CONCENTRATION INEQUALITIES
FOR PALEY–WIENER SPACES

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We consider how much of the mass of an element in a Paley–Wiener space can be concentrated on a given set. We seek bounds in terms of relative densities of the given set. We extend a result of Donoho and Logan from 1992 in one dimension and consider similar results in higher dimensions.

1. Introduction

Let M be a convex body in $\mathbb{R}^d$, and let $\mathcal{B}_p(M)$, $1 \le p \le \infty$, be the Paley–Wiener space of elements from $L^p(\mathbb{R}^d)$ with distributional Fourier transform supported in M. The Fourier transform $\mathcal{F}f$ is given by

$$\mathcal{F}f(\varphi) = \int_{\mathbb{R}} \hat{\varphi}(t) f(t)\, dt$$

for a Schwartz function φ. (We use $\hat{\varphi}(t) = \int \varphi(x) e^{-2\pi i x t}\, dx$.) We write $\mathcal{B}_p(\tau)$ if M is the ball with center at the origin and radius τ.

Let N and $W_\delta \subseteq \mathbb{R}^d$ be measurable and set $W_\delta(x) = x + W_\delta$. (In this article, W_δ is either a ball or a cube.) We consider the problem of finding a constant $C(M, \delta) > 0$ such that

(1) $\qquad \|G \chi_N\|_1 \le C(M, \delta) \sup_{x \in \mathbb{R}^d} |N \cap W_\delta(x)| \, \|G\|_1 \quad$ for all $G \in \mathcal{B}_1(M)$.

Here $|\,.\,|$ denotes Lebesgue measure and χ_N is the characteristic function of N. We emphasize that the constant is not allowed to depend on N.

This question was studied by Donoho and Logan [1992] in dimension $d = 1$ in connection with recovery of a bandlimited signal that is corrupted by noise. In their setting, an unknown noise $n \in L^1(\mathbb{R})$ is added to a known signal $F \in \mathcal{B}_1([-\tau, \tau])$, and they investigate sufficient conditions under which the best approximation $\widetilde{F} \in \mathcal{B}_1([-\tau, \tau])$ to $F + n$ satisfies $\widetilde{F} = F$, i.e., when F can be perfectly recovered from knowledge of $F + n$ through best L^1-approximation.

Denoting now by N the support of n, it is a useful fact that the concentration condition

$$(2) \qquad \frac{\|G\chi_N\|_1}{\|G\|_1} < \frac{1}{2} \quad \text{for all } G \in \mathcal{B}_1(M)$$

is sufficient to conclude that $F = \widetilde{F}$. The argument can be found in several places, e.g., Donoho and Stark [1989, Section 6.2], who refer to it as Logan's phenomenon. (Logan's thesis [1965] appears to contain the earliest record of this argument.) It was shown in [Donoho and Logan 1992, Theorem 7] that (1) holds for $W_\delta(x) = \left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right]$ with

$$(3) \qquad C([-\tau, \tau], \delta) = \frac{\pi\tau}{\sin(\pi\tau\delta)},$$

where $\tau\delta < 1$, and combining this with (2), it is evident that this gives $F = \widetilde{F}$ provided the relative density (or Nyquist density) of the support of the noise satisfies

$$\delta^{-1} \sup_{x \in \mathbb{R}} |N \cap [x, x + \delta]| < \frac{\sin(\pi\tau\delta)}{2\pi\tau\delta}.$$

A preprint of Baranov, Jaming, Kellay, and Speckbacher [Baranov et al. 2023] extends these results to the setting of model spaces (of which the Paley–Wiener spaces are essentially special cases). We describe their results in more detail in the next section.

We mention that conditions to recover an element of a closed subspace of an L^1-space that has been corrupted by a sparse L^1-noise have been investigated in many different settings, and concentration inequalities lead frequently to sufficient conditions. (This relies on the fact that if a set N satisfies an analogue of (2) for all G in a given closed subspace of an L^1-space, then the zero function is the closest element from the subspace to every L^1 function with support contained in N.) We highlight a few of these results here. Abreu and Speckbacher [2021] obtained estimates about the concentration on a given subset of R^2 for the L_p-norm for functions in modulation and polyanalytic Fock space using convex optimization methods. In [Abreu and Speckbacher 2022], they formulated the large sieve principle for the continuous wavelet transform on the Hardy space, adapting the concept of maximum Nyquist density to the hyperbolic geometry of the underlying space. Jaming and Speckbacher [2021] found concentration estimates for finite expansions of spherical harmonics on two-point homogeneous spaces via the large sieve principle. Speckbacher and Hrycak [2020] used estimates of the spherical harmonics coefficients of certain zonal filters to derive upper bounds for concentration in terms of the maximum Nyquist density on the unit sphere S^2 for band-limited spherical harmonics expansions. Herrmann and Hennenfent [2008] developed a curvelet-based recovery, which recovered seismic wavefields from seismic data volumes

with large percentages of traces missing. Candès, Romberg, and Tao [Candès et al. 2006] did reconstruction using the discrete Fourier transform of finite discrete-time signals belonging to space C^N. Benyamini, Kroó, and Pinkus [Benyamini et al. 2012] studied the phenomenon that the zero function is the best L_1-approximant to functions with small support.

