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**VISHIK EQUIVALENCE AND SIMILARITY OF QUASILINEAR
p-FORMS AND TOTALLY SINGULAR QUADRATIC FORMS**

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For quadratic forms over fields of characteristic different from two, there is a so-called Vishik criterion, giving a purely algebraic characterization of when two quadratic forms are motivically equivalent. In analogy to that, we define Vishik equivalence on quasilinear p-forms. We study the question whether Vishik equivalent p-forms must be similar. We prove that this is not true for quasilinear p-forms in general, but we find some families of totally singular quadratic forms (i.e., of quasilinear 2-forms) for which the question has a positive answer.

1. Introduction

Vishik [1997] defined an equivalence relation on quadratic forms over fields of characteristic other than 2 (see also [Karpenko 2000; Vishik 2004]): φ and ψ are equivalent if and only if

$$(1\text{-}1) \quad \dim \varphi = \dim \psi \quad \text{and} \quad i_W(\varphi_E) = i_W(\psi_E) \quad \text{for any field extension } E/F.$$

Vishik proved that this equivalence coincides with motivic equivalence. Hence, equation (1-1) is nowadays known as *Vishik's criterion* for motivic equivalence. Then the question was raised whether the equivalence defined by Vishik also coincides with similarity; in other words, the following question was asked:

Question A. Are quadratic forms satisfying Vishik's criterion (1-1) necessarily similar?

Izhboldin [1998; 2000] proved that the answer to Question A is positive for odd-dimensional quadratic forms, but negative for even-dimensional forms of dimension greater or equal to 8 (except possibly for the dimension 12). Hoffmann [2015] proves that, under some conditions on the base field, Question A has positive answer for even-dimensional quadratic forms as well.

In the case of fields of characteristic 2, we have to distinguish between different types of quadratic forms — nonsingular and totally singular (the two extreme cases) and singular (the mixed type). Totally singular quadratic forms over fields of characteristic 2 have been generalized to *quasilinear p-forms* over fields of characteristic p in [Hoffmann 2004].

Let F be a field of characteristic p, and let φ and ψ be quasilinear p-forms over F. Inspired by Vishik's criterion, we define *Vishik equivalence* of φ and ψ as

$$\dim \varphi = \dim \psi \qquad \text{and} \qquad \mathfrak{i}_d(\varphi_E) = \mathfrak{i}_d(\psi_E) \quad \text{for any field extension } E/F,$$

where $\mathfrak{i}_d(\tau)$ is the *defect* (sometimes also called the *quasilinear index*) of the quasilinear p-form τ. In analogy to Question A, we ask:

Question Q. Are Vishik equivalent quasilinear p-forms necessarily similar?

We show in Examples 3.14 and 3.15 that Question Q has a negative answer for p-forms if $p > 3$. Therefore, in Section 4, we will focus on totally singular quadratic forms, i.e., on the case when $p = 2$. In this case, we can give a positive answer to Question Q at least for some families of forms. For example, we prove:

Theorem 1.1 (see Theorem 4.6). *Let φ, ψ be totally singular quadratic forms over F such that φ_{an} is minimal over F. If φ and ψ are Vishik equivalent, then they are similar.*

See Theorem 4.17 and Corollary 4.18 for more families for which the answer to Question Q is positive. We would also like to point out Theorem 3.7 which shows that Vishik equivalence preserves similarity factors, even in the case of quasilinear p-forms.

This paper is based on the second and third chapters of the author's PhD thesis [Zemková 2022].

2. Preliminaries

All fields in this article are of characteristic $p > 0$. Whenever we talk about (totally singular) quadratic forms, we assume $p = 2$.

Quasilinear p-forms. Most of the necessary background on quasilinear p-forms can be found in [Hoffmann 2004]; we include it here for the readers' convenience.

Definition 2.1. Let F be a field and V a finite-dimensional F-vector space. A *quasilinear p-form* (or simply a *p-form*) *over F* is a map $\varphi : V \to F$ with the following properties:

(1) $\varphi(av) = a^p \varphi(v)$ for any $a \in F$ and $v \in V$, and

(2) $\varphi(v + w) = \varphi(v) + \varphi(w)$ for any $v, w \in V$.

The dimension of φ is defined as $\dim \varphi = \dim V$.

Any p-form φ on an F-vector space V can be associated with the polynomial $\sum_{i=1}^{n} a_i X_i^p \in F[X_1, \ldots, X_n]$, where $a_i = \varphi(v_i)$ with $\{v_1, \ldots, v_n\}$ a basis of V. In such case, we write φ as $\langle a_1, \ldots, a_n \rangle$.

We have $c\langle a_1, \ldots, a_n \rangle = \langle ca_1, \ldots, ca_n \rangle$ for any $c \in F^*$. If $\langle a_1, \ldots, a_n \rangle$ and $\langle b_1 \ldots, b_m \rangle$ are two p-forms over F, then we define

$$\langle a_1, \ldots, a_n \rangle \perp \langle b_1 \ldots, b_m \rangle = \langle a_1, \ldots, a_n, b_1 \ldots, b_m \rangle,$$
$$\langle a_1, \ldots, a_n \rangle \otimes \langle b_1 \ldots, b_m \rangle = a_1 \langle b_1 \ldots, b_m \rangle \perp \ldots \perp a_n \langle b_1 \ldots, b_m \rangle.$$

Moreover, for a positive integer k, we write $k \times \varphi$ for the p-form $\varphi \perp \ldots \perp \varphi$ consisting of k copies of φ.

Two p-forms $\varphi : V \to F$ and $\psi : W \to F$ are called *isometric* (denoted $\varphi \simeq \psi$) if there exists a bijective homomorphism $f : V \to W$ of vector spaces such that $\varphi(v) = \psi(f(v))$ for any $v \in V$. If f is not bijective but injective, then φ is called a *subform* of ψ (denoted $\varphi \subseteq \psi$); then there exists a p-form σ over F such that $\psi \simeq \varphi \perp \sigma$. If $\varphi \simeq c\psi$ for some $c \in F^*$, then φ and ψ are called *similar* (denoted $\varphi \overset{\text{sim}}{\sim} \psi$).

A p-form $\varphi : V \to F$ is called *isotropic* if $\varphi(v) = 0$ for some $v \in V \setminus \{0\}$; otherwise, φ is called *anisotropic*. The p-form φ can be written as $\varphi \simeq \sigma \perp k \times \langle 0 \rangle$ with σ an anisotropic p-form over F and k a nonnegative integer. Then σ is unique up to isometry, and it is called the *anisotropic part of φ* (denoted φ_{an}). The integer k is called the *defect* of φ (denoted $i_d(\varphi)$).

Let φ be a p-form on an F-vector space V; we set

$$D_F(\varphi) = \{\varphi(v) \mid v \in V\} \qquad \text{and} \quad D_F^*(\varphi) = D_F(\varphi) \setminus \{0\},$$
$$G_F^*(\varphi) = \{x \in F^* \mid x\varphi \simeq \varphi\} \quad \text{and} \quad G_F(\varphi) = G_F^*(\varphi) \cup \{0\}.$$

Note that $D_F(\varphi)$ is an F^p-vector space; in particular, if $\varphi \simeq \langle a_1, \ldots, a_n \rangle$, then $D_F(\varphi) = \text{span}_{F^p}\{a_1, \ldots, a_n\}$.

Lemma 2.2 [Hoffmann 2004, Proposition 2.6]. *Let φ be a p-form over F.*

(i) *Let $\{c_1, \ldots, c_k\}$ be any F^p-basis of the vector space $D_F(\varphi)$. Then we have*
$$\varphi_{\text{an}} \simeq \langle c_1, \ldots, c_k \rangle.$$

(ii) *If $a_1, \ldots, a_m \in D_F(\varphi)$, then $\langle a_1, \ldots, a_m \rangle_{\text{an}} \subseteq \varphi$.*

It follows that, for any $a, b \in F$ and $x \in F^*$, we have

$$\langle a, b \rangle \simeq \langle a + b, b \rangle \quad \text{and} \quad \langle a \rangle \simeq \langle ax^p \rangle.$$

For a p-form φ on an F-vector space V and a field extension E/F, we denote by φ_E the p-form on the E-vector space $V_E = E \otimes V$ defined by $\varphi_E(e \otimes v) = e^p \varphi(v)$ for any $e \in E$ and $v \in V$. It was proved in [Hoffmann 2004, Lemma 5.1] that if $\varphi \simeq \langle a_1, \ldots, a_n \rangle$, then $(\varphi_E)_{\text{an}} \simeq \langle a_{i_1}, \ldots, a_{i_k} \rangle$ for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

Quasi-Pfister forms and quasi-Pfister neighbors. For $n > 0$, an *n-fold quasi-Pfister form* is a p-form $\langle\langle a_1, \ldots, a_n \rangle\rangle = \langle\langle a_1 \rangle\rangle \otimes \cdots \otimes \langle\langle a_n \rangle\rangle$, where $\langle\langle a \rangle\rangle = \langle 1, a, \ldots, a^{p-1} \rangle$. Moreover, $\langle 1 \rangle$ is called the 0-*fold quasi-Pfister form.*

A p-form φ over F is called a *quasi-Pfister neighbor* if $c\varphi \subseteq \pi$ for some $c \in F^*$ and a quasi-Pfister form π over F such that $\dim \varphi > \frac{1}{p} \dim \pi$. The value $\dim \pi - \dim \varphi$ is called the *codimension* of φ.

In the following lemma, we summarize some of the most important properties of quasi-Pfister forms and quasi-Pfister neighbors.

Lemma 2.3 [Hoffmann 2004, Propositions 4.6 and 4.14; Scully 2013, Lemma 2.6]. *Let $\pi \simeq \langle\langle a_1, \ldots, a_n \rangle\rangle$ be a quasi-Pfister form over F, φ a quasi-Pfister neighbor of π, and E/F a field extension. Then*:

(i) $G_E(\pi) = D_E(\pi)$.

(ii) *There either exist $k > 0$ and a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $(\pi_E)_{\mathrm{an}} \simeq \langle\langle a_{i_1}, \ldots, a_{i_k} \rangle\rangle_E$, or $(\pi_E)_{\mathrm{an}} \simeq \langle 1 \rangle_E$.*

(iii) *φ_E is isotropic if and only if π_E is isotropic.*

p-bases, norm fields and norm forms. Let $a_1, \ldots, a_n \in F$. We call the set $\{a_1, \ldots, a_n\}$ *p-independent over F* if $[F^p(a_1, \ldots, a_n) : F^p] = p^n$, and *p-dependent over F* otherwise. The set $\{a_1, \ldots, a_n\}$ is p-independent over F if and only if the quasi-Pfister form $\langle\langle a_1, \ldots, a_n \rangle\rangle$ is anisotropic over F [Zemková 2024, Corollary 2.8]. A (possibly infinite) set $\mathcal{S} \subseteq F$ is called *p-independent over F* if any finite subset of $\mathcal{S}$ is p-independent over F.

Let E be a field such that $F^p \subseteq E \subseteq F$; a set $\mathcal{B} \subseteq E$ is called a *p-basis of E over F* if $\mathcal{B}$ is p-independent over F and $E = F^p(\mathcal{B})$. Such a p-basis always exists, and any subset of E that is p-independent over F can be extended into a p-basis of E over F [Gille and Szamuely 2006, Corollary A.8.9]. A set $\mathcal{B} = \{b_i \mid i \in I\}$ is a p-basis of E over F if and only if

$$\widehat{\mathcal{B}} = \left\{ \prod_{i \in I} b_i^{\lambda(i)} \;\middle|\; \lambda : I \to \{0, \ldots, p-1\}, \lambda(i) = 0 \text{ for almost all } i \in I \right\}$$

is an F^p-linear basis of E [Pickert 1949, page 27]; in such case, any $a \in E$ can be expressed *uniquely* as

$$a = \sum_\lambda x_\lambda^p \prod_{i \in I} b_i^{\lambda(i)}$$

for some $x_\lambda \in F$, almost all of them zero. By abuse of language, we say that *a can be expressed uniquely with respect to $\mathcal{B}$.*

Let φ be a p-form over F. We define the *norm field of φ over F* as the field

$$N_F(\varphi) = F^p \left(\frac{a}{b} \;\middle|\; a, b \in D_F^*(\varphi) \right).$$

By the definition, we have $N_F(\varphi) = N_F(\varphi_{an})$ and also that $N_F(\varphi) = N_F(c\varphi)$ for any $c \in F^*$. Furthermore, if ψ is another p-form over F satisfying $\psi \simeq \varphi$, then $N_F(\psi) = N_F(\varphi)$. Finally, if $\tau_{an} \subseteq c\varphi_{an}$ for some p-form τ over F and $c \in F^*$, then $N_F(\tau) \subseteq N_F(\varphi)$.

Lemma 2.4 [Hoffmann 2004, Lemma 4.2 and Corollary 4.3]. *Let φ be a p-form over F.*

(i) *If $\varphi \simeq \langle a_0, \ldots, a_n \rangle$ for some $n \geq 1$ and $a_i \in F$, $0 \leq i \leq n$, with $a_0 \neq 0$, then*
$$N_F(\varphi) = F^p\left(\tfrac{a_1}{a_0}, \ldots, \tfrac{a_n}{a_0}\right).$$

(ii) *Suppose $N_F(\varphi) = F^p(b_1, \ldots, b_m)$ for some $b_i \in F^*$, $1 \leq i \leq m$, and let E/F be a field extension. Then $N_E(\varphi) = E^p(b_1, \ldots, b_m)$.*

The following proposition provides an alternative possibility for a determination of the norm field.

Proposition 2.5. *Let $\varphi = \langle 1, a_1, \ldots, a_n \rangle$ be a p-form over F, and let $b_1, \ldots, b_m \in F$. Then*
$$(\varphi_{F(\sqrt[p]{b_1}, \ldots, \sqrt[p]{b_m})})_{an} \simeq \langle 1 \rangle \quad \Longleftrightarrow \quad N_F(\varphi) \subseteq F^p(b_1, \ldots, b_m).$$

If, moreover, $\{b_1, \ldots, b_m\} \subseteq \{a_1, \ldots, a_n\}$, then
$$(\varphi_{F(\sqrt[p]{b_1}, \ldots, \sqrt[p]{b_m})})_{an} \simeq \langle 1 \rangle \quad \Longleftrightarrow \quad N_F(\varphi) = F^p(b_1, \ldots, b_m).$$

In particular, if m is minimal with this property, then $\mathrm{ndeg}_F \varphi = p^m$.

