Pacific Journal of Mathematics

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MONICA NEVINS

Volume 329 No. 2 April 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 329, No. 2, 2024
https://doi.org/10.2140/pjm.2024.329.259

THE LOCAL CHARACTER EXPANSION AS BRANCHING RULES: NILPOTENT CONES AND THE CASE OF SL(2)

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We show there exist representations of each maximal compact subgroup K of the p-adic group $G = \mathrm{SL}(2, F)$, $p \neq 2$, for each nilpotent coadjoint orbit, such that every irreducible admissible (complex) representation of G, upon restriction to a suitable subgroup of K, is a sum of these five representations in the Grothendieck group. This is a representation-theoretic analogue of the analytic local character expansion due to Harish-Chandra and Howe. Moreover, we show for general connected reductive groups that the wave front set of many irreducible positive-depth representations of G are completely determined by the *nilpotent support* of their unrefined minimal K-types.

1. Introduction

The distribution character of an admissible (complex) representation of a p-adic group can be expressed, in a neighbourhood of the identity, as a linear combination of Fourier transforms of the finitely many nilpotent orbital integrals in the dual of the Lie algebra. This remarkable theorem, known as the Harish-Chandra–Howe local character expansion, has many variations (such as expansions on neighbourhoods of other semisimple elements, or expansions in terms of other collections of orbital integrals [Kim and Murnaghan 2003; 2006; Spice 2018]) and many applications (such as determining the Gelfand–Kirillov dimension of a representation, or relating to conjectural classifications such as the orbit method, or the local Langlands correspondence [Barbasch and Moy 1997; Ciubotaru et al. 2022a; 2022b; Jiang et al. 2022]). Though it is primarily considered in characteristic zero, it also holds when the characteristic is sufficiently large and a suitable substitute for the exponential map exists [Cluckers et al. 2014].

In this paper, we interpret the local character expansion as a statement in the Grothendieck group of representations of a maximal compact open subgroup, upon restriction to a subgroup of suitable depth, for the case that $G = \mathrm{SL}(2, F)$, where F

Supported by NSERC Discovery grant RGPIN-2020-05020.
MSC2020: 22E50.
Keywords: representation theory, nilpotent orbits, local character expansion, p-adic groups.

is a local nonarchimedean field of residual characteristic at least 3. In particular, we construct for each nilpotent orbit $\mathcal{O}$ of G in the dual of its Lie algebra $\mathfrak{g}^*$ a (highly reducible) representation $\tau_x(\mathcal{O})$ of each maximal compact open subgroup G_x with the following property.

Theorem 1.1. *Let π be an irreducible admissible representation of $G = \mathrm{SL}(2, F)$ of depth $r \geq 0$, and let x be a vertex in the building of G. Then there exist integers $c_{x,\mathcal{O}}(\pi)$ such that in the Grothendieck group of representations we have*

$$\mathrm{Res}^{G}_{G_{x,r+}} \pi = \sum_{\mathcal{O}} c_{x,\mathcal{O}}(\pi)\, \mathrm{Res}^{G_x}_{G_{x,r+}} \tau_x(\mathcal{O}), \tag{1-1}$$

where $G_{x,r+}$ is the Moy–Prasad filtration subgroup of G_x of depth $r+$, and the sum is over all nilpotent orbits in $\mathfrak{g}^*$.

Moreover, the coefficients corresponding to the regular nilpotent orbits in this expansion are nonnegative integers and agree with those of the Harish-Chandra–Howe local character expansion (subject to suitable normalizations). Note that while inherently expressing the same local nature of representations, our statement holds with fewer restrictions on F than does the local character expansion, because it does not depend on the existence of a G-equivariant map, such as the exponential or a Cayley transform, from the Lie algebra to the group.

If G is $\mathrm{SL}_2(F)$ or an inner form of $\mathrm{GL}_n(F)$, then Henniart and Vignéras have proven a different local expansion in the same spirit as (1-1), one that holds for representations over any field R of characteristic not p, in a sufficiently small neighbourhood of 1, but which constructs the right-hand side as restrictions of particular representations of G itself ([Henniart and Vignéras 2024, Theorem 6.18] and [Henniart and Vignéras 2023, Theorem 1.3], respectively). When $G = \mathrm{GL}_n(F)$, these representations are of the form $\mathrm{Ind}_P^G \mathbf{1}$, for a suitable parabolic subgroup attached to $\mathcal{O}$, vastly generalizing a result of Roger Howe [1974]. When $G = \mathrm{SL}_2(F)$, they are representations that occur in an L-packet of size 4 (called "special unipotent representations" in the complex case here); the distinguished role of these representations in the complex case was observed previously in [Nevins 2011, §4]. In Section 8 we explore applications of these ideas, and answer [Henniart and Vignéras 2023, Questions 1.1 and 1.2] for complex representations of $\mathrm{SL}(2, F)$.

Now suppose G is a general connected reductive group. In Section 3, we develop some theory towards establishing the direct relationship from the local character expansion to a decomposition like (1-1), as follows.

The set of maximal orbits appearing in the local character expansion for an admissible representation π is denoted by $\mathcal{WF}(\pi)$; the closure of the union of these orbits is the wave front set of π. For depth-zero representations π, Barbasch and Moy [1997] proved that $\mathcal{WF}(\pi)$ is determined by the depth-zero components of the restriction of π to various maximal compact subgroups, through the theory of Gelfand–Graev representations.

For a positive-depth representation with minimal K-type Γ (in the sense of Moy and Prasad [1994]), we should instead infer $\mathcal{WF}(\pi)$ from the *nilpotent support* $\mathrm{Nil}(\Gamma)$ (Definition 3.2) of Γ. This definition, of independent interest, depends strongly on the classification of nilpotent orbits using Bruhat–Tits theory [Barbasch and Moy 1997; DeBacker 2002b]. In fact, in Proposition 3.4 we show that the algebraic notion of nilpotent support can be characterized as the set of nonzero nilpotent orbits appearing in the asymptotic cone on Γ, as defined in [Adams and Vogan 2021]. In Theorem 3.5 (proof due to Fiona Murnaghan), we prove that $\mathcal{WF}(\pi)$ is the set of maximal orbits of $\mathrm{Nil}(\Gamma)$ whenever the Γ-asymptotic expansion [Kim and Murnaghan 2003] reduces to a single term.

This last result is similar to recent work of Ciubotaru and Okada [2023], who show that the depth-r components of the restriction to certain compact open subgroups determine the wave front set of π. The idea of the nilpotent support is also central to that work, where they develop it using, among other things, the geometry of the associated finite reductive group.

Now again suppose that $G = \mathrm{SL}(2, F)$. Our result gives a second characterization of $\mathcal{WF}(\pi)$: it can be entirely determined from the *nontypical* representations occurring in the restriction of π to a maximal compact open subgroup, for π of any depth. That is, the asymptotic decomposition of $\mathrm{Res}_{G_x} \pi$ unfolds exactly as the representations $\tau_x(\mathcal{O})$ for $\mathcal{O} \in \mathcal{WF}(\pi)$.

For the case of a positive-depth representation π, our main theorem is stated in Theorem 6.4, with the explicit values of the constant coefficient given in Proposition 6.7. To prove the theorem, we first show that the restriction of π to a maximal compact subgroup can be expressed entirely in terms of twists of the pair (Γ, χ) used in the construction of π (Theorem 6.2), using results from [Nevins 2005; 2013]. Here, χ is a character of a torus $T = \mathrm{Cent}_G(\Gamma)$ that is realized by $\Gamma \in \mathfrak{g}^*$, and the realization of the irreducible components of the restriction is framed in terms of a generalization (Proposition 5.4) of a construction due to Shalika in his thesis. From this characterization, and a key technical result (Lemma 5.5), it follows that the expansion (1-1) exists and has leading terms corresponding to the nilpotent support of Γ. Since Γ represents a minimal K-type of π in the sense of Moy and Prasad [1994], we independently recover from Theorem 3.5 that the maximal orbits in $\mathrm{Nil}(\Gamma)$ coincide with $\mathcal{WF}(\pi)$.

For representations of depth zero, the principal technical difficulties lie in matching the depth-zero components with nilpotent orbits, particularly in the case of the twelve "exceptional" representations: the reducible principal series, the principal series composed of the trivial and the Steinberg representation, and the four special supercuspidal representations. Once these are addressed, Theorem 7.4 follows by carefully extracting the necessary branching rules from [Nevins 2005; 2013]. Again, the orbits in $\mathcal{WF}(\pi)$ are obtained from both the depth-zero components (via [Barbasch and Moy 1997]) and the asymptotic development of the branching rules.

At two crucial junctures we use information that is currently only known for $G = \mathrm{SL}(2, F)$ and a handful of other small-rank groups: one is the explicit calculation of the asymptotic cone on any semisimple element of $\mathfrak{g}^*$ (Section 4); the other is the full knowledge of the representation theory of the maximal compact subgroups of G (Section 5). While the former seems a tractable and interesting question in general, the latter is quite daunting: it is not expected that we will achieve a classification of the representations of maximal compact open subgroups of p-adic reductive groups. Note that a full classification is not necessary to prove the theorem: what is needed is a construction of an appropriate representation of G_x attached to each nilpotent orbit, and we explore how this might be done in Section 5B.

There are many interesting applications and open directions left to pursue. Evidently the overarching goal is to establish a result like (1-1) for a large class of groups, using the tools presented here, or those developed in [Henniart and Vignéras 2023; 2024]. To extend the work here, it may be fruitful to build representations of the groups $G_{x,0+}$ directly, rather than to construct representations of $G_{x,0}$; this has the advantage of avoiding the difficulties inherent at depth zero. It may also allow for a more uniform treatment of all points x of the building; in this paper, we consider only vertices, and the union of all $G_{x,r+}$ as x runs over vertices is not equal to G_{r+} in general.

In another direction, the Γ-asymptotic expansions of [Kim and Murnaghan 2003; 2006] describe the character of a positive-depth representation in a larger neighbourhood than does the local character expansion, by incorporating a minimal K-type Γ. Then Theorem 6.2 can be interpreted as analogously formulating these expansions in terms of branching rules. It would be interesting to explore this idea further.

The paper is organized as follows. We set our notation in Section 2 and then present some background on the local character expansion that provides the motivation and context for our results. In Section 3 we consider a general connected reductive group G. We define the nilpotent support of an element Γ of $\mathfrak{g}^*$, show it defines the asymptotic cone of Γ, and relate this to the wave front set via the theory of Γ-asymptotic expansions.

We then specialize to $G = \mathrm{SL}(2, F)$. In Section 4 we characterize the nilpotent cones $\mathrm{Nil}(\Gamma)$ in many ways (Proposition 4.1) and compute them explicitly. In Section 5 we recall the construction of certain irreducible representations of $\mathrm{SL}(2, \mathcal{R})$ by Shalika in his 1966 thesis [Shalika 2004], and then rephrase it using Bruhat–Tits theory and derive some consequences. This allows us to define, for each vertex $x \in \mathcal{B}(G)$, each nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$, and each central character ζ a representation $\tau_x(\mathcal{O}, \zeta)$ of G_x.

We prove our main theorems for representations of positive depth in Section 6 and for representations of depth zero in Section 7. We conclude with two brief applications of Theorem 1.1 in Section 8: an explicit formula for the functions $\widehat{\mu_{\mathcal{O}}}$ in terms of the trace character of the representation $\tau_x(\mathcal{O})$ of the compact group G_x; and an explicit polynomial expression for $\dim(\pi^{G_{x,2n}})$ (in the spirit of [Henniart and Vignéras 2023]) whose existence is predicted by the local character expansion.

2. Notation and background

Let F be a local nonarchimedean field of residual characteristic $p \neq 2$, with integer ring $\mathcal{R}$, maximal ideal $\mathcal{P}$ and residue field $\mathfrak{f}$ of cardinality q. We impose additional hypotheses on p in Section 2B, below. Fix once and for all an additive character ψ of F that is trivial on $\mathcal{P}$ and nontrivial on $\mathcal{R}$. Fix a uniformizer ϖ and normalize the valuation on F (and any extension thereof) by $\mathrm{val}(\varpi) = 1$. We write $\mathrm{val}(0) := \infty$.

Let $\boldsymbol{G}$ denote a connected reductive algebraic group defined over F whose group of F-rational points is denoted by G; we use $\mathfrak{g} = \mathrm{Lie}(\boldsymbol{G})(F)$ to denote its Lie algebra over F. We simplify notation by referring to tori, Borel subgroups and parabolic subgroups of G when we mean the F-points of such algebraic F-subgroups of $\boldsymbol{G}$, and denote them in roman font. Let G^{reg}, respectively $\mathfrak{g}^{\mathrm{reg}}$, denote the set of regular semisimple elements of G, respectively $\mathfrak{g}$. The group G acts on $\mathfrak{g}$ via the adjoint action Ad and on its dual $\mathfrak{g}^*$ via the coadjoint action Ad^*; we abbreviate these by both $g \cdot X$ or ${}^gX$ for $g \in G$ and X in $\mathfrak{g}$ or $\mathfrak{g}^*$. Similarly, if H is a subgroup of G we write ${}^gH$ for the group gHg^{-1}.

An element $X \in \mathfrak{g}^*$ or $\mathfrak{g}$ is called *semisimple* (or *almost stable*) if its G-orbit is closed. We define $X \in \mathfrak{g}^*$ or $\mathfrak{g}$ to be *nilpotent* if there exists an F-rational one-parameter subgroup $\lambda \in X_*(\boldsymbol{G})$ such that $\lim_{t\to 0} {}^{\lambda(t)}X = 0$. By [Adler and DeBacker 2002, §2.5], this is equivalent to a more usual definition that the closure of the coadjoint orbit in the rational topology contains 0. We say the one-parameter subgroup λ is *adapted* to X [DeBacker 2002b, Definition 4.5.6] if ${}^{\lambda(t)}X = t^2X$. We write $\mathcal{N}^*$ for the set of nilpotent elements of $\mathfrak{g}^*$ and $\mathcal{O}(0)$ for the (finite) set of G-orbits in $\mathcal{N}^*$.

We sometimes specify a group of matrices merely by the sets in which its entries lie; in this case, that the resulting subgroup is the intersection of this set with G is understood. We write $\lceil t \rceil = \min\{n \in \mathbb{Z} \mid n \geq t\}$ and $\lfloor t \rfloor = \max\{n \in \mathbb{Z} \mid n \leq t\}$. Write $\mathrm{Cent}_G(S)$ for the centralizer in G of the element or set S. We may write $[\sigma]$ for the trace character of a representation σ of a finite or compact group. The trivial representation is denoted by $\mathbf{1}$, and the characteristic function of a subset S is denoted by $\mathbf{1}_S$.

2A. ***The Bruhat–Tits building and Moy–Prasad filtration subgroups.*** Let $\mathcal{B}(G) = \mathcal{B}(\boldsymbol{G}, F)$ denote the (enlarged) Bruhat–Tits building of G; then to each $x \in \mathcal{B}(G)$

we associate its stabilizer G_x, which is a compact subgroup of G containing the parahoric subgroup $G_{x,0}$. These admit a Moy–Prasad filtration by normal subgroups $G_{x,r}$ with $r \in \mathbb{R}_{\geq 0}$ defined relative to the valuation on F. We briefly recap the definition; see also [Fintzen 2021b, §2] and [Kaletha and Prasad 2023, §13].

To define $G_{x,r}$, choose an apartment $\mathcal{A} \subset \mathcal{B}(G)$ containing x; this is the affine space over $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ for some maximal split torus T of G and we write $\mathcal{A} = \mathcal{A}(G, T)$. Let $\Phi = \Phi(G, T)$ denote the corresponding root system and Ψ the set of affine roots, viewed as functions on $\mathcal{A}$. For each root $\alpha \in \Phi$, let U_α denote the corresponding root subgroup. The affine roots ψ with gradient α define a filtration of U_α by compact open subgroups U_ψ.

Let $\boldsymbol{C} = \mathrm{Cent}_{\boldsymbol{G}}(\boldsymbol{T})$. As summarized at the start of [Kaletha and Prasad 2023, §9.8], $C = \boldsymbol{C}(F)$ contains a parahoric subgroup C_0, and a filtration by compact open normal subgroups C_r, $r > 0$, that is independent of the point $x \in \mathcal{A}$. When $\boldsymbol{C}$ is not tamely ramified, this filtration can be very subtle; see the extended careful analysis in [Kaletha and Prasad 2023, §13.3, §B.10].

As summarized in [Kaletha and Prasad 2023, Proposition 13.2.5], for any $r \geq 0$ we define compact open subgroups

$$G_{x,r} = \langle C_r, U_\psi \mid \psi \in \Psi, \psi(x) \geq r \rangle;$$

if $r = 0$ this is the parahoric subgroup and for $r > 0$ it is a Moy–Prasad filtration subgroup of $G_{x,0}$. It is independent of the choice of apartment containing x. The Moy–Prasad filtration is G-equivariant; for example, ${}^g G_{x,r} = G_{gx,r}$ for all $x \in \mathcal{B}(G)$ and $r \geq 0$.

Similarly, the Lie algebra $\mathfrak{g}$ admits a filtration $\mathfrak{g}_{x,r}$ by $\mathcal{R}$-modules indexed by $r \in \mathbb{R}$, as follows. Let $\mathfrak{t}$ denote the Lie algebra of T, $\mathfrak{c}$ its centralizer in $\mathfrak{g}$ and for each $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ denote the corresponding root subspace. These subspaces admit filtrations by $\mathcal{R}$-submodules $\mathfrak{c}_r$ with $r \in \mathbb{R}$ and $\mathfrak{g}_\psi$ for $\psi \in \Psi$, respectively, such that

$$\mathfrak{g}_{x,r} = \mathfrak{c}_r \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha,x,r}, \tag{2-1}$$

where $\mathfrak{g}_{\alpha,x,r}$ is the union of the $\mathcal{R}$-submodules $\mathfrak{g}_\psi$ such that $\psi \in \Psi$, the gradient of ψ is α, and $\psi(x) \geq r$. We write

$$G_{x,r+} = \bigcup_{s>r} G_{x,s} \quad \text{and} \quad \mathfrak{g}_{x,r+} = \bigcup_{s>r} \mathfrak{g}_{x,s}.$$

If the maximally split maximal tori of G are weakly induced, as defined in [Kaletha and Prasad 2023, Definition B.6.2], then for all $r > 0$ we have the Moy–Prasad isomorphism $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,2r} \cong G_{x,r}/G_{x,2r}$ [Kaletha and Prasad 2023, Theorem 13.5.1], which can be realized by a mock exponential map $e = e_x : \mathfrak{g}_{x,0+} \to G_{x,0+}$ as in [Adler 1998, §1.6]. Writing $\langle X, Y \rangle$ for the natural pairing of $X \in \mathfrak{g}^*$ with $Y \in \mathfrak{g}$, the Moy–Prasad filtration on the dual of the Lie algebra is defined by

$$\mathfrak{g}^*_{x,r} = \{X \in \mathfrak{g}^* \mid \langle X, Y \rangle \in \mathcal{P} \text{ for all } Y \in \mathfrak{g}_{x,(-r)+}\}.$$

We again define $\mathfrak{g}^*_{x,r+} = \bigcup_{s>r} \mathfrak{g}^*_{x,s}$. For any $x \in \mathcal{B}(G)$, $r > 0$ and $s \in \mathbb{R}$, the adjoint (respectively coadjoint) action of G on $\mathfrak{g}$ induces an action of $G_x/G_{x,r}$ on $\mathfrak{g}_{x,s}/\mathfrak{g}_{x,s+r}$ (respectively, $\mathfrak{g}^*_{x,s}/\mathfrak{g}^*_{x,s+r}$) [Adler 1998, Proposition 1.4.3].

Finally, for any $r \geq 0$ we define G-stable subsets

$$G_r = \bigcup_{x \in \mathcal{B}(G)} G_{x,r} \quad \text{and} \quad G_{r+} = \bigcup_{x \in \mathcal{B}(G)} G_{x,r+}.$$

For any real number r we do the same to define $\mathfrak{g}_r$ and $\mathfrak{g}_{r+}$.

If (π, V) is an irreducible admissible representation of G, then its depth is defined as the least real number $r \geq 0$ such that there exists $x \in \mathcal{B}(G)$ for which $V^{G_{x,r+}} \neq \{0\}$. We define the depth of a smooth irreducible representation ρ of G_x, for fixed x, in the same way; this is equivalent to the least $r \geq 0$ for which ρ factors through $G_x/G_{x,r+}$.