2. Results

There are two questions that this article seeks to address. Our first result deals with reconstruction in higher dimensions. We investigate the case when M is a cube and $W_\delta(0)$ is a ball with center at the origin. We denote by J_ν the Bessel function of the first kind and by $j_\nu(k)$ its k-th positive zero.

Theorem 1. *Let* $\lambda, \alpha > 0$, $d \in \mathbb{N}$ *and let* $N \subseteq \mathbb{R}^d$ *be the support of* $n \in L^1(\mathbb{R}^d)$. *If*

$$\alpha\lambda < \frac{j_{d/2}(1)\, d^{-\frac{1}{2}}}{2\pi},$$

then for all $G \in \mathcal{B}_1([-\lambda, \lambda]^d)$

(4) $$\|G\chi_N\|_1 \leq \frac{(\sqrt{d}\lambda)^{d/2}}{\alpha^{d/2} J_{d/2}(2\pi\sqrt{d}\alpha\lambda)} \sup_{x\in\mathbb{R}^d} |N \cap B(x, \alpha)| \|G\|_1.$$

We discuss conditions when this constant is best possible at the end of Section 3.

Second, it is clear that the shape of the bound in (3) requires $\delta\tau < 1$. In contrast, it was shown for $p = 2$ in [Donoho and Logan 1992, Theorem 4] that for any positive τ and δ

$$\|G\chi_N\|_2^2 \leq (\tau + \delta^{-1}) \sup_{x\in\mathbb{R}} |N \cap [x, x+\delta]| \|G\|_2^2$$

for all $G \in \mathcal{B}_2(\tau)$ (with constants adjusted due to the different normalization of the Fourier transform) which suggests that an inequality with constant $c(\tau + \delta^{-1})$ should also be true for $p = 1$. The preprint [Baranov et al. 2023] mentioned in the introduction gives two approaches to establishing such an inequality for $1 \leq p < \infty$, one approach is through oversampling, and another relying on a Bernstein type inequality in model spaces. Our next theorem shows that such a result for $p = 1$ can also be obtained with the strategy of Donoho and Logan. The constants in the following theorem are worse than the constants obtainable through the Bernstein type inequality in [Baranov et al. 2023] and better than the constants obtainable through oversampling.

Theorem 2. *Let* $\tau, \delta > 0$ *and let* N *be the support of* $n \in L^1(\mathbb{R})$. *Then for all* $G \in \mathcal{B}_1(\tau)$

$$\|G\chi_N\|_1 \leq C_{\tau,\delta} \sup_{x\in\mathbb{R}} |N \cap [x, x+\delta]| \|G\|_1.$$

where $C_{\tau,\delta} \leq \frac{80}{13}(\tau + \delta^{-1})$ *for all positive* τ *and* δ. *The bound may be improved to* $C_{\tau,\delta} \leq \frac{5}{2}(\tau + \delta^{-1})$ *for* $\tau\delta \geq 2$.

As is usual with this method, the bounds only become effective when the density is a fraction of the reciprocal of the type τ. If one is interested in bounds for $\|G\chi_N\|_1/\|G\|_1$ at larger densities, a version of the Logvinenko–Sereda theorem from O. Kovrijkine [2001] gives nontrivial bounds whenever the density is smaller than 1.[1] (The constants in [Kovrijkine 2001] are not effective and don't yield concrete bounds to decide when the quotient is $< \frac{1}{2}$.)

3. Proof of Theorem 1

We briefly review a general approach to prove inequalities of the above form introduced by Donoho and Logan [1992]. Construct a kernel $K(x, y)$ so that $f \mapsto Tf$ given by

$$Tf(y) = \int K(x, y) f(x)\, dx$$

defines a bounded invertible transformation when restricted to $\mathcal{B}_1(\tau)$. Then a change of integration order gives

$$\int_N |G(x)|\, dx \le \int_N \int |K(x, y)| T^{-1} G(x)|\, dx\, dy$$

$$\le \left(\sup_x \int_N K(x, y)\, dy \right) \|T^{-1}\| \|G\|_1.$$

If $K(x, y) = g(x - y)$ for some $g \in L^\infty$ with $\operatorname{supp}(g) \subseteq W_\delta(0)$, then the supremum may be further estimated by $\|g\|_\infty \sup_x |N \cap W_\delta(x)|$, where $T = T_g$ is now the convolution operator $T_g f = f * g$ restricted to $\mathcal{B}_1(\tau)$. For given g the size of the constant depends then only on $\|g\|_\infty \|T_g^{-1}\|$, and (2) shows that we need