Proof. We set $E = F(\sqrt[p]{b_1}, \ldots, \sqrt[p]{b_m})$.

First, $(\varphi_E)_{an} \simeq \langle 1 \rangle$ is equivalent to $\mathrm{span}_{E^p}\{1\} = \mathrm{span}_{E^p}\{1, a_1, \ldots, a_n\}$, which holds if and only if $a_i \in E^p$ for all $1 \leq i \leq n$. As $E^p = F^p(b_1, \ldots, b_m)$, the latter condition is equivalent to $F^p(a_1, \ldots, a_n) \subseteq F^p(b_1, \ldots, b_m)$. But by Lemma 2.4 $F^p(a_1, \ldots, a_n) = N_F(\varphi)$, so we are done.

If $\{b_1, \ldots, b_m\} \subseteq \{a_1, \ldots, a_n\}$, then $F^p(b_1, \ldots, b_m) \subseteq N_F(\varphi)$, and the claim follows by the previous case. $\qquad\square$

Note that $N_F(\varphi)$ is a finite field extension of F^p, and hence there always exists a finite p-basis $\{b_1, \ldots, b_n\}$ of $N_F(\varphi)$ over F, i.e., $N_F(\varphi) = F^p(b_1, \ldots, b_n)$ with $[F^p(b_1, \ldots, b_n) : F^p] = p^n$. Then p^n is called the *norm degree of φ over F* and denoted by $\mathrm{ndeg}_F \varphi$. There is a relation between the norm degree and the dimension of a p-form.

Lemma 2.6 [Hoffmann 2004, Proposition 4.8]. *Let φ be a nonzero p-form with $\mathrm{ndeg}_F \varphi = p^n$. Then $n + 1 \leq \dim \varphi_{an} \leq p^n$.*

By [Gille and Szamuely 2006, Corollary A.8.9], any p-generating set contains a p-basis. Therefore, part (i) of Lemma 2.4 implies that:

Lemma 2.7. *Assume that $\varphi \simeq \langle a_0, \ldots, a_n \rangle$ for some $n \geq 1$ and $a_0, \ldots, a_n \in F$ with $a_0 \neq 0$. Moreover, suppose that $\mathrm{ndeg}_F \varphi = p^k$. Then there exists a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $\left\{ \frac{a_{i_1}}{a_0}, \ldots, \frac{a_{i_k}}{a_0} \right\}$ is a p-basis of $N_F(\varphi)$ over F.*

In particular, if $1 \in D_F^*(\varphi)$, then there exist $b_1, \ldots, b_n \in F$ such that $\varphi \simeq \langle 1, b_1, \ldots, b_n \rangle$ and $N_F(\varphi) = F^p(b_1, \ldots, b_k)$.

Let φ be a p-form over F with $\mathrm{ndeg}_F \varphi = p^n$ and $N_F(\varphi) = F^p(a_1, \ldots, a_n)$ (so, in particular, $\{a_1, \ldots, a_n\}$ is p-independent over F). Then we define the *norm form* of φ over F, denoted by $\hat{v}_F(\varphi)$, as the (necessarily anisotropic) quasi-Pfister form $\langle\!\langle a_1, \ldots, a_n \rangle\!\rangle$. It follows that $\hat{v}_F(\varphi)$ is the smallest quasi-Pfister form that contains a scalar multiple of φ_{an} as its subform. In particular, if φ_{an} is a quasi-Pfister form itself, then we have $\hat{v}_F(\varphi) \simeq \varphi_{\mathrm{an}}$.

Isotropy. Let φ be an anisotropic p-form over F, and let E/F be a field extension. If E/F is purely transcendental or separable, then φ_E remains anisotropic [Hoffmann 2004, Proposition 5.3].

Lemma 2.8 [Scully 2016, Lemma 2.27]. *Let φ be a p-form over F and let $a \in F \setminus F^p$. Then*:

(i) $D_{F(\sqrt[p]{a})}(\varphi) = D_F(\langle\!\langle a \rangle\!\rangle \otimes \varphi) = \sum_{i=0}^{p-1} a^i D_F(\varphi)$.

(ii) $\mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) = \frac{1}{p} \mathfrak{i}_{\mathrm{d}}(\langle\!\langle a \rangle\!\rangle \otimes \varphi)$.

(iii) *If φ is anisotropic and $\varphi_{F(\sqrt[p]{a})}$ is isotropic, then $a \in N_F(\varphi)$.*

(iv) $\dim(\varphi_{F(\sqrt[p]{a})})_{\mathrm{an}} \geq \frac{1}{p} \dim \varphi_{\mathrm{an}}$.

(v) *Equality holds in (iv) if and only if there exists a p-form γ over F such that $\varphi_{\mathrm{an}} \simeq \langle\!\langle a \rangle\!\rangle \otimes \gamma$.*

We want to point out that $N_F(\varphi)$ is the smallest field extension of F^p with property (iii) of Lemma 2.8. More precisely, we have:

Proposition 2.9. *Let φ be an anisotropic p-form over F and E/F^p be a field extension such that the following holds for any $a \in F^*$:*

$$(\divideontimes) \qquad\qquad \mathfrak{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) > 0 \Rightarrow a \in E.$$

Then $N_F(\varphi) \subseteq E$.

Proof. Let $\mathrm{ndeg}_F \varphi = p^k$. Since neither the isotropy nor the norm field depends on the choice of the representative of the similarity class, we can assume $1 \in D_F^*(\varphi)$. Invoking Lemma 2.7, we find $b_1, \ldots, b_n \in F$ with $n \geq k$ such that $\varphi \simeq \langle 1, b_1, \ldots, b_n \rangle$ and $\{b_1, \ldots, b_k\}$ is a p-basis of $N_F(\varphi)$ over F. Since $\varphi_{F(\sqrt[p]{b_i})}$ is obviously isotropic for each $1 \leq i \leq k$, we get by the assumption $(\divideontimes)$ that $b_1, \ldots, b_k \in E$. Since E is a field containing F^p, it follows that $N_F(\varphi) = F^p(b_1, \ldots, b_k) \subseteq E$. $\qquad\square$

Lemma 2.10. *Let φ be an anisotropic quasi-Pfister form or an anisotropic quasi-Pfister neighbor over F, and let $x \in F^*$. Then $\varphi \otimes \langle\langle x \rangle\rangle$ is isotropic if and only if $x \in N_F(\varphi)$.*

Proof. First, we prove the lemma in the case when $\varphi \simeq \pi$ is a quasi-Pfister form. Assume that $\pi \simeq \langle\langle a_1, \ldots, a_n \rangle\rangle$, i.e., $N_F(\pi) = F^p(a_1, \ldots, a_n)$. Note that π is anisotropic. Then the form $\pi \otimes \langle\langle x \rangle\rangle$ is isotropic if and only if $(\pi \otimes \langle\langle x \rangle\rangle)_{\mathrm{an}} \simeq \pi$ if and only if $F^p(a_1, \ldots, a_n) = F^p(a_1, \ldots, a_n, x)$ if and only if $x \in N_F(\pi)$.

Now suppose that φ is a quasi-Pfister neighbor. Then $c\varphi \subseteq \pi$ for $\pi \simeq \hat{v}_F(\varphi)$ and some $c \in F^*$. It follows that $c\varphi \otimes \langle\langle x \rangle\rangle$ is a quasi-Pfister neighbor of $\pi \otimes \langle\langle x \rangle\rangle$, and hence $c\varphi \otimes \langle\langle x \rangle\rangle$ is isotropic if and only if $\pi \otimes \langle\langle x \rangle\rangle$ is isotropic (Lemma 2.3). By the first part of the proof, the latter is equivalent to $x \in N_F(\pi)$. Since we have $N_F(\pi) = N_F(c\varphi) = N_F(\varphi)$, the claim follows. □

Minimal forms. In this paper, we will work with minimal forms; they are, in a way, a counterpart to quasi-Pfister forms.

Definition 2.11. Let φ be an anisotropic p-form over F. We call φ *minimal over F* if $\mathrm{ndeg}_F \varphi = p^{\dim \varphi - 1}$.

Note that any one-dimensional anisotropic p-form is minimal. Minimal p-forms of dimension at least two are described by the following lemma:

Lemma 2.12 [Zemková 2024, Lemma 2.16]. *Let φ, ψ be p-forms over F of dimension at least two.*

  (i) *The p-form $\langle 1, a_1, \ldots, a_n \rangle$ is minimal over F if and only if the set $\{a_1, \ldots, a_n\}$ is p-independent over F.*

 (ii) *Minimality is not invariant under field extensions.*

(iii) *If φ is minimal over F and $\psi \overset{\mathrm{sim}}{\sim} \varphi$, then ψ is minimal over F.*

(iv) *If φ is minimal over F and $\psi \subseteq c\varphi$ for some $c \in F^*$, then ψ is minimal over F.*

 (v) *If $\mathrm{ndeg}_F \varphi = p^k$ with $k \geq 1$, then φ contains a minimal subform of dimension $k + 1$.*

We can characterize elements represented by a given minimal p-form.

Lemma 2.13. *Let φ be a minimal p-form over F with $1 \in D_F(\varphi)$. Let $n \geq 2$ and $\{b_1, \ldots, b_n\} \subseteq D_F(\varphi)$ be p-independent over F. Denote*

$$S = \{ \lambda : \{1, \ldots, n\} \to \{0, \ldots, p-1\} \},$$

and consider

$$\beta = \sum_{\lambda \in S} x_\lambda^p \prod_{i=1}^n b_i^{\lambda(i)}$$

with $x_\lambda \in F$ for all $\lambda \in S$. Then the following are equivalent:

(i) $\beta \in D_F(\varphi)$.

(ii) $x_\lambda = 0$ *for all* $\lambda \in S$ *such that* $\sum_{i=1}^n \lambda(i) \geq 2$, *i.e.,* $\beta = y_0^p + \sum_{i=1}^n b_i y_i^p$ *for some* $y_i \in F$, $0 \leq i \leq n$.

Proof. Write $\tau \simeq \langle 1, b_1, \ldots, b_n \rangle$. Then (ii) holds if and only if $\beta \in D_F(\tau)$. Since $\tau \subseteq \varphi$ by the assumptions, the implication (ii) $\Rightarrow$ (i) is obvious.

To prove (i) $\Rightarrow$ (ii), suppose that (ii) does not hold (e.g., $\beta = b_1 b_2$). Then we can see that $\beta \notin D_F(\tau)$. Hence, the p-form $\tau \perp \langle \beta \rangle$ is anisotropic. On the other hand, $\beta \in F^p(b_1, \ldots, b_n)$, so $\tau \perp \langle \beta \rangle$ is not minimal over F by part (i) of Lemma 2.12. If $\beta \in D_F(\varphi)$, then $\tau \perp \langle \beta \rangle \subseteq \varphi$, in which case φ could not be minimal over F by part (iv) of the same lemma, which is a contradiction; therefore, $\beta \notin D_F(\varphi)$. $\square$

Vishik equivalence. We introduce Vishik equivalence of p-forms. The name originates from *Vishik criterion* that, in characteristic different from 2, gives an algebraic characterization of the motivic equivalence (see [Hoffmann 2015]).

Definition 2.14. Let φ, ψ be p-forms over F. We say that they are *Vishik equivalent* and write $\varphi \overset{v}{\sim} \psi$ if $\dim \varphi = \dim \psi$ and the following holds:

$$(v) \qquad\qquad \mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\psi_E) \quad \text{for all } E/F.$$

Remark 2.15. For quadratic forms φ and ψ over F (not necessarily totally singular) such that $\dim \varphi = \dim \psi$, we define Vishik equivalence as (see [Zemková 2022, Definition 1.57])

$$\varphi \overset{v}{\sim} \psi \quad \overset{\text{def}}{\Longleftrightarrow} \quad \mathfrak{i}_{\mathrm{W}}(\varphi_E) = \mathfrak{i}_{\mathrm{W}}(\psi_E) \text{ and } \mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\psi_E) \qquad \text{for all } E/F.$$

Note that this definition agree with Definition 2.14 in the special case of totally singular quadratic forms.

Lemma 2.16. *Let* φ, ψ *be p-forms over F such that* $\varphi \overset{v}{\sim} \psi$, *and let E/F be a field extension. Then* $(\varphi_E)_{\mathrm{an}} \overset{v}{\sim} (\psi_E)_{\mathrm{an}}$. *In particular,* $\varphi_{\mathrm{an}} \overset{v}{\sim} \psi_{\mathrm{an}}$.

Proof. Denote $\varphi' = (\varphi_E)_{\mathrm{an}}$ and $\psi' = (\psi_E)_{\mathrm{an}}$. Let K/E be another field extension. It holds that $\mathfrak{i}_{\mathrm{d}}(\varphi_K) = \mathfrak{i}_{\mathrm{d}}(\psi_K)$ and $\mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\psi_E)$. Thus,

$$\mathfrak{i}_{\mathrm{d}}(\varphi'_K) = \mathfrak{i}_{\mathrm{d}}(\varphi_K) - \mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\psi_K) - \mathfrak{i}_{\mathrm{d}}(\psi_E) = \mathfrak{i}_{\mathrm{d}}(\psi'_K).$$

Therefore, $\varphi' \overset{v}{\sim} \psi'$. $\square$

Since similar p-forms must be of the same dimension and have the same defects over any field, we get:

Lemma 2.17. *If* $\varphi \overset{\text{sim}}{\sim} \psi$, *then* $\varphi \overset{v}{\sim} \psi$.

Among others, Vishik equivalence implies that a p-form is isotropic over the function field of any Vishik equivalent p-form.

