Pacific Journal of Mathematics

Volume 329 No. 2 April 2024

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Matthias Aschenbrenner
Fakultät für Mathematik
Universität Wien
Vienna, Austria
matthias.aschenbrenner@univie.ac.at

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Atsushi Ichino
Department of Mathematics
Kyoto University
Kyoto 606-8502, Japan
atsushi.ichino@gmail.com

Robert Lipshitz
Department of Mathematics
University of Oregon
Eugene, OR 97403
lipshitz@uoregon.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Dimitri Shlyakhtenko
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
shlyakht@ipam.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Ruixiang Zhang
Department of Mathematics
University of California
Berkeley, CA 94720-3840
ruixiang@berkeley.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2024 is US $645/year for the electronic version, and $875/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2024 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS
Vol. 329, No. 2, 2024
https://doi.org/10.2140/pjm.2024.329.201

CONCENTRATION INEQUALITIES FOR PALEY–WIENER SPACES

SYED HUSAIN AND FRIEDRICH LITTMANN

We consider how much of the mass of an element in a Paley–Wiener space can be concentrated on a given set. We seek bounds in terms of relative densities of the given set. We extend a result of Donoho and Logan from 1992 in one dimension and consider similar results in higher dimensions.

1. Introduction

Let M be a convex body in $\mathbb{R}^d$, and let $\mathcal{B}_p(M)$, $1 \le p \le \infty$, be the Paley–Wiener space of elements from $L^p(\mathbb{R}^d)$ with distributional Fourier transform supported in M. The Fourier transform $\mathcal{F}f$ is given by

$$\mathcal{F}f(\varphi) = \int_{\mathbb{R}} \hat{\varphi}(t) f(t)\, dt$$

for a Schwartz function φ. (We use $\hat{\varphi}(t) = \int \varphi(x) e^{-2\pi i x t}\, dx$.) We write $\mathcal{B}_p(\tau)$ if M is the ball with center at the origin and radius τ.

Let N and $W_\delta \subseteq \mathbb{R}^d$ be measurable and set $W_\delta(x) = x + W_\delta$. (In this article, W_δ is either a ball or a cube.) We consider the problem of finding a constant $C(M, \delta) > 0$ such that

$$\|G\chi_N\|_1 \le C(M,\delta) \sup_{x\in\mathbb{R}^d} |N \cap W_\delta(x)|\, \|G\|_1 \quad \text{for all } G \in \mathcal{B}_1(M). \tag{1}$$

Here $|\,.\,|$ denotes Lebesgue measure and χ_N is the characteristic function of N. We emphasize that the constant is not allowed to depend on N.

This question was studied by Donoho and Logan [1992] in dimension $d = 1$ in connection with recovery of a bandlimited signal that is corrupted by noise. In their setting, an unknown noise $n \in L^1(\mathbb{R})$ is added to a known signal $F \in \mathcal{B}_1([-\tau, \tau])$, and they investigate sufficient conditions under which the best approximation $\widetilde{F} \in \mathcal{B}_1([-\tau, \tau])$ to $F + n$ satisfies $\widetilde{F} = F$, i.e., when F can be perfectly recovered from knowledge of $F + n$ through best L^1-approximation.

MSC2020: primary 42A05, 94A12; secondary 30D10, 42A38, 94A11.
Keywords: entire functions of exponential type, bandlimiting, signal recovery, L1 recovery method, Logan's phenomenon, Nyquist density.

© 2024 The Authors, under license to MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

Denoting now by N the support of n, it is a useful fact that the concentration condition

$$\frac{\|G\chi_N\|_1}{\|G\|_1} < \frac{1}{2} \quad \text{for all } G \in \mathcal{B}_1(M) \tag{2}$$

is sufficient to conclude that $F = \widetilde{F}$. The argument can be found in several places, e.g., Donoho and Stark [1989, Section 6.2], who refer to it as Logan's phenomenon. (Logan's thesis [1965] appears to contain the earliest record of this argument.) It was shown in [Donoho and Logan 1992, Theorem 7] that (1) holds for $W_\delta(x) = \left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right]$ with

$$C([-\tau, \tau], \delta) = \frac{\pi\tau}{\sin(\pi\tau\delta)}, \tag{3}$$

where $\tau\delta < 1$, and combining this with (2), it is evident that this gives $F = \widetilde{F}$ provided the relative density (or Nyquist density) of the support of the noise satisfies

$$\delta^{-1} \sup_{x\in\mathbb{R}} |N \cap [x, x+\delta]| < \frac{\sin(\pi\tau\delta)}{2\pi\tau\delta}.$$

A preprint of Baranov, Jaming, Kellay, and Speckbacher [Baranov et al. 2023] extends these results to the setting of model spaces (of which the Paley–Wiener spaces are essentially special cases). We describe their results in more detail in the next section.

We mention that conditions to recover an element of a closed subspace of an L^1-space that has been corrupted by a sparse L^1-noise have been investigated in many different settings, and concentration inequalities lead frequently to sufficient conditions. (This relies on the fact that if a set N satisfies an analogue of (2) for all G in a given closed subspace of an L^1-space, then the zero function is the closest element from the subspace to every L^1 function with support contained in N.) We highlight a few of these results here. Abreu and Speckbacher [2021] obtained estimates about the concentration on a given subset of R^2 for the L_p-norm for functions in modulation and polyanalytic Fock space using convex optimization methods. In [Abreu and Speckbacher 2022], they formulated the large sieve principle for the continuous wavelet transform on the Hardy space, adapting the concept of maximum Nyquist density to the hyperbolic geometry of the underlying space. Jaming and Speckbacher [2021] found concentration estimates for finite expansions of spherical harmonics on two-point homogeneous spaces via the large sieve principle. Speckbacher and Hrycak [2020] used estimates of the spherical harmonics coefficients of certain zonal filters to derive upper bounds for concentration in terms of the maximum Nyquist density on the unit sphere S^2 for band-limited spherical harmonics expansions. Herrmann and Hennenfent [2008] developed a curvelet-based recovery, which recovered seismic wavefields from seismic data volumes

with large percentages of traces missing. Candès, Romberg, and Tao [Candès et al. 2006] did reconstruction using the discrete Fourier transform of finite discrete-time signals belonging to space C^N. Benyamini, Kroó, and Pinkus [Benyamini et al. 2012] studied the phenomenon that the zero function is the best L_1-approximant to functions with small support.

2. Results

There are two questions that this article seeks to address. Our first result deals with reconstruction in higher dimensions. We investigate the case when M is a cube and $W_\delta(0)$ is a ball with center at the origin. We denote by J_ν the Bessel function of the first kind and by $j_\nu(k)$ its k-th positive zero.

Theorem 1. *Let $\lambda, \alpha > 0$, $d \in \mathbb{N}$ and let $N \subseteq \mathbb{R}^d$ be the support of $n \in L^1(\mathbb{R}^d)$. If*

$$\alpha\lambda < \frac{j_{d/2}(1)\, d^{-\frac{1}{2}}}{2\pi},$$

then for all $G \in \mathcal{B}_1([-\lambda, \lambda]^d)$

$$\|G\chi_N\|_1 \le \frac{(\sqrt{d}\lambda)^{d/2}}{\alpha^{d/2} J_{d/2}(2\pi\sqrt{d}\alpha\lambda)} \sup_{x\in\mathbb{R}^d} |N \cap B(x,\alpha)|\, \|G\|_1. \tag{4}$$

We discuss conditions when this constant is best possible at the end of Section 3.

Second, it is clear that the shape of the bound in (3) requires $\delta\tau < 1$. In contrast, it was shown for $p = 2$ in [Donoho and Logan 1992, Theorem 4] that for any positive τ and δ

$$\|G\chi_N\|_2^2 \le (\tau + \delta^{-1}) \sup_{x\in\mathbb{R}} |N \cap [x, x+\delta]|\, \|G\|_2^2$$

for all $G \in \mathcal{B}_2(\tau)$ (with constants adjusted due to the different normalization of the Fourier transform) which suggests that an inequality with constant $c(\tau + \delta^{-1})$ should also be true for $p = 1$. The preprint [Baranov et al. 2023] mentioned in the introduction gives two approaches to establishing such an inequality for $1 \le p < \infty$, one approach is through oversampling, and another relying on a Bernstein type inequality in model spaces. Our next theorem shows that such a result for $p = 1$ can also be obtained with the strategy of Donoho and Logan. The constants in the following theorem are worse than the constants obtainable through the Bernstein type inequality in [Baranov et al. 2023] and better than the constants obtainable through oversampling.

Theorem 2. *Let $\tau, \delta > 0$ and let N be the support of $n \in L^1(\mathbb{R})$. Then for all $G \in \mathcal{B}_1(\tau)$*

$$\|G\chi_N\|_1 \le C_{\tau,\delta} \sup_{x\in\mathbb{R}} |N \cap [x, x+\delta]|\, \|G\|_1.$$

where $C_{\tau,\delta} \le \frac{80}{13}(\tau + \delta^{-1})$ for all positive τ and δ. The bound may be improved to $C_{\tau,\delta} \le \frac{5}{2}(\tau + \delta^{-1})$ for $\tau\delta \ge 2$.

As is usual with this method, the bounds only become effective when the density is a fraction of the reciprocal of the type τ. If one is interested in bounds for $\|G\chi_N\|_1/\|G\|_1$ at larger densities, a version of the Logvinenko–Sereda theorem from O. Kovrijkine [2001] gives nontrivial bounds whenever the density is smaller than 1.[1] (The constants in [Kovrijkine 2001] are not effective and don't yield concrete bounds to decide when the quotient is $< \frac{1}{2}$.)

3. Proof of Theorem 1

We briefly review a general approach to prove inequalities of the above form introduced by Donoho and Logan [1992]. Construct a kernel $K(x, y)$ so that $f \mapsto Tf$ given by

$$Tf(y) = \int K(x, y) f(x)\, dx$$

defines a bounded invertible transformation when restricted to $\mathcal{B}_1(\tau)$. Then a change of integration order gives

$$\begin{aligned}\int_N |G(x)|\, dx &\le \int_N \int |K(x, y)| T^{-1} G(x)|\, dx\, dy \\ &\le \left(\sup_x \int_N K(x, y)\, dy\right) \|T^{-1}\|\, \|G\|_1.\end{aligned}$$

If $K(x, y) = g(x-y)$ for some $g \in L^\infty$ with $\operatorname{supp}(g) \subseteq W_\delta(0)$, then the supremum may be further estimated by $\|g\|_\infty \sup_x |N \cap W_\delta(x)|$, where $T = T_g$ is now the convolution operator $T_g f = f * g$ restricted to $\mathcal{B}_1(\tau)$. For given g the size of the constant depends then only on $\|g\|_\infty \|T_g^{-1}\|$, and (2) shows that we need

$$\sup_x |N \cap W_\delta(x)| < \frac{1}{2\|g\|_\infty \|T_g^{-1}\|}.$$

Thus, it is the task to construct g as above where $\|g\|_\infty \|T_g^{-1}\|$ is as small as possible. To create an auxiliary function g with computable product $\|g\|_\infty \|T_g^{-1}\|$, Logan and Donoho observed that if $1/\hat{g}$ is positive and concave on an interval $I = [-a, a]$ with center at the origin, then the periodic extension of $1/\hat{g}$ restricted to I is the Fourier transform of a measure ν that acts as the inverse operator of convolution with g on $\mathcal{B}_1(a)$ and has total variation $|\nu| = 1/\hat{g}(a)$. (In fact, ν is the minimal extrapolation of $1/\hat{g}$ restricted to I in the sense of Beurling.)

For $x \in \mathbb{R}^d$ we consider

$$g_\alpha(x) = \chi_{B(0,\alpha)}(x) \tag{5}$$

[1] The authors are grateful to Walton Green for drawing their attention to [Kovrijkine 2001].

whose Fourier transform for $t \in \mathbb{R}^d$ is

$$\hat{g}_\alpha(t) = \frac{\alpha^{d/2} J_{d/2}(2\pi\alpha|t|)}{|t|^{d/2}}.$$

To construct a minimal extrapolation of $1/\hat{g}_\alpha$ restricted to $[-\tau, \tau]^d$, we need a representation of the reciprocal of $\hat{g}_\alpha$ as a Laplace transform of totally positive function. The theory was originally developed by Schoenberg [1951]; our notation follows the book of Hirschman and Widder [1955]. An entire function E belongs to the Laguerre–Pólya class $\mathcal{E}$ if and only if it has the form

$$E(s) = e^{cs^2+bs} \prod_{k=1}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{\frac{s}{a_k}},$$

where $c \geq 0,\ b,\ a_k\ (k = 1, 2, \dots)$ are real, and

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty.$$

Lemma 3. *There exists an integrable function $G \geq 0$ such that for $x \in (-j_p(1), j_p(1))$*

$$\frac{x^p}{J_p(x)} = \int_0^\infty e^{x^2 t} G(t)\, dt. \tag{6}$$

Proof. The Bessel function $J_p(x)$ has an infinite product representation [Olver and Maximon 2022, Section 10.21(iii)]. Dividing each side by x^p gives us

$$\begin{aligned}\frac{J_p(x)}{x^p} &= \frac{1}{2^p \Gamma(p+1)} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{j_p^2(k)}\right) \\ &= \frac{1}{2^p \Gamma(p+1)} \prod_{k=1}^{\infty} \left(1 - \frac{x}{j_p(k)}\right) e^{x/j_p(k)} \prod_{k=1}^{\infty} \left(1 + \frac{x}{j_p(k)}\right) e^{-x/j_p(k)}\end{aligned}$$

Substituting $x = \sqrt{y}$ in the infinite product representation of $J_p(x)/x^p$ gives us

$$\frac{J_p(\sqrt{y})}{y^{p/2}} = \frac{1}{2^p \Gamma(p+1)} \prod_{k=1}^{\infty} \left(1 - \frac{y}{j_p(k)^2}\right)$$

which is an entire function and belongs to class $\mathcal{E}$. Let $E(x) = J_p(x)/x^p$. Then by [Hirschman and Widder 1955, Theorem 6.1] the function $1/E(\sqrt{y})$ has a Laplace transform representation given by

$$\frac{1}{E(\sqrt{y})} = \int_{\mathbb{R}} e^{-yt} G(t)\, dt,$$

where $G(t) \in C^\infty$ is a nonnegative, integrable function and the integral converges in the largest vertical strip which contains the origin and is free of zeroes of $E(\sqrt{y})$,

which is $-\infty < y < j_p^2(1)$. Next, we want to determine the values of t for which $G(t) > 0$. Note that $E(\sqrt{y})$ may be expressed as

$$\begin{aligned} E(\sqrt{y}) &= \frac{1}{2^p\Gamma(p+1)} \prod_{k=1}^{\infty} \left(1 - \frac{y}{j_p^2(k)}\right) \\ &= \frac{1}{2^p\Gamma(p+1)} \exp\left(-\sum_{k=1}^{\infty} \frac{y}{j_p(k)}\right) \prod_{k=1}^{\infty} \left(1 - \frac{y}{j_p^2(k)}\right) e^{y/j_p(k)} \end{aligned}$$

The function $E(\sqrt{y})$ has no negative zeroes. We apply [Hirschman and Widder 1955, Chapter 5, Corollary 3.1] with $\alpha_1 = -\infty$ and

$$b = -\sum_{k=1}^{\infty} \frac{1}{j_p(k)}$$

to obtain that $G(t) > 0$ if $t \in (-\infty, 0)$ and $G(t) = 0$ otherwise, giving us for $y \in (0, j_p^2(1))$,

$$\frac{1}{E(\sqrt{y})} = \int_{-\infty}^{0} e^{-yt} G(t)\, dt.$$

Substituting $y = x^2$ and $t \mapsto -t$ gives the claim. □

Let $\lambda > 0$ and $\alpha > 0$ with $2\pi\sqrt{d}\alpha\lambda < j_{d/2}(1)$. We construct a (signed) measure ν that is an inverse transform on $\mathcal{B}_1([-\lambda, \lambda]^d)$ of convolution with g_α satisfying

$$\|f * \nu\|_1 \le \frac{(\sqrt{d}\lambda)^{d/2}}{\alpha^{d/2} J_{d/2}(2\pi\sqrt{d}\lambda\alpha)} \|f\|_1,$$

and we show that the constant is best possible among all inverse transformations of convolution with g_α on $\mathcal{B}_1([-\lambda, \lambda]^d)$. We expand $1/\hat{g}_\alpha$ restricted to $[-\lambda, \lambda]^d$ into its Fourier series

$$\frac{1}{\hat{g}_\alpha(t)} = \sum_{n \in \mathbb{Z}^d} H_\alpha(n) e^{2\pi i n t},$$

where

$$H_\alpha(n) = \left(\frac{1}{2\lambda}\right)^d \int_{[-\lambda,\lambda]^d} \frac{|x|^{d/2}}{\alpha^{d/2} J_{d/2}(2\pi\alpha|x|)} e^{-i\frac{\pi}{\lambda} n x}\, dx.$$

Lemma 4. *The coefficients satisfy*

$$H(n_1, \ldots, n_d) = (-1)^{n_1 + \cdots + n_d} |H(n_1, \ldots, n_d)|$$

Proof. The restrictions on $\alpha\lambda$ imply that the following integrals converge absolutely. Inserting the Schoenberg representation (6) gives with $n = (n_1, \dots, n_d)$

$$H_\alpha(n) = \left(\frac{1}{2\sqrt{2\pi\alpha\lambda}}\right)^d \int_{-\infty}^0 G(t)\left(\prod_{j=1}^d \int_{[-\lambda,\lambda]} e^{-x_j^2 t} e^{-i\frac{\pi}{\lambda} n_j x_j}\, dx_j\right) dt.$$

Since $t < 0$, the function $x_j \mapsto e^{-tx_j^2}$ is positive, symmetric, and concave. Hence $e^{-i\frac{\pi}{\lambda} n_j x_j}$ may be replaced by $\cos\left(\frac{\pi}{\lambda} n_j x_j\right)$. A short argument involving two integration by parts may be used to show that

$$(-1)^{n_j} \int_{[-\lambda,\lambda]} e^{-x_j^2 t} \cos\left(\tfrac{\pi}{\lambda} n_j x_j\right) dx_j \geq 0,$$

which implies the claim of the lemma. □

We define a measure ν_α on $\mathbb{R}^d$ by

$$\nu_\alpha = \sum_{n\in\mathbb{Z}^d} H_\alpha(n)\delta_{\frac{n}{2\lambda}},$$

where δ_x is the point measure at x with $\delta_x(\mathbb{R}^d) = 1$.

Lemma 5. *Let λ and α be positive with $2\pi\sqrt{d}\lambda\alpha < j_{d/2}(1)$. Convolution with ν_α is the inverse operator of convolution with g_α on $\mathcal{B}_1([-\lambda,\lambda]^d)$ with*

$$\|f * \nu_\alpha\|_1 \leq \frac{(\sqrt{d}\lambda)^{d/2}}{\alpha^{d/2} J_{d/2}(2\pi\sqrt{d}\alpha\lambda)} \|f\|_1$$

for all $f \in \mathcal{B}_1([-\lambda,\lambda]^d)$.

Proof. By construction of ν_α we have

$$\hat{g}_\alpha(t)\hat{\nu}_\alpha(t) = 1$$

for all $t \in [-\lambda,\lambda]^d$, and we observe that the total variation measure $|\nu_\alpha|$ satisfies

$$|\nu_\alpha|(\mathbb{R}^d) = \sum_{n\in\mathbb{Z}^d} |H_\alpha(n)| = \sum_{n\in\mathbb{Z}^d} H_\alpha(n)(-1)^{n_1+\cdots+n_d} = \frac{1}{\hat{g}_\alpha(\lambda,\dots,\lambda)},$$

and Minkowski's inequality $\|f * \nu_\alpha\|_1 \leq |\nu_\alpha|\,\|f\|_1$ shows that convolution with ν_α defines a bounded operator on $\mathcal{B}_1([-\lambda,\lambda]^d)$ that inverts convolution with g_α. □

Lemma 5 gives a bound for the operator norm of the inverse of convolution with g_α, and the calculation at the beginning of the proof of Theorem 1 may be used to complete the proof of (4).

Optimality. Let ν_g be a measure with $\hat{\nu}_g = 1/\hat{g}$ on $[-\lambda, \lambda]^d$. Among auxiliary functions g that satisfy

$$|\hat{g}(\lambda, \dots, \lambda)| = |\hat{g}(-\lambda, \dots, -\lambda)| \tag{7}$$

the choice g_α is optimal in the range $\alpha\lambda < (2\pi)^{-1} j_{d/2}(1) d^{-\frac{1}{2}}$. To show this, we follow the strategy of [Donoho and Logan 1992, Lemma 11]. We define

$$I_\infty = \sup\{|\hat{g}(\lambda, \dots, \lambda)| : \operatorname{supp}(g) \subseteq B(0, \alpha),\ \|g\|_\infty = 1\}$$

and observe that $\|\nu_g\| \geq 1/I_\infty$. If g satisfies (7) then we may assume that the function g optimizing I_∞ is even. It follows that

$$\hat{g}(\lambda, \dots, \lambda) = \int_{B(0,\alpha)} g(y_1, \dots, y_d) \cos(\lambda y_1) \cdots \cos(\lambda y_d)\, d(y_1, \dots, y_d).$$

The cosine terms are nonnegative in the stated range, hence the value of the transform is maximized under the constraint $|g| \leq 1$ by taking g to be equal to 1. The constant in Theorem 1 is obtained by choosing this g in (5), hence the constant is optimal in this case.

As a final remark, if we construct ν_g through a periodic extension of $1/\hat{g}$ (as in the previous section), then the condition that ν has finite total variation implies that $\hat{\nu}_g$ is continuous. Since a periodic extension must satisfy $\hat{\nu}(\lambda, \dots, \lambda) = \hat{\nu}(-\lambda, \dots, -\lambda)$, the condition (7) is then necessary in order for ν_g of finite total variation to exist.

4. Window comparisons

Analogously to dimension one, for a convex body K we define the maximum Nyquist density of N (relatively to K) by

$$\rho(N, K) = \frac{1}{|K|} \sup_{u \in \mathbb{R}^d} |N \cap (u + K)|.$$

We compare the result of Theorem 1 to the case where the window K is a hypercube of side length δ, which is an extension of the L_1 reconstruction result by [Donoho and Logan 1992]. The zero $j_p(1)$ has an asymptotic expansion given in [Olver and Maximon 2022] by

$$j_p(1) \simeq \left(\frac{p}{2} + \frac{1}{4}\right)\pi.$$

Denote the ball of radius r centered at origin by $B(0, r)$, and the volume of a ball with radius α in d-dimensions by $V_d(\alpha)$. It is given by

$$V_d(\alpha) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}\, \alpha^d.$$

When the window is a ball of radius α, perfect reconstruction is possible if the maximum Nyquist density satisfies

$$\rho(N, B(0,\alpha)) < \frac{\Gamma(d/2+1) J_{d/2}(2\pi\sqrt{d}\alpha\lambda)}{2(\pi\alpha\sqrt{d}\lambda)^{d/2}},$$

where

$$\alpha < \frac{j_{d/2}(1)}{2\pi\sqrt{d}\lambda}.$$

For $\lambda\delta < 2\pi$, the corresponding density bound is

$$\rho(N, [-\delta/2, \delta/2]^d) < \frac{1}{2}\left(\frac{\sin(\lambda\delta/2)}{\lambda\delta/2}\right)^d.$$

The support of the Fourier transform for both the problems is same, that is, $[-\lambda, \lambda]^d$. In order to be closely compare the two windows, we consider the following three cases. First, consider the ball of radius $\alpha = \delta/2$, centered at the origin, such that the ball is inside the cube. In this case, for dimension $d > 119$, the Nyquist density threshold for the ball window is bigger than that of cube window.

In the second case, we consider the ball with radius $\alpha = \delta\sqrt{d}/2$, so that the cube is inside the ball. Let $\delta = 1/(2\pi^2)$. For large d, the Nyquist density threshold asymptotically satisfies

$$\begin{aligned}\rho(T, B(0,\alpha), F) &< \frac{\Gamma(d/2+1) J_{d/2}(d/2)}{2(d/4)^{d/2}} \\ &\sim \frac{\sqrt{\pi d}\left(\frac{d}{2e}\right)^{d/2}}{2\left(\frac{d}{4}\right)^{d/2}} \frac{\Gamma(1/3)}{2^{1/3}\cdot 3^{1/6}\cdot \pi . d^{1/3}} \\ &\sim d^{1/6}\left(\frac{2}{e}\right)^{d/2}\end{aligned}$$

The Nyquist density for the cube window satisfies

$$\rho(T, [-\delta/2, \delta/2]^d, F) < \tfrac{1}{2}(4\pi^2 \sin(1/4\pi^2))^d.$$

The bound for the Nyquist density of the cube window remains larger than the bound for the Nyquist density of ball window for any d in this case.

Third, we set the volume of the cube is equal to the volume of the ball. Then the radius α of the ball satisfies

$$\alpha = \delta \sqrt[d]{\frac{\Gamma(d/2+1)}{\pi^{d/2}}}.$$

Using Stirling's approximation, we get

$$\alpha \simeq \delta \sqrt[d]{\left(\frac{d}{2\pi e}\right)^{d/2} (\pi d)^{1/2}}.$$

Let $\delta = \sqrt{2\pi e}/(4\pi^2)$. For large d, the Bessel function in the Nyquist density of the ball window satisfies

$$J_{d/2}(2\pi^2\sqrt{\alpha}) = J_{d/2}(d \cdot \pi^{1/2d} \cdot d^{1/2d}/2) \to J_{d/2}(d/2),$$

since $\pi^{1/2d} \cdot d^{1/2d} \to 1$ for large d. The bound for the Nyquist density of the ball window is then

$$\begin{aligned}\rho(T, B(0,\alpha), F) &< \frac{\Gamma(d/2+1) J_{d/2}(d/2)}{2\left(\pi^2\sqrt{d}\,\frac{\sqrt{2\pi e}}{4\pi^2\sqrt{\pi}} \sqrt[d]{\Gamma(d/2+1)}\right)} \\ &\sim \frac{\Gamma(1/3)}{2^{4/3} \cdot 3^{1/6} \cdot \pi^{3/4} d^{1/12}} \left(\frac{4}{2e}\right)^{d/2}\end{aligned}$$

For the cube window, the sufficient bound for reconstruction is

$$\rho(T, [-\delta/2, \delta/2]^d, F) < \frac{1}{2}\left(\frac{\sin(\sqrt{2\pi e}/8\pi)}{\sqrt{2\pi e}/8\pi}\right)^d.$$

In this case also, the Nyquist density for the cube window remains larger than the Nyquist density of ball window for any dimension d. In conclusion, the bigger the Nyquist density threshold, the better, since it allows the signal to incorporate more noise but still be recovered. Only for the case when the ball is just inside the cube, the Nyquist density threshold for the ball window is larger than that of cube window for $d > 119$. For all other cases, the Nyquist density threshold for the cube window stays larger.

5. Proof of Theorem 2

Returning to the strategy described in Theorem 1, the choice in [Donoho and Logan 1992] was $g = \chi_{[-\frac{\delta}{2}, \frac{\delta}{2}]}$, which is optimal for $\delta\tau \le \frac{1}{2}$, gives a nonoptimal bound for $\frac{1}{2} < \delta\tau < 1$, and fails to give a bound for $\delta\tau \ge 1$. This can be traced back to the fact that $\hat{g}(\delta) = 0$.

We define for $\tau > 0$ and real x a function g_τ, supported on $[-1, 1]$, by

$$g_\tau(x) = -2(1 - |x|)\frac{\cos 2\pi(\tau+1)x - \cos 2\pi\tau x}{4\pi^2 x^2} \chi_{[-1,1]}(x).$$

The Fourier transform of g_τ has the useful property that the sum of its partials with respect to t and τ has a simple integral representation.

Proposition 6. *For any t and τ*

$$\frac{\partial}{\partial t}(\hat{g}_\tau(t)) + \frac{\partial}{\partial \tau}(\hat{g}_\tau(t)) = \int_{2\pi(t+\tau)}^{2\pi(t+\tau+1)} \frac{\sin^2 u}{u^2}\, du.$$

Proof. For ease of notation we set $G(t,\tau) = \hat{g}_\tau(t)$, and we denote first partials by G_t and G_τ. Writing

$$g_\tau(x) = 2(1-|x|)\frac{\cos 2\pi(\tau+1)x - \cos 2\pi\tau x}{(-2\pi i x)^2}\chi_{[-1,1]}(x)$$

and using that $g_\tau(x)$ is even, we have

$$\begin{aligned} G_t(t,\tau) &= \int_{-1}^{1} (-2\pi i x) g_\tau(x) e^{-2\pi i x t}\, dx \\ &= \int_{-1}^{1} (-2\pi i x) g_\tau(x)(-i\sin(2\pi x t))\, dx \\ &= \int_{-1}^{1} 2(1-|x|)\frac{\cos(2\pi(\tau+1)x) - \cos(2\pi\tau x)}{2\pi x}\sin(2\pi x t)\, dx \\ &= 4\int_0^1 (1-x)\frac{\cos(2\pi(\tau+1)x) - \cos(2\pi\tau x)}{2\pi x}\sin(2\pi x t)\, dx. \end{aligned}$$

Similarly,

$$\begin{aligned} G_\tau(t,\tau) &= \int_{-1}^{1} \frac{\partial}{\partial\tau}(g_\tau(x)) e^{-2\pi i x t}\, dx \\ &= 4\int_0^1 (1-x)\frac{\sin(2\pi(\tau+1)x) - \sin(2\pi\tau x)}{2\pi x}\cos(2\pi x t)\, dx. \end{aligned}$$

The integrals have representations in terms of the sine-integral

$$\mathrm{Si}(u) = \int_0^u \frac{\sin(w)}{w}\, dw.$$

A direct calculation gives

$$2\int_0^1 \cos(2\pi a x)\frac{\sin(2\pi b x)}{x}\, dx = \mathrm{Si}(2\pi(a+b)) - \mathrm{Si}(2\pi(a-b))$$

$$2\int_0^1 \cos(2\pi a x)\sin(2\pi b x)\, dx = -\frac{b}{\pi(a-b)(a+b)} + \frac{\cos(2\pi(a-b))}{2\pi(a-b)} - \frac{\cos(2\pi(a+b))}{2\pi(a+b)}.$$

We obtain

$$\begin{aligned} G_t(t,\tau) + G_\tau(t,\tau) &= \frac{2}{\pi^2}\left(\frac{\sin^2(\pi(t+\tau))}{(t+\tau)(t+\tau+1)} + \pi\,\mathrm{Si}(2\pi(t+\tau+1)) - \pi\,\mathrm{Si}(2\pi(t+\tau))\right) \\ &= \frac{2}{\pi}\int_{\pi(t+\tau)}^{\pi(t+\tau+1)}\left(-\frac{\partial}{\partial u}\frac{\sin^2 u}{u}\right)du + \frac{2}{\pi}\int_{2\pi(t+\tau)}^{2\pi(t+\tau+1)}\frac{\sin w}{w}\,dw \\ &= \frac{2}{\pi}\int_{\pi(t+\tau)}^{\pi(t+\tau+1)}\frac{\sin^2 u}{u^2}\,du \end{aligned}$$

after substituting $w = 2u$ and combining the integrands. □

Corollary 7. (i) *The function* $\tau \mapsto \hat{g}_\tau(0)$ *is positive, monotonically increasing, and has limit* 1 *as* $\tau \to \infty$. *Moreover,*

$$\hat{g}_0(0) > 0.65, \quad \hat{g}_1(1) > 0.8.$$

(ii) $t \mapsto (\hat{g}_\tau(t))^{-1}$ *is positive and concave on* $[-\tau, \tau]$.

(iii) $\|g_\tau\|_\infty = g_\tau(0) = 2\tau + 1$.

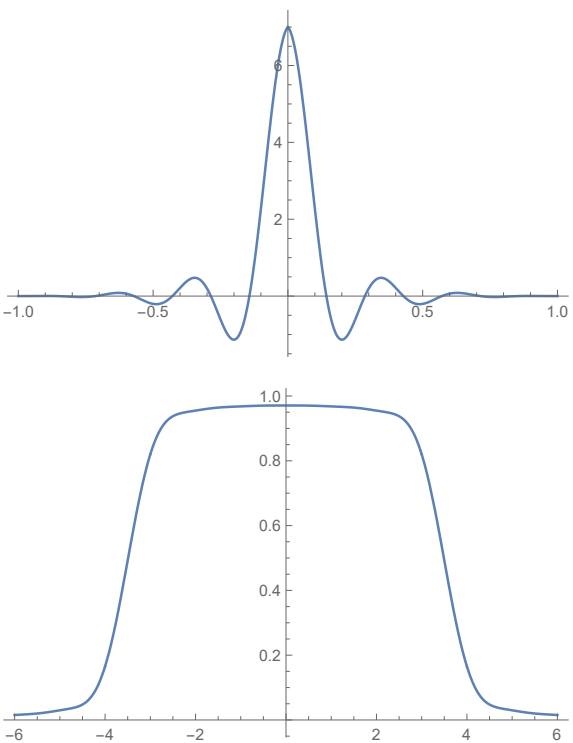

Figure 1. The transform pair $g_4(x)$ and $\hat{g}_4(t)$.

Proof. For the proof of (i) it follows from symmetry of $t \mapsto \hat{g}_\tau(t)$ that $\hat{g}'_\tau(0) = 0$, and hence Proposition 6 gives $\partial/\partial\tau \hat{g}_\tau(0) > 0$. Direct calculations give the claimed bounds.

Regarding (ii), we require an explicit representation of $\hat{g}''_\tau(t)$. It follows from

$$g_\tau(x) = 2(1-|x|)\frac{\cos 2\pi(\tau+1)x - \cos 2\pi\tau x}{(-2\pi i x)^2}\chi_{[-1,1]}(x)$$

that

$$\begin{aligned}(8)\qquad \hat{g}''_\tau(t) &= \int_{-1}^{1}(-2\pi i x)^2 g_\tau(x) e^{-2\pi i x t}\,dx \\ &= -\left(\frac{\sin\pi(t-\tau)}{\pi(t-\tau)}\right)^2 - \left(\frac{\sin\pi(t+\tau)}{\pi(t+\tau)}\right)^2 \\ &\qquad + \left(\frac{\sin\pi(t-\tau-1)}{\pi(t-\tau-1)}\right)^2 + \left(\frac{\sin\pi(t+\tau+1)}{\pi(t+\tau+1)}\right)^2 \\ &= \frac{\sin^2(\pi(t-\tau))(2(t-\tau)-1)}{(t-\tau)^2(t-\tau-1)^2} - \frac{\sin^2(\pi(t+\tau))(2(t+\tau)+1)}{(t+\tau)^2(t+\tau+1)^2}.\end{aligned}$$

Since the first term is negative for $t-\tau < \frac{1}{2}$ and the second term is positive for $t+\tau > -\frac{1}{2}$, it follows that

$$\hat{g}''_\tau(t) < 0 \text{ for } -\tau - \tfrac{1}{2} < t < \tau + \tfrac{1}{2}.$$

Multivariable chain rule and Proposition 6 show that

$$\frac{\partial}{\partial\tau}(\hat{g}_\tau(\tau)) > 0,$$

and since $\hat{g}_0(0) > 0$, it follows that $\hat{g}_\tau(\tau) > 0$ for all τ. Since $\hat{g}_\tau$ is concave down on $[-\tau, \tau]$, it follows that $\hat{g}_\tau(t) > 0$ for $t \in [-\tau, \tau]$. It follows that the second derivative of $t \mapsto (\hat{g}_\tau(t))^{-1}$ is positive for $|t| \le \tau$.

To prove (iii) we show $\hat{g}_\tau > 0$ on $\mathbb{R}$. Identity (8) implies that $\hat{g}''_\tau(t) = \mathcal{O}(|t|^{-2})$. Since $\hat{g}'_\tau(0) = 0$, the integral of $\hat{g}''_\tau$ on $(0, t)$ equals $\hat{g}'_\tau(t)$.

A (lengthy) calculation along the lines of the proof of Proposition 6 shows from (8) that $\hat{g}'_\tau < 0$ on $(0, \infty)$. Since $\hat{g}_\tau(t) \to 0$ as $t \to \infty$ (by the Riemann–Lebesgue lemma) it follows that $\hat{g}_\tau$ is positive on $(0, \infty)$ and by symmetry on $\mathbb{R}$. Hence $|g_\tau(x)| \le g_\tau(0)$ for all x. The second identity in (iii) is a direct evaluation. □

Proof of Theorem 2. Setting $g_{\tau,\delta}(x) = g_{\tau\delta/2}(2x/\delta)$, we observe that $g_{\tau,\delta}$ is supported on $[-\delta/2, \delta/2]$, and

$$\hat{g}_{\tau,\delta}(t) = \frac{\delta}{2}\hat{g}_{\tau\delta/2}(\delta t/2).$$

It follows that $t \mapsto (\hat{g}_{\tau,\delta}(t))^{-1}$ is positive and concave for $|t| \le \tau$. Let $a_n = a_n(\tau, \delta)$ be the Fourier coefficients satisfying

$$\frac{1}{\hat{g}_{\tau,\delta}(t)} = \sum_{n\in\mathbb{Z}} a_n e^{\pi i \frac{t}{\tau} n}$$

for $|t| \le \tau$. Positivity and convexity imply that $|a_n| = (-1)^n a_n$. Define a measure $\nu = \nu_{\tau,\delta}$ on $\mathbb{R}$ for any Borel set A by

$$\nu(A) = \sum_{n\in\mathbb{Z}} a_n \delta_{n/(2\tau)}(A),$$

where δ_b is the Dirac measure at $b \in \mathbb{R}$. We observe that $\hat{\nu}(t) = 1/\hat{g}_{\tau,\delta}(t)$ for $|t| \le \tau$, and the total variation satisfies

$$|\nu|(\mathbb{R}) = \sum_{n\in\mathbb{Z}} |a_n| = \sum_{n\in\mathbb{Z}} a_n (-1)^n = \frac{1}{\hat{g}_{\tau\delta/2}(\tau\delta/2)}.$$

It follows that convolution with ν is the inverse operator of convolution with $g_{\tau,\delta}$ when restricted to PW^1_τ. Moreover, for $g_{\tau,\delta}$ the choice of ν is optimal, since the value of the Fourier transform of ν is always a lower bound for the total variation.

It follows that

$$\|T^{-1}_{g_{\tau,\delta}}\| = \frac{1}{\hat{g}_{\tau\delta/2}(\tau\delta/2)}.$$

We observe the identities

$$\frac{\|g_{\tau,\delta}\|_\infty}{\hat{g}_{\tau,\delta}(\tau)} = \frac{2}{\delta}\frac{\|g_{\tau\delta/2}\|_\infty}{\hat{g}_{\tau\delta/2}(\tau\delta/2)} = \frac{2\tau + 2\delta^{-1}}{\hat{g}_{\tau\delta/2}(\tau\delta/2)}.$$

For $\tau > 0$ and $\delta > 0$ we use the inequality $\hat{g}_{\tau\delta/2}(\tau\delta/2) \ge \hat{g}_0(0) > 0.65$. For $\tau\delta \ge 2$, we may use the lower bound $\hat{g}_1(1) > 0.8$ instead. □

References

[Abreu and Speckbacher 2021] L. D. Abreu and M. Speckbacher, "Donoho–Logan large sieve principles for modulation and polyanalytic Fock spaces", *Bull. Sci. Math.* **171** (2021), art. id. 103032. MR Zbl

[Abreu and Speckbacher 2022] L. D. Abreu and M. Speckbacher, "Donoho–Logan large sieve principles for the continuous wavelet transform", 2022. arXiv 2210.13056

[Baranov et al. 2023] A. Baranov, P. Jaming, K. Kellay, and M. Speckbacher, "Carleson measures and Oversampling in model spaces", 2023. arXiv 2304.02385v1

[Benyamini et al. 2012] Y. Benyamini, A. Kroó, and A. Pinkus, "L^1-approximation and finding solutions with small support", *Constr. Approx.* **36**:3 (2012), 399–431. MR Zbl

[Candès et al. 2006] E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information", *IEEE Trans. Inform. Theory* **52**:2 (2006), 489–509. MR Zbl

[Donoho and Logan 1992] D. L. Donoho and B. F. Logan, "Signal recovery and the large sieve", *SIAM J. Appl. Math.* **52**:2 (1992), 577–591. MR Zbl

[Donoho and Stark 1989] D. L. Donoho and P. B. Stark, "Uncertainty principles and signal recovery", *SIAM J. Appl. Math.* **49**:3 (1989), 906–931. MR Zbl

[Herrmann and Hennenfent 2008] F. J. Herrmann and G. Hennenfent, "Non-parametric seismic data recovery with curvelet frames", *Geophys. J. Internat.* **173**:1 (2008), 233–248.

[Hirschman and Widder 1955] I. I. Hirschman and D. V. Widder, *The convolution transform*, Princeton University Press, 1955. MR Zbl

[Jaming and Speckbacher 2021] P. Jaming and M. Speckbacher, "Concentration estimates for finite expansions of spherical harmonics on two-point homogeneous spaces via the large sieve principle", *Sampl. Theory Signal Process. Data Anal.* **19**:1 (2021), art. id. 9. MR Zbl

[Kovrijkine 2001] O. Kovrijkine, "Some results related to the Logvinenko–Sereda theorem", *Proc. Amer. Math. Soc.* **129**:10 (2001), 3037–3047. MR Zbl

[Logan 1965] B. F. Logan, *Properties of high-pass signals*, Ph.D. thesis, Columbia University, 1965.

[Olver and Maximon 2022] F. W. J. Olver and L. C. Maximon, "Bessel functions", Digital Library of Mathematical Functions, 2022, available at http://dlmf.nist.gov/10. Version 1.1.6.

[Schoenberg 1951] I. J. Schoenberg, "On Pólya frequency functions, I: The totally positive functions and their Laplace transforms", *J. Analyse Math.* **1** (1951), 331–374. MR Zbl

[Speckbacher and Hrycak 2020] M. Speckbacher and T. Hrycak, "Concentration estimates for band-limited spherical harmonics expansions via the large sieve principle", *J. Fourier Anal. Appl.* **26**:3 (2020), art. id. 38. MR Zbl

Received January 6, 2024.

Syed Husain
Department of Mathematics
Truman State University
Kirksville, MO
United States
shusain@truman.edu

Friedrich Littmann
Department of Mathematics
North Dakota State University
Fargo, ND
United States
friedrich.littmann@ndsu.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 329, No. 2, 2024
https://doi.org/10.2140/pjm.2024.329.217

CHARACTERIZING THE FOURIER TRANSFORM BY ITS PROPERTIES

MATEUSZ KRUKOWSKI

It is common knowledge that the Fourier transform enjoys the convolution property, i.e., it turns convolution in the time domain into multiplication in the frequency domain. It is probably less known that this property characterizes the Fourier transform amongst all linear and bounded operators $T : L^1 \to C^b$. Thus, a natural question arises: are there other features characterizing the Fourier transform besides convolution property? We provide an affirmative answer by investigating the time differentiation property and its discrete counterpart, used to characterize discrete-time Fourier transform. Next, we move on to locally compact abelian groups, where differentiation becomes meaningless, but the Fourier transform can be characterized via time shifts. The penultimate section of the paper returns to the convolution characterization, this time in the context of compact (not necessarily abelian) groups. We demonstrate that the proof existing in the literature can be greatly simplified with the aid of representation theory techniques. Lastly, we hint at the possibility of other transforms being characterized by their properties and demonstrate that the Hankel transform may be characterized by a Bessel-type differential property.

1. Introduction

Fourier transform is one of the central topics in harmonic analysis. Its significance is boosted by numerous applications in fields such as the analysis of differential equations, nuclear magnetic resonance, infrared spectroscopy, signal/image processing or quantum mechanics. Amongst many attempts at explaining why the Fourier transform is so exceptional, there is a group of arguments which follow a similar pattern. They boil down to the reasoning that the Fourier transform is the only operator satisfying a specified set of properties. One may regard this approach as "axiomatic" in the sense that we do not start with a complete formula for the Fourier transform, but rather demand certain properties of an operator and derive the formula as a necessary conclusion.

MSC2020: 42A38, 43A25, 43A30.

Keywords: Fourier transform, convolution property, representation theory on compact groups, Hankel transform.

© 2024 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

To give an example of such an approach, in his paper "A characterization of Fourier transforms" (see [7]) Jaming proved that the convolution property

$$\forall_{f,g\in L^1} \quad \mathcal{F}(f \star g) = \mathcal{F}(f)\,\mathcal{F}(g)$$

essentially characterizes the Fourier transform on real numbers $\mathbb{R}$, the circle group S^1, the integers $\mathbb{Z}$ and the finite cyclic group $\mathbb{Z}_n$. His article inspired Lavanya and Thangavelu to show that any continuous *-homomorphism of $L^1(\mathbb{C}^d)$ (with twisted convolution as multiplication) into $B(L^2(\mathbb{R}^d))$ is essentially the Weyl transform and deduce a similar characterization for the Fourier transform on the Heisenberg group (see [8] and [9]). Furthermore, Kumar and Sivananthan went on to demonstrate that the convolution property characterizes the Fourier transform on all compact groups (see [11]).