$$\sup_x |N \cap W_\delta(x)| < \frac{1}{2\|g\|_\infty \|T_g^{-1}\|}.$$

Thus, it is the task to construct g as above where $\|g\|_\infty \|T_g^{-1}\|$ is as small as possible. To create an auxiliary function g with computable product $\|g\|_\infty \|T_g^{-1}\|$, Logan and Donoho observed that if $1/\hat{g}$ is positive and concave on an interval $I = [-a, a]$ with center at the origin, then the periodic extension of $1/\hat{g}$ restricted to I is the Fourier transform of a measure ν that acts as the inverse operator of convolution with g on $\mathcal{B}_1(a)$ and has total variation $|\nu| = 1/\hat{g}(a)$. (In fact, ν is the minimal extrapolation of $1/\hat{g}$ restricted to I in the sense of Beurling.)

For $x \in \mathbb{R}^d$ we consider

$$(5) \qquad\qquad g_\alpha(x) = \chi_{B(0,\alpha)}(x)$$

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[1]The authors are grateful to Walton Green for drawing their attention to [Kovrijkine 2001].

whose Fourier transform for $t \in \mathbb{R}^d$ is

$$\hat{g}_\alpha(t) = \frac{\alpha^{d/2} J_{d/2}(2\pi\alpha|t|)}{|t|^{d/2}}.$$

To construct a minimal extrapolation of $1/\hat{g}_\alpha$ restricted to $[-\tau, \tau]^d$, we need a representation of the reciprocal of $\hat{g}_\alpha$ as a Laplace transform of totally positive function. The theory was originally developed by Schoenberg [1951]; our notation follows the book of Hirschman and Widder [1955]. An entire function E belongs to the Laguerre–Pólya class $\mathcal{E}$ if and only if it has the form

$$E(s) = e^{cs^2 + bs} \prod_{k=1}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{\frac{s}{a_k}},$$

where $c \geq 0$, b, a_k $(k = 1, 2, \dots)$ are real, and

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty.$$

Lemma 3. *There exists an integrable function $G \geq 0$ such that for $x \in (-j_p(1), j_p(1))$*

(6)
$$\frac{x^p}{J_p(x)} = \int_0^{\infty} e^{x^2 t} G(t) \, dt.$$

Proof. The Bessel function $J_p(x)$ has an infinite product representation [Olver and Maximon 2022, Section 10.21(iii)]. Dividing each side by x^p gives us

$$\frac{J_p(x)}{x^p} = \frac{1}{2^p \Gamma(p+1)} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{j_p^2(k)}\right)$$

$$= \frac{1}{2^p \Gamma(p+1)} \prod_{k=1}^{\infty} \left(1 - \frac{x}{j_p(k)}\right) e^{x/j_p(k)} \prod_{k=1}^{\infty} \left(1 + \frac{x}{j_p(k)}\right) e^{-x/j_p(k)}$$

Substituting $x = \sqrt{y}$ in the infinite product representation of $J_p(x)/x^p$ gives us

$$\frac{J_p(\sqrt{y})}{y^{p/2}} = \frac{1}{2^p \Gamma(p+1)} \prod_{k=1}^{\infty} \left(1 - \frac{y}{j_p(k)^2}\right)$$

which is an entire function and belongs to class $\mathcal{E}$. Let $E(x) = J_p(x)/x^p$. Then by [Hirschman and Widder 1955, Theorem 6.1] the function $1/E(\sqrt{y})$ has a Laplace transform representation given by

$$\frac{1}{E(\sqrt{y})} = \int_{\mathbb{R}} e^{-yt} G(t) \, dt,$$

where $G(t) \in C^{\infty}$ is a nonnegative, integrable function and the integral converges in the largest vertical strip which contains the origin and is free of zeroes of $E(\sqrt{y})$,

which is $-\infty < y < j_p^2(1)$. Next, we want to determine the values of t for which $G(t) > 0$. Note that $E(\sqrt{y})$ may be expressed as

$$E(\sqrt{y}) = \frac{1}{2^p \Gamma(p+1)} \prod_{k=1}^{\infty} \left(1 - \frac{y}{j_p^2(k)}\right)$$

$$= \frac{1}{2^p \Gamma(p+1)} \exp\left(-\sum_{k=1}^{\infty} \frac{y}{j_p(k)}\right) \prod_{k=1}^{\infty} \left(1 - \frac{y}{j_p^2(k)}\right) e^{y/j_p(k)}$$