Lemma 2.18. *Let φ, ψ be p-forms over F of dimension at least two. If $\varphi \overset{v}{\sim} \psi$, then $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic.*

Proof. Since $\varphi_{F(\varphi)}$ is isotropic and we know that $i_d(\psi_{F(\varphi)}) = i_d(\varphi_{F(\varphi)})$, we get that $\psi_{F(\varphi)}$ is isotropic. Symmetrically, $\varphi_{F(\psi)}$ is isotropic. $\square$

Lemma 2.19. *Let φ, ψ be anisotropic p-forms over F such that $\varphi \overset{v}{\sim} \psi$. Then $\hat{v}_F(\varphi) \simeq \hat{v}_F(\psi)$, $N_F(\varphi) = N_F(\psi)$ and $\mathrm{ndeg}_F(\varphi) = \mathrm{ndeg}_F(\psi)$.*

Proof. The claim is trivial if $\dim \varphi = \dim \psi = 1$. Assume that the dimension is at least two. By Lemma 2.18, we know that $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic. Thus, this is a consequence of [Hoffmann 2004, Lemma 7.12]. $\square$

3. Vishik equivalence on p-forms

As an easy consequence of some know results, we get that Question Q has a positive answer for quasi-Pfister forms and quasi-Pfister neighbors of codimension one.

Proposition 3.1. *Let φ, ψ be anisotropic p-forms such that $\varphi \overset{v}{\sim} \psi$. If φ is a quasi-Pfister form or a quasi-Pfister neighbor of codimension one, then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. By Lemma 2.19, there exists a quasi-Pfister form π over F such that $\pi \simeq \hat{v}_F(\varphi) \simeq \hat{v}_F(\psi)$.

If φ is a quasi-Pfister form, then we can find $c, d \in F^*$ such that $c\varphi \simeq \pi \supseteq d\psi$; thus, $\varphi \overset{\mathrm{sim}}{\sim} \psi$ for dimension reasons.

Now assume that φ is a quasi-Pfister neighbor of π of codimension 1. Then $\dim \psi = \dim \varphi = \dim \pi - 1$. Moreover, $c\psi \subseteq \hat{v}_F(\psi) \simeq \pi$ for some $c \in F^*$. So, ψ is a quasi-Pfister neighbor of π of codimension 1. Using [Hoffmann 2004, Proposition 4.15], any two such p-forms are similar, i.e., $\varphi \overset{\mathrm{sim}}{\sim} \psi$ as claimed. $\square$

Weak Vishik equivalence. Proving that two p-forms are Vishik equivalent might be difficult. Moreover, we do not always need the full strength of the Vishik equivalence; therefore, we define a weaker version.

Definition 3.2. Let φ, ψ be p-forms over F. We say that they are *weakly Vishik equivalent* and write $\varphi \overset{v_0}{\sim} \psi$ if $\dim \varphi = \dim \psi$ and the following holds:

$$(v_0) \qquad\qquad i_d(\varphi_{F(\sqrt[p]{a})}) = i_d(\psi_{F(\sqrt[p]{a})}) \quad \text{for all } a \in F.$$

The weak Vishik equivalence has three (rather obvious) properties.

Lemma 3.3. *Let φ, ψ be p-forms over F. Then*:

(i) *If $\varphi \overset{v_0}{\sim} \psi$, then $i_d(\varphi) = i_d(\psi)$.*

(ii) *$\varphi \overset{v_0}{\sim} \psi$ if and only if $\varphi_{\mathrm{an}} \overset{v_0}{\sim} \psi_{\mathrm{an}}$.*

(iii) *If $\varphi \overset{v}{\sim} \psi$, then $\varphi \overset{v_0}{\sim} \psi$.*

Proof. Part (i) follows directly from the definition, because we include the equality of the defects over the field $F(\sqrt[p]{1}) \simeq F$. Part (ii) is then an easy consequence of part (i). Finally, part (iii) is trivial. $\qquad\square$

To prove that Vishik equivalent forms have the same norm field, we used the isotropy over the function fields of each other (see Lemma 2.19). Another possibility to determine the norm field is to use Proposition 2.5, which only needs one particular purely inseparable field extension of exponent one. But for weakly Vishik equivalent forms, neither of these is at our disposal; nevertheless, thanks to Proposition 2.9, we are still able to prove that they have the same norm field.

Proposition 3.4. *Let φ, ψ be p-forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Then $\hat{v}_F(\varphi) \simeq \hat{v}_F(\psi)$, $N_F(\varphi) = N_F(\psi)$ and $\mathrm{ndeg}_F(\varphi) = \mathrm{ndeg}_F(\psi)$.*

Proof. We prove only $N_F(\varphi) = N_F(\psi)$, then the rest follows. Invoking Lemma 3.3 and recalling that the norm field takes into account only the anisotropic part of a p-form, we can assume that φ and ψ are anisotropic. By the definition of the weak Vishik equivalence and by part (iii) of Lemma 2.8, we have for any $a \in F$:

$$\mathfrak{i}_d(\varphi_{F(\sqrt[p]{a})}) > 0 \quad \Rightarrow \quad \mathfrak{i}_d(\psi_{F(\sqrt[p]{a})}) > 0 \quad \Rightarrow \quad a \in N_F(\psi).$$

We can apply Proposition 2.9 on the p-form φ and the field $E = N_F(\psi)$; we obtain $N_F(\varphi) \subseteq N_F(\psi)$. By the symmetry of the argument, we get $N_F(\varphi) = N_F(\psi)$. $\quad\square$

Since (weakly) Vishik equivalent forms are always of the same dimension, we immediately get:

Corollary 3.5. *Let φ, ψ be p-forms over F such that $\varphi \overset{v_0}{\sim} \psi$.*

(i) *If φ is minimal over F, then ψ is also minimal over F.*

(ii) *If φ is a quasi-Pfister form or a quasi-Pfister neighbor of codimension one over F, then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. By Proposition 3.4, we have $\mathrm{ndeg}_F \varphi = \mathrm{ndeg}_F \psi$; since $\dim \varphi = \dim \psi$, part (i) follows immediately. To prove part (ii), note that $\hat{v}_F(\varphi) \simeq \hat{v}_F(\psi)$, and so we can proceed as in the proof of Proposition 3.1. $\qquad\square$

Similarity factors. In this subsection, we will show that weak Vishik equivalence (and hence Vishik equivalence as well) preserves divisibility by quasi-Pfister forms. In particular, we will prove that if $\varphi \overset{v_0}{\sim} \psi$ for some p-forms φ, ψ with φ divisible by a quasi-Pfister form π, then ψ is divisible by π, too.

Let φ be a p-form defined over F. Recall that we write

$$G_F(\varphi) = \{x \in F^* \mid x\varphi \simeq \varphi\} \cup \{0\},$$

we call the nonzero elements of this set the *similarity factors* of φ. As observed in [Hoffmann 2004, Proposition 6.4], the set $G_F(\varphi)$ together with the usual operations

is a finite field extension of F^p inside $N_F(\varphi)$; in particular, there exists a p-independent set $\{a_1, \ldots, a_m\} \subseteq F^*$ such that $G_F(\varphi) = F^p(a_1, \ldots, a_m)$. We denote $\hat{\sigma}_F(\varphi) \simeq \langle\!\langle a_1, \ldots, a_m \rangle\!\rangle$ and call it the *similarity form* of φ over F. Moreover, again by [Hoffmann 2004, Proposition 6.4], there exists a p-form γ over F such that $\varphi_{\mathrm{an}} \simeq \hat{\sigma}_F(\varphi) \otimes \gamma$. It holds that $D_F(\hat{\sigma}_F(\varphi)) = G_F(\varphi)$.

We will show that weak Vishik equivalent p-forms have the same similarity factors. But first, we need a simple lemma.

Lemma 3.6. *Let π_1, π_2 be anisotropic quasi-Pfister forms. Then $\pi_1 \subseteq \pi_2$ if and only if $\pi_2 \simeq \pi_1 \otimes \gamma$ for some p-form γ over F. In that case, γ can be chosen to be a quasi-Pfister form.*

Proof. Write $\pi_1 \simeq \langle\!\langle a_1, \ldots, a_r \rangle\!\rangle$ and $\pi_2 \simeq \langle\!\langle b_1, \ldots, b_s \rangle\!\rangle$, which means that we have $N_F(\pi_1) = F^p(a_1, \ldots, a_r)$ and $N_F(\pi_2) = F^p(b_1, \ldots, b_s)$. By [Hoffmann 2004, Proposition 4.6], there is a bijection between finite field extensions of F^p inside F and Pfister p-forms over F, which implies that $\pi_1 \subseteq \pi_2$ if and only if $N_F(\pi_1) \subseteq N_F(\pi_2)$.

If this holds, then we can extend $\{a_1, \ldots, a_r\}$ to a p-basis of $F^p(b_1, \ldots, b_s)$; thus, $F^p(b_1, \ldots, b_s) = F^p(a_1, \ldots, a_r, a_{r+1}, \ldots, a_s)$ for some $a_{r+1}, \ldots, a_s \in F^*$, and so $\langle\!\langle b_1, \ldots, b_s \rangle\!\rangle \simeq \langle\!\langle a_1, \ldots, a_r \rangle\!\rangle \otimes \langle\!\langle a_{r+1}, \ldots, a_s \rangle\!\rangle$.

On the other hand, assume $\pi_2 \simeq \pi_1 \otimes \gamma$ for a p-form γ. If $1 \in D_F^*(\gamma)$, then we can write $\gamma \simeq \langle 1 \rangle \perp \gamma'$ for a suitable p-form γ'; in that case, $\pi_2 \simeq \pi_1 \perp \pi_1 \otimes \gamma'$ and we are done. More generally, if $c \in D_F^*(\gamma)$, then $c \in D_F^*(\pi_2) = G_F^*(\pi_2)$. Now $\pi_2 \simeq c\pi_2 \simeq \pi_1 \otimes c\gamma$ with $1 \in D_F^*(c\gamma)$, so we are done by the previous case. $\square$

Theorem 3.7. *Let φ, ψ be anisotropic p-forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Then $G_F(\varphi) = G_F(\psi)$.*

Proof. The inclusion $F^p \subseteq G_F(\psi)$ is obvious. Hence, pick $a \in G_F(\varphi) \setminus F^p$. Since $G_F(\varphi) = D_F(\hat{\sigma}_F(\varphi))$ is a field, we get that $a^2, \ldots, a^{p-1} \in D_F(\hat{\sigma}_F(\varphi))$ and $1 \in D_F(\hat{\sigma}_F(\varphi))$. As the p-form $\langle 1, a, \ldots, a^{p-1} \rangle \simeq \langle\!\langle a \rangle\!\rangle$ is anisotropic, we get $\langle\!\langle a \rangle\!\rangle \subseteq \hat{\sigma}_F(\varphi)$ by Lemma 2.2; therefore, $\langle\!\langle a \rangle\!\rangle$ divides $\hat{\sigma}_F(\varphi)$ by Lemma 3.6. Since further $\hat{\sigma}_F(\varphi)$ divides φ, we can find a p-form φ' defined over F such that $\varphi \simeq \langle\!\langle a \rangle\!\rangle \otimes \varphi'$.

Set $E = F(\sqrt[p]{a})$; it holds that $(\varphi_E)_{\mathrm{an}} \simeq (\varphi_E')_{\mathrm{an}}$. So, we have $\dim(\varphi_E')_{\mathrm{an}} \leq \frac{1}{p} \dim \varphi$. But the reverse inequality is true by Lemma 2.8; hence, φ_E' is anisotropic and $\dim(\varphi_E)_{\mathrm{an}} = \frac{1}{p} \dim \varphi$.

Since $\varphi \overset{v_0}{\sim} \psi$, we have $\dim(\psi_E)_{\mathrm{an}} = \frac{1}{p} \dim \psi$. Let ψ' be a p-form over F such that $\psi_E' \simeq (\psi_E)_{\mathrm{an}}$. As it holds that

$$D_F(\psi) \subseteq D_E(\psi) = D_E(\psi') = D_F(\langle\!\langle a \rangle\!\rangle \otimes \psi'),$$

we get $\psi \subseteq \langle\langle a \rangle\rangle \otimes \psi'$ by Lemma 2.2. Comparing the dimensions implies that $\psi \simeq \langle\langle a \rangle\rangle \otimes \psi'$, and hence $a \in G_F(\psi)$.

All in all, we have proved $G_F(\varphi) \subseteq G_F(\psi)$. The other inclusion follows by the symmetry of the argument. $\qquad \square$

Lemma 3.8. *Let τ be a p-form defined over F and K/F be a field extension such that $K^p \subseteq G_F(\tau)$. Let φ, ψ be p-forms defined over F, anisotropic over K and such that $\varphi_K \simeq \psi_K$. Then $\varphi \otimes \tau \simeq \psi \otimes \tau$ over F.*

Proof. Let $\varphi \simeq \langle b_1, \ldots, b_m \rangle$ and $\psi \simeq \langle c_1, \ldots, c_m \rangle$. It follows from the assumptions that $\{b_1, \ldots, b_m\}$ and $\{c_1, \ldots, c_m\}$ are two bases of the same K^p-vector space. Recall that we can get one basis from the other by a finite series of operations of the following type: an exchange of two basis elements, scalar multiplication of one basis element, and adding one basis element to another basis element. Thus, it is sufficient to show that the following hold for any $0 \le i, j \le m$, $i \ne j$, and $a \in K^p$:

(i) $\langle b_0, \ldots, b_i, \ldots, b_j, \ldots, b_m \rangle \otimes \tau \simeq \langle b_0, \ldots, b_j, \ldots, b_i, \ldots, b_m \rangle \otimes \tau.$

(ii) $\langle b_0, \ldots, b_i, \ldots, b_m \rangle \otimes \tau \simeq \langle b_0, \ldots, ab_i, \ldots, b_m \rangle \otimes \tau.$

(iii) $\langle b_0, \ldots, b_i, \ldots, b_j, \ldots, b_m \rangle \otimes \tau \simeq \langle b_0, \ldots, b_i + b_j, \ldots, b_j, \ldots, b_m \rangle \otimes \tau.$

Part (i) is obvious. Part (ii) follows from the fact that $b_i \tau \simeq ab_i \tau$ since $a \in G_F(\tau)$. To prove part (iii), note that $b_i \tau \perp b_j \tau \simeq (b_i + b_j) \tau \perp b_j \tau$. $\qquad \square$