It is easy to recognize a common thread in the articles cited above — they all focus on the convolution property as the key ingredient in characterizing the Fourier transform. It is thus quite natural to ponder the question: are there any other properties that distinguish the Fourier transform? Our goal is to answer this question with a resounding "yes!"

The paper is comprised of the introduction, followed by four thematic sections and a bibliography. Section 2 provides a brief summary of the classical Fourier transform, with an emphasis on its time differentiation property. We extend this property to a larger class of integral transforms and introduce the Dirac delta property, which is a kind of "boundary condition". Theorem 1 is the climactic point of the section, where we demonstrate that the time differentiation property (coupled with Dirac delta property) characterizes the classical Fourier transform on $\mathbb{R}$. We go on to prove that a similar technique (suitably adjusted to $\mathbb{Z}$ instead of $\mathbb{R}$) can be used to characterize the discrete-time Fourier transform.

Section 3 is preoccupied with the Fourier transform on an arbitrary locally compact abelian group. Needless to say, it makes little sense to talk of differentiation in this context so we study the time shift property instead. We demonstrate that this condition characterizes the Fourier transform with the help of (suitably reformulated) Dirac delta property (see Theorem 3).

Section 4 commences with a recap of representation theory on compact groups. We work towards describing a bijection between irreducible, unitary representations of G and irreducible, unitary *-representations of $L^1(G)$. This correspondence becomes the fundamental tool in demonstrating that the convolution property characterizes the Fourier transform on compact groups. Arguably, our proof simplifies earlier works of Kumar and Sivananthan (see [11]).

Section 5 is meant to stimulate further research regarding other integral transforms. We prove that the Fourier transform is not the only operator which is characterized by some differentiation property. After a brief summary of the Hankel transform,

Theorem 5 demonstrates that the Bessel's differentiation property (together with a suitable "boundary condition") characterizes the Hankel transform.

2. Fourier transform on reals and integers

It is common knowledge that the Fourier transform

$$\mathcal{F}(f) := \int_{\mathbb{R}} f(x)\, e^{-2\pi i x y}\, dx$$

satisfies the equality

$$\mathcal{F}(f')(y) = 2\pi i y \mathcal{F}(f)(y) \tag{1}$$

for every $y \in \mathbb{R}$ and sufficiently "good" function f. The property is so elemental that it permeates to the realm of quantum mechanics, where it states that the Fourier transform of the momentum operator

$$\hat{p} := -i\hbar \frac{\partial}{\partial x}$$

is (up to a constant) the position operator

$$\hat{x}\psi(x,t) := x\psi(x,t).$$

Our aim in the current section is to argue that the time differentiation property (1) is so fundamental that it characterizes the Fourier transform (given an appropriate "boundary condition").

To begin with, let:

- $C^b(\mathbb{R})$ be the Banach space of complex-valued, continuous and bounded functions on $\mathbb{R}$.
- $C^1_c(\mathbb{R})$ be the vector space of complex-valued, compactly supported and C^1-functions on $\mathbb{R}$.
- $L^1(\mathbb{R})$ be the Banach space of complex-valued, integrable functions on $\mathbb{R}$.

We focus on integral transforms $T : L^1(\mathbb{R}) \to C^b(\mathbb{R})$, i.e.,

$$T(f)(y) := \int_{\mathbb{R}} K(x, y)\, f(x)\, dx,$$

where $K \in C^b(\mathbb{R} \times \mathbb{R})$ is differentiable with respect to the first variable. Amongst all these operators we distinguish those that satisfy the time differentiation property, i.e.,

$$T(f')(y) = 2\pi i y T(f)(y) \tag{2}$$

for every $f \in C^1_c(\mathbb{R})$ and $y \in \mathbb{R}$. The condition is plainly molded in the image of (1), although we cannot expect it to characterize the Fourier transform alone. This is

because the zero map is a perfect example of an integral transform satisfying (2), which is not the Fourier transform. Thus, we need some sort of "boundary condition" accompanying the time differentiation property.

In order to introduce such a condition, we say that $(\delta_n) \subset C_c^1(\mathbb{R})$ is a Dirac delta sequence if every δ_n

- has compact support contained in $\left[-\frac{1}{n}, \frac{1}{n}\right]$,
- is nonnegative and symmetric, i.e., $\delta_n(x) = \delta_n(-x)$ for every $x \in \mathbb{R}$,
- satisfies the equality
$$\int_{\mathbb{R}} \delta_n(x)\,dx = 1.$$

An example of a Dirac delta sequence is

$$\delta_n(x) := \begin{cases} Cne^{-\frac{1}{1-(nx)^2}} & \text{if } x \in \left(-\frac{1}{n}, \frac{1}{n}\right), \\ 0, & \text{otherwise,} \end{cases}$$

where

$$C := \left(\int_{-1}^{1} e^{-\frac{1}{1-x^2}}\,dx\right)^{-1}.$$

The choice of constant C is such that

$$\int_{\mathbb{R}} \delta_n(x)\,dx = 1$$

for every $n \in \mathbb{N}$.

We are ready to introduce the "boundary condition" accompanying the time differentiation property. We say that $T : L^1(\mathbb{R}) \to C^b(\mathbb{R})$ satisfies the Dirac delta property if there exists a Dirac delta sequence $(\delta_n) \subset C_c^1(\mathbb{R})$ such that for every $y \in \mathbb{R}$ we have

$$\lim_{n\to\infty} T(\delta_n)(y) = 1. \tag{3}$$

Intuitively, the property means that T maps (certain) @functions "compressed in time" to functions "spread in frequency". It is not surprising that the Fourier transform has the Dirac delta property (which is a good indicator that we have not strayed from the set path). Indeed, for every $y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \mathcal{F}(\delta_n)(y) = \int_{\mathbb{R}} \delta_n(x)\,e^{-2\pi i x y}\,dx &= \int_{(-\infty,0)} \delta_n(x)\,e^{-2\pi i x y}\,dx + \int_{(0,\infty)} \delta_n(x)\,e^{-2\pi i x y}\,dx \\ &= \int_{(0,\infty)} \delta_n(-x)\,e^{2\pi i x y}\,dx + \int_{(0,\infty)} \delta_n(x)\,e^{-2\pi i x y}\,dx \\ &= 2\int_{(0,\infty)} \delta_n(x)\cos(2\pi x y)\,dx, \end{aligned}$$

due to the fact that every δ_n is symmetric. Since

$$1 \geqslant \cos(2\pi xy) \geqslant \cos\left(2\pi \cdot \frac{y}{n}\right)$$

for $x \in \left[0, \frac{1}{n}\right)$ and n large enough then

$$1 = 2\int_{(0,\infty)} \delta_n(x)\,dx \geqslant \mathcal{F}(\delta_n)(y) \geqslant 2\cos\left(2\pi \cdot \frac{y}{n}\right)\int_{(0,\infty)} \delta_n(x)\,dx = \cos\left(2\pi \cdot \frac{y}{n}\right).$$

Taking the limit $n \to \infty$ we conclude that $\mathcal{F}(\delta_n)(y) \to 1$ for every $y \in \mathbb{R}$.

Theorem 1. *Suppose that $T : L^1(\mathbb{R}) \to C^b(\mathbb{R})$ is an integral transform with kernel $K \in C^b(\mathbb{R} \times \mathbb{R})$, which is differentiable with respect to the first variable. If T satisfies*

- *the time differentiation property (2), and*
- *the Dirac delta property (3)*

then T is the Fourier transform.

Proof. For every $f \in C_c^1(\mathbb{R})$ and $y \in \mathbb{R}$ we have (due to integration by parts)

$$T(f')(y) = \int_{\mathbb{R}} K(x, y)\, f'(x)\,dx = -\int_{\mathbb{R}} \frac{\partial}{\partial x} K(x, y)\, f(x)\,dx. \tag{4}$$

Using the time differentiation property we write

$$2\pi i y \int_{\mathbb{R}} K(x, y)\, f(x)\,dx = -\int_{\mathbb{R}} \frac{\partial}{\partial x} K(x, y)\, f(x)\,dx$$

or, equivalently

$$0 = \int_{\mathbb{R}} \left(\frac{\partial}{\partial x} K(x, y) + 2\pi i y K(x, y)\right) f(x)\,dx$$

for every $f \in C_c^1(\mathbb{R})$ and $y \in \mathbb{R}$. By the fundamental theorem of calculus of variations, we conclude that

$$0 = \frac{\partial}{\partial x} K(x, y) + 2\pi i y K(x, y) = \frac{\partial}{\partial x}\left(e^{2\pi i y x} K(x, y)\right)$$

for every $x, y \in \mathbb{R}$. This implies that

$$K(x, y) = g(y)\, e^{-2\pi i y x}$$

where $g \in C^b(\mathbb{R})$.

At this stage we know that

$$T(f)(y) = g(y)\, \mathcal{F}(f)(y) \quad \text{for every } f \in C_c^1(\mathbb{R}) \text{ and } y \in \mathbb{R}.$$

We fix $y \in \mathbb{R}$ and pick any Dirac delta sequence (δ_n) for the operator T. We have

$$1 = \lim_{n\to\infty} T(\delta_n)(y) = \lim_{n\to\infty} g(y)\,\mathcal{F}(\delta_n)(y) = g(y),$$

which proves that $T = \mathcal{F}$ on $C_c^1(\mathbb{R})$. However, since $C_c^1(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, the two linear and bounded transforms (T and $\mathcal{F}$) must coincide on the whole space $L^1(\mathbb{R})$. □

Motivated by the theorem above, we try to adapt the proof to characterize the Fourier transform on $\mathbb{Z}$, i.e., the discrete-time Fourier transform $\mathcal{F}: \ell^1(\mathbb{Z}) \to C(S^1)$ given by

$$\mathcal{F}(f)(y) := \sum_{n\in\mathbb{Z}} f(n)\, e^{-2\pi i n y}.$$

S^1 usually stands for the unit circle group, but in the context of the discrete-time Fourier transform, it is convenient (or customary) to understand it as an interval $[0, 1]$ with addition mod 1.

Functions (sequences) in $\ell^1(\mathbb{Z})$ are not differentiable per se, but we may imitate differentiability with forward and backward difference operators:

$$\Delta^+ f(n) := f(n+1) - f(n), \quad \Delta^- f(n) := f(n) - f(n-1).$$

This allows for the following characterization:

Theorem 2. *If a linear and bounded operator $T: \ell^1(\mathbb{Z}) \to C(S^1)$ satisfies*

- *the time difference property, i.e., for every $f \in \ell^1(\mathbb{Z})$ and $y \in S^1$ we have*

$$T(\Delta^+ f)(y) = (e^{2\pi i y} - 1)\, T(f)(y), \tag{5}$$

- *$T(\mathbb{1}_0)(y) = 1$ for every $y \in S^1$ ($\mathbb{1}_0$ stands for an indicator function/sequence of the singleton $\{0\}$),*

then it is the discrete-time Fourier transform.

Proof. Since both $\mathbb{Z}$ and S^1 are σ-finite spaces, we use $L^1 - L^\infty$ duality to establish the existence of a kernel $K \in L^\infty(\mathbb{Z} \times S^1)$ such that

$$T(f)(y) = \sum_{n\in\mathbb{Z}} K(n, y)\, f(n)$$

for every $y \in S^1$ (see Theorem 1.3 in [2]). Next, for every $f \in \ell^1(\mathbb{Z})$ and $y \in S^1$ we use summation by parts to get

$$\begin{aligned} T(\Delta^+ f)(y) &= \lim_{N\to\infty} \sum_{n=-N}^{N} K(n, y)\big(f(n+1) - f(n)\big) \\ &= \lim_{N\to\infty} \Big(K(N, y)\, f(N+1) - K(-N, y)\, f(-N)\Big) \\ &\qquad - \lim_{N\to\infty} \sum_{n=-N}^{N-1} f(n+1)\big(K(n+1, y) - K(n, y)\big) \\ &= -\sum_{n\in\mathbb{Z}} f(n+1)\, \Delta^+ K(n, y) = -\sum_{n\in\mathbb{Z}} f(n) \Delta^- K(n, y). \end{aligned}$$

By (5) we have

$$(e^{2\pi i y} - 1)\, T(f)(y) = -\sum_{n\in\mathbb{Z}} f(n) \Delta^- K(n, y)$$

for every $f \in \ell^1(\mathbb{Z})$ and $y \in S^1$. A simple rearrangement of terms yields

$$\sum_{n\in\mathbb{Z}} \big(e^{2\pi i y} K(n, y) - K(n-1, y)\big)\, f(n) = 0,$$

which implies

$$e^{2\pi i y} K(n, y) = K(n-1, y) \tag{6}$$

for every $n \in \mathbb{Z}$ and $y \in S^1$. Finally, since

$$K(0, y) = T(\mathbb{1}_0)(y) = 1$$

we may iterate (6) to obtain $K(n, y) = e^{-2\pi i n y}$. □

3. Shifts on locally compact abelian groups

Previous section focused on characterizing the Fourier transform on $\mathbb{R}$ and $\mathbb{Z}$. In the current section we establish a "broader" characterization which applies to all locally compact abelian groups, of which $\mathbb{R}$ and $\mathbb{Z}$ are just particular examples. Naturally, we need to shift our perspective and focus on other properties of the Fourier transform, since differential or difference operators are meaningless for LCA groups in general. Thus, we bring our attention to the time shift property, which says that a time shift of signal f by x corresponds to frequency spectrum being modified by a linear phase shift $e^{-2\pi i x y}$. More formally:

$$\mathcal{F}(L_x f)(y) = e^{-2\pi i x y} \mathcal{F}(f)(y),$$

where $L_x : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by $L_x f(t) := f(t - x)$. Our goal is to prove that a generalized version of this property characterizes the Fourier transform on any LCA group.

Let G be a locally compact abelian group with dual group $\widehat{G}$. We denote the set of all open, nonempty neighborhoods of the neutral element of G by N_0. Next, we say that a family $(\delta_U)_{U\in N_0}$ is called a Dirac delta family, if every δ_U

- is continuous and compactly supported, i.e., $\delta_U \in C_c(G)$,
- is nonnegative and symmetric, i.e., $\delta_U(x) = \delta_U(-x)$ for every $x \in G$,
- satisfies the equality $\int_G \delta_U(x)\,dx = 1$.

We say that $T : L^1(G) \to C^b(\widehat{G})$ satisfies Dirac delta property if there exists a Dirac delta family (δ_U) such that for every $\chi \in \widehat{G}$ we have

$$\lim_{U\to 0} T(\delta_U)(\chi) = 1, \tag{7}$$

or, more explicitly,

$$\forall_{\varepsilon>0}\,\exists_{U\in N_0}\forall_{\substack{V\subset U\\ V\in N_0}} \quad |T(\delta_V)(\chi) - 1| < \varepsilon.$$

Theorem 3. *If a linear and bounded operator $T : L^1(G) \to C^b(\widehat{G})$ satisfies*

- *the time shift property, i.e., for every $f \in L^1(G)$ and $x \in G$, $\chi \in \widehat{G}$ we have*

$$T(L_x f)(\chi) = \overline{\chi(x)}\,T(f)(\chi), \tag{8}$$

- *the Dirac delta property,*

then T is the Fourier transform.

Proof. Let (δ_U) be a Dirac delta family for T. For every $f \in L^1(G)$, $\chi \in \widehat{G}$ and $U \in N_0$ we use Lemma 11.45 in [1, p. 427] (or Proposition 7 in [4, p. 123], to get

$$\begin{aligned} T(\delta_U \star f)(\chi) &= T\left(\int_G f(x)\,L_x\,\delta_U\,dx\right)(\chi) \\ &= \int_G f(x)\,T(L_x\delta_U)(\chi)\,dx \\ &\overset{(8)}{=} T(\delta_U)(\chi)\int_G f(x)\,\overline{\chi(x)}\,dx = T(\delta_U)(\chi)\,\mathcal{F}(f)(\chi). \end{aligned} \tag{9}$$

On the other hand, $\|\delta_U \star f - f\|_1 \to 0$ by Proposition 2.42 in [6, p. 53], so the continuity of T implies

$$\|T(\delta_U \star f) - T(f)\|_\infty \to 0.$$

Consequently, we have

$$T(\delta_U \star f)(\chi) \to T(f)(\chi)$$

for every $f \in L^1(G)$ and $\chi \in \widehat{G}$. Finally,

$$T(f)(\chi) = \lim_{U\to 0} T(\delta_U \star f)(\chi) \overset{(9)}{=} \lim_{U\to 0} T(\delta_U)(\chi)\,\mathcal{F}(f)(\chi) \overset{(7)}{=} \mathcal{F}(f)(\chi). \qquad \square$$

Let us remark that Theorems 1 and 2 are independent from Theorem 3. This is because Theorems 1 and 2 use time differentiation and difference properties, respectively, whereas Theorem 3 focuses on time shift property.

4. Fourier transform on compact groups

We have already established novel characterizations of the Fourier transform on $\mathbb{R}$, $\mathbb{Z}$ and all locally compact abelian groups in general. The current section takes a step back and reviews the existing literature concerning the Fourier transform characterization on compact groups. With that goal in mind, let us briefly summarize the historical background.

In 2010, Jaming proved that the convolution property characterizes the Fourier transform on four canonical groups: $\mathbb{R}$, S^1, $\mathbb{Z}$, $\mathbb{Z}_n$ (see [7]). Within next four years, Lavanya and Thangavelu proved that the Heisenberg group admits a similar characterization (see [8] and [9]). Their approach was adapted by Kumar and Sivananthan, who investigated the convolution property of the Fourier transform on arbitrary compact group (see [11]). However, studying the proof of their main result, it feels like Kumar and Sivananthan take an unnecessary long detour in order to reach the final destination. It is even difficult to lay down a summary of their proof, which goes through the space of all coefficient functions and establishes invariance of carefully crafted Hilbert subspaces in order to construct unitary operators relating the operator in question to the Fourier transform. An interested Reader is encouraged (at their own risk) to consult [11] for meticulous details.

Our goal is to show that there exists a more direct path than the one chosen by Kumar and Sivananthan. The proof we lay down is rooted in representation theory and circumvents the technicalities of the earlier approach by using the correspondence between representations of a compact group and its L^1-space. With the clarity of our exposition in mind, we take the liberty of recalling the basic concepts of representation theory.

Let G be a compact group. A map $\pi : G \to U(H_\pi)$ is called a unitary representation of G if:

- H_π is a nonzero Hilbert space and $U(H_\pi)$ is a space of unitary operators on H_π.
- π is a homomorphism, i.e., $\pi(xy) = \pi(x)\,\pi(y)$ and $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$ for every $x, y \in G$.
- π is continuous in the strong operator topology, i.e., the map $x \mapsto \pi(x)\,u$ if continuous for any $u \in H_\pi$.

An instructive example of a representation is the left regular representation $\pi_L : G \to U(L^2(G))$ defined by

$$\pi_L(x)(f)(y) := L_x f(y) = f(x^{-1}y).$$

Two representations π_1, π_2 are said to be unitarily equivalent if there exists a unitary operator $Q : H_{\pi_1} \to H_{\pi_2}$ such that

$$\pi_2(x) = Q\pi_1(x)\,Q^{-1}$$

for every $x \in G$. For instance, the right regular representation $\pi_R : G \to U(L^2(G))$ defined by

$$\pi_R(x)(f)(y) := R_x f(y) = f(yx)$$

is unitarily equivalent to π_L. Indeed, the unitary operator $Q : L^2(G) \to L^2(G)$ given by $Q(f)(y) := f(y^{-1})$ satisfies

$$\begin{aligned} Q\pi_L(x)\,Q^{-1} f(y) &= Q\pi_L(x)\, f(y^{-1}) \\ &= Qf(x^{-1}y^{-1}) \\ &= Qf((yx)^{-1}) = f(yx) = R_x f(y) = \pi_R(x)(f)(y) \end{aligned}$$

for every $x, y \in G$. Given a representation π, the set of all unitarily equivalent representations is called the unitary equivalence class and formally denoted by $[\pi]$. However, such a symbol is rather cumbersome and we usually abuse the notation by writing π instead of $[\pi]$.

A closed subspace S of H_π is called invariant if $\pi(x)S \subset S$ for every $x \in G$. Obviously, the trivial space $\{0\}$ as well as the whole space H_π are invariant. If it happens that these are the only invariant subspaces of π, we say that the representation is irreducible. The set of all unitary equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$.

For our purposes we cannot restrict ourselves only to representations of compact groups, but rather extend the theory to L^1-spaces. In general, a map $\rho : A \to B(H_\rho)$ is called a *-representation of a Banach *-algebra A if

- H_π is a nonzero Hilbert space and $B(H_\pi)$ is a space of linear and bounded operators on H_π,
- $\rho(a+b) = \rho(a) + \rho(b)$,
- $\rho(\lambda a) = \lambda\rho(a)$,
- $\rho(ab) = \rho(a)\rho(b)$,
- $\rho(a^*) = \rho(a)^*$

for every $a, b \in A$ and $\lambda \in \mathbb{C}$. We do not need to assume that a *-representation is continuous, since this is always the case due to Proposition 1.3.7 in [5, p. 9]. We say that a *-representation ρ is nondegenerate (see Proposition 9.2 and Definition 9.3 in [12, p. 36]) if

- for every $u \in H_\rho$ there exists $a \in A$ such that $\rho(a)\,u \neq 0$, or equivalently
- the vector space $\rho(A)H_\rho$ is dense in H_ρ.

At this point we are ready to describe the correspondence between representations on compact groups and L^1-spaces. For every unitary representation π of G we define $\Pi : L^1(G) \to B(H_\pi)$ with the formula

$$\forall_{u,v\in H_\pi} \quad \langle \Pi(f)\, u \mid v\rangle := \int_G f(x)\langle \pi(x)\, u \mid v\rangle\, dx, \tag{10}$$

where $\langle \cdot \mid \cdot \rangle$ is the inner product in the Hilbert space H_π. By Theorem 3.9 in [6, p. 73], (or Proposition 6.2.1 in [3, p. 131]) the map Π is a nondegenerate *-representation of $L^1(G)$. Furthermore, by Theorem 3.11 in [6, p. 74] (or Proposition 6.2.3 in [3, p. 133]) every *-representation is of the form (10), i.e., the map $\pi \mapsto \Pi$ is a bijection between unitary representations of G and nondegenerate *-representation of $L^1(G)$. Since a unitary representation π is irreducible if and only if Π is irreducible (see Theorem 3.12(b) in [6, p. 75]) then the map $\pi \mapsto \Pi$ determines a bijection between $\widehat{G}$ and the set of irreducible *-representations of $L^1(G)$, denoted by $\widehat{L^1(G)}$ (see Remark 6.2.4 in [3, p. 134]).

Let

$$\ell^\infty - \oplus_{\pi\in\widehat{G}} B(H_\pi) := \left\{ (A_\pi) \in \prod_{\pi\in\widehat{G}} B(H_\pi) : \sup_{\pi\in\widehat{G}} \sup_{\|u\|=1} \|A_\pi\, u\|_\pi < \infty \right\}$$

be a Banach *-algebra with the norm

$$\|(A_\pi)\|_\infty := \sup_{\pi\in\widehat{G}} \sup_{\|u\|=1} \|A_\pi\, u\|_\pi$$

and involution $(A_\pi)^* = (A_\pi^*)$. The Fourier transform on compact group G is the map $\mathcal{F} : L^1(G) \to \ell^\infty - \oplus_{\pi\in\widehat{G}} B(H_\pi)$ given by

$$\langle \mathcal{F}(f)(\pi)\, u \mid v\rangle := \int_G f(x)\langle \pi(x)^* u \mid v\rangle\, dx$$

for every $u, v \in H_\pi$.

Theorem 4. *Let $T : L^1(G) \to \ell^\infty - \oplus_{\pi\in\widehat{G}} B(H_\pi)$ be a linear and bounded operator, which is*

- **-preserving, i.e., $T(f)^* = T(f^*)$ for every $f \in L^1(G)$, and*
- *pointwise irreducible, i.e., for every $\pi \in \widehat{G}$ the map $f \mapsto T(f)(\pi)$ is irreducible.*

If T satisfies

- *the convolution property, i.e., for every $f, g \in L^1(G)$ and $\sigma \in \widehat{G}$ we have*

$$T(f \star g)(\sigma) = T(f)(\sigma)\, T(g)(\sigma), \tag{11}$$

- *the time shift property, i.e., for every $f \in L^1(G)$, $x \in G$ and $\pi \in \widehat{G}$ we have*

$$T(L_x f)(\pi) = T(f)(\pi)\, \pi(x)^*, \tag{12}$$

then T is the Fourier transform.

Proof. We choose $\pi \in \widehat{G}$ and consider a linear map $T_\pi : L^1(G) \to B(H_\pi)$ given by $T_\pi(f) := T(f)(\pi)$. Since T is *-preserving, pointwise irreducible and it satisfies the convolution property then T_π is an irreducible *-representation of $L^1(G)$. Consequently, there exists an irreducible unitary representation $\sigma : G \to U(H_\pi)$ such that

$$\langle T_\pi(f)\, u \mid v\rangle = \int_G f(x)\langle \sigma(x)^* u \mid v\rangle\, dx$$

for every $u, v \in H_\pi$.

Next, for every $f \in L^1(G)$, $x \in G$ and $u, v \in H_\pi$ we have

$$\begin{aligned}\int_G f(y)\langle \sigma(y)^*\pi(x)^* u \mid v\rangle\, dy = \langle T_\pi(f)\,\pi(x)^* u \mid v\rangle &\overset{(12)}{=} \langle T_\pi(L_x f)\, u \mid v\rangle \\ &= \int_G L_x f(y)\langle \sigma(y)^* u \mid v\rangle\, dy \\ &= \int_G f(x^{-1}y)\langle \sigma(y)^* u \mid v\rangle\, dy \\ &\overset{y\mapsto xy}{=} \int_G f(y)\langle \sigma(xy)^* u \mid v\rangle\, dy \\ &= \int_G f(y)\langle \sigma(y)^*\sigma(x)^* u \mid v\rangle\, dy.\end{aligned}$$

A simple rearrangement yields

$$\int_G f(y)\langle \sigma(y)^*(\sigma(x)^* - \pi(x)^*)\, u \mid v\rangle = 0$$

for every $f \in L^1(G)$, $x \in G$ and $u, v \in H_\pi$. Putting

$$f(y) := \overline{\langle \sigma(y)^*(\sigma(x)^* - \pi(x)^*)\, u \mid v\rangle}$$

we arrive at the conclusion that

$$\langle \sigma(y)^*(\sigma(x)^* - \pi(x)^*)\, u \mid v\rangle = 0$$

for every $x, y \in G$ and $u, v \in H_\pi$. Finally, we take y to be the identity element in G and $v = (\sigma(x)^* - \pi(x)^*)\, u$ to obtain $(\sigma(x)^* - \pi(x)^*)\, u = 0$ for every $x \in G$ and $u \in H_\pi$. This means that $\sigma = \pi$. □

Our proof of Theorem 4 takes slightly more than half of a page. For comparison, Kumar and Sivananthan's original proof of nearly identical result (Theorem 3.1 in [11]) runs for almost two pages. To be fair, Kumar and Sivananthan do not assume pointwise irreducibility and prove that $T(f) = \mathcal{F}(f)$ only on the set

$$E := \{\pi \in \widehat{G} : \exists_{f \in L^1(G)}\, T(f)(\pi) \neq 0\}.$$

This, however, is far from the crux of the proof and we feel that our assumption makes for a much "cleaner" thesis in Theorem 4: T is the Fourier transform (without invoking any set E).

5. Beyond the Fourier transform

This last section is meant to stimulate further research regarding integral transforms and their characterizations. As the title suggests, our goal is to prove that the Fourier transform is not the only well-known operator characterized by a differential property. We hope that our next result will become a catalyst for future study of Fourier-like integral transform and their characterizations.

To begin with, let J_α be the Bessel's function of the first kind solving the Bessel's differential equation (of order α):

$$r^2 f''(r) + r f'(r) + (r^2 - \alpha^2) f(r) = 0 \tag{13}$$

for every $r \in \mathbb{R}_+$. One form of expressing J_α (see Chapter 9.4 in [13, p. 230] or Chapter VI in [14, p. 176]) is

$$\forall_{r\in\mathbb{R}_+} \quad J_\alpha(r) = \frac{1}{\pi}\int_0^\pi \cos(\alpha t - r\sin(t))\,dt - \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-r\sinh(t)-\alpha t}\,dt,$$

which implies that J_α is a bounded function. This feature distinguishes J_α from Y_α, which is another solution of (13) called the Bessel's function of the second kind. Y_α is unbounded at the origin $r = 0$.

Next, we focus on the Hankel transform $\mathcal{H}_\alpha : C_c(\mathbb{R}_+) \to C_0(\mathbb{R}_+)$ of order α:

$$\mathcal{H}_\alpha(f)(y) := \int_{\mathbb{R}_+} J_\alpha(yx)\, f(x)\, x\, dx, \tag{14}$$

whose elementary properties (see Chapter 2 in [10]) include continuity and vanishing at infinity, i.e.,

$$\lim_{y\to\infty} \mathcal{H}_\alpha(f)(y) = 0.$$

We are now ready to prove the final result of the paper:

Theorem 5. *Let $T : C_c(\mathbb{R}_+) \to C_0(\mathbb{R}_+)$ be an integral transform given by the formula*

$$T(f)(y) := \int_{\mathbb{R}_+} K(yx)\, f(x)\, x\, dx, \tag{15}$$

where $K \in C^2(\mathbb{R}_+)$ is a bounded function. If

- *T satisfies the Bessel's differential property (of order $\alpha > 0$)*

$$\forall_{f\in C_c^2(\mathbb{R}_+)} \quad T\left(f'' + \frac{1}{x}f' - \frac{\alpha^2}{x^2}f\right)(y) = -y^2 T(f)(y), \tag{16}$$

and

- *there exists a function* $f_* \in C_c^2(\mathbb{R}_+)$ *and* $y_* \in \mathbb{R}_+$ *such that*

$$T(f_*)(y_*) = \mathcal{H}_\alpha(f_*)(y_*) \neq 0, \tag{17}$$

then T *is the Hankel transform* $\mathcal{H}_\alpha$.

Proof. In order to facilitate the computations throughout the proof, let us begin with a pair of auxiliary integrations by parts. For every $f \in C_c^2(\mathbb{R}_+)$ and $y \in \mathbb{R}_+$ we have

$$\int_{\mathbb{R}_+} xK(yx)\, f''(x)\, dx = -\int_{\mathbb{R}_+} K(yx)\, f'(x)\, dx - \int_{\mathbb{R}_+} yxK'(yx)\, f'(x)\, dx, \tag{18}$$

$$\int_{\mathbb{R}_+} yxK'(yx)\, f'(x)\, dx = -\int_{\mathbb{R}_+} yK'(yx)\, f(x)\, dx - \int_{\mathbb{R}_+} y^2xK''(yx)\, f(x)\, dx. \tag{19}$$

Due to Bessel's differential property, for every $f \in C_c^2(\mathbb{R}_+)$ and $y \in \mathbb{R}_+$ we have

$$\begin{aligned}
-y^2T(f)(y) &\overset{(16)}{=} T\left(f'' + \frac{1}{x}f' - \frac{\alpha^2}{x^2}f\right)(y) \\
&= \int_{\mathbb{R}_+} K(yx)\left(f''(x) + \frac{1}{x}f'(x) - \frac{\alpha^2}{x^2}f(x)\right) x\, dx \\
&= \int_{\mathbb{R}_+} xK(yx)\, f''(x)\, dx + \int_{\mathbb{R}_+} K(yx)\, f'(x)\, dx - \int_{\mathbb{R}_+} K(yx)\frac{\alpha^2}{x}f(x)\, dx \\
&\overset{(18)}{=} -\int_{\mathbb{R}_+} yxK'(yx)\, f'(x)\, dx - \int_{\mathbb{R}_+} K(yx)\frac{\alpha^2}{x}f(x)\, dx \\
&\overset{(19)}{=} \int_{\mathbb{R}_+}\left(y^2xK''(yx) + yK'(yx) - \frac{\alpha^2}{x}K(yx)\right) f(x)\, dx,
\end{aligned}$$

which can be rewritten as

$$0 = \int_{\mathbb{R}_+}\left((yx)^2K''(yx) + yxK'(yx) + \left((yx)^2 - \alpha^2\right)K(yx)\right)\frac{f(x)}{x}dx.$$

By the fundamental theorem of calculus of variations we have

$$r^2K''(r) + rK'(r) + (r^2 - \alpha^2)\,K(r) = 0$$

for every $r \in \mathbb{R}_+$. This implies that K is a linear combination of Bessel's functions J_α and Y_α of first and second kind, respectively, so

$$K(r) = C_1J_\alpha(r) + C_2Y_\alpha(r)$$

for some constants $C_1, C_2 \in \mathbb{R}$. Since Y_α is unbounded near the origin (and J_α is bounded), it must be the case that $C_2 = 0$, as the kernel K is (by assumption) bounded. This means that $T = C_1\mathcal{H}_\alpha$.

Finally, we have

$$\mathcal{H}_\alpha(f_*)(y_*) \overset{(17)}{=} T(f_*)(y_*) = C_1 \mathcal{H}_\alpha(f_*)(y_*),$$

which leads to the conclusion that $C_1 = 1$ and completes the proof. □

Acknowledgement

I would love to express my deep gratitude towards the reviewer, who has painstakingly scrutinized my work pointing out various inconsistencies in the text and whose comments and suggestions have palpably enhanced the readability of the article. I feel honored to have had my paper reviewed by such a supportive and proficient person.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*: *a Hitchhiker's guide*, 3rd ed., Springer, Berlin, 2006. MR Zbl

[2] W. Arendt and A. V. Bukhvalov, "Integral representations of resolvents and semigroups", *Forum Math.* **6**:1 (1994), 111–135. MR Zbl

[3] A. Deitmar and S. Echterhoff, *Principles of harmonic analysis*, Springer, New York, 2009. MR Zbl

[4] N. Dinculeanu, *Vector measures*, International Series of Monographs in Pure and Applied Mathematics **95**, Pergamon Press, Berlin, 1967. MR Zbl

[5] J. Dixmier, *C∗-algebras*, North-Holland Mathematical Library **15**, North-Holland Publishing Co., New York, 1977. MR Zbl

[6] G. B. Folland, *A course in abstract harmonic analysis*, CRC Press, Boca Raton, FL, 1995. MR Zbl

[7] P. Jaming, "A characterization of Fourier transforms", *Colloq. Math.* **118**:2 (2010), 569–580. MR Zbl

[8] R. Lakshmi Lavanya and S. Thangavelu, "A characterisation of the Fourier transform on the Heisenberg group", *Ann. Funct. Anal.* **3**:1 (2012), 109–120. MR Zbl

[9] R. Lakshmi Lavanya and S. Thangavelu, "Revisiting the Fourier transform on the Heisenberg group", *Publ. Mat.* **58**:1 (2014), 47–63. MR Zbl

[10] S. S. Platonov, "Bessel harmonic analysis and the approximation of functions on a half-line", *Izv. Ross. Akad. Nauk Ser. Mat.* **71**:5 (2007), 1001–1048. MR Zbl

[11] N. Shravan Kumar and S. Sivananthan, "Characterisation of the Fourier transform on compact groups", *Bull. Aust. Math. Soc.* **93**:3 (2016), 467–472. MR Zbl

[12] M. Takesaki, *Theory of operator algebras*, *I*, Encyclopaedia of Mathematical Sciences **124**, Springer, Berlin, 2002. MR Zbl

[13] N. M. Temme, *Special functions: an introduction to the classical functions of mathematical physics*, John Wiley & Sons, New York, 1996. MR Zbl

[14] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, 1944. MR Zbl

Received February 20, 2024. Revised June 22, 2024.

MATEUSZ KRUKOWSKI
INSTITUTE OF MATHEMATICS
ŁÓDŹ UNIVERSITY OF TECHNOLOGY
ŁÓDŹ
POLAND
mateusz.krukowski@p.lodz.pl

PACIFIC JOURNAL OF MATHEMATICS
Vol. 329, No. 2, 2024
https://doi.org/10.2140/pjm.2024.329.233

REDUCTION TYPES OF CM CURVES

MENTZELOS MELISTAS

We study the reduction properties of low genus curves whose Jacobian has complex multiplication. In the elliptic curve case, we classify the possible Kodaira types of reduction that can occur. Moreover, we investigate the possible Namikawa–Ueno types that can occur for genus 2 curves whose Jacobian has complex multiplication which is defined over the base field. We also produce bounds on the torsion subgroup of abelian varieties with complex multiplication defined over local fields.

1. Introduction

Let $g \geq 1$ be an integer, let K be a CM field, i.e., K is a totally imaginary quadratic extension of a totally real number field, and assume that K has degree $2g$ over $\mathbb{Q}$. Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and finite residue field. If A/L is an abelian variety, then we will denote by $\mathrm{End}_L(A)$ the ring of endomorphisms of A/L which are defined over L. An abelian variety A/L with complex multiplication by K over the field L is an abelian variety A/L of dimension g together with an embedding $\iota : K \hookrightarrow \mathrm{End}^0_L(A) := \mathbb{Q} \otimes \mathrm{End}_L(A)$. Our definition implies that A/L is isotypic (see [2, Theorem 1.3.1.1]). We also require that K injects into $\mathrm{End}^0_L(A)$ and not just in $\mathrm{End}^0(A) := \mathbb{Q} \otimes \mathrm{End}_{\overline{\mathbb{Q}}}(A)$. If K injects into $\mathrm{End}^0(A)$, then we will say that A/L has potential complex multiplication by the field K. If C/L is a curve, then we will say that C/L has complex multiplication over L (or is a CM curve over L) if the Jacobian $\mathrm{Jac}(C)/L$ has complex multiplication over the field L.

The study of the reduction properties of abelian varieties with complex multiplication is a classical topic with a rich history. Serre and Tate in [33] proved, as a consequence of the Néron–Ogg–Shafarevich criterion, that every abelian variety with complex multiplication defined over a complete discrete valuation ring with finite residue field has potentially good reduction. More generally, Oort in [27], proved the same result under the weaker assumption that the residue field is perfect. Lorenzini in [19], among other results, proved, in the case where the residue field is algebraically closed, that if C/L is a curve with potentially good reduction with

MSC2020: 11G05, 11G07, 14H25, 14K15, 14K22.
Keywords: complex multiplication, elliptic curve, genus 2 curve, Kodaira-type, reduction-type.

© 2024 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

simple Jacobian $\mathrm{Jac}(C)/L$ that has complex multiplication over L, then the degree of the minimal extension over which $\mathrm{Jac}(C)/L$ acquires good reduction has at most three prime divisors. In this paper, we study the possible configurations of the special fiber of the minimal proper regular model of elliptic curves with complex multiplication and of genus 2 curves with complex multiplication.

Our first result is the following.

Theorem 1.1. *Let R be a complete discrete valuation ring with valuation v, fraction field L of characteristic 0, and algebraically closed residue field k_L of characteristic $p > 0$. Let E/L be an elliptic curve with j-invariant j_E that has complex multiplication by an imaginary quadratic field K and let $\mathcal{O}_K$ be the ring of integers of K. If $K = \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, then assume that $End_L(E) \cong \mathcal{O}_K$. Then, depending on p, $v(p)$, j_E, and K, the possible reduction types of E/L are as follows.*

p	$v(p)$	j_E	*possible reduction types*
$\neq 2$	*any*	$\neq 0, 1728$	I_0 *or* I_0^*
2	1	$\neq 0, 1728$	I_0, I_4^*, I_8^*, II, *or* II*
$\neq 2$	*any*	1728	I_0, III, III*, *or* I_0^*
$\neq 3$	*any*	0	I_0, II, II*, IV, IV* *or* I_0^*

Keeping the same notation as above, if we do not assume that the complex multiplication is defined over L, then, as Theorem 1.2 below shows, in addition to the Kodaira types of Theorem 1.1, a few more Kodaira types can also occur.

Theorem 1.2. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and algebraically closed residue field k_L of characteristic $p > 0$. Let E/L be an elliptic curve with potential complex multiplication by an imaginary quadratic field K and denote by j_E the j-invariant of E/L. If $K = \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, then assume that $End_{LK}(E) \cong \mathcal{O}_K$. Then, depending on p, $v(p)$, j_E, and K, the possible reduction types of E/L are as follows.*

p	$v(p)$	j_E	*possible reduction types*
$\neq 2$	*any*	$\neq 0, 1728$	I_0, III, III*, *or* I_0^*.
$\neq 2$	*any*	1728	I_0, III, III*, *or* I_0^*
2	1	1728	I_0, II, III, III*, I_2^* *or* I_3^*
$\neq 3$	*any*	0	I_0, II, II*, IV, IV* *or* I_0^*

In Section 2 below, we present examples showing that all the reduction types of Theorem 1.1 and of Theorem 1.2 do indeed occur.

Curves of genus 2 with complex multiplication have received a lot of interest lately, especially due to their cryptographic applications. If an elliptic curve has complex multiplication, then it has potentially good reduction. However, it is not true that every curve of genus 2 whose Jacobian has complex multiplication has

potentially good reduction. For a primitive CM field, Goren and Lauter in [12] proved a bound on the primes of geometric bad reduction for curves of genus 2 whose Jacobian has complex multiplication.

Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and algebraically closed residue field of characteristic $p > 0$. Let C/L be a projective, smooth, and geometrically connected curve of genus 2 and let $C^{\min}/R$ be its minimal proper regular model. There exists a complete classification of the possibilities for the special fiber of $C^{\min}/R$ (see [26]). In this classification, there are more than 120 possibilities, referred to as reduction types. Moreover, Liu in [16] has produced an algorithm that computes the special fiber of $C^{\min}/R$ under the assumption that the extension of minimal degree over which C/L acquires stable reduction is tame, i.e., the degree of this extension is not divisible by p. We note that this tameness assumption is automatically satisfied if $p > 5$.

Throughout this article, if A/L is any variety over a field L and M/L is a field extension, then we will denote by A_M/M the base change of A/L to M. Among other results in Section 4, we prove the following theorem (for the reduction types we follow Liu's notation in [16]).

Theorem 1.3. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and finite residue field k_L of characteristic $p > 5$. Let C/L be a projective, smooth, and geometrically connected curve of genus 2 with simple Jacobian $\mathrm{Jac}(C)/L$ that has complex multiplication by a quartic CM field K over the field L. Let $\mu' = |\mu(K)|$, where $\mu(K)$ is the group of roots of unity in K, and let L^{unr} be the maximal unramified extension of L.*

(i) *Assume that C/L has potentially good reduction. Then the possible special fibers of the minimal proper regular model of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are as follows.*

μ'	*possible reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$*
2	$[\mathrm{I}_{0-0-0}]$, $[\mathrm{I}^*_{0-0-0}]$
4	$[\mathrm{I}_{0-0-0}]$, $[\mathrm{I}^*_{0-0-0}]$, [VI]
6	$[\mathrm{I}_{0-0-0}]$, $[\mathrm{I}^*_{0-0-0}]$, [III], [IV]
8	$[\mathrm{I}_{0-0-0}]$, $[\mathrm{I}^*_{0-0-0}]$, [VI], [VII], [VII*]
10	$[\mathrm{I}_{0-0-0}]$, $[\mathrm{I}^*_{0-0-0}]$, [IX − 1], [IX − 2], [IX − 3], [IX − 4], [VIII − 1], [VIII − 2], [VIII − 3], [VIII − 4]
12	$[\mathrm{I}_{0-0-0}]$, $[\mathrm{I}^*_{0-0-0}]$, [III], [IV], [VI]

(ii) *Assume that C/L does not have potentially good reduction. Then the possible special fibers of the minimal proper regular model of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are given below, where d and r are defined as in* [16, Section 4.3].