The function $E(\sqrt{y})$ has no negative zeroes. We apply [Hirschman and Widder 1955, Chapter 5, Corollary 3.1] with $\alpha_1 = -\infty$ and

$$b = -\sum_{k=1}^{\infty} \frac{1}{j_p(k)}$$

to obtain that $G(t) > 0$ if $t \in (-\infty, 0)$ and $G(t) = 0$ otherwise, giving us for $y \in (0, j_p^2(1))$,

$$\frac{1}{E(\sqrt{y})} = \int_{-\infty}^{0} e^{-yt} G(t) \, dt.$$

Substituting $y = x^2$ and $t \mapsto -t$ gives the claim. $\qquad\square$

Let $\lambda > 0$ and $\alpha > 0$ with $2\pi\sqrt{d}\alpha\lambda < j_{d/2}(1)$. We construct a (signed) measure ν that is an inverse transform on $\mathcal{B}_1([-\lambda, \lambda]^d)$ of convolution with g_α satisfying

$$\|f * \nu\|_1 \le \frac{(\sqrt{d}\lambda)^{d/2}}{\alpha^{d/2} J_{d/2}(2\pi\sqrt{d}\lambda\alpha)} \|f\|_1,$$

and we show that the constant is best possible among all inverse transformations of convolution with g_α on $\mathcal{B}_1([-\lambda, \lambda]^d)$. We expand $1/\hat{g}_\alpha$ restricted to $[-\lambda, \lambda]^d$ into its Fourier series

$$\frac{1}{\hat{g}_\alpha(t)} = \sum_{n \in \mathbb{Z}^d} H_\alpha(n) e^{2\pi i n t},$$

where

$$H_\alpha(n) = \left(\frac{1}{2\lambda}\right)^d \int_{[-\lambda, \lambda]^d} \frac{|x|^{d/2}}{\alpha^{d/2} J_{d/2}(2\pi\alpha|x|)} e^{-i\frac{\pi}{\lambda} n x} \, dx.$$

Lemma 4. *The coefficients satisfy*

$$H(n_1, \ldots, n_d) = (-1)^{n_1 + \cdots + n_d} |H(n_1, \ldots, n_d)|$$

Proof. The restrictions on $\alpha\lambda$ imply that the following integrals converge absolutely. Inserting the Schoenberg representation (6) gives with $n = (n_1, \ldots, n_d)$

$$H_\alpha(n) = \left(\frac{1}{2\sqrt{2\pi\alpha\lambda}}\right)^d \int_{-\infty}^0 G(t)\left(\prod_{j=1}^d \int_{[-\lambda,\lambda]} e^{-x_j^2 t} e^{-i\frac{\pi}{\lambda}n_j x_j} \, dx_j\right) dt.$$

Since $t < 0$, the function $x_j \mapsto e^{-tx_j^2}$ is positive, symmetric, and concave. Hence $e^{-i\frac{\pi}{\lambda}n_j x_j}$ may be replaced by $\cos\left(\frac{\pi}{\lambda}n_j x_j\right)$. A short argument involving two integration by parts may be used to show that

$$(-1)^{n_j} \int_{[-\lambda,\lambda]} e^{-x_j^2 t} \cos\left(\frac{\pi}{\lambda}n_j x_j\right) dx_j \geq 0,$$

which implies the claim of the lemma. $\square$

We define a measure ν_α on $\mathbb{R}^d$ by

$$\nu_\alpha = \sum_{n\in\mathbb{Z}^d} H_\alpha(n)\delta_{\frac{n}{2\lambda}},$$

where δ_x is the point measure at x with $\delta_x(\mathbb{R}^d) = 1$.

Lemma 5. *Let λ and α be positive with $2\pi\sqrt{d}\lambda\alpha < j_{d/2}(1)$. Convolution with ν_α is the inverse operator of convolution with g_α on $\mathcal{B}_1([-\lambda,\lambda]^d)$ with*

$$\|f * \nu_\alpha\|_1 \leq \frac{(\sqrt{d}\lambda)^{d/2}}{\alpha^{d/2} J_{d/2}(2\pi\sqrt{d}\alpha\lambda)}\|f\|_1$$

for all $f \in \mathcal{B}_1([-\lambda,\lambda]^d)$.

Proof. By construction of ν_α we have

$$\hat{g}_\alpha(t)\hat{\nu}_\alpha(t) = 1$$

for all $t \in [-\lambda,\lambda]^d$, and we observe that the total variation measure $|\nu_\alpha|$ satisfies

$$|\nu_\alpha|(\mathbb{R}^d) = \sum_{n\in\mathbb{Z}^d} |H_\alpha(n)| = \sum_{n\in\mathbb{Z}^d} H_\alpha(n)(-1)^{n_1+\cdots+n_d} = \frac{1}{\hat{g}_\alpha(\lambda,\ldots,\lambda)},$$

and Minkowski's inequality $\|f * \nu_\alpha\|_1 \leq |\nu_\alpha|\|f\|_1$ shows that convolution with ν_α defines a bounded operator on $\mathcal{B}_1([-\lambda,\lambda]^d)$ that inverts convolution with g_α. $\square$

Lemma 5 gives a bound for the operator norm of the inverse of convolution with g_α, and the calculation at the beginning of the proof of Theorem 1 may be used to complete the proof of (4).