Proposition 3.9. *Let $\pi \simeq \langle\langle a_1, \ldots, a_n \rangle\rangle$ be an anisotropic quasi-Pfister form over F and $K = F(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_n})$. Assume that $\varphi \simeq \pi \otimes \varphi'$ and $\psi \simeq \pi \otimes \psi'$ are anisotropic p-forms over F such that $\varphi'_K \overset{\text{sim}}{\sim} \psi'_K$. Then $\varphi \overset{\text{sim}}{\sim} \psi$.*

Proof. Without loss of generality, assume that $1 \in D_F(\varphi') \cap D_F(\psi')$. Write $\varphi' \simeq \langle 1, b_1, \ldots, b_m \rangle$, and let $c \in K^*$ be such that $c\psi'_K \simeq \varphi'_K$. Then

$$c \in D_K(\varphi') = \operatorname{span}_{K^p}\{1, b_1, \ldots, b_m\} \subseteq F,$$

thus, $c\psi'$ is defined over F. By [Zemková 2024, Theorem 3.3], we have that $p^n \mathfrak{i}_\mathrm{d}(\varphi'_K) = \mathfrak{i}_\mathrm{d}(\varphi) = 0$, and hence φ'_K is anisotropic; analogously, we get that $c\psi'_K$ is anisotropic. Now Lemma 3.8 implies that $\pi \otimes \varphi' \simeq \pi \otimes c\psi'$. As $\pi \otimes c\psi' \simeq c(\pi \otimes \psi')$, the claim follows. $\qquad \square$

Remark 3.10. Theorem 3.7 and Proposition 3.9 can be used for a simplification of the problem, whether Vishik equivalence implies the similarity. Namely, we can restrict ourselves to the case of forms with the similarity factors F^p: Let φ, ψ be two anisotropic p-forms over F such that $\varphi \overset{v}{\sim} \psi$. Since $G_F(\varphi) = G_F(\psi)$ by Theorem 3.7, we can write $\varphi \simeq \pi \otimes \varphi'$ and $\psi \simeq \pi \otimes \psi'$ with the quasi-Pfister form $\pi \simeq \hat{\sigma}_F(\varphi) \simeq \hat{\sigma}_F(\psi)$ and some anisotropic p-forms φ', ψ' defined over F. Let $\pi \simeq \langle\langle a_1, \ldots, a_n \rangle\rangle$, and put $K = F(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_n})$. Then φ'_K, ψ'_K are anisotropic by [Hoffmann 2004, Proposition 5.7]. Hence $(\varphi_K)_\mathrm{an} \simeq \varphi'_K$ and $(\psi_K)_\mathrm{an} \simeq \psi'_K$,

and we get $\varphi'_K \overset{v}{\sim} \psi'_K$ by Lemma 2.16 (note that to apply this lemma, we need the full strength of the Vishik equivalence, the weak Vishik equivalence does not have to be sufficient here). If we knew that $\varphi'_K \overset{\text{sim}}{\sim} \psi'_K$, it would follow by Proposition 3.9 that $\varphi \overset{\text{sim}}{\sim} \psi$.

Counterexamples. Recall that by Proposition 3.1, Question Q has a positive answer for quasi-Pfister forms and quasi-Pfister neighbors of codimension one (i.e., Vishik equivalence is sufficient for the p-forms to be similar). It would be nice if Question Q had a positive answer for all p-forms. Unfortunately, this is not true in general — as we will see, it is not even true for all quasi-Pfister neighbors. We will provide two examples of pairs of p-forms that are Vishik equivalent but not similar. However, both of the counterexamples have two things in common: First, the considered p-forms are subforms of $\langle\!\langle a \rangle\!\rangle$, and so they have norm degree one. Second, they do not work for $p = 2$ and $p = 3$. We will focus on the characteristic two case in Section 4, but the case $p = 3$ remains open as well as p-forms of higher norm degrees.

Before we can get to the counterexamples, we need some lemmas.

Lemma 3.11. *Let $1 \le k, l \le p - 1$, $a \in F \setminus F^p$, and let E/F be a field extension. Then the following are equivalent*:

  (i) $\langle 1, a^k \rangle$ *is isotropic over E.*

  (ii) $\langle 1, a^l \rangle$ *is isotropic over E.*

(iii) $a^k \in E^p$.

(iv) $a^l \in E^p$.

Proof. Assume that $\langle 1, a^k \rangle$ is isotropic over E. This is equivalent to the existence of $x, y \in E$, at least one (and hence both) of them nonzero, such that $x^p + a^k y^p = 0$, which is equivalent to $a^k = \frac{(-x)^p}{y^p} \in E^p$. This proves both (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv).

Now let $1 \le t \le p - 1$ be such that $kt \equiv l \pmod p$. If $a^k \in E^p$, then $a^{kt} \in E^p$, and hence also $a^l \in E^p$. This proves (iii) $\Rightarrow$ (iv) and, by the symmetry of the argument, also (iv) $\Rightarrow$ (iii). $\square$

Lemma 3.12. *Let $a \in F \setminus F^p$ and $k, l \in \mathbb{Z}$ with $k, l \not\equiv 0 \pmod p$. Then we have $\langle 1, a^k \rangle \overset{\text{sim}}{\sim} \langle 1, a^l \rangle$ if and only if $k \equiv \pm l \pmod p$.*

Proof. If $k \equiv l \pmod p$, then $k = pm + l$ for some $m \in \mathbb{Z}$, and so we have $\langle 1, a^k \rangle \simeq \langle 1, a^{pm+l} \rangle \simeq \langle 1, a^l \rangle$. If $k \equiv -l \pmod p$, then we can write $k = pn - l$ for some $n \in \mathbb{Z}$, and then we have

$$\langle 1, a^k \rangle = \langle 1, a^{pn-l} \rangle \simeq \langle 1, a^{-l} \rangle \overset{\text{sim}}{\sim} a^l \langle 1, a^{-l} \rangle \simeq \langle a^l, 1 \rangle \simeq \langle 1, a^l \rangle.$$

To prove the opposite direction, assume that $\langle 1, a^k \rangle \overset{\text{sim}}{\sim} \langle 1, a^l \rangle$, i.e., there exists $c \in F^*$ such that $c\langle 1, a^k \rangle \simeq \langle 1, a^l \rangle$. Then necessarily $c \in D_F(\langle 1, a^l \rangle)$, so we can find $x, y \in F$ such that $c = x^p + a^l y^p$. On the other hand, $1 \in D_F(c\langle 1, a^k \rangle)$, and thus $1 = cu^p + ca^k v^p$ for some $u, v \in F$. Putting these together, we have

$$1 = (x^p + a^l y^p) u^p + (x^p + a^l y^p) a^k u^p = (xu)^p + a^l (yu)^p + a^k (xv)^p + a^{k+l} (yv)^p.$$

Suppose $k \not\equiv \pm l \pmod{p}$. Then $\tau = \langle 1, a^k, a^l, a^{k+l} \rangle$ is a subform of $\langle\langle a \rangle\rangle$, and hence anisotropic. Thus, τ represents every element of $D_F(\tau)$ uniquely, so we get $xu \neq 0$ and $yu = xv = yv = 0$. This implies $x, u \neq 0$ and $y = v = 0$; hence, $c = x^p$ and $\langle 1, a^k \rangle \simeq \langle 1, a^l \rangle$. Thus, $a^l \in D_F(\langle 1, a^k \rangle)$, which is impossible for $l \not\equiv 0, k \pmod{p}$. $\qquad\square$

Lemma 3.13. *Let $a \in F \setminus F^p$ and φ, ψ be quasi-Pfister neighbors of $\langle\langle a \rangle\rangle$ of the same dimension. Then we have $\varphi \overset{v}{\sim} \psi$.*

Proof. Obviously, both φ and ψ are anisotropic. Let E/F be any field extension; it holds $(\varphi_E)_{\text{an}}, (\psi_E)_{\text{an}} \subseteq (\langle\langle a \rangle\rangle_E)_{\text{an}}$. Then either $\langle\langle a \rangle\rangle_E$ is anisotropic, and hence both φ_E and ψ_E are anisotropic, or we have $(\langle\langle a \rangle\rangle_E)_{\text{an}} \simeq \langle 1 \rangle_E$ by Lemma 2.3, in which case $(\varphi_E)_{\text{an}} \simeq \langle 1 \rangle_E \simeq (\psi_E)_{\text{an}}$. Consequently, $\varphi \overset{v}{\sim} \psi$. $\qquad\square$

Example 3.14. Let $p > 3$, and let $\varphi \simeq \langle 1, a \rangle$, $\psi \simeq \langle 1, a^2 \rangle$ be p-forms with $a \in F \setminus F^p$. Note that φ and ψ are quasi-Pfister neighbors of $\langle\langle a \rangle\rangle$; therefore, $\varphi \overset{v}{\sim} \psi$ by Lemma 3.13. On the other hand, Lemma 3.12 ensures that φ and ψ are not similar for any $p > 3$.

Note that the problems with characteristics two and three are different. If $p = 2$, then φ is anisotropic while ψ is isotropic. If $p = 3$, it holds that $a^2 \varphi \simeq \psi$.

Example 3.15. Let $p = 5$ and $a \in F \setminus F^5$. Set $\pi \simeq \langle 1, a, a^2, a^3, a^4 \rangle \simeq \langle\langle a \rangle\rangle$, $\varphi \simeq \langle 1, a, a^2 \rangle$ and $\psi \simeq \langle 1, a, a^3 \rangle$. It follows by Lemma 3.13 that $\varphi \overset{v}{\sim} \psi$.

Assume that $c \in F$ is such that $c\psi \simeq \varphi$. Note that

$$D_F(\varphi) = \{x_0^5 + ax_1^5 + a^2 x_2^5 \mid x_i \in F, \ 0 \leq i \leq 2\},$$
$$D_F(\pi) = \{x_0^5 + ax_1^5 + a^2 x_2^5 + a^3 x_3^5 + a^4 x_4^5 \mid x_i \in F, \ 0 \leq i \leq 4\},$$

and the expression of any element of $D_F(\varphi)$ (resp. $D_F(\pi)$) is unique thanks to the anisotropy of φ (resp. π). In particular, φ does not represent any term of the form $a^3 x^5$ or $a^4 x^5$ with $x \in F^*$.

As $c \in D_F(\varphi)$, we can write $c = x^5 + ay^5 + a^2 z^5$ for some $x, y, z \in F$. Since $ca = ax^5 + a^2 y^5 + a^3 z^5 \in D_F(\varphi)$, it follows that $z = 0$. Furthermore, we have that $ca^3 = a^3 x^5 + a^4 y^5 + (az)^5 \in D_F(\varphi)$ implies that $x = y = 0$. But then $c = 0$, which is absurd. Therefore, φ and ψ are not similar.

4. Vishik equivalence on totally singular quadratic forms

In this section, we will answer Question Q for some families of totally singular
quadratic forms. Note that totally singular quadratic forms are the special case of
p-forms with $p = 2$, so we will use some results from the previous section. In
particular, by Proposition 3.1, we already know that Question Q has a positive answer
in case of quasi-Pfister forms and quasi-Pfister neighbors of codimension one.

There is one tool which is not available for $p > 2$ (at least not this strong version
of it, see the note behind Question 4.7 in [Scully 2013]) that we will use repeatedly:

Proposition 4.1 [Laghribi 2006, Proposition 2.11]. *Let $a \in F \setminus F^2$ and let φ be an
anisotropic totally singular quadratic form over F with $\dim \varphi \geq 2$. If φ becomes
isotropic over $E = F(\sqrt{a})$, then there exists a totally singular quadratic form τ
over F such that $\dim \tau = i_d(\varphi_E)$ and $\tau \otimes \langle 1, a \rangle \subseteq \varphi$.*

Minimal quadratic forms. Recall that a minimal quadratic form is an anisotropic
totally singular quadratic form φ over F such that $\operatorname{ndeg}_F \varphi = 2^{\dim \varphi - 1}$.

We start with some preparatory lemmas on forms of dimensions two and three.
Note that all anisotropic totally singular quadratic forms of dimensions two and
three are minimal by Lemma 2.6.

Lemma 4.2. *Let $b, c \in F \setminus F^2$. Then $c \in D_F(\langle 1, b \rangle)$ if and only if $b \in D_F(\langle 1, c \rangle)$
if and only if $\langle 1, b \rangle \simeq \langle 1, c \rangle$.*

Proof. $c \in D_F(\langle 1, b \rangle)$ holds if we can find $x, y \in F$ such that $x^2 + by^2 = c$; as
$c \notin F^2$, we have $y \neq 0$. It follows $\langle 1, b \rangle \simeq \langle 1, by^2 \rangle \simeq \langle 1, x^2 + by^2 \rangle \simeq \langle 1, c \rangle$, and
thus also $b \in D_F(\langle 1, c \rangle)$. $\qquad \square$

Lemma 4.3. *Let $a, b, x, y \in F$ and $y \neq 0$. Then*

$$\langle 1, a, bx^2 + aby^2 \rangle \simeq \left(a + \left(\frac{x}{y} \right)^2 \right) \langle 1, a, b \rangle.$$

Proof. We have

$$\langle 1, a, bx^2 + aby^2 \rangle \simeq \left\langle 1, a + \left(\frac{x}{y} \right)^2, b \left(a + \left(\frac{x}{y} \right)^2 \right) \right\rangle$$
$$\simeq \left(a + \left(\frac{x}{y} \right)^2 \right) \left\langle a + \left(\frac{x}{y} \right)^2, 1, b \right\rangle \simeq \left(a + \left(\frac{x}{y} \right)^2 \right) \langle 1, a, b \rangle. \quad \square$$

Lemma 4.4. *Let $a, b, c \in F \setminus F^2$, and suppose that $a \in D_F(\langle 1, c, bc \rangle)$. Then there
exists $s \in F$ such that $\langle 1, c, bc \rangle \simeq (a + s^2)\langle 1, a, b \rangle$.*

Proof. Let $x, y, z \in F$ be such that

$$(4\text{-}1) \qquad\qquad\qquad a = x^2 + cy^2 + bcz^2.$$

Suppose first that $z = 0$; then $a \in D_F(\langle 1, c \rangle)$, and we have $\langle 1, c \rangle \simeq \langle 1, a \rangle$ by Lemma 4.2. Moreover, since $a \notin F^2$, it must be $y \neq 0$, and (4-1) can be rewritten as $c = \left(\frac{x}{y}\right)^2 + a\left(\frac{1}{y}\right)^2$. Putting these together, we obtain

$$\langle 1, c, bc \rangle \simeq \left\langle 1, a, b\left(\left(\frac{x}{y}\right)^2 + a\left(\frac{1}{y}\right)^2\right)\right\rangle \simeq \langle 1, a, b(x^2 + a) \rangle,$$

hence, $\langle 1, c, bc \rangle \simeq (a + x^2)\langle 1, a, b \rangle$ by Lemma 4.3.