μ'	*possible reduction types of* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$
2 *or* 10	$[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$
4	$[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{III} - \mathrm{III} - (d-2)/4]$, $[\mathrm{III} - \mathrm{III}^* - (d-4)/4]$, $[\mathrm{III}^* - \mathrm{III}^* - (d-6)/4]$, $[2\mathrm{I}_0^* - (r-1)/2]$
6	$[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{IV} - \mathrm{IV} - (d-2)/3]$, $[\mathrm{IV} - \mathrm{IV}^* - (d-3)/3]$, $[\mathrm{IV}^* - \mathrm{IV}^* - (d-4)/3]$, $[\mathrm{II} - \mathrm{II} - (d-2)/6]$, $[\mathrm{II} - \mathrm{II}^* - (d-6)/6]$, $[\mathrm{II}^* - \mathrm{II}^* - (d-10)/6]$, $[\mathrm{I}_0^* - \mathrm{II} - (d-4)/6]$, $[\mathrm{I}_0^* - \mathrm{II}^* - (d-8)/6]$, $[2\mathrm{IV} - (r-1)/3]$, $[2\mathrm{IV}^* - (r-2)/3]$
8	$[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{III} - \mathrm{III} - (d-2)/4]$, $[\mathrm{III} - \mathrm{III}^* - (d-4)/4]$, $[\mathrm{III}^* - \mathrm{III}^* - (d-6)/4]$, $[2\mathrm{I}_0^* - (r-1)/2]$, $[2\mathrm{III} - (r-1)/4]$, $[2\mathrm{III}^* - (r-3)/4]$
12	$[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{III} - \mathrm{III} - (d-2)/4]$, $[\mathrm{III} - \mathrm{III}^* - (d-4)/4]$, $[\mathrm{III}^* - \mathrm{III}^* - (d-6)/4]$, $[2\mathrm{I}_0^* - (r-1)/2]$, $[\mathrm{IV} - \mathrm{IV} - (d-2)/3]$, $[\mathrm{IV} - \mathrm{IV}^* - (d-3)/3]$, $[\mathrm{IV}^* - \mathrm{IV}^* - (d-4)/3]$, $[\mathrm{II} - \mathrm{II} - (d-2)/6]$, $[\mathrm{II} - \mathrm{II}^* - (d-6)/6]$, $[\mathrm{II}^* - \mathrm{II}^* - (d-10)/6]$, $[\mathrm{I}_0^* - \mathrm{II} - (d-4)/6]$, $[\mathrm{I}_0^* - \mathrm{II}^* - (d-8)/6]$, $[2\mathrm{IV} - (r-1)/3]$, $[2\mathrm{IV}^* - (r-2)/3]$, $[2\mathrm{II} - (r-1)/6]$, $[2\mathrm{II}^* - (r-5)/6]$

When $\mu' = 8$ or 10, assuming that C/L has potentially good reduction and the special fiber of the stable model is not isomorphic to either the curve C_0 or the curve C_1 of Remark 4.3, we obtain, in Section 4, a more precise list of possible reduction types (see Theorem 4.4 below).

In Section 5, we focus on component groups and torsion points of CM abelian varieties. Using ideas of Clark and Xarles from [3] we prove the next proposition.

Proposition 1.4. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and finite residue field k_L which has characteristic p and cardinality q. Denote by e the absolute ramification index of L. Let A/L be an abelian variety with complex multiplication by a CM field K over L. Then*

$$|A(L)_{\mathrm{tors}}| \leq \max\left\{|\mu(K)| \cdot p^{2g\gamma_p(e|\mu(K)|)}, \lfloor(1+\sqrt{q})^2\rfloor^g \cdot p^{2g\gamma_p(e)}\right\},$$

where $\gamma_p(m) = \left\lfloor \log_p\left(\frac{pm}{p-1}\right)\right\rfloor$.

This article is organized as follows. In Section 2 we consider the elliptic curve case and prove Theorems 1.1 and 1.2. Section 3 mostly contains background material on reduction of abelian varieties used in the last two sections. After briefly recalling some basic background on reduction of genus 2 curves, we prove Theorem 1.3 in Section 4. Finally, in Section 5 we study the possible geometric component groups of abelian varieties and prove Proposition 1.4.

2. Kodaira types of CM elliptic curves

In this section we prove Theorems 1.1 and 1.2, and present a few examples. We first prove Theorem 1.1 as it will be needed in the proof of Theorem 1.2.

Our starting point for our proofs is Lemma 2.1 below, whose proof can be found in [7, Lemma 2.4]. We note that when E/L is an elliptic curve defined over a number field, then Lemma 2.1 was originally due to Serre and Tate [33, page 507] (see also [30, Corollary 5.22]).

Lemma 2.1. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and algebraically closed residue field k_L. Let E/L be an elliptic curve with complex multiplication by an imaginary quadratic field $K \subset L$. If $K = \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, then assume that $End_L(E) \cong \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K. Then there exists an elliptic curve E'/L such that E'/L has good reduction and the curves E/L and E'/L become isomorphic over an algebraic closure of L.*

We also record here the following useful lemma, which gives the reduction type of a quadratic twist of an elliptic curve with good reduction.

Lemma 2.2. *Let R be a complete discrete valuation ring with valuation v, fraction field L of characteristic 0, and algebraically closed residue field k_L of characteristic $p > 0$. Let E/L be an elliptic curve good reduction and let E'/L be a quadratic twist of E/L.*

(i) *If $p > 2$, then E'/L has either good reduction or reduction of type I_0^*.*

(ii) *If $p = 2$ and $v(2) = 1$, then E'/L has either good reduction, or reduction of type I_8^*, I_4^*, II, or II^*.*

Proof. Part (i) is well known (see, e.g., [4, Proposition 1]). Part (ii) follows by combining explicit formulas for quadratic twists of elliptic curves (see [5, Proposition 5.7.1]) along with [28, Tableau IV]. Alternatively, one can use [21, Theorem 4.2] together with [41]. □

Proof of Theorem 1.1. Assume that $p \neq 2$ and that $j_E \neq 0, 1728$. Lemma 2.1 tells us that there exists an elliptic curve E'/L with good reduction which becomes isomorphic to E/L over an algebraic closure of L. Since $j_E \neq 0, 1728$, we see that E'/L is a quadratic twist of E/L (see [38, Section X.5]). Therefore, since $p \neq 2$, using (i) of Lemma 2.2, we find that E/L has good reduction or reduction of type I_0^*.

Assume that $p = 2$, $v(2) = 1$, and that $j_E \neq 0, 1728$. Proceeding in the same way as in the previous paragraph we see that there exists a quadratic twist E'/L of E/L which has good reduction. Since $v(2) = 1$, by (ii) of Lemma 2.2 we obtain that E/L has good reduction or reduction of type I_8^*, I_4^*, II, or II^*.

Assume that $p \neq 2$ and that $j_E = 1728$. Using [38, Proposition III.1.4] and [38, Proposition X.5.4] we find that the elliptic curve E/L has a short Weierstrass equation of the form

$$y^2 = x^3 + Ax$$

for some $A \in L^*$. The discriminant of this Weierstrass equation is $\Delta = -64A^3$. Since $p \neq 2$ by assumption, we have that $v(\Delta) = 3v(A)$, and hence 3 divides $v(\Delta)$. Let $\Delta_{\min}$ be the discriminant of a minimal Weierstrass equation for E/L. Since when a change of Weierstrass equation is performed the valuation of the discriminant changes by a multiple of 12, we find that $v(\Delta) - v(\Delta_{\min})$ is a multiple of 12. Since 3 divides $v(\Delta)$, we obtain that 3 divides $v(\Delta_{\min})$.

Case 1. $p \geq 5$. Using Tate's algorithm [39] (see also [37, page 365]) and using the fact that E/L has potentially good reduction, we find that E/L can only have good reduction or reduction of type I_0^*, III, or III*.

Case 2. $p = 3$. Choose a minimal Weierstrass equation for E/L and let c_4, c_6, and $\Delta_{\min}$ be the c_4-invariant, the c_6-invariant, and the discriminant of this equation, respectively. Since $j_E = 1728$, $j_E = c_4^3/\Delta_{\min}$, and $1728\Delta_{\min} = c_4^3 - c_6^2$ (see [38, page 42]), we find that $c_6 = 0$. Therefore, using [28, Tableau III] we find that E/L can only have good reduction or reduction of type I_0^*, III, or III*.

Assume that $j_E = 0$. Using [38, Proposition III.1.4] and [38, Proposition X.5.4] we find that the curve E/L has a short Weierstrass equation of the form

$$y^2 = x^3 + B$$

for some $B \in L^*$. The discriminant of this Weierstrass equation is $\Delta = -432B^2$. Since $p \neq 3$ by assumption, we have that $v(\Delta) = 4v(2) + 2v(B)$, and hence 2 divides $v(\Delta)$. Let $\Delta_{\min}$ be the discriminant of a minimal Weierstrass equation for E/L. Since when a change of Weierstrass equation is performed the valuation of the discriminant changes by a multiple of 12, we find that $v(\Delta) - v(\Delta_{\min})$ is a multiple of 12. Since 2 divides $v(\Delta)$, we obtain that 2 divides $v(\Delta_{\min})$.

Case 3. $p \geq 5$. Using Tate's algorithm [39], we obtain that E/L cannot have reduction type III or III*. Therefore, it can only have good reduction or reduction of type II, II*, IV, IV*, or I_0^*.

Case 4. $p = 2$. Choose a minimal Weierstrass equation for E/L and let c_4 and $\Delta_{\min}$ be the c_4-invariant and the discriminant of this equation, respectively. Since $j_E = 0$ and $j_E = c_4^3/\Delta_{\min}$, we find that $c_4 = 0$. Therefore, using [28, Tableau V] we find that E/L can only have good reduction or reduction of type II, II*, IV, IV*, or I_0^*. This completes the proof of Theorem 1.1. □

We now present examples showing that all the possible reduction types of Theorem 1.1 do indeed occur.

Example 2.3. Consider the elliptic curve $E/\mathbb{Q}(\sqrt{-11})$ given by the Weierstrass equation

$$y^2 + y = x^3 + ax + (a-3)\,x - 2,$$

where $a = \frac{1}{2}(1+\sqrt{-11})$. This is the curve with LMFDB [18] label 2.0.11.1-9.1-CMa1, which has complex multiplication over $\mathbb{Q}(\sqrt{-11})$, j-invariant equal to -32768, and only one prime of additive reduction with Kodaira type I_0^*.

Example 2.4. Consider the curve $E_1/\mathbb{Q}(\sqrt{-7})$ given by the Weierstrass equation

$$y^2 + axy = x^3 + (-a-1)\,x + 1,$$

where $a = \frac{1}{2}(1+\sqrt{-7})$. Note that the prime (a) of $\mathbb{Q}(\sqrt{-7})$ lies above (2). The curve $E_1/\mathbb{Q}(\sqrt{-7})$ has LMFDB [18] label 2.0.7.1-16.1-CMa1, j-invariant equal to -3375, and additive reduction of Kodaira type I_4^* at (a).

Consider the curve $E_2/\mathbb{Q}(\sqrt{-7})$ given by the Weierstrass equation

$$y^2 + axy + ay = x^3 + (-a-1)\,x^2 + (2a+2)\,x - 2a + 3.$$

The curve $E_2/\mathbb{Q}(\sqrt{-7})$ has LMFDB [18] label 2.0.7.1-64.1-CMa1, j-invariant equal to -3375, and additive reduction of Kodaira type I_8^* at (a).

Example 2.5. Consider the curve $E_1/\mathbb{Q}(\sqrt{-11})$ given by the Weierstrass equation

$$y^2 = x^3 + (a+1)\,x^2 + (a+2)\,x + 1,$$

where $a = \frac{1}{2}(1+\sqrt{-11})$. The curve $E_1/\mathbb{Q}(\sqrt{-11})$ has LMFDB [18] label 2.0.11.1-4096.1-CMb1, j-invariant equal to -32768, and additive reduction of Kodaira type II at (2).

Consider the curve $E_2/\mathbb{Q}(\sqrt{-11})$ given by the Weierstrass equation

$$y^2 = x^3 + (a+1)\,x^2 + (a+10)\,x + 12a - 1.$$

The curve $E_2/\mathbb{Q}(\sqrt{-11})$ has LMFDB [18] label 2.0.11.1-256.1-CMb1, j-invariant equal to -32768, and additive reduction of Kodaira type II^* at (2).

Example 2.6. Consider the elliptic curves $E_1/\mathbb{Q}(i)$, $E_2/\mathbb{Q}(i)$, $E_3/\mathbb{Q}(i)$ given by LMFDB [18] labels 2.0.4.1-2025.1-CMa1, 2.0.4.1-2025.1-CMb1, and 2.0.4.1-2025.1-CMc1, respectively. All these elliptic curves have complex multiplication by $\mathbb{Q}(i)$ (and j-invariant equal to 1728). Moreover, they have bad reduction at 3 of Kodaira type III, I_0^*, and III^*, respectively.

Example 2.7. Consider the elliptic curves $E_1/\mathbb{Q}(\sqrt{-3})$, $E_2/\mathbb{Q}(\sqrt{-3})$, $E_3/\mathbb{Q}(\sqrt{-3})$, $E_4/\mathbb{Q}(\sqrt{-3})$ given by LMFDB [18] labels 2.0.3.1-256.1-CMa1, 2.0.3.1-784.1-CMa1, 2.0.3.1-784.3-CMb1, and 2.0.3.1-4096.1-CMb1, respectively. All these elliptic curves have complex multiplication by $\mathbb{Q}(\sqrt{-3})$ (and j-invariant equal to 0). Moreover, they have bad reduction at 2 of Kodaira type II, IV, IV^*, and I_0^*.

We can also find an elliptic curve $E_5/\mathbb{Q}(\sqrt{-3})$ complex multiplication by $\mathbb{Q}(\sqrt{-3})$ having bad reduction at 2 of Kodaira type II* as follows. Start with an elliptic curve $E/\mathbb{Q}$ that has potential complex multiplication by $\mathbb{Q}(\sqrt{-3})$ and bad reduction at 2 of Kodaira type II* (e.g., the curve with LMFDB label 1728.m1). Then consider the base change of $E/\mathbb{Q}$ to $\mathbb{Q}(\sqrt{-3})$ which has complex multiplication by $\mathbb{Q}(\sqrt{-3})$. Since 2 is unramified in $\mathbb{Q}(\sqrt{-3})$, the base change will still have bad reduction at 2 of Kodaira type II*.

Example 2.8. Consider the elliptic curve $E/\mathbb{Q}(\sqrt{-3})$ given by the Weierstrass equation

$$y^2 + y = x^3 - 30x + 63.$$

This is the curve with LMFDB [18] label 2.0.3.1-81.1-CMa2 and we have that $\mathrm{End}_L(E) \cong \mathbb{Z}\left[\frac{1}{2}(1+\sqrt{-27})\right]$. Moreover, $E/\mathbb{Q}(\sqrt{-3})$ has only one prime of additive reduction with Kodaira type IV* and j-invariant equal to -12288000. We note that in this example $L = K = \mathbb{Q}(\sqrt{-3})$.

We now proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that $p \neq 2$ and that $j_E \neq 0, 1728$. Let $\mathcal{O}_K$ be the ring of integers of K, let F be the compositum LK, and let E_F/F be the base change of E/L to F. Since $K \subset F$, the curve E_F/F has complex multiplication, and hence it follows from Theorem 1.1 that E_F/F has good reduction or reduction of type I_0^*. Since the ramification index $e(F/L)$ is either 1 or 2, the extension F/L is tame because we assume that $p \neq 2$. Therefore, since E_F/F has either good reduction or reduction of type I_0^*, using [9, Theorem 3], we find that E/L can only have good reduction or reduction of type III, III*, or I_0^*.

Assume now that $j_E = 1728$ and that $v(2) = 1$. Proceeding in the same way as in the proof of Theorem 1.1 we see that E/L has a short Weierstrass equation of the form

$$y^2 = x^3 + Ax$$

for some $A \in L^*$. The discriminant and c_6 invariant of this Weierstrass equation are $\Delta = -64A^3$ and 0. Since $v(2) = 1$ by assumption, we have $v(\Delta) = 6 + 3v(A)$, and hence 3 divides $v(\Delta)$. Let $\Delta_{\min}$ be the discriminant of a minimal Weierstrass equation for E/L. Since when a change of Weierstrass equation is performed the valuation of the discriminant changes by a multiple of 12, we find that $v(\Delta) - v(\Delta_{\min})$ is a multiple of 12. Since 3 divides $v(\Delta)$, we obtain that 3 divides $v(\Delta_{\min})$. Therefore, since $v(2) = 1$, using [28, Tableau V] we find that E/L can only have good reduction or reduction of type I_0, II, III, III*, I_2^* or I_3^*. Finally, the proof of other two cases is exactly the same as in Theorem 1.1. □

Example 2.9. Consider the elliptic curve $E_1/\mathbb{Q}$ given by the Weierstrass equation

$$y^2 + xy = x^3 - x^2 - 2x - 1.$$

This is the curve with LMFDB [18] label 49.a4 and j-invariant equal to -3375. The curve $E_1/\mathbb{Q}$ has potential complex multiplication by $K = \mathbb{Q}(\sqrt{-7})$ and it has bad reduction modulo 7 of Kodaira type III.

Consider now the elliptic curve $E_2/\mathbb{Q}$ given by the Weierstrass equation

$$y^2 + xy = x^3 - x^2 - 1822x + 30393.$$

This is the curve with LMFDB [18] label 49.a1 and j-invariant equal to 16581375. The curve $E_2/\mathbb{Q}$ has potential complex multiplication by $K = \mathbb{Q}(\sqrt{-7})$ and it has bad reduction modulo 7 of Kodaira type III*.

Thus, both the additional reduction types of Theorem 1.2 do occur. We note that our examples are isogenous, so both reduction types can also occur in the same isogeny class.

Example 2.10. Consider the elliptic curves $E_1/\mathbb{Q}$, $E_2/\mathbb{Q}$, $E_3/\mathbb{Q}$, $E_4/\mathbb{Q}$, and $E_5/\mathbb{Q}$ given by LMFDB [18] labels 32.a3, 32.a4, 64.a3, 64a4, 256.b2, respectively. All these elliptic curves have potential complex multiplication by $\mathbb{Q}(i)$ (and j-invariant equal to 1728). Moreover, they have reduction at 2 of type III, I_3^*, I_2^*, II, and III*, respectively.

Proposition 2.11. *Let R be a complete discrete valuation ring with valuation v, fraction field L of characteristic 0, and algebraically closed residue field k_L of characteristic p. Let E/L be an elliptic curve with j-invariant j_E.*

(i) *If $p = 2$ and $j_E = 1728$, then E/L cannot have reduction of type* IV *or* IV*.

(ii) *If $p = 3$, $j_E = 0$, and $v(3)$ is even, then E/L cannot have reduction of type* III *or* III*.

Proof. Assume that $p = 2$ and that $j_E = 1728$. Proceeding in the same way as in the proof of Theorem 1.1 we see that E/L has a short Weierstrass equation of the form

$$y^2 = x^3 + Ax$$

for some $A \in L^*$. The discriminant of this Weierstrass equation is $\Delta = -64A^3$. We have that $v(\Delta) = 6v(2) + 3v(A)$, and hence 3 divides $v(\Delta)$. Let $\Delta_{\min}$ be the discriminant of a minimal Weierstrass equation for E/L. Since when a change of Weierstrass equation is performed the valuation of the discriminant changes by a multiple of 12, we find that $v(\Delta) - v(\Delta_{\min})$ is a multiple of 12. Since 3 divides $v(\Delta)$, we obtain that 3 divides $v(\Delta_{\min})$. Therefore, using [28, Tableau V] we find that E/L cannot have reduction of type IV or IV*.

Assume that $p = 3$, $j_E = 0$, and $v(3)$ is even. Proceeding in the same way as in the proof of Theorem 1.1 we see that E/L has a short Weierstrass equation of the form

$$y^2 = x^3 + B$$

for some $B \in L^*$. The discriminant of this Weierstrass equation is $\Delta = -432B^2$. Since $p = 3$ is even by assumption, we have that $v(\Delta) = 3v(3) + 2v(B)$, and hence 2 divides $v(\Delta)$ because $v(3)$ is even. Let $\Delta_{\min}$ be the discriminant of a minimal Weierstrass equation for E/L. Since when a change of Weierstrass equation is performed the valuation of the discriminant changes by a multiple of 12, we find that $v(\Delta) - v(\Delta_{\min})$ is a multiple of 12. Since 2 divides $v(\Delta)$, we obtain that 2 divides $v(\Delta_{\min})$. Therefore, using [28, Tableau III] we find that E/L cannot have reduction of type III or III*. □

3. Abelian varieties with complex multiplication

In this section we first prove a general lemma which is a consequence of the Néron–Ogg–Shafarevich criterion and is essentially due to Serre and Tate [33]. Then we recall a few basic facts concerning reduction of abelian varieties and reduction of genus 2 curves.

Lemma 3.1. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and finite residue field k_L of characteristic $p > 0$. Let A/L be an abelian variety with complex multiplication by K over L and let $\mu' = |\mu(K)|$, where $\mu(K)$ are the roots of unity contained in K. Then there exists a finite extension M/L^{unr}, where L^{unr} is the maximal unramified extension of L, which has degree dividing μ' and such that the base change A_M/M has good reduction.*

Proof. This is a consequence of results of Serre and Tate in [33]. We include some of the details for completeness. Fix a separable closure $\bar{L}$ of L. Let $\ell \neq p$ be a prime and let $\rho_\ell : \mathrm{Gal}(\bar{L}/L) \to \mathrm{Aut}(T_\ell(A))$ be the ℓ-adic Galois representation of A/L. Let v be the valuation of L, let $\bar{v}$ be the extension of v to $\bar{L}$, and let $I(\bar{v})$ be the inertia group of $\bar{v}$. Note that the extension of v to $\bar{L}$ is unique because R is complete. Since A/L has complex multiplication by K over L, the image $\rho_\ell(I(\bar{v}))$ is contained in $\mu(K)$ (see [33, Theorem 6]). Let now M/L^{unr} be the minimal Galois extension over which $A_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ acquires good reduction. Such an extension exists by [33, Theorem 6 and Corollary 3]. Then, we have that $\ker(\rho_\ell|_{I(\bar{v})}) = \mathrm{Gal}(\bar{L}/M)$ (by [33, Corollary 3]) and that $|\mathrm{Gal}(M/L^{\mathrm{unr}})| = |\rho_\ell(I(\bar{v}))|$ which divides μ'. □

Example 3.2 (see also [25, Example 3.1]). Let p be an odd prime and s be an integer with $1 \leq s \leq p-2$. Consider the smooth projective curve $C_{p,s}/\mathbb{Q}$ birational to

$$y^p = x^s(1-x).$$

The curve $C_{p,s}/\mathbb{Q}$ has genus $\frac{1}{2}(p-1)$. The Jacobian $J_{p,s}/\mathbb{Q}$ of $C_{p,s}/\mathbb{Q}$ has complex multiplication by $K = \mathbb{Q}(\zeta_p)$ defined over $\mathbb{Q}(\zeta_p)$ (see [14, page 202]). It turns out that $J_{p,s}/\mathbb{Q}$ has good reduction away from p and potentially good reduction modulo p. When $C_{p,s}/\mathbb{Q}$ is tame (see [20, Example 5.1] for the definition) $J_{p,s}/\mathbb{Q}$

has purely additive reduction modulo p and achieves good reduction after a totally ramified extension of degree $2(p-1)$ (see [23, page 339] for the last statement).

Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and perfect residue field k_L. Let A/L be an abelian variety of dimension g. We denote by $\mathcal{A}/R$ the Néron model of A/L (see [1] for the definition as well as the basic properties of Néron models). The special fiber $\mathcal{A}_{k_L}/k_L$ of $\mathcal{A}/R$ is a smooth commutative group scheme. We denote by $\mathcal{A}^0_{k_L}/k_L$ the connected component of the identity of $\mathcal{A}_{k_L}/k_L$. Since k_L is perfect, by a theorem of Chevalley (see [6, Theorem 1.1]) we have a short exact sequence

$$0 \to T \times U \to \mathcal{A}^0_{k_L} \to B \to 0,$$

where T/k_L is a torus, U/k_L is a unipotent group, and B/k_L is an abelian variety. The number $\dim(U)$ (resp. $\dim(T)$, $\dim(B)$) is called the unipotent (resp. toric, abelian) rank of A/L. By construction, $g = \dim(U) + \dim(T) + \dim(B)$. We say that A/L has purely additive reduction if $g = \dim(U)$, or equivalently, if $\dim(T) = \dim(B) = 0$.

The following two theorems will be very useful in the next section.

Theorem 3.3 (see [27, Lemma 2.4]). *Let R be a complete discrete valuation ring with fraction field L and perfect residue field. Let A/L be a simple abelian variety with complex multiplication by K over the field L. Then A/L has either purely additive or good reduction.*

Theorem 3.4 (see [19, Proposition 2.7]). *Let R be a complete discrete valuation ring with fraction field L and algebraically closed residue field of characteristic $p > 5$. Let C/L be a projective, smooth, and geometrically connected curve of genus 2 with Jacobian $\mathrm{Jac}(C)/L$. Assume that the Jacobian $\mathrm{Jac}(C)/L$ has purely additive and potentially good reduction. Then $[M : L] \leq 10$.*

We now recall some background material on reduction of algebraic curves. The reader is referred to [17, Chapter 10] for more information on this topic. A projective, connected, and reduced curve $C/\bar{k}$ over an algebraically closed field $\bar{k}$ is called stable if it has arithmetic genus greater or equal to 2, its singular points are ordinary double points, and all of its irreducible components that are isomorphic to $\mathbb{P}^1_{\bar{k}}$ meet the other components in at least 3 points. Let R be a complete discrete valuation ring with fraction field L and algebraically closed residue field, and let C/L be a smooth, projective, and geometrically connected curve of genus $g \geq 2$. Recall that a stable model of C/L is a proper and flat scheme $\mathcal{C}/R$ whose generic fiber is isomorphic to the curve C/L and whose special fiber is a stable curve. We will say that C/L has stable reduction if it has a model whose special fiber is a stable curve over the residue field.

Theorem 3.5 (see [17, Theorem 10.4.44]). *Let R be a complete discrete valuation ring with fraction field L and algebraically closed residue field. Let C/L be a smooth, projective, and geometrically connected curve of genus $g \geq 2$. Then there exists a (unique) finite extension M/L such that the curve C_M/M has stable reduction that has the following minimality property; for every other finite extension N/L the base change C_N/N has stable reduction if and only if $M \subseteq N$.*

Using the same assumptions as in the preceding theorem, we denote the stable model of C_M/M by $\mathcal{C}^{\mathrm{st}}$ which is unique by [17, Theorem 10.3.34]. Below we will refer to $\mathcal{C}^{\mathrm{st}}$ as the stable model, but the reader should keep in mind that $\mathcal{C}^{\mathrm{st}}$ is the stable model of C_M/M. If C/L has potentially good reduction, then $\mathcal{C}^{\mathrm{st}}$ is smooth. On the other hand, if C/L does not have potentially good reduction but the Jacobian $\mathrm{Jac}(C)/L$ of C/L has potentially good reduction, then the special fiber of $\mathcal{C}^{\mathrm{st}}$ is a union of two elliptic curves meeting at a single point (see the paragraph before [15, Proposition 2]).

The following lemma will be very useful in the proofs of the next section because it will help us exclude certain reduction types of the special fiber of the minimal proper regular model of our curve.

Lemma 3.6. *Let R be a discrete valuation ring with fraction field L and algebraically closed residue field k. Let C/L be a projective, smooth, and geometrically connected curve of genus 2. Assume that the Jacobian $\mathrm{Jac}(C)/L$ has purely additive reduction. Then the special fiber of the minimal proper regular model of C/L contains only rational curves.*

Proof. This is well known to the experts. Nevertheless, we provide some details of the proof. Let $\mathcal{C}^{\min}/R$ be the minimal proper regular model of C/L. By performing a sequence of blowups of closed points of $\mathcal{C}^{\min}/R$ we can find a new regular model that has the properties (1) and (2) of [22, Section 6]. Moreover, the condition $r = 1$ of [22, Theorem 6.1] is automatically satisfied because $g = 2$ (see [29, Proposition 9.5.1]). Since blowing up at a closed point does not introduce curves of genus bigger than 0 and $\mathrm{Jac}(C)/L$ has purely additive reduction by assumption, it follows from [22, Theorem 6.1, part (a)] that the special fiber of the minimal proper regular model of C/L can only contain rational curves. □

4. Genus 2 CM curves

In this section we prove Theorem 1.3. For the convenience of the reader, we have repeated the statements to be proved and we have split the proof of the theorem into parts.

Note. Recall that a quartic CM field K is a totally imaginary quadratic extension of a totally real quadratic number field. Since the degree of the cyclotomic field $\mathbb{Q}(\zeta_m)$

over $\mathbb{Q}$ is $\phi(m)$, we find that if K is a quartic CM field which contains a primitive root of unity of order m, then $m = 2, 3, 4, 5, 6, 8, 10$ or 12. Note that $\mathbb{Q}(\zeta_5) \cong \mathbb{Q}(\zeta_{10})$, $\mathbb{Q}(\zeta_8)$, and $\mathbb{Q}(\zeta_{12})$ are all quartic CM fields and that $\mathbb{Q}(\zeta_8)$, $\mathbb{Q}(\zeta_{10})$, and $\mathbb{Q}(\zeta_{12})$ are the only quartic CM fields that have 8, 10, and 12 roots of unity, respectively. From the above discussion, it follows that the number of roots of unity for a quartic CM field is 2, 4, 6, 8, 10, or 12.

Lemma 4.1. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and algebraically closed residue field k_L of characteristic $p > 5$. Let C/L be a projective, smooth, and geometrically connected curve of genus 2 with Jacobian $\mathrm{Jac}(C)/L$ that has complex multiplication over the field L. Assume that C/L has potentially good reduction. Then C/L cannot have reduction type* [II], [V], *or* [V^*].

Proof. If C/L has reduction type [II], then we see from [26, page 155] that the special fiber of the minimal regular model of C/L contains a curve of genus 1. Therefore, Lemma 3.6 implies that the Jacobian $\mathrm{Jac}(C)/L$ cannot have purely additive reduction. However, this contradicts Theorem 3.3.

Assume now that C/L has reduction type [V] or [V^*]. We will find a contradiction. The idea is that after a cubic base extension, C/L will acquire reduction of type [II], so we get a contradiction by the previous paragraph. If C/L has reduction of [V] or [V^*], then, since $p > 5$, from [16, Table 1] we find that C/L acquires good reduction after a cyclic extension M/L of degree 6. Let now N/L be the cubic field subextension of M/L. The base change C_N/N acquires good reduction after an extension of degree 2. Therefore, using [16, Table 1], we find that C_N/N has either reduction of type [II] or [I^*_{0-0-0}]. Let r be the r-invariant defined by [16, Théorème 1] corresponding to C/L and let r' be the r-invariant defined by [16, Théorème 1] corresponding to C_N/N. We will not introduce Liu's notation because we will only use it very briefly here, but the interested reader is referred to [16] or [31, Section 5.1] for more information. It follows from [16, Table 1] that $r \equiv 1$ or 5 (mod 6), because $n = 6$ for C/L. Let v_L and v_N be the associated (normalized) valuations for L and N respectively. By looking at the expression for r' and keeping in mind that $v_N|_L = 3v_L$, we find that $r' \equiv 1 \pmod 2$. Therefore, using [16, Table 1], we find that C_N/N has reduction type [II], which is a contradiction. □

We now prove part (i) of Theorem 1.3.

Theorem 4.2. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and finite residue field k_L of characteristic $p > 5$. Let C/L be a projective, smooth, and geometrically connected curve of genus 2 with Jacobian $\mathrm{Jac}(C)/L$ that has complex multiplication by a quartic CM field K over the field L. Let $\mu' = |\mu(K)|$, where $\mu(K)$ are the roots of unity in K, and let L^{unr} be the maximal unramified extension of L. Assume that C/L has potentially good reduction. Then:*

(i) *If* $\mu' = 2$, *then* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ *can only have reduction of type* $[\mathrm{I}^*_{0-0-0}]$ *or good reduction* $[\mathrm{I}_{0-0-0}]$.

(ii) *If* $\mu' = 4$, *then* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ *can only have reduction of type* $[\mathrm{I}^*_{0-0-0}]$, [VI], *or good reduction* $[\mathrm{I}_{0-0-0}]$.

(iii) *If* $\mu' = 6$, *then* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ *can only have reduction of type* $[\mathrm{I}^*_{0-0-0}]$, [III], [IV], *or good reduction* $[\mathrm{I}_{0-0-0}]$.

(iv) *If* $\mu' = 8$, *then* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ *can only have reduction of type* $[\mathrm{I}^*_{0-0-0}]$, [VI], [VII], [VII*], *or good reduction* $[\mathrm{I}_{0-0-0}]$.

(v) *If* $\mu' = 10$, *then* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ *can only have reduction of type* $[\mathrm{I}^*_{0-0-0}]$, $[\mathrm{IX}-1]$, $[\mathrm{IX}-2]$, $[\mathrm{IX}-3]$, $[\mathrm{IX}-4]$, $[\mathrm{VIII}-1]$, $[\mathrm{VIII}-2]$, $[\mathrm{VIII}-3]$, $[\mathrm{VIII}-4]$, *or good reduction* $[\mathrm{I}_{0-0-0}]$.

(vi) *If* $\mu' = 12$, *then* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ *can only have reduction of type* $[\mathrm{I}^*_{0-0-0}]$, [III], [IV], [VI], *or good reduction* $[\mathrm{I}_{0-0-0}]$.

Proof. Let M/L^{unr} be the extension of minimal degree over which $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ acquires stable reduction, which is provided by Theorem 3.5. By [8, Theorem 2.4] and Lemma 3.1 we find that $[M : L^{\mathrm{unr}}]$ divides $\mu' = |\mu(K)|$. The idea of the proof is to use this divisibility combined with Liu's algorithm [16] and Lemma 4.1 to compute the possible reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$.

Since we assume that C/L has potentially good reduction, it follows from [15, Proposition 3] that the stable model $\mathcal{C}^{\mathrm{st}}$ has good reduction. Moreover, since $p > 5$, the extension M/L^{unr} is tame (see [16, Proposition 4.1.2]). Therefore, by [16, Table 1] we find that the possible reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are as follows.

degree of M/L^{unr}	possible reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$
1	$[\mathrm{I}_{0-0-0}]$
2	$[\mathrm{I}^*_{0-0-0}]$, [II]
3	[III]
4	[VI]
5	$[\mathrm{IX}-1]$, $[\mathrm{IX}-2]$, $[\mathrm{IX}-3]$, $[\mathrm{IX}-4]$
6	[IV], $[V]$, $[V^*]$
8	[VII], [VII*]
10	$[\mathrm{VIII}-1]$, $[\mathrm{VIII}-2]$, $[\mathrm{VIII}-3]$, $[\mathrm{VIII}-4]$

(i) If $\mu' = 2$, then we see that $[M : L^{\mathrm{unr}}]$ is either 1 or 2. So, the curve $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ has either reduction of type $[\mathrm{I}_{0-0-0}]$, $[\mathrm{I}^*_{0-0-0}]$, or [II]. However, Lemma 4.1 excludes the case of type [II].

(ii) If $\mu' = 4$, then we see that $[M : L^{\mathrm{unr}}]$ divides 4. If $[M : L^{\mathrm{unr}}] \leq 2$, then we find that the reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are the types that appear in (i). On the other hand, if $[M : L^{\mathrm{unr}}] = 4$, then $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ has reduction of type [VI].

(iii) If $\mu' = 6$, then we see that $[M : L^{\mathrm{unr}}]$ divides 6. Therefore, $[M : L^{\mathrm{unr}}] = 1, 2, 3,$ or 6. Using the table above we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ has reduction of type $[\mathrm{I}^*_{0-0-0}]$, [II], [III], [IV], $[V]$, or $[V^*]$. However, using Lemma 4.1, we find that the types [II], $[V]$, and $[V^*]$ cannot occur.

(iv) If $\mu' = 8$, then we see that $[M : L^{\mathrm{unr}}]$ divides 8. If $[M : L^{\mathrm{unr}}]$ divides 4, then we find that the reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are the types that appear in (ii). On the other hand, if $[M : L^{\mathrm{unr}}] = 8$, then we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ has reduction of type [VII] or [VII*].

(v) If $\mu' = 10$, then we see that $[M : L^{\mathrm{unr}}]$ divides 10. If $[M : L^{\mathrm{unr}}] = 1$ or 2, then $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ has reduction of type $[\mathrm{I}^*_{0-0-0}]$ or $[\mathrm{I}_{0-0-0}]$. On the other hand, if $[M : L^{\mathrm{unr}}] = 5$ or 10, then we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ has reduction of type [IX $-$ 1], [IX $-$ 2], [IX $-$ 3], [IX $-$ 4], [VIII $-$ 1], [VIII $-$ 2], [VIII $-$ 3], [VIII $-$ 4].

(vi) If $\mu' = 12$, the we see that $[M : L^{\mathrm{unr}}]$ divides 12. Since $[M : L^{\mathrm{unr}}] \leq 10$ by Theorem 3.4, we find that $[M : L^{\mathrm{unr}}] = 1, 2, 3, 4,$ or 6. If $[M : L^{\mathrm{unr}}]$ divides 6, then we find that the reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are the types that appear in (iii). On the other hand, if $[M : L^{\mathrm{unr}}] = 4$, then $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type [VI]. □

Remark 4.3. The following two curves, C_0/k and C_1/k, will play a special role in Theorem 4.4 below. Let k be an algebraically closed field of characteristic $p > 5$. Let C_0/k be the smooth projective geometrically connected curve given by the following affine equation

$$C_0 : y^2 = x^5 - 1.$$

Moreover, let C_1/k be the smooth projective geometrically connected curve given by the affine equation

$$C_1 : y^2 = x^5 - x.$$

We now show that under some extra assumptions on the special fiber of the stable model of C/L, we can achieve more precise results.

Theorem 4.4. *Keep the same assumptions and notation as in Theorem 4.2 and assume in addition that the special fiber C_s of $\mathcal{C}^{st}$ is not isomorphic to either the curve C_0/k or the curve C_1/k of Remark 4.3. Then*:

(i) *If $\mu' = 8$, then $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type* $[\mathrm{I}^*_{0-0-0}]$, [VI], *or good reduction* $[\mathrm{I}_{0-0-0}]$.

(ii) *If $\mu' = 10$, then $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type* $[\mathrm{I}^*_{0-0-0}]$ *or good reduction* $[\mathrm{I}_{0-0-0}]$.

Proof. Let M/L^{unr} be the extension of minimal degree over which $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ acquires stable reduction, which is provided by Theorem 3.5. Since C/L has potentially good reduction, it follows from [15, Corollaire 4.1] that $[M : L^{\mathrm{unr}}]$ divides 4 or 6, except if $C_s \cong C_0, C_1$, where C_s is the stable reduction of C/L.

(i) Assume that $\mu' = 8$. Lemma 3.1 implies that $[M : L^{\mathrm{unr}}]$ divides 8. Since $[M : L^{\mathrm{unr}}]$ also divides 4 or 6, we find that $[M : L^{\mathrm{unr}}] = 1, 2$, or 4. Therefore, proceeding in a similar way as in the proof of (ii) of Theorem 4.2, we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type $[\mathrm{I}^*_{0-0-0}]$, [VI], or good reduction $[\mathrm{I}_{0-0-0}]$.

(ii) Assume that $\mu' = 10$. Lemma 3.1 implies that $[M : L^{\mathrm{unr}}]$ divides 10. Since $[M : L^{\mathrm{unr}}]$ also divides 4 or 6, we find that $[M : L^{\mathrm{unr}}] = 1$ or 2. Therefore, proceeding in a similar way as in the proof of (i) of Theorem 4.2, we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type $[\mathrm{I}^*_{0-0-0}]$ or good reduction $[\mathrm{I}_{0-0-0}]$. □

We now consider the possible types that can occur when C/L does not have potentially good reduction.

Remark 4.5. Let R be a complete discrete valuation ring with valuation v_L, fraction field L of characteristic 0, and algebraically closed residue field k_L of characteristic $p > 5$. Let C/L be a projective, smooth, and geometrically connected curve of genus 2 with simple Jacobian $\mathrm{Jac}(C)/L$ that has complex multiplication by a quartic CM field K over the field L. Assume also that C/L does not have potentially good reduction. Let M/L be the extension of minimal degree over which C/L acquires stable reduction, which exists by Theorem 3.5. Recall that we denote by $\mathcal{C}^{\mathrm{st}}$ the stable model of C_M/M. It follows that the special fiber of the stable model $\mathcal{C}^{\mathrm{st}}$ is a union of two elliptic curves E_1 and E_2 intersecting at a point, see the paragraph before [15, Proposition 2] and note that the Jacobian of C/L has potentially good reduction. Let

$$d_L := v_L \tfrac{1}{12}(J_{10}\, J_2^{-5}),$$

where J_{10} and J_2 are the (Igusa) J_{10}- and J_2-invariants associated to C/L, see [16, Section 2.2] for the relevant definitions. The number $d := [M : L]\, d_L$ is called the degree of singularity of the point of intersection $E_1 \cap E_2$ in $\mathcal{C}^{\mathrm{st}}$. We note that this d is the same d that appears in the reduction type of the second part of Theorem 1.3 (as well as in its restatement below).

Before we proceed to the proof of the second part of Theorem 1.3, we need to prove a lemma that will significantly simplify our proof. In the lemma and in the theorem below the number r corresponds to the r-invariant defined in [16, Théorème 3].

Lemma 4.6. *Let R be a complete discrete valuation ring with (normalized) valuation v_L, with fraction field L of characteristic 0, and algebraically closed residue field k_L of characteristic $p > 5$. Let C/L be a projective, smooth, and geometrically connected curve of genus 2 with Jacobian $\mathrm{Jac}(C)/L$ that has complex multiplication over the field L.*

(i) *The curve C/L cannot have any of the following reduction types*:

$[\mathrm{I}_0-\mathrm{I}_0-d]$, $[\mathrm{I}_0-\mathrm{I}_0^*-(d-1)/2]$, $[2\mathrm{I}_0-r]$,
$[\mathrm{I}_0-\mathrm{IV}-(d-1)/3]$, $[\mathrm{I}_0-\mathrm{IV}^*-(d-2)/3]$, $[\mathrm{I}_0-\mathrm{III}-(d-1)/4]$,
$[\mathrm{I}_0-\mathrm{III}^*-(d-3)/4]$, $[\mathrm{I}_0^*-\mathrm{III}^*-(d-5)/4]$, $[\mathrm{I}_0^*-\mathrm{III}-(d-3)/4]$,
$[2\mathrm{IV}-(r-1)/3]$, $[2\mathrm{IV}^*-(r-2)/3]$, $[\mathrm{I}_0-\mathrm{II}-(d-1)/6]$,
$[\mathrm{I}_0-\mathrm{II}^*-(d-5)/6]$, $[\mathrm{I}_0^*-\mathrm{IV}^*-(d-7)/6]$, $[\mathrm{I}_0^*-\mathrm{IV}-(d-5)/6]$,
$[\mathrm{II}^*-\mathrm{IV}-(d-7)/6]$, $[\mathrm{II}-\mathrm{IV}-(d-3)/6]$, $[\mathrm{II}-\mathrm{IV}^*-(d-5)/6]$,
$[\mathrm{II}^*-\mathrm{IV}^*-(d-9)/6]$,

(ii) *If C/L acquires semistable reduction after an extension of degree* 6 *or* 12, *then* $2\nmid v_L(J_2)$ *except if C/L has reduction type* $[\mathrm{II}-\mathrm{II}-(d-2)/6]$, $[\mathrm{II}-\mathrm{II}^*-(d-6)/6]$, $[\mathrm{II}^*-\mathrm{II}^*-(d-10)/6]$, $[\mathrm{I}_0^*-\mathrm{II}-(d-4)/6]$, *or* $[\mathrm{I}_0^*-\mathrm{II}^*-(d-8)/6]$.

Proof. (i) If C/L has reduction of type $[\mathrm{I}_0-\mathrm{I}_0-d]$, $[\mathrm{I}_0-\mathrm{I}_0^*-(d-1)/2]$, $[2\mathrm{I}_0-r]$, $[\mathrm{I}_0-\mathrm{IV}-(d-1)/3]$, or $[\mathrm{I}_0-\mathrm{IV}^*-(d-2)/3]$, then we see from [26] that the special fiber of the minimal regular model of C/L contains a smooth curve of genus 1. Therefore, Lemma 3.6 implies that the Jacobian $\mathrm{Jac}(C)/L$ cannot have purely additive reduction. However, this contradicts Theorem 3.3.

Assume that C/L has reduction of type $[\mathrm{I}_0-\mathrm{III}-(d-1)/4]$, $[\mathrm{I}_0-\mathrm{III}^*-(d-3)/4]$, $[\mathrm{I}_0^*-\mathrm{III}^*-(d-5)/4]$, or $[\mathrm{I}_0^*-\mathrm{III}-(d-3)/4]$, and we will find a contradiction. We know from [16, Table 3.1] that C/L acquires stable reduction after a cyclic extension M/L of degree 4. Let now N/L be a quadratic field extension contained in M/L. The base change C_N/N acquires stable reduction after an extension of degree 2. Let d be the integer defined in Remark 4.5 computed with respect to the field L. Looking at [16, Table 3.1], we find that $d\equiv 1$ or 3 (mod 4). Let v_L and v_N are the corresponding valuations for L and N respectively. Since $v_N|_L = 2v_L$ and $[M:N]=\frac{1}{2}[M:N]$, we see that the value of d remains the same when computed over N. Therefore, using [16, Table 3.1], we see that C_N/N has reduction $[\mathrm{I}_0-\mathrm{I}_0^*-(d-1)/2]$, because $d\equiv 1$ (mod 2). This contradicts the previous paragraph because $\mathrm{Jac}(C_N)/N$ has complex multiplication over N.

