$
$$\left(\frac{l''(0)}{2!}\right)^2=\left(\frac{T_x(D^2F(0)(x_0^2))}{2!}\right)^2=(Df(0)(x_0))^2,$$

and

$$\frac{D^3F(0)(x_0^3)}{3!}=\frac{D^2f(0)(x_0^2)}{2!}x_0,\quad \tfrac{1}{2}D^2F(0)\left(x_0,\frac{D^2F(0)(x_0^2)}{2!}\right)=(Df(0)(x_0))^2x_0.$$

Using the above equalities, (3-8) and (3-9), we obtain all of the desired conclusions about Theorems 3.1. The example which shows the sharpness of Theorem 3.1 is the same as the mapping defined in [37]. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $f:\mathbb{B}\to\mathbb{C}$, $h:\mathbb{B}\to\mathbb{C}$ be holomorphic functions, and let $H(x)=xh(x)\in\mathcal{S}^*(\mathbb{B})$. Suppose that $F(x)=xf(x)$ is a close-to-starlike mapping with respect to $H(x)$. Then, for $x\in X\setminus\{0\}$, $T_x\in T(x)$ and $\lambda\in[0,1]$, we have the same conclusions as in* Theorem 3.1.

Proof. Fix $x \in X\backslash\{0\}$ and denote $x_0 = \frac{x}{\|x\|}$. Let $p : \mathbb{U} \to \mathbb{C}$ be given by

$$p(\xi) = \begin{cases} \frac{T_x((DH(\xi x_0))^{-1}H(\xi x_0))}{\xi}, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Then $p \in H(\mathbb{U})$, $p(0) = 1$, and

$$p(\xi) = \frac{T_x((DH(\xi x_0))^{-1}H(\xi x_0))}{\xi} = \frac{T_{\xi x_0}((DH(\xi x_0))^{-1}H(\xi x_0))}{\|\xi x_0\|}, \quad \xi \in \mathbb{U} \setminus \{0\}.$$

Since $H \in \mathcal{S}^*(\mathbb{B})$, by Lemma 2.1, we have

$$\Re e(p(\xi)) > 0, \quad \xi \in \mathbb{U}. \tag{3-10}$$

At the same time, a short computation yields the relation

$$(DH(x))^{-1}H(x) = \frac{h(x)}{h(x) + Dh(x)\,x}x.$$

Hence the above relations imply that

$$p(\xi) = \frac{h(\xi x_0)}{h(\xi x_0) + Dh(\xi x_0)(\xi x_0)}, \quad \xi \in \mathbb{U}. \tag{3-11}$$

Let

$$k(\xi) = T_x(H(\xi x_0)) = \xi h(\xi x_0), \quad \xi \in \mathbb{U}.$$

Then, we have

$$k'(\xi) = h(\xi x_0) + Dh(\xi x_0)(\xi x_0)$$

and

$$\frac{\xi k'(\xi)}{k(\xi)} = \frac{h(\xi x_0) + Dh(\xi x_0)(\xi x_0)}{h(\xi x_0)}. \tag{3-12}$$

By using (3-10), (3-11) and (3-12), we obtain $k \in \mathcal{S}^*$.

Let $s(x) = (DF(x))^{-1}H(x)$, and let

$$r(\xi) = \begin{cases} \frac{T_x(s(\xi x_0))}{\xi}, & \xi \in \mathbb{U} \setminus \{0\}, \\ 1, & \xi = 0. \end{cases}$$

Then r is holomorphic on $\mathbb{U}$, $r(0) = 1$ and

$$r(\xi) = \frac{T_x(s(\xi x_0))}{\xi} = \frac{T_{x_0}(s(\xi x_0))}{\xi} = \frac{T_{\xi x_0}(s(\xi x_0))}{\|\xi x_0\|}, \quad \xi \in \mathbb{U} \setminus \{0\}.$$