Optimality. Let ν_g be a measure with $\hat{\nu}_g = 1/\hat{g}$ on $[-\lambda, \lambda]^d$. Among auxiliary functions g that satisfy

$$(7) \qquad\qquad |\hat{g}(\lambda, \ldots, \lambda)| = |\hat{g}(-\lambda, \ldots, -\lambda)|$$

the choice g_α is optimal in the range $\alpha\lambda < (2\pi)^{-1} j_{d/2}(1) d^{-\frac{1}{2}}$. To show this, we follow the strategy of [Donoho and Logan 1992, Lemma 11]. We define

$$I_\infty = \sup\{|\hat{g}(\lambda, \ldots, \lambda)| : \operatorname{supp}(g) \subseteq B(0, \alpha), \|g\|_\infty = 1\}$$

and observe that $\|\nu_g\| \geq 1/I_\infty$. If g satisfies (7) then we may assume that the function g optimizing I_∞ is even. It follows that

$$\hat{g}(\lambda, \ldots, \lambda) = \int_{B(0,\alpha)} g(y_1, \ldots, y_d) \cos(\lambda y_1) \cdots \cos(\lambda y_d) \, d(y_1, \ldots, y_d).$$

The cosine terms are nonnegative in the stated range, hence the value of the transform is maximized under the constraint $|g| \leq 1$ by taking g to be equal to 1. The constant in Theorem 1 is obtained by choosing this g in (5), hence the constant is optimal in this case.

As a final remark, if we construct ν_g through a periodic extension of $1/\hat{g}$ (as in the previous section), then the condition that ν has finite total variation implies that $\hat{\nu}_g$ is continuous. Since a periodic extension must satisfy $\hat{\nu}(\lambda, \ldots, \lambda) = \hat{\nu}(-\lambda, \ldots, -\lambda)$, the condition (7) is then necessary in order for ν_g of finite total variation to exist.

4. Window comparisons

Analogously to dimension one, for a convex body K we define the maximum Nyquist density of N (relatively to K) by

$$\rho(N, K) = \frac{1}{|K|} \sup_{u \in \mathbb{R}^d} |N \cap (u + K)|.$$

We compare the result of Theorem 1 to the case where the window K is a hypercube of side length δ, which is an extension of the L_1 reconstruction result by [Donoho and Logan 1992]. The zero $j_p(1)$ has an asymptotic expansion given in [Olver and Maximon 2022] by

$$j_p(1) \simeq \left(\frac{p}{2} + \frac{1}{4}\right)\pi.$$

Denote the ball of radius r centered at origin by $B(0, r)$, and the volume of a ball with radius α in d-dimensions by $V_d(\alpha)$. It is given by

$$V_d(\alpha) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} \alpha^d.$$

When the window is a ball of radius α, perfect reconstruction is possible if the maximum Nyquist density satisfies

$$\rho(N, B(0, \alpha)) < \frac{\Gamma(d/2+1)J_{d/2}(2\pi\sqrt{d}\alpha\lambda)}{2(\pi\alpha\sqrt{d}\lambda)^{d/2}},$$

where

$$\alpha < \frac{j_{d/2}(1)}{2\pi\sqrt{d}\lambda}.$$

For $\lambda\delta < 2\pi$, the corresponding density bound is

$$\rho(N, [-\delta/2, \delta/2]^d) < \frac{1}{2}\left(\frac{\sin(\lambda\delta/2)}{\lambda\delta/2}\right)^d.$$

The support of the Fourier transform for both the problems is same, that is, $[-\lambda, \lambda]^d$. In order to be closely compare the two windows, we consider the following three cases. First, consider the ball of radius $\alpha = \delta/2$, centered at the origin, such that the ball is inside the cube. In this case, for dimension $d > 119$, the Nyquist density threshold for the ball window is bigger than that of cube window.

In the second case, we consider the ball with radius $\alpha = \delta\sqrt{d}/2$, so that the cube is inside the ball. Let $\delta = 1/(2\pi^2)$. For large d, the Nyquist density threshold asymptotically satisfies

$$\rho(T, B(0, \alpha), F) < \frac{\Gamma(d/2+1)J_{d/2}(d/2)}{2(d/4)^{d/2}}$$

$$\sim \frac{\sqrt{\pi d}\left(\frac{d}{2e}\right)^{d/2}}{2\left(\frac{d}{4}\right)^{d/2}}\frac{\Gamma(1/3)}{2^{1/3}\cdot 3^{1/6}\cdot\pi.d^{1/3}}$$

$$\sim d^{1/6}\left(\frac{2}{e}\right)^{d/2}$$

The Nyquist density for the cube window satisfies

$$\rho(T, [-\delta/2, \delta/2]^d, F) < \tfrac{1}{2}(4\pi^2\sin(1/4\pi^2))^d.$$

The bound for the Nyquist density of the cube window remains larger than the bound for the Nyquist density of ball window for any d in this case.