2B. *Restrictions on p.* We impose the restriction that G splits over a tamely ramified extension of F and that p does not divide the order of the absolute Weyl group of $\boldsymbol{G}$. One of the main results of [Fintzen 2021c] is that this is sufficient to ensure that all irreducible admissible representations are tame. It furthermore ensures that all maximally split maximal tori over F are weakly induced, so that our parahoric subgroups are defined relative to the standard filtration sketched above [Kaletha and Prasad 2023, §B.5, Proposition B.10.5] and the Moy–Prasad isomorphism holds. Combining [Fintzen 2021c, Lemma 2.2, Table 1] and [Adler and Roche 2000, §1], one sees that the hypotheses of [Adler 1998, Hypothesis 2.1.1] or [Adler and Roche 2000, Proposition 4.1] hold, so that there is a nondegenerate G-invariant bilinear form on $\mathfrak{g}$ under which $\mathfrak{g}^*_{x,r}$ and $\mathfrak{g}_{x,r}$ are identified for all x and r. For $\boldsymbol{G} = \mathrm{SL}(2)$ and $p \neq 2$ we may take the trace form, and define for each $\dot{X} \in \mathfrak{g}$ the element $X \in \mathfrak{g}^*$ by $\langle X, \cdot \rangle = \mathrm{tr}(\dot{X} \cdot)$.

We also impose the hypotheses of [DeBacker 2002b, §4] to obtain the classification of nilpotent orbits; this requires the use of $\mathfrak{sl}_2(F)$ triples over the residue field as well as some properties of a mock exponential map. By recent work of Stewart and Thomas [2018] the former condition is satisfied for $p > h$, where h is the Coxeter number of G. To satisfy all hypotheses for $G = \mathrm{SL}(2, F)$, it suffices to take $p \neq 2$.

In contrast, to state the local character expansion, which relates a function on the group to one on the Lie algebra, one needs a G-equivariant map $\mathfrak{g}_{0+} \to G_{0+}$ satisfying [Debacker 2002a, Hypothesis 3.2.1]. Such a map, which we'll simply denote by exp, can exist in large positive characteristic (see, for example, the discussion in [Cluckers et al. 2014, §2]); in characteristic zero, [DeBacker and

Reeder 2009, Lemma B.0.3] gives an effective lower bound on p. For $G = \mathrm{SL}(2, F)$, this entails in characteristic zero that $p > e + 1$, where e is the ramification index of F over $\mathbb{Q}_p$, for example.

2C. ***The local character expansion.*** As detailed in the expanded notes [Harish-Chandra 1999], Harish-Chandra proved in the 1970s that the distribution character of an irreducible admissible representation π of G, which is given on $f \in C_c^\infty(G)$ by

$$\Theta_\pi(f) = \operatorname{tr} \int f(g)\pi(g)\, dg,$$

is well defined and representable by a function, which we also denote by Θ_π, that is locally integrable on G and locally constant on the set G^{reg} of regular semisimple elements of G (see [Adler and Korman 2007, §13] and the discussion therein).

Similarly, to each coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ we associate its orbital integral, given on $f \in C_c^\infty(\mathfrak{g}^*)$ by

$$\mu_{\mathcal{O}}(f) = \int_{\mathcal{O}} f(X)\, d\mu_{\mathcal{O}}(X), \tag{2-2}$$

where $d\mu_{\mathcal{O}}$ is a Radon measure [Ranga Rao 1972]. Relative to ψ, the fixed additive character of F, the Fourier transform of $f \in C_c^\infty(\mathfrak{g})$ is a function $\widehat{f} \in C_c^\infty(\mathfrak{g}^*)$. The Fourier transform of the orbital integral $\mu_{\mathcal{O}}$ is the distribution given on $f \in C_c^\infty(\mathfrak{g})$ by $\widehat{\mu_{\mathcal{O}}}(f) = \mu_{\mathcal{O}}(\widehat{f})$. Then $\widehat{\mu_{\mathcal{O}}}$ is representable by a locally integrable function on $\mathfrak{g}$ that is locally constant on $\mathfrak{g}^{\mathrm{reg}}$ [Harish-Chandra 1999, Theorem 4.4]. We set $\mathfrak{g}_{r+}^{\mathrm{reg}} := \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r+} \cap \mathfrak{g}^{\mathrm{reg}}$.

The local character expansion expresses that these finitely many functions $\widehat{\mu_{\mathcal{O}}}$, for $\mathcal{O} \in \mathscr{O}(0)$, form a basis, in a neighbourhood of 0, for the space of locally integrable G-invariant functions that are locally constant on $\mathfrak{g}^{\mathrm{reg}}$. The nature of the expansion was first proven for $G = \mathrm{GL}(n, F)$ in characteristic 0 by Howe [1974] and then in the generality of connected reductive groups in characteristic zero by Harish-Chandra [1999]. Cluckers, Gordon and Halupczok [Cluckers et al. 2014] proved its validity in large positive characteristic; Adler and Korman [2007] proved an analogous result for expansions centred at other semisimple elements.

The precise domain on which the local character expansion holds was conjectured by Hales, Moy and Prasad [Moy and Prasad 1994] and proven in [Waldspurger 1995] for a large class of groups and by [Debacker 2002a] in the following generality.

Theorem 2.1 (the local character expansion). *If π is an irreducible admissible representation of G of depth r, then there exist unique $c_{\mathcal{O}}(\pi) \in \mathbb{C}$ such that for all $X \in \mathfrak{g}_{r+}^{\mathrm{reg}}$, we have*

$$\Theta_\pi(\exp(X)) = \sum_{\mathcal{O} \in \mathscr{O}(0)} c_{\mathcal{O}}(\pi)\widehat{\mu_{\mathcal{O}}}(X). \tag{2-3}$$

We denote by $\mathcal{WF}(\pi)$ the set of *maximal* nilpotent orbits $\mathcal{O}$ such that $c_{\mathcal{O}}(\pi) \neq 0$, where the ordering is taken in the local topology; this is the set denoted by $\mathrm{WF}^{\mathrm{rat}}(\pi)$ in [Tsai 2023a]. Heifetz [1985] defined and developed the analytic notion of the *wave front set* of a representation of a p-adic group, in analogy with the work of Howe [1981] in the real case. Przebinda [1990] proved that the wave front set coincides with the support of the right side of (2-3), which is the closure of the union of these orbits. Recent work of Cheng-Chiang Tsai [2024; 2023a; 2023b] has shown that the orbits of $\mathcal{WF}(\pi)$ may fail to be stably conjugate.

Finally, note that for $G = \mathrm{GL}(n, F)$, Howe proved that for each $\mathcal{O} \in \mathscr{O}(0)$, there is a corresponding parabolic subgroup P such that in a neighbourhood O of $0 \in \mathfrak{g}$,

$$\widehat{\mu_{\mathcal{O}}}|_O = \Theta_\pi \circ \exp|_O,$$

where $\pi = \mathrm{Ind}_P^G \mathbf{1}$ [Howe 1974, Lemma 5]. In the same vein, for $\mathrm{SL}(2, F)$, the functions $\widehat{\mu_{\mathcal{O}}}$ are almost equal to the characters of special unipotent representations (see (8-1)). In recent work, Henniart and Vignéras [2023; 2024] have proven that these in turn correspond to representation-theoretic expansions in a small enough neighbourhood of the identity, for all inner forms of $\mathrm{GL}_n(F)$ as well as for the group $\mathrm{SL}_2(F)$. We do not, however, expect such representations to exist in general as, for example, for classical groups nonspecial orbits cannot occur in $\mathcal{WF}(\pi)$ for any π [Mœglin 1996, Theorem 1.4]), yet can occur with nonzero coefficients in a local character expansion. The main goal in this paper is to propose an example of a weaker form of the Howe–Henniart–Vignéras theorem, based on representations of a maximal compact open subgroup, that one may hope can hold true in general.

3. Nilpotent orbits and nilpotent support

In this section, G is an arbitrary connected reductive group, subject to the hypothesis on p of Section 2B. We define the (local) nilpotent support of an element of $\mathfrak{g}^*$, and relate this both to the asymptotic cone and to the wave front set of a representation of positive depth.

3A. ***Degenerate cosets and nilpotent orbits.*** In [Adler and DeBacker 2002, §3], the authors generalize ideas of Moy and Prasad to establish for connected reductive groups that for all $r \in \mathbb{R}$,

$$\mathfrak{g}_r^* = \bigcap_{x \in \mathcal{B}(G)} (\mathfrak{g}_{x,r}^* + \mathcal{N}^*),$$

where $\mathfrak{g}_r^* := \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r}^*$. They further show that

$$\mathcal{N}^* = \bigcap_{r \in \mathbb{R}} \mathfrak{g}_r^*.$$

Given $x \in \mathcal{B}(G)$ and $X \in \mathfrak{g}^* \setminus \{0\}$, the *depth of X at x* is the unique value $t = d_x(X)$

such that $X \in \mathfrak{g}^*_{x,t} \smallsetminus \mathfrak{g}^*_{x,t+}$. When X is not nilpotent, they prove that the *depth of* X, given by

$$d(X) = \max\{d_x(X) \mid x \in \mathcal{B}(G)\} = \max\{r \mid X \in \mathfrak{g}^*_r\}$$

is well defined and rational. For X nilpotent, we set $d(X) = \infty$. Depth is G-invariant.

For semisimple $\Gamma \in \mathfrak{g}^*$, let $T \subset \mathrm{Cent}_G(\Gamma)$ be a maximal torus with associated absolute root system $\Phi(G, T)$. Then Γ is called *good* if for all $\alpha \in \Phi(G, T)$, we have $\mathrm{val}\big(\Gamma(d\alpha^\vee(1))\big) \in \{d(\Gamma), \infty\}$. By [Kim and Murnaghan 2003, Theorem 2.3.1], if Γ is good then the set of points $x \in \mathcal{B}(G)$ at which $d_x(\Gamma)$ attains its maximum value $d(\Gamma)$ is exactly $\mathcal{B}(\mathrm{Cent}_G(\Gamma)) \subset \mathcal{B}(G)$.

For any $\Gamma \in \mathfrak{g}^*$ set $d = d_x(\Gamma)$. The coset $\Gamma + \mathfrak{g}^*_{x,d+}$ is called *degenerate* if it contains a nilpotent element $X \in \mathcal{N}^*$. From the relations above it follows that this happens if and only if $d < d(\Gamma)$. DeBacker [2002b, §5] proved that the set of nilpotent G-orbits meeting a degenerate coset $\Gamma + \mathfrak{g}^*_{x,d+}$ has a unique minimal element with respect to the (rational) closure relation on orbits, which we'll denote by $\mathcal{O}(\Gamma, x)$. This generalizes a result of Barbasch and Moy [1997, Proposition 3.1.6] for $d = 0$, which was integral to their determination of the wave front set of a depth-zero representation.

To classify nilpotent orbits in this way, DeBacker proceeds as follows. Identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Given a nilpotent element $X \in \mathfrak{g}$, complete X to an $\mathfrak{sl}_2(F)$ triple (X, H, Y). Choose $r \in \mathbb{R}$ and create the building set

$$\mathcal{B}_r(X, H, Y) = \{x \in \mathcal{B}(G) \mid X \in \mathfrak{g}_{x,r},\ H \in \mathfrak{g}_{x,0},\ Y \in \mathfrak{g}_{x,-r}\}.$$

He proves this set is a nonempty, closed, convex subset of $\mathcal{B}(G)$ with the property that for all $x \in \mathcal{B}_r(X, H, Y)$ we have $\mathcal{O}(X, x) = G \cdot X$.

Remark 3.1. For each $g \in G$, we have $\mathcal{B}_r({}^gX, {}^gH, {}^gY) = {}^g\mathcal{B}_r(X, H, Y)$, and for fixed X the union of these need not cover $\mathcal{B}(G)$. Moreover, if μ is a one-parameter subgroup adapted to this triple, then by [DeBacker 2002b, Remark 5.1.5], $\mathcal{B}_r(X, H, Y) = \mathcal{B}_0(X, H, Y) + \frac{1}{2}r\mu$, where this sum is taken in any apartment in $\mathcal{B}(C_G(\mu))$. It follows that (if the rank of G is greater than 1) there exist orbits $\mathcal{O}$ (such as ones for which $\mathcal{B}_r(X, H, Y)$ is a point) for which there exist $y \in \mathcal{B}(G)$ such that $\mathcal{O} \neq \mathcal{O}(X, y)$ for any $X \in \mathcal{O}$. For example, in $\mathrm{Sp}(4, F)$, the principal nilpotent orbits are only obtained along certain lines emanating from vertices.

3B. ***Nilpotent support and nilpotent cones.*** We now explore different ways to understand the asymptotic nilpotent support of a general element $\Gamma \in \mathfrak{g}^*$ and show their equivalence.

Definition 3.2. Let $\Gamma \in \mathfrak{g}^*$. If $x \in \mathcal{B}(G)$, then the *local nilpotent support at* x of Γ is

$$\mathrm{Nil}_x(\Gamma) = \{\mathcal{O}({}^g\Gamma, x) \mid g \in G,\ d_x(g \cdot \Gamma) < d(\Gamma)\},$$

which is the set of nilpotent orbits defined by degenerate cosets at x of elements of the G-orbit of Γ. On the other hand, the *nilpotent support* of Γ is

$$\mathrm{Nil}(\Gamma) = \{\mathcal{O}(\Gamma, x) \mid x \in \mathcal{B}(G), d_x(\Gamma) < d(\Gamma)\},$$

the set of nilpotent orbits corresponding to any (nontrivial) degenerate coset of Γ.

Note that if Γ is nilpotent, then $\mathrm{Nil}(\Gamma) \ni G \cdot \Gamma$. More generally, for any $g \in G$, $d_x(\Gamma) = d_{gx}({}^g\Gamma)$ and ${}^g(\Gamma + \mathfrak{g}^*_{x,d+}) = {}^g\Gamma + \mathfrak{g}^*_{gx,d+}$. Thus $\mathcal{O}(\Gamma, x) = \mathcal{O}({}^g\Gamma, gx)$, and

$$\mathrm{Nil}(\Gamma) = \bigcup_{x \in \mathcal{B}(G)} \mathrm{Nil}_x(\Gamma),$$

that is, the nilpotent support is the union of the local nilpotent supports, and $\mathrm{Nil}(\Gamma)$ is an invariant of the G-orbit of Γ. One may alternately restrict this union to one over the points in a fundamental domain for the action of G on $\mathcal{B}(G)$.

By Remark 3.1, when the rank of G is greater than 1, not all nilpotent orbits will occur as some $\mathcal{O}(\Gamma, x)$ for a given point $x \in \mathcal{B}(G)$, so $\mathrm{Nil}_x(\Gamma) \neq \mathrm{Nil}(\Gamma)$ in general. Even when these sets are equal, as for $\mathrm{SL}(2, F)$ (see Proposition 4.1), they are interesting subsets of the nilpotent cone (see Lemma 4.2).

On the other hand, the asymptotic cone on an element Γ is defined in [Adams and Vogan 2021, Definition 3.9] analytically as follows.

Definition 3.3. Let $\Gamma \in \mathfrak{g}^*$. The *asymptotic cone* on Γ is the set

$$\mathrm{Cone}(\Gamma) = \bigl\{X \in \mathfrak{g}^* \mid \exists \varepsilon_i \to 0, \varepsilon_i \in F^\times, \exists g_i \in G, \lim_{i\to\infty} \varepsilon_i^2 \,\mathrm{Ad}^*(g_i)\Gamma = X\bigr\}.$$

This is a closed, nonempty union of nilpotent orbits of G on $\mathfrak{g}^*$.

Proposition 3.4. *Let $\Gamma \in \mathfrak{g}^*$. Then the nonzero G-orbits occurring in the asymptotic cone of Γ are those in its nilpotent support, that is,*

$$\mathrm{Cone}(\Gamma) = \bigcup_{\mathcal{O} \in \mathrm{Nil}(\Gamma)} \mathcal{O} \cup \{0\}.$$

Proof. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and prove this result for $\Gamma \in \mathfrak{g}$, where we may apply the theory of $\mathfrak{sl}_2(F)$ triples.

Let $\Gamma \in \mathfrak{g}$ have depth $r \leq \infty$ and let $\mathcal{O} \in \mathrm{Nil}(\Gamma)$. Then there exists $x \in \mathcal{B}(G)$ and $d < r$ such that $d_x(\Gamma) = d$ and $\mathcal{O} = \mathcal{O}(\Gamma, x)$. Choose a representative

$$X \in \mathcal{O}(\Gamma, x) \cap (\Gamma + \mathfrak{g}_{x,d+}).$$

Choose an $\mathfrak{sl}_2(F)$ triple (X, H, Y) and the corresponding one-parameter subgroup μ adapted to X. By [DeBacker 2002b, Lemma 5.2.1], we have

$$X + \mathfrak{g}_{x,d+} = \mathrm{Ad}(G_{x,0+})(X + C_{\mathfrak{g}_{x,d+}}(Y)).$$

Therefore there exist $g \in G_{x,0+}$ and $C \in C_{\mathfrak{g}_{x,d+}}(Y)$ for which

$$\Gamma = \mathrm{Ad}(g^{-1})(X + C).$$

Note that $C_{\mathfrak{g}}(Y)$ is spanned by the lowest weight vectors of $\mathrm{ad}(H)$, so we may decompose C as $C = \sum_{i\le 0} C_i$, where $\mathrm{Ad}(\mu(t))C_i = t^i C_i$ for all $t \in F^\times$. Similarly, for all $t \in F^\times$ we have $\mathrm{Ad}(\mu(t))X = t^2 X$. Therefore

$$\lim_{t\to 0} t^2 \,\mathrm{Ad}(\mu(t^{-1})g)\Gamma = \lim_{t\to 0} t^2 \,\mathrm{Ad}(\mu(t^{-1}))(X + C) = X,$$

so $X \in \mathrm{Cone}(\Gamma)$. Since $\mathrm{Cone}(\Gamma)$ is G-invariant, we deduce $\mathcal{O} \subset \mathrm{Cone}(\Gamma)$.

Conversely, let $X \in \mathrm{Cone}(\Gamma)$ be nonzero, so that there exists a sequence of elements $\varepsilon_i \in F^\times$, with $\varepsilon_i \to 0$, and a sequence of elements $g_i \in G$, such that

$$\lim_{i\to\infty} \varepsilon_i^2 \,\mathrm{Ad}(g_i)\Gamma = X.$$

Complete X to an $\mathfrak{sl}_2(F)$ triple (X, H, Y) and choose a point $x \in \mathcal{B}_0(X, H, Y)$. Since the given sequence converges to X, it enters the neighbourhood $X + \mathfrak{g}_{x,0+}$ so we may choose $i \in \mathbb{N}$ such that

$$\varepsilon_i^2 \,\mathrm{Ad}(g_i)\Gamma \in X + \mathfrak{g}_{x,0+}.$$

So $\mathrm{Ad}(g_i)\Gamma \in \varepsilon_i^{-2} X + \mathfrak{g}_{x,-2\,\mathrm{val}(\varepsilon_i)+}$, a nontrivial degenerate coset of depth $-2\,\mathrm{val}(\varepsilon_i)$. Because $(\varepsilon_i^{-2}X, H, \varepsilon_i^2 Y)$ is again an $\mathfrak{sl}_2(F)$ triple and $\mathcal{B}_{-2\,\mathrm{val}(\varepsilon_i)}(\varepsilon_i^{-2}X, H, \varepsilon_i^2 Y) = \mathcal{B}_0(X, H, Y)$, we can infer that the minimal nilpotent orbit meeting this coset is $\mathrm{Ad}(G)(\varepsilon_i^{-2}X) = \mathrm{Ad}(G)X$. Thus $\mathrm{Ad}(G)X = \mathcal{O}({}^{g_i}\Gamma, x) \in \mathrm{Nil}_x(\Gamma) \subset \mathrm{Nil}(\Gamma)$, as required. □

3C. ***Connection with the wave front set of a positive-depth representation.*** Suppose now that π is an irreducible admissible representation of G of depth r with good minimal K-type Γ of depth $-r$ (in the sense of [Kim and Murnaghan 2003, Definitions 2.4.3 and 2.4.6]). Then, under suitable hypotheses (that are satisfied if F has characteristic zero and the exponential map converges on $\mathfrak{g}_{0+}$), Kim and Murnaghan prove a version of the local character expansion that is valid on the strictly larger neighbourhood $\mathfrak{g}_r^{\mathrm{reg}}$. The Γ-asymptotic expansion [Kim and Murnaghan 2003, Theorem 5.3] asserts that there exist complex coefficients $c_{\mathcal{O}'}(\pi)$ such that for any $X \in \mathfrak{g}_r^{\mathrm{reg}}$ we have

$$\Theta_\pi(\exp(X)) = \sum_{\mathcal{O}' \in \mathscr{O}(\Gamma)} c_{\mathcal{O}'}(\pi)\widehat{\mu_{\mathcal{O}'}}(X), \tag{3-1}$$

where $\mathscr{O}(\Gamma)$ denotes the set of G-orbits in $\mathfrak{g}^*$ with Γ in their closure, and for $\mathcal{O}' \in \mathscr{O}(\Gamma)$, $\widehat{\mu_{\mathcal{O}'}}$ denotes the Fourier transform of the corresponding orbital integral (2-2).