Now let $z \neq 0$; then

$$\langle 1, c, bc \rangle \simeq \langle 1, c, x^2 + cy^2 + bcz^2 \rangle \simeq \langle 1, a, c \rangle.$$

Note that $z \neq 0$ and $b \notin F^2$ imply $y^2 + bz^2 \neq 0$; hence, we can rewrite (4-1) as

$$c(y^2 + bz^2)^2 = (a + x^2)(y^2 + bz^2) = (xy)^2 + ay^2 + b(xz)^2 + abz^2.$$

Thus, we have

$$\langle 1, a, c \rangle \simeq \langle 1, a, b(xz)^2 + abz^2 \rangle \simeq (a + x^2)\langle 1, a, b \rangle,$$

where the latter isometry follows from Lemma 4.3. Therefore, in this case we get $\langle 1, c, bc \rangle \simeq (a + x^2)\langle 1, a, b \rangle$, too. $\square$

The following lemma mimics the situation we end up with after applying Proposition 4.1.

Lemma 4.5. *Let ψ be a totally singular quadratic form over F such that $1 \in D_F(\psi)$. Let $a, c \in F^*$ be such that $c\langle 1, a \rangle$ is anisotropic over F, and suppose $c\langle 1, a \rangle \subseteq \psi$. Then*

(i) *either $\langle 1, c, ac \rangle \subseteq \psi$ (this occurs if and only if $1 \notin D_F(c\langle 1, a \rangle)$),*

(ii) *or $\langle 1, a \rangle \subseteq \psi$ and $c \in D_F(\langle 1, a \rangle)$.*

Proof. If $1 \notin D_F(c\langle 1, a \rangle)$, then $\langle 1, a, ca \rangle$ is anisotropic, and so $\langle 1, c, ca \rangle \subseteq \psi$ according to Lemma 2.2. On the other hand, if $1 \in D_F(c\langle 1, a \rangle)$, then it follows that $c \in D_F(\langle 1, a \rangle) = G_F(\langle 1, a \rangle)$; thus, $\langle 1, a \rangle \simeq c\langle 1, a \rangle \subseteq \psi$. $\square$

Now we prove that (weakly) Vishik equivalent minimal quadratic forms are always similar. The proof is based mainly on the 2-independence of the coefficients as explored in Lemma 2.13.

Theorem 4.6. *Let φ, ψ be totally singular quadratic forms over F such that φ_{an} is minimal over F. If $\varphi \overset{v_0}{\sim} \psi$, then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. Invoking Lemma 3.3 (and as obviously $\varphi \overset{\mathrm{sim}}{\sim} \psi$ if and only if $\varphi_{\mathrm{an}} \overset{\mathrm{sim}}{\sim} \psi_{\mathrm{an}}$), we can assume that the forms φ and ψ are anisotropic. By Lemma 2.12, we can

suppose that $\varphi \simeq \langle 1, a_1, \ldots, a_n \rangle$ for some $\{a_1, \ldots, a_n\} \subseteq F$ that is 2-independent over F; it follows that $N_F(\varphi) = F^2(a_1, \ldots, a_n)$, and

$$\mathcal{B}_{(0)} = \{a_1, \ldots, a_n\}$$

is a 2-basis of $N_F(\varphi)$ over F. Moreover, $N_F(\varphi) = N_F(\psi)$ by Proposition 3.4, and ψ is minimal over F by Corollary 3.5.

We start with an observation:

($\diamond$) For any $a \in D_F(\varphi) \setminus F^2$, the form $\varphi_{F(\sqrt{a})}$ is isotropic. As $\varphi \overset{v_0}{\sim} \psi$, ψ must be isotropic over $F(\sqrt{a})$ as well. By Proposition 4.1, we can find $c \in F^*$ such that $c\langle 1, a \rangle \subseteq \psi$.

The main idea is to look at such ca for some $a \in D_F(\varphi)$, and express it with respect to an appropriate 2-basis of $N_F(\psi)$ over F. Applying Lemma 2.13, we get that almost all coefficients must be zero. Via a combination of different values of a, we usually end up with the conclusion that c must be a square, which means that $a \in D_F(\psi)$. In particular, we will prove that there is a scalar multiple of ψ which represents all the values $1, a_1, \ldots, a_n$.

We divide the proof into several steps and cases. To simplify the notation and omit multiple indices, we use the same letters repeatedly — the meaning of s_i, u_i, x_i, y_i, z_i changes in each subcase (although they are usually used in similar situations). On the other hand, the meaning of a_i, c_i, d_i, e_i is "global", i.e., does not change during the proof. Moreover, we consider only 2-independence and 2-bases over F, and so we omit to repeat "over F" each time. We also would like to recall the notation (see page 330): By a "unique expression with respect to a 2-basis" we actually mean the unique expression with respect to the corresponding F^2-linear basis.

(1) First of all, note that ($\diamond$) proves the claim completely if $\dim \psi = 2$. Therefore, suppose $\dim \psi \geq 3$.

As the first step, we will prove that ψ contains a subform similar to $\langle 1, a_1, a_2 \rangle$: By ($\diamond$), we find $c_1 \in F^*$ such that $c_1 \langle 1, a_1 \rangle \subseteq \psi$. Since we are interested in ψ only up to similarity, we can assume without loss of generality that $c_1 = 1$. Now, $c_2 \langle 1, a_2 \rangle \subseteq \psi$ for some $c_2 \in F^*$. By Lemma 4.5, there are two possibilities: Either $\langle 1, a_2 \rangle \subseteq \psi$ or $\langle 1, c_2, c_2 a_2 \rangle \subseteq \psi$.

(1A) If $\langle 1, a_2 \rangle \subseteq \psi$, we get $\langle 1, a_1, a_2 \rangle \subseteq \psi$ immediately (by Lemma 2.2).

(1B) Suppose that $\langle 1, c_2, c_2 a_2 \rangle \subseteq \psi$ holds. We have to further distinguish two cases, depending on whether a_1 is represented by the form $\langle 1, c_2, c_2 a_2 \rangle$ or not.

(1Bi) Assume that $a_1 \in D_F(\langle 1, c_2, c_2 a_2 \rangle)$. Then $\langle 1, c_2, c_2 a_2 \rangle \overset{\text{sim}}{\sim} \langle 1, a_1, a_2 \rangle$ by Lemma 4.4.

(1Bii) Let $a_1 \notin D_F(\langle 1, c_2, c_2 a_2 \rangle)$; then $\langle 1, a_1, c_2, c_2 a_2 \rangle$ is an anisotropic subform of ψ, and hence $\psi \simeq \langle 1, a_1, c_2, c_2 a_2, s_4, \ldots, s_n \rangle$ for some suitable $s_4, \ldots, s_n \in F^*$. Since ψ is minimal, the set

$$\mathcal{B}_{(1\mathrm{Bii})} = \{a_1, c_2, c_2 a_2, s_4, \ldots, s_n\}$$

is 2-independent, and hence it is a 2-basis of $N_F(\psi)$. Since $a_2 \in N_F(\psi)$, the element a_2 has a unique expression with respect to $\mathcal{B}_{(1\mathrm{Bii})}$:

$$(4\text{-}2) \qquad a_2 = c_2 \cdot c_2 a_2 \cdot \left(\frac{1}{c_2} \right)^2.$$

It follows by Lemma 2.13 that $a_2 \notin D_F(\psi)$.

Furthermore, as $a_1 + a_2 \in D_F(\varphi) \setminus F^2$, we use ($\diamond$) to find $d_2 \in F^*$ such that $d_2 \langle 1, a_1 + a_2 \rangle \subseteq \psi$. In particular, $d_2 \in D_F(\psi)$; hence,

$$d_2 = x_0^2 + a_1 \cdot x_1^2 + c_2 \cdot x_2^2 + c_2 a_2 \cdot x_3^2 + \sum_{i=4}^{n} s_i \cdot x_i^2$$

for some suitable $x_0, \ldots, x_n \in F$. Multiplying d_2 by $(a_1 + a_2)$ and using (4-2) (i.e., expressing $d_2(a_1 + a_2)$ with respect to the basis $\mathcal{B}_{(1\mathrm{Bii})}$), we get

$$\begin{aligned} d_2(a_1 + a_2) = {} & (a_1 x_1)^2 + a_1 \cdot x_0^2 + c_2 \cdot (a_2 x_3)^2 + c_2 a_2 \cdot x_2^2 \\ & + a_1 \cdot c_2 \cdot x_2^2 + a_1 \cdot c_2 a_2 \cdot x_3^2 + \sum_{i=4}^{n} a_1 \cdot s_i \cdot x_i^2 + c_2 \cdot c_2 a_2 \cdot \left(\frac{x_0}{c_2} \right)^2 \\ & + a_1 \cdot c_2 \cdot c_2 a_2 \cdot \left(\frac{x_1}{c_2} \right)^2 + \sum_{i=4}^{n} c_2 \cdot c_2 a_2 \cdot s_i \cdot \left(\frac{x_i}{c_2} \right)^2. \end{aligned}$$

We know that $d_2(a_1 + a_2)$ is represented by ψ; but that is impossible by Lemma 2.13 unless all the terms composed from at least two elements of $\mathcal{B}_{(1\mathrm{Bii})}$ (i.e., all but the first four) are zero. It follows $x_i = 0$ for all $0 \le i \le n$; hence, $d_2 = 0$ which is absurd. Therefore, this case cannot happen at all.

We can conclude step (1): We have proved that ψ has a subform similar to $\langle 1, a_1, a_2 \rangle$. If $\dim \psi = 3$, then there is nothing more to prove. From now on, we will assume that $\dim \psi \ge 4$ and $\langle 1, a_1, a_2 \rangle \subseteq \psi$.

(2) Now let $k \in \{3, \ldots, n\}$. By ($\diamond$), we find $c_k, d_k, e_k \in F^*$ such that $c_k \langle 1, a_k \rangle$, $d_k \langle 1, a_1 + a_k \rangle$ and $e_k \langle 1, a_2 + a_k \rangle$ are subforms of ψ. The case distinction is slightly different than in (1); here it depends on whether c_k is represented by $D_F(\langle 1, a_1, a_2 \rangle)$ or not.

(2A) Assume $c_k \in D_F(\langle 1, a_1, a_2 \rangle)$. Then

$$(4\text{-}3) \qquad c_k = u_0^2 + a_1 \cdot u_1^2 + a_2 \cdot u_2^2$$

for some $u_0, u_1, u_2 \in F$ (here we slightly abuse the notation; technically, u_0, u_1, u_2 depend on k), and so

$$c_k a_k = a_k \cdot u_0^2 + a_1 \cdot a_k \cdot u_1^2 + a_2 \cdot a_k \cdot u_2^2.$$

By the uniqueness of the expression of $c_k a_k$ with respect to $\mathcal{B}_{(0)}$, it follows by Lemma 2.13 that $c_k a_k \notin D_F(\langle 1, a_1, a_2 \rangle)$. On the other hand, we know that $c_k a_k \in D_F(\psi)$; therefore, $\psi \simeq \langle 1, a_1, a_2, c_k a_k, s_4, \dots, s_n \rangle$ for some $s_4, \dots, s_n \in F^*$ (possibly different from the s_i's in case (1Bii) above), and

$$\mathcal{B}_{(2A)} = \{a_1, a_2, c_k a_k, s_4, \dots, s_n\}$$

is a 2-basis of $N_F(\psi)$ by the minimality of ψ. Obviously, $a_k = c_k \cdot c_k a_k \cdot c_k^{-2}$; rewriting c_k via (4-3), we get the unique expression of a_k with respect to $\mathcal{B}_{(2A)}$:

$$(4\text{-}4) \qquad a_k = c_k a_k \cdot \left(\frac{u_0}{c_k}\right)^2 + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k}\right)^2 + a_2 \cdot c_k a_k \cdot \left(\frac{u_2}{c_k}\right)^2.$$

Furthermore, we have $d_k \in D_F(\psi)$, and hence

$$d_k = x_0^2 + a_1 \cdot x_1^2 + a_2 \cdot x_2^2 + c_k a_k \cdot x_3^2 + \sum_{i=4}^{n} s_i \cdot x_i^2$$

for suitable $x_0, \dots, x_n \in F$ (again, we omit to express the dependence on k). Multiplying d_k by $(a_1 + a_k)$ and using (4-4), we can express $d_k(a_1 + a_k)$ with respect to $\mathcal{B}_{(2A)}$ as

$$
\begin{aligned}
d_k(a_1 + a_k) = {} & (a_1 x_1 + a_k u_0 x_3)^2 + a_1 \cdot (x_0 + a_k u_1 x_3)^2 + a_2 \cdot (a_k u_2 x_3)^2 \\
& + c_k a_k \cdot \left(\frac{u_0}{c_k} x_0 + \frac{a_1 u_1}{c_k} x_1 + \frac{a_2 u_2}{c_k} x_2\right)^2 + a_1 \cdot a_2 \cdot x_2^2 \\
& + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k} x_0 + \frac{u_0}{c_k} x_1 + x_3\right)^2 + \sum_{i=4}^{n} a_1 \cdot s_i \cdot x_i^2 \\
& + a_2 \cdot c_k a_k \cdot \left(\frac{u_2}{c_k} x_0 + \frac{u_0}{c_k} x_2\right)^2 + \sum_{i=4}^{n} c_k a_k \cdot s_i \cdot \left(\frac{u_0}{c_k} x_i\right)^2 \\
& + a_1 \cdot a_2 \cdot c_k a_k \cdot \left(\frac{u_2}{c_k} x_1 + \frac{u_1}{c_k} x_2\right)^2 + \sum_{i=4}^{n} a_1 \cdot c_k a_k \cdot s_i \cdot \left(\frac{u_1}{c_k} x_i\right)^2 \\
& + \sum_{i=4}^{n} a_2 \cdot c_k a_k \cdot s_i \cdot \left(\frac{u_2}{c_k} x_i\right)^2.
\end{aligned}
$$

Similarly as before, since $d_k(a_1 + a_k) \in D_F(\psi)$ and ψ is minimal, it follows from Lemma 2.13 that all the "composed" terms, i.e., all the terms except for the first four, must be zero. In particular:

- The coefficients by $c_k a_k \cdot s_i$ and $a_1 \cdot c_k a_k \cdot s_i$ and $a_2 \cdot c_k a_k \cdot s_i$ are zero for each $4 \leq i \leq n$; since at least one of u_0, u_1 and u_2 is nonzero, we get that $x_i = 0$ for each $4 \leq i \leq n$.