Assume that C/L has reduction of type $[2\mathrm{IV}-(r-1)/3]$ or $[2\mathrm{IV}^*-(r-2)/3]$, and we will find a contradiction. Let v_L be the (normalized) valuation of L and let J_2 be the J_2-invariant associated to C/L (see [16, Section 2.2]). Since C/L has reduction of type $[2\mathrm{IV}-(r-1)/3]$ or $[2\mathrm{IV}^*-(r-2)/3]$, we know that $2\nmid v_L(J_2)$ by [16, Table 3.2]. Moreover, from [16, Table 3.1] we find that C/L acquires good reduction after a cyclic extension M/L of degree 6. Let now N/L be a cubic extension contained in M/L and denote by v_N the corresponding valuation. Since $v_N|_L = 3v_L$, we see that $2\nmid v_N(J_2)$. Therefore, C_N/N must have reduction of type $[2\mathrm{I}_0-r]$ by [16, Table 3.2]. This is a contradiction by Lemma 4.6, because $\mathrm{Jac}(C_N)/N$ has complex multiplication over N.

Assume that C/L has reduction of type $[\mathrm{I}_0-\mathrm{II}-(d-1)/6]$, $[\mathrm{I}_0-\mathrm{II}^*-(d-5)/6]$, $[\mathrm{I}_0^*-\mathrm{IV}^*-(d-7)/6]$, $[\mathrm{I}_0^*-\mathrm{IV}-(d-5)/6]$, $[\mathrm{II}^*-\mathrm{IV}-(d-7)/6]$, $[\mathrm{II}-\mathrm{IV}-(d-3)/6]$,

$[\mathrm{II}-\mathrm{IV}^*-(d-5)/6]$, or $[\mathrm{II}^*-\mathrm{IV}^*-(d-9)/6]$, and we will find a contradiction. We know from [16, Table 3.1] that C/L acquires good reduction after a cyclic extension M/L of degree 6. Let d be the integer defined in Remark 4.5 with respect to L. Looking at [16, Table 3.1], we find that $d \equiv 1, 3,$ or 5 (mod 6). Let now N/L be a cubic extension contained in M/L and denote by v_N the corresponding valuation. Since $v_N|_L = 3v_L$ and $[M : N] = \frac{1}{3}[M : N]$, we see that the value of d remains the same when computed over N. Therefore, using [16, Table 3.1], we see that C_N/N has reduction $[\mathrm{I}_0-\mathrm{I}_0^*-(d-1)/2]$, because $d \equiv 1$ (mod 2). Finally, using Lemma 4.6 we find a contradiction because $\mathrm{Jac}(C_N)/N$ has complex multiplication over N.

(ii) We now assume that $2 \mid v_L(J_2)$ and that C/L does not have reduction type $[\mathrm{II}-\mathrm{II}-(d-2)/6]$, $[\mathrm{II}-\mathrm{II}^*-(d-6)/6]$, $[\mathrm{II}^*-\mathrm{II}^*-(d-10)/6]$, $[\mathrm{I}_0^*-\mathrm{II}-(d-4)/6]$, or $[\mathrm{I}_0^*-\mathrm{II}^*-(d-8)/6]$. We proceed by contradiction. Let M/L be the extension of minimal degree over which C/L acquires stable reduction. Note that the special fiber of the stable model of C_M/M is a union of two elliptic curves meeting at one point (see the paragraph right before [15, Proposition 2]). By assumption $[M : L] = 6$ or 12. Let now N/L be the extension of degree 3 contained in M/L and denote by v_N the corresponding valuation. Then C_N/N acquires good reduction after an extension of degree $\frac{1}{3}[M : L]$ and $2 \mid v_N(J_2)$. Let d be the integer defined in Remark 4.5 with respect to L. It follows from [16, Table 3.1] that either $d \equiv 1, 3,$ or 5 (mod 6) or $d \equiv 1, 3,$ or 5 (mod 6). The curve C_N/N has stable reduction after an extension of degree 2 or 4 while d is odd. Let v_N be the valuation of N. As $v_N|_L = \left(\frac{1}{3}[M : L]\right) v_L$ and $[M : N] = 3$, we see that the value of d remains the same when computed over N. Therefore, using [16, Table 3.1], we see that C_N/N has reduction type $[\mathrm{I}_0-\mathrm{I}_0^*-(d-1)/2]$, $[\mathrm{I}_0-\mathrm{III}-(d-1)/4]$, $[\mathrm{I}_0-\mathrm{III}^*-(d-3)/4]$, $[\mathrm{I}_0^*-\mathrm{III}-(d-3)/4]$, or $[\mathrm{I}_0^*-\mathrm{III}^*-(d-5)/4]$. This is a contradiction by Lemma 4.6 because $\mathrm{Jac}(C_N)$ has complex multiplication over the field N. □

We now prove the second part of Theorem 1.3. For the convenience of the reader we have repeated the statement to be proved.

Theorem 4.7. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and finite residue field k_L of characteristic $p > 5$. Let C/L be a projective, smooth, and geometrically connected curve of genus 2 with Jacobian $\mathrm{Jac}(C)/L$ that has complex multiplication by a quartic CM field K over the field L. Let $\mu' = |\mu(K)|$, where $\mu(K)$ is the group of roots of unity in K. Assume that C/L does not have potentially good reduction. Then*:

(i) *If $\mu' = 2$ or 10, then $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type $[\mathrm{I}_0^*-\mathrm{I}_0^*-(d-2)/2]$.*

(ii) *If $\mu' = 4$, then $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type $[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{III} - \mathrm{III} - (d-2)/4]$, $[\mathrm{III} - \mathrm{III}^* - (d-4)/4]$, $[\mathrm{III}^* - \mathrm{III}^* - (d-6)/4]$, or $[2\mathrm{I}_0^* - (r-1)/2]$.*

(iii) *If* $\mu' = 6$, *then* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ *can only have reduction of type* $[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{IV}-\mathrm{IV}-(d-2)/3]$, $[\mathrm{IV}-\mathrm{IV}^*-(d-3)/3]$, $[\mathrm{IV}^*-\mathrm{IV}^*-(d-4)/3]$, $[\mathrm{II}-\mathrm{II}-(d-2)/6]$, $[\mathrm{II}-\mathrm{II}^*-(d-6)/6]$, $[\mathrm{II}^*-\mathrm{II}^*-(d-10)/6]$, $[\mathrm{I}_0^*-\mathrm{II}-(d-4)/6]$, $[\mathrm{I}_0^*-\mathrm{II}^*-(d-8)/6]$, $[2\mathrm{IV} - (r-1)/3]$, *or* $[2\mathrm{IV}^* - (r-2)/3]$.

(iv) *If* $\mu' = 8$, *then* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ *can only have reduction of type* $[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{III}-\mathrm{III}-(d-2)/4]$, $[\mathrm{III}-\mathrm{III}^*-(d-4)/4]$, $[\mathrm{III}^*-\mathrm{III}^*-(d-6)/4]$, $[2\mathrm{I}_0^*-(r-1)/2]$, $[2\mathrm{III} - (r-1)/4]$, $[2\mathrm{III}^* - (r-3)/4]$.

(v) *If* $\mu' = 12$, *then the possible reduction types of* $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ *are as follows*:

$[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{III} - \mathrm{III} - (d-2)/4]$, $[\mathrm{III} - \mathrm{III}^* - (d-4)/4]$,
$[\mathrm{III}^* - \mathrm{III}^* - (d-6)/4]$, $[2\mathrm{I}_0^* - (r-1)/2]$, $[\mathrm{IV} - \mathrm{IV} - (d-2)/3]$,
$[\mathrm{IV} - \mathrm{IV}^* - (d-3)/3]$, $[\mathrm{IV}^* - \mathrm{IV}^* - (d-4)/3]$, $[\mathrm{II} - \mathrm{II} - (d-2)/6]$,
$[\mathrm{II} - \mathrm{II}^* - (d-6)/6]$, $[\mathrm{II}^* - \mathrm{II}^* - (d-10)/6]$, $[\mathrm{I}_0^* - \mathrm{II} - (d-4)/6]$,
$[\mathrm{I}_0^* - \mathrm{II}^* - (d-8)/6]$, $[2\mathrm{IV} - (r-1)/3]$, $[2\mathrm{IV}^* - (r-2)/3]$,
$[2\mathrm{II} - (r-1)/6]$, $[2\mathrm{II}^* - (r-5)/6]$.

Proof. Let M/L^{unr} be the extension of minimal degree over which $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ acquires stable reduction, which is provided by Theorem 3.5. By [8, Theorem (2.4)] and Lemma 3.1 we find that $[M : L^{\mathrm{unr}}]$ divides $\mu' = |\mu(K)|$. Moreover, since $p > 5$, the extension M/L^{unr} is tame (see [16, Proposition 4.1.2]). It follows from [15, Proposition 3] that C_M/M has stable reduction. Moreover, since the stable model of C_M/M is assumed to be singular, $[M : L^{\mathrm{unr}}]$ divides 8 or 12 by [15, Corollaire 4.1] and the special fiber of the stable model of C_M/M is a union of two elliptic curves meeting at one point (see the paragraph right before [15, Proposition 2]). We will combine the above observations with Liu's algorithm [16] along with Lemma 4.6 to compute the special fiber of the minimal proper regular model.

(i) If $\mu' = 2$ or 10, then we see that $[M : L^{\mathrm{unr}}]$ divides 2 or 10. Since $[M : L^{\mathrm{unr}}]$ also divides 8 or 12, we find that $[M : L^{\mathrm{unr}}] = 1$ or 2. Therefore, we find, using [16, Tables 3.1 and 3.2], that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ has either reduction of type $[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{I}_0 - \mathrm{I}_0 - d]$, $[\mathrm{I}_0 - \mathrm{I}_0^* - (d-1)/2]$, or $[2\mathrm{I}_0 - r]$. However, the last three types are excluded by (i) of Lemma 4.6.

(ii) If $\mu' = 4$, then we see that $[M : L^{\mathrm{unr}}]$ divides 4. If $[M : L^{\mathrm{unr}}] = 1$ or 2, then we find that the reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are the types that appear in (i). Therefore, we assume from now on that $[M : L^{\mathrm{unr}}] = 4$. Using [16, Tables 3.1 and 3.2], we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type $[\mathrm{I}_0^* - \mathrm{I}_0^* - (d-2)/2]$, $[\mathrm{III}-\mathrm{III}-(d-2)/4]$, $[\mathrm{III}-\mathrm{III}^*-(d-4)/4]$, $[\mathrm{III}^*-\mathrm{III}^*-(d-6)/4]$, $[2\mathrm{I}_0^*-(r-1)/2]$, $[\mathrm{I}_0-\mathrm{III}-(d-1)/4]$, $[\mathrm{I}_0 - \mathrm{III}^* - (d-3)/4]$, $[\mathrm{I}_0^* - \mathrm{III}^* - (d-5)/4]$, $[\mathrm{I}_0^* - \mathrm{III} - (d-3)/4]$. However, the last four types are excluded by (i) of Lemma 4.6.

(iii) If $\mu' = 6$, then we see that $[M : L^{\mathrm{unr}}]$ divides 6. If $[M : L^{\mathrm{unr}}] = 1$ or 2, then we find that the reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are the types that appear in (i). If

$[M : L^{\mathrm{unr}}] = 3$, then, using [16, Table 3.1], we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type [IV−IV−$(d-2)/3$], [IV−IV*−$(d-3)/3$], [IV*−IV*−$(d-4)/3$], [I_0 − IV − $(d-1)/3$], or [I_0 − IV* − $(d-2)/3$]. However, the last two types are excluded by Lemma 4.6. On the other hand, if $[M : L^{\mathrm{unr}}] = 6$, then (ii) of Lemma 4.6 below shows that either $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ has reduction type [II−II−$(d-2)/6$], [II−II*−$(d-6)/6$], [II*−II*−$(d-10)/6$], [I_0^*−II−$(d-4)/6$], [I_0^*−II*−$(d-8)/6$], or we have that $2 \nmid v_L(J_2)$. Assume from now on that $2 \nmid v_L(J_2)$. Using [16, Table 3.2], we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type [2IV − $(r-1)/3$] or [2IV* − $(r-2)/3$].

(iv) If $\mu' = 8$, then we see that $[M : L^{\mathrm{unr}}]$ divides 8. If $[M : L^{\mathrm{unr}}]$ divides 4, then we find that the reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are the types that appear in (ii). On the other hand, if $[M : L^{\mathrm{unr}}] = 8$, then, using [16, Tables 3.1 and 3.2], we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type [2III − $(r-1)/4$] or [2III* − $(r-3)/4$].

(v) If $\mu' = 12$, then we see that $[M : L^{\mathrm{unr}}]$ divides 12. If $[M : L^{\mathrm{unr}}]$ divides 4 or 6, then we find that the reduction types of $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ are the types that appear in (ii) or in (iii). Assume now that $[M : L^{\mathrm{unr}}] = 12$. Part (ii) of Lemma 4.6 shows that $2 \nmid v_L(J_2)$. Therefore, using [16, Table 3.2], we find that $C_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ can only have reduction of type [2II − $(r-1)/6$] or [2II* − $(r-5)/6$]. □

Note. In Theorem 1.3, if a reduction type appears in the tables, then the reduction types corresponding to quadratic twists of the original curve also appear in the tables, as we explain here. Keep the same assumptions and notation as in Theorem 1.3. Let $L_d = L(\sqrt{d})$ be a quadratic extension of L and let $\chi : \mathrm{Gal}(\bar{L}/L) \to \{\pm 1\}$ with $\chi(\sigma) = (\sqrt{d})^\sigma/\sqrt{d}$ be the associated quadratic character. Denote by j the hyperelliptic involution of C/L. Consider the cocycle $\xi \in \mathrm{H}^1(\mathrm{Gal}(\bar{L}/L), \mathrm{Aut}_{\bar{L}}(C))$ given by $\xi(\sigma) = [j]$ if $\chi(\sigma) = -1$, and $\xi(\sigma) = [id]$ otherwise. Let C^ξ/L be the twist of C/L corresponding to the cocycle ξ. We claim that $\mathrm{Jac}(C^\xi)/L$ also has complex multiplication by K over the field L. To justify this, note that the embedding $\iota : K \hookrightarrow \mathrm{End}_L^0(\mathrm{Jac}(C))$ induces an embedding $\iota_\xi : K \hookrightarrow \mathrm{End}_L^0(\mathrm{Jac}(C^\xi))$ (the proof of [34, Lemma 2.2] carries over in our case). Sadek in [31] has computed the reduction type of C^ξ/L based on the reduction type of C/L.

We do not know whether all the types allowed by Theorem 1.3 actually occur. We present some examples of reduction types of CM curves below.

Example 4.8 (see [13, Example 3.6.2] and [40, Table 1]). Consider the hyperelliptic curve $C/\mathbb{Q}$ given by

$$y^2 = -8x^6 - 64x^5 + 1120x^4 + 4760x^3 - 48400x^2 + 22627x - 91839.$$

The Jacobian of this curve does not have complex multiplication over $\mathbb{Q}$ but it acquires complex multiplication after a finite extension of $\mathbb{Q}$.

Using SAGE [32] we see that the curve $C/\mathbb{Q}$ has bad reduction modulo 2, 5, 11, and 13, it has reduction type $[\mathrm{I}_0^* - \mathrm{I}_0^* - 0]$ modulo 11, and it has reduction type [VI] modulo 13. Therefore, $C/\mathbb{Q}$ has potentially good reduction modulo 13, and the potential stable reduction of $C/\mathbb{Q}$ modulo 11 is the union of two elliptic curves intersecting transversely at a point. Let $N > 1$ be a positive integer not divisible by 2, 5, 11, or 13, and let $L = \mathbb{Q}(\mathrm{Jac}(C)[N])$. Then $\mathrm{Jac}(C)_L/L$ has good reduction, by [33, Corollary 3], and it has all of its endomorphisms defined over L, by [36, Theorem 2.4]. Therefore, C_L/L has good reduction modulo primes of L above 13 and it has reduction of type $[\mathrm{I}_0 - \mathrm{I}_0 - 2]$ modulo primes of L above 11.

Example 4.9 (see [13, Example 3.2] and [40, Table 1]). Consider the hyperelliptic curve $C/\mathbb{Q}$ given by

$$y^2 = x^5 + 1.$$

The curve $C/\mathbb{Q}$ has good reduction outside of 2, 5 and it has reduction type [VII] modulo 5. Moreover, the base extension $\mathrm{Jac}(C)_{\mathbb{Q}(\zeta_5)}/\mathbb{Q}(\zeta_5)$ of the Jacobian of $C/\mathbb{Q}$ to $\mathbb{Q}(\zeta_5)$ has complex multiplication by $\mathbb{Q}(\zeta_5)$.

5. Geometric component groups and torsion of CM abelian varieties

In this section, we focus on component groups and torsion points of CM abelian varieties. We first prove Proposition 5.1 below which provides a bound for the component group of CM abelian varieties and a list of the possible component groups of elliptic curves with complex multiplication. We then prove Proposition 1.4 using Proposition 5.1.

Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and perfect residue field k_L of characteristic $p > 0$, and let A/K be an abelian variety. We denote by $\mathcal{A}/R$ the Néron model of A/K (see [1] for the definition as well as the basic properties of Néron models). The special fiber $\mathcal{A}_{k_L}/k_L$ of $\mathcal{A}/R$ is a smooth commutative group scheme. We denote by $\mathcal{A}^0_{k_L}/k_L$ the connected component of the identity of $\mathcal{A}_{k_L}/k_L$. The finite étale group scheme defined by $\Phi := \mathcal{A}_{k_L}/\mathcal{A}^0_{k_L}$ is called the component group of $\mathcal{A}/R$.

Proposition 5.1. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and finite residue field k_L of characteristic p. Let L^{unr} be the maximal unramified extension of L and denote the residue field of L^{unr} by $\overline{k_L}$.*

(i) *Assume that $p \neq 2$ and that $K \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$. Let E/L be an elliptic curve with complex multiplication by K over L such that $j_E \neq 0, 1728$. Then E/L has geometric component group $\Phi(\overline{k_L})$ isomorphic (as an abelian group) to* (0) *or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

(ii) *Let A/L be an abelian variety with complex multiplication by a CM field K over L. Then the geometric component group $\Phi(\overline{k_L})$ of $A_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ is killed by $|\mu(K)|$, where $\mu(K)$ is the group of roots of unity in K.*

Proof. (i) Theorem 1.1 implies that E/L can only have good reduction or reduction of Kodaira type I_0^*. Therefore, part (i) follows from a simple application of Tate's algorithm (see [37, page 365]).

(ii) Let M/L^{unr} be the extension of minimal degree over which $A_{L^{\mathrm{unr}}}/L^{\mathrm{unr}}$ acquires semistable reduction. It follows from Lemma 3.1 that the degree $[M : L^{\mathrm{unr}}]$ divides $|\mu(K)|$. On the other hand, work of McCallum [24] and Edixhoven, Liu, and Lorenzini [10, Theorem 1] tells us that $[M : L^{\mathrm{unr}}]$ kills $\Phi(\overline{k_L})$. Therefore, since $[M : L^{\mathrm{unr}}]$ divides $|\mu(K)|$, we find that $\Phi(\overline{k_L})$ is killed by $|\mu(K)|$. □

Note. Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and finite residue field k_L of characteristic $p > 0$. Lorenzini in [20, Corollary 3.25] has provided a list of the possible prime-to-p parts of the geometric component group of abelian varieties defined over L that have purely additive and potentially good reduction. When $p > 5$ using Theorem 1.3, together with [16, Section 8], we find that the cases where the group is $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ cannot occur among Jacobian surfaces with complex multiplication defined over the base field. By Theorem 3.3 Jacobian surfaces with complex multiplication have either purely additive or good reduction.

We now proceed to the proof of Proposition 1.4. For every positive integer m we let $\gamma_p(m) = \lfloor \log_p\left(\frac{pm}{p-1}\right) \rfloor$.

Proposition 5.2. *Let R be a complete discrete valuation ring with fraction field L of characteristic 0 and finite residue field k_L of characteristic $p > 0$. Denote by e the absolute ramification index of L. Let A/L be an abelian variety with complex multiplication by a CM field K over L and assume that A/L does not have good reduction. Then*

$$|A(L)_{\mathrm{tors}}| \leq |\mu(K)| \cdot p^{2g\gamma_p(e|\mu(K)|)}.$$

Proof. By Theorem 3.3 we find that A/K has purely additive reduction. Then the proof of [3, part (iii) of Main Theorem] carries over verbatim in our case. The only extra input is that instead of using [3, Corollary 3.8] in the proof of [3, part (iii) of Main Theorem] is that we can use the more precise bound provided by Proposition 5.1. □

Combining the previous proposition with [3, Main Theorem] below we prove Proposition 1.4, which improves slightly [3, Main Theorem] in the case where the abelian variety has complex multiplication.

Proof of Proposition 1.4. If A/L does not have good reduction, then we have by Proposition 5.2,

$$|A(L)_{\mathrm{tors}}| \le |\mu(K)| \cdot p^{2g\gamma_p(e|\mu(K)|)}.$$

On the other hand, if A/K has good reduction then, since the toric and unipotent ranks are zero, i.e., $\mu = \alpha = 0$ in their notation, while the abelian rank is g, i.e., $\beta = g$ in their notation, [3, part (ii) of Main Theorem] implies that

$$|A(L)_{\mathrm{tors}}| \le \lfloor (1+\sqrt{q})^2 \rfloor^g \cdot p^{2g\gamma_p(e)}. \qquad \square$$

Note. In the literature there exist global bounds on the torsion of CM abelian varieties over number fields (see [11] and [35]), which rely on the main theorem of complex multiplication and class field theory. On the other hand, our result is local, i.e., it only depends on local invariants, it is much more elementary, and it is of most interest when A/L has bad reduction while $|\mu(K)|$ and e are small.

Acknowledgements

The author would like to thank Pete L. Clark for suggesting the study of the properties of bad reduction of elliptic curves with complex multiplication. I would like to thank Dino J. Lorenzini for some very helpful email correspondence and some insightful comments on an earlier version of this manuscript. I would also like to thank a referee for many valuable comments and suggestions that improved the manuscript. The author was supported by Czech Science Foundation (GAČR) grant 21-00420M and by Charles University Research Center program no. UNCE/SCI/022.

References

[1] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Results in Mathematics and Related Areas **21**, Springer, Berlin, 1990. MR Zbl

[2] C.-L. Chai, B. Conrad, and F. Oort, *Complex multiplication and lifting problems*, Mathematical Surveys and Monographs **195**, American Mathematical Society, Providence, RI, 2014. MR Zbl

[3] P. L. Clark and X. Xarles, "Local bounds for torsion points on abelian varieties", *Canad. J. Math.* **60**:3 (2008), 532–555. MR Zbl

[4] S. Comalada, "Twists and reduction of an elliptic curve", *J. Number Theory* **49**:1 (1994), 45–62. MR Zbl

[5] I. Connell, "Elliptic curve handbook", preprint, 1999, available at https://webs.ucm.es/BUCM/mat/doc8354.pdf.

[6] B. Conrad, "A modern proof of Chevalley's theorem on algebraic groups", *J. Ramanujan Math. Soc.* **17**:1 (2002), 1–18. MR Zbl

[7] B. Conrad, "Gross–Zagier revisited", pp. 67–163 in *Heegner points and Rankin L-series*, Math. Sci. Res. Inst. Publ. **49**, Cambridge Univ. Press, Cambridge, 2004. MR Zbl

[8] P. Deligne and D. Mumford, "The irreducibility of the space of curves of given genus", *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), 75–109. MR Zbl

[9] T. Dokchitser and V. Dokchitser, "A remark on Tate's algorithm and Kodaira types", *Acta Arith.* **160**:1 (2013), 95–100. MR Zbl

[10] B. Edixhoven, Q. Liu, and D. Lorenzini, "The p-part of the group of components of a Néron model", *J. Algebraic Geom.* **5**:4 (1996), 801–813. MR Zbl

[11] E. Gaudron and G. Rémond, "Torsion des variétés abéliennes CM", *Proc. Amer. Math. Soc.* **146**:7 (2018), 2741–2747. MR Zbl

[12] E. Z. Goren and K. E. Lauter, "Class invariants for quartic CM fields", *Ann. Inst. Fourier* (*Grenoble*) **57**:2 (2007), 457–480. MR Zbl

[13] E. Z. Goren and K. E. Lauter, "Genus 2 curves with complex multiplication", *Int. Math. Res. Not.* **2012**:5 (2012), 1068–1142. MR Zbl

[14] B. H. Gross and D. E. Rohrlich, "Some results on the Mordell–Weil group of the Jacobian of the Fermat curve", *Invent. Math.* **44**:3 (1978), 201–224. MR Zbl

[15] Q. Liu, "Courbes stables de genre 2 et leur schéma de modules", *Math. Ann.* **295**:2 (1993), 201–222. MR Zbl

[16] Q. Liu, "Modèles minimaux des courbes de genre deux", *J. Reine Angew. Math.* **453** (1994), 137–164. MR Zbl

[17] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics **6**, Oxford University Press, Oxford, 2002. MR Zbl

[18] The LMFDB collaboration, "The L-functions and modular forms database", electronic reference, 2024, available at https://www.lmfdb.org.

[19] D. J. Lorenzini, "Dual graphs of degenerating curves", *Math. Ann.* **287**:1 (1990), 135–150. MR Zbl

[20] D. J. Lorenzini, "On the group of components of a Néron model", *J. Reine Angew. Math.* **445** (1993), 109–160. MR Zbl

[21] D. Lorenzini, "Wild quotient singularities of surfaces", *Math. Z.* **275**:1-2 (2013), 211–232. MR Zbl

[22] D. Lorenzini, "Néron models", *Eur. J. Math.* **3**:2 (2017), 171–198. MR Zbl

[23] H. Maeda, "Die stabile Reduktion der Fermatkurve über einem Zahlkörper", *Manuscripta Math.* **56**:3 (1986), 333–342. MR Zbl

[24] W. G. McCallum, "The component group of a Néron model", preprint, 1986, available at https://www.math.arizona.edu/∼wmc/Research/ComponentGroup.pdf.

[25] M. Melistas, "Purely additive reduction of abelian varieties with torsion", *J. Number Theory* **239** (2022), 21–39. MR Zbl

[26] Y. Namikawa and K. Ueno, "The complete classification of fibres in pencils of curves of genus two", *Manuscripta Math.* **9** (1973), 143–186. MR Zbl

[27] F. Oort, "Good and stable reduction of abelian varieties", *Manuscripta Math.* **11** (1974), 171–197. MR Zbl

[28] I. Papadopoulos, "Sur la classification de Néron des courbes elliptiques en caractéristique résiduelle 2 et 3", *J. Number Theory* **44**:2 (1993), 119–152. MR Zbl

[29] M. Raynaud, "Spécialisation du foncteur de Picard", *Inst. Hautes Études Sci. Publ. Math.* **38** (1970), 27–76. MR Zbl

[30] K. Rubin, "Elliptic curves with complex multiplication and the conjecture of Birch and Swinnerton-Dyer", preprint, Arizona Winter School, 1999, available at https://swc-math.github.io/aws/1999/99RubinCM.pdf.

[31] M. Sadek, "Minimal regular models of quadratic twists of genus two curves", *Acta Arith.* **183**:4 (2018), 317–337. MR Zbl

[32] The Sage Developers, *SageMath (version 9.0)*, 2020, available at https://www.sagemath.org.

[33] J.-P. Serre and J. Tate, "Good reduction of abelian varieties", *Ann. of Math.* (2) **88** (1968), 492–517. MR Zbl

[34] A. Shnidman, "Quadratic twists of abelian varieties with real multiplication", *Int. Math. Res. Not.* **2021**:5 (2021), 3267–3298. MR Zbl

[35] A. Silverberg, "Torsion points on abelian varieties of CM-type", *Compositio Math.* **68**:3 (1988), 241–249. MR Zbl

[36] A. Silverberg, "Fields of definition for homomorphisms of abelian varieties", *J. Pure Appl. Algebra* **77**:3 (1992), 253–262. MR Zbl

[37] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics **151**, Springer, New York, 1994. MR Zbl

[38] J. H. Silverman, *The arithmetic of elliptic curves*, 2nd ed., Graduate Texts in Mathematics **106**, Springer, Dordrecht, 2009. MR Zbl

[39] J. Tate, "Algorithm for determining the type of a singular fiber in an elliptic pencil", pp. 33–52 in *Modular functions of one variable*, *IV* (Antwerp, 1972), Lecture Notes in Math. **476**, Springer, Berlin, 1975. MR Zbl

[40] P. van Wamelen, "Examples of genus two CM curves defined over the rationals", *Math. Comp.* **68**:225 (1999), 307–320. MR Zbl

[41] H. Wang, "On the Kodaira types of elliptic curves with potentially good supersingular reduction", preprint, 2024. arXiv 2406.01985

Received January 1, 2023. Revised January 19, 2024.

MENTZELOS MELISTAS
DEPARTMENT OF ALGEBRA
FACULTY OF MATHEMATICS AND PHYSICS
CHARLES UNIVERSITY
PRAHA
CZECH REPUBLIC

and

DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF TWENTE
ENSCHEDE
NETHERLANDS

mentzmel@gmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 329, No. 2, 2024
https://doi.org/10.2140/pjm.2024.329.259

THE LOCAL CHARACTER EXPANSION AS BRANCHING RULES: NILPOTENT CONES AND THE CASE OF SL(2)

MONICA NEVINS

We show there exist representations of each maximal compact subgroup K of the p-adic group $G = \mathrm{SL}(2, F)$, $p \neq 2$, for each nilpotent coadjoint orbit, such that every irreducible admissible (complex) representation of G, upon restriction to a suitable subgroup of K, is a sum of these five representations in the Grothendieck group. This is a representation-theoretic analogue of the analytic local character expansion due to Harish-Chandra and Howe. Moreover, we show for general connected reductive groups that the wave front set of many irreducible positive-depth representations of G are completely determined by the *nilpotent support* of their unrefined minimal K-types.

1. Introduction

The distribution character of an admissible (complex) representation of a p-adic group can be expressed, in a neighbourhood of the identity, as a linear combination of Fourier transforms of the finitely many nilpotent orbital integrals in the dual of the Lie algebra. This remarkable theorem, known as the Harish-Chandra–Howe local character expansion, has many variations (such as expansions on neighbourhoods of other semisimple elements, or expansions in terms of other collections of orbital integrals [Kim and Murnaghan 2003; 2006; Spice 2018]) and many applications (such as determining the Gelfand–Kirillov dimension of a representation, or relating to conjectural classifications such as the orbit method, or the local Langlands correspondence [Barbasch and Moy 1997; Ciubotaru et al. 2022a; 2022b; Jiang et al. 2022]). Though it is primarily considered in characteristic zero, it also holds when the characteristic is sufficiently large and a suitable substitute for the exponential map exists [Cluckers et al. 2014].

In this paper, we interpret the local character expansion as a statement in the Grothendieck group of representations of a maximal compact open subgroup, upon restriction to a subgroup of suitable depth, for the case that $G = \mathrm{SL}(2, F)$, where F

Supported by NSERC Discovery grant RGPIN-2020-05020.
MSC2020: 22E50.
Keywords: representation theory, nilpotent orbits, local character expansion, p-adic groups.

© 2024 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

is a local nonarchimedean field of residual characteristic at least 3. In particular, we construct for each nilpotent orbit $\mathcal{O}$ of G in the dual of its Lie algebra $\mathfrak{g}^*$ a (highly reducible) representation $\tau_x(\mathcal{O})$ of each maximal compact open subgroup G_x with the following property.

Theorem 1.1. *Let π be an irreducible admissible representation of $G = \mathrm{SL}(2, F)$ of depth $r \geq 0$, and let x be a vertex in the building of G. Then there exist integers $c_{x,\mathcal{O}}(\pi)$ such that in the Grothendieck group of representations we have*

$$\mathrm{Res}^{G}_{G_{x,r+}} \pi = \sum_{\mathcal{O}} c_{x,\mathcal{O}}(\pi)\, \mathrm{Res}^{G_x}_{G_{x,r+}} \tau_x(\mathcal{O}), \tag{1-1}$$

where $G_{x,r+}$ is the Moy–Prasad filtration subgroup of G_x of depth $r+$, and the sum is over all nilpotent orbits in $\mathfrak{g}^*$.

Moreover, the coefficients corresponding to the regular nilpotent orbits in this expansion are nonnegative integers and agree with those of the Harish-Chandra–Howe local character expansion (subject to suitable normalizations). Note that while inherently expressing the same local nature of representations, our statement holds with fewer restrictions on F than does the local character expansion, because it does not depend on the existence of a G-equivariant map, such as the exponential or a Cayley transform, from the Lie algebra to the group.

If G is $\mathrm{SL}_2(F)$ or an inner form of $\mathrm{GL}_n(F)$, then Henniart and Vignéras have proven a different local expansion in the same spirit as (1-1), one that holds for representations over any field R of characteristic not p, in a sufficiently small neighbourhood of 1, but which constructs the right-hand side as restrictions of particular representations of G itself ([Henniart and Vignéras 2024, Theorem 6.18] and [Henniart and Vignéras 2023, Theorem 1.3], respectively). When $G = \mathrm{GL}_n(F)$, these representations are of the form $\mathrm{Ind}_P^G \mathbf{1}$, for a suitable parabolic subgroup attached to $\mathcal{O}$, vastly generalizing a result of Roger Howe [1974]. When $G = \mathrm{SL}_2(F)$, they are representations that occur in an L-packet of size 4 (called "special unipotent representations" in the complex case here); the distinguished role of these representations in the complex case was observed previously in [Nevins 2011, §4]. In Section 8 we explore applications of these ideas, and answer [Henniart and Vignéras 2023, Questions 1.1 and 1.2] for complex representations of $\mathrm{SL}(2, F)$.

Now suppose G is a general connected reductive group. In Section 3, we develop some theory towards establishing the direct relationship from the local character expansion to a decomposition like (1-1), as follows.

The set of maximal orbits appearing in the local character expansion for an admissible representation π is denoted by $\mathcal{WF}(\pi)$; the closure of the union of these orbits is the wave front set of π. For depth-zero representations π, Barbasch and Moy [1997] proved that $\mathcal{WF}(\pi)$ is determined by the depth-zero components of

the restriction of π to various maximal compact subgroups, through the theory of Gelfand–Graev representations.

For a positive-depth representation with minimal K-type Γ (in the sense of Moy and Prasad [1994]), we should instead infer $\mathcal{WF}(\pi)$ from the *nilpotent support* $\mathrm{Nil}(\Gamma)$ (Definition 3.2) of Γ. This definition, of independent interest, depends strongly on the classification of nilpotent orbits using Bruhat–Tits theory [Barbasch and Moy 1997; DeBacker 2002b]. In fact, in Proposition 3.4 we show that the algebraic notion of nilpotent support can be characterized as the set of nonzero nilpotent orbits appearing in the asymptotic cone on Γ, as defined in [Adams and Vogan 2021]. In Theorem 3.5 (proof due to Fiona Murnaghan), we prove that $\mathcal{WF}(\pi)$ is the set of maximal orbits of $\mathrm{Nil}(\Gamma)$ whenever the Γ-asymptotic expansion [Kim and Murnaghan 2003] reduces to a single term.

This last result is similar to recent work of Ciubotaru and Okada [2023], who show that the depth-r components of the restriction to certain compact open subgroups determine the wave front set of π. The idea of the nilpotent support is also central to that work, where they develop it using, among other things, the geometry of the associated finite reductive group.

Now again suppose that $G = \mathrm{SL}(2, F)$. Our result gives a second characterization of $\mathcal{WF}(\pi)$: it can be entirely determined from the *nontypical* representations occurring in the restriction of π to a maximal compact open subgroup, for π of any depth. That is, the asymptotic decomposition of $\mathrm{Res}_{G_x}\pi$ unfolds exactly as the representations $\tau_x(\mathcal{O})$ for $\mathcal{O} \in \mathcal{WF}(\pi)$.

For the case of a positive-depth representation π, our main theorem is stated in Theorem 6.4, with the explicit values of the constant coefficient given in Proposition 6.7. To prove the theorem, we first show that the restriction of π to a maximal compact subgroup can be expressed entirely in terms of twists of the pair (Γ, χ) used in the construction of π (Theorem 6.2), using results from [Nevins 2005; 2013]. Here, χ is a character of a torus $T = \mathrm{Cent}_G(\Gamma)$ that is realized by $\Gamma \in \mathfrak{g}^*$, and the realization of the irreducible components of the restriction is framed in terms of a generalization (Proposition 5.4) of a construction due to Shalika in his thesis. From this characterization, and a key technical result (Lemma 5.5), it follows that the expansion (1-1) exists and has leading terms corresponding to the nilpotent support of Γ. Since Γ represents a minimal K-type of π in the sense of Moy and Prasad [1994], we independently recover from Theorem 3.5 that the maximal orbits in $\mathrm{Nil}(\Gamma)$ coincide with $\mathcal{WF}(\pi)$.

For representations of depth zero, the principal technical difficulties lie in matching the depth-zero components with nilpotent orbits, particularly in the case of the twelve "exceptional" representations: the reducible principal series, the principal series composed of the trivial and the Steinberg representation, and the four special supercuspidal representations. Once these are addressed, Theorem 7.4 follows

by carefully extracting the necessary branching rules from [Nevins 2005; 2013]. Again, the orbits in $\mathcal{WF}(\pi)$ are obtained from both the depth-zero components (via [Barbasch and Moy 1997]) and the asymptotic development of the branching rules.

At two crucial junctures we use information that is currently only known for $G = \mathrm{SL}(2, F)$ and a handful of other small-rank groups: one is the explicit calculation of the asymptotic cone on any semisimple element of $\mathfrak{g}^*$ (Section 4); the other is the full knowledge of the representation theory of the maximal compact subgroups of G (Section 5). While the former seems a tractable and interesting question in general, the latter is quite daunting: it is not expected that we will achieve a classification of the representations of maximal compact open subgroups of p-adic reductive groups. Note that a full classification is not necessary to prove the theorem: what is needed is a construction of an appropriate representation of G_x attached to each nilpotent orbit, and we explore how this might be done in Section 5B.

There are many interesting applications and open directions left to pursue. Evidently the overarching goal is to establish a result like (1-1) for a large class of groups, using the tools presented here, or those developed in [Henniart and Vignéras 2023; 2024]. To extend the work here, it may be fruitful to build representations of the groups $G_{x,0+}$ directly, rather than to construct representations of $G_{x,0}$; this has the advantage of avoiding the difficulties inherent at depth zero. It may also allow for a more uniform treatment of all points x of the building; in this paper, we consider only vertices, and the union of all $G_{x,r+}$ as x runs over vertices is not equal to G_{r+} in general.

In another direction, the Γ-asymptotic expansions of [Kim and Murnaghan 2003; 2006] describe the character of a positive-depth representation in a larger neighbourhood than does the local character expansion, by incorporating a minimal K-type Γ. Then Theorem 6.2 can be interpreted as analogously formulating these expansions in terms of branching rules. It would be interesting to explore this idea further.

The paper is organized as follows. We set our notation in Section 2 and then present some background on the local character expansion that provides the motivation and context for our results. In Section 3 we consider a general connected reductive group G. We define the nilpotent support of an element Γ of $\mathfrak{g}^*$, show it defines the asymptotic cone of Γ, and relate this to the wave front set via the theory of Γ-asymptotic expansions.

We then specialize to $G = \mathrm{SL}(2, F)$. In Section 4 we characterize the nilpotent cones $\mathrm{Nil}(\Gamma)$ in many ways (Proposition 4.1) and compute them explicitly. In Section 5 we recall the construction of certain irreducible representations of $\mathrm{SL}(2, \mathcal{R})$ by Shalika in his 1966 thesis [Shalika 2004], and then rephrase it using Bruhat–Tits theory and derive some consequences. This allows us to define, for each vertex $x \in \mathcal{B}(G)$, each nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$, and each central character ζ a representation $\tau_x(\mathcal{O}, \zeta)$ of G_x.

We prove our main theorems for representations of positive depth in Section 6 and for representations of depth zero in Section 7. We conclude with two brief applications of Theorem 1.1 in Section 8: an explicit formula for the functions $\widehat{\mu_{\mathcal{O}}}$ in terms of the trace character of the representation $\tau_x(\mathcal{O})$ of the compact group G_x; and an explicit polynomial expression for $\dim(\pi^{G_{x,2n}})$ (in the spirit of [Henniart and Vignéras 2023]) whose existence is predicted by the local character expansion.

2. Notation and background

Let F be a local nonarchimedean field of residual characteristic $p \neq 2$, with integer ring $\mathcal{R}$, maximal ideal $\mathcal{P}$ and residue field $\mathfrak{f}$ of cardinality q. We impose additional hypotheses on p in Section 2B, below. Fix once and for all an additive character ψ of F that is trivial on $\mathcal{P}$ and nontrivial on $\mathcal{R}$. Fix a uniformizer ϖ and normalize the valuation on F (and any extension thereof) by $\mathrm{val}(\varpi) = 1$. We write $\mathrm{val}(0) := \infty$.

Let $\boldsymbol{G}$ denote a connected reductive algebraic group defined over F whose group of F-rational points is denoted by G; we use $\mathfrak{g} = \mathrm{Lie}(\boldsymbol{G})(F)$ to denote its Lie algebra over F. We simplify notation by referring to tori, Borel subgroups and parabolic subgroups of G when we mean the F-points of such algebraic F-subgroups of $\boldsymbol{G}$, and denote them in roman font. Let G^{reg}, respectively $\mathfrak{g}^{\mathrm{reg}}$, denote the set of regular semisimple elements of G, respectively $\mathfrak{g}$. The group G acts on $\mathfrak{g}$ via the adjoint action Ad and on its dual $\mathfrak{g}^*$ via the coadjoint action Ad^*; we abbreviate these by both $g \cdot X$ or ${}^gX$ for $g \in G$ and X in $\mathfrak{g}$ or $\mathfrak{g}^*$. Similarly, if H is a subgroup of G we write ${}^gH$ for the group gHg^{-1}.

An element $X \in \mathfrak{g}^*$ or $\mathfrak{g}$ is called *semisimple* (or *almost stable*) if its G-orbit is closed. We define $X \in \mathfrak{g}^*$ or $\mathfrak{g}$ to be *nilpotent* if there exists an F-rational one-parameter subgroup $\lambda \in X_*(\boldsymbol{G})$ such that $\lim_{t\to 0} {}^{\lambda(t)}X = 0$. By [Adler and DeBacker 2002, §2.5], this is equivalent to a more usual definition that the closure of the coadjoint orbit in the rational topology contains 0. We say the one-parameter subgroup λ is *adapted* to X [DeBacker 2002b, Definition 4.5.6] if ${}^{\lambda(t)}X = t^2X$. We write $\mathcal{N}^*$ for the set of nilpotent elements of $\mathfrak{g}^*$ and $\mathscr{O}(0)$ for the (finite) set of G-orbits in $\mathcal{N}^*$.

We sometimes specify a group of matrices merely by the sets in which its entries lie; in this case, that the resulting subgroup is the intersection of this set with G is understood. We write $\lceil t \rceil = \min\{n \in \mathbb{Z} \mid n \geq t\}$ and $\lfloor t \rfloor = \max\{n \in \mathbb{Z} \mid n \leq t\}$. Write $\mathrm{Cent}_G(S)$ for the centralizer in G of the element or set S. We may write $[\sigma]$ for the trace character of a representation σ of a finite or compact group. The trivial representation is denoted by $\mathbf{1}$, and the characteristic function of a subset S is denoted by $\mathbf{1}_S$.

2A. ***The Bruhat–Tits building and Moy–Prasad filtration subgroups.*** Let $\mathcal{B}(G) = \mathcal{B}(\boldsymbol{G}, F)$ denote the (enlarged) Bruhat–Tits building of G; then to each $x \in \mathcal{B}(G)$

we associate its stabilizer G_x, which is a compact subgroup of G containing the parahoric subgroup $G_{x,0}$. These admit a Moy–Prasad filtration by normal subgroups $G_{x,r}$ with $r \in \mathbb{R}_{\geq 0}$ defined relative to the valuation on F. We briefly recap the definition; see also [Fintzen 2021b, §2] and [Kaletha and Prasad 2023, §13].

To define $G_{x,r}$, choose an apartment $\mathcal{A} \subset \mathcal{B}(G)$ containing x; this is the affine space over $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ for some maximal split torus T of G and we write $\mathcal{A} = \mathcal{A}(G, T)$. Let $\Phi = \Phi(G, T)$ denote the corresponding root system and Ψ the set of affine roots, viewed as functions on $\mathcal{A}$. For each root $\alpha \in \Phi$, let U_α denote the corresponding root subgroup. The affine roots ψ with gradient α define a filtration of U_α by compact open subgroups U_ψ.