This yields a special case of interest: that of the expansion (3-1) having a single nonzero term $c_{\mathcal{O}'}(\pi)\widehat{\mu_{\mathcal{O}'}}$ corresponding to $\mathcal{O}' = G \cdot \Gamma$. We claim this happens, for example, when $G' = \mathrm{Cent}_G(\Gamma)$ is compact-mod-centre, such as when Γ is a regular semisimple element. Namely, let $\mathfrak{g}'$ denote the Lie algebra of G'. Then the set $\mathscr{O}(\Gamma)$ indexing the sum in (3-1) is in bijective correspondence with the set of nilpotent G'-orbits in $(\mathfrak{g}')^*$, which is the singleton $\{G \cdot \Gamma\}$ under this hypothesis.

Theorem 3.5. *Let π be an irreducible admissible representation of G of depth $r > 0$, and let $\Gamma \in \mathfrak{g}^*$ be a good minimal K-type of π such that π admits a Γ-asymptotic expansion. Suppose further that this expansion has a unique nonzero term, corresponding to the Fourier transform of the orbital integral corresponding to Γ itself. Then $\mathcal{WF}(\pi)$ coincides with the maximal elements of* $\mathrm{Nil}(\Gamma)$; *that is, the asymptotic cone on Γ is the wave front set of π.*

The following proof was communicated to me by Fiona Murnaghan.

Proof. Combining the Γ-asymptotic and local character expansions yields that for some scalar t we have the equality $t\widehat{\mu_{G\cdot\Gamma}} = \sum_{\mathcal{O}\in\mathscr{O}(0)} c_{\mathcal{O}}(\pi)\widehat{\mu_{\mathcal{O}}}$ of functions on $\mathfrak{g}^{\mathrm{reg}} \cap \mathfrak{g}_{r+}$, which can be viewed equivalently as an equality of distributions upon restriction to the set $C_c^\infty(\mathfrak{g}_{r+})$. Since $f \in C_c^\infty(\mathfrak{g}_{r+})$ implies $\widehat{f} \in \mathcal{D}_{-r} := \sum_{x\in\mathcal{B}(G)} C_c(\mathfrak{g}^*/\mathfrak{g}^*_{x,r})$, taking the inverse Fourier transform yields the equality of distributions

$$t\mu_{G\cdot\Gamma} = \sum_{\mathcal{O}\in\mathscr{O}(0)} c_{\mathcal{O}}(\pi)\mu_{\mathcal{O}} \tag{3-2}$$

on $\mathcal{D}_{-r}$. (See [Kim and Murnaghan 2003, Proof of Theorem 5.3.1] or [Debacker 2002a, Proof of Theorem 3.5.2]; note that our r is ρ in the former and $\rho(\pi)$ in the latter.)

So let $x \in \mathcal{B}(G)$ and let d be such that $\mathfrak{g}^*_{x,d+} \supset \mathfrak{g}^*_{x,-r}$. Given a nonzero coset $\xi \in \mathfrak{g}^*_{x,d}/\mathfrak{g}^*_{x,d+}$ let $\mathbf{1}_\xi$ denote the characteristic function of this subset of $\mathfrak{g}^*$; then $\mathbf{1}_\xi \in \mathcal{D}_{-r}$. Note that if $X \in \xi \cap \mathcal{O}$ for some (not necessarily nilpotent) G-orbit $\mathcal{O}$, then this intersection contains the open set $G_{x,0+} \cdot X$ as well. Thus we have

$$\mu_{\mathcal{O}}(\mathbf{1}_\xi) = 0 \iff \xi \cap \mathcal{O} = \varnothing. \tag{3-3}$$

Now suppose that $\mathcal{O} \in \mathscr{O}(0)$, and choose $x \in \mathcal{B}(G)$ and $\xi = X + \mathfrak{g}^*_{x,d+}$ with $\mathfrak{g}^*_{x,d+} \supset \mathfrak{g}^*_{x,-r}$ with the property that $\mathcal{O} = \mathcal{O}(X,x)$. The minimality of $\mathcal{O}(X,x)$ proven by DeBacker implies that any nilpotent orbit $\mathcal{O}'$ meeting ξ (or equivalently, by (3-3), satisfying $\mu_{\mathcal{O}'}(\mathbf{1}_\xi) \neq 0$) must contain $\mathcal{O}$ in its closure.

Suppose first that $\mathcal{O}$ is not in the wave front set $\bigcup_{\mathcal{O}'\in\mathcal{WF}(\pi)} \overline{\mathcal{O}'}$ of π. Let $\mathcal{O}' \in \mathscr{O}(0)$ be such that $c_{\mathcal{O}'}(\pi) \neq 0$; then $\mathcal{O}'$ is in the wave front set, so $\mathcal{O} \not\subset \overline{\mathcal{O}'}$. This implies by the preceding paragraph that $\mu_{\mathcal{O}'}(\mathbf{1}_\xi) = 0$. As this holds for all such $\mathcal{O}'$, we conclude from (3-2) that $\mu_{G\cdot\Gamma}(\mathbf{1}_\xi) = 0$, whence by (3-3) we have $\xi \cap G \cdot \Gamma = \varnothing$, and thus $\mathcal{O} \notin \mathrm{Nil}(\Gamma)$. Therefore every $\mathcal{O} \in \mathrm{Nil}(\Gamma)$ lies in the wave front set of π.

Now suppose $\mathcal{O} \in \mathcal{WF}(\pi)$; that is, it is maximal among nilpotent orbits with nonzero coefficient in (3-2). Thus the preceding argument implies $\mu_{\mathcal{O}'}(\mathbf{1}_\xi) = 0$ for all $\mathcal{O}' \neq \mathcal{O}$ in the wave front set. So (3-2) yields $t\mu_{G\cdot\Gamma}(\mathbf{1}_\xi) = c_{\mathcal{O}}(\pi)\mu_{\mathcal{O}}(\mathbf{1}_\xi) \neq 0$, and therefore, by (3-3), ξ must meet $G \cdot \Gamma$ and thus $\mathcal{O} \in \mathrm{Nil}(\Gamma)$. Hence, the maximal elements of $\mathrm{Nil}(\Gamma)$ coincide with $\mathcal{WF}(\pi)$. □

In fact, the key to the proof is that the maximal nilpotent orbits occurring in the Shalika germ expansion of $\mu_{G\cdot\Gamma}$ are the maximal orbits of $\mathrm{Nil}(\Gamma)$.

Ciubotaru and Okada [2023] obtained a similar result directly, by analysing the asymptotic nilpotent cone of the characters of $G_{x,r}/G_{x,r+}$ appearing in $\pi^{G_{x,r+}}$.

Remark 3.6. One might ask if Theorem 3.5 could be extended to show that $\mathcal{WF}(\pi)$ is the union of the nilpotent supports of the maximal orbits occurring in the Γ-asymptotic expansion (3-1). The answer is expected to be negative. In the supercuspidal case, the key result is [Spice 2022, Corollary 10.2.3(1)], which implies that this latter set of orbits (in $\mathscr{O}(\Gamma)$) corresponds exactly to $\mathcal{WF}(\pi^0)$ (in $\mathscr{O}(0)$ for $G^0 = \mathrm{Cent}_G(\Gamma)$), where π^0 is the associated depth-zero supercuspidal representation of G^0. Tsai [2023b] has constructed explicit examples of supercuspidal representations where the wave front set does not follow such a pleasant inductive structure. In effect, one expects that when substituting Shalika germ expansions into the Γ-asymptotic expansion, cancellations among coefficients may occur.

While the proof of Theorem 3.5 entails some additional hypotheses on F, a consequence of the main theorem of Section 6 is that, for $G = \mathrm{SL}(2, F)$, the conclusion of the theorem holds whenever the characteristic and residual characteristic of F are not 2.

4. Nilpotent orbits and nilpotent cones of $G = \mathrm{SL}(2, F)$

For the rest of this paper we suppose that $\boldsymbol{G} = \mathrm{SL}(2)$ and $\mathfrak{g} = \mathfrak{sl}(2, F)$. In this section, we derive some additional properties of the nilpotent support of an element $\Gamma \in \mathfrak{g}^*$. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ with the trace form.

There are five nilpotent orbits: the zero orbit, and four two-dimensional principal (or regular) orbits that are in bijection with the rational square classes $F^\times/(F^\times)^2$. Representatives of these five orbits in $\mathfrak{g}$ are

$$\dot{X}_u = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}, \tag{4-1}$$

where u runs over the set $\{0, 1, \varepsilon, \varpi, \varepsilon\varpi\}$ modulo $(F^\times)^2$ and $\varepsilon \in \mathcal{R}^\times$ is a fixed nonsquare. For each u, write $\mathcal{O}_u$ for the orbit in $\mathfrak{g}^*$ corresponding to $\dot{X}_u$. The following proposition relaxes the conditions for identifying the orbits in the nilpotent support of an element Γ.

Proposition 4.1. *Let* $\mathfrak{g} = \mathfrak{sl}_2(F)$ *and* $\Gamma \in \mathfrak{g}^* \setminus \{0\}$. *Set* $r = d(\Gamma) \in \mathbb{R} \cup \{\infty\}$. *Then*

(a) *every* $\mathcal{O} \in \mathrm{Nil}(\Gamma)$ *meets* $\Gamma + \mathfrak{g}^*_{x,r}$ *for some* $x \in \mathcal{B}(G)$ *such that* $d_x(\Gamma) < r$;

(b) *for each* $x \in \mathcal{B}(G)$ *such that* $d_x(\Gamma) < r$, *if* $\Gamma + \mathfrak{g}^*_{x,r}$ *meets a nilpotent orbit* $\mathcal{O}$, *then* $\mathcal{O} \in \mathrm{Nil}(\Gamma)$;

(c) *for each* $x \in \mathcal{B}(G)$, $\mathrm{Nil}(\Gamma) = \{\mathcal{O}({}^g\Gamma, x) \mid g \in G\} = \mathrm{Nil}_x(\Gamma)$, *that is*, *every nonzero nilpotent orbit in* $\mathrm{Cone}(\Gamma)$ *appears in the local nilpotent support at every* x.

Proof. The first two statements use that there are no closure relations between the principal orbits of $\mathfrak{sl}_2(F)$, and so the uniqueness of the minimal nilpotent orbit meeting any degenerate coset implies that any nontrivial degenerate coset meets only one nilpotent orbit.

For (a), suppose $\mathcal{O} \in \mathrm{Nil}(\Gamma)$; then $\mathcal{O} = \mathcal{O}(\Gamma, x)$ for some $x \in \mathcal{B}(G)$, implying that $d_x(\Gamma) < r$. Since $\Gamma \in \mathfrak{g}^*_r \subset \mathfrak{g}^*_{x,r} + \mathcal{N}^*$, the set $\Gamma + \mathfrak{g}^*_{x,r}$ contains a (nonzero) nilpotent element Y. Since Y is an element of $\Gamma + \mathfrak{g}^*_{x,r} \subset \Gamma + \mathfrak{g}^*_{x,d_x(\Gamma)}$, it lies in $\mathcal{O}$, so $\mathcal{O}$ meets the smaller coset, as required.

For (b), note that if $d_x(\Gamma) < r$ then $0 \notin \Gamma + \mathfrak{g}^*_{x,r} \subset \Gamma + \mathfrak{g}^*_{x,d(\Gamma)+}$; any nilpotent orbit meeting the smaller set meets the larger one, and thus by uniqueness this orbit is $\mathcal{O}(\Gamma, x) \in \mathrm{Nil}(\Gamma)$.

To prove (c), let $x \in \mathcal{B}(G)$ and let $\mathrm{Nil}_x(\Gamma)$ be the local nilpotent support of Γ at x; we have already noted that $\mathrm{Nil}_x(\Gamma) \subset \mathrm{Nil}(\Gamma)$. The reverse inclusion follows from the one-dimensionality of $\mathcal{B}(G)$. Let $\mathcal{O} \in \mathrm{Nil}(\Gamma)$; then $\mathcal{O} = \mathcal{O}(\Gamma, y)$ for some $y \in \mathcal{B}(G)$. Let S be a split torus with associated root system $\Phi(G, S) = \{\pm\alpha\}$ such that $y \in \mathcal{A}(G, S)$.

Set $d = d_y(\Gamma)$ and let $\dot{\Gamma} \in \mathfrak{g}$ correspond to Γ via the trace form. Choose $\dot{X} \in \mathcal{O}$ such that $\dot{\Gamma} \in \dot{X} + \mathfrak{g}_{y,d+}$. Conjugating both $\dot{\Gamma}$ and $\dot{X}$ by G_y as necessary we may assume $\dot{X} \in \mathfrak{g}_\alpha$. Relative to the pinning of a fixed base point, we have the decomposition of $\mathcal{R}$-modules

$$\mathfrak{g}_{y,d} = \mathfrak{g}_{-\alpha,d+\alpha(y)} \oplus \mathfrak{s}_d \oplus \mathfrak{g}_{\alpha,d-\alpha(y)}.$$

Let $\alpha^\vee$ denote the positive coroot, and choose $g \in G$ so that $gx \in \mathcal{A}(G, S)$ and $gx = y - \ell\alpha^\vee$ for some $\ell \geq 0$. Therefore if $d' = d - 2\ell$ then $\mathfrak{g}_{y,d} \subset \mathfrak{g}_{gx,d'}$. Because $\dot{X} \in \mathfrak{g}_{\alpha,d-\alpha(y)} \smallsetminus \mathfrak{g}_{\alpha,(d-\alpha(y))+}$ and $d' - \alpha(gx) = d - \alpha(y)$, we conclude that $d_{gx}(\dot{\Gamma}) = d_{gx}(\dot{X}) = d'$ and $\dot{\Gamma} - \dot{X} \in \mathfrak{g}_{gx,d'+}$. By uniqueness, we infer that $\mathcal{O} = \mathcal{O}(\dot{\Gamma}, gx) = \mathcal{O}({}^{g^{-1}}\dot{\Gamma}, x) \in \mathrm{Nil}_x(\dot{\Gamma})$, yielding the result. □

We next determine $\mathrm{Nil}(\Gamma)$ explicitly, for any $\Gamma \in \mathfrak{g} = \mathfrak{sl}_2(F)$ (identified with its dual via the trace form). There is nothing to do if Γ is nilpotent. If $\Gamma \neq 0$ is semisimple, then it is G-conjugate to a matrix of the form

$$\dot{X}(u, v) = \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix}, \tag{4-2}$$

for some $u, v \in F^\times$. Its centralizer is a maximal torus. There is one G-conjugacy class of split torus, represented by any diagonal element, and two classes of unramified anisotropic tori, represented by $\dot{X}(1, \varepsilon) \in \mathfrak{g}$ and $\dot{X}(\varpi^{-1}, \varepsilon\varpi) \in \mathfrak{g}$, respectively. The classes of ramified tori are represented by $\dot{X}(1, t) \in \mathfrak{g}$ with $t \in \{\varpi, \varepsilon\varpi, \varepsilon^2\varpi, \varepsilon^3\varpi\}$, noting that if $-\varepsilon \in (F^\times)^2$ then there are only two classes.

We can now describe the nilpotent support of each such element, using the parametrization given in (4-1).

Lemma 4.2. *Let* $G = \mathrm{SL}(2, F)$ *and* $\Gamma \in \mathfrak{g} \smallsetminus \{0\}$ *semisimple. If* Γ *splits over* F,

$$\mathrm{Nil}(\Gamma) = \{\mathcal{O}_1, \mathcal{O}_\varepsilon, \mathcal{O}_\varpi, \mathcal{O}_{\varepsilon\varpi}\}.$$

Otherwise, Γ *is conjugate to* $\dot{X}(u, v)$ *for some* $u, v \in F^\times$ *and splits over* $E = F[\sqrt{uv}]$. *Let* $\mathrm{Norm}_{E/F}(E^\times)/(F^\times)^2$ *be represented by* $\{1, \gamma\}$. *Then* u *and* v *are uniquely defined mod* $\mathrm{Norm}_{E/F}(E^\times)$ *and*

$$\mathrm{Nil}(\Gamma) = \{\mathcal{O}_u, \mathcal{O}_{u\gamma}\}.$$

Proof. By Proposition 4.1, we may fix the choice $x = x_0 \in \mathcal{B}(G)$ to be the vertex such that $\mathfrak{g}_{x,r}$ is the set of traceless 2×2 matrices with entries in $\mathcal{P}^{\lceil r \rceil}$, and replace Γ by any G-conjugate.

First suppose $\Gamma = \mathrm{diag}(a, -a)$ with $\mathrm{val}(a) = r$. Let $u \in F^\times$ and note that if $g_u = \left[\begin{smallmatrix} 1 & -a^{-1}u/2 \\ 0 & 1 \end{smallmatrix}\right] \in G$ then ${}^{g_u}\Gamma = \left[\begin{smallmatrix} a & u \\ 0 & -a \end{smallmatrix}\right]$. Therefore, for any u such that $\mathrm{val}(u) = d < r$, we have ${}^{g_u}\Gamma \in \dot{X}_u + \mathfrak{g}_{x,d+}$. Thus $\mathrm{Nil}(\Gamma)$ contains every nonzero nilpotent orbit.

Now suppose $\Gamma = \dot{X}(u, v)$ for some $u, v \in F^\times$ such that $uv \notin (F^\times)^2$ and set $E = F[\sqrt{uv}]$. We calculate directly that the upper triangular entry of any G-conjugate of Γ takes the form

$$u' = a^2u - b^2v = u(a^2 - b^2vu^{-1}) \in u\,\mathrm{Norm}_{E/F}(E^\times)$$

for some $a, b \in F$, not both zero, from which it follows that $\mathrm{Nil}(\Gamma) \subset \{\mathcal{O}_u, \mathcal{O}_{u\gamma}\}$.

For the reverse inclusion, first note $\dot{X}(u, v)$ is G-conjugate to $\dot{X}(u\varpi^{-2n}, v\varpi^{2n})$ for all $n \in \mathbb{Z}$ and for n sufficiently large $\dot{X}(u\varpi^{-2n}, v\varpi^{2n}) - \dot{X}_{u\varpi^{-2n}} \in \mathfrak{g}_{x,r}$. Thus $\mathcal{O}_u \in \mathrm{Nil}(\Gamma)$.

Now note that when E is ramified, we may take $\gamma = -uv$ so $\mathcal{O}_{u\gamma} = \mathcal{O}_{-v}$; since $\dot{X}(u, v)$ is G-conjugate to $\dot{X}(-v, -u)$ we are done by the preceding. If E is unramified, we have instead $\gamma = uv$, whence $\mathcal{O}_{u\gamma} = \mathcal{O}_v$. As -1 is a norm, we may choose $\alpha, \beta \in F$ such that $-1 = \beta^2 - \alpha^2uv^{-1}$; then $g = \left[\begin{smallmatrix} \alpha & \beta \\ \beta & \alpha uv^{-1} \end{smallmatrix}\right] \in G$ satisfies ${}^{g}\dot{X}(u, v) = \dot{X}(v, u)$, and again by the preceding we may conclude $\mathcal{O}_v \in \mathrm{Nil}(\Gamma)$. $\square$

5. Representations of G_x associated to nilpotent orbits

5A. ***Shalika's representations of*** **SL(2, $\mathcal{R}$).** In his thesis, Shalika constructed all irreducible smooth representations of $K = \mathrm{SL}(2, \mathcal{R})$. In this section we recap his explicit construction for the so-called ramified case, which attaches an irreducible representation of K to certain K-orbits in $\mathfrak{g}^*$; we'll then provide a coordinate-free generalization more suited to our needs in the next section.

Let S be the diagonal split torus, B the upper triangular Borel subgroup and U its unipotent radical. We use a subscript 0 to indicate their intersections with K: $S_0 = S \cap K$, $B_0 = B \cap K$ and $U_0 = U \cap K$. Let $x_0 \in \mathcal{A}(G, S)$ be such that $K = G_{x_0}$ and z_0 the barycentre of the positive alcove adjacent to x_0 (relative to B).

Let d be a positive integer. Choose $u \in \mathcal{P}^{-d} \setminus \mathcal{P}^{-d+1}$ and $v \in \mathcal{P}^{-d+1}$ and consider the antidiagonal matrix $\dot{X} := \dot{X}(u, v) \in \mathfrak{g}_{x_0,-d}$ of (4-2). Identify this with the element $X \in \mathfrak{g}^*_{x_0,-d}$ by the rule $X(Z) = \operatorname{tr}(\dot{X}Z)$ for all $Z \in \mathfrak{g}$. If $v = 0$ then X is nilpotent and its centralizer $C_K(X)$ in K coincides with ZU_0, where $Z = \{\pm I\}$. Otherwise, X is semisimple and $C_K(X)$ is a torus. Note that every $X \in \mathfrak{g}^*_{x_0,-d}$ that represents a degenerate coset is K-conjugate to one of this form.

Define an open subgroup of K by

$$J_d = \begin{bmatrix} 1+\mathcal{P}^{\lceil d/2\rceil} & \mathcal{P}^{\lceil d/2\rceil} \\ \mathcal{P}^{\lceil (d+1)/2\rceil} & 1+\mathcal{P}^{\lceil d/2\rceil} \end{bmatrix} \cap K. \tag{5-1}$$

It is straightforward to verify that X gives a well-defined character η_X of J_d, trivial on $G_{x_0,d+}$, by the rule

$$\eta_X(g) = \psi\big(\operatorname{tr}(\dot{X}(g-I))\big). \tag{5-2}$$

This character depends only on the classes $u + \mathcal{P}^{\lceil (-d+1)/2\rceil}$ and $v + \mathcal{P}^{\lceil -d/2\rceil}$. For any choice of character θ of $C_K(X)$ agreeing with $\eta_X(g)$ on $C_K(X) \cap J_d$, write $\eta(X, \theta)$ for the resulting extension to a character of $C_K(X)J_d$.