- The coefficient by $a_1 \cdot a_2 \cdot$ equals zero; thus, $x_2 = 0$.

- The coefficient by $a_2 \cdot c_k a_k$ must be zero, i.e., $\frac{u_2}{c_k} x_0 + \frac{u_0}{c_k} x_2 = 0$. As $x_2 = 0$, we get $u_2 x_0 = 0$.

- The coefficient by $a_1 \cdot a_2 \cdot c_k a_k$ is zero, so $\frac{u_2}{c_k} x_1 + \frac{u_1}{c_k} x_2 = 0$. Again, as $x_2 = 0$, it must hold $u_2 x_1 = 0$.

- The coefficient by $a_1 \cdot c_k a_k$ must be zero, thus $x_3 + \frac{u_1}{c_k} x_0 + \frac{u_0}{c_k} x_1 = 0$.

If $u_2 \neq 0$, then $x_0 = x_1 = 0$, and in that case also $x_3 = 0$ (by the last bullet point); but then $d_k = 0$, which is absurd. Therefore, $u_2 = 0$.

We proceed analogously for e_k and $e_k(a_2 + a_k)$, only now we know $u_2 = 0$, which means that

$$c_k = u_0^2 + a_1 \cdot u_1^2 \quad \text{and} \quad a_k = c_k a_k \cdot \left(\frac{u_0}{c_k}\right)^2 + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k}\right)^2.$$

Since $e_k \in D_F(\psi)$, we have

$$e_k = y_0^2 + a_1 \cdot y_1^2 + a_2 \cdot y_2^2 + c_k a_k \cdot y_3^2 + \sum_{i=4}^{n} s_i \cdot y_i^2$$

for some $y_o, \ldots, y_n \in F$, and

$$
\begin{aligned}
e_k(a_2 + a_k) = {}& \left(a_2 y_2 + c_k a_k \frac{u_0}{c_k} y_3\right)^2 + a_1 \cdot \left(c_k a_k \frac{u_1}{c_k} y_3\right)^2 + a_2 \cdot y_0^2 \\
&+ c_k a_k \cdot \left(\frac{u_0}{c_k} y_0 + a_1 \frac{u_1}{c_k} y_1\right)^2 + a_1 \cdot a_2 \cdot y_1^2 \\
&+ a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k} y_0 + \frac{u_0}{c_k} y_1\right)^2 + a_2 \cdot c_k a_k \cdot \left(y_3 + \frac{u_0}{c_k} y_2\right)^2 \\
&+ \sum_{i=4}^{n} a_2 \cdot s_i \cdot y_i^2 + \sum_{i=4}^{n} c_k a_k \cdot s_i \cdot \left(\frac{u_0}{c_k} y_i\right)^2 \\
&+ a_1 \cdot a_2 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k} y_2\right)^2 + \sum_{i=4}^{n} a_1 \cdot c_k a_k \cdot s_i \cdot \left(\frac{u_1}{c_k} y_i\right)^2.
\end{aligned}
$$

Again, all the coefficients by the composed terms must be zero, so in particular:

- $y_i = 0$ for all $4 \leq i \leq n$ because of the coefficients by $a_2 \cdot s_i$.

- $y_1 = 0$ because of the coefficient by $a_1 \cdot a_2$.

- $u_1 y_0 = 0$ because of the coefficient by $a_1 \cdot c_k a_k$.

- $u_1 y_2 = 0$ because of the coefficient by $a_1 \cdot a_2 \cdot c_k a_k$.
- $y_3 + \frac{u_0}{c_k} y_2 = 0$ because of the coefficient by $a_2 \cdot c_k a_k$.

If $u_1 \neq 0$, then necessarily $y_0 = 0$ and $y_2 = 0$, which implies $y_3 = 0$; it would follow $e_k = 0$, which is absurd. Thus, we have $u_1 = 0$. Therefore, $c_k = u_0^2$, and we have $\langle 1, a_k \rangle \subseteq \psi$; in particular, $a_k \in D_F(\psi)$.

(2B) Suppose $c_k \notin D_F(\langle 1, a_1, a_2 \rangle)$; then $\langle 1, a_1, a_2, c_k \rangle \subseteq \psi$. Here we distinguish two subcases:

(2Bi) Let $c_k a_k \in D_F(\langle 1, a_1, a_2, c_k \rangle)$; then

$$c_k a_k = u_0^2 + a_1 \cdot u_1^2 + a_2 \cdot u_2^2 + c_k \cdot u_3^2$$

for some $u_0, \ldots, u_3 \in F$, and hence

$$a_k = u_3^2 + c_k \cdot \left(\frac{u_0}{c_k}\right)^2 + a_1 \cdot c_k \cdot \left(\frac{u_1}{c_k}\right)^2 + a_2 \cdot c_k \cdot \left(\frac{u_2}{c_k}\right)^2.$$

We have

$$\psi \simeq \langle 1, a_1, a_2, c_k, s_4, \ldots, s_n \rangle$$

for some $s_4, \ldots, s_n \in F$; then

$$\mathcal{B}_{(2\mathrm{Bi})} = \{a_1, a_2, c_k, s_4, \ldots, s_n\}$$

is a 2-basis of $N_F(\psi)$. As before, we consider the unique representations of d_k and e_k by ψ, and express $d_k(a_1 + a_k)$ and $e_k(a_2 + a_k)$ with respect to $\mathcal{B}_{(2\mathrm{Bi})}$. We obtain that $u_1 = u_2 = 0$; therefore, $a_k = u_3^2 + c_k \left(\frac{u_0}{c_k}\right)^2$, so in particular $a_k \in D_F(\psi)$.

(2Bii) If $c_k a_k \notin D_F(\langle 1, a_1, a_2, c_k \rangle)$, then we have $\dim \psi \geq 5$ and

$$\psi \simeq \langle 1, a_1, a_2, c_k, c_k a_k, s_5, \ldots, s_n \rangle$$

for some $s_5, \ldots, s_n \in F$, and the corresponding 2-basis of $N_F(\psi)$ is

$$\mathcal{B}_{(2\mathrm{Bii})} = \{a_1, a_2, c_k, c_k a_k, s_5, \ldots, s_n\}.$$

Again, we consider the unique representation of d_k by ψ. In this case, we express a_k with respect to $\mathcal{B}_{(2\mathrm{Bii})}$ as

$$a_k = c_k \cdot c_k a_k \cdot \left(\frac{1}{c_k}\right)^2.$$

This time, the consideration of the element $d_k(a_1 + a_k)$ with respect to $\mathcal{B}_{(2\mathrm{Bii})}$ already implies $d_k = 0$; that is absurd, and hence this case cannot happen.

(3) We have proved that, up to multiplying ψ by a constant from F^*, we have $\langle 1, a_1, a_2 \rangle \subseteq \psi$ (step (1)), and $a_k \in D_F(\psi)$ for all $3 \leq k \leq n$ (step (2)). Invoking Lemma 2.2, we get $\varphi \overset{\mathrm{sim}}{\sim} \psi$. $\qquad \square$

Special quasi-Pfister neighbors. The definition of a special quasi-Pfister neighbor is motivated by its counterpart in characteristic different from 2, which appeared in [Ahmad and Ohm 1995].

Definition 4.7. We call a totally singular quadratic form φ over F a *special quasi-Pfister neighbor* if $\varphi \overset{\text{sim}}{\sim} \pi \perp b\sigma$ with π a quasi-Pfister form over F, $b \in F^*$ and $\sigma \subseteq \pi$. In such situation, we also say that φ is given by the triple (π, b, σ).

For a totally singular quadratic form φ over F, we define its *full splitting pattern* as

$$\text{fSP}(\varphi) = \{\dim(\varphi_E)_{\text{an}} \mid E/F \text{ a field extension}\}.$$

We will describe all special quasi-Pfister neighbors of dimension up to eleven and their full splitting pattern. In particular, we will see that all quasi-Pfister neighbors up to dimension eight are special. First, we recall a proposition from [Hoffmann and Laghribi 2004].

Lemma 4.8 [Hoffmann and Laghribi 2004, Proposition 8.12]. *Let φ be an anisotropic totally singular quadratic form over F with $\dim \varphi \leq 8$. Then φ is a quasi-Pfister neighbor if and only if*

(i) $\dim \varphi \leq 3$, *or*

(ii) $\dim \varphi = 2^n$ *for some $n \geq 1$ and φ is similar to an n-fold quasi-Pfister form, or*

(iii) *there exist $x, y, z \in F^*$ such that*

    (a) $\varphi \overset{\text{sim}}{\sim} \langle 1, x, y, xy, z \rangle$ *in the case of $\dim \varphi = 5$,*

    (b) $\varphi \overset{\text{sim}}{\sim} \langle 1, x, y, xy, z, xz \rangle$ *in the case of $\dim \varphi = 6$,*

    (c) $\varphi \overset{\text{sim}}{\sim} \langle x, y, z, xy, xz, yz, xyz \rangle$ *in the case of $\dim \varphi = 7$.*

With the previous lemma, it is easy to classify all special quasi-Pfister neighbors of dimensions up to 8: Any quasi-Pfister neighbor of dimension 2^n for some $n \geq 0$ is similar to an n-fold quasi-Pfister form, and hence special. For the other small dimensions, we have

- $\langle 1, x, y \rangle \simeq \langle\!\langle x \rangle\!\rangle \perp y\langle 1 \rangle$,

- $\langle 1, x, y, xy, z \rangle \simeq \langle\!\langle x, y \rangle\!\rangle \perp z\langle 1 \rangle$,

- $\langle 1, x, y, xy, z, xz \rangle \simeq \langle\!\langle x, y \rangle\!\rangle \perp z\langle 1, x \rangle$,

- $\langle x, y, z, xy, xz, yz, xyz \rangle \simeq xyz(\langle\!\langle x, y \rangle\!\rangle \perp z\langle 1, x, y \rangle)$;

therefore, we get the following corollary.

Corollary 4.9. *All anisotropic quasi-Pfister neighbors of dimensions up to 8 are special.*

Proposition 4.10. *Let φ be an anisotropic special quasi-Pfister neighbor.*

(i) *If* $\dim \varphi = 3$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle\!\langle a \rangle\!\rangle \perp d\langle 1 \rangle$ *for some* $a, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3\}$.

(ii) *If* $\dim \varphi = 5$, *then we have* $\varphi \overset{\mathrm{sim}}{\sim} \langle\!\langle a, b \rangle\!\rangle \perp d\langle 1 \rangle$ *for some* $a, b, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 5\}$.

(iii) *If* $\dim \varphi = 6$, *then we have* $\varphi \overset{\mathrm{sim}}{\sim} \langle\!\langle a, b \rangle\!\rangle \perp d\langle 1, a \rangle$ *for some* $a, b, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 6\}$.

(iv) *If* $\dim \varphi = 7$, *then we have* $\varphi \overset{\mathrm{sim}}{\sim} \langle\!\langle a, b \rangle\!\rangle \perp d\langle 1, a, b \rangle$ *for some* $a, b, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 4, 7\}$.

(v) *If* $\dim \varphi = 9$, *then we have* $\varphi \overset{\mathrm{sim}}{\sim} \langle\!\langle a, b, c \rangle\!\rangle \perp d\langle 1 \rangle$ *for some* $a, b, c, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 5, 8, 9\}$.

(vi) *If* $\dim \varphi = 10$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle\!\langle a, b, c \rangle\!\rangle \perp d\langle 1, a \rangle$ *for some* $a, b, c, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 5, 6, 8, 10\}$.

(vii) *If* $\dim \varphi = 11$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle\!\langle a, b, c \rangle\!\rangle \perp d\langle 1, a, b \rangle$ *for some* $a, b, c, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 6, 7, 8, 11\}$.

Proof. All special quasi-Pfister neighbors up to dimension 8 have been described by Lemma 4.8 and Corollary 4.9. If $\varphi \overset{\mathrm{sim}}{\sim} \pi \perp d\sigma$ is a special quasi-Pfister neighbor and $9 \leq \dim \varphi \leq 11$, then necessarily $\dim \pi = 2^3$ and $1 \leq \dim \sigma \leq 3$. Recall that all totally singular quadratic forms of dimensions two or three are minimal; thus, without loss of generality, $\langle 1 \rangle \subseteq \sigma \subseteq \langle 1, a, b \rangle$ for some 2-independent set $\{a, b\} \subseteq F$. Then we can find $c \in F^*$ such that $\{a, b, c\}$ is a 2-basis of $N_F(\pi)$ over F, and it follows that $\pi \simeq \langle\!\langle a, b, c \rangle\!\rangle$.

The full splitting pattern of φ follows directly from [Zemková 2024, Theorem 4.11]. $\qquad\square$

In the following example, we show that not all quasi-Pfister neighbors are special.

Example 4.11. Let $\varphi \simeq \langle 1, a, b, c, d, ab, ac, ad, bc \rangle$ be a totally singular quadratic form such that $\{a, b, c, d\} \subseteq F$ is 2-independent over F. Then φ is a quasi-Pfister neighbor of $\langle\!\langle a, b, c, d \rangle\!\rangle$.