Let $\boldsymbol{C} = \mathrm{Cent}_{\boldsymbol{G}}(\boldsymbol{T})$. As summarized at the start of [Kaletha and Prasad 2023, §9.8], $C = \boldsymbol{C}(F)$ contains a parahoric subgroup C_0, and a filtration by compact open normal subgroups C_r, $r > 0$, that is independent of the point $x \in \mathcal{A}$. When $\boldsymbol{C}$ is not tamely ramified, this filtration can be very subtle; see the extended careful analysis in [Kaletha and Prasad 2023, §13.3, §B.10].

As summarized in [Kaletha and Prasad 2023, Proposition 13.2.5], for any $r \geq 0$ we define compact open subgroups

$$G_{x,r} = \langle C_r, U_\psi \mid \psi \in \Psi, \psi(x) \geq r \rangle;$$

if $r = 0$ this is the parahoric subgroup and for $r > 0$ it is a Moy–Prasad filtration subgroup of $G_{x,0}$. It is independent of the choice of apartment containing x. The Moy–Prasad filtration is G-equivariant; for example, ${}^g G_{x,r} = G_{gx,r}$ for all $x \in \mathcal{B}(G)$ and $r \geq 0$.

Similarly, the Lie algebra $\mathfrak{g}$ admits a filtration $\mathfrak{g}_{x,r}$ by $\mathcal{R}$-modules indexed by $r \in \mathbb{R}$, as follows. Let $\mathfrak{t}$ denote the Lie algebra of T, $\mathfrak{c}$ its centralizer in $\mathfrak{g}$ and for each $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ denote the corresponding root subspace. These subspaces admit filtrations by $\mathcal{R}$-submodules $\mathfrak{c}_r$ with $r \in \mathbb{R}$ and $\mathfrak{g}_\psi$ for $\psi \in \Psi$, respectively, such that

$$\mathfrak{g}_{x,r} = \mathfrak{c}_r \oplus \bigoplus_\alpha \mathfrak{g}_{\alpha,x,r}, \tag{2-1}$$

where $\mathfrak{g}_{\alpha,x,r}$ is the union of the $\mathcal{R}$-submodules $\mathfrak{g}_\psi$ such that $\psi \in \Psi$, the gradient of ψ is α, and $\psi(x) \geq r$. We write

$$G_{x,r+} = \bigcup_{s>r} G_{x,s} \quad \text{and} \quad \mathfrak{g}_{x,r+} = \bigcup_{s>r} \mathfrak{g}_{x,s}.$$

If the maximally split maximal tori of G are weakly induced, as defined in [Kaletha and Prasad 2023, Definition B.6.2], then for all $r > 0$ we have the Moy–Prasad isomorphism $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,2r} \cong G_{x,r}/G_{x,2r}$ [Kaletha and Prasad 2023, Theorem 13.5.1], which can be realized by a mock exponential map $e = e_x : \mathfrak{g}_{x,0+} \to G_{x,0+}$ as in [Adler 1998, §1.6]. Writing $\langle X, Y \rangle$ for the natural pairing of $X \in \mathfrak{g}^*$ with $Y \in \mathfrak{g}$,

the Moy–Prasad filtration on the dual of the Lie algebra is defined by

$$\mathfrak{g}^*_{x,r} = \{X \in \mathfrak{g}^* \mid \langle X, Y\rangle \in \mathcal{P} \text{ for all } Y \in \mathfrak{g}_{x,(-r)+}\}.$$

We again define $\mathfrak{g}^*_{x,r+} = \bigcup_{s>r} \mathfrak{g}^*_{x,s}$. For any $x \in \mathcal{B}(G)$, $r > 0$ and $s \in \mathbb{R}$, the adjoint (respectively coadjoint) action of G on $\mathfrak{g}$ induces an action of $G_x/G_{x,r}$ on $\mathfrak{g}_{x,s}/\mathfrak{g}_{x,s+r}$ (respectively, $\mathfrak{g}^*_{x,s}/\mathfrak{g}^*_{x,s+r}$) [Adler 1998, Proposition 1.4.3].

Finally, for any $r \geq 0$ we define G-stable subsets

$$G_r = \bigcup_{x\in\mathcal{B}(G)} G_{x,r} \quad \text{and} \quad G_{r+} = \bigcup_{x\in\mathcal{B}(G)} G_{x,r+}.$$

For any real number r we do the same to define $\mathfrak{g}_r$ and $\mathfrak{g}_{r+}$.

If (π, V) is an irreducible admissible representation of G, then its depth is defined as the least real number $r \geq 0$ such that there exists $x \in \mathcal{B}(G)$ for which $V^{G_{x,r+}} \neq \{0\}$. We define the depth of a smooth irreducible representation ρ of G_x, for fixed x, in the same way; this is equivalent to the least $r \geq 0$ for which ρ factors through $G_x/G_{x,r+}$.

2B. ***Restrictions on p.*** We impose the restriction that G splits over a tamely ramified extension of F and that p does not divide the order of the absolute Weyl group of $\boldsymbol{G}$. One of the main results of [Fintzen 2021c] is that this is sufficient to ensure that all irreducible admissible representations are tame. It furthermore ensures that all maximally split maximal tori over F are weakly induced, so that our parahoric subgroups are defined relative to the standard filtration sketched above [Kaletha and Prasad 2023, §B.5, Proposition B.10.5] and the Moy–Prasad isomorphism holds. Combining [Fintzen 2021c, Lemma 2.2, Table 1] and [Adler and Roche 2000, §1], one sees that the hypotheses of [Adler 1998, Hypothesis 2.1.1] or [Adler and Roche 2000, Proposition 4.1] hold, so that there is a nondegenerate G-invariant bilinear form on $\mathfrak{g}$ under which $\mathfrak{g}^*_{x,r}$ and $\mathfrak{g}_{x,r}$ are identified for all x and r. For $\boldsymbol{G} = \mathrm{SL}(2)$ and $p \neq 2$ we may take the trace form, and define for each $\dot{X} \in \mathfrak{g}$ the element $X \in \mathfrak{g}^*$ by $\langle X, \cdot\,\rangle = \mathrm{tr}(\dot{X}\,\cdot\,)$.

We also impose the hypotheses of [DeBacker 2002b, §4] to obtain the classification of nilpotent orbits; this requires the use of $\mathfrak{sl}_2(F)$ triples over the residue field as well as some properties of a mock exponential map. By recent work of Stewart and Thomas [2018] the former condition is satisfied for $p > h$, where h is the Coxeter number of G. To satisfy all hypotheses for $G = \mathrm{SL}(2, F)$, it suffices to take $p \neq 2$.

In contrast, to state the local character expansion, which relates a function on the group to one on the Lie algebra, one needs a G-equivariant map $\mathfrak{g}_{0+} \to G_{0+}$ satisfying [Debacker 2002a, Hypothesis 3.2.1]. Such a map, which we'll simply denote by exp, can exist in large positive characteristic (see, for example, the discussion in [Cluckers et al. 2014, §2]); in characteristic zero, [DeBacker and

Reeder 2009, Lemma B.0.3] gives an effective lower bound on p. For $G = \mathrm{SL}(2, F)$, this entails in characteristic zero that $p > e + 1$, where e is the ramification index of F over $\mathbb{Q}_p$, for example.

2C. ***The local character expansion.*** As detailed in the expanded notes [Harish-Chandra 1999], Harish-Chandra proved in the 1970s that the distribution character of an irreducible admissible representation π of G, which is given on $f \in C_c^\infty(G)$ by

$$\Theta_\pi(f) = \mathrm{tr} \int f(g)\pi(g)\, dg,$$

is well defined and representable by a function, which we also denote by Θ_π, that is locally integrable on G and locally constant on the set G^{reg} of regular semisimple elements of G (see [Adler and Korman 2007, §13] and the discussion therein).

Similarly, to each coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ we associate its orbital integral, given on $f \in C_c^\infty(\mathfrak{g}^*)$ by

$$\mu_{\mathcal{O}}(f) = \int_{\mathcal{O}} f(X)\, d\mu_{\mathcal{O}}(X), \tag{2-2}$$

where $d\mu_{\mathcal{O}}$ is a Radon measure [Ranga Rao 1972]. Relative to ψ, the fixed additive character of F, the Fourier transform of $f \in C_c^\infty(\mathfrak{g})$ is a function $\widehat{f} \in C_c^\infty(\mathfrak{g}^*)$. The Fourier transform of the orbital integral $\mu_{\mathcal{O}}$ is the distribution given on $f \in C_c^\infty(\mathfrak{g})$ by $\widehat{\mu_{\mathcal{O}}}(f) = \mu_{\mathcal{O}}(\widehat{f})$. Then $\widehat{\mu_{\mathcal{O}}}$ is representable by a locally integrable function on $\mathfrak{g}$ that is locally constant on $\mathfrak{g}^{\mathrm{reg}}$ [Harish-Chandra 1999, Theorem 4.4]. We set $\mathfrak{g}^{\mathrm{reg}}_{r+} := \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r+} \cap \mathfrak{g}^{\mathrm{reg}}$.

The local character expansion expresses that these finitely many functions $\widehat{\mu_{\mathcal{O}}}$, for $\mathcal{O} \in \mathscr{O}(0)$, form a basis, in a neighbourhood of 0, for the space of locally integrable G-invariant functions that are locally constant on $\mathfrak{g}^{\mathrm{reg}}$. The nature of the expansion was first proven for $G = \mathrm{GL}(n, F)$ in characteristic 0 by Howe [1974] and then in the generality of connected reductive groups in characteristic zero by Harish-Chandra [1999]. Cluckers, Gordon and Halupczok [Cluckers et al. 2014] proved its validity in large positive characteristic; Adler and Korman [2007] proved an analogous result for expansions centred at other semisimple elements.

The precise domain on which the local character expansion holds was conjectured by Hales, Moy and Prasad [Moy and Prasad 1994] and proven in [Waldspurger 1995] for a large class of groups and by [Debacker 2002a] in the following generality.

Theorem 2.1 (the local character expansion). *If π is an irreducible admissible representation of G of depth r, then there exist unique $c_{\mathcal{O}}(\pi) \in \mathbb{C}$ such that for all $X \in \mathfrak{g}^{\mathrm{reg}}_{r+}$, we have*

$$\Theta_\pi(\exp(X)) = \sum_{\mathcal{O} \in \mathscr{O}(0)} c_{\mathcal{O}}(\pi)\widehat{\mu_{\mathcal{O}}}(X). \tag{2-3}$$

We denote by $\mathcal{WF}(\pi)$ the set of *maximal* nilpotent orbits $\mathcal{O}$ such that $c_{\mathcal{O}}(\pi) \neq 0$, where the ordering is taken in the local topology; this is the set denoted by $\mathrm{WF}^{\mathrm{rat}}(\pi)$ in [Tsai 2023a]. Heifetz [1985] defined and developed the analytic notion of the *wave front set* of a representation of a p-adic group, in analogy with the work of Howe [1981] in the real case. Przebinda [1990] proved that the wave front set coincides with the support of the right side of (2-3), which is the closure of the union of these orbits. Recent work of Cheng-Chiang Tsai [2024; 2023a; 2023b] has shown that the orbits of $\mathcal{WF}(\pi)$ may fail to be stably conjugate.

Finally, note that for $G = \mathrm{GL}(n, F)$, Howe proved that for each $\mathcal{O} \in \mathscr{O}(0)$, there is a corresponding parabolic subgroup P such that in a neighbourhood O of $0 \in \mathfrak{g}$,

$$\widehat{\mu_{\mathcal{O}}}|_O = \Theta_\pi \circ \exp|_O,$$

where $\pi = \mathrm{Ind}_P^G \mathbf{1}$ [Howe 1974, Lemma 5]. In the same vein, for $\mathrm{SL}(2, F)$, the functions $\widehat{\mu_{\mathcal{O}}}$ are almost equal to the characters of special unipotent representations (see (8-1)). In recent work, Henniart and Vignéras [2023; 2024] have proven that these in turn correspond to representation-theoretic expansions in a small enough neighbourhood of the identity, for all inner forms of $\mathrm{GL}_n(F)$ as well as for the group $\mathrm{SL}_2(F)$. We do not, however, expect such representations to exist in general as, for example, for classical groups nonspecial orbits cannot occur in $\mathcal{WF}(\pi)$ for any π [Mœglin 1996, Theorem 1.4]), yet can occur with nonzero coefficients in a local character expansion. The main goal in this paper is to propose an example of a weaker form of the Howe–Henniart–Vignéras theorem, based on representations of a maximal compact open subgroup, that one may hope can hold true in general.

3. Nilpotent orbits and nilpotent support

In this section, G is an arbitrary connected reductive group, subject to the hypothesis on p of Section 2B. We define the (local) nilpotent support of an element of $\mathfrak{g}^*$, and relate this both to the asymptotic cone and to the wave front set of a representation of positive depth.

3A. ***Degenerate cosets and nilpotent orbits.*** In [Adler and DeBacker 2002, §3], the authors generalize ideas of Moy and Prasad to establish for connected reductive groups that for all $r \in \mathbb{R}$,

$$\mathfrak{g}_r^* = \bigcap_{x \in \mathcal{B}(G)} (\mathfrak{g}_{x,r}^* + \mathcal{N}^*),$$

where $\mathfrak{g}_r^* := \bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r}^*$. They further show that

$$\mathcal{N}^* = \bigcap_{r \in \mathbb{R}} \mathfrak{g}_r^*.$$

Given $x \in \mathcal{B}(G)$ and $X \in \mathfrak{g}^* \smallsetminus \{0\}$, the *depth of X at x* is the unique value $t = d_x(X)$

such that $X \in \mathfrak{g}^*_{x,t} \smallsetminus \mathfrak{g}^*_{x,t+}$. When X is not nilpotent, they prove that the *depth of* X, given by

$$d(X) = \max\{d_x(X) \mid x \in \mathcal{B}(G)\} = \max\{r \mid X \in \mathfrak{g}^*_r\}$$

is well defined and rational. For X nilpotent, we set $d(X) = \infty$. Depth is G-invariant.

For semisimple $\Gamma \in \mathfrak{g}^*$, let $T \subset \mathrm{Cent}_G(\Gamma)$ be a maximal torus with associated absolute root system $\Phi(G, T)$. Then Γ is called *good* if for all $\alpha \in \Phi(G, T)$, we have $\mathrm{val}\big(\Gamma(d\alpha^\vee(1))\big) \in \{d(\Gamma), \infty\}$. By [Kim and Murnaghan 2003, Theorem 2.3.1], if Γ is good then the set of points $x \in \mathcal{B}(G)$ at which $d_x(\Gamma)$ attains its maximum value $d(\Gamma)$ is exactly $\mathcal{B}(\mathrm{Cent}_G(\Gamma)) \subset \mathcal{B}(G)$.

For any $\Gamma \in \mathfrak{g}^*$ set $d = d_x(\Gamma)$. The coset $\Gamma + \mathfrak{g}^*_{x,d+}$ is called *degenerate* if it contains a nilpotent element $X \in \mathcal{N}^*$. From the relations above it follows that this happens if and only if $d < d(\Gamma)$. DeBacker [2002b, §5] proved that the set of nilpotent G-orbits meeting a degenerate coset $\Gamma + \mathfrak{g}^*_{x,d+}$ has a unique minimal element with respect to the (rational) closure relation on orbits, which we'll denote by $\mathcal{O}(\Gamma, x)$. This generalizes a result of Barbasch and Moy [1997, Proposition 3.1.6] for $d = 0$, which was integral to their determination of the wave front set of a depth-zero representation.

To classify nilpotent orbits in this way, DeBacker proceeds as follows. Identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Given a nilpotent element $X \in \mathfrak{g}$, complete X to an $\mathfrak{sl}_2(F)$ triple (X, H, Y). Choose $r \in \mathbb{R}$ and create the building set

$$\mathcal{B}_r(X, H, Y) = \{x \in \mathcal{B}(G) \mid X \in \mathfrak{g}_{x,r},\ H \in \mathfrak{g}_{x,0},\ Y \in \mathfrak{g}_{x,-r}\}.$$

He proves this set is a nonempty, closed, convex subset of $\mathcal{B}(G)$ with the property that for all $x \in \mathcal{B}_r(X, H, Y)$ we have $\mathcal{O}(X, x) = G \cdot X$.

Remark 3.1. For each $g \in G$, we have $\mathcal{B}_r({}^gX, {}^gH, {}^gY) = {}^g\mathcal{B}_r(X, H, Y)$, and for fixed X the union of these need not cover $\mathcal{B}(G)$. Moreover, if μ is a one-parameter subgroup adapted to this triple, then by [DeBacker 2002b, Remark 5.1.5], $\mathcal{B}_r(X, H, Y) = \mathcal{B}_0(X, H, Y) + \frac{1}{2}r\mu$, where this sum is taken in any apartment in $\mathcal{B}(C_G(\mu))$. It follows that (if the rank of G is greater than 1) there exist orbits $\mathcal{O}$ (such as ones for which $\mathcal{B}_r(X, H, Y)$ is a point) for which there exist $y \in \mathcal{B}(G)$ such that $\mathcal{O} \neq \mathcal{O}(X, y)$ for any $X \in \mathcal{O}$. For example, in $\mathrm{Sp}(4, F)$, the principal nilpotent orbits are only obtained along certain lines emanating from vertices.

3B. ***Nilpotent support and nilpotent cones.*** We now explore different ways to understand the asymptotic nilpotent support of a general element $\Gamma \in \mathfrak{g}^*$ and show their equivalence.

Definition 3.2. Let $\Gamma \in \mathfrak{g}^*$. If $x \in \mathcal{B}(G)$, then the *local nilpotent support at* x of Γ is

$$\mathrm{Nil}_x(\Gamma) = \{\mathcal{O}({}^g\Gamma, x) \mid g \in G, d_x(g \cdot \Gamma) < d(\Gamma)\},$$

which is the set of nilpotent orbits defined by degenerate cosets at x of elements of the G-orbit of Γ. On the other hand, the *nilpotent support* of Γ is

$$\mathrm{Nil}(\Gamma) = \{\mathcal{O}(\Gamma, x) \mid x \in \mathcal{B}(G), d_x(\Gamma) < d(\Gamma)\},$$

the set of nilpotent orbits corresponding to any (nontrivial) degenerate coset of Γ.

Note that if Γ is nilpotent, then $\mathrm{Nil}(\Gamma) \ni G \cdot \Gamma$. More generally, for any $g \in G$, $d_x(\Gamma) = d_{gx}({}^g\Gamma)$ and ${}^g(\Gamma + \mathfrak{g}^*_{x,d+}) = {}^g\Gamma + \mathfrak{g}^*_{gx,d+}$. Thus $\mathcal{O}(\Gamma, x) = \mathcal{O}({}^g\Gamma, gx)$, and

$$\mathrm{Nil}(\Gamma) = \bigcup_{x \in \mathcal{B}(G)} \mathrm{Nil}_x(\Gamma),$$

that is, the nilpotent support is the union of the local nilpotent supports, and $\mathrm{Nil}(\Gamma)$ is an invariant of the G-orbit of Γ. One may alternately restrict this union to one over the points in a fundamental domain for the action of G on $\mathcal{B}(G)$.

By Remark 3.1, when the rank of G is greater than 1, not all nilpotent orbits will occur as some $\mathcal{O}(\Gamma, x)$ for a given point $x \in \mathcal{B}(G)$, so $\mathrm{Nil}_x(\Gamma) \neq \mathrm{Nil}(\Gamma)$ in general. Even when these sets are equal, as for $\mathrm{SL}(2, F)$ (see Proposition 4.1), they are interesting subsets of the nilpotent cone (see Lemma 4.2).

On the other hand, the asymptotic cone on an element Γ is defined in [Adams and Vogan 2021, Definition 3.9] analytically as follows.

Definition 3.3. Let $\Gamma \in \mathfrak{g}^*$. The *asymptotic cone* on Γ is the set

$$\mathrm{Cone}(\Gamma) = \left\{X \in \mathfrak{g}^* \mid \exists \varepsilon_i \to 0, \varepsilon_i \in F^\times, \exists g_i \in G, \lim_{i \to \infty} \varepsilon_i^2 \,\mathrm{Ad}^*(g_i)\Gamma = X\right\}.$$

This is a closed, nonempty union of nilpotent orbits of G on $\mathfrak{g}^*$.

Proposition 3.4. *Let $\Gamma \in \mathfrak{g}^*$. Then the nonzero G-orbits occurring in the asymptotic cone of Γ are those in its nilpotent support, that is,*

$$\mathrm{Cone}(\Gamma) = \bigcup_{\mathcal{O} \in \mathrm{Nil}(\Gamma)} \mathcal{O} \cup \{0\}.$$

Proof. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and prove this result for $\Gamma \in \mathfrak{g}$, where we may apply the theory of $\mathfrak{sl}_2(F)$ triples.

Let $\Gamma \in \mathfrak{g}$ have depth $r \leq \infty$ and let $\mathcal{O} \in \mathrm{Nil}(\Gamma)$. Then there exists $x \in \mathcal{B}(G)$ and $d < r$ such that $d_x(\Gamma) = d$ and $\mathcal{O} = \mathcal{O}(\Gamma, x)$. Choose a representative

$$X \in \mathcal{O}(\Gamma, x) \cap (\Gamma + \mathfrak{g}_{x,d+}).$$

Choose an $\mathfrak{sl}_2(F)$ triple (X, H, Y) and the corresponding one-parameter subgroup μ adapted to X. By [DeBacker 2002b, Lemma 5.2.1], we have

$$X + \mathfrak{g}_{x,d+} = \mathrm{Ad}(G_{x,0+})(X + C_{\mathfrak{g}_{x,d+}}(Y)).$$

Therefore there exist $g \in G_{x,0+}$ and $C \in C_{\mathfrak{g}_{x,d+}}(Y)$ for which

$$\Gamma = \mathrm{Ad}(g^{-1})(X + C).$$

Note that $C_{\mathfrak{g}}(Y)$ is spanned by the lowest weight vectors of $\mathrm{ad}(H)$, so we may decompose C as $C = \sum_{i \le 0} C_i$, where $\mathrm{Ad}(\mu(t))C_i = t^i C_i$ for all $t \in F^\times$. Similarly, for all $t \in F^\times$ we have $\mathrm{Ad}(\mu(t))X = t^2 X$. Therefore

$$\lim_{t \to 0} t^2 \,\mathrm{Ad}(\mu(t^{-1})g)\Gamma = \lim_{t \to 0} t^2 \,\mathrm{Ad}(\mu(t^{-1}))(X + C) = X,$$

so $X \in \mathrm{Cone}(\Gamma)$. Since $\mathrm{Cone}(\Gamma)$ is G-invariant, we deduce $\mathcal{O} \subset \mathrm{Cone}(\Gamma)$.

Conversely, let $X \in \mathrm{Cone}(\Gamma)$ be nonzero, so that there exists a sequence of elements $\varepsilon_i \in F^\times$, with $\varepsilon_i \to 0$, and a sequence of elements $g_i \in G$, such that

$$\lim_{i \to \infty} \varepsilon_i^2 \,\mathrm{Ad}(g_i)\Gamma = X.$$

Complete X to an $\mathfrak{sl}_2(F)$ triple (X, H, Y) and choose a point $x \in \mathcal{B}_0(X, H, Y)$. Since the given sequence converges to X, it enters the neighbourhood $X + \mathfrak{g}_{x,0+}$ so we may choose $i \in \mathbb{N}$ such that

$$\varepsilon_i^2 \,\mathrm{Ad}(g_i)\Gamma \in X + \mathfrak{g}_{x,0+}.$$

So $\mathrm{Ad}(g_i)\Gamma \in \varepsilon_i^{-2} X + \mathfrak{g}_{x,-2\,\mathrm{val}(\varepsilon_i)+}$, a nontrivial degenerate coset of depth $-2\,\mathrm{val}(\varepsilon_i)$. Because $(\varepsilon_i^{-2} X, H, \varepsilon_i^2 Y)$ is again an $\mathfrak{sl}_2(F)$ triple and $\mathcal{B}_{-2\,\mathrm{val}(\varepsilon_i)}(\varepsilon_i^{-2} X, H, \varepsilon_i^2 Y) = \mathcal{B}_0(X, H, Y)$, we can infer that the minimal nilpotent orbit meeting this coset is $\mathrm{Ad}(G)(\varepsilon_i^{-2} X) = \mathrm{Ad}(G)X$. Thus $\mathrm{Ad}(G)X = \mathcal{O}({}^{g_i}\Gamma, x) \in \mathrm{Nil}_x(\Gamma) \subset \mathrm{Nil}(\Gamma)$, as required. □

3C. ***Connection with the wave front set of a positive-depth representation.*** Suppose now that π is an irreducible admissible representation of G of depth r with good minimal K-type Γ of depth $-r$ (in the sense of [Kim and Murnaghan 2003, Definitions 2.4.3 and 2.4.6]). Then, under suitable hypotheses (that are satisfied if F has characteristic zero and the exponential map converges on $\mathfrak{g}_{0+}$), Kim and Murnaghan prove a version of the local character expansion that is valid on the strictly larger neighbourhood $\mathfrak{g}_r^{\mathrm{reg}}$. The Γ-asymptotic expansion [Kim and Murnaghan 2003, Theorem 5.3] asserts that there exist complex coefficients $c_{\mathcal{O}'}(\pi)$ such that for any $X \in \mathfrak{g}_r^{\mathrm{reg}}$ we have

$$\Theta_\pi(\exp(X)) = \sum_{\mathcal{O}' \in \mathscr{O}(\Gamma)} c_{\mathcal{O}'}(\pi)\widehat{\mu_{\mathcal{O}'}}(X), \tag{3-1}$$

where $\mathscr{O}(\Gamma)$ denotes the set of G-orbits in $\mathfrak{g}^*$ with Γ in their closure, and for $\mathcal{O}' \in \mathscr{O}(\Gamma)$, $\widehat{\mu_{\mathcal{O}'}}$ denotes the Fourier transform of the corresponding orbital integral (2-2).

This yields a special case of interest: that of the expansion (3-1) having a single nonzero term $c_{\mathcal{O}'}(\pi)\widehat{\mu_{\mathcal{O}'}}$ corresponding to $\mathcal{O}' = G \cdot \Gamma$. We claim this happens, for

example, when $G' = \mathrm{Cent}_G(\Gamma)$ is compact-mod-centre, such as when Γ is a regular semisimple element. Namely, let $\mathfrak{g}'$ denote the Lie algebra of G'. Then the set $\mathscr{O}(\Gamma)$ indexing the sum in (3-1) is in bijective correspondence with the set of nilpotent G'-orbits in $(\mathfrak{g}')^*$, which is the singleton $\{G \cdot \Gamma\}$ under this hypothesis.

Theorem 3.5. *Let π be an irreducible admissible representation of G of depth $r > 0$, and let $\Gamma \in \mathfrak{g}^*$ be a good minimal K-type of π such that π admits a Γ-asymptotic expansion. Suppose further that this expansion has a unique nonzero term, corresponding to the Fourier transform of the orbital integral corresponding to Γ itself. Then $\mathcal{WF}(\pi)$ coincides with the maximal elements of* Nil(Γ); *that is, the asymptotic cone on Γ is the wave front set of π.*

The following proof was communicated to me by Fiona Murnaghan.

Proof. Combining the Γ-asymptotic and local character expansions yields that for some scalar t we have the equality $t\widehat{\mu_{G\cdot\Gamma}} = \sum_{\mathcal{O}\in\mathscr{O}(0)} c_{\mathcal{O}}(\pi)\widehat{\mu_{\mathcal{O}}}$ of functions on $\mathfrak{g}^{\mathrm{reg}} \cap \mathfrak{g}_{r+}$, which can be viewed equivalently as an equality of distributions upon restriction to the set $C_c^\infty(\mathfrak{g}_{r+})$. Since $f \in C_c^\infty(\mathfrak{g}_{r+})$ implies $\widehat{f} \in \mathcal{D}_{-r} := \sum_{x\in\mathcal{B}(G)} C_c(\mathfrak{g}^*/\mathfrak{g}^*_{x,r})$, taking the inverse Fourier transform yields the equality of distributions

$$t\mu_{G\cdot\Gamma} = \sum_{\mathcal{O}\in\mathscr{O}(0)} c_{\mathcal{O}}(\pi)\mu_{\mathcal{O}} \tag{3-2}$$

on $\mathcal{D}_{-r}$. (See [Kim and Murnaghan 2003, Proof of Theorem 5.3.1] or [Debacker 2002a, Proof of Theorem 3.5.2]; note that our r is ρ in the former and $\rho(\pi)$ in the latter.)

So let $x \in \mathcal{B}(G)$ and let d be such that $\mathfrak{g}^*_{x,d+} \supset \mathfrak{g}^*_{x,-r}$. Given a nonzero coset $\xi \in \mathfrak{g}^*_{x,d}/\mathfrak{g}^*_{x,d+}$ let $\mathbf{1}_\xi$ denote the characteristic function of this subset of $\mathfrak{g}^*$; then $\mathbf{1}_\xi \in \mathcal{D}_{-r}$. Note that if $X \in \xi \cap \mathcal{O}$ for some (not necessarily nilpotent) G-orbit $\mathcal{O}$, then this intersection contains the open set $G_{x,0+} \cdot X$ as well. Thus we have

$$\mu_{\mathcal{O}}(\mathbf{1}_\xi) = 0 \iff \xi \cap \mathcal{O} = \varnothing. \tag{3-3}$$

Now suppose that $\mathcal{O} \in \mathscr{O}(0)$, and choose $x \in \mathcal{B}(G)$ and $\xi = X + \mathfrak{g}^*_{x,d+}$ with $\mathfrak{g}^*_{x,d+} \supset \mathfrak{g}^*_{x,-r}$ with the property that $\mathcal{O} = \mathcal{O}(X, x)$. The minimality of $\mathcal{O}(X, x)$ proven by DeBacker implies that any nilpotent orbit $\mathcal{O}'$ meeting ξ (or equivalently, by (3-3), satisfying $\mu_{\mathcal{O}'}(\mathbf{1}_\xi) \neq 0$) must contain $\mathcal{O}$ in its closure.

Suppose first that $\mathcal{O}$ is not in the wave front set $\bigcup_{\mathcal{O}'\in\mathcal{WF}(\pi)} \overline{\mathcal{O}'}$ of π. Let $\mathcal{O}' \in \mathscr{O}(0)$ be such that $c_{\mathcal{O}'}(\pi) \neq 0$; then $\mathcal{O}'$ is in the wave front set, so $\mathcal{O} \not\subset \overline{\mathcal{O}'}$. This implies by the preceding paragraph that $\mu_{\mathcal{O}'}(\mathbf{1}_\xi) = 0$. As this holds for all such $\mathcal{O}'$, we conclude from (3-2) that $\mu_{G\cdot\Gamma}(\mathbf{1}_\xi) = 0$, whence by (3-3) we have $\xi \cap G \cdot \Gamma = \varnothing$, and thus $\mathcal{O} \notin$ Nil(Γ). Therefore every $\mathcal{O} \in$ Nil(Γ) lies in the wave front set of π.

Now suppose $\mathcal{O} \in \mathcal{WF}(\pi)$; that is, it is maximal among nilpotent orbits with nonzero coefficient in (3-2). Thus the preceding argument implies $\mu_{\mathcal{O}'}(\mathbf{1}_\xi) = 0$ for all $\mathcal{O}' \neq \mathcal{O}$ in the wave front set. So (3-2) yields $t\mu_{G\cdot\Gamma}(\mathbf{1}_\xi) = c_{\mathcal{O}}(\pi)\mu_{\mathcal{O}}(\mathbf{1}_\xi) \neq 0$, and therefore, by (3-3), ξ must meet $G\cdot\Gamma$ and thus $\mathcal{O} \in \mathrm{Nil}(\Gamma)$. Hence, the maximal elements of $\mathrm{Nil}(\Gamma)$ coincide with $\mathcal{WF}(\pi)$. □

In fact, the key to the proof is that the maximal nilpotent orbits occurring in the Shalika germ expansion of $\mu_{G\cdot\Gamma}$ are the maximal orbits of $\mathrm{Nil}(\Gamma)$.

Ciubotaru and Okada [2023] obtained a similar result directly, by analysing the asymptotic nilpotent cone of the characters of $G_{x,r}/G_{x,r+}$ appearing in $\pi^{G_{x,r+}}$.

Remark 3.6. One might ask if Theorem 3.5 could be extended to show that $\mathcal{WF}(\pi)$ is the union of the nilpotent supports of the maximal orbits occurring in the Γ-asymptotic expansion (3-1). The answer is expected to be negative. In the supercuspidal case, the key result is [Spice 2022, Corollary 10.2.3(1)], which implies that this latter set of orbits (in $\mathcal{O}(\Gamma)$) corresponds exactly to $\mathcal{WF}(\pi^0)$ (in $\mathcal{O}(0)$ for $G^0 = \mathrm{Cent}_G(\Gamma)$), where π^0 is the associated depth-zero supercuspidal representation of G^0. Tsai [2023b] has constructed explicit examples of supercuspidal representations where the wave front set does not follow such a pleasant inductive structure. In effect, one expects that when substituting Shalika germ expansions into the Γ-asymptotic expansion, cancellations among coefficients may occur.

While the proof of Theorem 3.5 entails some additional hypotheses on F, a consequence of the main theorem of Section 6 is that, for $G = \mathrm{SL}(2, F)$, the conclusion of the theorem holds whenever the characteristic and residual characteristic of F are not 2.

4. Nilpotent orbits and nilpotent cones of $G = \mathbf{SL}(2, F)$

For the rest of this paper we suppose that $\boldsymbol{G} = \mathrm{SL}(2)$ and $\mathfrak{g} = \mathfrak{sl}(2, F)$. In this section, we derive some additional properties of the nilpotent support of an element $\Gamma \in \mathfrak{g}^*$. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ with the trace form.

There are five nilpotent orbits: the zero orbit, and four two-dimensional principal (or regular) orbits that are in bijection with the rational square classes $F^\times/(F^\times)^2$. Representatives of these five orbits in $\mathfrak{g}$ are

$$\dot{X}_u = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}, \tag{4-1}$$

where u runs over the set $\{0, 1, \varepsilon, \varpi, \varepsilon\varpi\}$ modulo $(F^\times)^2$ and $\varepsilon \in \mathcal{R}^\times$ is a fixed nonsquare. For each u, write $\mathcal{O}_u$ for the orbit in $\mathfrak{g}^*$ corresponding to $\dot{X}_u$. The following proposition relaxes the conditions for identifying the orbits in the nilpotent support of an element Γ.

Proposition 4.1. *Let $\mathfrak{g} = \mathfrak{sl}_2(F)$ and $\Gamma \in \mathfrak{g}^* \smallsetminus \{0\}$. Set $r = d(\Gamma) \in \mathbb{R} \cup \{\infty\}$. Then*

(a) *every* $\mathcal{O} \in \mathrm{Nil}(\Gamma)$ *meets* $\Gamma + \mathfrak{g}^*_{x,r}$ *for some* $x \in \mathcal{B}(G)$ *such that* $d_x(\Gamma) < r$;

(b) *for each* $x \in \mathcal{B}(G)$ *such that* $d_x(\Gamma) < r$, *if* $\Gamma + \mathfrak{g}^*_{x,r}$ *meets a nilpotent orbit* $\mathcal{O}$, *then* $\mathcal{O} \in \mathrm{Nil}(\Gamma)$;

(c) *for each* $x \in \mathcal{B}(G)$, $\mathrm{Nil}(\Gamma) = \{\mathcal{O}({}^g\Gamma, x) \mid g \in G\} = \mathrm{Nil}_x(\Gamma)$, *that is*, *every nonzero nilpotent orbit in* $\mathrm{Cone}(\Gamma)$ *appears in the local nilpotent support at every* x.

Proof. The first two statements use that there are no closure relations between the principal orbits of $\mathfrak{sl}_2(F)$, and so the uniqueness of the minimal nilpotent orbit meeting any degenerate coset implies that any nontrivial degenerate coset meets only one nilpotent orbit.

For (a), suppose $\mathcal{O} \in \mathrm{Nil}(\Gamma)$; then $\mathcal{O} = \mathcal{O}(\Gamma, x)$ for some $x \in \mathcal{B}(G)$, implying that $d_x(\Gamma) < r$. Since $\Gamma \in \mathfrak{g}^*_r \subset \mathfrak{g}^*_{x,r} + \mathcal{N}^*$, the set $\Gamma + \mathfrak{g}^*_{x,r}$ contains a (nonzero) nilpotent element Y. Since Y is an element of $\Gamma + \mathfrak{g}^*_{x,r} \subset \Gamma + \mathfrak{g}^*_{x,d_x(\Gamma)}$, it lies in $\mathcal{O}$, so $\mathcal{O}$ meets the smaller coset, as required.

For (b), note that if $d_x(\Gamma) < r$ then $0 \notin \Gamma + \mathfrak{g}^*_{x,r} \subset \Gamma + \mathfrak{g}^*_{x,d(\Gamma)+}$; any nilpotent orbit meeting the smaller set meets the larger one, and thus by uniqueness this orbit is $\mathcal{O}(\Gamma, x) \in \mathrm{Nil}(\Gamma)$.

To prove (c), let $x \in \mathcal{B}(G)$ and let $\mathrm{Nil}_x(\Gamma)$ be the local nilpotent support of Γ at x; we have already noted that $\mathrm{Nil}_x(\Gamma) \subset \mathrm{Nil}(\Gamma)$. The reverse inclusion follows from the one-dimensionality of $\mathcal{B}(G)$. Let $\mathcal{O} \in \mathrm{Nil}(\Gamma)$; then $\mathcal{O} = \mathcal{O}(\Gamma, y)$ for some $y \in \mathcal{B}(G)$. Let S be a split torus with associated root system $\Phi(G, S) = \{\pm\alpha\}$ such that $y \in \mathcal{A}(G, S)$.

Set $d = d_y(\Gamma)$ and let $\dot{\Gamma} \in \mathfrak{g}$ correspond to Γ via the trace form. Choose $\dot{X} \in \mathcal{O}$ such that $\dot{\Gamma} \in \dot{X} + \mathfrak{g}_{y,d+}$. Conjugating both $\dot{\Gamma}$ and $\dot{X}$ by G_y as necessary we may assume $\dot{X} \in \mathfrak{g}_\alpha$. Relative to the pinning of a fixed base point, we have the decomposition of $\mathcal{R}$-modules

$$\mathfrak{g}_{y,d} = \mathfrak{g}_{-\alpha,d+\alpha(y)} \oplus \mathfrak{s}_d \oplus \mathfrak{g}_{\alpha,d-\alpha(y)}.$$

Let $\alpha^\vee$ denote the positive coroot, and choose $g \in G$ so that $gx \in \mathcal{A}(G, S)$ and $gx = y - \ell\alpha^\vee$ for some $\ell \geq 0$. Therefore if $d' = d - 2\ell$ then $\mathfrak{g}_{y,d} \subset \mathfrak{g}_{gx,d'}$. Because $\dot{X} \in \mathfrak{g}_{\alpha,d-\alpha(y)} \smallsetminus \mathfrak{g}_{\alpha,(d-\alpha(y))+}$ and $d' - \alpha(gx) = d - \alpha(y)$, we conclude that $d_{gx}(\dot{\Gamma}) = d_{gx}(\dot{X}) = d'$ and $\dot{\Gamma} - \dot{X} \in \mathfrak{g}_{gx,d'+}$. By uniqueness, we infer that $\mathcal{O} = \mathcal{O}(\dot{\Gamma}, gx) = \mathcal{O}({}^{g^{-1}}\dot{\Gamma}, x) \in \mathrm{Nil}_x(\dot{\Gamma})$, yielding the result. □

We next determine $\mathrm{Nil}(\Gamma)$ explicitly, for any $\Gamma \in \mathfrak{g} = \mathfrak{sl}_2(F)$ (identified with its dual via the trace form). There is nothing to do if Γ is nilpotent. If $\Gamma \neq 0$ is semisimple, then it is G-conjugate to a matrix of the form

$$\dot{X}(u, v) = \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix}, \tag{4-2}$$

for some $u, v \in F^\times$. Its centralizer is a maximal torus. There is one G-conjugacy class of split torus, represented by any diagonal element, and two classes of unramified anisotropic tori, represented by $\dot{X}(1, \varepsilon) \in \mathfrak{g}$ and $\dot{X}(\varpi^{-1}, \varepsilon\varpi) \in \mathfrak{g}$, respectively. The classes of ramified tori are represented by $\dot{X}(1, t) \in \mathfrak{g}$ with $t \in \{\varpi, \varepsilon\varpi, \varepsilon^2\varpi, \varepsilon^3\varpi\}$, noting that if $-\varepsilon \in (F^\times)^2$ then there are only two classes.

We can now describe the nilpotent support of each such element, using the parametrization given in (4-1).

Lemma 4.2. *Let $G = \mathrm{SL}(2, F)$ and $\Gamma \in \mathfrak{g} \smallsetminus \{0\}$ semisimple. If Γ splits over F,*

$$\mathrm{Nil}(\Gamma) = \{\mathcal{O}_1, \mathcal{O}_\varepsilon, \mathcal{O}_\varpi, \mathcal{O}_{\varepsilon\varpi}\}.$$

Otherwise, Γ is conjugate to $\dot{X}(u, v)$ for some $u, v \in F^\times$ and splits over $E = F[\sqrt{uv}]$. Let $\mathrm{Norm}_{E/F}(E^\times)/(F^\times)^2$ be represented by $\{1, \gamma\}$. Then u and v are uniquely defined mod $\mathrm{Norm}_{E/F}(E^\times)$ and

$$\mathrm{Nil}(\Gamma) = \{\mathcal{O}_u, \mathcal{O}_{u\gamma}\}.$$

Proof. By Proposition 4.1, we may fix the choice $x = x_0 \in \mathcal{B}(G)$ to be the vertex such that $\mathfrak{g}_{x,r}$ is the set of traceless 2×2 matrices with entries in $\mathcal{P}^{\lceil r \rceil}$, and replace Γ by any G-conjugate.

First suppose $\Gamma = \mathrm{diag}(a, -a)$ with $\mathrm{val}(a) = r$. Let $u \in F^\times$ and note that if $g_u = \left[\begin{smallmatrix} 1 & -a^{-1}u/2 \\ 0 & 1 \end{smallmatrix}\right] \in G$ then ${}^{g_u}\Gamma = \left[\begin{smallmatrix} a & u \\ 0 & -a \end{smallmatrix}\right]$. Therefore, for any u such that $\mathrm{val}(u) = d < r$, we have ${}^{g_u}\Gamma \in \dot{X}_u + \mathfrak{g}_{x,d+}$. Thus $\mathrm{Nil}(\Gamma)$ contains every nonzero nilpotent orbit.

Now suppose $\Gamma = \dot{X}(u, v)$ for some $u, v \in F^\times$ such that $uv \notin (F^\times)^2$ and set $E = F[\sqrt{uv}]$. We calculate directly that the upper triangular entry of any G-conjugate of Γ takes the form

$$u' = a^2u - b^2v = u(a^2 - b^2vu^{-1}) \in u\,\mathrm{Norm}_{E/F}(E^\times)$$

for some $a, b \in F$, not both zero, from which it follows that $\mathrm{Nil}(\Gamma) \subset \{\mathcal{O}_u, \mathcal{O}_{u\gamma}\}$.

For the reverse inclusion, first note $\dot{X}(u, v)$ is G-conjugate to $\dot{X}(u\varpi^{-2n}, v\varpi^{2n})$ for all $n \in \mathbb{Z}$ and for n sufficiently large $\dot{X}(u\varpi^{-2n}, v\varpi^{2n}) - \dot{X}_{u\varpi^{-2n}} \in \mathfrak{g}_{x,r}$. Thus $\mathcal{O}_u \in \mathrm{Nil}(\Gamma)$.

Now note that when E is ramified, we may take $\gamma = -uv$ so $\mathcal{O}_{u\gamma} = \mathcal{O}_{-v}$; since $\dot{X}(u, v)$ is G-conjugate to $\dot{X}(-v, -u)$ we are done by the preceding. If E is unramified, we have instead $\gamma = uv$, whence $\mathcal{O}_{u\gamma} = \mathcal{O}_v$. As -1 is a norm, we may choose $\alpha, \beta \in F$ such that $-1 = \beta^2 - \alpha^2uv^{-1}$; then $g = \left[\begin{smallmatrix} \alpha & \beta \\ \beta & \alpha uv^{-1} \end{smallmatrix}\right] \in G$ satisfies ${}^g\dot{X}(u, v) = \dot{X}(v, u)$, and again by the preceding we may conclude $\mathcal{O}_v \in \mathrm{Nil}(\Gamma)$. □

5. Representations of G_x associated to nilpotent orbits

5A. ***Shalika's representations of* $\mathrm{SL}(2, \mathcal{R})$.** In his thesis, Shalika constructed all irreducible smooth representations of $K = \mathrm{SL}(2, \mathcal{R})$. In this section we recap his explicit construction for the so-called ramified case, which attaches an irreducible

representation of K to certain K-orbits in $\mathfrak{g}^*$; we'll then provide a coordinate-free generalization more suited to our needs in the next section.