Shalika [2004, Theorems 4.2.1 and 4.2.5, §4.3] proves the following result with an intricate elementary argument. To briefly translate from the notation of that work: K_n denotes $\mathrm{SL}(\mathcal{R}/\mathcal{P}^n) \cong G_{x_0}/G_{x_0,n}$ and its primitive representations inflate to the representations of G_{x_0} of depth $d = n-1$. Shalika's group $T_{\overline{X},n}$ inflates to $C_K(X)G_{x_0,n}$ in our notation and our J_d is the inflation of Shalika's N_k when n is even and B_n when n is odd. Though Shalika's X has depth 0, his additive characters η, ξ are normalized such that the characters denoted by $\eta_{\mathbf{x}}$ and $\eta_{X,\xi}$ match K-equivariantly with our η_X, for those "ramified" orbits of [Shalika 2004, Lemma 4.2.2(ii)] considered here. Nilpotent X fall under this ramified case, by choosing $\nu > n$ in [Shalika 2004, §4.3].

Proposition 5.1 (Shalika). *Set $K = G_{x_0}$ and $K_n = G_{x_0,n}$ for $n > 0$. For any $d > 0$, let $\dot{X} = \dot{X}(u, v)$ as above, corresponding to $X \in \mathfrak{g}^*_{x_0,-d}$. Then for any character θ of its centralizer $C_K(X)$ agreeing with η_X on $C_K(X) \cap J_d$, the representation*

$$\mathcal{S}_{x_0}(X, \theta) = \operatorname{Ind}_{C_K(X)J_d}^{K} \eta(X, \theta)$$

is irreducible, of degree $\frac{1}{2}q^{d-1}(q^2-1)$ and of depth d, meaning it is nontrivial on K_d but trivial on K_{d+}. Its equivalence class is independent of the choice of *representative of the* K*-orbit of* $\dot{X}(u+\mathcal{P}^{\lfloor (d+1)/2\rfloor}, v+\mathcal{P}^{\lceil (d+1)/2\rceil})$*, and if* $\mathcal{S}_{x_0}(X,\theta)\cong \mathcal{S}_{x_0}(X',\theta')$ *then there is some* $g\in K$ *such that* $X'={}^gX$ *and* $\theta'={}^g\theta$.

5B. ***Irreducible representations of*** G_x ***parametrized by degenerate cosets at*** x**.** Our goal in this section is to give a coordinate-free interpretation of Shalika's construction that allows us to unambiguously attach representations of G_x to any degenerate coset of negative depth.

Note that $\mathrm{GL}(2,F)$ acts on $\mathcal{B}(G)$, and all vertices are conjugate under this action. This conjugacy does not in general preserve the $\mathrm{SL}(2,F)$-orbit of Γ or X.

Example. Let x_0, z_0 be as in Section 5A and x_1 the other vertex of the chamber containing z_0 in its closure. The element $\omega=\left[\begin{smallmatrix}0&1\\ \varpi&0\end{smallmatrix}\right]$ used in [Nevins 2005] is an affine reflection such that $\omega\cdot x_0=x_1$, and ${}^{\omega}\dot{X}(u,v)=\dot{X}(\varpi^{-1}v,\varpi u)$. Therefore, in particular, in the case of nilpotent orbits, where $\dot{X}(0,1)\sim\dot{X}(-1,0)$, we have ${}^{\omega}\mathcal{O}_1=\mathcal{O}_{-\varpi}$. On the other hand, the element $\eta=\left[\begin{smallmatrix}1&0\\ 0&\varpi\end{smallmatrix}\right]$ used in [Nevins 2013] is a translation such that $\eta\cdot x_0=x_1$, but now ${}^{\eta}\dot{X}(u,v)=\dot{X}(\varpi^{-1}u,\varpi v)$ and thus ${}^{\eta}\mathcal{O}_1=\mathcal{O}_{\varpi}$ instead.

We begin by showing that any degenerate coset determines a chamber of $\mathcal{B}(G)$ adjacent to x.

Lemma 5.2. *Let* $G=\mathrm{SL}(2,F)$*. Let* $x\in\mathcal{B}(G)$ *be any vertex and let* $\Gamma\in\mathfrak{g}^*_{x,-d}\setminus\mathfrak{g}^*_{x,-d+}$ *represent a degenerate coset for some* $d\in\mathbb{Z}_{>0}$*. Then there exists a unique chamber* $\mathcal{C}=\mathcal{C}_\Gamma$ *of* $\mathcal{B}(G)$ *adjacent to* x*, independent of the choice of representative of* $\Gamma+\mathfrak{g}^*_{x,-d+}$*, such that for any* $z\in\mathcal{C}$ *we have* $\Gamma\in\mathfrak{g}^*_{x,-d}\cap\mathfrak{g}^*_{z,-d+}$*. Moreover, we have* $\mathrm{Cent}_{G_x}(\Gamma)=\mathrm{Cent}_{G_z}(\Gamma)$.

Proof. Uniqueness is immediate: given z' in any other chamber adjacent to x, the geodesic from z to z' contains x; hence $\mathfrak{g}^*_{z,-d+}\cap\mathfrak{g}^*_{z',-d+}$ is a subset of $\mathfrak{g}^*_{x,-d+}$ and therefore does not contain Γ. Identify Γ with an element $\dot{\Gamma}\in\mathfrak{g}_{x,-d}$ via the trace form. Choose a nilpotent element $\dot{X}\in\dot{\Gamma}+\mathfrak{g}_{x,-d+}$. By [DeBacker 2002b, §5], we may complete $\dot{X}$ to an $\mathfrak{sl}_2(F)$-triple $\{\dot{X},\dot{H}\in\mathfrak{g}_{x,0},\dot{Y}\in\mathfrak{g}_{x,d}\}$ and find a split torus S and corresponding apartment $\mathcal{A}(G,S)$ containing x, such that if $\Phi(G,S)=\{\pm\alpha\}$, then $\dot{X}\in\mathfrak{g}_\alpha$ and $\dot{Y}\in\mathfrak{g}_{-\alpha}$. Let $\mathcal{C}$ be the positive alcove adjacent to x in this apartment.

Note that $\mathrm{Cent}_{\mathfrak{g}}(\dot{Y})=\mathfrak{g}_{-\alpha}$. From [DeBacker 2002b, Lemma 5.2.1] we know that

$$\dot{X}+\mathfrak{g}_{x,-d+}={}^{G_{x,0+}}(\dot{X}+\mathrm{Cent}_{\mathfrak{g}_{x,-d+}}(\dot{Y}));$$

thus there exists $g\in G_{x,0+}$ such that $\dot{\Gamma}\in{}^g(\dot{X}+\mathfrak{g}_{-\alpha}\cap\mathfrak{g}_{x,-d+})$. Since $G_{x,0+}$ fixes $\mathcal{C}$ and the coset $\dot{\Gamma}+\mathfrak{g}_{x,-d+}$, we may without loss of generality replace the Lie triple and torus of the preceding paragraph with their g-conjugate, so that we have $\dot{\Gamma}\in\dot{X}+\mathfrak{g}_{-\alpha}\cap\mathfrak{g}_{x,-d+}$. For any $z\in\mathcal{C}$ we have $0<\alpha(z-x)<1$; thus since $\alpha(x), d\in\mathbb{Z}$ we may conclude

$$\mathfrak{g}_\alpha\cap\mathfrak{g}_{x,-d}=\mathfrak{g}_\alpha\cap\mathfrak{g}_{z,-d+}\quad\text{and}\quad\mathfrak{g}_{-\alpha}\cap\mathfrak{g}_{x,-d+}=\mathfrak{g}_{-\alpha}\cap\mathfrak{g}_{z,-d+}.$$

Since $\dot{\Gamma}$ lies in the sum of these two spaces we have $\dot{\Gamma} \in \mathfrak{g}_{z,-d+}$, whence $\Gamma \in \mathfrak{g}^*_{x,-d} \cap \mathfrak{g}^*_{z,-d+}$.

Finally, note that $\mathrm{Cent}_G(\dot{X}) = U_\alpha$ and $U_\alpha \cap G_x = U_\alpha \cap G_z$. Since $\dot{\Gamma} \in \dot{X} + \mathfrak{g}_{x,-d+}$, we have $\mathrm{Cent}_{G_x}(\dot{\Gamma}) \subset \mathrm{Cent}_{G_x}(\dot{X}) G_{x,0+} = \mathrm{Cent}_{G_z}(\dot{X}) G_{x,0+} \subset G_z$. □

Definition 5.3. Let $d = -d_x(\Gamma) > 0$ be such that $\Gamma + \mathfrak{g}^*_{x,-d+}$ is a degenerate coset. Let z be the barycentre of the associated alcove $\mathcal{C}_\Gamma$. Define the subgroup $J_{x,\Gamma}$ by

$$J_{x,\Gamma} = \begin{cases} G_{x,d/2} & \text{if } d = -d_x(\Gamma) \text{ is odd}, \\ G_{z,d/2} & \text{if } d \text{ is even}. \end{cases} \tag{5-3}$$

Note that when $x = x_0$ and $z = z_0$ this group coincides with J_d in (5-1).

Since $G_{x,n+} \subseteq G_{z,n} \subseteq G_{x,n}$ for any positive integer n, it follows directly that

$$G_{x,d/2+} \subseteq J_{x,\Gamma} \subseteq G_{x,d/2}.$$

Since $\Gamma \in \mathfrak{g}^*_{x,-d} \cap \mathfrak{g}^*_{z,-d+}$, it defines a character η_Γ of $J_{x,\Gamma}$ that is trivial on $G_{x,d+}$ via the corresponding Moy–Prasad isomorphism. The character depends only on the coset $\Gamma + \mathfrak{g}^*_{x,-d/2}$ if d is odd and on $\Gamma + \mathfrak{g}^*_{z,-d/2+}$ otherwise. Moreover, since $\mathrm{Cent}_{G_x}(\Gamma) = \mathrm{Cent}_{G_z}(\Gamma)$ we deduce directly that $J_{x,\Gamma}$ is normalized by $C_x(\Gamma) := \mathrm{Cent}_{G_x}(\Gamma)$.

Thus, for any character θ of $C_x(\Gamma)$ coinciding with η_Γ on the intersection of their domains there is a unique extension $\eta(\Gamma, \theta)$ of η_Γ to $C_x(\Gamma) J_{x,\Gamma}$. Define

$$\mathcal{S}_x(\Gamma, \theta) = \mathrm{Ind}_{C_x(\Gamma) J_{x,\Gamma}}^{G_x} \eta(\Gamma, \theta).$$

Proposition 5.4. *Suppose Γ represents a degenerate coset at a vertex $x \in \mathcal{B}(G)$ and $-d = d_x(\Gamma) < 0$. Suppose θ is a character of the centralizer $C_x(\Gamma)$ of Γ in G_x defining a character $\eta(\Gamma, \theta)$ of $C_x(\Gamma) J_{x,\Gamma}$ as above. Then*

(a) *$\mathcal{S}_x(\Gamma, \theta)$ is an irreducible representation of G_x of depth d and of degree $\frac{1}{2} q^{d-1}(q^2 - 1)$;*

(b) *$\mathcal{S}_x(\Gamma, \theta) \cong \mathcal{S}_x(\Gamma', \theta')$ if and only if there exists $g \in G_x$ such that $\eta(\Gamma, \theta) = {}^g\eta(\Gamma', \theta')$; and*

(c) *for any $\nu \in \mathrm{GL}(2, F)$ we have*

$$ {}^\nu\mathcal{S}_x(\Gamma, \theta) \cong \mathcal{S}_{\nu \cdot x}({}^\nu\Gamma, {}^\nu\theta). \tag{5-4}$$

Proof. When $x = x_0$ and $\Gamma \in \mathfrak{g}^*$ corresponds to some $\dot{X}(u, v) \in \mathfrak{g}_{x_0,-d} \smallsetminus \mathfrak{g}_{x_0,-d+}$, then this construction coincides with Shalika's. If $g \in G_x$, then ${}^g C_x(\Gamma) = C_x({}^g\Gamma)$ and ${}^g J_{x,\Gamma} = J_{x,{}^g\Gamma}$, so we obtain the invariance of $\mathcal{S}_x(\Gamma, \theta)$ under G_x-conjugacy and the choice of representative of the appropriate coset of Γ. More generally, for any $\nu \in \mathrm{GL}(2, F)$ such that $\nu \cdot x_0 = x$, we have ${}^\nu(\mathfrak{g}^*_{x_0,d}) = \mathfrak{g}^*_{x,d}$, ${}^\nu C_{x_0}(\Gamma) = C_x({}^\nu\Gamma)$ and ${}^\nu J_{x_0,\Gamma} = J_{x,{}^\nu\Gamma}$. Thus

$${}^\nu\mathcal{S}_{x_0}(\Gamma, \theta) \cong \mathcal{S}_x({}^\nu\Gamma, {}^\nu\theta),$$

where we have identified a ν-conjugate of a representation of G_{x_0} with a representation of G_x under the group isomorphism ${}^{\nu}G_{x_0} \cong G_x$. Since $\mathrm{GL}(2, F)$ acts transitively on the set of vertices of $\mathcal{B}(\mathrm{SL}(2, F))$, the rest of the statements follow from Proposition 5.1. $\square$

The simple nature of the representations $\mathcal{S}_x(\Gamma, \theta)$ is revealed as follows.

Lemma 5.5. *Suppose x is a vertex of $\mathcal{B}(G)$ and $\Gamma_1, \Gamma_2 \in \mathfrak{g}^*_{x,-d}$ represent nonzero but degenerate cosets of $\mathfrak{g}^*_{x,-d}/\mathfrak{g}^*_{x,-d+}$ for some $d \in \mathbb{Z}_{>0}$. Suppose $s \in \mathbb{R}_{>0}$ satisfies $\Gamma_1 \in \Gamma_2 + \mathfrak{g}^*_{x,-s}$. Then for any choice of characters θ_i of $C_x(\Gamma_i)$ such that the characters $\eta(\Gamma_i, \theta_i)$ agree for $i \in \{1, 2\}$ upon restriction to $C_x(\Gamma_1)J_{x,\Gamma_1} \cap G_{x,s+} = C_x(\Gamma_2)J_{x,\Gamma_2} \cap G_{x,s+}$, we have*

$$\mathrm{Res}_{G_{x,s+}} \mathcal{S}_x(\Gamma_1, \theta_1) \cong \mathrm{Res}_{G_{x,s+}} \mathcal{S}_x(\Gamma_2, \theta_2). \tag{5-5}$$

In particular, if $s \geq \frac{1}{2}d$ then (5-5) *holds independent of θ_i.*

Proof. For any Γ_i, the two representations have the same degree $\frac{1}{2}q^{d-1}(q^2-1)$ and the same depth d. If $s \geq d$ then both sides are **1**-isotypic of the same degree hence equivalent.

Suppose $s < d$. Then $G_x/G_{x,d-s}$ acts on $\mathfrak{g}^*_{x,-d}/\mathfrak{g}^*_{x,-s}$. The stabilizer of Γ_i in G_x stabilizes its coset in $\mathfrak{g}^*_{x,-d}/\mathfrak{g}^*_{x,-s}$; the full stabilizer of the coset is $C_x(\Gamma_i)G_{x,d-s}$. Since $\Gamma_1 \in \Gamma_2 + \mathfrak{g}^*_{x,-s}$, we thus have $C_x(\Gamma_1) \subset C_x(\Gamma_2)G_{x,d-s}$. Because $\Gamma_1 \in \Gamma_2 + \mathfrak{g}^*_{x,-d+}$, Lemma 5.2 yields $J_{x,\Gamma_1} = J_{x,\Gamma_2}$; let us denote this group by J. Thus η_{Γ_i} for $i \in \{1, 2\}$ are characters of J that agree on $J \cap G_{x,s+}$. We consider two cases.

If $s \geq \frac{1}{2}d$ then $G_{x,s+} \subset J$, and so $\mathrm{Res}_{G_{x,s+} \cap C_x(\Gamma_i)J}\, \eta(\Gamma_i, \theta_i) = \eta_{\Gamma_i}$ is independent of θ_i. Mackey theory thus yields the decomposition

$$\mathrm{Res}_{G_{x,s+}} \mathcal{S}_x(\Gamma_i, \theta_i) \cong \bigoplus_{\gamma \in G_x/C_x(\Gamma_i)J} {}^{\gamma}\eta_{\Gamma_i}|_{G_{x,s+}}. \tag{5-6}$$

Each $\gamma \in C_x(\Gamma_i)G_{x,d-s}/C_x(\Gamma_i)J$ fixes the character $\eta_{\Gamma_i}|_{G_{x,s+}}$. The elements $\gamma' \in G_x/C_x(\Gamma_1)G_{x,d-s} = G_x/C_x(\Gamma_2)G_{x,d-s}$ parametrize the orbit of the coset $\Gamma_1 + \mathfrak{g}^*_{x,-s} = \Gamma_2 + \mathfrak{g}^*_{x,-s}$. Thus (5-6) gives the same sum of characters for $i \in \{1, 2\}$.

If instead $s < \frac{1}{2}d$, then $G_{x,d-s} \subseteq J$ so $C_x(\Gamma_1)J = C_x(\Gamma_2)J$. Since $J \subseteq G_{x,s+}$, the double coset space $G_{x,s+}\backslash G_x/C_x(\Gamma_i)J$ is now equal to $G_x/C_x(\Gamma_i)G_{x,s+}$, and is independent of i. So again by Mackey theory we have

$$\begin{aligned}\mathrm{Res}_{G_{x,s+}} \mathcal{S}_x(\Gamma_i, \theta_i) &= \bigoplus_{\gamma \in G_x/C_x(\Gamma_i)G_{x,s+}} \mathrm{Ind}^{G_{x,s+}}_{G_{x,s+} \cap {}^{\gamma}(C_x(\Gamma_i)J)} {}^{\gamma}(\eta(\Gamma_i, \theta_i)) \\ &= \bigoplus_{\gamma \in G_x/C_x(\Gamma_i)G_{x,s+}} {}^{\gamma}\big(\mathrm{Ind}^{G_{x,s+}}_{G_{x,s+} \cap C_x(\Gamma_i)J}(\eta(\Gamma_i, \theta_i))\big).\end{aligned}$$

When the restriction of $\eta(\Gamma_i, \theta_i)$ to $G_{x,s+} \cap C_x(\Gamma_1)J = G_{x,s+} \cap C_x(\Gamma_2)J$ is independent of i, we infer (5-5). $\square$

5C. ***Representations attached to nilpotent orbits.*** Let $X \in \mathcal{N}^* \setminus \{0\}$ and let λ be a corresponding adapted one-parameter subgroup. Its centralizer in G is a maximal split torus S. In fact, S is generated by S_0 and $\lambda(\varpi)$, and $\mathrm{Cent}_G(X) = ZU$, where Z is the centre of G and $B = SU$ is a Borel subgroup. For any vertex $x \in \mathcal{B}(G)$, applying the Iwasawa decomposition yields $G = G_x S\, \mathrm{Cent}_G(X)$. Consequently,

$$\mathcal{O} = G \cdot X = \bigsqcup_{n\in\mathbb{Z}} G_x \cdot (\lambda(\varpi)^n \cdot X) = \bigsqcup_{n\in\mathbb{Z}} G_x \cdot (\varpi^{2n} X) \tag{5-7}$$

is the decomposition of the G-orbit of X into disjoint G_x-orbits. It follows that the parity of $d_x(Y)$, for any $Y \in \mathcal{O}$, is an invariant of the G-orbit, and we call this the *parity depth of $\mathcal{O}$ at x*, denoted by $\mathrm{pd}_x(\mathcal{O})$. Furthermore, for each $d \in \mathbb{Z}_{>0}$ of this parity there exists exactly one G_x-orbit in $\mathcal{O} \setminus \mathfrak{g}^*_{x,0}$ whose elements have depth equal to $-d$.

Let Y be a representative of such a G_x orbit. We claim that any choice of central character ζ of Z defines a character (also denoted by ζ) of $C_x(Y)$ coinciding with η_Y on the intersection of their domains. Namely, the chamber $\mathcal{C}_Y$ associated to (Y, x) by Lemma 5.2 defines a Borel subgroup with unipotent radical U. Setting $U_x = G_x \cap U$ we have $C_x(Y) = ZU_x$. Since η_Y is trivial on $ZU_x \cap J_{x,Y}$, the character ζ of ZU_x defined by $\zeta(zu) = \zeta(z)$ for all $z \in Z$ and $u \in U_x$ extends η_Y. We abbreviatedly denote this representation by $\mathcal{S}_x(Y, \zeta)$.