Third, we set the volume of the cube is equal to the volume of the ball. Then the radius α of the ball satisfies

$$\alpha = \delta\sqrt[d]{\frac{\Gamma(d/2+1)}{\pi^{d/2}}}.$$

Using Stirling's approximation, we get

$$\alpha \simeq \delta \sqrt[d]{\left(\frac{d}{2\pi e}\right)^{d/2} (\pi d)^{1/2}}.$$

Let $\delta = \sqrt{2\pi e}/(4\pi^2)$. For large d, the Bessel function in the Nyquist density of the ball window satisfies

$$J_{d/2}(2\pi^2 \sqrt{\alpha}) = J_{d/2}(d \cdot \pi^{1/2d} \cdot d^{1/2d}/2) \to J_{d/2}(d/2),$$

since $\pi^{1/2d} \cdot d^{1/2d} \to 1$ for large d. The bound for the Nyquist density of the ball window is then

$$\rho(T, B(0, \alpha), F) < \frac{\Gamma(d/2+1)J_{d/2}(d/2)}{2\left(\pi^2 \sqrt{d} \frac{\sqrt{2\pi e}}{4\pi^2 \sqrt{\pi}} \sqrt[d]{\Gamma(d/2+1)}\right)}$$

$$\sim \frac{\Gamma(1/3)}{2^{4/3} \cdot 3^{1/6} \cdot \pi^{3/4} d^{1/12}} \left(\frac{4}{2e}\right)^{d/2}$$

For the cube window, the sufficient bound for reconstruction is

$$\rho(T, [-\delta/2, \delta/2]^d, F) < \frac{1}{2}\left(\frac{\sin(\sqrt{2\pi e}/8\pi)}{\sqrt{2\pi e}/8\pi}\right)^d.$$

In this case also, the Nyquist density for the cube window remains larger than the Nyquist density of ball window for any dimension d. In conclusion, the bigger the Nyquist density threshold, the better, since it allows the signal to incorporate more noise but still be recovered. Only for the case when the ball is just inside the cube, the Nyquist density threshold for the ball window is larger than that of cube window for $d > 119$. For all other cases, the Nyquist density threshold for the cube window stays larger.

5. Proof of Theorem 2

Returning to the strategy described in Theorem 1, the choice in [Donoho and Logan 1992] was $g = \chi_{[-\frac{\delta}{2}, \frac{\delta}{2}]}$, which is optimal for $\delta\tau \leq \frac{1}{2}$, gives a nonoptimal bound for $\frac{1}{2} < \delta\tau < 1$, and fails to give a bound for $\delta\tau \geq 1$. This can be traced back to the fact that $\hat{g}(\delta) = 0$.

We define for $\tau > 0$ and real x a function g_τ, supported on $[-1, 1]$, by

$$g_\tau(x) = -2(1 - |x|)\frac{\cos 2\pi(\tau+1)x - \cos 2\pi\tau x}{4\pi^2 x^2}\chi_{[-1,1]}(x).$$

The Fourier transform of g_τ has the useful property that the sum of its partials with respect to t and τ has a simple integral representation.

Proposition 6. *For any t and τ*

$$\frac{\partial}{\partial t}(\hat{g}_\tau(t)) + \frac{\partial}{\partial \tau}(\hat{g}_\tau(t)) = \int_{2\pi(t+\tau)}^{2\pi(t+\tau+1)} \frac{\sin^2 u}{u^2}\, du.$$

Proof. For ease of notation we set $G(t, \tau) = \hat{g}_\tau(t)$, and we denote first partials by G_t and G_τ. Writing

$$g_\tau(x) = 2(1 - |x|)\frac{\cos 2\pi(\tau + 1)x - \cos 2\pi \tau x}{(-2\pi ix)^2}\chi_{[-1,1]}(x)$$

and using that $g_\tau(x)$ is even, we have

$$
\begin{aligned}
G_t(t, \tau) &= \int_{-1}^{1} (-2\pi ix)g_\tau(x)e^{-2\pi ixt}\, dx \\
&= \int_{-1}^{1} (-2\pi ix)g_\tau(x)(-i\sin(2\pi xt))\, dx \\
&= \int_{-1}^{1} 2(1 - |x|)\frac{\cos(2\pi(\tau + 1)x) - \cos(2\pi \tau x)}{2\pi x}\sin(2\pi xt)\, dx \\
&= 4\int_{0}^{1} (1 - x)\frac{\cos(2\pi(\tau + 1)x) - \cos(2\pi \tau x)}{2\pi x}\sin(2\pi xt)\, dx.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
G_\tau(t, \tau) &= \int_{-1}^{1} \frac{\partial}{\partial \tau}(g_\tau(x))e^{-2\pi ixt}\, dx \\
&= 4\int_{0}^{1} (1 - x)\frac{\sin(2\pi(\tau + 1)x) - \sin(2\pi \tau x)}{2\pi x}\cos(2\pi xt)\, dx.
\end{aligned}
$$