Let S be the diagonal split torus, B the upper triangular Borel subgroup and U its unipotent radical. We use a subscript 0 to indicate their intersections with K: $S_0 = S \cap K$, $B_0 = B \cap K$ and $U_0 = U \cap K$. Let $x_0 \in \mathcal{A}(G, S)$ be such that $K = G_{x_0}$ and z_0 the barycentre of the positive alcove adjacent to x_0 (relative to B).

Let d be a positive integer. Choose $u \in \mathcal{P}^{-d} \smallsetminus \mathcal{P}^{-d+1}$ and $v \in \mathcal{P}^{-d+1}$ and consider the antidiagonal matrix $\dot{X} := \dot{X}(u, v) \in \mathfrak{g}_{x_0,-d}$ of (4-2). Identify this with the element $X \in \mathfrak{g}^*_{x_0,-d}$ by the rule $X(Z) = \mathrm{tr}(\dot{X}Z)$ for all $Z \in \mathfrak{g}$. If $v = 0$ then X is nilpotent and its centralizer $C_K(X)$ in K coincides with ZU_0, where $Z = \{\pm I\}$. Otherwise, X is semisimple and $C_K(X)$ is a torus. Note that every $X \in \mathfrak{g}^*_{x_0,-d}$ that represents a degenerate coset is K-conjugate to one of this form.

Define an open subgroup of K by

$$J_d = \begin{bmatrix} 1+\mathcal{P}^{\lceil d/2 \rceil} & \mathcal{P}^{\lceil d/2 \rceil} \\ \mathcal{P}^{\lceil (d+1)/2 \rceil} & 1+\mathcal{P}^{\lceil d/2 \rceil} \end{bmatrix} \cap K. \tag{5-1}$$

It is straightforward to verify that X gives a well-defined character η_X of J_d, trivial on $G_{x_0,d+}$, by the rule

$$\eta_X(g) = \psi\big(\mathrm{tr}(\dot{X}(g - I))\big). \tag{5-2}$$

This character depends only on the classes $u + \mathcal{P}^{\lceil (-d+1)/2 \rceil}$ and $v + \mathcal{P}^{\lceil -d/2 \rceil}$. For any choice of character θ of $C_K(X)$ agreeing with $\eta_X(g)$ on $C_K(X) \cap J_d$, write $\eta(X, \theta)$ for the resulting extension to a character of $C_K(X)J_d$.

Shalika [2004, Theorems 4.2.1 and 4.2.5, §4.3] proves the following result with an intricate elementary argument. To briefly translate from the notation of that work: K_n denotes $\mathrm{SL}(\mathcal{R}/\mathcal{P}^n) \cong G_{x_0}/G_{x_0,n}$ and its primitive representations inflate to the representations of G_{x_0} of depth $d = n - 1$. Shalika's group $T_{\overline{X},n}$ inflates to $C_K(X)G_{x_0,n}$ in our notation and our J_d is the inflation of Shalika's N_k when n is even and B_n when n is odd. Though Shalika's X has depth 0, his additive characters η, ξ are normalized such that the characters denoted by $\eta_{\mathbf{x}}$ and $\eta_{X,\tilde{\xi}}$ match K-equivariantly with our η_X, for those "ramified" orbits of [Shalika 2004, Lemma 4.2.2(ii)] considered here. Nilpotent X fall under this ramified case, by choosing $\nu > n$ in [Shalika 2004, §4.3].

Proposition 5.1 (Shalika). *Set $K = G_{x_0}$ and $K_n = G_{x_0,n}$ for $n > 0$. For any $d > 0$, let $\dot{X} = \dot{X}(u, v)$ as above, corresponding to $X \in \mathfrak{g}^*_{x_0,-d}$. Then for any character θ of its centralizer $C_K(X)$ agreeing with η_X on $C_K(X) \cap J_d$, the representation*

$$\mathcal{S}_{x_0}(X, \theta) = \mathrm{Ind}_{C_K(X)J_d}^K \eta(X, \theta)$$

is irreducible, of degree $\frac{1}{2}q^{d-1}(q^2 - 1)$ and of depth d, meaning it is nontrivial on K_d but trivial on K_{d+}. Its equivalence class is independent of the choice of

representative of the K-orbit of $\dot{X}(u+\mathcal{P}^{\lfloor (d+1)/2\rfloor}, v+\mathcal{P}^{\lceil (d+1)/2\rceil})$, and if $\mathcal{S}_{x_0}(X,\theta)\cong \mathcal{S}_{x_0}(X',\theta')$ then there is some $g\in K$ such that $X'={}^gX$ and $\theta'={}^g\theta$.

5B. ***Irreducible representations of G_x parametrized by degenerate cosets at x.*** Our goal in this section is to give a coordinate-free interpretation of Shalika's construction that allows us to unambiguously attach representations of G_x to any degenerate coset of negative depth.

Note that $\mathrm{GL}(2,F)$ acts on $\mathcal{B}(G)$, and all vertices are conjugate under this action. This conjugacy does not in general preserve the $\mathrm{SL}(2,F)$-orbit of Γ or X.

Example. Let x_0, z_0 be as in Section 5A and x_1 the other vertex of the chamber containing z_0 in its closure. The element $\omega=\left[\begin{smallmatrix}0&1\\ \varpi&0\end{smallmatrix}\right]$ used in [Nevins 2005] is an affine reflection such that $\omega\cdot x_0=x_1$, and ${}^\omega\dot{X}(u,v)=\dot{X}(\varpi^{-1}v,\varpi u)$. Therefore, in particular, in the case of nilpotent orbits, where $\dot{X}(0,1)\sim\dot{X}(-1,0)$, we have ${}^\omega\mathcal{O}_1=\mathcal{O}_{-\varpi}$. On the other hand, the element $\eta=\left[\begin{smallmatrix}1&0\\ 0&\varpi\end{smallmatrix}\right]$ used in [Nevins 2013] is a translation such that $\eta\cdot x_0=x_1$, but now ${}^\eta\dot{X}(u,v)=\dot{X}(\varpi^{-1}u,\varpi v)$ and thus ${}^\eta\mathcal{O}_1=\mathcal{O}_\varpi$ instead.

We begin by showing that any degenerate coset determines a chamber of $\mathcal{B}(G)$ adjacent to x.

Lemma 5.2. *Let $G=\mathrm{SL}(2,F)$. Let $x\in\mathcal{B}(G)$ be any vertex and let $\Gamma\in\mathfrak{g}^*_{x,-d}\smallsetminus\mathfrak{g}^*_{x,-d+}$ represent a degenerate coset for some $d\in\mathbb{Z}_{>0}$. Then there exists a unique chamber $\mathcal{C}=\mathcal{C}_\Gamma$ of $\mathcal{B}(G)$ adjacent to x, independent of the choice of representative of $\Gamma+\mathfrak{g}^*_{x,-d+}$, such that for any $z\in\mathcal{C}$ we have $\Gamma\in\mathfrak{g}^*_{x,-d}\cap\mathfrak{g}^*_{z,-d+}$. Moreover, we have $\mathrm{Cent}_{G_x}(\Gamma)=\mathrm{Cent}_{G_z}(\Gamma)$.*

Proof. Uniqueness is immediate: given z' in any other chamber adjacent to x, the geodesic from z to z' contains x; hence $\mathfrak{g}^*_{z,-d+}\cap\mathfrak{g}^*_{z',-d+}$ is a subset of $\mathfrak{g}^*_{x,-d+}$ and therefore does not contain Γ. Identify Γ with an element $\dot{\Gamma}\in\mathfrak{g}_{x,-d}$ via the trace form. Choose a nilpotent element $\dot{X}\in\dot{\Gamma}+\mathfrak{g}_{x,-d+}$. By [DeBacker 2002b, §5], we may complete $\dot{X}$ to an $\mathfrak{sl}_2(F)$-triple $\{\dot{X},\dot{H}\in\mathfrak{g}_{x,0},\dot{Y}\in\mathfrak{g}_{x,d}\}$ and find a split torus S and corresponding apartment $\mathcal{A}(G,S)$ containing x, such that if $\Phi(G,S)=\{\pm\alpha\}$, then $\dot{X}\in\mathfrak{g}_\alpha$ and $\dot{Y}\in\mathfrak{g}_{-\alpha}$. Let $\mathcal{C}$ be the positive alcove adjacent to x in this apartment.

Note that $\mathrm{Cent}_{\mathfrak{g}}(\dot{Y})=\mathfrak{g}_{-\alpha}$. From [DeBacker 2002b, Lemma 5.2.1] we know that

$$\dot{X}+\mathfrak{g}_{x,-d+}={}^{G_{x,0+}}(\dot{X}+\mathrm{Cent}_{\mathfrak{g}_{x,-d+}}(\dot{Y}));$$

thus there exists $g\in G_{x,0+}$ such that $\dot{\Gamma}\in{}^g(\dot{X}+\mathfrak{g}_{-\alpha}\cap\mathfrak{g}_{x,-d+})$. Since $G_{x,0+}$ fixes $\mathcal{C}$ and the coset $\dot{\Gamma}+\mathfrak{g}_{x,-d+}$, we may without loss of generality replace the Lie triple and torus of the preceding paragraph with their g-conjugate, so that we have $\dot{\Gamma}\in\dot{X}+\mathfrak{g}_{-\alpha}\cap\mathfrak{g}_{x,-d+}$. For any $z\in\mathcal{C}$ we have $0<\alpha(z-x)<1$; thus since $\alpha(x),d\in\mathbb{Z}$ we may conclude

$$\mathfrak{g}_\alpha\cap\mathfrak{g}_{x,-d}=\mathfrak{g}_\alpha\cap\mathfrak{g}_{z,-d+}\quad\text{and}\quad\mathfrak{g}_{-\alpha}\cap\mathfrak{g}_{x,-d+}=\mathfrak{g}_{-\alpha}\cap\mathfrak{g}_{z,-d+}.$$

Since $\dot\Gamma$ lies in the sum of these two spaces we have $\dot\Gamma \in \mathfrak{g}_{z,-d+}$, whence $\Gamma \in \mathfrak{g}^*_{x,-d} \cap \mathfrak{g}^*_{z,-d+}$.

Finally, note that $\mathrm{Cent}_G(\dot X) = U_\alpha$ and $U_\alpha \cap G_x = U_\alpha \cap G_z$. Since $\dot\Gamma \in \dot X + \mathfrak{g}_{x,-d+}$, we have $\mathrm{Cent}_{G_x}(\dot\Gamma) \subset \mathrm{Cent}_{G_x}(\dot X) G_{x,0+} = \mathrm{Cent}_{G_z}(\dot X) G_{x,0+} \subset G_z$. □

Definition 5.3. Let $d = -d_x(\Gamma) > 0$ be such that $\Gamma + \mathfrak{g}^*_{x,-d+}$ is a degenerate coset. Let z be the barycentre of the associated alcove $\mathcal{C}_\Gamma$. Define the subgroup $J_{x,\Gamma}$ by

$$J_{x,\Gamma} = \begin{cases} G_{x,d/2} & \text{if } d = -d_x(\Gamma) \text{ is odd,} \\ G_{z,d/2} & \text{if } d \text{ is even.} \end{cases} \tag{5-3}$$

Note that when $x = x_0$ and $z = z_0$ this group coincides with J_d in (5-1).

Since $G_{x,n+} \subseteq G_{z,n} \subseteq G_{x,n}$ for any positive integer n, it follows directly that

$$G_{x,d/2+} \subseteq J_{x,\Gamma} \subseteq G_{x,d/2}.$$

Since $\Gamma \in \mathfrak{g}^*_{x,-d} \cap \mathfrak{g}^*_{z,-d+}$, it defines a character η_Γ of $J_{x,\Gamma}$ that is trivial on $G_{x,d+}$ via the corresponding Moy–Prasad isomorphism. The character depends only on the coset $\Gamma + \mathfrak{g}^*_{x,-d/2}$ if d is odd and on $\Gamma + \mathfrak{g}^*_{z,-d/2+}$ otherwise. Moreover, since $\mathrm{Cent}_{G_x}(\Gamma) = \mathrm{Cent}_{G_z}(\Gamma)$ we deduce directly that $J_{x,\Gamma}$ is normalized by $C_x(\Gamma) := \mathrm{Cent}_{G_x}(\Gamma)$.

Thus, for any character θ of $C_x(\Gamma)$ coinciding with η_Γ on the intersection of their domains there is a unique extension $\eta(\Gamma, \theta)$ of η_Γ to $C_x(\Gamma) J_{x,\Gamma}$. Define

$$\mathcal{S}_x(\Gamma, \theta) = \mathrm{Ind}_{C_x(\Gamma) J_{x,\Gamma}}^{G_x} \eta(\Gamma, \theta).$$

Proposition 5.4. *Suppose Γ represents a degenerate coset at a vertex $x \in \mathcal{B}(G)$ and $-d = d_x(\Gamma) < 0$. Suppose θ is a character of the centralizer $C_x(\Gamma)$ of Γ in G_x defining a character $\eta(\Gamma, \theta)$ of $C_x(\Gamma) J_{x,\Gamma}$ as above. Then*

(a) *$\mathcal{S}_x(\Gamma, \theta)$ is an irreducible representation of G_x of depth d and of degree $\frac{1}{2} q^{d-1}(q^2 - 1)$;*

(b) *$\mathcal{S}_x(\Gamma, \theta) \cong \mathcal{S}_x(\Gamma', \theta')$ if and only if there exists $g \in G_x$ such that $\eta(\Gamma, \theta) = {}^g\eta(\Gamma', \theta')$; and*

(c) *for any $\nu \in \mathrm{GL}(2, F)$ we have*

$${}^\nu\mathcal{S}_x(\Gamma, \theta) \cong \mathcal{S}_{\nu\cdot x}({}^\nu\Gamma, {}^\nu\theta). \tag{5-4}$$

Proof. When $x = x_0$ and $\Gamma \in \mathfrak{g}^*$ corresponds to some $\dot X(u, v) \in \mathfrak{g}_{x_0,-d} \smallsetminus \mathfrak{g}_{x_0,-d+}$, then this construction coincides with Shalika's. If $g \in G_x$, then ${}^g C_x(\Gamma) = C_x({}^g\Gamma)$ and ${}^g J_{x,\Gamma} = J_{x,{}^g\Gamma}$, so we obtain the invariance of $\mathcal{S}_x(\Gamma, \theta)$ under G_x-conjugacy and the choice of representative of the appropriate coset of Γ. More generally, for any $\nu \in \mathrm{GL}(2, F)$ such that $\nu \cdot x_0 = x$, we have ${}^\nu(\mathfrak{g}^*_{x_0,d}) = \mathfrak{g}^*_{x,d}$, ${}^\nu C_{x_0}(\Gamma) = C_x({}^\nu\Gamma)$ and ${}^\nu J_{x_0,\Gamma} = J_{x,{}^\nu\Gamma}$. Thus

$${}^\nu\mathcal{S}_{x_0}(\Gamma, \theta) \cong \mathcal{S}_x({}^\nu\Gamma, {}^\nu\theta),$$

where we have identified a ν-conjugate of a representation of G_{x_0} with a representation of G_x under the group isomorphism ${}^{\nu}G_{x_0} \cong G_x$. Since $\mathrm{GL}(2, F)$ acts transitively on the set of vertices of $\mathcal{B}(\mathrm{SL}(2, F))$, the rest of the statements follow from Proposition 5.1. □

The simple nature of the representations $\mathcal{S}_x(\Gamma, \theta)$ is revealed as follows.

Lemma 5.5. *Suppose x is a vertex of $\mathcal{B}(G)$ and $\Gamma_1, \Gamma_2 \in \mathfrak{g}^*_{x,-d}$ represent nonzero but degenerate cosets of $\mathfrak{g}^*_{x,-d}/\mathfrak{g}^*_{x,-d+}$ for some $d \in \mathbb{Z}_{>0}$. Suppose $s \in \mathbb{R}_{>0}$ satisfies $\Gamma_1 \in \Gamma_2 + \mathfrak{g}^*_{x,-s}$. Then for any choice of characters θ_i of $C_x(\Gamma_i)$ such that the characters $\eta(\Gamma_i, \theta_i)$ agree for $i \in \{1, 2\}$ upon restriction to $C_x(\Gamma_1)J_{x,\Gamma_1} \cap G_{x,s+} = C_x(\Gamma_2)J_{x,\Gamma_2} \cap G_{x,s+}$, we have*

$$\mathrm{Res}_{G_{x,s+}} \mathcal{S}_x(\Gamma_1, \theta_1) \cong \mathrm{Res}_{G_{x,s+}} \mathcal{S}_x(\Gamma_2, \theta_2). \tag{5-5}$$

In particular, if $s \geq \frac{1}{2}d$ then (5-5) holds independent of θ_i.

Proof. For any Γ_i, the two representations have the same degree $\frac{1}{2}q^{d-1}(q^2-1)$ and the same depth d. If $s \geq d$ then both sides are $\mathbf{1}$-isotypic of the same degree hence equivalent.

Suppose $s < d$. Then $G_x/G_{x,d-s}$ acts on $\mathfrak{g}^*_{x,-d}/\mathfrak{g}^*_{x,-s}$. The stabilizer of Γ_i in G_x stabilizes its coset in $\mathfrak{g}^*_{x,-d}/\mathfrak{g}^*_{x,-s}$; the full stabilizer of the coset is $C_x(\Gamma_i)G_{x,d-s}$. Since $\Gamma_1 \in \Gamma_2 + \mathfrak{g}^*_{x,-s}$, we thus have $C_x(\Gamma_1) \subset C_x(\Gamma_2)G_{x,d-s}$. Because $\Gamma_1 \in \Gamma_2 + \mathfrak{g}^*_{x,-d+}$, Lemma 5.2 yields $J_{x,\Gamma_1} = J_{x,\Gamma_2}$; let us denote this group by J. Thus η_{Γ_i} for $i \in \{1, 2\}$ are characters of J that agree on $J \cap G_{x,s+}$. We consider two cases.

If $s \geq \frac{1}{2}d$ then $G_{x,s+} \subset J$, and so $\mathrm{Res}_{G_{x,s+} \cap C_x(\Gamma_i)J}\, \eta(\Gamma_i, \theta_i) = \eta_{\Gamma_i}$ is independent of θ_i. Mackey theory thus yields the decomposition

$$\mathrm{Res}_{G_{x,s+}} \mathcal{S}_x(\Gamma_i, \theta_i) \cong \bigoplus_{\gamma \in G_x/C_x(\Gamma_i)J} {}^{\gamma}\eta_{\Gamma_i}|_{G_{x,s+}}. \tag{5-6}$$

Each $\gamma \in C_x(\Gamma_i)G_{x,d-s}/C_x(\Gamma_i)J$ fixes the character $\eta_{\Gamma_i}|_{G_{x,s+}}$. The elements $\gamma' \in G_x/C_x(\Gamma_1)G_{x,d-s} = G_x/C_x(\Gamma_2)G_{x,d-s}$ parametrize the orbit of the coset $\Gamma_1 + \mathfrak{g}^*_{x,-s} = \Gamma_2 + \mathfrak{g}^*_{x,-s}$. Thus (5-6) gives the same sum of characters for $i \in \{1, 2\}$.

If instead $s < \frac{1}{2}d$, then $G_{x,d-s} \subseteq J$ so $C_x(\Gamma_1)J = C_x(\Gamma_2)J$. Since $J \subseteq G_{x,s+}$, the double coset space $G_{x,s+}\backslash G_x/C_x(\Gamma_i)J$ is now equal to $G_x/C_x(\Gamma_i)G_{x,s+}$, and is independent of i. So again by Mackey theory we have

$$\begin{aligned}\mathrm{Res}_{G_{x,s+}} \mathcal{S}_x(\Gamma_i, \theta_i) &= \bigoplus_{\gamma \in G_x/C_x(\Gamma_i)G_{x,s+}} \mathrm{Ind}_{G_{x,s+} \cap {}^{\gamma}(C_x(\Gamma_i)J)}^{G_{x,s+}} {}^{\gamma}(\eta(\Gamma_i, \theta_i)) \\ &= \bigoplus_{\gamma \in G_x/C_x(\Gamma_i)G_{x,s+}} {}^{\gamma}\big(\mathrm{Ind}_{G_{x,s+} \cap C_x(\Gamma_i)J}^{G_{x,s+}}(\eta(\Gamma_i, \theta_i))\big).\end{aligned}$$

When the restriction of $\eta(\Gamma_i, \theta_i)$ to $G_{x,s+} \cap C_x(\Gamma_1)J = G_{x,s+} \cap C_x(\Gamma_2)J$ is independent of i, we infer (5-5). □

5C. ***Representations attached to nilpotent orbits.*** Let $X \in \mathcal{N}^* \smallsetminus \{0\}$ and let λ be a corresponding adapted one-parameter subgroup. Its centralizer in G is a maximal split torus S. In fact, S is generated by S_0 and $\lambda(\varpi)$, and $\mathrm{Cent}_G(X) = ZU$, where Z is the centre of G and $B = SU$ is a Borel subgroup. For any vertex $x \in \mathcal{B}(G)$, applying the Iwasawa decomposition yields $G = G_x S \,\mathrm{Cent}_G(X)$. Consequently,

$$\mathcal{O} = G \cdot X = \bigsqcup_{n\in\mathbb{Z}} G_x \cdot (\lambda(\varpi)^n \cdot X) = \bigsqcup_{n\in\mathbb{Z}} G_x \cdot (\varpi^{2n} X) \tag{5-7}$$

is the decomposition of the G-orbit of X into disjoint G_x-orbits. It follows that the parity of $d_x(Y)$, for any $Y \in \mathcal{O}$, is an invariant of the G-orbit, and we call this the *parity depth of $\mathcal{O}$ at x*, denoted by $\mathrm{pd}_x(\mathcal{O})$. Furthermore, for each $d \in \mathbb{Z}_{>0}$ of this parity there exists exactly one G_x-orbit in $\mathcal{O} \smallsetminus \mathfrak{g}^*_{x,0}$ whose elements have depth equal to $-d$.

Let Y be a representative of such a G_x orbit. We claim that any choice of central character ζ of Z defines a character (also denoted by ζ) of $C_x(Y)$ coinciding with η_Y on the intersection of their domains. Namely, the chamber $\mathcal{C}_Y$ associated to (Y, x) by Lemma 5.2 defines a Borel subgroup with unipotent radical U. Setting $U_x = G_x \cap U$ we have $C_x(Y) = ZU_x$. Since η_Y is trivial on $ZU_x \cap J_{x,Y}$, the character ζ of ZU_x defined by $\zeta(zu) = \zeta(z)$ for all $z \in Z$ and $u \in U_x$ extends η_Y. We abbreviatedly denote this representation by $\mathcal{S}_x(Y, \zeta)$.

Then, applying Proposition 5.4, we may conclude the following.

Proposition 5.6. *Let x be a vertex in $\mathcal{B}(G)$, $\mathcal{O}$ a nonzero nilpotent G-orbit in $\mathfrak{g}^*$ and ζ a character of Z. For each $d \in \mathbb{Z}_{>0}$ of parity $\mathrm{pd}_x(\mathcal{O})$, fix a representative X_{-d} of the corresponding G_x-orbit in $\mathcal{O} \smallsetminus \mathfrak{g}^*_{x,0}$. Then the* ***representation of G_x attached to $\mathcal{O}$ with central character ζ****, given by*

$$\tau_x(\mathcal{O}, \zeta) = \bigoplus_{d\in\mathbb{Z}_{>0},\, d\equiv \mathrm{pd}_x(\mathcal{O}) \bmod 2} \mathcal{S}_x(X_{-d}, \zeta), \tag{5-8}$$

is independent of choices up to G_x-equivalence.

The depths d of the components of $\tau_x(\mathcal{O}, \zeta)$ all have parity equal to the parity depth of $\mathcal{O}$ at x. Furthermore, for any $X \in \mathcal{O}$ such that $d_x(X) \in \{0, -1\}$, one set of representatives for the G_x-orbits in $\mathcal{O} \smallsetminus \mathfrak{g}^*_{x,0}$ is $\{\varpi^{-2t} X \mid t \in \mathbb{Z}_{\geq 0}\}$.

Since the restriction of $\tau_x(\mathcal{O}, \zeta)$ to any subgroup of $G_{x,0+}$ is independent of the choice of ζ, we may (and do) drop ζ from the notation in such cases. As needed, we associate to the zero nilpotent orbit the trivial representation of G_x, and denote by it $\tau_x(\{0\})$.

6. The case of positive-depth representations of SL(2, F)

The irreducible admissible representations of $\mathrm{SL}(2, F)$ come in exactly two flavours: the irreducible subquotients of the principal series; and the irreducible supercuspidal

representations. A classification of the former is nicely developed in [Tadić 1994, §7]; the original classification of the latter is in the 1966 thesis of Shalika [2004].

In this section, we focus on those representations of positive depth. We begin by phrasing the classification of positive-depth irreducible admissible representations π of $\mathrm{SL}(2, F)$ in a way that emphasizes their construction from characters of tori. We then establish that their explicit branching to a maximal compact open subgroup G_x can be described as twists of the datum (χ, Γ) defining π. This allows us to state and prove our main theorem in this case, and to explicitly compute the constant terms that arise.

6A. ***Representations of*** **SL(2, *F*)** ***of positive depth.*** All principal series of positive depth are irreducible. We classify the positive-depth supercuspidal representations using the parametrization of Adler and Yu [Adler 1998; Yu 2001; Fintzen 2021a], which applies since $p > 2$; this was done explicitly in [Nevins 2013]. Because the tori in $\mathrm{SL}(2, F)$ are one-dimensional, the correcting twist to this construction given by Fintzen, Kaletha and Spice in [Fintzen et al. 2023, Definition 3.1] is trivial.

Proposition 6.1. *Up to isomorphism, the irreducible admissible representations of* $\mathrm{SL}(2, F)$ *of positive depth r are parametrized by the G-conjugacy classes of pairs (T, χ), where T is a maximal torus of G and χ is a character of T of depth r.*

To construct the representations explicitly, we first recall some facts about the maximal tori and their characters. Let T be a maximal torus of G and let χ be a character of T of depth $r > 0$. The building $\mathcal{B}(T)$ of T embeds into $\mathcal{B}(G)$ as the apartment $\mathcal{A}(G, T)$ if T is split and as a single point $\{x_T\}$ otherwise. This point x_T is a vertex if T is unramified and the midpoint of a chamber if T is ramified. It follows that the depth $r > 0$, which is in particular a value for which $T_r \neq T_{r+}$, is an integer if T is split or unramified and is an element of $\frac{1}{2} + \mathbb{Z}$ otherwise.

To each pair (T, χ) we associate an element $\Gamma = \Gamma_\pi$ as follows. If $\mathfrak{t}$ denotes the Lie algebra of T, then via the Moy–Prasad isomorphism $e : \mathfrak{t}_{r/2+}/\mathfrak{t}_{r+} \to T_{r/2+}/T_{r+}$ there exists a nonzero element $\Gamma \in \mathfrak{t}^*_{-r}$, uniquely defined modulo $\mathfrak{t}^*_{-r/2}$, such that

$$\chi(t) = \psi\big(\Gamma(e^{-1}(t))\big).$$

We identify Γ with an element of $\mathfrak{g}^*$ that is zero on the T-invariant complement of $\mathfrak{t}$ in $\mathfrak{g}$. Then $\Gamma \in \mathfrak{g}^*_{x,-r}$ for any $x \in \mathcal{B}(T)$ and we recover T as $\mathrm{Cent}_G(\Gamma)$. Moreover, Γ thus defines a character of $G_{x,r}/G_{x,r+} \cong \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$, and following the work of Moy and Prasad, the pair $(G_{x,r}, \Gamma)$ is called an *unrefined minimal K-type*.

Proof of Proposition 6.1. The construction of the representations $\pi = \pi(T, \chi)$ varies.

If T is a split torus, then choose a Borel subgroup $B = TU$ of G containing T and extend χ trivially across U to a character of B. Set

$$\mathrm{Ind}_{TU}^{G}(\chi) = \{f : G \to \mathbb{C} \mid f(tug) = \chi(t)\nu(t)f(g) \text{ for all } t \in T, u \in U, g \in G\},$$

where ν is the square root of the modular character and is given on $T \cong F^\times$ by the p-adic norm. Then $\pi(T, \chi) = \mathrm{Ind}_B^G(\chi)$ is an irreducible principal series representation.

If T is anisotropic, with associated point $y = x_T \in \mathcal{B}(G)$, then we first extend χ to a character of $TG_{y,r/2+}$, by setting

$$\chi'(tg) = \chi(t)\psi\big(\Gamma(e^{-1}(g))\big),$$

where $e : \mathfrak{g}_{y,r/2+}/\mathfrak{g}_{y,r+} \to G_{y,r/2+}/G_{y,r+}$ is the Moy–Prasad isomorphism. When $G_{y,r/2} = G_{y,r/2+}$, we set $\kappa = \chi'$. When $G_{y,r/2} \neq G_{y,r/2+}$ (which will happen only if T is unramified and $r \in 2\mathbb{Z}$), we take a certain Weil–Heisenberg lift of $\chi'|_{G_{y,r}}$ to form a q-dimensional representation ω of $T \ltimes G_{y,r/2}$, and set $\kappa(tg) = \chi(t)\omega(t, g)$. Then $\pi(T, \chi) = \text{c-Ind}_{TG_{y,r/2}}^G \kappa$ is an irreducible supercuspidal representation. □

Given $\pi = \pi(T, \chi)$, we let $\Gamma = \Gamma_\pi$ denote a choice of minimal K-type realizing the character χ, as preceded the proof. Because $T = \mathrm{Cent}_G(\Gamma)$ we may also say that (χ, Γ) is the datum defining π.

6B. ***Branching rules obtained as twists of the inducing datum.*** We begin by proving that the branching rules obtained in [Nevins 2005, Theorem 7.4; 2013, Theorem 6.2] are in fact constructible from twists of the datum (χ, Γ).

Theorem 6.2. *Let $\pi = \pi(T, \chi)$ be an irreducible admissible representation of G of depth $r > 0$. Let $\Gamma = \Gamma_\pi \in \mathfrak{g}^*$ realize χ as above, so that $T = \mathrm{Cent}_G(\Gamma)$. Then for any vertex $x \in \mathcal{B}(G)$ we have*

$$\mathrm{Res}_{G_x} \pi = \pi^{G_{x,r+}} \oplus \bigoplus_{g \in [G_x \backslash G / T]^{dg}} \mathcal{S}_x({}^g\Gamma, {}^g\chi), \tag{6-1}$$

where $[G_x \backslash G / T]^{dg}$ denotes a parameter set for the G_x-orbits in $G \cdot \Gamma$ that do not meet $\mathfrak{g}^*_{x,-r}$, that is, such that the coset ${}^g\Gamma + \mathfrak{g}^*_{x,d_x({}^g\Gamma)+}$ is degenerate.

Proof. Let us first show that the proof may be reduced to the special case that $x = x_0$. Suppose that $x \in \mathcal{B}(G)$ is an arbitrary vertex. Then there exists $k \in \mathrm{GL}(2, F)$ such that $kx = x_0$, yielding ${}^kG_x = G_{x_0}$. If T is anisotropic, choose $h \in \mathrm{SL}(2, F)$ such that $hx_T \in k^{-1}\bar{\mathcal{C}}$, the closure of the fundamental alcove. If T is split, choose h instead so that ${}^hT = S$ where $x_0 \in \mathcal{A}(G, S)$. Then we have

$$\mathrm{Res}_{{}^{kh}G_{x_0}} {}^{kh}\pi \cong \mathrm{Res}_{{}^hG_x} {}^h\pi \cong \mathrm{Res}_{G_x} \pi,$$

where the first two representations are isomorphic under the identification of the groups G_{x_0} and G_x via conjugation by k, and the second two are isomorphic as representations of G_x since ${}^h\pi \cong \pi$. Even when $kh \notin \mathrm{SL}(2, F)$, the datum defining the representation ${}^{kh}\pi$ is simply $({}^{kh}T, {}^{kh}\chi, {}^{kh}\Gamma, khx_T)$, where the term khx_T is only for the supercuspidal case.

Suppose that we have proven the decomposition (6-1) of $\mathrm{Res}_{{}^{kh}G_{x_0}} {}^{kh}\pi$. Via the identification ${}^kG_{x,r+} \cong G_{x_0,r+}$, we have $({}^{kh}\pi)^{G_{x_0,r+}} \cong ({}^h\pi)^{G_{x,r+}}$. Moreover, for

each $g \in [G_{x_0}\backslash G/\mathrm{Cent}_G({}^{kh}\Gamma)]^{\mathrm{dg}}$, which is an element of G such that the G_{x_0}-orbit of ${}^{gkh}\Gamma$ does not meet $\mathfrak{g}^*_{x_0,-r}$, we have

$$\mathcal{S}_{x_0}({}^{gkh}\Gamma, {}^{gkh}\chi) = {}^{k}(\mathcal{S}_{x_0}({}^{(k^{-1}gk)h}\Gamma, {}^{(k^{-1}gk)h}\chi)) \cong \mathcal{S}_x({}^{g'}\Gamma, {}^{g'}\chi),$$

where we set $g' = k^{-1}gkh$. Then $g' \in G$ is such that the G_x-orbit of ${}^{g'}\Gamma$ does not meet $\mathfrak{g}^*_{x,-r}$. It follows that the map $g \mapsto k^{-1}gkh$ takes $[G_{x_0}\backslash G/\mathrm{Cent}_G({}^{kh}\Gamma)]^{\mathrm{dg}}$ bijectively onto $[G_x\backslash G/T]^{\mathrm{dg}}$, as required.

We now prove the theorem in the special case that $x = x_0$, which was considered in [Nevins 2005; 2013].

Suppose first that $T = \mathrm{Cent}_G(\Gamma)$ is anisotropic. Set $y = x_T$, which we assume lies in $\overline{C}$, and let $\pi(T, \chi) = \text{c-Ind}_{TG_{y,r/2}}^{G} \kappa$ be the corresponding supercuspidal representation. By [Nevins 2013, Proposition 4.4], the double coset space $G_x\backslash G/TG_{y,r/2}$ that arises in the Mackey decomposition

$$\mathrm{Res}_{G_x}\pi = \bigoplus_{g \in G_x\backslash G/TG_{y,r/2}} \mathrm{Ind}_{G_x \cap {}^g(TG_{y,r/2})}^{G_x} {}^g\kappa$$

is independent of r and is given by $G_x\backslash G/T$. Since $T = \mathrm{Cent}_G(\Gamma)$, this latter space parametrizes the G_x-orbits in the G-orbit of Γ in $\mathfrak{g}^*$. By [Nevins 2013, Theorem 6.1], each of these Mackey components is irreducible.

The element Γ has depth $-r$ and depth is G-invariant. Thus ${}^g\Gamma$ meets $\mathfrak{g}^*_{x,-r}$ if and only if $d_x({}^g\Gamma) = d_{g^{-1}x}(\Gamma) = -r$, which by [Kim and Murnaghan 2003, Theorem 2.3.1] happens if and only if $g^{-1}x = x_T = y$. If this is the case, then T is an unramified torus attached to a vertex in the G-orbit of x; the corresponding Mackey component has depth r and so lies in $\pi^{G_{x,r+}}$. In all other cases $\pi^{G_{x,r+}} = \{0\}$.

Thus the elements $g \in [G_x\backslash G/T]^{\mathrm{dg}}$ parametrize all Mackey components except $\pi^{G_{x,r+}}$ (if it is nonzero). Furthermore, by [Nevins 2013, Theorem 6.2],[1] they satisfy

$$\mathrm{Ind}_{G_x \cap {}^g(TG_{y,r/2})}^{G_x} {}^g\kappa \cong \mathcal{S}_x({}^g\Gamma, {}^g\chi),$$

as required, yielding (6-1) for the supercuspidal representations.

Now suppose that $T = S$ is the split torus, and that $\pi = \pi(T, \chi) = \mathrm{Ind}_B^G \chi$ for some Borel subgroup $B = TU$ containing T having U as its unipotent radical. Since $G = G_x B$, there is a unique (highly reducible) Mackey component in this case. Instead, the decomposition of $\mathrm{Res}_{G_x}\pi$ into irreducible subrepresentations is found in [Nevins 2005] (in the case that $x \in \{x_0, x_1\}$) by explicitly decomposing the G_x-subrepresentations $\pi^{G_{x,n}}$ as $n \to \infty$. We need to show that this decomposition is in fact of the form (6-1).

First note that as π has depth r at x, the subrepresentation $\pi^{G_{x,r+}}$ is nonzero, and in fact is irreducible as a representation of G_x by [Nevins 2005, Proposition 4.4]

[1] Correction to [Nevins 2013, Theorem 6.2]: the decomposition in the case $y = 1$ is missing the term corresponding to the double coset representative e^{η}.

(where this space is denoted by V^{K_m}, for $m = r+1$ the conductor of χ). Again by [Kim and Murnaghan 2003, Theorem 2.3.1] we know that ${}^g\Gamma \in \mathfrak{g}^*_{x,-r}$ if and only if ${}^gT \subset G_x$, so the G_x-orbit in $G \cdot \Gamma$ meeting $\mathfrak{g}^*_{x,-r}$ corresponds to the trivial double coset of $G_x \backslash G / T$.

The irreducible representations of G_x of depth greater than r appearing in $\mathrm{Res}_{G_x} \pi$ are classified in [Nevins 2005, Theorem 7.4]. The notation in that paper relates to ours as follows. Identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the trace form. For any $d \in \mathbb{Z}_{>0}$, $u \in \mathcal{R}^\times$, $v \in \mathcal{P}$, let $X = \varpi^{-d} \dot{X}(u, v) \in \mathfrak{g}_{x,-d}$; then for any character θ of $C_x(X)$ agreeing with η_X on $C_x(X) \cap J_{x,X}$, we had set

$$\mathcal{D}_{d-1}(\theta, \varpi^d X) := \mathcal{S}_x(X, \theta), \tag{6-2}$$

which is of depth d. Thus [Nevins 2005, Theorem 7.4] asserts that for each integer $d > r$, $\mathrm{Res}_{G_x} \pi$ has two irreducible components of depth d, denoted by $W^{\pm}_{d-1}$.

Explicitly, if $\dot{\Gamma}$ is a diagonal matrix $\mathrm{diag}(a, -a) \in \mathfrak{g}_{x,-r} \smallsetminus \mathfrak{g}_{x,-r+}$, then for each fixed $d > r$ define $\gamma_0 = a\varpi^d \in \mathcal{P}^{d-r}$, $\gamma_1 = a\varepsilon^{-1}\varpi^d \in \mathcal{P}^{d-r}$, $Y_0 = \dot{X}(1, \gamma_0^2)$ and $Y_1 = \dot{X}(\varepsilon, \varepsilon\gamma_1^2)$. Then $\varpi^{-d} Y_i \in \mathfrak{g}_{x,-d}$ and the theorem asserts there are characters ρ_i of $C_x(Y_i)$ such that

$$W^+_{d-1} \oplus W^-_{d-1} = \mathcal{S}_x(\varpi^{-d} Y_0, \rho_0) \oplus \mathcal{S}_x(\varpi^{-d} Y_1, \rho_1).$$

We claim this is the desired expression. Namely, if we choose, for $i \in \{0, 1\}$,

$$g_{i,d} = \begin{bmatrix} 1 & -\frac{1}{2}\gamma_i^{-1} \\ \gamma_i & \frac{1}{2} \end{bmatrix},$$

then $\varpi^{-d} Y_i = {}^{g_{i,d}}\Gamma$. Moreover, the characters ρ_i were defined in [Nevins 2005, Definition 7.3] on

$$\begin{bmatrix} b & c \\ c\gamma_i^2 & b \end{bmatrix} \in C_x(Y_i)$$

as $\chi(b + c\gamma_i)$; we can compute directly that therefore $\rho_i = {}^{g_{i,d}}\chi$.

It remains to show $\{g_{i,d} \mid i \in \{0, 1\}, d > r\}$ is a set of double coset representatives for $(G_x \backslash G / T) \smallsetminus G_x T$. Because $G = G_x U T$, these double cosets are represented by T-conjugacy classes of unipotent upper triangular matrices of strictly negative depth. Noting the factorization

$$g = \begin{bmatrix} 1 & -\frac{1}{2}\gamma^{-1} \\ \gamma & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2}\gamma^{-1} \\ 0 & 1 \end{bmatrix} \in G_x \begin{bmatrix} 1 & -\frac{1}{2}\gamma^{-1} \\ 0 & 1 \end{bmatrix} T,$$

we see that as $-\frac{1}{2}\gamma^{-1}$ runs over the distinct square classes in $\mathcal{P}^{r-d} \smallsetminus \mathcal{P}^{r-d+1}$, for all $d > r$, we obtain representatives of the distinct nontrivial cosets of $G_x \backslash G / T$, which is the index set $[G_x \backslash G / T]^{\mathrm{dg}}$, as required. □

6C. ***Main theorem for representations of positive depth.*** Before stating the next theorem, we require a short lemma about filtrations of tori.

Lemma 6.3. *Let $T = \mathrm{Cent}_G(\Gamma)$, where Γ is a semisimple element of depth $-r$. Suppose that at $x \in \mathcal{B}(G)$ we have $d_x(\Gamma) = -d < -r$. Then $T \cap G_x = ZT_{d-r}$, so that $T \cap G_{x,\ell} = T_{d-r+\ell}$ for any $\ell \geq 0$.*

Proof. Let $\mathfrak{t}$ be the Lie algebra of T. The hypotheses imply that $d_x(\varpi^k\Gamma) = d(\varpi^k\Gamma) - (d-r)$ for any $k \in \mathbb{Z}$. Thus, since $\mathfrak{t}$ is one-dimensional, for any element $\dot{X} \in \mathfrak{t}_\ell \smallsetminus \mathfrak{t}_{\ell+}$ we have $d_x(\dot{X}) = \ell - (d-r)$, yielding $\mathfrak{t} \cap \mathfrak{g}_{x,\ell} = \mathfrak{t}_{d-r+\ell}$. Passage to the group yields the desired result, where at depth zero, we observe that $Z \subset T \cap G_x$ for all x. □

Theorem 6.4. *Let $\pi = \pi(T, \chi)$ be an irreducible admissible representation of $G = \mathrm{SL}(2, F)$ of depth $r > 0$ and let $\Gamma = \Gamma_\pi \in \mathfrak{g}^*$ be an associated K-type. Then for each maximal compact subgroup G_x, there is an integer $n_x(\pi)$ such that*

$$\mathrm{Res}_{G_{x,r+}} \pi \cong n_x(\pi)\mathbf{1} + \sum_{\mathcal{O} \in \mathrm{Nil}(\Gamma)} \mathrm{Res}_{G_{x,r+}} \tau_x(\mathcal{O}) \tag{6-3}$$

in the Grothendieck group of representations. In particular, up to some copies of the trivial representation, π is locally completely determined by the nilpotent support of Γ.

Proof. The restriction to $G_{x,r+}$ will be trivial on any irreducible G_x-representations of depth less than or equal to r, so our first step is to match components of depth $d > r$ in $\mathrm{Res}_{G_x} \pi$ and in $\sum_{\mathcal{O} \in \mathrm{Nil}(\Gamma)} \tau_x(\mathcal{O})$. Note that the restriction to $G_{x,r+}$ is independent of the choice of central character ζ so it is omitted from the notation.

Theorem 6.2 gives the decomposition (6-1) of the left side: the irreducible components of depth greater than r are parametrized by the degenerate G_x-orbits of Γ at x. From (5-8) we infer the decomposition of the right side: the components are parametrized by nilpotent G_x-orbits in $\mathcal{O} \smallsetminus \mathfrak{g}^*_{x,0}$ for each $\mathcal{O} \in \mathrm{Nil}(\Gamma)$.

By Proposition 4.1, each degenerate coset $\xi = {}^g\Gamma + \mathfrak{g}^*_{x,-d+}$, where $d = -d_x({}^g\Gamma)$, is represented by a nilpotent element $X \in \mathcal{O}({}^g\Gamma, x)$ such that also ${}^g\Gamma - X \in \mathfrak{g}^*_{x,-r}$. The G_x-orbit of ξ determines the G_x-orbit of X and by definition $G \cdot X \in \mathrm{Nil}(\Gamma)$. Thus for each $d > r$ there is a one-to-one correspondence between the G_x-orbits in $G \cdot \Gamma$ whose depth at x is $-d$ and the G_x-orbits in $\mathrm{Nil}(\Gamma)$ whose depth at x is $-d$.

To complete the proof we need to show the corresponding representations are isomorphic upon restriction to $G_{x,r+}$.

Let $\zeta = \chi|_Z$ be the central character of π. If $r < d \leq 2\lfloor r \rfloor + 1$, then applying Lemma 5.5 to the pair $\Gamma_1 = {}^g\Gamma$ and $\Gamma_2 = X$, with $s = r$, gives $\mathrm{Res}_{G_{x,r+}} \mathcal{S}_x({}^g\Gamma, {}^g\chi) \cong \mathrm{Res}_{G_{x,r+}} \mathcal{S}_x(X, \zeta)$, as required.