Then, applying Proposition 5.4, we may conclude the following.

Proposition 5.6. *Let x be a vertex in $\mathcal{B}(G)$, $\mathcal{O}$ a nonzero nilpotent G-orbit in $\mathfrak{g}^*$ and ζ a character of Z. For each $d \in \mathbb{Z}_{>0}$ of parity $\mathrm{pd}_x(\mathcal{O})$, fix a representative X_{-d} of the corresponding G_x-orbit in $\mathcal{O} \setminus \mathfrak{g}^*_{x,0}$. Then the* ***representation of G_x attached to $\mathcal{O}$ with central character ζ****, given by*

$$\tau_x(\mathcal{O}, \zeta) = \bigoplus_{d\in\mathbb{Z}_{>0},\ d\equiv \mathrm{pd}_x(\mathcal{O}) \bmod 2} \mathcal{S}_x(X_{-d}, \zeta), \tag{5-8}$$

is independent of choices up to G_x-equivalence.

The depths d of the components of $\tau_x(\mathcal{O}, \zeta)$ all have parity equal to the parity depth of $\mathcal{O}$ at x. Furthermore, for any $X \in \mathcal{O}$ such that $d_x(X) \in \{0, -1\}$, one set of representatives for the G_x-orbits in $\mathcal{O} \setminus \mathfrak{g}^*_{x,0}$ is $\{\varpi^{-2t} X \mid t \in \mathbb{Z}_{\geq 0}\}$.

Since the restriction of $\tau_x(\mathcal{O}, \zeta)$ to any subgroup of $G_{x,0+}$ is independent of the choice of ζ, we may (and do) drop ζ from the notation in such cases. As needed, we associate to the zero nilpotent orbit the trivial representation of G_x, and denote by it $\tau_x(\{0\})$.

6. The case of positive-depth representations of SL(2, F)

The irreducible admissible representations of SL(2, F) come in exactly two flavours: the irreducible subquotients of the principal series; and the irreducible supercuspidal representations. A classification of the former is nicely developed in [Tadić 1994, §7]; the original classification of the latter is in the 1966 thesis of Shalika [2004].

In this section, we focus on those representations of positive depth. We begin by phrasing the classification of positive-depth irreducible admissible representations π of $\mathrm{SL}(2, F)$ in a way that emphasizes their construction from characters of tori. We then establish that their explicit branching to a maximal compact open subgroup G_x can be described as twists of the datum (χ, Γ) defining π. This allows us to state and prove our main theorem in this case, and to explicitly compute the constant terms that arise.

6A. ***Representations of*** $\mathbf{SL(2,}$ ***F*****)** ***of positive depth.*** All principal series of positive depth are irreducible. We classify the positive-depth supercuspidal representations using the parametrization of Adler and Yu [Adler 1998; Yu 2001; Fintzen 2021a], which applies since $p > 2$; this was done explicitly in [Nevins 2013]. Because the tori in $\mathrm{SL}(2, F)$ are one-dimensional, the correcting twist to this construction given by Fintzen, Kaletha and Spice in [Fintzen et al. 2023, Definition 3.1] is trivial.

Proposition 6.1. *Up to isomorphism, the irreducible admissible representations of* $\mathrm{SL}(2, F)$ *of positive depth r are parametrized by the G-conjugacy classes of pairs (T, χ), where T is a maximal torus of G and χ is a character of T of depth r.*

To construct the representations explicitly, we first recall some facts about the maximal tori and their characters. Let T be a maximal torus of G and let χ be a character of T of depth $r > 0$. The building $\mathcal{B}(T)$ of T embeds into $\mathcal{B}(G)$ as the apartment $\mathcal{A}(G, T)$ if T is split and as a single point $\{x_T\}$ otherwise. This point x_T is a vertex if T is unramified and the midpoint of a chamber if T is ramified. It follows that the depth $r > 0$, which is in particular a value for which $T_r \neq T_{r+}$, is an integer if T is split or unramified and is an element of $\frac{1}{2} + \mathbb{Z}$ otherwise.

To each pair (T, χ) we associate an element $\Gamma = \Gamma_\pi$ as follows. If $\mathfrak{t}$ denotes the Lie algebra of T, then via the Moy–Prasad isomorphism $e : \mathfrak{t}_{r/2+}/\mathfrak{t}_{r+} \to T_{r/2+}/T_{r+}$ there exists a nonzero element $\Gamma \in \mathfrak{t}^*_{-r}$, uniquely defined modulo $\mathfrak{t}^*_{-r/2}$, such that

$$\chi(t) = \psi\big(\Gamma(e^{-1}(t))\big).$$

We identify Γ with an element of $\mathfrak{g}^*$ that is zero on the T-invariant complement of $\mathfrak{t}$ in $\mathfrak{g}$. Then $\Gamma \in \mathfrak{g}^*_{x,-r}$ for any $x \in \mathcal{B}(T)$ and we recover T as $\mathrm{Cent}_G(\Gamma)$. Moreover, Γ thus defines a character of $G_{x,r}/G_{x,r+} \cong \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$, and following the work of Moy and Prasad, the pair $(G_{x,r}, \Gamma)$ is called an *unrefined minimal K-type*.

Proof of Proposition 6.1. The construction of the representations $\pi = \pi(T, \chi)$ varies.

If T is a split torus, then choose a Borel subgroup $B = TU$ of G containing T and extend χ trivially across U to a character of B. Set

$$\mathrm{Ind}_{TU}^{G}(\chi) = \{f : G \to \mathbb{C} \mid f(tug) = \chi(t)\nu(t)f(g) \text{ for all } t \in T, u \in U, g \in G\},$$

where ν is the square root of the modular character and is given on $T \cong F^\times$ by the p-adic norm. Then $\pi(T, \chi) = \mathrm{Ind}_B^G(\chi)$ is an irreducible principal series representation.

If T is anisotropic, with associated point $y = x_T \in \mathcal{B}(G)$, then we first extend χ to a character of $TG_{y,r/2+}$, by setting

$$\chi'(tg) = \chi(t)\psi\big(\Gamma(e^{-1}(g))\big),$$

where $e : \mathfrak{g}_{y,r/2+}/\mathfrak{g}_{y,r+} \to G_{y,r/2+}/G_{y,r+}$ is the Moy–Prasad isomorphism. When $G_{y,r/2} = G_{y,r/2+}$, we set $\kappa = \chi'$. When $G_{y,r/2} \neq G_{y,r/2+}$ (which will happen only if T is unramified and $r \in 2\mathbb{Z}$), we take a certain Weil–Heisenberg lift of $\chi'|_{G_{y,r}}$ to form a q-dimensional representation ω of $T \ltimes G_{y,r/2}$, and set $\kappa(tg) = \chi(t)\omega(t, g)$. Then $\pi(T, \chi) = \text{c-Ind}_{TG_{y,r/2}}^G \kappa$ is an irreducible supercuspidal representation. □

Given $\pi = \pi(T, \chi)$, we let $\Gamma = \Gamma_\pi$ denote a choice of minimal K-type realizing the character χ, as preceded the proof. Because $T = \mathrm{Cent}_G(\Gamma)$ we may also say that (χ, Γ) is the datum defining π.

6B. ***Branching rules obtained as twists of the inducing datum.*** We begin by proving that the branching rules obtained in [Nevins 2005, Theorem 7.4; 2013, Theorem 6.2] are in fact constructible from twists of the datum (χ, Γ).

Theorem 6.2. *Let $\pi = \pi(T, \chi)$ be an irreducible admissible representation of G of depth $r > 0$. Let $\Gamma = \Gamma_\pi \in \mathfrak{g}^*$ realize χ as above, so that $T = \mathrm{Cent}_G(\Gamma)$. Then for any vertex $x \in \mathcal{B}(G)$ we have*

$$\mathrm{Res}_{G_x} \pi = \pi^{G_{x,r+}} \oplus \bigoplus_{g \in [G_x \backslash G/T]^{dg}} \mathcal{S}_x({}^g\Gamma, {}^g\chi), \tag{6-1}$$

where $[G_x \backslash G/T]^{dg}$ denotes a parameter set for the G_x-orbits in $G \cdot \Gamma$ that do not meet $\mathfrak{g}^*_{x,-r}$, that is, such that the coset ${}^g\Gamma + \mathfrak{g}^*_{x,d_x({}^g\Gamma)+}$ is degenerate.

Proof. Let us first show that the proof may be reduced to the special case that $x = x_0$. Suppose that $x \in \mathcal{B}(G)$ is an arbitrary vertex. Then there exists $k \in \mathrm{GL}(2, F)$ such that $kx = x_0$, yielding ${}^kG_x = G_{x_0}$. If T is anisotropic, choose $h \in \mathrm{SL}(2, F)$ such that $hx_T \in k^{-1}\bar{\mathcal{C}}$, the closure of the fundamental alcove. If T is split, choose h instead so that ${}^hT = S$ where $x_0 \in \mathcal{A}(G, S)$. Then we have

$$\mathrm{Res}_{G_{x_0}} {}^{kh}\pi \cong \mathrm{Res}_{G_x} {}^h\pi \cong \mathrm{Res}_{G_x} \pi,$$

where the first two representations are isomorphic under the identification of the groups G_{x_0} and G_x via conjugation by k, and the second two are isomorphic as representations of G_x since ${}^h\pi \cong \pi$. Even when $kh \notin \mathrm{SL}(2, F)$, the datum defining the representation ${}^{kh}\pi$ is simply $({}^{kh}T, {}^{kh}\chi, {}^{kh}\Gamma, khx_T)$, where the term khx_T is only for the supercuspidal case.

Suppose that we have proven the decomposition (6-1) of $\mathrm{Res}_{G_{x_0}} {}^{kh}\pi$. Via the identification ${}^kG_{x,r+} \cong G_{x_0,r+}$, we have $({}^{kh}\pi)^{G_{x_0,r+}} \cong ({}^h\pi)^{G_{x,r+}}$. Moreover, for each $g \in [G_{x_0}\backslash G/\operatorname{Cent}_G({}^{kh}\Gamma)]^{\mathrm{dg}}$, which is an element of G such that the G_{x_0}-orbit of ${}^{gkh}\Gamma$ does not meet $\mathfrak{g}^*_{x_0,-r}$, we have

$$\mathcal{S}_{x_0}({}^{gkh}\Gamma, {}^{gkh}\chi) = {}^{k}(\mathcal{S}_{x_0}({}^{(k^{-1}gk)h}\Gamma, {}^{(k^{-1}gk)h}\chi)) \cong \mathcal{S}_x({}^{g'}\Gamma, {}^{g'}\chi),$$

where we set $g' = k^{-1}gkh$. Then $g' \in G$ is such that the G_x-orbit of ${}^{g'}\Gamma$ does not meet $\mathfrak{g}^*_{x,-r}$. It follows that the map $g \mapsto k^{-1}gkh$ takes $[G_{x_0}\backslash G/\operatorname{Cent}_G({}^{kh}\Gamma)]^{\mathrm{dg}}$ bijectively onto $[G_x\backslash G/T]^{\mathrm{dg}}$, as required.

We now prove the theorem in the special case that $x = x_0$, which was considered in [Nevins 2005; 2013].

Suppose first that $T = \operatorname{Cent}_G(\Gamma)$ is anisotropic. Set $y = x_T$, which we assume lies in $\overline{C}$, and let $\pi(T,\chi) = \text{c-Ind}_{TG_{y,r/2}}^{G}\kappa$ be the corresponding supercuspidal representation. By [Nevins 2013, Proposition 4.4], the double coset space $G_x\backslash G/TG_{y,r/2}$ that arises in the Mackey decomposition

$$\operatorname{Res}_{G_x}\pi = \bigoplus_{g\in G_x\backslash G/TG_{y,r/2}} \operatorname{Ind}_{G_x\cap {}^g(TG_{y,r/2})}^{G_x} {}^{g}\kappa$$

is independent of r and is given by $G_x\backslash G/T$. Since $T = \operatorname{Cent}_G(\Gamma)$, this latter space parametrizes the G_x-orbits in the G-orbit of Γ in $\mathfrak{g}^*$. By [Nevins 2013, Theorem 6.1], each of these Mackey components is irreducible.

The element Γ has depth $-r$ and depth is G-invariant. Thus ${}^{g}\Gamma$ meets $\mathfrak{g}^*_{x,-r}$ if and only if $d_x({}^{g}\Gamma) = d_{g^{-1}x}(\Gamma) = -r$, which by [Kim and Murnaghan 2003, Theorem 2.3.1] happens if and only if $g^{-1}x = x_T = y$. If this is the case, then T is an unramified torus attached to a vertex in the G-orbit of x; the corresponding Mackey component has depth r and so lies in $\pi^{G_{x,r+}}$. In all other cases $\pi^{G_{x,r+}} = \{0\}$.

Thus the elements $g \in [G_x\backslash G/T]^{\mathrm{dg}}$ parametrize all Mackey components except $\pi^{G_{x,r+}}$ (if it is nonzero). Furthermore, by [Nevins 2013, Theorem 6.2],[1] they satisfy

$$\operatorname{Ind}_{G_x\cap {}^g(TG_{y,r/2})}^{G_x} {}^{g}\kappa \cong \mathcal{S}_x({}^{g}\Gamma, {}^{g}\chi),$$

as required, yielding (6-1) for the supercuspidal representations.

Now suppose that $T = S$ is the split torus, and that $\pi = \pi(T,\chi) = \operatorname{Ind}_B^G\chi$ for some Borel subgroup $B = TU$ containing T having U as its unipotent radical. Since $G = G_xB$, there is a unique (highly reducible) Mackey component in this case. Instead, the decomposition of $\operatorname{Res}_{G_x}\pi$ into irreducible subrepresentations is found in [Nevins 2005] (in the case that $x \in \{x_0, x_1\}$) by explicitly decomposing the G_x-subrepresentations $\pi^{G_{x,n}}$ as $n \to \infty$. We need to show that this decomposition is in fact of the form (6-1).

First note that as π has depth r at x, the subrepresentation $\pi^{G_{x,r+}}$ is nonzero, and in fact is irreducible as a representation of G_x by [Nevins 2005, Proposition 4.4]

[1] Correction to [Nevins 2013, Theorem 6.2]: the decomposition in the case $y = 1$ is missing the term corresponding to the double coset representative e^{η}.

(where this space is denoted by V^{K_m}, for $m = r + 1$ the conductor of χ). Again by [Kim and Murnaghan 2003, Theorem 2.3.1] we know that ${}^{g}\Gamma \in \mathfrak{g}^*_{x,-r}$ if and only if ${}^{g}T \subset G_x$, so the G_x-orbit in $G \cdot \Gamma$ meeting $\mathfrak{g}^*_{x,-r}$ corresponds to the trivial double coset of $G_x \backslash G/T$.

The irreducible representations of G_x of depth greater than r appearing in $\operatorname{Res}_{G_x} \pi$ are classified in [Nevins 2005, Theorem 7.4]. The notation in that paper relates to ours as follows. Identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the trace form. For any $d \in \mathbb{Z}_{>0}$, $u \in \mathcal{R}^\times$, $v \in \mathcal{P}$, let $X = \varpi^{-d} \dot{X}(u, v) \in \mathfrak{g}_{x,-d}$; then for any character θ of $C_x(X)$ agreeing with η_X on $C_x(X) \cap J_{x,X}$, we had set

$$\mathcal{D}_{d-1}(\theta, \varpi^d X) := \mathcal{S}_x(X, \theta), \tag{6-2}$$

which is of depth d. Thus [Nevins 2005, Theorem 7.4] asserts that for each integer $d > r$, $\operatorname{Res}_{G_x} \pi$ has two irreducible components of depth d, denoted by $W^{\pm}_{d-1}$.

Explicitly, if $\dot{\Gamma}$ is a diagonal matrix $\operatorname{diag}(a, -a) \in \mathfrak{g}_{x,-r} \setminus \mathfrak{g}_{x,-r+}$, then for each fixed $d > r$ define $\gamma_0 = a\varpi^d \in \mathcal{P}^{d-r}$, $\gamma_1 = a\varepsilon^{-1}\varpi^d \in \mathcal{P}^{d-r}$, $Y_0 = \dot{X}(1, \gamma_0^2)$ and $Y_1 = \dot{X}(\varepsilon, \varepsilon\gamma_1^2)$. Then $\varpi^{-d} Y_i \in \mathfrak{g}_{x,-d}$ and the theorem asserts there are characters ρ_i of $C_x(Y_i)$ such that

$$W^+_{d-1} \oplus W^-_{d-1} = \mathcal{S}_x(\varpi^{-d} Y_0, \rho_0) \oplus \mathcal{S}_x(\varpi^{-d} Y_1, \rho_1).$$

We claim this is the desired expression. Namely, if we choose, for $i \in \{0, 1\}$,

$$g_{i,d} = \begin{bmatrix} 1 & -\frac{1}{2}\gamma_i^{-1} \\ \gamma_i & \frac{1}{2} \end{bmatrix},$$

then $\varpi^{-d} Y_i = {}^{g_{i,d}}\Gamma$. Moreover, the characters ρ_i were defined in [Nevins 2005, Definition 7.3] on

$$\begin{bmatrix} b & c \\ c\gamma_i^2 & b \end{bmatrix} \in C_x(Y_i)$$

as $\chi(b + c\gamma_i)$; we can compute directly that therefore $\rho_i = {}^{g_{i,d}}\chi$.

It remains to show $\{g_{i,d} \mid i \in \{0, 1\}, d > r\}$ is a set of double coset representatives for $(G_x \backslash G/T) \setminus G_x T$. Because $G = G_x U T$, these double cosets are represented by T-conjugacy classes of unipotent upper triangular matrices of strictly negative depth. Noting the factorization

$$g = \begin{bmatrix} 1 & -\frac{1}{2}\gamma^{-1} \\ \gamma & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2}\gamma^{-1} \\ 0 & 1 \end{bmatrix} \in G_x \begin{bmatrix} 1 & -\frac{1}{2}\gamma^{-1} \\ 0 & 1 \end{bmatrix} T,$$

we see that as $-\frac{1}{2}\gamma^{-1}$ runs over the distinct square classes in $\mathcal{P}^{r-d} \setminus \mathcal{P}^{r-d+1}$, for all $d > r$, we obtain representatives of the distinct nontrivial cosets of $G_x \backslash G/T$, which is the index set $[G_x \backslash G/T]^{\text{dg}}$, as required. □

6C. ***Main theorem for representations of positive depth.*** Before stating the next theorem, we require a short lemma about filtrations of tori.

Lemma 6.3. *Let $T = \mathrm{Cent}_G(\Gamma)$, where Γ is a semisimple element of depth $-r$. Suppose that at $x \in \mathcal{B}(G)$ we have $d_x(\Gamma) = -d < -r$. Then $T \cap G_x = ZT_{d-r}$, so that $T \cap G_{x,\ell} = T_{d-r+\ell}$ for any $\ell \geq 0$.*

Proof. Let $\mathfrak{t}$ be the Lie algebra of T. The hypotheses imply that $d_x(\varpi^k\Gamma) = d(\varpi^k\Gamma) - (d-r)$ for any $k \in \mathbb{Z}$. Thus, since $\mathfrak{t}$ is one-dimensional, for any element $\dot{X} \in \mathfrak{t}_\ell \smallsetminus \mathfrak{t}_{\ell+}$ we have $d_x(\dot{X}) = \ell - (d-r)$, yielding $\mathfrak{t} \cap \mathfrak{g}_{x,\ell} = \mathfrak{t}_{d-r+\ell}$. Passage to the group yields the desired result, where at depth zero, we observe that $Z \subset T \cap G_x$ for all x. □

Theorem 6.4. *Let $\pi = \pi(T, \chi)$ be an irreducible admissible representation of $G = \mathrm{SL}(2, F)$ of depth $r > 0$ and let $\Gamma = \Gamma_\pi \in \mathfrak{g}^*$ be an associated K-type. Then for each maximal compact subgroup G_x, there is an integer $n_x(\pi)$ such that*

$$\mathrm{Res}_{G_{x,r+}} \pi \cong n_x(\pi)\mathbf{1} + \sum_{\mathcal{O} \in \mathrm{Nil}(\Gamma)} \mathrm{Res}_{G_{x,r+}} \tau_x(\mathcal{O}) \tag{6-3}$$

in the Grothendieck group of representations. In particular, up to some copies of the trivial representation, π is locally completely determined by the nilpotent support of Γ.

Proof. The restriction to $G_{x,r+}$ will be trivial on any irreducible G_x-representations of depth less than or equal to r, so our first step is to match components of depth $d > r$ in $\mathrm{Res}_{G_x} \pi$ and in $\sum_{\mathcal{O} \in \mathrm{Nil}(\Gamma)} \tau_x(\mathcal{O})$. Note that the restriction to $G_{x,r+}$ is independent of the choice of central character ζ so it is omitted from the notation.

Theorem 6.2 gives the decomposition (6-1) of the left side: the irreducible components of depth greater than r are parametrized by the degenerate G_x-orbits of Γ at x. From (5-8) we infer the decomposition of the right side: the components are parametrized by nilpotent G_x-orbits in $\mathcal{O} \smallsetminus \mathfrak{g}^*_{x,0}$ for each $\mathcal{O} \in \mathrm{Nil}(\Gamma)$.