The integrals have representations in terms of the sine-integral

$$\text{Si}(u) = \int_{0}^{u} \frac{\sin(w)}{w}\, dw.$$

A direct calculation gives

$$2\int_{0}^{1} \cos(2\pi ax)\frac{\sin(2\pi bx)}{x}\, dx = \text{Si}(2\pi(a + b)) - \text{Si}(2\pi(a - b))$$

$$2\int_{0}^{1} \cos(2\pi ax)\sin(2\pi bx)\, dx = -\frac{b}{\pi(a - b)(a + b)} + \frac{\cos(2\pi(a - b))}{2\pi(a - b)}$$
$$-\frac{\cos(2\pi(a + b))}{2\pi(a + b)}.$$

We obtain

$$G_t(t, \tau) + G_\tau(t, \tau)$$
$$= \frac{2}{\pi^2} \left(\frac{\sin^2(\pi(t + \tau))}{(t + \tau)(t + \tau + 1)} + \pi \operatorname{Si}(2\pi(t + \tau + 1)) - \pi \operatorname{Si}(2\pi(t + \tau)) \right)$$
$$= \frac{2}{\pi} \int_{\pi(t+\tau)}^{\pi(t+\tau+1)} \left(-\frac{\partial}{\partial u} \frac{\sin^2 u}{u} \right) du + \frac{2}{\pi} \int_{2\pi(t+\tau)}^{2\pi(t+\tau+1)} \frac{\sin w}{w} \, dw$$
$$= \frac{2}{\pi} \int_{\pi(t+\tau)}^{\pi(t+\tau+1)} \frac{\sin^2 u}{u^2} \, du$$

after substituting $w = 2u$ and combining the integrands. $\qquad \square$

Corollary 7. (i) *The function* $\tau \mapsto \hat{g}_\tau(0)$ *is positive, monotonically increasing, and has limit* 1 *as* $\tau \to \infty$. *Moreover*,

$$\hat{g}_0(0) > 0.65, \quad \hat{g}_1(1) > 0.8.$$

(ii) $t \mapsto (\hat{g}_\tau(t))^{-1}$ *is positive and concave on* $[-\tau, \tau]$.

(iii) $\|g_\tau\|_\infty = g_\tau(0) = 2\tau + 1$.
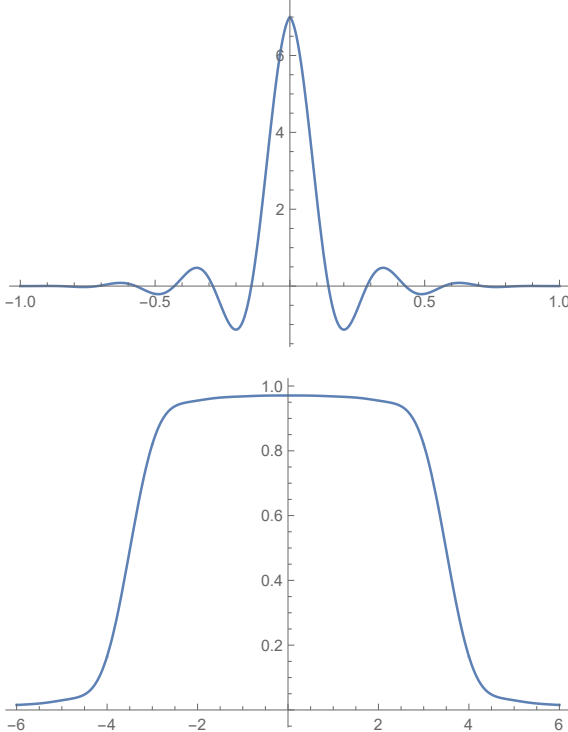
Figure 1. The transform pair $g_4(x)$ and $\hat{g}_4(t)$.

Proof. For the proof of (i) it follows from symmetry of $t \mapsto \hat{g}_\tau(t)$ that $\hat{g}'_\tau(0) = 0$, and hence Proposition 6 gives $\partial/\partial\tau\, \hat{g}_\tau(0) > 0$. Direct calculations give the claimed bounds.

Regarding (ii), we require an explicit representation of $\hat{g}''_\tau(t)$. It follows from

$$g_\tau(x) = 2(1 - |x|) \frac{\cos 2\pi(\tau + 1)x - \cos 2\pi\tau x}{(-2\pi i x)^2} \chi_{[-1,1]}(x)$$

that

$$(8) \quad \hat{g}''_\tau(t) = \int_{-1}^1 (-2\pi i x)^2 g_\tau(x) e^{-2\pi i x t}\, dx$$

$$= -\left(\frac{\sin \pi(t - \tau)}{\pi(t - \tau)}\right)^2 - \left(\frac{\sin \pi(t + \tau)}{\pi(t + \tau)}\right)^2$$

$$+ \left(\frac{\sin \pi(t - \tau - 1)}{\pi(t - \tau - 1)}\right)^2 + \left(\frac{\sin \pi(t + \tau + 1)}{\pi(t + \tau + 1)}\right)^2$$

$$= \frac{\sin^2(\pi(t - \tau))(2(t - \tau) - 1)}{(t - \tau)^2(t - \tau - 1)^2} - \frac{\sin^2(\pi(t + \tau))(2(t + \tau) + 1)}{(t + \tau)^2(t + \tau + 1)^2}.$$