Since $F(x)$ is a close-to-starlike mapping with respect to $H(x)$, from Definition 1.3, we obtain

$$\Re e(r(\xi)) > 0, \quad \xi \in \mathbb{U}.$$

A simple computation shows that

$$r(\xi) = \frac{T_x(s(\xi x_0))}{\xi} = \frac{T_x((DF(\xi x_0))^{-1} H(\xi x_0))}{\xi} = \frac{h(\xi x_0)}{f(\xi x_0) + Df(\xi x_0)\,\xi x_0}, \quad \xi \in \mathbb{U} \setminus \{0\}.$$

Letting $l(\xi) = T_x(F(\xi x_0))$, $\xi \in \mathbb{U}$ again, we obtain

$$l'(\xi) = T_x(DF(\xi x_0)\, x_0) = f(\xi x_0) + Df(\xi x_0)\,\xi x_0.$$

Consequently, combining this observation with the preceding relations, we have

$$\Re e\left(\frac{\xi l'(\xi)}{k(\xi)}\right) = \Re e\left(\frac{f(\xi x_0) + Df(\xi x_0)\,\xi x_0}{h(\xi x_0)}\right) > 0,$$

which implies that $l \in \mathcal{C}$. The remaining part of the proof of Theorem 3.2 is similar to that in the proof of Theorem 3.1, so we omit the details. □

4. Simplified proofs of coefficient inequalities for a subclass of g-starlike mappings of complex order λ

By using Lemmas 2.2, 2.4 and 2.6, we establish bounds of all terms of homogeneous expansions and the Fekete–Szegő inequality for a subclass of g-starlike mappings of complex order λ on the open unit ball of a complex Banach space, which generalize the corresponding results appeared in [25; 26; 28; 33; 36].

Theorem 4.1. *Let $g : \mathbb{U} \to \mathbb{C}$ be a convex function which satisfies the conditions of* Definition 1.9, *and $f \in H(\mathbb{B}, C)$, $f(0) = 1$. Suppose that $F(x) = xf(x) \in \mathcal{S}^*_{g,\lambda}(\mathbb{B})$. Then for $x \in \mathbb{B}$, we have*

$$\frac{\|D^m F(0)(x^m)\|}{m!} \leq \frac{\prod_{r=2}^{m}\left[r - 2 + \frac{1}{1-\lambda}|g'(0)|\right]}{(m-1)!}\|x\|^m, \quad m = 2, 3, 4, \ldots.$$

Proof. Fix $x \in \mathbb{B} \setminus \{0\}$ and denote $x_0 = \frac{x}{\|x\|}$. Let $l(\xi) = T_x(F(\xi x_0))$, $\xi \in \mathbb{U}$. In view of Lemma 2.2, we have $l \in \mathcal{S}^*_{g,\lambda}(\mathbb{U})$. Since $l(\xi) = T_x(F(\xi x_0)) = \xi f(\xi x_0)$, we obtain

$$\frac{D^m F(0)(x_0^m)}{m!} = x_0 \frac{D^{m-1} f(0)(x_0^{m-1})}{(m-1)!} = \frac{l^{(m)}(0)}{m!}.$$

By using Lemma 2.6, we have

$$\frac{\|D^m F(0)(x^m)\|}{m!} \leq \frac{\prod_{r=2}^{m}\left[r - 2 + \frac{1}{1-\lambda}|g'(0)|\right]}{(m-1)!}\|x\|^m, \quad m = 2, 3, 4, \ldots,$$

as desired. □

Remark 4.2. Theorem 4.1 generalizes many known results. In Theorem 4.3 if we set $\lambda = 0$ and $g(\xi) = \frac{1+\xi}{1-\xi}$, $\xi \in \mathbb{U}$, $\lambda = 0$ and $\lambda = -\frac{\alpha}{1-\alpha}$, $0 \le \alpha < 1$, we can readily deduce the corresponding results of [28], [26] and [33], respectively. Moreover, the proofs presented here are simpler than those given in [28], [26] and [33].