By Proposition 4.1, each degenerate coset $\xi = {}^g\Gamma + \mathfrak{g}^*_{x,-d+}$, where $d = -d_x({}^g\Gamma)$, is represented by a nilpotent element $X \in \mathcal{O}({}^g\Gamma, x)$ such that also ${}^g\Gamma - X \in \mathfrak{g}^*_{x,-r}$. The G_x-orbit of ξ determines the G_x-orbit of X and by definition $G \cdot X \in \mathrm{Nil}(\Gamma)$. Thus for each $d > r$ there is a one-to-one correspondence between the G_x-orbits in $G \cdot \Gamma$ whose depth at x is $-d$ and the G_x-orbits in $\mathrm{Nil}(\Gamma)$ whose depth at x is $-d$.

To complete the proof we need to show the corresponding representations are isomorphic upon restriction to $G_{x,r+}$.

Let $\zeta = \chi|_Z$ be the central character of π. If $r < d \leq 2\lfloor r \rfloor + 1$, then applying Lemma 5.5 to the pair $\Gamma_1 = {}^g\Gamma$ and $\Gamma_2 = X$, with $s = r$, gives $\mathrm{Res}_{G_{x,r+}} \mathcal{S}_x({}^g\Gamma, {}^g\chi) \cong \mathrm{Res}_{G_{x,r+}} \mathcal{S}_x(X, \zeta)$, as required.

If $d > 2r$ (which includes all $d > 2\lfloor r \rfloor + 1$), we have a stronger result. Lemma 6.3 implies that $\mathrm{Cent}_{G_x}({}^g\Gamma) = Z\,{}^gT_{d-r} \subseteq Z\,{}^gT_{r+}$, and therefore ${}^g\chi$ is given on this subgroup by the central character ζ. Since the chambers attached to ${}^g\Gamma$ and to X by Lemma 5.2 coincide, we have $J := J_{x,{}^g\Gamma} = J_{x,X}$. Since ${}^g\Gamma - X \in \mathfrak{g}^*_{x,-r} \subset \mathfrak{g}^*_{x,-d/2+}$, we have $\eta_{{}^g\Gamma} = \eta_X$ as characters of J. Moreover, since ${}^g\Gamma \in X + \mathfrak{g}^*_{x,-r}$, we have $C_x({}^g\Gamma) \subseteq C_x(X)G_{x,d-r} \subseteq C_x(X)J$. Therefore $C_x(X)J_{x,X} = C_x({}^g\Gamma)J_{x,{}^g\Gamma}$ and $\eta({}^g\Gamma, {}^g\chi) = \eta(X, \zeta)$ as characters of this common group. Thus $\mathcal{S}_x({}^g\Gamma, {}^g\chi) = \mathcal{S}_x(X, \zeta)$ as representations of G_x. □

In the course of the proof we established that the components of depth $d > 2r$ occurring in $\mathrm{Res}_{G_x} \pi$ coincide as representations of G_x (not just as representations of $G_{x,r+}$), with the components of depth $d > 2r$ in $\sum_{\mathcal{O}\in \mathrm{Nil}(\Gamma)} \tau_x(\mathcal{O}, \zeta)$, where ζ is the central character of π. This was proven case by case in [Nevins 2005, Remark 7.5; 2013, Proposition 7.6].

For convenience, we recap these sets $\mathrm{Nil}(\Gamma)$ in Table 1.

Remark 6.5. By Theorem 3.5, we know that $\mathrm{Nil}(\Gamma_\pi) = \mathcal{WF}(\pi)$. Moreover, with the standard normalization chosen in [Mœglin and Waldspurger 1987, I.8], the coefficients of the leading terms of the local character expansion agree with those of (6-3); namely $c_{\mathcal{O}}(\pi) = 1$ for all $\mathcal{O} \in \mathcal{WF}(\pi)$. Thus Theorem 6.4 is a representation-theoretic analogue of the analytic local character expansion.

On the other hand, the constant term $n_x(\pi)$ of the decomposition (6-3) does not (and could not) agree with the constant term $c_0(\pi)$ of the local character expansion. For one, $n_x(\pi) \in \mathbb{Z}$, whereas $c_0(\pi)$ may be half-integral; see Table 6. For another, $n_x(\pi)$ depends on the dimension of $\pi^{G_{x,r+}}$, which may vary based on the G-conjugacy class of the vertex $x \in \mathcal{B}(G)$.

Let us compute the constant terms $n_x(\pi)$ explicitly. We begin with a lemma.

Lemma 6.6. *Let $\ell \geq 0$. Then we have*

$$\dim(\tau_x(\mathcal{O})^{G_{x,\ell+}}) = \begin{cases} \frac{1}{2}q(q^{2\lfloor \ell/2\rfloor} - 1) & \text{if } \mathrm{pd}_x(\mathcal{O}) \text{ is even,} \\ \frac{1}{2}(q^{2\lceil \ell/2\rceil} - 1) & \text{if } \mathrm{pd}_x(\mathcal{O}) \text{ is odd.} \end{cases}$$

In particular, $\dim(\tau_x(\mathcal{O}) \oplus \tau_x(\mathcal{O}'))^{G_{x,\ell+}} = \frac{1}{2}(q+1)(q^{\lfloor \ell\rfloor} - 1)$ *when the G_x-orbits in $\mathcal{O}$ and $\mathcal{O}'$ have opposite parity depths at x.*

Proof. The space of $G_{x,\ell+}$-fixed vectors of $\tau_x(\mathcal{O})$ is exactly the sum of its irreducible components of depth $d \leq \ell$. By Proposition 5.4(a), each of these has dimension $\frac{1}{2}q^{d-1}(q^2 - 1)$ and correspond to the G_x-orbits of $\mathcal{O}$ whose depths $-d$ at x satisfy $-\ell \leq -d \leq -1$. Thus if the parity depth of $\mathcal{O}$ at x is even then with $d = 2e$ we have

$$\dim(\tau_x(\mathcal{O})^{G_{x,\ell+}}) = \sum_{e=1}^{\lfloor \ell/2\rfloor} \tfrac{1}{2}q^{2e-1}(q^2 - 1) = \tfrac{1}{2}q(q^{2\lfloor \ell/2\rfloor} - 1),$$

representation $\pi(T,\chi)$ *of positive depth*	$\mathrm{Nil}(\Gamma_\pi)=\mathcal{WF}(\pi)$
principal series, T split	$\{\mathcal{O}_1,\mathcal{O}_\varepsilon,\mathcal{O}_\varpi,\mathcal{O}_{\varepsilon\varpi}\}$
supercuspidal, T splits over $F[\sqrt{\varepsilon}]$	$\{\mathcal{O}_1,\mathcal{O}_\varepsilon\}$ *if* $d_{x_0}(\Gamma_\pi)$ *is even* $\{\mathcal{O}_\varpi,\mathcal{O}_{\varepsilon\varpi}\}$ *if* $d_{x_1}(\Gamma_\pi)$ *is even*
supercuspidal, T splits over $F[\sqrt{\varpi}]$	$\{\mathcal{O}_u,\mathcal{O}_{-u\varpi}\}$ *for some* $u\in\{1,\varepsilon\}$
supercuspidal, T splits over $F[\sqrt{\varepsilon\varpi}]$	$\{\mathcal{O}_u,\mathcal{O}_{-u\varepsilon\varpi}\}$ *for some* $u\in\{1,\varepsilon\}$

Table 1. The forms of the sets $\mathrm{Nil}(\Gamma_\pi)=\mathcal{WF}(\pi)$ for each type of positive-depth representation; the precise sets are determined from Γ_π in Lemma 4.2.

whereas if it is odd then with $d=2e-1$ we have

$$\dim(\tau_x(\mathcal{O})^{G_{x,\ell+}})=\sum_{e=1}^{\lceil \ell/2\rceil}\tfrac12 q^{2e-2}(q^2-1)=\tfrac12(q^{2\lceil \ell/2\rceil}-1).$$

If $\ell=2k$ the sum of these is $\frac12(q+1)(q^\ell-1)$ and if $\ell=2k+1$ then the sum is $\frac12 q(q^{2k}-1)+\frac12(q^{2(k+1)}-1)=\frac12(q+1)(q^\ell-1)$. The result follows since $G_{x,\ell+}=G_{x,\lfloor\ell\rfloor+}$ for each vertex x. □

Proposition 6.7. *Let $\pi=\pi(T,\chi)$ be an irreducible representation of depth $r>0$ as in Theorem 6.2. Then, for each vertex $x\in\mathcal{B}(G)$, the dimension of the subspace of $G_{x,r+}$-fixed vectors and the value of the coefficient $n_x(\pi)$ appearing in* (6-3) *are as given in Table 2.*

Proof. Let $\pi=\pi(T,\chi)$ have depth $r>0$, with associated minimal K-type $\Gamma=\Gamma_\pi$. From Theorem 6.2 we have the equality

$$n_x(\pi)=\dim(\pi^{G_{x,r+}})-\sum_{\mathcal{O}\in\mathrm{Nil}(\Gamma)}\dim(\tau_x(\mathcal{O}))^{G_{x,r+}}. \tag{6-4}$$

If π is a principal series representation and B is a Borel subgroup containing T, then $\pi^{G_{x,r+}}=\mathrm{Ind}_{(B\cap G_x)G_{x,r+}}^{G_x}\chi$, whence $\dim(\pi^{G_{x,r+}})=|G_x/(B\cap G_x)G_{x,r+}|=(q+1)q^r$. Moreover, in this case Γ_π is split and all principal nilpotent orbits occur in $\mathrm{Nil}(\Gamma)$ (Table 1). Thus using (6-4) and Lemma 6.6, we compute

$$n_x(\pi)=(q+1)q^r-2\big(\tfrac12(q+1)(q^r-1)\big)=q+1.$$

If π is a supercuspidal representation corresponding to a ramified torus, then its depth is half-integral, whence for a vertex x we have $G_{x,r}=G_{x,r+}$, and thus by definition of depth $\pi^{G_{x,r+}}=\{0\}$. On the other hand, by Table 1, $\mathrm{Nil}(\Gamma)$ consists of two nonzero orbits which will be of opposite parity depth at any vertex x. Since $\lfloor r\rfloor=r-\frac12$, Lemma 6.6 yields $n_x(\pi)=0-\frac12(q+1)(q^{r-1/2}-1)$.

type of torus T	split	unramified		ramified
		$x_T \sim x$	$x_T \not\sim x$	
depth r	$r \in \mathbb{Z}_{>0}$	$r \in \mathbb{Z}_{>0}$	$r \in \mathbb{Z}_{>0}$	$r \in \frac{1}{2}+\mathbb{Z}_{\geq 0}$
$\dim(\pi(T, \chi)^{G_{x,r+}})$	$(q+1)q^r$	$(q-1)q^r$	0	0
$n_x(\pi)$	$q+1$	$q-q^r$ (r even) $1-q^r$(r odd)	$1-q^r$ (r even) $q-q^r$ (r odd)	$\frac{1}{2}(1-q^{r-1/2})(q+1)$

Table 2. The values of $n_x(\pi)$ appearing in (6-3) for each irreducible admissible representation of SL(2, F) of depth $r > 0$.

Finally, suppose that π is a supercuspidal representation corresponding to an unramified torus. Then Nil(Γ) consists of two nonzero orbits of the same parity depth, and by Lemma 4.2, at any vertex x, the parity of the depths of elements of these orbits is that of $d_x(\Gamma)$. There are thus two cases.

If some G-conjugate of T is contained in G_x, then (replacing T and π by this conjugate) we have $x_T = x$ and $\pi^{G_{x,r+}} = \mathrm{Ind}_{TG_{x,r/2}}^{G_x} \kappa$. It follows from a calculation in [Nevins 2013, Proposition 4.8] that independently of the parity of $r \in \mathbb{Z}$ we have $\dim(\pi^{G_{x,r+}}) = (q-1)q^r$. Since the orbits that occur in Nil(Γ) have the same parity as $-r = d_{x_T}(\Gamma)$, we have by (6-4) and Lemma 6.6 that

$$n_x(\pi) = \begin{cases} (q-1)q^r - 2\big(\frac{1}{2}q(q^r-1)\big) = q - q^r & \text{if } r \text{ is even,} \\ (q-1)q^r - 2\big(\frac{1}{2}(q^{r+1}-1)\big) = 1 - q^r & \text{if } r \text{ is odd.} \end{cases}$$

On the other hand, if T is not conjugate to a torus contained in G_x, then $d_x(\Gamma)$ and $-r = d_{x_T}(\Gamma)$ have opposite parity, and $\mathrm{Res}_{G_x} \pi(T, \chi)$ has no components of depth r. Thus $\dim(\pi^{G_{x,r+}}) = 0$ and we compute instead that

$$n_x(\pi) = \begin{cases} 0 - 2\big(\frac{1}{2}(q^r-1)\big) = 1 - q^r & \text{if } r \text{ is even,} \\ 0 - 2\big(\frac{1}{2}q(q^{r-1}-1)\big) = q - q^r & \text{if } r \text{ is odd.} \end{cases} \qquad \square$$

7. The case of depth-zero representations of SL(2, F)

To establish the theorem for a depth-zero representation π of SL(2, F), we apply a result by Barbasch and Moy [1997] relating the wave front set of π to that of $\pi^{G_{x,0+}}$, viewed as a representation of SL(2, $\mathfrak{f}$) $\cong G_{x,0}/G_{x,0+}$ (Proposition 7.2). We begin by recalling the representation theory of SL(2, $\mathfrak{f}$) and then the classification of depth-zero representations of SL(2, F).

7A. ***Representations of*** **SL(2, $\mathfrak{f}$).** This theory is well known and is beautifully recapped in [Digne and Michel 1991, §15]. Let G = SL(2, $\mathfrak{f}$), T a maximal torus of G and $\bar{\chi}$ a character of T (which is assumed to be nontrivial if T is anisotropic).

The irreducible representations of G, when $p \neq 2$, are parametrized by these pairs $(\mathsf{T}, \bar{\chi})$ as follows.

If T is split and $\bar{\chi}^2 \neq \mathbf{1}$ then $\sigma(\mathsf{T}, \bar{\chi})$ is an irreducible principal series representation; if T is anisotropic and $\bar{\chi}^2 \neq \mathbf{1}$ then $\sigma(\mathsf{T}, \bar{\chi})$ is a (Deligne–Lusztig) cuspidal representation. If T is split and $\bar{\chi} = \mathbf{1}$ then $\sigma(\mathsf{T}, \bar{\chi}) = \mathbf{1} \oplus \overline{\mathrm{St}}$, where $\overline{\mathrm{St}}$ denotes the Steinberg representation of G.

For either T, when $\bar{\chi}$ is a strictly quadratic character, we obtain two irreducible representations $\sigma^u(\mathsf{T}, \bar{\chi})$ for $u \in \{1, \varepsilon\}$ (as the components of the restriction $\sigma(\mathsf{T}, \bar{\chi})$ to $\mathrm{SL}(2, \mathfrak{f})$ of a corresponding (irreducible) representation of $\mathrm{GL}(2, \mathfrak{f})$). They are distinguished by the theory of Gelfand–Graev representations, as follows.

Let $X \in \mathfrak{g}(\mathfrak{f})^* \setminus \{0\}$ be nilpotent, and identify X with a nilpotent element $\dot{X} \in \mathfrak{g}(\mathfrak{f})$. Complete this to an $\mathfrak{sl}(2, \mathfrak{f})$ triple $\{\dot{Y}, \dot{H}, \dot{X}\}$ and let $\mathfrak{u}(\mathfrak{f}) = \mathfrak{f}\dot{Y}$. Then X defines a character of $\mathsf{U} = \exp(\mathfrak{u}(\mathfrak{f}))$ by $\psi_X(\exp(W)) = \psi(X(W))$ for all $W \in \mathfrak{u}(\mathfrak{f})$. The (highly reducible) representation of $\mathrm{SL}(2, \mathfrak{f})$ given by

$$\gamma_{\mathcal{O}} = \mathrm{Ind}_{\mathsf{U}}^{\mathsf{G}} \psi_X \tag{7-1}$$

depends (up to equivalence) only on the nonzero orbit $\mathcal{O} = \mathsf{G} \cdot X$, and is called the Gelfand–Graev representation of G associated to $\mathcal{O}$.

Contrary to convention, we parametrize our nonzero nilpotent orbits by upper triangular matrices $\dot{X}_u \in \mathfrak{g}(\mathfrak{f})$ as in (4-1), where $u \in \mathfrak{f}^\times/(\mathfrak{f}^\times)^2 \sim \{1, \varepsilon\}$. We compute the character $[\gamma_{\mathcal{O}_u}]$ directly, noting that $[\gamma_{\mathcal{O}_u}](g) = 0$ if g is not conjugate to an element of $\mathfrak{u}(\mathfrak{f})$, and that for any $s \neq 0$,

$$\psi_{X_u}\left(\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\right) = \psi(-us).$$

This yields

$$[\gamma_{\mathcal{O}_u}](g) = \begin{cases} q^2 - 1 & \text{if } g = I, \\ 2\sum_{t \in (\mathfrak{f}^\times)^2} \psi(-ust) & \text{if } g \sim \left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right], \\ 0 & \text{otherwise.} \end{cases}$$

By [Digne and Michel 1991, Theorem 14.30], the decomposition into irreducible subrepresentations of $\gamma_{\mathcal{O}_u}$ is multiplicity-free. Using character tables it is straightforward to compute that $\gamma_{\mathcal{O}_u}$ contains all irreducible principal series representations, all Deligne–Lusztig cuspidal representations, the Steinberg representation, and exactly one from each pair of representations arising from quadratic characters. Our parametrization is therefore as follows: for $u \in \mathfrak{f}^\times/(\mathfrak{f}^\times)^2$, and a quadratic character $\bar{\chi}$ of T, let $\sigma^u(\mathsf{T}, \bar{\chi})$ denote the component of $\sigma(\mathsf{T}, \bar{\chi})$ occurring in $\gamma_{\mathcal{O}_u}$. In the notation of [Digne and Michel 1991, §15], where $\sigma_a := \sum_{t \in (\mathfrak{f}^\times)^2} \psi(ta)$, the quadratic character of the split torus is denoted by χ_{α_0} and that of the anisotropic torus is denoted by χ_{ω_0}. For $\rho \in \{\chi_{\alpha_0}, \chi_{\omega_0}\}$, the characters of these components are correspondingly labelled $\rho^{\pm}$, where $[\sigma^{-1}(\mathsf{T}, \rho)] = \rho^{+}$ and $[\sigma^{-\varepsilon}(\mathsf{T}, \rho)] = \rho^{-}$. In the sequel, we sometimes denote the quadratic character of the split torus T by sgn.

7B. ***Depth-zero representations of* SL(2, *F*).** Now let $G = \mathrm{SL}(2, F)$ and let χ be a depth-zero character of a maximal split or unramified torus. Assume χ is nontrivial if the torus is nonsplit. There are two nonconjugate choices for an unramified anisotropic torus. If x_0 and x_1 are the vertices of the standard alcove, as before, then we can choose representatives T^i of the conjugacy classes of maximal tori such that $T^i \subset G_{x_i}$, for $i \in \{0, 1\}$. Then $\mathsf{T}_i = T^i_0 / T^i_{0+}$ is a maximal anisotropic torus of $G_{x_i,0}/G_{x_i,0+} =: \mathsf{G}_i \cong \mathrm{SL}(2, \mathfrak{f})$. Let T denote the split torus corresponding to the standard apartment and set $\mathsf{T} = T_0/T_{0+}$, which is a maximal split torus of both G_0 and G_1. In each case, the character χ factors to a character $\overline{\chi}$ of the quotient.

When T^i is anisotropic, for each $i \in \{0, 1\}$ inflate the representation $\sigma(\mathsf{T}_i, \overline{\chi})$ of G_i to a representation of G_{x_i} and, if $\chi^2 \neq \mathbf{1}$, define $\pi(T^i, \chi) = \text{c-Ind}_{G_{x_i}}^{G} \sigma(\mathsf{T}_i, \overline{\chi})$. When $\chi^2 = \mathbf{1}$ but $\chi \neq \mathbf{1}$, set $\pi^u(T^i, \chi) = \text{c-Ind}_{G_{x_i}}^{G} \sigma^u(\mathsf{T}_i, \overline{\chi})$ for $u \in \{1, \varepsilon\}$, using the notation of Section 7A. These representations are supercuspidal and irreducible [Moy and Prasad 1996, Proposition 6.6]; the latter four were called the special representations. Note that they are related via $\eta = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right] \in \mathrm{GL}(2, F)$ as follows: we have ${}^{\eta}\pi^*(T^0, \chi) = \pi^*(T^1, {}^{\eta}\chi)$, where $*$ indicates that this applies both to the special and nonspecial representations. It follows from [Nevins 2013, Theorem 5.3] that the depth-zero component $\pi^*(T^i, \chi)^{G_{x_i,0+}}$ is the inflation to G_{x_i} of $\sigma^*(\mathsf{T}_i, \overline{\chi})$, but $\pi^*(T^i, \chi)^{G_{x,0+}} = \{0\}$ if $x \in \{x_0, x_1\} \smallsetminus \{x_i\}$.