Since the first term is negative for $t - \tau < \frac{1}{2}$ and the second term is positive for $t + \tau > -\frac{1}{2}$, it follows that

$$\hat{g}''_\tau(t) < 0 \text{ for } -\tau - \tfrac{1}{2} < t < \tau + \tfrac{1}{2}.$$

Multivariable chain rule and Proposition 6 show that

$$\frac{\partial}{\partial \tau}(\hat{g}_\tau(\tau)) > 0,$$

and since $\hat{g}_0(0) > 0$, it follows that $\hat{g}_\tau(\tau) > 0$ for all τ. Since $\hat{g}_\tau$ is concave down on $[-\tau, \tau]$, it follows that $\hat{g}_\tau(t) > 0$ for $t \in [-\tau, \tau]$. It follows that the second derivative of $t \mapsto (\hat{g}_\tau(t))^{-1}$ is positive for $|t| \leq \tau$.

To prove (iii) we show $\hat{g}_\tau > 0$ on $\mathbb{R}$. Identity (8) implies that $\hat{g}''_\tau(t) = \mathcal{O}(|t|^{-2})$. Since $\hat{g}'_\tau(0) = 0$, the integral of $\hat{g}''_\tau$ on $(0, t)$ equals $\hat{g}'_\tau(t)$.

A (lengthy) calculation along the lines of the proof of Proposition 6 shows from (8) that $\hat{g}'_\tau < 0$ on $(0, \infty)$. Since $\hat{g}_\tau(t) \to 0$ as $t \to \infty$ (by the Riemann–Lebesgue lemma) it follows that $\hat{g}_\tau$ is positive on $(0, \infty)$ and by symmetry on $\mathbb{R}$. Hence $|g_\tau(x)| \leq g_\tau(0)$ for all x. The second identity in (iii) is a direct evaluation. $\square$

Proof of Theorem 2. Setting $g_{\tau,\delta}(x) = g_{\tau\delta/2}(2x/\delta)$, we observe that $g_{\tau,\delta}$ is supported on $[-\delta/2, \delta/2]$, and

$$\hat{g}_{\tau,\delta}(t) = \frac{\delta}{2}\hat{g}_{\tau\delta/2}(\delta t/2).$$

It follows that $t \mapsto (\hat{g}_{\tau,\delta}(t))^{-1}$ is positive and concave for $|t| \leq \tau$. Let $a_n = a_n(\tau, \delta)$ be the Fourier coefficients satisfying

$$\frac{1}{\hat{g}_{\tau,\delta}(t)} = \sum_{n \in \mathbb{Z}} a_n e^{\pi i \frac{t}{\tau} n}$$

for $|t| \leq \tau$. Positivity and convexity imply that $|a_n| = (-1)^n a_n$. Define a measure $\nu = \nu_{\tau,\delta}$ on $\mathbb{R}$ for any Borel set A by

$$\nu(A) = \sum_{n \in \mathbb{Z}} a_n \delta_{n/(2\tau)}(A),$$

where δ_b is the Dirac measure at $b \in \mathbb{R}$. We observe that $\hat{\nu}(t) = 1/\hat{g}_{\tau,\delta}(t)$ for $|t| \leq \tau$, and the total variation satisfies

$$|\nu|(\mathbb{R}) = \sum_{n \in \mathbb{Z}} |a_n| = \sum_{n \in \mathbb{Z}} a_n(-1)^n = \frac{1}{\hat{g}_{\tau\delta/2}(\tau\delta/2)}.$$

It follows that convolution with ν is the inverse operator of convolution with $g_{\tau,\delta}$ when restricted to PW_τ^1. Moreover, for $g_{\tau,\delta}$ the choice of ν is optimal, since the value of the Fourier transform of ν is always a lower bound for the total variation.

It follows that

$$\|T_{g_{\tau,\delta}}^{-1}\| = \frac{1}{\hat{g}_{\tau\delta/2}(\tau\delta/2)}.$$

We observe the identities

$$\frac{\|g_{\tau,\delta}\|_\infty}{\hat{g}_{\tau,\delta}(\tau)} = \frac{2}{\delta} \frac{\|g_{\tau\delta/2}\|_\infty}{\hat{g}_{\tau\delta/2}(\tau\delta/2)} = \frac{2\tau + 2\delta^{-1}}{\hat{g}_{\tau\delta/2}(\tau\delta/2)}.$$

For $\tau > 0$ and $\delta > 0$ we use the inequality $\hat{g}_{\tau\delta/2}(\tau\delta/2) \geq \hat{g}_0(0) > 0.65$. For $\tau\delta \geq 2$, we may use the lower bound $\hat{g}_1(1) > 0.8$ instead. $\qquad\square$

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