Theorem 4.3. *Let* $g : \mathbb{U} \to \mathbb{C}$ *satisfy the conditions of* Definition 1.9, *and let* $f \in H(\mathbb{B}, C)$, $f(0) = 1$. *Suppose that* $F(x) = xf(x) \in \mathcal{S}^*_{g,\lambda}(\mathbb{B})$. *Then for* $\nu \in \mathbb{C}$, $x \in \mathbb{B} \setminus \{0\}$, *we have*

$$\left\| \frac{D^3F(0)(x^3)}{3!\,\|x\|^3} - \nu \frac{1}{2\|x\|^3} D^2F(0)\left(x, \frac{D^2F(0)(x^2)}{2!}\right) \right\| \le \frac{|g'(0)|}{2|1-\lambda|} \max\left\{1, \left|\frac{1}{2}\frac{g''(0)}{g'(0)} + \frac{1-2\nu}{1-\lambda} g'(0)\right|\right\}$$

and

$$\left| \frac{T_x(D^3F(0)(x^3))}{3!\,\|x\|^3} - \nu \left(\frac{T_x(D^2F(0)(x^2))}{2!\,\|x\|^2}\right)^2 \right| \le \frac{|g'(0)|}{2|1-\lambda|} \max\left\{1, \left|\frac{1}{2}\frac{g''(0)}{g'(0)} + \frac{1-2\nu}{1-\lambda} g'(0)\right|\right\}.$$

The above estimates are sharp.

Proof. Fix $x \in \mathbb{B} \setminus \{0\}$ and denote $x_0 = \frac{x}{\|x\|}$. Let $l(\xi) = T_x(F(\xi x_0))$, $\xi \in \mathbb{U}$. From Lemma 2.2, we have $l \in \mathcal{S}^*_{g,\lambda}(\mathbb{U})$. Since $l(\xi) = T_x(F(\xi x_0)) = \xi f(\xi x_0)$, we have

$$\frac{l'''(0)}{3!} = \frac{T_x(D^3F(0)(x_0^3))}{3!} = \frac{D^2f(0)(x_0^2)}{2},$$

$$\left(\frac{l''(0)}{2!}\right)^2 = \left(\frac{T_x(D^2F(0)(x_0^2))}{2!}\right)^2 = (Df(0)(x_0))^2,$$

and

$$\frac{D^3F(0)(x_0^3)}{3!} = \frac{D^2f(0)(x_0^2)}{2!} x_0, \quad \frac{1}{2} D^2F(0)\left(x_0, \frac{D^2F(0)(x_0^2)}{2!}\right) = (Df(0)(x_0))^2 x_0.$$

Using the above equalities and Lemma 2.4, we obtain the desired conclusion.

In order to prove that the estimates of Theorem 4.3 are sharp, it suffices to consider the following examples.

If $\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + \frac{1-2\nu}{1-\lambda} g'(0)\right| \ge 1$, we consider the example

$$F(x) = x \exp \frac{1}{1-\lambda} \int_0^{T_u(x)} (g(t) - 1) \frac{dt}{t}, \quad x \in \mathbb{B},\ \|u\| = 1.$$

If $\left|\frac{1}{2}\frac{g''(0)}{g'(0)} + \frac{1-2\nu}{1-\lambda} g'(0)\right| \le 1$, we consider the example

$$F(x) = x \exp \frac{1}{1-\lambda} \int_0^{T_u(x)} (g(t^2) - 1) \frac{dt}{t}, \quad x \in \mathbb{B},\ \|u\| = 1. \qquad \square$$

Remark 4.4. If we set $\lambda = -i \tan \beta$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ and $\lambda = 0$ in Theorem 4.3, we obtain the corresponding results of [25] and [36], respectively. Moreover, the proofs presented here are simpler than those given in [25] and [36].

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