If T is split, contained in a Borel subgroup B, then $\pi(T, \chi) = \mathrm{Ind}_B^G \chi$ is again in the principal series. It is immediate to see that for any vertex x, $\pi(T, \chi)^{G_{x,0+}} \cong \sigma(\mathsf{T}, \overline{\chi})$ under the isomorphism $G_{x,0}/G_{x,0+} \cong \mathrm{SL}(2, \mathfrak{f})$; note that this will be reducible whenever $\overline{\chi}$ is a quadratic character.

We summarize the results of the preceding two paragraphs in Table 3, and then address the remaining depth-zero irreducible representations, which are the reducible principal series [Tadić 1994, §7], below.

When $\chi \in \{\nu, \nu^{-1}\}$, where ν is the square root of the modular character, the Jordan–Hölder factors of $\pi(T, \chi)$ are the trivial representation and the Steinberg representation St, and we have $\mathrm{St}^{G_{x,0+}} = \overline{\mathrm{St}}$ and $\mathbf{1}^{G_{x,0+}} = \mathbf{1}$.

In the remaining cases, χ is quadratic. There are three such, corresponding to the distinct quadratic extensions $E = F[\sqrt{\tau}]$ of F. We write $\chi = \mathsf{sgn}_\tau$ for the quadratic character whose kernel is the image of the corresponding norm map. As described in [Nevins 2005, §8], there is for each such χ a realization of $\pi(T, \chi)$ on the space $L^2(F^\times)$ such that its irreducible summands are $H^\tau_\pm$, where H^τ_+ consists of the functions supported $N^\tau_+ = \ker(\mathsf{sgn}_\tau)$ and H^τ_- those supported on its complement. Note that $\varpi \in \ker(\mathsf{sgn}_{-\varpi})$, for example; thus if we parametrize the quadratic extensions by $\tau \in \{\varepsilon, -\varpi, -\varepsilon\varpi\}$, then we obtain the more pleasing uniform description below.

T	type	π	$\pi^{G_{x,0+}}$
T split	principal series Steinberg trivial	$\pi(T,\chi)$ St $\mathbf{1}$	$\sigma(\mathsf{T},\bar\chi)$ (possibly reducible) $\overline{\mathrm{St}}$ $\mathbf{1}$
T^i unramified $i \in \{0,1\}$	supercuspidal	$\pi(T^i,\chi)$	$\sigma(\mathsf{T}_i,\bar\chi)$ if $x \sim x_i$ $\{0\}$ if $x \not\sim x_i$
	special supercuspidal $u \in \{1,\varepsilon\}$	$\pi^u(T^i,\chi)$	$\sigma^u(\mathsf{T}_i,\bar\chi)$ if $x \sim x_i$ $\{0\}$ if $x \not\sim x_i$

Table 3. The depth-zero representations of G_x occurring in the restriction to G_x of the irreducible principal series, Steinberg, trivial and supercuspidal representations, for any vertex $x \in \mathcal{B}(G)$.

Proposition 7.1. *For each $\tau \in \{\varepsilon, -\varpi, -\varepsilon\varpi\}$ and $i \in \{0,1\}$, the G_i-representations $(H_\pm^\tau)^{G_{x_i,0+}}$ are irreducible and their isomorphism classes are given in Table 4.*

Proof. Without loss of generality we may assume T is such that $x_0, x_1 \in \mathcal{A}(G,T)$. Note that if the extension $F[\sqrt{\tau}]$ is unramified then $\overline{\mathsf{sgn}}_\tau = \mathbf{1}$ but in the remaining cases $\overline{\mathsf{sgn}}_\tau = \mathsf{sgn}$, the quadratic character of the split torus T in G. Since $\pi(T,\chi)^{G_{x,0+}} \cong \sigma(\mathsf{T},\bar\chi)$, which is reducible in all of these cases, we infer

$$\begin{aligned}\pi(T,\mathsf{sgn}_\tau)^{G_{x_i,0+}} &= (H_+^\tau)^{G_{x_i,0+}} \oplus (H_-^\tau)^{G_{x_i,0+}}\\ &\cong \begin{cases} \mathbf{1} \oplus \mathrm{St} & \text{if } \tau = \varepsilon,\\ \sigma^1(\mathsf{T},\mathsf{sgn}) \oplus \sigma^\varepsilon(\mathsf{T},\mathsf{sgn}) & \text{otherwise.}\end{cases}\end{aligned}$$

The character of $(H_\pm^\tau)^{G_{x_0,0+}}$ was computed in [Nevins 2005, Theorem 9.1]. In the unramified case, this gives $(H_+^\varepsilon)^{G_{x_0,0+}} = \overline{\mathrm{St}}$ and $(H_-^\varepsilon)^{G_{x_0,0+}} = \mathbf{1}$. For the ramified case, first note that in the notation of that paper, of [Digne and Michel 1991, §15], and ours, respectively, we have $\Xi_{\mathsf{sgn}}^+ = \chi_{\alpha_0}^- = [\sigma^{-\varepsilon}(\mathsf{T},\mathsf{sgn})]$ and $\Xi_{\mathsf{sgn}}^- = \chi_{\alpha_0}^+ = [\sigma^{-1}(\mathsf{T},\mathsf{sgn})]$. The theorem states, for the ramified case, that the character of $(H_+^\tau)^{G_{x_0,0+}}$ is Ξ_{sgn}^+ when $-1 \notin (F^\times)^2$ and Ξ_{sgn}^- otherwise. In our notation this is exactly the character of $\sigma^1(\mathsf{T},\mathsf{sgn})$. This completes the first row of Table 4.

For the character of $(H_\pm^\tau)^{G_{x_1,0+}}$, the proof of [Nevins 2005, Corollary 9.3] showed that twisting $\pi(T,\mathsf{sgn}_\tau)$ by $\omega = \left[\begin{smallmatrix}0&1\\ \varpi&0\end{smallmatrix}\right] \in \mathrm{GL}(2,F)$, which interchanges these vertices, preserves $H_\pm^\tau$ when $\mathsf{sgn}_\tau(-\varpi) = 1$ and interchanges them otherwise (including in the unramified case). Applying this twist to $\pi^{G_{x_0,0+}}$ yields $\pi^{G_{x_1,0+}}$. The entries for $(H_\pm^\varepsilon)^{G_{x_1,0+}}$ follow.

For the ramified case, note that twisting by ω takes $\mathcal{O}_u$ to $\mathcal{O}_{-u\varpi}$ and so it maps the Gelfand–Graev representation $\gamma_{\mathcal{O}_u}$ of G_{x_0} to the representation $\gamma_{\mathcal{O}_{-u}}$ of G_{x_1}.[2]

[2]This calculation was neglected in the proof of [Nevins 2005, Corollary 9.3], yielding an incorrect statement for the depth-zero components.

π	H_+^ε	H_-^ε	$H_+^{-\varpi}$	$H_-^{-\varpi}$	$H_+^{-\varepsilon\varpi}$	$H_-^{-\varepsilon\varpi}$
$\pi^{G_{x_0,0+}}$	$\overline{\mathrm{St}}$	$\mathbf{1}$	$\sigma^1(\mathsf{T}, \mathsf{sgn})$	$\sigma^\varepsilon(\mathsf{T}, \mathsf{sgn})$	$\sigma^1(\mathsf{T}, \mathsf{sgn})$	$\sigma^\varepsilon(\mathsf{T}, \mathsf{sgn})$
$\pi^{G_{x_1,0+}}$	$\mathbf{1}$	$\overline{\mathrm{St}}$	$\sigma^1(\mathsf{T}, \mathsf{sgn})$	$\sigma^\varepsilon(\mathsf{T}, \mathsf{sgn})$	$\sigma^\varepsilon(\mathsf{T}, \mathsf{sgn})$	$\sigma^1(\mathsf{T}, \mathsf{sgn})$

Table 4. The isomorphism classes of the depth-zero representations of G_{x_i} occurring in the restriction to G_{x_i} of the decomposable principal series.

Therefore, twisting by ω sends the inflation of the representation $\sigma^u(\mathsf{T}, \mathsf{sgn})$ of G_{x_0} to the inflation of the representation $\sigma^{-u}(\mathsf{T}, \mathsf{sgn})$ of G_{x_1}.

Thus for example, if $-1 \in (F^\times)^2$, then ${}^\omega H_+^{-\varpi} = H_+^{-\varpi}$ and $\sigma^{-1}(\mathsf{T}, \mathsf{sgn}) = \sigma^1(\mathsf{T}, \mathsf{sgn})$, whereas if -1 is not a square, then ${}^\omega H_+^{-\varpi} = H_-^{-\varpi}$ and $\sigma^{-1}(\mathsf{T}, \mathsf{sgn}) = \sigma^\varepsilon(\mathsf{T}, \mathsf{sgn})$. A careful accounting of signs completes the second row of the table. □

7C. *Wave front sets.* The wave front set is determined with the following result, which is based on [Barbasch and Moy 1997, Theorem 4.5].

Proposition 7.2. *Let π be an irreducible admissible nontrivial representation of depth zero of* $\mathrm{SL}(2, F)$. *Suppose* $\mathrm{char}(F) = 0$ *and $p > 3e + 1$, where e is the absolute ramification index of F over $\mathbb{Q}_p$. Then we have*

$$\mathcal{WF}(\pi) = \left\{ \mathcal{O} \in \mathscr{O}(0) \;\middle|\; \begin{array}{l} \text{there exists } x \text{, a vertex of } \mathcal{B}(G) \text{, such that } \mathrm{pd}_x(\mathcal{O}) \text{ is} \\ \text{even, and there exists } \sigma \text{, an irreducible constituent} \\ \text{of } \pi^{G_{x,0+}} \text{, such that } \bar{\sigma} \text{ occurs in } \gamma_{\overline{\mathcal{O}}} \end{array} \right\},$$

where $\gamma_{\overline{\mathcal{O}}}$ is the Gelfand–Graev representation (7-1) *attached to the nonzero nilpotent orbit in $\mathfrak{g}_{x,0}/\mathfrak{g}_{x,0+}$ under $\mathsf{G}_x \cong \mathrm{SL}(2, \mathfrak{f})$ whose inflation to $\mathfrak{g}_{x,0}$ meets $\mathcal{O}$.*

Proof. The hypotheses imply that exp converges on $\mathfrak{g}_{0+}$ and that the local character expansion holds. Barbasch and Moy [1997] used (generalized) Gelfand–Graev characters as test functions to determine the wave front set of π. For each nilpotent orbit $\mathcal{O}$ that is represented by a depth-zero coset at the vertex x (meaning, its parity depth at x is even), let $[\gamma_{\overline{\mathcal{O}}}]$ denote the lift to $G_{x,0}$ of the character of the corresponding Gelfand–Graev representation of $\mathsf{G}_x = G_{x,0}/G_{x,0+}$, viewed as a function on G. It is supported on the subset $G_{0+} \cap G_{x,0}$ of topologically unipotent elements. Let $f_{x,\mathcal{O}}$ be the function on $\mathfrak{g}$, with support in $\mathfrak{g}_{0+} \cap \mathfrak{g}_{x,0}$, that is given by $f_{x,\mathcal{O}} = [\gamma_{\overline{\mathcal{O}}}] \circ \exp$. They then show that $\widehat{\mu_{\mathcal{O}'}}(f_{x,\mathcal{O}}) = 0$ if $\mathcal{O}$ is not contained in the closure of $\mathcal{O}'$ and is nonzero if $\mathcal{O} = \mathcal{O}'$. Thus $\Theta_\pi([\gamma_{\overline{\mathcal{O}}}]) = 0$ for all $\mathcal{O}$ that do not meet the wave front set of π and $\Theta_\pi([\gamma_{\overline{\mathcal{O}}}]) \neq 0$ when $\mathcal{O} \in \mathcal{WF}(\pi)$.

For any irreducible representation $\bar{\sigma}$ of G_x, let $m(\sigma, \pi)$ denote the multiplicity of its inflation σ in $\pi^{G_{x,0+}}$ and $m(\bar{\sigma}, \gamma_{\overline{\mathcal{O}}})$ the multiplicity of $\bar{\sigma}$ in $\gamma_{\overline{\mathcal{O}}}$. Then [Barbasch and Moy 1997, Theorem 4.5(4)] becomes

$$\Theta_\pi([\gamma_{\bar{\mathcal{O}}}]) = \sum_\sigma m(\sigma, \pi) m(\bar{\sigma}, \gamma_{\bar{\mathcal{O}}}),$$

whence our result for the case of $\mathrm{SL}(2, F)$. □

Corollary 7.3. *Under the hypothesis of Proposition 7.2, the wave front sets corresponding to the depth-zero representations of* $\mathrm{SL}(2, F)$ *are as given in Table 5.*

Proof. For $u \in \{1, \varepsilon\}$ and $i, j \in \{0, 1\}$, the nilpotent orbit $\mathcal{O}_{u\varpi^j}$ is represented by a depth-zero coset at x_i if and only if $i = j$, and in this case it corresponds to the nilpotent orbit in the quotient $\mathfrak{g}_{x_i,0}/\mathfrak{g}_{x_i,0+} \cong \mathfrak{sl}(2, \mathfrak{f})$ under $\mathsf{G}_i \cong \mathrm{SL}(2, \mathfrak{f})$ that we denoted by simply $\mathcal{O}_u$. Therefore the Gelfand–Graev representations $\gamma_{\bar{\mathcal{O}}}$ referred to in Proposition 7.2 are $\gamma_{\bar{\mathcal{O}}_1}$ and $\gamma_{\bar{\mathcal{O}}_\varepsilon}$ for $x = x_0$, and $\gamma_{\bar{\mathcal{O}}_\varpi}$ and $\gamma_{\bar{\mathcal{O}}_{\varepsilon\varpi}}$ for $x = x_1$. By conjugacy, these two vertices suffice. The decomposition of $\pi^{G_{x,0+}}$ for $x \in \{x_0, x_1\}$ was given in Tables 3 and 4 for all irreducible depth-zero representations π, and matching these with the decomposition of the Gelfand–Graev representations of the corresponding groups G_i as in Section 7A yields Table 5. □

This table is consistent with the computation of the coefficients of the local character expansion for $\mathrm{SL}(2, F)$ using the existence of Whittaker models in [Assem 1994, §3].

Claim. *For each depth-zero irreducible representation π of* $\mathrm{SL}(2, F)$ *there exists an element $\Gamma \in \mathfrak{g}^*_{x,0}$, for some $x \in \mathcal{B}(G)$, such that $\mathcal{WF}(\pi) = \mathrm{Nil}(\Gamma)$.*

Proof of claim. Existence follows immediately from Table 5 and Lemma 4.2, though the elements Γ for which $\mathcal{WF}(\pi) = \mathrm{Nil}(\Gamma)$ do not correspond to minimal K-types for π (as these latter are not realized by elements on the Lie algebra). However, on an ad hoc basis, we can make this association of π with Γ more explicit, as follows.

For T unramified or split, and $x \in \mathcal{B}(T) \subset \mathcal{B}(G)$, we can in the same spirit attach to any *regular* $\pi(T, \chi)$ (in the sense of Kaletha [2019, Proposition 3.4.27] in the first case and [Tadić 1994, §7] in the second) *any* regular depth-zero element of $\mathfrak{t}^*$, that is, an element $\Gamma \in \mathfrak{g}^*_{x,0} \setminus \mathfrak{g}^*_{x,0+}$ whose centralizer in G is T. The same holds for $\pi = \mathrm{St}$, whereas we associate $\Gamma = 0$ to $\mathbf{1}$.

When $\pi^u(T^i, \chi)$ is a special representation (for some $u \in \{1, \varepsilon\}$ and $i \in \{0, 1\}$ and χ quadratic) then it is a supercuspidal unipotent representation and Γ is chosen to be a nilpotent element in the lift to $\mathfrak{g}^*_{x_i,0}$ of the nilpotent orbit corresponding to $\sigma^u(\mathsf{T}_i, \bar{\chi})$.

When $\pi \in \{H^\tau_\pm \mid \tau \in \{\varepsilon, -\varpi, -\varepsilon\varpi\}\}$, Γ is a choice of element of an anisotropic torus T that splits over $F[\sqrt{\tau}]$. However, while the orbit of Γ satisfies $\mathrm{Nil}(\Gamma) = \mathcal{WF}(\pi)$, when $-1 \in (F^\times)^2$ and $\tau \in \{\varpi, \varepsilon\varpi\}$, the centralizer may be one of two possible tori $T = \mathrm{Cent}_G(\Gamma)$ up to conjugacy, and neither one is expressly associated to π. □

representation π	$\mathcal{WF}(\pi)$
$\pi(T,\chi)$ irreducible principal series	$\{1, \varepsilon, \varpi, \varepsilon\varpi\}$
$\pi(T^0,\chi)$ irreducible supercuspidal	$\{1, \varepsilon\}$
$\pi(T^1,\chi)$ irreducible supercuspidal	$\{\varpi, \varepsilon\varpi\}$

π	$\mathcal{WF}(\pi)$	π	$\mathcal{WF}(\pi)$
$\mathbf{1}$	$\{0\}$	St	$\{1, \varepsilon, \varpi, \varepsilon\varpi\}$
H_+^{ε}	$\{1, \varepsilon\}$	H_-^{ε}	$\{\varpi, \varepsilon\varpi\}$
$H_+^{-\varpi}$	$\{1, \varpi\}$	$H_-^{-\varpi}$	$\{\varepsilon, \varepsilon\varpi\}$
$H_+^{-\varepsilon\varpi}$	$\{1, \varepsilon\varpi\}$	$H_-^{-\varepsilon\varpi}$	$\{\varepsilon, \varpi\}$
$\pi^1(T^0,\chi)$	$\{1\}$	$\pi^{\varepsilon}(T^0,\chi)$	$\{\varepsilon\}$
$\pi^1(T^1,\chi)$	$\{\varpi\}$	$\pi^{\varepsilon}(T^1,\chi)$	$\{\varepsilon\varpi\}$

Table 5. For each irreducible depth-zero representation of $\mathrm{SL}(2, F)$, we list under the heading $\mathcal{WF}(\pi)$ the set of elements $u \in \{0, 1, \varepsilon, \varpi, \varepsilon\varpi\}$ such that $\mathcal{O}_u \in \mathcal{WF}(\pi)$.

Note we may define $\mathcal{WF}(\pi)$ by Table 5, even over fields where Proposition 7.2 does not apply. Our main result, below, expresses that, just as in the positive-depth case, this is consistent (for all fields with residual characteristic different from 2).

Theorem 7.4. *Let π be an irreducible admissible representation of G of depth zero with central character ζ. For any vertex $x \in \mathcal{B}(G)$, we have*

$$\mathrm{Res}_{G_x} \pi \cong \pi^{G_{x,0+}} \oplus \bigoplus_{\mathcal{O}\in\mathcal{WF}(\pi)} \tau_x(\mathcal{O}, \zeta), \tag{7-2}$$

where $\mathcal{WF}(\pi)$ is as in Table 5. It follows that $\mathrm{Res}_{G_{x,0+}} \pi$ *takes the form of* (6-3) *with constant coefficient $n_x(\pi) = \dim(\pi^{G_{x,0+}})$.*

Proof. The decomposition will follow from the main results of [Nevins 2005; 2013], applied to $x \in \{x_0, x_1\}$, as in the proof of Theorem 6.2. Let π be a depth-zero representation of G with central character ζ.

For irreducible depth-zero principal series, one has $\gamma_0 = \gamma_1 = 0$ in [Nevins 2005, Theorem 7.4]. Matching notation as in (6-2), we conclude that $\mathcal{S}_x(\varpi^{-d}X_u, \zeta)$ occurs in $\mathrm{Res}_{G_x} \pi$, for $u \in \{1, \varepsilon\}$, for each $d > 0$, and that these exhaust the irreducible summands. Therefore the summands can be regrouped as the sum of $\tau_x(\mathcal{O}, \zeta)$, as defined in (5-8), over all regular nilpotent orbits, as required. As the positive-depth summands of $\mathrm{Res}_{G_x} \pi$ are identical for all depth-zero irreducible principal series, the case of $\pi = \mathrm{St}$ follows since $\mathrm{Res}_{G_x} \mathbf{1}$ has no positive-depth components.