On the other hand, we have

$$\varphi_{F(\sqrt{b})} \simeq \langle 1, a, 1, c, d, a, ac, ad, c \rangle_{F(\sqrt{b})} \simeq \langle 1, a, c, d, ac, ad, 0, 0, 0 \rangle_{F(\sqrt{b})},$$

since $\langle 1, a, c, d, ac, ad \rangle \subseteq \langle\!\langle a, c, d \rangle\!\rangle$ and $(\langle\!\langle a, c, d \rangle\!\rangle)_{F(\sqrt{b})} \simeq (\langle\!\langle a, b, c, d \rangle\!\rangle_{F(\sqrt{b})})_{\mathrm{an}}$ is anisotropic, we get

$$(\varphi_{F(\sqrt{b})})_{\mathrm{an}} \simeq \langle 1, a, c, d, ac, ad \rangle_{F(\sqrt{b})},$$

so we have in particular $6 \in \mathrm{fSP}(\varphi)$. As the full splitting pattern of any 9-dimensional special quasi-Pfister neighbor equals to $\{1, 2, 3, 4, 5, 8, 9\}$ by Proposition 4.10, it follows that φ cannot be any special quasi-Pfister neighbor.

Ideally, we would like to prove that if two totally singular forms are Vishik equivalent and at least one of them is a special quasi-Pfister neighbor, then they are similar. Unfortunately, we will need some additional assumptions. We start with a few lemmas.

Lemma 4.12 [Zemková 2024, Lemma 4.8]. *Let π be a quasi-Pfister form over F, $\sigma \subseteq \pi$ and $d \in F^*$ be such that the totally singular quadratic form $\varphi \simeq \pi \perp d\sigma$ is anisotropic. Let E/F be a field extension.*

(i) *If $d \in D_E(\pi)$, then $(\varphi_E)_{\mathrm{an}} \simeq (\pi_E)_{\mathrm{an}}$; in particular, $\mathfrak{i}_d(\varphi_E) = \mathfrak{i}_d(\pi_E) + \dim \sigma$.*

(ii) *If $d \notin D_E(\pi)$, then $(\varphi_E)_{\mathrm{an}} \simeq (\pi_E)_{\mathrm{an}} \perp d(\sigma_E)_{\mathrm{an}}$; in particular, $\mathfrak{i}_d(\varphi_E) = \mathfrak{i}_d(\pi_E) + \mathfrak{i}_d(\sigma_E)$.*

We prove that if a quasi-Pfister neighbor of norm degree 2^n contains a quasi-Pfister form of dimension 2^{n-1}, then it must be a special quasi-Pfister neighbor.

Lemma 4.13. *Let π be an anisotropic quasi-Pfister form over F. Moreover, let ψ be an anisotropic totally singular quadratic form over F such that $\pi \subseteq \psi$ and $\hat{v}_F(\psi) \simeq \pi \otimes \langle\!\langle b \rangle\!\rangle$ for some $b \in F^*$. Then there exists a totally singular form $\rho \subseteq \pi$ such that $\psi \simeq \pi \perp b\rho$.*

Proof. First, recall that a norm form is always anisotropic; hence, $\pi \otimes \langle\!\langle b \rangle\!\rangle$ is anisotropic, and we get by Lemma 2.10 that $b \notin N_F(\pi) = D_F(\pi)$.

Let ρ' be a totally singular quadratic form over F such that $\psi \simeq \pi \perp b\rho'$. Denote $s = \dim \rho'$ and $\rho' = \langle d_1', \ldots, d_s' \rangle$. For each $k \in \{1, \ldots, s\}$, we proceed as follows: Since $bD_F(\rho') \subseteq D_F(\psi) \subseteq D_F(\pi \otimes \langle\!\langle b \rangle\!\rangle)$, we can write

$$bd_k' = \pi(\boldsymbol{\xi}_k) + b\pi(\boldsymbol{\zeta}_k)$$

for some appropriate vectors $\boldsymbol{\xi}_k, \boldsymbol{\zeta}_k$. If $\boldsymbol{\zeta}_k = 0$, then $bd_k' = \pi(\boldsymbol{\xi}_k) \in D_F(\pi)$, which contradicts the anisotropy of ψ; thus, the vector $\boldsymbol{\zeta}_k$ must be nonzero. It follows that $\pi(\boldsymbol{\zeta}_k) \neq 0$, and so we set $d_k = \pi(\boldsymbol{\zeta}_k)$; then

$$\pi \perp b\langle d_k' \rangle \simeq \pi \perp \langle \pi(\boldsymbol{\xi}_k) + bd_k \rangle \simeq \pi \perp b\langle d_k \rangle.$$

It follows that

$$\pi \perp b\langle d_1', \ldots, d_k' \rangle \perp b\langle d_{k+1}, \ldots, d_s \rangle \simeq \pi \perp b\langle d_1', \ldots, d_{k-1}' \rangle \perp b\langle d_k, \ldots, d_s \rangle$$

for any $1 \leq k \leq s$. Therefore,

$$\pi \perp b\langle d_1', \ldots, d_s' \rangle \simeq \pi \perp b\langle d_1, \ldots, d_s \rangle,$$

where $\rho \simeq \langle d_1, \ldots, d_s \rangle$ is a subform of π, because $d_k \in D_F(\pi)$ for each k. $\square$

Proposition 4.14. *Let φ, ψ be anisotropic totally singular quadratic forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Assume that φ is a special quasi-Pfister neighbor given by a triple (π, b, σ). Moreover, suppose that $c\pi \subseteq \psi$ for some $c \in F^*$. Then ψ is a special*

quasi-Pfister neighbor given by a triple (π, b, ρ) *for some form* ρ *over* F *such that* $\rho \overset{v_0}{\sim} \sigma$.

Proof. Let $d \in D_F(\sigma)$. Then $d \in D_F(\pi) = G_F(\pi)$; hence, $d\varphi \simeq \pi \perp b(d\sigma)$ and $1 \in D_F(d\sigma)$. Therefore, we can assume that $1 \in D_F(\sigma)$. Then, since φ is anisotropic, we must have $b \notin D_F(\pi)$. It follows that $\hat{v}_F(\varphi) \simeq \pi \otimes \langle\!\langle b \rangle\!\rangle$. Since $\hat{v}_F(\psi) \simeq \hat{v}_F(\varphi)$ by [Proposition 3.4](), we also have $\hat{v}_F(\psi) \simeq \pi \otimes \langle\!\langle b \rangle\!\rangle$. Finally, we can suppose that $\pi \subseteq \psi$. Therefore, we can apply [Lemma 4.13]() to find a totally singular quadratic form $\rho \subseteq \pi$ such that $\psi \simeq \pi \perp b\rho$.

It remains to show that $\sigma \overset{v_0}{\sim} \rho$: The equality $\dim \sigma = \dim \rho$ follows directly from $\dim \varphi = \dim \psi$. So, consider the field $E = F(\sqrt{a})$ for some $a \in F^*$. If $a \notin N_F(\pi)$, then π_E is anisotropic by [Lemma 2.8](), and so are its subforms σ_E and ρ_E; in particular, $\mathrm{i_d}(\sigma_E) = \mathrm{i_d}(\rho_E)$. On the other hand, if $a \in N_F(\pi)$, then $(\pi \otimes \langle\!\langle a \rangle\!\rangle)_{\mathrm{an}} \simeq \pi$ by [Lemma 2.10](); together with [Lemma 2.8](), we obtain

$$D_E(\pi_E) = D_F(\pi \otimes \langle\!\langle a \rangle\!\rangle) = D_F(\pi).$$

It follows that $b \notin D_E(\pi_E)$, and so we get by [Lemma 4.12]() that

$$\mathrm{i_d}(\varphi_E) = \mathrm{i_d}(\pi_E) + \mathrm{i_d}(\sigma_E) \quad \text{and} \quad \mathrm{i_d}(\psi_E) = \mathrm{i_d}(\pi_E) + \mathrm{i_d}(\rho_E).$$

Since $\mathrm{i_d}(\varphi_E) = \mathrm{i_d}(\psi_E)$ by the assumption, we get $\mathrm{i_d}(\sigma_E) = \mathrm{i_d}(\rho_E)$. $\qquad \square$

Remark 4.15. With the same notation as in [Proposition 4.14](), we can consider a stronger assumption $\varphi \overset{v}{\sim} \psi$ and ask whether it implies $\sigma \overset{v}{\sim} \rho$.

First, note that any field E with $b \in D_E(\pi)$ is problematic: In this case, we have

$$b\varphi_E \simeq b\pi_E \perp \sigma_E \simeq (\pi \perp \sigma)_E.$$

Hence, $(\varphi_E)_{\mathrm{an}} \simeq (\pi_E)_{\mathrm{an}}$, so we do not get any information about $\mathrm{i_d}(\sigma_E)$.

We can still give a more specific characterization of the problematic fields: As in the proof of [Proposition 4.14](), we can assume that $1 \in D_F(\sigma)$. First, let T and S be fields such that $F \subseteq T \subseteq S \subseteq E$, where T/F is purely transcendental, S/T is separable and E/S is purely inseparable.

If $b \in D_S(\pi)$, then $(\pi \perp \langle b \rangle)_S$ is isotropic, and hence $\pi \perp \langle b \rangle$ must be isotropic over F. But that is impossible because $\pi \perp \langle b \rangle \subseteq \pi \perp b\sigma$ and $\pi \perp b\sigma$ is anisotropic. Thus, $b \notin D_S(\pi)$.

Furthermore, isotropy is a finite problem; thus, we can construct a field L with $S \subseteq L \subseteq E$ such that L/S is finite, and $\mathrm{i_d}(\sigma_L) = \mathrm{i_d}(\sigma_E)$ and $\mathrm{i_d}(\rho_L) = \mathrm{i_d}(\rho_E)$. Then $L = S\left(\sqrt[2^{n_1}]{s_1}, \ldots, \sqrt[2^{n_k}]{s_k}\right)$ for some $k \geq 0$, $s_i \in S$ and $n_i \geq 1$. Set $K = S(\sqrt{s_1}, \ldots, \sqrt{s_k})$; then $\mathrm{i_d}(\sigma_K) = \mathrm{i_d}(\sigma_L)$ and $\mathrm{i_d}(\rho_K) = \mathrm{i_d}(\rho_L)$ by [Zemková 2024, Theorem 3.3]().

If $b \notin D_K(\pi)$, then we can apply [Lemma 4.12]() to get $\mathrm{i_d}(\sigma_K) = \mathrm{i_d}(\rho_K)$; then $\mathrm{i_d}(\sigma_E) = \mathrm{i_d}(\rho_E)$ by the construction of K, and we are done. Therefore, assume $b \in D_K(\pi)$.

Now we can construct a field M such that $S \subseteq M \subseteq K$, it holds that $b \notin D_M(\pi)$, and for each $\varepsilon \in K \setminus M$, we have $b \in D_{M(\varepsilon)}(\pi)$. Then $(\varphi_M)_{\mathrm{an}} \simeq (\pi_M)_{\mathrm{an}} \perp b(\sigma_M)_{\mathrm{an}}$ and $(\psi_M)_{\mathrm{an}} \simeq (\pi_M)_{\mathrm{an}} \perp b(\rho_M)_{\mathrm{an}}$ and $\mathsf{i}_{\mathrm{d}}(\sigma_M) = \mathsf{i}_{\mathrm{d}}(\rho_M)$ by Lemma 4.12. Thus, let φ', ψ', π', σ', ρ' be forms over F such that $\varphi'_M \simeq (\varphi_M)_{\mathrm{an}}$, $\psi'_M \simeq (\psi_M)_{\mathrm{an}}$, etc. In particular, π'_M is a quasi-Pfister form by Lemma 2.3.

Let $\varepsilon \in K \setminus M$, and $e \in S$ be such that $\varepsilon^2 = e$. Then we know

$$b \in D_{M(\sqrt{e})}(\pi') \setminus D_M(\pi') = D_M(\pi' \otimes \langle\langle e \rangle\rangle) \setminus D_M(\pi'),$$

in particular, $D_M(\pi') \subsetneq D_M(\pi' \otimes \langle\langle e \rangle\rangle)$, which is only possible if $(\pi' \otimes \langle\langle e \rangle\rangle)_M$ is anisotropic. It follows by Lemma 2.8 that $\pi'_{M(\sqrt{e})}$ is anisotropic. In that case, its subforms $\sigma'_{M(\sqrt{e})}$, $\rho'_{M(\sqrt{e})}$ must be anisotropic, too. It follows that

$$\mathsf{i}_{\mathrm{d}}(\sigma_{M(\sqrt{e})}) = \mathsf{i}_{\mathrm{d}}(\sigma_M) = \mathsf{i}_{\mathrm{d}}(\rho_M) = \mathsf{i}_{\mathrm{d}}(\rho_{M(\sqrt{e})}).$$

However, the field extension K/M does not have to be simple, as we show in the following example: Let $\pi'_M \simeq \langle\langle a_1, \ldots, a_n \rangle\rangle_M$ for some $a_1, \ldots, a_n \in F^*$, and assume that $n \geq 2$. Set $K = M(\sqrt{b}, \sqrt{a_1 + a_2 b})$. It is easy to see that $\{b, a_1 + a_2 b\}$ is 2-independent over M, and hence $[K : M] = 4$. Since K does not contain any element of degree four over M, it follows that K/M is not simple. Now consider $\varepsilon \in K \setminus M$; then

$$\varepsilon = w + x\sqrt{b} + y\sqrt{b}\sqrt{a_1 + a_2 b} + z\sqrt{a_1 + a_2 b},$$

with $w, x, y, z \in M$, at least one of x, y, z nonzero. Then

$$D_{M(\varepsilon)}(\pi') = D_M(\pi' \otimes \langle\langle \varepsilon^2 \rangle\rangle) = M^2(a_1, \ldots, a_n, \varepsilon^2).$$

Considering ε^2 as an element of $K^2/M^2(a_1, \ldots, a_n)$, we get

$$\begin{aligned} \varepsilon^2 &= w^2 + bx^2 + a_1 by^2 + a_2(by)^2 + a_1 z^2 + a_2 bz^2 \\ &\equiv bx^2 + a_1 by^2 + a_2 bz^2 = b(x^2 + a_1 y^2 + a_2 z^2) \equiv b \mod M^2(a_1, \ldots, a_n), \end{aligned}$$

where we used that $x^2 + a_1 y^2 + a_2 z^2 \neq 0$ (this holds because $\{a_1, a_2\}$ is 2-independent over M and at least one of x, y, z is nonzero by the assumption). Therefore, we have $D_{M(\varepsilon)}(\pi') = M^2(a_1, \ldots, a_n, b)$. In particular, $b \in D_{M(\varepsilon)}(\pi')$ for any $\varepsilon \in K \setminus M$, and so we cannot find any field M' with $M \subsetneq M' \subseteq K$ such that $b \notin D_{M'}(\pi')$.