If $d > 2r$ (which includes all $d > 2\lfloor r \rfloor + 1$), we have a stronger result. Lemma 6.3 implies that $\mathrm{Cent}_{G_x}({}^g\Gamma) = Z\,{}^gT_{d-r} \subseteq Z\,{}^gT_{r+}$, and therefore ${}^g\chi$ is given on this

subgroup by the central character ζ. Since the chambers attached to ${}^g\Gamma$ and to X by Lemma 5.2 coincide, we have $J := J_{x,{}^g\Gamma} = J_{x,X}$. Since ${}^g\Gamma - X \in \mathfrak{g}^*_{x,-r} \subset \mathfrak{g}^*_{x,-d/2+}$, we have $\eta_{{}^g\Gamma} = \eta_X$ as characters of J. Moreover, since ${}^g\Gamma \in X + \mathfrak{g}^*_{x,-r}$, we have $C_x({}^g\Gamma) \subseteq C_x(X)G_{x,d-r} \subseteq C_x(X)J$. Therefore $C_x(X)J_{x,X} = C_x({}^g\Gamma)J_{x,{}^g\Gamma}$ and $\eta({}^g\Gamma, {}^g\chi) = \eta(X, \zeta)$ as characters of this common group. Thus $\mathcal{S}_x({}^g\Gamma, {}^g\chi) = \mathcal{S}_x(X, \zeta)$ as representations of G_x. □

In the course of the proof we established that the components of depth $d > 2r$ occurring in $\mathrm{Res}_{G_x}\pi$ coincide as representations of G_x (not just as representations of $G_{x,r+}$), with the components of depth $d > 2r$ in $\sum_{\mathcal{O}\in\mathrm{Nil}(\Gamma)} \tau_x(\mathcal{O}, \zeta)$, where ζ is the central character of π. This was proven case by case in [Nevins 2005, Remark 7.5; 2013, Proposition 7.6].

For convenience, we recap these sets $\mathrm{Nil}(\Gamma)$ in Table 1.

Remark 6.5. By Theorem 3.5, we know that $\mathrm{Nil}(\Gamma_\pi) = \mathcal{WF}(\pi)$. Moreover, with the standard normalization chosen in [Mœglin and Waldspurger 1987, I.8], the coefficients of the leading terms of the local character expansion agree with those of (6-3); namely $c_{\mathcal{O}}(\pi) = 1$ for all $\mathcal{O} \in \mathcal{WF}(\pi)$. Thus Theorem 6.4 is a representation-theoretic analogue of the analytic local character expansion.

On the other hand, the constant term $n_x(\pi)$ of the decomposition (6-3) does not (and could not) agree with the constant term $c_0(\pi)$ of the local character expansion. For one, $n_x(\pi) \in \mathbb{Z}$, whereas $c_0(\pi)$ may be half-integral; see Table 6. For another, $n_x(\pi)$ depends on the dimension of $\pi^{G_{x,r+}}$, which may vary based on the G-conjugacy class of the vertex $x \in \mathcal{B}(G)$.

Let us compute the constant terms $n_x(\pi)$ explicitly. We begin with a lemma.

Lemma 6.6. *Let $\ell \geq 0$. Then we have*

$$\dim(\tau_x(\mathcal{O})^{G_{x,\ell+}}) = \begin{cases} \frac{1}{2}q(q^{2\lfloor \ell/2 \rfloor} - 1) & \text{if } \mathrm{pd}_x(\mathcal{O}) \text{ is even,} \\ \frac{1}{2}(q^{2\lceil \ell/2 \rceil} - 1) & \text{if } \mathrm{pd}_x(\mathcal{O}) \text{ is odd.} \end{cases}$$

In particular, $\dim(\tau_x(\mathcal{O}) \oplus \tau_x(\mathcal{O}'))^{G_{x,\ell+}} = \frac{1}{2}(q+1)(q^{\lfloor \ell \rfloor} - 1)$ *when the G_x-orbits in $\mathcal{O}$ and $\mathcal{O}'$ have opposite parity depths at x.*

Proof. The space of $G_{x,\ell+}$-fixed vectors of $\tau_x(\mathcal{O})$ is exactly the sum of its irreducible components of depth $d \leq \ell$. By Proposition 5.4(a), each of these has dimension $\frac{1}{2}q^{d-1}(q^2-1)$ and correspond to the G_x-orbits of $\mathcal{O}$ whose depths $-d$ at x satisfy $-\ell \leq -d \leq -1$. Thus if the parity depth of $\mathcal{O}$ at x is even then with $d = 2e$ we have

$$\dim(\tau_x(\mathcal{O})^{G_{x,\ell+}}) = \sum_{e=1}^{\lfloor \ell/2 \rfloor} \tfrac{1}{2}q^{2e-1}(q^2-1) = \tfrac{1}{2}q(q^{2\lfloor \ell/2 \rfloor} - 1),$$

representation $\pi(T,\chi)$ *of positive depth*	$\mathrm{Nil}(\Gamma_\pi) = \mathcal{WF}(\pi)$
principal series, T *split*	$\{\mathcal{O}_1, \mathcal{O}_\varepsilon, \mathcal{O}_\varpi, \mathcal{O}_{\varepsilon\varpi}\}$
supercuspidal, T *splits over* $F[\sqrt{\varepsilon}]$	$\{\mathcal{O}_1, \mathcal{O}_\varepsilon\}$ *if* $d_{x_0}(\Gamma_\pi)$ *is even* $\{\mathcal{O}_\varpi, \mathcal{O}_{\varepsilon\varpi}\}$ *if* $d_{x_1}(\Gamma_\pi)$ *is even*
supercuspidal, T *splits over* $F[\sqrt{\varpi}]$	$\{\mathcal{O}_u, \mathcal{O}_{-u\varpi}\}$ *for some* $u \in \{1, \varepsilon\}$
supercuspidal, T *splits over* $F[\sqrt{\varepsilon\varpi}]$	$\{\mathcal{O}_u, \mathcal{O}_{-u\varepsilon\varpi}\}$ *for some* $u \in \{1, \varepsilon\}$

Table 1. The forms of the sets $\mathrm{Nil}(\Gamma_\pi) = \mathcal{WF}(\pi)$ for each type of positive-depth representation; the precise sets are determined from Γ_π in Lemma 4.2.

whereas if it is odd then with $d = 2e - 1$ we have

$$\dim(\tau_x(\mathcal{O})^{G_{x,\ell+}}) = \sum_{e=1}^{\lceil \ell/2 \rceil} \tfrac{1}{2} q^{2e-2}(q^2 - 1) = \tfrac{1}{2}(q^{2\lceil \ell/2 \rceil} - 1).$$

If $\ell = 2k$ the sum of these is $\frac{1}{2}(q+1)(q^\ell - 1)$ and if $\ell = 2k+1$ then the sum is $\frac{1}{2}q(q^{2k} - 1) + \frac{1}{2}(q^{2(k+1)} - 1) = \frac{1}{2}(q+1)(q^\ell - 1)$. The result follows since $G_{x,\ell+} = G_{x,\lfloor \ell \rfloor +}$ for each vertex x. □

Proposition 6.7. *Let* $\pi = \pi(T,\chi)$ *be an irreducible representation of depth* $r > 0$ *as in Theorem 6.2. Then, for each vertex* $x \in \mathcal{B}(G)$*, the dimension of the subspace of* $G_{x,r+}$*-fixed vectors and the value of the coefficient* $n_x(\pi)$ *appearing in (6-3) are as given in Table 2.*

Proof. Let $\pi = \pi(T,\chi)$ have depth $r > 0$, with associated minimal K-type $\Gamma = \Gamma_\pi$. From Theorem 6.2 we have the equality

$$n_x(\pi) = \dim(\pi^{G_{x,r+}}) - \sum_{\mathcal{O} \in \mathrm{Nil}(\Gamma)} \dim(\tau_x(\mathcal{O}))^{G_{x,r+}}. \tag{6-4}$$

If π is a principal series representation and B is a Borel subgroup containing T, then $\pi^{G_{x,r+}} = \mathrm{Ind}_{(B \cap G_x)G_{x,r+}}^{G_x} \chi$, whence $\dim(\pi^{G_{x,r+}}) = |G_x/(B \cap G_x)G_{x,r+}| = (q+1)q^r$. Moreover, in this case Γ_π is split and all principal nilpotent orbits occur in $\mathrm{Nil}(\Gamma)$ (Table 1). Thus using (6-4) and Lemma 6.6, we compute

$$n_x(\pi) = (q+1)q^r - 2\big(\tfrac{1}{2}(q+1)(q^r - 1)\big) = q + 1.$$

If π is a supercuspidal representation corresponding to a ramified torus, then its depth is half-integral, whence for a vertex x we have $G_{x,r} = G_{x,r+}$, and thus by definition of depth $\pi^{G_{x,r+}} = \{0\}$. On the other hand, by Table 1, $\mathrm{Nil}(\Gamma)$ consists of two nonzero orbits which will be of opposite parity depth at any vertex x. Since $\lfloor r \rfloor = r - \frac{1}{2}$, Lemma 6.6 yields $n_x(\pi) = 0 - \frac{1}{2}(q+1)(q^{r-1/2} - 1)$.

type of torus T	split	unramified		ramified
		$x_T \sim x$	$x_T \not\sim x$	
depth r	$r \in \mathbb{Z}_{>0}$	$r \in \mathbb{Z}_{>0}$	$r \in \mathbb{Z}_{>0}$	$r \in \frac{1}{2}+\mathbb{Z}_{\geq 0}$
$\dim(\pi(T,\chi)^{G_{x,r+}})$	$(q+1)q^r$	$(q-1)q^r$	0	0
$n_x(\pi)$	$q+1$	$q-q^r$ (r even) $1-q^r$ (r odd)	$1-q^r$ (r even) $q-q^r$ (r odd)	$\frac{1}{2}(1-q^{r-1/2})(q+1)$

Table 2. The values of $n_x(\pi)$ appearing in (6-3) for each irreducible admissible representation of $\mathrm{SL}(2, F)$ of depth $r > 0$.

Finally, suppose that π is a supercuspidal representation corresponding to an unramified torus. Then $\mathrm{Nil}(\Gamma)$ consists of two nonzero orbits of the same parity depth, and by Lemma 4.2, at any vertex x, the parity of the depths of elements of these orbits is that of $d_x(\Gamma)$. There are thus two cases.

If some G-conjugate of T is contained in G_x, then (replacing T and π by this conjugate) we have $x_T = x$ and $\pi^{G_{x,r+}} = \mathrm{Ind}_{TG_{x,r/2}}^{G_x} \kappa$. It follows from a calculation in [Nevins 2013, Proposition 4.8] that independently of the parity of $r \in \mathbb{Z}$ we have $\dim(\pi^{G_{x,r+}}) = (q-1)q^r$. Since the orbits that occur in $\mathrm{Nil}(\Gamma)$ have the same parity as $-r = d_{x_T}(\Gamma)$, we have by (6-4) and Lemma 6.6 that

$$n_x(\pi) = \begin{cases} (q-1)q^r - 2\big(\frac{1}{2}q(q^r-1)\big) = q-q^r & \text{if } r \text{ is even,} \\ (q-1)q^r - 2\big(\frac{1}{2}(q^{r+1}-1)\big) = 1-q^r & \text{if } r \text{ is odd.} \end{cases}$$

On the other hand, if T is not conjugate to a torus contained in G_x, then $d_x(\Gamma)$ and $-r = d_{x_T}(\Gamma)$ have opposite parity, and $\mathrm{Res}_{G_x}\pi(T,\chi)$ has no components of depth r. Thus $\dim(\pi^{G_{x,r+}}) = 0$ and we compute instead that

$$n_x(\pi) = \begin{cases} 0 - 2\big(\frac{1}{2}(q^r-1)\big) = 1-q^r & \text{if } r \text{ is even,} \\ 0 - 2\big(\frac{1}{2}q(q^{r-1}-1)\big) = q-q^r & \text{if } r \text{ is odd.} \end{cases}$$ □

7. The case of depth-zero representations of SL(2, F)

To establish the theorem for a depth-zero representation π of $\mathrm{SL}(2, F)$, we apply a result by Barbasch and Moy [1997] relating the wave front set of π to that of $\pi^{G_{x,0+}}$, viewed as a representation of $\mathrm{SL}(2, \mathfrak{f}) \cong G_{x,0}/G_{x,0+}$ (Proposition 7.2). We begin by recalling the representation theory of $\mathrm{SL}(2, \mathfrak{f})$ and then the classification of depth-zero representations of $\mathrm{SL}(2, F)$.

7A. ***Representations of*** **SL(2, $\mathfrak{f}$).** This theory is well known and is beautifully recapped in [Digne and Michel 1991, §15]. Let $\mathsf{G} = \mathrm{SL}(2, \mathfrak{f})$, T a maximal torus of G and $\bar{\chi}$ a character of T (which is assumed to be nontrivial if T is anisotropic).

The irreducible representations of G, when $p \neq 2$, are parametrized by these pairs $(\mathsf{T}, \bar{\chi})$ as follows.

If T is split and $\bar{\chi}^2 \neq \mathbf{1}$ then $\sigma(\mathsf{T}, \bar{\chi})$ is an irreducible principal series representation; if T is anisotropic and $\bar{\chi}^2 \neq \mathbf{1}$ then $\sigma(\mathsf{T}, \bar{\chi})$ is a (Deligne–Lusztig) cuspidal representation. If T is split and $\bar{\chi} = \mathbf{1}$ then $\sigma(\mathsf{T}, \bar{\chi}) = \mathbf{1} \oplus \overline{\mathrm{St}}$, where $\overline{\mathrm{St}}$ denotes the Steinberg representation of G.

For either T, when $\bar{\chi}$ is a strictly quadratic character, we obtain two irreducible representations $\sigma^u(\mathsf{T}, \bar{\chi})$ for $u \in \{1, \varepsilon\}$ (as the components of the restriction $\sigma(\mathsf{T}, \bar{\chi})$ to $\mathrm{SL}(2, \mathfrak{f})$ of a corresponding (irreducible) representation of $\mathrm{GL}(2, \mathfrak{f})$). They are distinguished by the theory of Gelfand–Graev representations, as follows.

Let $X \in \mathfrak{g}(\mathfrak{f})^* \smallsetminus \{0\}$ be nilpotent, and identify X with a nilpotent element $\dot{X} \in \mathfrak{g}(\mathfrak{f})$. Complete this to an $\mathfrak{sl}(2, \mathfrak{f})$ triple $\{\dot{Y}, \dot{H}, \dot{X}\}$ and let $\mathfrak{u}(\mathfrak{f}) = \mathfrak{f}\dot{Y}$. Then X defines a character of $\mathsf{U} = \exp(\mathfrak{u}(\mathfrak{f}))$ by $\psi_X(\exp(W)) = \psi(X(W))$ for all $W \in \mathfrak{u}(\mathfrak{f})$. The (highly reducible) representation of $\mathrm{SL}(2, \mathfrak{f})$ given by

$$\gamma_{\mathcal{O}} = \mathrm{Ind}_{\mathsf{U}}^{\mathsf{G}} \psi_X \tag{7-1}$$

depends (up to equivalence) only on the nonzero orbit $\mathcal{O} = \mathsf{G} \cdot X$, and is called the Gelfand–Graev representation of G associated to $\mathcal{O}$.

Contrary to convention, we parametrize our nonzero nilpotent orbits by upper triangular matrices $\dot{X}_u \in \mathfrak{g}(\mathfrak{f})$ as in (4-1), where $u \in \mathfrak{f}^\times/(\mathfrak{f}^\times)^2 \sim \{1, \varepsilon\}$. We compute the character $[\gamma_{\mathcal{O}_u}]$ directly, noting that $[\gamma_{\mathcal{O}_u}](g) = 0$ if g is not conjugate to an element of $\mathfrak{u}(\mathfrak{f})$, and that for any $s \neq 0$,

$$\psi_{X_u}\left(\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\right) = \psi(-us).$$

This yields

$$[\gamma_{\mathcal{O}_u}](g) = \begin{cases} q^2 - 1 & \text{if } g = I, \\ 2\sum_{t \in (\mathfrak{f}^\times)^2} \psi(-ust) & \text{if } g \sim \left[\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right], \\ 0 & \text{otherwise.} \end{cases}$$

By [Digne and Michel 1991, Theorem 14.30], the decomposition into irreducible subrepresentations of $\gamma_{\mathcal{O}_u}$ is multiplicity-free. Using character tables it is straightforward to compute that $\gamma_{\mathcal{O}_u}$ contains all irreducible principal series representations, all Deligne–Lusztig cuspidal representations, the Steinberg representation, and exactly one from each pair of representations arising from quadratic characters. Our parametrization is therefore as follows: for $u \in \mathfrak{f}^\times/(\mathfrak{f}^\times)^2$, and a quadratic character $\bar{\chi}$ of T, let $\sigma^u(\mathsf{T}, \bar{\chi})$ denote the component of $\sigma(\mathsf{T}, \bar{\chi})$ occurring in $\gamma_{\mathcal{O}_u}$. In the notation of [Digne and Michel 1991, §15], where $\sigma_a := \sum_{t \in (\mathfrak{f}^\times)^2} \psi(ta)$, the quadratic character of the split torus is denoted by χ_{α_0} and that of the anisotropic torus is denoted by χ_{ω_0}. For $\rho \in \{\chi_{\alpha_0}, \chi_{\omega_0}\}$, the characters of these components are

correspondingly labelled $\rho^{\pm}$, where $[\sigma^{-1}(\mathsf{T}, \rho)] = \rho^{+}$ and $[\sigma^{-\varepsilon}(\mathsf{T}, \rho)] = \rho^{-}$. In the sequel, we sometimes denote the quadratic character of the split torus T by sgn.

7B. ***Depth-zero representations of*** **SL(2, *F*).** Now let $G = \mathrm{SL}(2, F)$ and let χ be a depth-zero character of a maximal split or unramified torus. Assume χ is nontrivial if the torus is nonsplit. There are two nonconjugate choices for an unramified anisotropic torus. If x_0 and x_1 are the vertices of the standard alcove, as before, then we can choose representatives T^i of the conjugacy classes of maximal tori such that $T^i \subset G_{x_i}$, for $i \in \{0, 1\}$. Then $\mathsf{T}_i = T^i_0 / T^i_{0+}$ is a maximal anisotropic torus of $G_{x_i,0}/G_{x_i,0+} =: \mathsf{G}_i \cong \mathrm{SL}(2, \mathfrak{f})$. Let T denote the split torus corresponding to the standard apartment and set $\mathsf{T} = T_0/T_{0+}$, which is a maximal split torus of both G_0 and G_1. In each case, the character χ factors to a character $\overline{\chi}$ of the quotient.

When T^i is anisotropic, for each $i \in \{0, 1\}$ inflate the representation $\sigma(\mathsf{T}_i, \overline{\chi})$ of G_i to a representation of G_{x_i} and, if $\chi^2 \neq \mathbf{1}$, define $\pi(T^i, \chi) = \text{c-Ind}_{G_{x_i}}^{G} \sigma(\mathsf{T}_i, \overline{\chi})$. When $\chi^2 = \mathbf{1}$ but $\chi \neq \mathbf{1}$, set $\pi^u(T^i, \chi) = \text{c-Ind}_{G_{x_i}}^{G} \sigma^u(\mathsf{T}_i, \overline{\chi})$ for $u \in \{1, \varepsilon\}$, using the notation of Section 7A. These representations are supercuspidal and irreducible [Moy and Prasad 1996, Proposition 6.6]; the latter four were called the special representations. Note that they are related via $\eta = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right] \in \mathrm{GL}(2, F)$ as follows: we have ${}^{\eta}\pi^*(T^0, \chi) = \pi^*(T^1, {}^{\eta}\chi)$, where $*$ indicates that this applies both to the special and nonspecial representations. It follows from [Nevins 2013, Theorem 5.3] that the depth-zero component $\pi^*(T^i, \chi)^{G_{x_i,0+}}$ is the inflation to G_{x_i} of $\sigma^*(\mathsf{T}_i, \overline{\chi})$, but $\pi^*(T^i, \chi)^{G_{x,0+}} = \{0\}$ if $x \in \{x_0, x_1\} \smallsetminus \{x_i\}$.

If T is split, contained in a Borel subgroup B, then $\pi(T, \chi) = \mathrm{Ind}_B^G \chi$ is again in the principal series. It is immediate to see that for any vertex x, $\pi(T, \chi)^{G_{x,0+}} \cong \sigma(\mathsf{T}, \overline{\chi})$ under the isomorphism $G_{x,0}/G_{x,0+} \cong \mathrm{SL}(2, \mathfrak{f})$; note that this will be reducible whenever $\overline{\chi}$ is a quadratic character.

We summarize the results of the preceding two paragraphs in Table 3, and then address the remaining depth-zero irreducible representations, which are the reducible principal series [Tadić 1994, §7], below.

When $\chi \in \{\nu, \nu^{-1}\}$, where ν is the square root of the modular character, the Jordan–Hölder factors of $\pi(T, \chi)$ are the trivial representation and the Steinberg representation St, and we have $\mathrm{St}^{G_{x,0+}} = \overline{\mathrm{St}}$ and $\mathbf{1}^{G_{x,0+}} = \mathbf{1}$.

In the remaining cases, χ is quadratic. There are three such, corresponding to the distinct quadratic extensions $E = F[\sqrt{\tau}]$ of F. We write $\chi = \mathsf{sgn}_{\tau}$ for the quadratic character whose kernel is the image of the corresponding norm map. As described in [Nevins 2005, §8], there is for each such χ a realization of $\pi(T, \chi)$ on the space $L^2(F^{\times})$ such that its irreducible summands are $H^{\tau}_{\pm}$, where H^{τ}_{+} consists of the functions supported $N^{\tau}_{+} = \ker(\mathsf{sgn}_{\tau})$ and H^{τ}_{-} those supported on its complement. Note that $\varpi \in \ker(\mathsf{sgn}_{-\varpi})$, for example; thus if we parametrize the quadratic extensions by $\tau \in \{\varepsilon, -\varpi, -\varepsilon\varpi\}$, then we obtain the more pleasing uniform description below.

T	type	π	$\pi^{G_{x,0+}}$
T split	principal series Steinberg trivial	$\pi(T,\chi)$ St **1**	$\sigma(\mathsf{T},\bar{\chi})$ $\overline{\mathrm{St}}$ **1** (possibly reducible)
T^i unramified $i\in\{0,1\}$	supercuspidal	$\pi(T^i,\chi)$	$\sigma(\mathsf{T}_i,\bar{\chi})$ if $x\sim x_i$ $\{0\}$ if $x\not\sim x_i$
	special supercuspidal $u\in\{1,\varepsilon\}$	$\pi^u(T^i,\chi)$	$\sigma^u(\mathsf{T}_i,\bar{\chi})$ if $x\sim x_i$ $\{0\}$ if $x\not\sim x_i$

Table 3. The depth-zero representations of G_x occurring in the restriction to G_x of the irreducible principal series, Steinberg, trivial and supercuspidal representations, for any vertex $x\in\mathcal{B}(G)$.

Proposition 7.1. *For each $\tau\in\{\varepsilon,-\varpi,-\varepsilon\varpi\}$ and $i\in\{0,1\}$, the G_i-representations $(H^\tau_\pm)^{G_{x_i,0+}}$ are irreducible and their isomorphism classes are given in Table 4.*

Proof. Without loss of generality we may assume T is such that $x_0, x_1\in\mathcal{A}(G,T)$. Note that if the extension $F[\sqrt{\tau}]$ is unramified then $\overline{\mathsf{sgn}}_\tau=\mathbf{1}$ but in the remaining cases $\overline{\mathsf{sgn}}_\tau=\mathsf{sgn}$, the quadratic character of the split torus T in G. Since $\pi(T,\chi)^{G_{x,0+}}\cong\sigma(\mathsf{T},\bar{\chi})$, which is reducible in all of these cases, we infer

$$\pi(T,\mathsf{sgn}_\tau)^{G_{x_i,0+}}=(H^\tau_+)^{G_{x_i,0+}}\oplus(H^\tau_-)^{G_{x_i,0+}}$$
$$\cong\begin{cases}\mathbf{1}\oplus\mathrm{St} & \text{if } \tau=\varepsilon,\\ \sigma^1(\mathsf{T},\mathsf{sgn})\oplus\sigma^\varepsilon(\mathsf{T},\mathsf{sgn}) & \text{otherwise.}\end{cases}$$

The character of $(H^\tau_\pm)^{G_{x_0,0+}}$ was computed in [Nevins 2005, Theorem 9.1]. In the unramified case, this gives $(H^\varepsilon_+)^{G_{x_0,0+}}=\overline{\mathrm{St}}$ and $(H^\varepsilon_-)^{G_{x_0,0+}}=\mathbf{1}$. For the ramified case, first note that in the notation of that paper, of [Digne and Michel 1991, §15], and ours, respectively, we have $\Xi^+_{\mathsf{sgn}}=\chi^-_{\alpha_0}=[\sigma^{-\varepsilon}(\mathsf{T},\mathsf{sgn})]$ and $\Xi^-_{\mathsf{sgn}}=\chi^+_{\alpha_0}=[\sigma^{-1}(\mathsf{T},\mathsf{sgn})]$. The theorem states, for the ramified case, that the character of $(H^\tau_+)^{G_{x_0,0+}}$ is Ξ^+_{sgn} when $-1\notin(F^\times)^2$ and Ξ^-_{sgn} otherwise. In our notation this is exactly the character of $\sigma^1(\mathsf{T},\mathsf{sgn})$. This completes the first row of Table 4.

For the character of $(H^\tau_\pm)^{G_{x_1,0+}}$, the proof of [Nevins 2005, Corollary 9.3] showed that twisting $\pi(T,\mathsf{sgn}_\tau)$ by $\omega=\left[\begin{smallmatrix}0&1\\ \varpi&0\end{smallmatrix}\right]\in\mathrm{GL}(2,F)$, which interchanges these vertices, preserves $H^\tau_\pm$ when $\mathsf{sgn}_\tau(-\varpi)=1$ and interchanges them otherwise (including in the unramified case). Applying this twist to $\pi^{G_{x_0,0+}}$ yields $\pi^{G_{x_1,0+}}$. The entries for $(H^\varepsilon_\pm)^{G_{x_1,0+}}$ follow.

For the ramified case, note that twisting by ω takes $\mathcal{O}_u$ to $\mathcal{O}_{-u\varpi}$ and so it maps the Gelfand–Graev representation $\gamma_{\mathcal{O}_u}$ of G_{x_0} to the representation $\gamma_{\mathcal{O}_{-u}}$ of G_{x_1}.[2]

[2]This calculation was neglected in the proof of [Nevins 2005, Corollary 9.3], yielding an incorrect statement for the depth-zero components.

π	H_+^ε	H_-^ε	$H_+^{-\varpi}$	$H_-^{-\varpi}$	$H_+^{-\varepsilon\varpi}$	$H_-^{-\varepsilon\varpi}$
$\pi^{G_{x_0,0+}}$	$\overline{\mathrm{St}}$	$\mathbf{1}$	$\sigma^1(\mathsf{T},\mathsf{sgn})$	$\sigma^\varepsilon(\mathsf{T},\mathsf{sgn})$	$\sigma^1(\mathsf{T},\mathsf{sgn})$	$\sigma^\varepsilon(\mathsf{T},\mathsf{sgn})$
$\pi^{G_{x_1,0+}}$	$\mathbf{1}$	$\overline{\mathrm{St}}$	$\sigma^1(\mathsf{T},\mathsf{sgn})$	$\sigma^\varepsilon(\mathsf{T},\mathsf{sgn})$	$\sigma^\varepsilon(\mathsf{T},\mathsf{sgn})$	$\sigma^1(\mathsf{T},\mathsf{sgn})$

Table 4. The isomorphism classes of the depth-zero representations of G_{x_i} occurring in the restriction to G_{x_i} of the decomposable principal series.

Therefore, twisting by ω sends the inflation of the representation $\sigma^u(\mathsf{T},\mathsf{sgn})$ of G_{x_0} to the inflation of the representation $\sigma^{-u}(\mathsf{T},\mathsf{sgn})$ of G_{x_1}.

Thus for example, if $-1 \in (F^\times)^2$, then ${}^\omega H_+^{-\varpi} = H_+^{-\varpi}$ and $\sigma^{-1}(\mathsf{T},\mathsf{sgn}) = \sigma^1(\mathsf{T},\mathsf{sgn})$, whereas if -1 is not a square, then ${}^\omega H_+^{-\varpi} = H_-^{-\varpi}$ and $\sigma^{-1}(\mathsf{T},\mathsf{sgn}) = \sigma^\varepsilon(\mathsf{T},\mathsf{sgn})$. A careful accounting of signs completes the second row of the table. □

7C. ***Wave front sets.*** The wave front set is determined with the following result, which is based on [Barbasch and Moy 1997, Theorem 4.5].

Proposition 7.2. *Let π be an irreducible admissible nontrivial representation of depth zero of* $\mathrm{SL}(2,F)$. *Suppose* $\mathrm{char}(F) = 0$ *and* $p > 3e+1$, *where e is the absolute ramification index of F over $\mathbb{Q}_p$. Then we have*

$$\mathcal{WF}(\pi) = \left\{ \mathcal{O} \in \mathscr{O}(0) \;\middle|\; \begin{array}{l} \text{there exists } x, \text{ a vertex of } \mathcal{B}(G), \text{ such that } \mathrm{pd}_x(\mathcal{O}) \text{ is} \\ \text{even, and there exists } \sigma, \text{ an irreducible constituent} \\ \text{of } \pi^{G_{x,0+}}, \text{ such that } \bar\sigma \text{ occurs in } \gamma_{\overline{\mathcal{O}}} \end{array} \right\},$$

where $\gamma_{\overline{\mathcal{O}}}$ is the Gelfand–Graev representation (7-1) attached to the nonzero nilpotent orbit in $\mathfrak{g}_{x,0}/\mathfrak{g}_{x,0+}$ under $\mathsf{G}_x \cong \mathrm{SL}(2,\mathfrak{f})$ whose inflation to $\mathfrak{g}_{x,0}$ meets $\mathcal{O}$.

Proof. The hypotheses imply that exp converges on $\mathfrak{g}_{0+}$ and that the local character expansion holds. Barbasch and Moy [1997] used (generalized) Gelfand–Graev characters as test functions to determine the wave front set of π. For each nilpotent orbit $\mathcal{O}$ that is represented by a depth-zero coset at the vertex x (meaning, its parity depth at x is even), let $[\gamma_{\overline{\mathcal{O}}}]$ denote the lift to $G_{x,0}$ of the character of the corresponding Gelfand–Graev representation of $\mathsf{G}_x = G_{x,0}/G_{x,0+}$, viewed as a function on G. It is supported on the subset $G_{0+} \cap G_{x,0}$ of topologically unipotent elements. Let $f_{x,\mathcal{O}}$ be the function on $\mathfrak{g}$, with support in $\mathfrak{g}_{0+} \cap \mathfrak{g}_{x,0}$, that is given by $f_{x,\mathcal{O}} = [\gamma_{\overline{\mathcal{O}}}] \circ \exp$. They then show that $\widehat{\mu_{\mathcal{O}'}}(f_{x,\mathcal{O}}) = 0$ if $\mathcal{O}$ is not contained in the closure of $\mathcal{O}'$ and is nonzero if $\mathcal{O} = \mathcal{O}'$. Thus $\Theta_\pi([\gamma_{\overline{\mathcal{O}}}]) = 0$ for all $\mathcal{O}$ that do not meet the wave front set of π and $\Theta_\pi([\gamma_{\overline{\mathcal{O}}}]) \neq 0$ when $\mathcal{O} \in \mathcal{WF}(\pi)$.

For any irreducible representation $\bar\sigma$ of G_x, let $m(\sigma,\pi)$ denote the multiplicity of its inflation σ in $\pi^{G_{x,0+}}$ and $m(\bar\sigma, \gamma_{\overline{\mathcal{O}}})$ the multiplicity of $\bar\sigma$ in $\gamma_{\overline{\mathcal{O}}}$. Then [Barbasch

and Moy 1997, Theorem 4.5(4)] becomes

$$\Theta_\pi([\gamma_{\overline{\mathcal{O}}}]) = \sum_\sigma m(\sigma,\pi)m(\overline{\sigma},\gamma_{\overline{\mathcal{O}}}),$$

whence our result for the case of $\mathrm{SL}(2,F)$. □

Corollary 7.3. *Under the hypothesis of Proposition 7.2, the wave front sets corresponding to the depth-zero representations of* $\mathrm{SL}(2,F)$ *are as given in Table 5.*

Proof. For $u \in \{1,\varepsilon\}$ and $i,j \in \{0,1\}$, the nilpotent orbit $\mathcal{O}_{u\varpi^j}$ is represented by a depth-zero coset at x_i if and only if $i=j$, and in this case it corresponds to the nilpotent orbit in the quotient $\mathfrak{g}_{x_i,0}/\mathfrak{g}_{x_i,0+} \cong \mathfrak{sl}(2,\mathfrak{f})$ under $\mathsf{G}_i \cong \mathrm{SL}(2,\mathfrak{f})$ that we denoted by simply $\mathcal{O}_u$. Therefore the Gelfand–Graev representations $\gamma_{\overline{\mathcal{O}}}$ referred to in Proposition 7.2 are $\gamma_{\overline{\mathcal{O}}_1}$ and $\gamma_{\overline{\mathcal{O}}_\varepsilon}$ for $x=x_0$, and $\gamma_{\overline{\mathcal{O}}_\varpi}$ and $\gamma_{\overline{\mathcal{O}}_{\varepsilon\varpi}}$ for $x=x_1$. By conjugacy, these two vertices suffice. The decomposition of $\pi^{G_{x,0+}}$ for $x \in \{x_0,x_1\}$ was given in Tables 3 and 4 for all irreducible depth-zero representations π, and matching these with the decomposition of the Gelfand–Graev representations of the corresponding groups G_i as in Section 7A yields Table 5. □

This table is consistent with the computation of the coefficients of the local character expansion for $\mathrm{SL}(2,F)$ using the existence of Whittaker models in [Assem 1994, §3].

Claim. *For each depth-zero irreducible representation* π *of* $\mathrm{SL}(2,F)$ *there exists an element* $\Gamma \in \mathfrak{g}^*_{x,0}$, *for some* $x \in \mathcal{B}(G)$, *such that* $\mathcal{WF}(\pi) = \mathrm{Nil}(\Gamma)$.

Proof of claim. Existence follows immediately from Table 5 and Lemma 4.2, though the elements Γ for which $\mathcal{WF}(\pi) = \mathrm{Nil}(\Gamma)$ do not correspond to minimal K-types for π (as these latter are not realized by elements on the Lie algebra). However, on an ad hoc basis, we can make this association of π with Γ more explicit, as follows.

For T unramified or split, and $x \in \mathcal{B}(T) \subset \mathcal{B}(G)$, we can in the same spirit attach to any *regular* $\pi(T,\chi)$ (in the sense of Kaletha [2019, Proposition 3.4.27] in the first case and [Tadić 1994, §7] in the second) *any* regular depth-zero element of $\mathfrak{t}^*$, that is, an element $\Gamma \in \mathfrak{g}^*_{x,0} \smallsetminus \mathfrak{g}^*_{x,0+}$ whose centralizer in G is T. The same holds for $\pi = \mathrm{St}$, whereas we associate $\Gamma = 0$ to $\mathbf{1}$.

When $\pi^u(T^i,\chi)$ is a special representation (for some $u \in \{1,\varepsilon\}$ and $i \in \{0,1\}$ and χ quadratic) then it is a supercuspidal unipotent representation and Γ is chosen to be a nilpotent element in the lift to $\mathfrak{g}^*_{x_i,0}$ of the nilpotent orbit corresponding to $\sigma^u(\mathsf{T}_i,\overline{\chi})$.

When $\pi \in \{H^\tau_\pm \mid \tau \in \{\varepsilon,-\varpi,-\varepsilon\varpi\}\}$, Γ is a choice of element of an anisotropic torus T that splits over $F[\sqrt{\tau}]$. However, while the orbit of Γ satisfies $\mathrm{Nil}(\Gamma) = \mathcal{WF}(\pi)$, when $-1 \in (F^\times)^2$ and $\tau \in \{\varpi,\varepsilon\varpi\}$, the centralizer may be one of two possible tori $T = \mathrm{Cent}_G(\Gamma)$ up to conjugacy, and neither one is expressly associated to π. □

representation π	$\mathcal{WF}(\pi)$
$\pi(T, \chi)$ irreducible principal series	$\{1, \varepsilon, \varpi, \varepsilon\varpi\}$
$\pi(T^0, \chi)$ irreducible supercuspidal	$\{1, \varepsilon\}$
$\pi(T^1, \chi)$ irreducible supercuspidal	$\{\varpi, \varepsilon\varpi\}$

π	$\mathcal{WF}(\pi)$	π	$\mathcal{WF}(\pi)$
$\mathbf{1}$	$\{0\}$	St	$\{1, \varepsilon, \varpi, \varepsilon\varpi\}$
H_+^{ε}	$\{1, \varepsilon\}$	H_-^{ε}	$\{\varpi, \varepsilon\varpi\}$
$H_+^{-\varpi}$	$\{1, \varpi\}$	$H_-^{-\varpi}$	$\{\varepsilon, \varepsilon\varpi\}$
$H_+^{-\varepsilon\varpi}$	$\{1, \varepsilon\varpi\}$	$H_-^{-\varepsilon\varpi}$	$\{\varepsilon, \varpi\}$
$\pi^1(T^0, \chi)$	$\{1\}$	$\pi^{\varepsilon}(T^0, \chi)$	$\{\varepsilon\}$
$\pi^1(T^1, \chi)$	$\{\varpi\}$	$\pi^{\varepsilon}(T^1, \chi)$	$\{\varepsilon\varpi\}$

Table 5. For each irreducible depth-zero representation of $\mathrm{SL}(2, F)$, we list under the heading $\mathcal{WF}(\pi)$ the set of elements $u \in \{0, 1, \varepsilon, \varpi, \varepsilon\varpi\}$ such that $\mathcal{O}_u \in \mathcal{WF}(\pi)$.

Note we may define $\mathcal{WF}(\pi)$ by Table 5, even over fields where Proposition 7.2 does not apply. Our main result, below, expresses that, just as in the positive-depth case, this is consistent (for all fields with residual characteristic different from 2).

Theorem 7.4. *Let π be an irreducible admissible representation of G of depth zero with central character ζ. For any vertex $x \in \mathcal{B}(G)$, we have*

$$\mathrm{Res}_{G_x} \pi \cong \pi^{G_{x,0+}} \oplus \bigoplus_{\mathcal{O} \in \mathcal{WF}(\pi)} \tau_x(\mathcal{O}, \zeta), \tag{7-2}$$

where $\mathcal{WF}(\pi)$ is as in Table 5. It follows that $\mathrm{Res}_{G_{x,0+}} \pi$ takes the form of (6-3) with constant coefficient $n_x(\pi) = \dim(\pi^{G_{x,0+}})$.

Proof. The decomposition will follow from the main results of [Nevins 2005; 2013], applied to $x \in \{x_0, x_1\}$, as in the proof of Theorem 6.2. Let π be a depth-zero representation of G with central character ζ.

For irreducible depth-zero principal series, one has $\gamma_0 = \gamma_1 = 0$ in [Nevins 2005, Theorem 7.4]. Matching notation as in (6-2), we conclude that $\mathcal{S}_x(\varpi^{-d} X_u, \zeta)$ occurs in $\mathrm{Res}_{G_x} \pi$, for $u \in \{1, \varepsilon\}$, for each $d > 0$, and that these exhaust the irreducible summands. Therefore the summands can be regrouped as the sum of $\tau_x(\mathcal{O}, \zeta)$, as defined in (5-8), over all regular nilpotent orbits, as required. As the positive-depth summands of $\mathrm{Res}_{G_x} \pi$ are identical for all depth-zero irreducible principal series, the case of $\pi = \mathrm{St}$ follows since $\mathrm{Res}_{G_x} \mathbf{1}$ has no positive-depth components.

For the remaining reducible principal series, we use [Nevins 2005, Theorem 9.2], together with Proposition 7.1. Noting that $\mathrm{sgn}_{-\varpi}(\varpi) = 1$ and $\mathrm{sgn}_{-\varepsilon\varpi}(\varpi) = -1$,

the theorem states

$$\begin{aligned}&\mathrm{Res}_{G_{x_0}} H_+^\tau\\ &= \begin{cases} \mathrm{St} \oplus \bigoplus_{d>0} \big(\mathcal{S}_{x_0}(\varpi^{-2d} X_1, \zeta) \oplus \mathcal{S}_{x_0}(\varpi^{-2d} X_\varepsilon, \zeta)\big) & \text{if } \tau = \varepsilon, \\ \sigma^1(\mathsf{T}, \mathrm{sgn}) \oplus \bigoplus_{d>0} \mathcal{S}_{x_0}(\varpi^{-d} X_1, \zeta) & \text{if } \tau = -\varpi, \\ \sigma^1(\mathsf{T}, \mathrm{sgn}) \oplus \bigoplus_{d>0} \big(\mathcal{S}_{x_0}(\varpi^{-2d} X_1, \zeta) \oplus \mathcal{S}_{x_0}(\varpi^{-2d+1} X_\varepsilon, \zeta)\big) & \text{if } \tau = -\varepsilon\varpi. \end{cases}\end{aligned}$$

Regrouping the positive-depth summands yields the decomposition

$$\mathrm{Res}_{G_{x_0}} H_+^\tau = \begin{cases} \mathrm{St} \oplus \tau_{x_0}(\mathcal{O}_1, \zeta) \oplus \tau_{x_0}(\mathcal{O}_\varepsilon, \zeta) & \text{if } \tau = \varepsilon, \\ \sigma^1(\mathsf{T}, \mathrm{sgn}) \oplus \tau_{x_0}(\mathcal{O}_1, \zeta) \oplus \tau_{x_0}(\mathcal{O}_\varpi, \zeta) & \text{if } \tau = -\varpi, \\ \sigma^1(\mathsf{T}, \mathrm{sgn}) \oplus \tau_{x_0}(\mathcal{O}_1, \zeta) \oplus \tau_{x_0}(\mathcal{O}_{\varepsilon\varpi}, \zeta) & \text{if } \tau = -\varepsilon\varpi, \end{cases} \tag{7-3}$$

which is consistent with the wave front set computed in Table 5. As noted above, the positive-depth summands of $H_+^\tau \oplus H_-^\tau$ form $\bigoplus_{\mathcal{O} \in \mathscr{O}(0) \smallsetminus \{0\}} \tau_{x_0}(\mathcal{O}, \zeta)$, and the wave front sets of these representations are complementary, so this yields the result for $\mathrm{Res}_{G_{x_0}} H_-^\tau$ as well.

To determine $\mathrm{Res}_{G_{x_1}} \pi$ we proceed as in the proof of Proposition 7.1. Conjugation by ω interchanges the components of the principal series if and only if $\mathrm{sgn}_\tau(\varpi) = -1$, and ${}^\omega\tau_{x_0}(\mathcal{O}_u, \zeta) = \tau_{x_1}(\mathcal{O}_{-u\varpi}, \zeta)$. Thus when $\mathrm{sgn}_\tau(\varpi) = -1$ (the first three lines below) we obtain the decomposition of H_-^τ as the ω-conjugate of (7-3); when $\mathrm{sgn}_\tau(\varpi) = 1$ (the last two lines), then $H_-^\tau = {}^\omega H_-^\tau$ so we first take the complement of (7-3). (The depth-zero components are taken from Proposition 7.1.) This yields

$$\begin{aligned}&\mathrm{Res}_{G_{x_1}} H_-^\tau\\ &= \begin{cases} \mathrm{St} \oplus \tau_{x_1}(\mathcal{O}_\varpi, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{\varepsilon\varpi}, \zeta) & \text{if } \tau = \varepsilon, \\ \sigma^{-1}(\mathsf{T}, \mathrm{sgn}) \oplus \tau_{x_1}(\mathcal{O}_{-\varpi}, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{-\varpi^2}, \zeta) & \text{if } \tau = -\varpi \text{ and } -1 \notin (F^\times)^2, \\ \sigma^{-1}(\mathsf{T}, \mathrm{sgn}) \oplus \tau_{x_1}(\mathcal{O}_{-\varpi}, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{-\varepsilon\varpi^2}, \zeta) & \text{if } \tau = -\varepsilon\varpi \text{ and } -1 \in (F^\times)^2, \\ \sigma^{-\varepsilon}(\mathsf{T}, \mathrm{sgn}) \oplus \tau_{x_1}(\mathcal{O}_{-\varepsilon\varpi}, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{-\varepsilon\varpi^2}, \zeta) & \text{if } \tau = -\varpi \text{ and } -1 \in (F^\times)^2, \\ \sigma^{-\varepsilon}(\mathsf{T}, \mathrm{sgn}) \oplus \tau_{x_1}(\mathcal{O}_{-\varepsilon\varpi}, \zeta) \oplus \tau_{x_1}(\mathcal{O}_{-\varpi^2}, \zeta) & \text{if } \tau = -\varepsilon\varpi \text{ and } -1 \notin (F^\times)^2. \end{cases}\end{aligned}$$

Therefore, in any case, the nilpotent orbits arising in $\mathrm{Res}_{G_{x_1}} H_-^\varepsilon$ are $\{\mathcal{O}_\varpi, \mathcal{O}_{\varepsilon\varpi}\}$; those arising in $\mathrm{Res}_{G_{x_1}} H_-^{-\varpi}$ are $\{\mathcal{O}_\varepsilon, \mathcal{O}_{\varepsilon\varpi}\}$; and those arising in $\mathrm{Res}_{G_{x_1}} H_-^{-\varepsilon\varpi}$ are $\{\mathcal{O}_\varpi, \mathcal{O}_\varepsilon\}$, which again is consistent with Table 5, as required.