For the remaining reducible principal series, we use [Nevins 2005, Theorem 9.2], together with Proposition 7.1. Noting that $\mathsf{sgn}_{-\varpi}(\varpi) = 1$ and $\mathsf{sgn}_{-\varepsilon\varpi}(\varpi) = -1$, the theorem states

$$\operatorname{Res}_{G_{x_0}} H_+^\tau = \begin{cases} \mathrm{St} \oplus \bigoplus_{d>0} \big(\mathcal{S}_{x_0}(\varpi^{-2d} X_1, \zeta) \oplus \mathcal{S}_{x_0}(\varpi^{-2d} X_\varepsilon, \zeta)\big) & \text{if } \tau = \varepsilon, \\ \sigma^1(\mathsf{T}, \mathsf{sgn}) \oplus \bigoplus_{d>0} \mathcal{S}_{x_0}(\varpi^{-d} X_1, \zeta) & \text{if } \tau = -\varpi, \\ \sigma^1(\mathsf{T}, \mathsf{sgn}) \oplus \bigoplus_{d>0} \big(\mathcal{S}_{x_0}(\varpi^{-2d} X_1, \zeta) \oplus \mathcal{S}_{x_0}(\varpi^{-2d+1} X_\varepsilon, \zeta)\big) & \text{if } \tau = -\varepsilon\varpi. \end{cases}$$

Regrouping the positive-depth summands yields the decomposition

$$\operatorname{Res}_{G_{x_0}} H_+^\tau = \begin{cases} \mathrm{St} \oplus \tau_{x_0}(\mathcal{O}_1, \zeta) \oplus \tau_{x_0}(\mathcal{O}_\varepsilon, \zeta) & \text{if } \tau = \varepsilon, \\ \sigma^1(\mathsf{T}, \mathsf{sgn}) \oplus \tau_{x_0}(\mathcal{O}_1, \zeta) \oplus \tau_{x_0}(\mathcal{O}_\varpi, \zeta) & \text{if } \tau = -\varpi, \\ \sigma^1(\mathsf{T}, \mathsf{sgn}) \oplus \tau_{x_0}(\mathcal{O}_1, \zeta) \oplus \tau_{x_0}(\mathcal{O}_{\varepsilon\varpi}, \zeta) & \text{if } \tau = -\varepsilon\varpi, \end{cases} \tag{7-3}$$

which is consistent with the wave front set computed in Table 5. As noted above, the positive-depth summands of $H_+^\tau \oplus H_-^\tau$ form $\bigoplus_{\mathcal{O}\in\mathscr{O}(0)\smallsetminus\{0\}} \tau_{x_0}(\mathcal{O}, \zeta)$, and the wave front sets of these representations are complementary, so this yields the result for $\operatorname{Res}_{G_{x_0}} H_-^\tau$ as well.

To determine $\operatorname{Res}_{G_{x_1}} \pi$ we proceed as in the proof of Proposition 7.1. Conjugation by ω interchanges the components of the principal series if and only if $\mathsf{sgn}_\tau(\varpi) = -1$, and ${}^\omega\tau_{x_0}(\mathcal{O}_u, \zeta) = \tau_{x_1}(\mathcal{O}_{-u\varpi}, \zeta)$. Thus when $\mathsf{sgn}_\tau(\varpi) = -1$ (the first three lines below) we obtain the decomposition of H_-^τ as the ω-conjugate of (7-3); when $\mathsf{sgn}_\tau(\varpi) = 1$ (the last two lines), then $H_-^\tau = {}^\omega H_-^\tau$ so we first take the complement of (7-3). (The depth-zero components are taken from Proposition 7.1.) This yields

$$\operatorname{Res}_{G_{x_1}} H_-^\tau = \begin{cases} \mathrm{St} \oplus \tau_{x_1}(\mathcal{O}_\varpi, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{\varepsilon\varpi}, \zeta) & \text{if } \tau = \varepsilon, \\ \sigma^{-1}(\mathsf{T}, \mathsf{sgn}) \oplus \tau_{x_1}(\mathcal{O}_{-\varpi}, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{-\varpi^2}, \zeta) & \text{if } \tau = -\varpi \text{ and } -1 \notin (F^\times)^2, \\ \sigma^{-1}(\mathsf{T}, \mathsf{sgn}) \oplus \tau_{x_1}(\mathcal{O}_{-\varpi}, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{-\varepsilon\varpi^2}, \zeta) & \text{if } \tau = -\varepsilon\varpi \text{ and } -1 \in (F^\times)^2, \\ \sigma^{-\varepsilon}(\mathsf{T}, \mathsf{sgn}) \oplus \tau_{x_1}(\mathcal{O}_{-\varepsilon\varpi}, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{-\varepsilon\varpi^2}, \zeta) & \text{if } \tau = -\varpi \text{ and } -1 \in (F^\times)^2, \\ \sigma^{-\varepsilon}(\mathsf{T}, \mathsf{sgn}) \oplus \tau_{x_1}(\mathcal{O}_{-\varepsilon\varpi}, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{-\varpi^2}, \zeta) & \text{if } \tau = -\varepsilon\varpi \text{ and } -1 \notin (F^\times)^2. \end{cases}$$

Therefore, in any case, the nilpotent orbits arising in $\operatorname{Res}_{G_{x_1}} H_-^\varepsilon$ are $\{\mathcal{O}_\varpi, \mathcal{O}_{\varepsilon\varpi}\}$; those arising in $\operatorname{Res}_{G_{x_1}} H_-^{-\varpi}$ are $\{\mathcal{O}_\varepsilon, \mathcal{O}_{\varepsilon\varpi}\}$; and those arising in $\operatorname{Res}_{G_{x_1}} H_-^{-\varepsilon\varpi}$ are $\{\mathcal{O}_\varpi, \mathcal{O}_\varepsilon\}$, which again is consistent with Table 5, as required.

Now suppose that $\pi_i = \text{c-Ind}_{G_{x_i}}^G \sigma$ is a nonspecial supercuspidal representation. Translating the notation of [Nevins 2013, Proposition 5.2], we have $\pi_d^+(\theta) := \mathcal{S}_{x_0}(\varpi^{-d} X_{-1}, \theta)$ and $\pi_d^-(\theta) := \mathcal{S}_{x_0}(\varpi^{-d} X_{-\varepsilon}, \theta)$. Theorem 5.3 of [Nevins 2013] yields

$$\begin{aligned} \operatorname{Res}_{G_{x_0}} \pi_i &= \begin{cases} \sigma \oplus \bigoplus_{t>0} \big(\mathcal{S}_{x_0}(\varpi^{-2t} X_{-1}, \zeta) \oplus \mathcal{S}_{x_0}(\varpi^{-2t} X_{-\varepsilon}, \zeta)\big) & \text{if } i = 0, \\ \bigoplus_{t>0} \big(\mathcal{S}_{x_0}(\varpi^{-2t+1} X_{-1}, \zeta) \oplus \mathcal{S}_{x_0}(\varpi^{-2t+1} X_{-\varepsilon}, \zeta)\big) & \text{if } i = 1, \end{cases} \\ &= \begin{cases} \sigma \oplus \tau_{x_0}(\mathcal{O}_1, \zeta) \oplus \tau_{x_0}(\mathcal{O}_\varepsilon, \zeta) & \text{if } i = 0, \\ \tau_{x_0}(\mathcal{O}_\varpi, \zeta) \oplus \tau_{x_0}(\mathcal{O}_{\varepsilon\varpi}, \zeta) & \text{if } i = 1. \end{cases} \end{aligned}$$

Here, we have used that in [Nevins 2013, Theorem 5.3], $\eta = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right]$ and thus ${}^{\eta}\mathcal{O}_u = \mathcal{O}_{u\varpi}$.

Finally, for the special supercuspidal representations, corresponding to a quadratic character χ, note from [Nevins 2013, Proof of Proposition 5.2] that σ_0^+ corresponds to the character $\chi^{\pm}_{\omega_0}$ of [Digne and Michel 1991, §15] (see Section 7A) so in our notation here, $\sigma_0^+ := \sigma^{-1}(T^0, \chi)$ and $\sigma_0^- = \sigma^{-\varepsilon}(T^0, \chi)$. Since ${}^{\eta}\mathcal{O}_u = \mathcal{O}_{u\varpi}$, twisting by η sends the inflation of the representation $\sigma^u(\mathsf{T}, \overline{\chi})$ of G_{x_0} to the inflation of the representation $\sigma^u(\mathsf{T}, \overline{\chi})$ of G_{x_1}. We thus infer, for $u \in \{1, \varepsilon\}$, the decompositions

$$\begin{aligned}\operatorname{Res}_{G_{x_0}} \pi^u(T^i, \chi) &= \begin{cases} \sigma \oplus \bigoplus_{t>0} \mathcal{S}_{x_0}(-\varpi^{-2t} X_{-u}, \zeta) & \text{if } i = 0, \\ \bigoplus_{t>0} \mathcal{S}_{x_0}(-\varpi^{-2t+1} X_{-u}, \zeta) & \text{if } i = 1, \end{cases} \\ &= \begin{cases} \sigma \oplus \tau_{x_0}(\mathcal{O}_u, \zeta) & \text{if } i = 0, \\ \tau_{x_0}(\mathcal{O}_{u\varpi}, \zeta) & \text{if } i = 1, \end{cases}\end{aligned}$$

where again ζ is the corresponding central character. Comparing with Table 5, we conclude that (7-2) holds for $\operatorname{Res}_{G_{x_0}} \pi^u(T^i, \chi)$ in each case. The result for x_1 is obtained by conjugating by η.

Finally, the value of $n_x(\pi) = \dim(\pi^{G_{x,0+}})$ can be deduced from Tables 3 and 4: it is $q + 1$ for irreducible principal series, $q - 1$ for Deligne–Lusztig cuspidal representations, q for $\overline{\mathrm{St}}$, $(q - 1)/2$ for the special unipotent representations and $(q + 1)/2$ for the components of the reducible principal series. □

8. Applications

8A. *The Fourier transform of a nilpotent orbital integral.* As a first application, we derive a formula for the Fourier transform of a nilpotent orbital integral in any open set of the form $\mathfrak{g}_{x,0+}$ in terms of the trace characters of the representations $\tau_x(\mathcal{O}, \zeta)$.

Proposition 8.1. *Let $x \in \mathcal{B}(G)$ be a vertex. Let $[\tau_x(\mathcal{O})]$ denote the restriction to $G^{\mathrm{reg}}_{x,0+}$ of the trace character of the representation $\tau_x(\mathcal{O}, \zeta)$, for either choice of central character ζ. Assume* exp *converges on $\mathfrak{g}_{x,0+}$. Then for each nonzero nilpotent orbit $\mathcal{O}$ and $X \in \mathfrak{g}^{\mathrm{reg}}_{x,0+}$ we have*

$$\widehat{\mu_{\mathcal{O}}}(X) = \begin{cases} \tfrac{1}{2}q + [\tau_x(\mathcal{O})](\exp X) & \text{if } \mathcal{O} \text{ has even parity depth at } x, \\ \tfrac{1}{2} + [\tau_x(\mathcal{O})](\exp X) & \text{if } \mathcal{O} \text{ has odd parity depth at } x. \end{cases}$$

As x ranges over the vertices of $\mathcal{B}(G)$, these expressions determine the function $\widehat{\mu_{\mathcal{O}}}$ on $\mathfrak{g}^{\mathrm{reg}}_{1/2+}$.

Proof. Let π be a nontrivial irreducible admissible representation of depth $r \geq 0$, and let Θ_π denote its character. We assume the functions $\widehat{\mu_{\mathcal{O}}}$ are normalized as in [Mœglin and Waldspurger 1987], so that the coefficients $c_{\mathcal{O}}$ corresponding

type	representation of SL(2, F) of depth $r \geq 0$	coefficient c_0 of $\mu_{\{0\}}$ in local character expansion
principal series	irreducible	0
	irreducible summand	0
	St	-1
	1	1
supercuspidal	ramified case	$-q^{r-1/2}(q+1)/2$
	unramified, nonspecial	$-q^r$
	special unipotent	$-\frac{1}{2}$

Table 6. Values of the constant term in the local character expansion of irreducible admissible representations of SL(2, F).

to $\mathcal{O} \in \mathcal{WF}(\pi)$ in the local character expansion of $\Theta_\pi \circ \exp$ are all equal to 1. Thus on $\mathfrak{g}_{x,r+}^{\mathrm{reg}}$ we have

$$\Theta_\pi \circ \exp = c_0(\pi) + \sum_{\mathcal{O} \in \mathcal{WF}(\pi)} \widehat{\mu_{\mathcal{O}}}.$$

These constant terms in the case of SL(2, F) are well known and are summarized in Table 6. For example, for principal series, see [Assem 1994, Propositions 2.1 and 3.3.6], and for supercuspidal representations, see [DeBacker and Sally 2000, Tables 1–4].

Theorem 7.4, on the other hand, gives a formula for the character of any irreducible depth-zero representation on $G_{x,0+}$. Matching these for the special unipotent representations $\pi = \pi^u(T^i, \chi)$ (where $i \in \{0, 1\}$ and $u \in \{1, \varepsilon\}$) yields the given formula. It is moreover direct to verify the consistency of this expression across the local character expansions of all irreducible representations, including those of positive depth (on $G_{x,r+}$ as in Theorem 6.4). The result therefore holds on $\mathfrak{g}_{1/2+} = G \cdot (\mathfrak{g}_{x_0,0+} \cup \mathfrak{g}_{x_1,0+})$. □

Note that $\mathfrak{g}_{1/2+} \subsetneq \mathfrak{g}_{0+}$. One anticipates that Proposition 8.1 holds on all of $\mathfrak{g}_{0+}$, and that the restriction on the G-domain is an artefact of having considered only vertices in the present work.

Remark 8.2. Far more explicit formulae for the functions $\widehat{\mu_{\mathcal{O}}}$ have been computed for the group SL(2, F) in [Assem 1994; DeBacker and Sally 2000] among others. They have also noted that, under the exponential map, the characters of the five representations $\{1, \pi^u(T^i, \chi) \mid u \in \{1, \varepsilon\}, i \in \{0, 1\}\}$ (where χ denotes a quadratic character) form another basis for the span of the functions $\widehat{\mu_{\mathcal{O}}}$ on $\mathfrak{g}_{0+}^{\mathrm{reg}}$. It is these representations (and their generalizations for representations over arbitrary fields of characteristic different from p) that arise in the local representation-theoretic expansion of $\mathrm{SL}_2(F)$ given in [Henniart and Vignéras 2024, §6].

In fact the special unipotent representations have local character expansions of the form

$$\Theta_\pi(\exp(X)) = \widehat{\mu_{\mathcal{O}}}(X) - \tfrac{1}{2}, \tag{8-1}$$

for the single corresponding orbit $\mathcal{O}$, and this holds on the strictly larger set $\mathfrak{g}_{0+}^{\mathrm{reg}}$.

An advantage to Proposition 8.1 is the simplicity and explicitness of the construction, which uses no more than a vertex and a representative of the orbit as input. In this, it recalls some of the original formulae for these Fourier transforms of nilpotent orbital integrals in [Harish-Chandra 1999].

8B. *Computing the polynomial* $\dim(\pi^{G_{x,2n}})$. This arose from a question posed to me by Marie-France Vignéras in 2022 and gives a (not surprisingly) negative answer to [Henniart and Vignéras 2023, Question 1.1]: Is there a polynomial with *integer* coefficients such that $\dim(\pi^{G_{x,r+j}}) = P(p^j)$ for large enough j?

If π is an irreducible admissible representation of G, the local character expansion implies that $\dim(\pi^{G_{x,m}})$, for m even, is expressible as a polynomial in q, as described in [Barbasch and Moy 1997, §5.1]; see also [Henniart and Vignéras 2023, Remark 11.8]. Here we can obtain this polynomial as a corollary of Theorems 6.4 and 7.4, using the explicit values computed in Proposition 6.7.

Corollary 8.3. *Let π be an irreducible representation of $G = \mathrm{SL}(2, F)$ of depth r. Then for each* **even** *integer $m > r$, we have*

$$\dim(\pi^{G_{x,m}}) = \begin{cases} q^m + q^{m-1} & \text{if } \pi \text{ is an irreducible principal series,} \\ q^{m-1} - q^r & \text{if } \pi \text{ is supercuspidal nonspecial, from a vertex} \sim x, \\ q^m - q^r & \text{if } \pi \text{ is supercuspidal nonspecial, from a vertex} \not\sim x, \\ \tfrac{1}{2}(q+1)(q^{m-1} - q^{r-\frac{1}{2}}) & \text{if } \pi \text{ is supercuspidal, from a nonvertex.} \end{cases}$$

On the other hand, if $\pi_s = H_s^\varepsilon$ then $\dim(\pi_s^{G_{x,m}}) = q^{m-1}$ *when the parity depth at x of the orbits in $\mathcal{WF}(\pi_s)$ is even, and equals q^m otherwise*; *and if* $\pi = \mathrm{St}$, *then* $\dim(\pi^{G_{x,m}}) = q^m + q^{m-1} - 1$. *In all other cases*, $\dim(\pi^{G_{x,m}})$ *is exactly half of that of a corresponding* (*irreducible principal series or nonspecial supercuspidal*) *representation.*

Proof. Let π be of depth r. By Theorem 6.4 (and Remark 6.5) in positive depth and Theorem 7.4 in depth zero, we have for $m > r$ that

$$\dim(\pi^{G_{x,m}}) = n + \sum_{\mathcal{O} \in \mathcal{WF}(\pi)} \dim(\tau_x(\mathcal{O})^{G_{x,m}}),$$

where $n = n_x(\pi)$ if $r > 0$ (Proposition 6.7) and $n = \dim(\pi^{G_{x,0+}})$ if $r = 0$ (Tables 3 and 4). We obtain $\dim(\tau_x(\mathcal{O})^{G_{x,m}})$ from Lemma 6.6 by setting $\ell = m-1$, and refer to Tables 1 and 5 for the sets $\mathcal{WF}(\pi)$.

If π is an irreducible principal series, then $\mathcal{WF}(\pi)$ consists of all nilpotent orbits and $n = (q+1)$ for all r, yielding a total $(q+1)+(q+1)(q^{m-1}-1) = q^{m-1}(q+1)$. For the reducible principal series, we have $\dim(\mathrm{St}) = q$ and $\dim(\mathbf{1}) = 1$, so

$$\dim((H_+^{\varepsilon})^{G_{x,m}}) = \begin{cases} q+q(q^{m-2}-1) = q^{m-1} & \text{if } x \sim x_0, \\ 1+(q^m-1) = q^m & \text{if } x \sim x_1, \end{cases}$$

with $\dim((H_-^{\varepsilon})^{G_{x,m}}) = (q^{m-1}+q^m) - \dim((H_+^{\varepsilon})^{G_{x,m}})$. On the other hand, for $\tau \in \{-\varpi, -\varepsilon\varpi\}$ the decomposition is symmetric and therefore $\dim((H_\pm^{\tau})^{G_{x,m}}) = \frac{1}{2}(q^m+q^{m-1})$.

If π is a ramified supercuspidal representation, then r is a positive half-integer, and $\mathcal{WF}(\pi)$ consists of two orbits of opposite parity depth. We compute

$$\dim(\pi^{G_{x,m}}) = \tfrac{1}{2}(1-q^{r-\frac{1}{2}})(q+1)+\tfrac{1}{2}(q+1)(q^{m-1}-1) = \tfrac{1}{2}(q+1)(q^{m-1}-q^{r-\frac{1}{2}}).$$

Suppose π is an unramified supercuspidal representation of depth $r > 0$. Note that $-r = d(\Gamma)$ has the same parity as $d_x(\Gamma)$ (which coincides with $\mathrm{pd}_x(\mathcal{O})$ for each $\mathcal{O} \in \mathcal{WF}(\pi)$) if and only if $x \sim x_T$. Rephrasing the conditions in Table 2 yields that if $d_x(\Gamma)$ is even then $n_x(\pi) = q - q^r$ for all x, whereas if $d_x(\Gamma)$ is odd then $n_x(\pi) = 1 - q^r$. Lemma 6.6 now gives

$$\dim(\pi^{G_{x,m}}) = \begin{cases} (q-q^r)+q(q^{m-2}-1) = q^{m-1}-q^r & \text{if } d_x(\Gamma) \text{ is even}, \\ (1-q^r)+(q^m-1) = q^m-q^r & \text{if } d_x(\Gamma) \text{ is odd}. \end{cases}$$

The same holds for nonspecial supercuspidal representations of depth $r = 0$, since there $n = q-1$ if $x \sim x_T$ and $n = 0$ otherwise. Finally, for each of the special representations, the dimension will be half the corresponding value for an unramified supercuspidal from the same vertex, by symmetry. □

Acknowledgements

This work was instigated by a question posed to the author by David Vogan and has benefitted enormously from many conversations with him in the online research community Representation Theory and Noncommutative Geometry sponsored by the American Institute of Mathematics. The approach to $\mathrm{Nil}(\Gamma)$ given here was significantly refined through conversations with Fiona Murnaghan and Loren Spice. This work progressed over a period of visits to many colleagues, and benefitted from their comments and interest: Vincent Sécherre, Laboratoire de Mathématiques de Versailles, Université Paris-Saclay; Anne-Marie Aubert, Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université de Paris/Sorbonne Université and Jessica Fintzen, Universität Bonn. It is a true pleasure to thank all of these generous people.

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Received November 10, 2023. Revised May 20, 2024.

MONICA NEVINS
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF OTTAWA
OTTAWA, ON
CANADA
mnevins@uottawa.ca

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 329 No. 2 April 2024