With the notation and assumptions of Proposition 4.14, we know that $\sigma \overset{v_0}{\sim} \rho$, and hence $N_F(\sigma) \simeq N_F(\rho)$ by Proposition 3.4. To conclude this section, we prove $N_F(\sigma) \simeq N_F(\rho)$ by a different approach.

Lemma 4.16. *Let $\varphi = \pi \perp b\sigma$, $\psi = \pi \perp b\rho$ be anisotropic totally singular quadratic forms over F with π a quasi-Pfister form, $b \in F^*$ and $\sigma, \rho \subseteq \pi$. Moreover, suppose that $1 \in D_F(\sigma) \cap D_F(\rho)$. If $\varphi \overset{v}{\sim} \psi$, then $N_F(\sigma) = N_F(\rho)$.*

Proof. First, note that the anisotropy of φ together with the assumption $1 \in D_F(\sigma)$ implies that $b \notin D_F(\pi)$.

By Lemma 2.7, we find $c_1, \ldots, c_s \in F^*$ and $0 \le k \le s$ such that $\sigma = \langle 1, c_1, \ldots, c_s \rangle$ and $\{c_1, \ldots, c_k\}$ is a 2-basis of $N_F(\sigma)$ over F. Set $K = F(\sqrt{c_1}, \ldots, \sqrt{c_k})$. Then $(\sigma_K)_{\mathrm{an}} \simeq \langle 1 \rangle$ by Proposition 2.5. Write $\pi \simeq \langle\!\langle a_1, \ldots, a_n \rangle\!\rangle$ and $\pi' \simeq (\pi_K)_{\mathrm{an}}$. Then

$$D_K(\pi') = K^2(a_1, \ldots, a_n)$$
$$= F^2(c_1, \ldots, c_k, a_1, \ldots, a_n) \stackrel{N_F(\sigma) \subseteq N_F(\pi)}{=\!=\!=} F^2(a_1, \ldots, a_n) = D_F(\pi);$$

hence, $b \notin D_K(\pi')$, and so $(\varphi_K)_{\mathrm{an}} \simeq \pi' \perp b\langle 1 \rangle$. We will show that this is isometric to $(\psi_K)_{\mathrm{an}}$: Obviously, $\pi' \subseteq (\psi_K)_{\mathrm{an}}$. From the Vishik equivalence of φ and ψ we know $\dim(\psi_K)_{\mathrm{an}} = \dim \pi' + 1$. Set $\rho' \simeq (\rho_K)_{\mathrm{an}}$. Since $b \notin D_K(\pi')$, we get by Lemma 4.12 that $\pi' \perp b\rho'$ is anisotropic. It means that $\pi' \perp b\rho' \simeq (\psi_K)_{\mathrm{an}}$, and hence $\dim \rho' = 1$. As $1 \in D_F(\rho) \subseteq D_K(\rho')$, it follows that $\rho' \simeq \langle 1 \rangle$. Consequently, again by Proposition 2.5, we get $N_F(\rho) \subseteq F^2(c_1, \ldots, c_k) = N_F(\sigma)$. The other inclusion can be proved analogously. Therefore, $N_F(\sigma) = N_F(\rho)$. $\square$

Conclusion. In this final part, we put together all our results on the Vishik equivalence of totally singular quadratic forms.

Before stating the main theorem, note that any anisotropic totally singular quadratic form of dimension less or equal to four is either minimal or it is similar to a quasi-Pfister form.

Theorem 4.17. *Let φ, ψ be totally singular quadratic forms over F such that $\varphi \stackrel{v_0}{\sim} \psi$. Assume that φ is a special quasi-Pfister neighbor given by the triple (π, b, σ), and $c\pi \subseteq \psi$ for some $c \in F^*$. Moreover, suppose that σ is either a quasi-Pfister form, a quasi-Pfister neighbor of codimension one, or a minimal form. Then $\varphi \stackrel{\mathrm{sim}}{\sim} \psi$.*

Proof. By Proposition 4.14, there exists a totally singular quadratic form $\rho \subseteq \pi$ and $c' \in F^*$ such that $c'\psi \simeq \pi \perp b\rho$ and $\sigma \stackrel{v_0}{\sim} \rho$. Then, by Corollary 3.5, resp. by Theorem 4.6, we have $\sigma \stackrel{\mathrm{sim}}{\sim} \rho$.

Let $d \in F^*$ be such that $\rho \simeq d\sigma$, and let $a \in D_F^*(\sigma)$. Then

$$da \in D_F^*(d\sigma) = D_F^*(\rho) \subseteq D_F^*(\pi) = G_F^*(\pi).$$

As $a \in G_F^*(\pi)$ and $G_F^*(\pi)$ is a group, it follows that $d \in G_F^*(\pi)$. Hence,

$$d\varphi \simeq d\pi \perp b(d\sigma) \simeq \pi \perp b\rho \simeq c'\psi,$$

i.e., $\varphi \stackrel{\mathrm{sim}}{\sim} \psi$. $\square$

Corollary 4.18. *Let φ, ψ be totally singular quadratic forms over F such that $\varphi \stackrel{v}{\sim} \psi$. Let K/F be a field extension such that $K^2 \simeq G_F(\varphi)$. Assume that $(\varphi_K)_{\mathrm{an}}$ is a special quasi-Pfister neighbor given by the triple (π, b, σ), and that $c\pi \subseteq \psi_K$ for some $c \in K^*$. Moreover, suppose that σ is either a quasi-Pfister form, a quasi-Pfister neighbor of codimension one, or a minimal form (over K). Then $\varphi \stackrel{\mathrm{sim}}{\sim} \psi$.*

Proof. Without loss of generality, assume that φ and ψ are anisotropic. Denote $\tau \simeq \hat{\sigma}_F(\varphi)$, and let $a_1, \ldots, a_n \in F^*$ be such that $\tau \simeq \langle\langle a_1, \ldots, a_n \rangle\rangle$. Then there exists a totally singular quadratic form φ' over F such that $\varphi \simeq \tau \otimes \varphi'$. Furthermore, as $G_F(\varphi) = G_F(\tau) = D_F(\tau) = F^2(a_1, \ldots, a_n)$, we have $K = F(\sqrt{a_1}, \ldots, \sqrt{a_n})$. By [Zemková 2024, Theorem 3.3], $\mathfrak{i}_d(\varphi'_K) = \mathfrak{i}_d(\tau \otimes \varphi') = 0$. Therefore, φ'_K is anisotropic, and we have $(\varphi_K)_{an} \simeq \varphi'_K$. Since $G_F(\varphi) \simeq G_F(\psi)$ by Theorem 3.7, we can use analogous arguments to find a form ψ' over F such that $\psi \simeq \tau \otimes \psi'$ and $\psi'_K \simeq (\psi_K)_{an}$.

Since $\varphi \overset{v}{\sim} \psi$, we get by Lemma 2.16 that $\varphi'_K \overset{v}{\sim} \psi'_K$. Since $c\pi \subseteq \psi_K$ and π is anisotropic over K, it follows that $c\pi \subseteq \psi'_K$. Note that $\varphi'_K \simeq (\varphi_K)_{an}$ is a special quasi-Pfister neighbor given by the triple (π, b, σ). Hence, by Theorem 4.17, we have $\varphi'_K \overset{sim}{\sim} \psi'_K$. Finally, Proposition 3.9 implies that $\varphi \overset{sim}{\sim} \psi$. $\square$

Remark 4.19. We need in the proof of Corollary 4.18 that $\varphi'_K \overset{v_0}{\sim} \psi'_K$. To be able to conclude that, we need the full strength of $\varphi \overset{v}{\sim} \psi$, it would not be sufficient to assume $\varphi \overset{v_0}{\sim} \psi$.

We would like to conclude the article with a determination of the smallest dimension in which the answer to Question Q is not fully known. To do that, we first need to characterize totally singular quadratic forms of low dimension.

Proposition 4.20. *Let φ be a totally singular quadratic form over F. In each of the following cases, there exists a 2-independent set $\{a, b, c, d, e\}$ over F such that*:

(i) *Let $\dim \varphi = 1$ and $\operatorname{ndeg}_F \varphi = 1$, then $\varphi \overset{sim}{\sim} \langle 1 \rangle$.*

(ii) *Let $\dim \varphi = 2$ and $\operatorname{ndeg}_F \varphi = 2$, then $\varphi \overset{sim}{\sim} \langle 1, a \rangle$.*

(iii) *Let $\dim \varphi = 3$ and $\operatorname{ndeg}_F \varphi = 4$, then $\varphi \overset{sim}{\sim} \langle 1, a, b \rangle$.*

(iv) *Let $\dim \varphi = 4$;*

- *if $\operatorname{ndeg}_F \varphi = 4$, then $\varphi \overset{sim}{\sim} \langle\langle a, b \rangle\rangle$,*
- *if $\operatorname{ndeg}_F \varphi = 8$, then $\varphi \overset{sim}{\sim} \langle 1, a, b, c \rangle$.*

(v) *Let $\dim \varphi = 5$;*

- *if $\operatorname{ndeg}_F \varphi = 8$, then $\varphi \overset{sim}{\sim} \langle\langle a, b \rangle\rangle \perp c\langle 1 \rangle$,*
- *if $\operatorname{ndeg}_F \varphi = 16$, then $\varphi \overset{sim}{\sim} \langle 1, a, b, c, d \rangle$.*

(vi) *Let $\dim \varphi = 6$;*

- *if $\operatorname{ndeg}_F \varphi = 8$, then $\varphi \overset{sim}{\sim} \langle\langle a, b \rangle\rangle \perp c\langle 1, a \rangle$,*
- *if $\operatorname{ndeg}_F \varphi = 16$, then there are two possibilities: either $\varphi \overset{sim}{\sim} \langle\langle a, b \rangle\rangle \perp \langle c, d \rangle$, or $\varphi \overset{sim}{\sim} \langle 1, a, b, c, d, t + ad \rangle$ for some $t \in F^2(a, b, c)$,*
- *if $\operatorname{ndeg}_F \varphi = 32$, then $\varphi \overset{sim}{\sim} \langle 1, a, b, c, d, e \rangle$.*

Proof. Cases (i)–(iv) are obvious.

(v) Let $\dim \varphi = 5$. If $\operatorname{ndeg}_F \varphi = 8$, then φ is a quasi-Pfister neighbor, which is special by Corollary 4.9; the claim then follows from Proposition 4.10. If $\operatorname{ndeg}_F \varphi = 16 = 2^{\dim \varphi - 1}$, then φ is minimal, and the claim follows from Lemma 2.12.

(vi) Assume that $\dim \varphi = 6$. If $\operatorname{ndeg}_F \varphi = 8$, then φ is a quasi-Pfister neighbor, special by Corollary 4.9, and we obtain the required form from Proposition 4.10.

If $\dim \varphi = 6$ and $\operatorname{ndeg}_F \varphi = 16$, then consider $\tau \subseteq \varphi$ with $\dim \tau = 5$, and let $x \in F$ be such that $\varphi \simeq \tau \perp \langle x \rangle$. By (v), we can assume that either $\tau \simeq \langle\!\langle a, b \rangle\!\rangle \perp \langle c \rangle$, or $\tau \simeq \langle 1, a, b, c, d \rangle$. In the former case, we must have $x \in F^2(a, b, c, d) \setminus F^2(a, b, c)$, as otherwise $\operatorname{ndeg}_F \varphi = 8$. Therefore, $N_F(\varphi) = F^2(a, b, c, d) = F^2(a, b, c, x)$, so we can exchange the d in the 2-basis of $N_F(\varphi)$ for x. Up to renaming, we get that $\varphi \simeq \langle\!\langle a, b \rangle\!\rangle \perp \langle c, d \rangle$.

Now, let $\tau \simeq \langle 1, a, b, c, d \rangle$, i.e., $\varphi \simeq \langle 1, a, b, c, d, x \rangle$ for some $x \in F^2(a, b, c, d)$. Write $x = y + dz$ with $y, z \in F^2(a, b, c)$. If $z \in F^2$, then $\varphi \simeq \langle 1, a, b, c, d, y \rangle$. Denote $\sigma \simeq \langle 1, a, b, c, y \rangle$; then $\dim \sigma = 5$ and $\operatorname{ndeg}_F \sigma = 8$, so (up to renaming) $\sigma \simeq \langle\!\langle a, b \rangle\!\rangle \perp \langle c \rangle$ by (v), and hence $\varphi \simeq \langle\!\langle a, b \rangle\!\rangle \perp \langle c, d \rangle$. Now, assume that $z \notin F^2$. Then we can write $z = z_1 + a z_2$ with $z_1, z_2 \in F^2(b, c)$; we can assume that $z_2 \neq 0$ (otherwise exchange a with b or c). Hence, $F^2(a, b, c) = F^2(z, b, c)$; without loss of generality, we can assume $a = z$. We get $\varphi \simeq \langle 1, a, b, c, d, y + ad \rangle$ with $y \in F^2(a, b, c)$ as claimed.

In the remaining case, $\dim \varphi = 6$ and $\operatorname{ndeg}_F \varphi = 32$, we have $\operatorname{ndeg}_F \varphi = 2^{\dim \varphi - 1}$. Hence, φ is minimal, and we conclude by applying Lemma 2.12. $\square$

Corollary 4.21. *The answer to Question Q is positive for all totally singular quadratic forms of dimension ≤ 5.*

Proof. Let φ be a totally singular form of dimension ≤ 5. By Proposition 4.20, φ is minimal, or a quasi-Pfister form, or a special quasi-Pfister neighbor given by a triple (π, b, σ) with σ minimal. Hence, the claim follows by Theorem 4.6 (minimal forms), Proposition 3.1 (quasi-Pfister forms), and Theorem 4.17 (special quasi-Pfister neighbors). $\square$

Remark 4.22. In dimension six, the only case that remains open is if the norm degree of the form is 16. This case will be covered in a forthcoming article.

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