Now suppose that $\pi_i = \text{c-Ind}_{G_{x_i}}^G \sigma$ is a nonspecial supercuspidal representation. Translating the notation of [Nevins 2013, Proposition 5.2], we have $\pi_d^+(\theta) := \mathcal{S}_{x_0}(\varpi^{-d} X_{-1}, \theta)$ and $\pi_d^-(\theta) := \mathcal{S}_{x_0}(\varpi^{-d} X_{-\varepsilon}, \theta)$. Theorem 5.3 of [Nevins 2013] yields

$$\begin{aligned}\mathrm{Res}_{G_{x_0}} \pi_i &= \begin{cases} \sigma \oplus \bigoplus_{t>0} \big(\mathcal{S}_{x_0}(\varpi^{-2t} X_{-1}, \zeta) \oplus \mathcal{S}_{x_0}(\varpi^{-2t} X_{-\varepsilon}, \zeta)\big) & \text{if } i = 0, \\ \bigoplus_{t>0} \big(\mathcal{S}_{x_0}(\varpi^{-2t+1} X_{-1}, \zeta) \oplus \mathcal{S}_{x_0}(\varpi^{-2t+1} X_{-\varepsilon}, \zeta)\big) & \text{if } i = 1, \end{cases}\\ &= \begin{cases} \sigma \oplus \tau_{x_0}(\mathcal{O}_1, \zeta) \oplus \tau_{x_0}(\mathcal{O}_\varepsilon, \zeta) & \text{if } i = 0, \\ \tau_{x_0}(\mathcal{O}_\varpi, \zeta) \oplus \tau_{x_0}(\mathcal{O}_{\varepsilon\varpi}, \zeta) & \text{if } i = 1. \end{cases}\end{aligned}$$

Here, we have used that in [Nevins 2013, Theorem 5.3], $\eta = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right]$ and thus ${}^{\eta}\mathcal{O}_u = \mathcal{O}_{u\varpi}$.

Finally, for the special supercuspidal representations, corresponding to a quadratic character χ, note from [Nevins 2013, Proof of Proposition 5.2] that σ_0^+ corresponds to the character $\chi_{\omega_0}^{\pm}$ of [Digne and Michel 1991, §15] (see Section 7A) so in our notation here, $\sigma_0^+ := \sigma^{-1}(T^0, \chi)$ and $\sigma_0^- = \sigma^{-\varepsilon}(T^0, \chi)$. Since ${}^{\eta}\mathcal{O}_u = \mathcal{O}_{u\varpi}$, twisting by η sends the inflation of the representation $\sigma^u(\mathsf{T}, \bar{\chi})$ of G_{x_0} to the inflation of the representation $\sigma^u(\mathsf{T}, \bar{\chi})$ of G_{x_1}. We thus infer, for $u \in \{1, \varepsilon\}$, the decompositions

$$\begin{aligned}\mathrm{Res}_{G_{x_0}} \pi^u(T^i, \chi) &= \begin{cases} \sigma \oplus \bigoplus_{t>0} \mathcal{S}_{x_0}(-\varpi^{-2t} X_{-u}, \zeta) & \text{if } i = 0, \\ \bigoplus_{t>0} \mathcal{S}_{x_0}(-\varpi^{-2t+1} X_{-u}, \zeta) & \text{if } i = 1, \end{cases} \\ &= \begin{cases} \sigma \oplus \tau_{x_0}(\mathcal{O}_u, \zeta) & \text{if } i = 0, \\ \tau_{x_0}(\mathcal{O}_{u\varpi}, \zeta) & \text{if } i = 1, \end{cases}\end{aligned}$$

where again ζ is the corresponding central character. Comparing with Table 5, we conclude that (7-2) holds for $\mathrm{Res}_{G_{x_0}} \pi^u(T^i, \chi)$ in each case. The result for x_1 is obtained by conjugating by η.

Finally, the value of $n_x(\pi) = \dim(\pi^{G_{x,0+}})$ can be deduced from Tables 3 and 4: it is $q+1$ for irreducible principal series, $q-1$ for Deligne–Lusztig cuspidal representations, q for $\overline{\mathrm{St}}$, $(q-1)/2$ for the special unipotent representations and $(q+1)/2$ for the components of the reducible principal series. □

8. Applications

8A. ***The Fourier transform of a nilpotent orbital integral.*** As a first application, we derive a formula for the Fourier transform of a nilpotent orbital integral in any open set of the form $\mathfrak{g}_{x,0+}$ in terms of the trace characters of the representations $\tau_x(\mathcal{O}, \zeta)$.

Proposition 8.1. *Let $x \in \mathcal{B}(G)$ be a vertex. Let $[\tau_x(\mathcal{O})]$ denote the restriction to $G_{x,0+}^{\mathrm{reg}}$ of the trace character of the representation $\tau_x(\mathcal{O}, \zeta)$, for either choice of central character ζ. Assume* exp *converges on $\mathfrak{g}_{x,0+}$. Then for each nonzero nilpotent orbit $\mathcal{O}$ and $X \in \mathfrak{g}_{x,0+}^{\mathrm{reg}}$ we have*

$$\widehat{\mu_{\mathcal{O}}}(X) = \begin{cases} \frac{1}{2} q + [\tau_x(\mathcal{O})](\exp X) & \text{if } \mathcal{O} \text{ has even parity depth at } x, \\ \frac{1}{2} + [\tau_x(\mathcal{O})](\exp X) & \text{if } \mathcal{O} \text{ has odd parity depth at } x. \end{cases}$$

As x ranges over the vertices of $\mathcal{B}(G)$, these expressions determine the function $\widehat{\mu_{\mathcal{O}}}$ on $\mathfrak{g}_{1/2+}^{\mathrm{reg}}$.

Proof. Let π be a nontrivial irreducible admissible representation of depth $r \geq 0$, and let Θ_π denote its character. We assume the functions $\widehat{\mu_{\mathcal{O}}}$ are normalized as in [Mœglin and Waldspurger 1987], so that the coefficients $c_{\mathcal{O}}$ corresponding

type	representation of $\mathrm{SL}(2, F)$ of depth $r \geq 0$	coefficient c_0 of $\mu_{\{0\}}$ in local character expansion
principal series	irreducible	0
	irreducible summand	0
	St	-1
	1	1
supercuspidal	ramified case	$-q^{r-1/2}(q+1)/2$
	unramified, nonspecial	$-q^r$
	special unipotent	$-\frac{1}{2}$

Table 6. Values of the constant term in the local character expansion of irreducible admissible representations of $\mathrm{SL}(2, F)$.

to $\mathcal{O} \in \mathcal{WF}(\pi)$ in the local character expansion of $\Theta_\pi \circ \exp$ are all equal to 1. Thus on $\mathfrak{g}^{\mathrm{reg}}_{x,r+}$ we have

$$\Theta_\pi \circ \exp = c_0(\pi) + \sum_{\mathcal{O} \in \mathcal{WF}(\pi)} \widehat{\mu_{\mathcal{O}}}.$$

These constant terms in the case of $\mathrm{SL}(2, F)$ are well known and are summarized in Table 6. For example, for principal series, see [Assem 1994, Propositions 2.1 and 3.3.6], and for supercuspidal representations, see [DeBacker and Sally 2000, Tables 1–4].

Theorem 7.4, on the other hand, gives a formula for the character of any irreducible depth-zero representation on $G_{x,0+}$. Matching these for the special unipotent representations $\pi = \pi^u(T^i, \chi)$ (where $i \in \{0, 1\}$ and $u \in \{1, \varepsilon\}$) yields the given formula. It is moreover direct to verify the consistency of this expression across the local character expansions of all irreducible representations, including those of positive depth (on $G_{x,r+}$ as in Theorem 6.4). The result therefore holds on $\mathfrak{g}_{1/2+} = G \cdot (\mathfrak{g}_{x_0,0+} \cup \mathfrak{g}_{x_1,0+})$. □

Note that $\mathfrak{g}_{1/2+} \subsetneq \mathfrak{g}_{0+}$. One anticipates that Proposition 8.1 holds on all of $\mathfrak{g}_{0+}$, and that the restriction on the G-domain is an artefact of having considered only vertices in the present work.

Remark 8.2. Far more explicit formulae for the functions $\widehat{\mu_{\mathcal{O}}}$ have been computed for the group $\mathrm{SL}(2, F)$ in [Assem 1994; DeBacker and Sally 2000] among others. They have also noted that, under the exponential map, the characters of the five representations $\{1, \pi^u(T^i, \chi) \mid u \in \{1, \varepsilon\}, i \in \{0, 1\}\}$ (where χ denotes a quadratic character) form another basis for the span of the functions $\widehat{\mu_{\mathcal{O}}}$ on $\mathfrak{g}^{\mathrm{reg}}_{0+}$. It is these representations (and their generalizations for representations over arbitrary fields of characteristic different from p) that arise in the local representation-theoretic expansion of $\mathrm{SL}_2(F)$ given in [Henniart and Vignéras 2024, §6].

In fact the special unipotent representations have local character expansions of the form

$$\Theta_\pi(\exp(X)) = \widehat{\mu_{\mathcal{O}}}(X) - \tfrac{1}{2}, \tag{8-1}$$

for the single corresponding orbit $\mathcal{O}$, and this holds on the strictly larger set $\mathfrak{g}_{0+}^{\mathrm{reg}}$.

An advantage to Proposition 8.1 is the simplicity and explicitness of the construction, which uses no more than a vertex and a representative of the orbit as input. In this, it recalls some of the original formulae for these Fourier transforms of nilpotent orbital integrals in [Harish-Chandra 1999].

8B. ***Computing the polynomial*** $\dim(\pi^{G_{x,2n}})$**.** This arose from a question posed to me by Marie-France Vignéras in 2022 and gives a (not surprisingly) negative answer to [Henniart and Vignéras 2023, Question 1.1]: Is there a polynomial with *integer* coefficients such that $\dim(\pi^{G_{x,r+j}}) = P(p^j)$ for large enough j?

If π is an irreducible admissible representation of G, the local character expansion implies that $\dim(\pi^{G_{x,m}})$, for m even, is expressible as a polynomial in q, as described in [Barbasch and Moy 1997, §5.1]; see also [Henniart and Vignéras 2023, Remark 11.8]. Here we can obtain this polynomial as a corollary of Theorems 6.4 and 7.4, using the explicit values computed in Proposition 6.7.

Corollary 8.3. *Let π be an irreducible representation of $G = \mathrm{SL}(2, F)$ of depth r. Then for each* ***even*** *integer $m > r$, we have*

$$\dim(\pi^{G_{x,m}}) = \begin{cases} q^m + q^{m-1} & \text{if } \pi \text{ is an irreducible principal series,} \\ q^{m-1} - q^r & \text{if } \pi \text{ is supercuspidal nonspecial, from a vertex } \sim x, \\ q^m - q^r & \text{if } \pi \text{ is supercuspidal nonspecial, from a vertex } \not\sim x, \\ \frac{1}{2}(q+1)(q^{m-1} - q^{r-\frac{1}{2}}) & \text{if } \pi \text{ is supercuspidal, from a nonvertex.} \end{cases}$$

On the other hand, if $\pi_s = H_s^\varepsilon$ then $\dim(\pi_s^{G_{x,m}}) = q^{m-1}$ when the parity depth at x of the orbits in $\mathcal{WF}(\pi_s)$ is even, and equals q^m otherwise; and if $\pi = \mathrm{St}$, then $\dim(\pi^{G_{x,m}}) = q^m + q^{m-1} - 1$. In all other cases, $\dim(\pi^{G_{x,m}})$ is exactly half of that of a corresponding (irreducible principal series or nonspecial supercuspidal) representation.

Proof. Let π be of depth r. By Theorem 6.4 (and Remark 6.5) in positive depth and Theorem 7.4 in depth zero, we have for $m > r$ that

$$\dim(\pi^{G_{x,m}}) = n + \sum_{\mathcal{O} \in \mathcal{WF}(\pi)} \dim(\tau_x(\mathcal{O})^{G_{x,m}}),$$

where $n = n_x(\pi)$ if $r > 0$ (Proposition 6.7) and $n = \dim(\pi^{G_{x,0+}})$ if $r = 0$ (Tables 3

and 4). We obtain $\dim(\tau_x(\mathcal{O})^{G_{x,m}})$ from Lemma 6.6 by setting $\ell = m-1$, and refer to Tables 1 and 5 for the sets $\mathcal{WF}(\pi)$.

If π is an irreducible principal series, then $\mathcal{WF}(\pi)$ consists of all nilpotent orbits and $n = (q+1)$ for all r, yielding a total $(q+1)+(q+1)(q^{m-1}-1) = q^{m-1}(q+1)$. For the reducible principal series, we have $\dim(\mathrm{St}) = q$ and $\dim(\mathbf{1}) = 1$, so

$$\dim((H_+^{\varepsilon})^{G_{x,m}}) = \begin{cases} q + q(q^{m-2}-1) = q^{m-1} & \text{if } x \sim x_0, \\ 1 + (q^m - 1) = q^m & \text{if } x \sim x_1, \end{cases}$$

with $\dim((H_-^{\varepsilon})^{G_{x,m}}) = (q^{m-1}+q^m) - \dim((H_+^{\varepsilon})^{G_{x,m}})$. On the other hand, for $\tau \in \{-\varpi, -\varepsilon\varpi\}$ the decomposition is symmetric and therefore $\dim((H_{\pm}^{\tau})^{G_{x,m}}) = \frac{1}{2}(q^m + q^{m-1})$.

If π is a ramified supercuspidal representation, then r is a positive half-integer, and $\mathcal{WF}(\pi)$ consists of two orbits of opposite parity depth. We compute

$$\dim(\pi^{G_{x,m}}) = \tfrac{1}{2}(1-q^{r-\frac{1}{2}})(q+1) + \tfrac{1}{2}(q+1)(q^{m-1}-1) = \tfrac{1}{2}(q+1)(q^{m-1}-q^{r-\frac{1}{2}}).$$

Suppose π is an unramified supercuspidal representation of depth $r > 0$. Note that $-r = d(\Gamma)$ has the same parity as $d_x(\Gamma)$ (which coincides with $\mathrm{pd}_x(\mathcal{O})$ for each $\mathcal{O} \in \mathcal{WF}(\pi)$) if and only if $x \sim x_T$. Rephrasing the conditions in Table 2 yields that if $d_x(\Gamma)$ is even then $n_x(\pi) = q - q^r$ for all x, whereas if $d_x(\Gamma)$ is odd then $n_x(\pi) = 1 - q^r$. Lemma 6.6 now gives

$$\dim(\pi^{G_{x,m}}) = \begin{cases} (q-q^r) + q(q^{m-2}-1) = q^{m-1} - q^r & \text{if } d_x(\Gamma) \text{ is even}, \\ (1-q^r) + (q^m - 1) = q^m - q^r & \text{if } d_x(\Gamma) \text{ is odd}. \end{cases}$$

The same holds for nonspecial supercuspidal representations of depth $r = 0$, since there $n = q-1$ if $x \sim x_T$ and $n = 0$ otherwise. Finally, for each of the special representations, the dimension will be half the corresponding value for an unramified supercuspidal from the same vertex, by symmetry. □

Acknowledgements

This work was instigated by a question posed to the author by David Vogan and has benefitted enormously from many conversations with him in the online research community Representation Theory and Noncommutative Geometry sponsored by the American Institute of Mathematics. The approach to $\mathrm{Nil}(\Gamma)$ given here was significantly refined through conversations with Fiona Murnaghan and Loren Spice. This work progressed over a period of visits to many colleagues, and benefitted from their comments and interest: Vincent Sécherre, Laboratoire de Mathématiques de Versailles, Université Paris-Saclay; Anne-Marie Aubert, Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université de Paris/Sorbonne Université and Jessica Fintzen, Universität Bonn. It is a true pleasure to thank all of these generous people.

References

[Adams and Vogan 2021] J. Adams and D. A. Vogan, Jr., "Associated varieties for real reductive groups", *Pure Appl. Math. Q.* **17**:4 (2021), 1191–1267. MR Zbl

[Adler 1998] J. D. Adler, "Refined anisotropic K-types and supercuspidal representations", *Pacific J. Math.* **185**:1 (1998), 1–32. MR Zbl

[Adler and DeBacker 2002] J. D. Adler and S. DeBacker, "Some applications of Bruhat–Tits theory to harmonic analysis on the Lie algebra of a reductive p-adic group", *Michigan Math. J.* **50**:2 (2002), 263–286. MR Zbl

[Adler and Korman 2007] J. D. Adler and J. Korman, "The local character expansion near a tame, semisimple element", *Amer. J. Math.* **129**:2 (2007), 381–403. MR Zbl

[Adler and Roche 2000] J. D. Adler and A. Roche, "An intertwining result for p-adic groups", *Canad. J. Math.* **52**:3 (2000), 449–467. MR Zbl

[Assem 1994] M. Assem, "The Fourier transform and some character formulae for p-adic SL_l, l a prime", *Amer. J. Math.* **116**:6 (1994), 1433–1467. MR Zbl

[Barbasch and Moy 1997] D. Barbasch and A. Moy, "Local character expansions", *Ann. Sci. École Norm. Sup.* (4) **30**:5 (1997), 553–567. MR Zbl

[Ciubotaru and Okada 2023] D. Ciubotaru and E. Okada, "Local character expansions via positive depth Barbasch–Moy theory", 2023. arXiv 2307.06780

[Ciubotaru et al. 2022a] D. Ciubotaru, L. Mason-Brown, and E. Okada, "The wavefront sets of Iwahori-spherical representations of reductive p-adic groups", 2022. arXiv 2112.14354v4

[Ciubotaru et al. 2022b] D. Ciubotaru, L. Mason-Brown, and E. Okada, "The wavefront sets of unipotent supercuspidal representations", 2022. arXiv 2206.08628v2

[Cluckers et al. 2014] R. Cluckers, J. Gordon, and I. Halupczok, "Local integrability results in harmonic analysis on reductive groups in large positive characteristic", *Ann. Sci. Éc. Norm. Supér.* (4) **47**:6 (2014), 1163–1195. MR Zbl

[Debacker 2002a] S. Debacker, "Homogeneity results for invariant distributions of a reductive p-adic group", *Ann. Sci. École Norm. Sup.* (4) **35**:3 (2002), 391–422. MR Zbl

[DeBacker 2002b] S. DeBacker, "Parametrizing nilpotent orbits via Bruhat–Tits theory", *Ann. of Math.* (2) **156**:1 (2002), 295–332. MR Zbl

[DeBacker and Reeder 2009] S. DeBacker and M. Reeder, "Depth-zero supercuspidal L-packets and their stability", *Ann. of Math.* (2) **169**:3 (2009), 795–901. MR Zbl

[DeBacker and Sally 2000] S. DeBacker and P. J. Sally, Jr., "Germs, characters, and the Fourier transforms of nilpotent orbits", pp. 191–221 in *The mathematical legacy of Harish-Chandra*, edited by R. S. Doran and V. S. Varadarajan, Proc. Sympos. Pure Math. **68**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[Digne and Michel 1991] F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts **21**, Cambridge Univ. Press, 1991. MR Zbl

[Fintzen 2021a] J. Fintzen, "On the construction of tame supercuspidal representations", *Compos. Math.* **157**:12 (2021), 2733–2746. MR Zbl

[Fintzen 2021b] J. Fintzen, "On the Moy–Prasad filtration", *J. Eur. Math. Soc.* **23**:12 (2021), 4009–4063. MR Zbl

[Fintzen 2021c] J. Fintzen, "Types for tame p-adic groups", *Ann. of Math.* (2) **193**:1 (2021), 303–346. MR Zbl

[Fintzen et al. 2023] J. Fintzen, T. Kaletha, and L. Spice, "A twisted Yu construction, Harish-Chandra characters, and endoscopy", *Duke Math. J.* **172**:12 (2023), 2241–2301. MR Zbl

[Harish-Chandra 1999] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, University Lecture Series **16**, Amer. Math. Soc., Providence, RI, 1999. With a preface and notes by Stephen DeBacker and Paul J. Sally, Jr. MR Zbl

[Heifetz 1985] D. B. Heifetz, "p-adic oscillatory integrals and wave front sets", *Pacific J. Math.* **116**:2 (1985), 285–305. MR Zbl

[Henniart and Vignéras 2023] G. Henniart and M.-F. Vignéras, "Representations of $\mathrm{GL}_n(D)$ near the identity", 2023. arXiv 2305.06581

[Henniart and Vignéras 2024] G. Henniart and M.-F. Vignéras, "Representations of $\mathrm{SL}_2(F)$", 2024. arXiv 2404.11188

[Howe 1974] R. Howe, "The Fourier transform and germs of characters (case of Gl_n over a p-adic field)", *Math. Ann.* **208** (1974), 305–322. MR Zbl

[Howe 1981] R. Howe, "Wave front sets of representations of Lie groups", pp. 117–140 in *Automorphic forms, representation theory and arithmetic* (Bombay, 1979), Tata Inst. Fundam. Res. Stud. Math. **10**, Springer, 1981. MR Zbl

[Jiang et al. 2022] D. Jiang, D. Liu, and L. Zhang, "Arithmetic wavefront sets and generic L-packets", 2022. arXiv 2207.04700v2

[Kaletha 2019] T. Kaletha, "Regular supercuspidal representations", *J. Amer. Math. Soc.* **32**:4 (2019), 1071–1170. MR Zbl

[Kaletha and Prasad 2023] T. Kaletha and G. Prasad, *Bruhat–Tits theory—a new approach*, New Mathematical Monographs **44**, Cambridge Univ. Press, 2023. MR Zbl

[Kim and Murnaghan 2003] J.-L. Kim and F. Murnaghan, "Character expansions and unrefined minimal K-types", *Amer. J. Math.* **125**:6 (2003), 1199–1234. MR Zbl

[Kim and Murnaghan 2006] J.-L. Kim and F. Murnaghan, "K-types and Γ-asymptotic expansions", *J. Reine Angew. Math.* **592** (2006), 189–236. MR Zbl

[Mœglin 1996] C. Mœglin, "Front d'onde des représentations des groupes classiques p-adiques", *Amer. J. Math.* **118**:6 (1996), 1313–1346. MR Zbl

[Mœglin and Waldspurger 1987] C. Mœglin and J.-L. Waldspurger, "Modèles de Whittaker dégénérés pour des groupes p-adiques", *Math. Z.* **196**:3 (1987), 427–452. MR Zbl

[Moy and Prasad 1994] A. Moy and G. Prasad, "Unrefined minimal K-types for p-adic groups", *Invent. Math.* **116**:1-3 (1994), 393–408. MR Zbl

[Moy and Prasad 1996] A. Moy and G. Prasad, "Jacquet functors and unrefined minimal K-types", *Comment. Math. Helv.* **71**:1 (1996), 98–121. MR Zbl

[Nevins 2005] M. Nevins, "Branching rules for principal series representations of SL(2) over a p-adic field", *Canad. J. Math.* **57**:3 (2005), 648–672. MR Zbl

[Nevins 2011] M. Nevins, "Patterns in branching rules for irreducible representations of $\mathrm{SL}_2(k)$, for k a p-adic field", pp. 185–199 in *Harmonic analysis on reductive, p-adic groups*, edited by R. S. Doran et al., Contemp. Math. **543**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[Nevins 2013] M. Nevins, "Branching rules for supercuspidal representations of $\mathrm{SL}_2(k)$, for k a p-adic field", *J. Algebra* **377** (2013), 204–231. MR Zbl

[Przebinda 1990] T. Przebinda, "The wave front set and the asymptotic support for p-adic groups", *Pacific J. Math.* **141**:2 (1990), 383–389. MR Zbl

[Ranga Rao 1972] R. Ranga Rao, "Orbital integrals in reductive groups", *Ann. of Math.* (2) **96** (1972), 505–510. MR Zbl

[Shalika 2004] J. A. Shalika, "Representation of the two by two unimodular group over local fields", pp. 1–38 in *Contributions to automorphic forms, geometry, and number theory*, Johns Hopkins Univ. Press, Baltimore, MD, 2004. Zbl

[Spice 2018] L. Spice, "Explicit asymptotic expansions for tame supercuspidal characters", *Compos. Math.* **154**:11 (2018), 2305–2378. MR Zbl

[Spice 2022] L. Spice, "Explicit asymptotic expansions in p-adic harmonic analysis, II", 2022. arXiv 2108.12935v2

[Stewart and Thomas 2018] D. I. Stewart and A. R. Thomas, "The Jacobson–Morozov theorem and complete reducibility of Lie subalgebras", *Proc. Lond. Math. Soc.* (3) **116**:1 (2018), 68–100. MR Zbl

[Tadić 1994] M. Tadić, "Representations of classical p-adic groups", pp. 129–204 in *Representations of Lie groups and quantum groups* (Trento, 1993), edited by V. Baldoni and M. A. Picardello, Pitman Res. Notes Math. Ser. **311**, Longman Sci. Tech., Harlow, 1994. MR Zbl

[Tsai 2023a] C.-C. Tsai, "On two definitions of wave-front sets for p-adic groups", 2023. arXiv 2306.09536

[Tsai 2023b] C.-C. Tsai, "Wave-front sets for p-adic Lie algebras", 2023. arXiv 2311.08078

[Tsai 2024] C.-C. Tsai, "Geometric wave-front set may not be a singleton", *J. Amer. Math. Soc.* **37**:1 (2024), 281–304. MR Zbl

[Waldspurger 1995] J.-L. Waldspurger, "Homogénéité de certaines distributions sur les groupes p-adiques", *Inst. Hautes Études Sci. Publ. Math.* **81** (1995), 25–72. MR Zbl

[Yu 2001] J.-K. Yu, "Construction of tame supercuspidal representations", *J. Amer. Math. Soc.* **14**:3 (2001), 579–622. MR Zbl

Received November 10, 2023. Revised May 20, 2024.

Monica Nevins
Department of Mathematics and Statistics
University of Ottawa
Ottawa, ON
Canada

mnevins@uottawa.ca

PACIFIC JOURNAL OF MATHEMATICS
Vol. 329, No. 2, 2024
https://doi.org/10.2140/pjm.2024.329.303

EXTREMELY CLOSED SUBGROUPS AND A VARIANT ON GLAUBERMAN'S Z^*-THEOREM

HUNG P. TONG-VIET

Let G be a finite group and let H be a subgroup of G. We say that H is extremely closed in G if $\langle H, H^g\rangle \cap N_G(H) = H$ for all $g \in G$. We determine the structure of finite groups with an extremely closed abelian p-subgroup for some prime p. In particular, we show that if G contains such a subgroup H, then $G = N_G(H)\, O_{p'}(G)$. This is a variant on the celebrated Glauberman's Z^*-theorem.

1. Introduction

It is an important problem in finite group theory to determine whether a finite group is simple or not. Many nonsimplicity criteria have been obtained in the literature. Among those is the celebrated Glauberman's Z^*-theorem. To state this theorem, we need some definitions. Let G be a finite group and let p be a prime. Let $x \in G$ be a p-element and let P be a Sylow p-subgroup of G containing x. We say that x is isolated in P with respect to G if $x^G \cap P = \{x\}$, that is, x is not conjugate in G to any element in P except for x itself. Here, x^G denotes the conjugacy class of G containing x. We say that x is isolated in G if x is isolated in some Sylow p-subgroup of G containing it. Glauberman's Z^*-theorem [18] states that if $x \in G$ is an isolated involution in G, then $G = C_G(x)\, O_{2'}(G)$. The proof of this theorem depends on the modular representation theory and is independent of the classification of finite simple groups. Recall that for a prime p, $O_{p'}(G)$ is the largest normal p'-subgroup of G. Extending this fundamental theorem to all primes, Glauberman's Z_p^*-theorem states that if $x \in G$ is an isolated p-element, then $G = C_G(x)\, O_{p'}(G)$. For various proofs of this theorem, see [1; 19; 27; 40]. Note that all of these proofs depend on the classification of finite simple groups. For many equivalent statements of this theorem, see [30]. Also, see [14] for some variant of Glauberman's Z_p^*-theorem.

In this paper, we introduce the so-called *extremely closed subgroup* and obtain some new factorization of finite groups similar to Glauberman's Z_p^*-theorem which gives some nonsimplicity criteria for finite groups. Let G be a finite group and let

MSC2020: 20D05, 20D15.
Keywords: extremely closed, strongly closed, weakly closed.

© 2024 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

$H \leq M$ be subgroups of G. We say that H is extremely closed in M with respect to G if $\langle H, H^g \rangle \cap M = H$ for all $g \in G - M$, and that H is extremely closed in G if H is extremely closed in $N_G(H)$ with respect to G. Trivially, if H is a normal or self-normalizing subgroup of G, then H is extremely closed in G.

If G is a finite group, we write $Z(G)$ for the center of G and $\Phi(G)$ for the Frattini subgroup of G, that is, the intersection of all maximal subgroups of G. Furthermore, if H is a subgroup of G, then $\langle H^G \rangle$ is the normal closure of H in G. We next compare our definition of extremely closed subgroups with other known embedding properties of subgroups of finite groups.

The first motivation for our definition comes from work of Flavell [10] on the generation of finite groups with maximal subgroups of maximal subgroups. In particular, a triple (G, M, H) with $H \leq M \leq G$ is called a γ-triple if $H < M < G$ and $\langle H, g \rangle \cap M = H$ for all $g \in G - M$. If H is maximal in M and M is maximal in G and moreover, G cannot be generated by any two conjugates of H, then (G, M, H) is a γ-triple. Clearly, if (G, M, H) is a γ-triple, then H is extremely closed in M with respect to G. The converse is not true by the example below.

Example 1.1. Let $G = \mathrm{S}_4$ be the symmetric group of degree 4. Let $H = \langle (1, 2, 3) \rangle$ and $M = N_G(H) \cong \mathrm{S}_3$. Then $\langle H^G \rangle = \mathrm{A}_4$ and $\langle H^G \rangle \cap M = H$. So $\langle H, H^g \rangle \cap M = H$ for all $g \in G$, and hence H is extremely closed in M with respect to G. However, let $g = (1, 3, 2, 4) \in G - M$. Then $\langle H, g \rangle = G$ and so $\langle H, g \rangle \cap M = M \neq H$. Therefore (G, M, H) is not a γ-triple.

For the second motivation, following Hawkes and Humphreys [31], a subgroup M of a finite group G is said to have property CR (character restriction) if every irreducible complex character of M is the restriction of a character of G. In [31], the authors studied finite solvable groups with a CR-subgroup and the general cases were considered by Isaacs in [32]. One important property of a CR-subgroup M of a finite group G is that if $H \trianglelefteq M$, then $\langle H^G \rangle \cap M = H$ [32, Proposition 1.1]. Berkovich [3] called a triple (G, M, H) with $H \trianglelefteq M \leq G$ *special* in G if $\langle H^G \rangle \cap M = H$. (Li [36] calls H an NE-subgroup of G if $(G, N_G(H), H)$ is special in G.) Isaacs [32] showed that if P is a Sylow p-subgroup of G, where p is a prime and assume that $N_G(P)$ satisfies CR in G, then $N_G(P)$ has a normal complement in G. This result was extended by Berkovich [3], where he showed that if both triples $(G, N_G(P), P)$ and $(G, N_G(P), \Phi(P))$ are special in G, then $N_G(P)$ has a normal complement in G. This gives a character theory free proof of Isaacs' result mentioned earlier. Observe that if a triple $(G, N_G(H), H)$ is special in G, then H is extremely closed in G. However, the converse is not true.

Example 1.2. Let $G = P : H$ be a semidirect product of H and P, where $H = \langle a \rangle$ is a cyclic group of order 2 and $P \cong 3^{1+2}_{+}$ is an extraspecial group of order 27 with

exponent 3, so

$$P = \langle x, y, z \mid z = [x, y],\ x^3 = y^3 = z^3 = 1 = [x, z] = [y, z] \rangle$$

and H acts on P via $x^a = x^{-1}$, $y^a = y^{-1}$ and $z^a = z$. Then $N_G(H) = C_G(H) = H\langle z \rangle$, and $\langle H^G \rangle = G$. For every $g \in G - N_G(H)$, we have $T = \langle H, H^g \rangle$ is a dihedral group of order 6, so $N_T(H) = H$, and hence H is extremely closed in G but $(G, N_G(H), H)$ is not a special triple since $\langle H^G \rangle \cap N_G(H) = H\langle z \rangle \neq H$.

Finally, we mention the last inspiration for our new embedding property. Let $H \leq M$ be subgroups of a finite group G. Recall that H is said to be *strongly closed* in M with respect to G if, whenever $a^g \in M$, where $a \in H$, $g \in G$, then $a^g \in H$. This is equivalent to saying that $M \cap H^g \leq H$ for all $g \in G$. Furthermore, we say that H is strongly closed in G if H is strongly closed in $N_G(H)$ with respect to G. Noting that in [4], H is called an $\mathscr{H}$-subgroup of G if H is strongly closed in G. If $H = \langle x \rangle$ is cyclic of order 2, then H is strongly closed in G if and only if x is isolated in G. Finite groups with a strongly closed p-subgroup are determined in [15; 16; 20]. It is easy to see that if H is extremely closed in G, then H is strongly closed in G.

Example 1.3. Let $G = \mathrm{U}_3(4)$. By [20], if P is a Sylow 2-subgroup of G, then $H = Z(P) = \Phi(P)$ is a strongly closed abelian 2-subgroup of G. Using GAP [17], we can find $g \in G$ of order 15 such that $\langle H, H^g \rangle \cong \mathrm{A}_5 \cong \mathrm{SU}_2(4)$, $N_G(H) \cong P\langle g \rangle$ and $\langle H, H^g \rangle \cap N_G(H) \cong \mathrm{A}_4 \neq H$. Thus H is not extremely closed in G.

Finally, we recall the definition of weakly closed subgroups. Let $H \leq M$ be subgroups of a finite group G. We say that H is *weakly closed* in M with respect to G if, whenever $H^g \subseteq M$, where $g \in G$, then $H^g = H$. It is easy to see that if H is strongly closed in M with respect to G, then H is weakly closed in M with respect to G. Moreover, when H is cyclic of prime order, these two concepts coincide. We know that if $H \trianglelefteq M \leq G$ and $\langle H^G \rangle \cap M = H$, then H is extremely closed in M with respect to G. In our first result, we show that in certain cases, the converse holds.

Theorem 1.4. *Let G be a finite group and let $H \leq M$ be subgroups of G. Suppose that $\langle H, H^g \rangle \cap M = H$ for all $g \in G$. If H is maximal in M and M is maximal in G, then $\langle H^G \rangle \cap M = H$.*

Let $H \leq M$ be subgroups of a finite group G. Recall that H is called a weak second maximal subgroup of G if there exists a maximal subgroup M of G such that H is maximal in M. Moreover, H is called a second maximal subgroup of G if $H \neq G$ and it is maximal in every maximal subgroup of G containing it. Flavell [10] shows that if G is a finite nonabelian simple group and H is a weak second maximal subgroup of G, then $G = \langle H, g \rangle$ for some $g \in G$. And in [9], it is shown that

if H is a second maximal subgroup of a finite nonabelian simple group G, then $G = \langle H, H^g \rangle$ for some $g \in G$. As a corollary to Theorem 1.4, we obtain the following generation result for finite nonabelian simple groups.

Corollary 1.5. *Let G be a finite nonabelian simple group. Let M be a maximal subgroup of G and let H be a normal subgroup of M of prime index. Then $G = \langle H, H^g \rangle$ for some $g \in G$.*

Note that if $G = \mathrm{A}_5$, $M = \mathrm{S}_3$ and H is a Sylow 2-subgroup of M, then M is a maximal subgroup of G and $|M : H| = 3$ is a prime, but G cannot be generated by any two conjugates of H. So we cannot drop the hypothesis that $H \trianglelefteq M$ in Corollary 1.5. We should mention that Flavell [13] asks whether a finite nonabelian simple group can be generated by two conjugates of a self-normalizing subgroup.

Let G be a finite group and let p be a prime. We now focus on extremely closed p-subgroups. Let H be an extremely closed p-subgroup of G. We first assume that $H = \langle x \rangle$ is cyclic of order p. If $p = 2$, then it is not hard to see that x is isolated in G or equivalently H is strongly closed in G and so $G = C_G(x)\, O_{2'}(G)$ by Glauberman's Z^*-theorem. In particular, G is not simple. When p is odd, we note that there exists a simple group with a weakly closed or strongly closed subgroup of order p, for instance, when a Sylow p-subgroup of G is cyclic. However, when H is extremely closed in G, the subgroups generated by any two distinct conjugates of H are Frobenius groups. By applying a result due to B. Fischer [8] concerning Frobenius automorphisms, we can show that $(G, N_G(H), H)$ is special in G, and then we obtain a factorization similar to that of Glauberman's Z_p^*-theorem.

Theorem 1.6. *Let G be a finite group and let p be an odd prime divisor of the order of G. Let H be a cyclic subgroup of order p of G. If $\langle H, H^g \rangle \cap N_G(H) = H$ for all $g \in G$, then $\langle H^G \rangle \cap N_G(H) = H$. In particular, $G = N_G(H)\, O_{p'}(G)$.*

It would be nice to have a proof of Theorem 1.6 which does not rely on the classification of finite simple groups. For an arbitrary prime p, if H is an abelian extremely closed p-subgroup of a finite group G, we also obtain a similar factorization $G = N_G(H)\, O_{p'}(G)$ as in Theorem 1.6. For even prime, the proof depends only on the classification of finite groups with an abelian strongly closed subgroup by Goldschmidt [20] which is independent of the classification of finite simple groups. For odd primes, we make use of a result due to Guest [25] on the characterization of solvable radical of finite groups and the classification of finite groups with strongly closed subgroups by Flores and Foote [15].

Theorem 1.7. *Let G be a finite group and let p be prime. Let H be an abelian p-subgroup of G. If $\langle H, H^g \rangle \cap N_G(H) = H$ for all $g \in G$, then $G = N_G(H)\, O_{p'}(G)$.*

Remark that we cannot drop the hypothesis that H is abelian when $p = 2$ in Theorem 1.7, since the simple group $G = \mathrm{L}_2(17)$ has a self-normalizing Sylow 2-subgroup P which is nonabelian and so P is clearly an extremely closed 2-subgroup of G. Note that if $|H| = 2$, then Theorem 1.7 is just Glauberman's Z^*- theorem. Example 1.2 above shows that an abelian extremely closed 2-subgroup H may not satisfy the condition $\langle H^G\rangle \cap N_G(H) = H$. However, this holds true for odd primes. We obtain the following as a corollary to Theorem 1.7.

Corollary 1.8. *Let G be a finite group and let p be an odd prime. Let H be an abelian p-subgroup of G. If $\langle H, H^g\rangle \cap N_G(H) = H$ for all $g \in G$, then $\langle H^G\rangle \cap N_G(H) = H$.*

Recall that for a finite group G, the solvable radical of G, denoted by $R(G)$, is the largest normal solvable subgroup of G. Theorem 1.7 and Corollary 1.8 now yield the following.

Corollary 1.9. *Let G be a finite group and let p be a prime. If H is an extremely closed abelian p-subgroup of G, then $H \subseteq R(G)$.*

By an application of Burnside's normal p-complement theorem and the solvability of finite groups admitting a fixed point free coprime group action, if H satisfies the hypothesis of the corollary, then $\langle H, H^g\rangle$ is solvable for all $g \in G$. Thus if $x \in H$, then $\langle x, x^g\rangle$ is solvable for all $g \in G$. By the main results in [21; 25], if $p \geq 5$, then $x \in R(G)$ and hence $H \leq R(G)$. Thus the above corollary only provides new result when $p = 2$ or 3.

In general, if x is a p-element and $\langle x, x^g\rangle$ is p-solvable for all $g \in G$, then it is not true that $x \in R_p(G)$, where $R_p(G)$ is the p-solvable radical of G, that is, $R_p(G)$ is the largest normal p-solvable subgroup of G. For a counterexample, consider $G = \mathrm{U}_3(3)$ and $x \in G$ a transvection, so x has order 3 and the conjugacy class of G containing x has size 56, then we can check that $\langle x, x^g\rangle$ is isomorphic to either $\langle x\rangle$ or $\mathrm{SL}_2(3)$ for every $g \in G$. Clearly $\langle x, x^g\rangle$ is 3-nilpotent and hence it is 3-solvable for every $g \in G$. There is also a counterexample when $p = 2$ since if $x \in G$ is an involution, then $\langle x, x^g\rangle$ is 2-nilpotent for every $g \in G$. Recall that a finite group G is p-nilpotent if it has a normal p-complement for some prime p. On the other hand, it is proved in [11] that if P is a Sylow p-subgroup of a finite group G for some prime p, then G is p-solvable if and only if $\langle P, g\rangle$ is p-solvable for all $g \in G$. Generalizing this result, we can prove the following.

Theorem 1.10. *Let G be a finite group and let p be a prime. Let P be a Sylow p-subgroup of G. Then G is p-solvable if and only if $\langle P, P^g\rangle$ is p-solvable for all $g \in G$.*

Our notation is standard. For finite group theory, we follow [22] and [35] and for finite simple groups, we follow the notation in [34].

For the organization of the paper, we collect some useful results in Section 2. We will prove Theorems 1.4–1.7 and the corollaries in Section 3 and the last theorem will be proved in Section 4.

2. Preliminaries

Let G be a finite group. Recall that the Fitting subgroup of G, denoted by $F(G)$, is the largest nilpotent normal subgroup of G. The layer of G, denoted by $E(G)$, is the product of all components of G, where a component of G is a subnormal quasisimple subgroup of G. A finite group L is quasisimple if L is perfect and $L/Z(L)$ is a nonabelian simple group. The generalized Fitting subgroup of G, denoted by $F^*(G)$, is defined by $F^*(G) = F(G)E(G)$. As usual, if $H \leq G$, then $N_G(H)$ and $C_G(H)$ denote the normalizer and centralizer of H in G, respectively. Finally, a finite group G is almost simple with socle S if there exists a finite nonabelian simple group S such that $S \trianglelefteq G \leq \mathrm{Aut}(S)$.

Recall that a subgroup H of G is called *pronormal* (resp. *abnormal*) in G if for any $g \in G$, $H^g = H^u$ for some $u \in \langle H, H^g\rangle$, (resp. $\langle H, H^g\rangle = \langle H, g\rangle$). The first lemma is obvious, for completeness, we will include a proof here.

Lemma 2.1. *Let G be a finite group. Let H be a pronormal subgroup of G and let $N \trianglelefteq G$. Then the following hold.*

(i) *If $N \trianglelefteq G$ and $P \in \mathrm{Syl}_p(N)$, then P is pronormal in G.*

(ii) *$N_G(H)$ is abnormal in G.*

(iii) *If $H \leq N \trianglelefteq G$, then $G = N_G(H)N$.*

(iv) *If $H \leq L \leq G$, then H is pronormal in L.*

(v) *If H is subnormal in K, where $K \leq G$, then $H \trianglelefteq K$.*

(vi) *If $L = \langle H^G\rangle$, then $\langle H^L\rangle = L$.*

Proof. (i) Let $g \in G$. Since $P \leq N \trianglelefteq G$, we have $\langle P, P^g\rangle \leq N$. As $P \in \mathrm{Syl}_p(N)$, it follows that $P \in \mathrm{Syl}_p(\langle P, P^g\rangle)$ and hence by Sylow's theorem, $P^g = P^u$ for some $u \in \langle P, P^g\rangle$. Thus P is pronormal in G.

(ii) Assume that H is pronormal in G. Let $M = N_G(H)$ and let $g \in G$. By definition, $H^g = H^u$, for some $u \in \langle H, H^g\rangle$, whence $gu^{-1} \in N_G(H) = M$. Since $u \in \langle H, H^g\rangle \leq \langle M, M^g\rangle$, we have $g \in \langle M, M^g\rangle$ and so $\langle M, M^g\rangle = \langle M, g\rangle$. Hence M is abnormal in G.

(iii) Let $g \in G$. We have $H^g = H^u$, where $u \in \langle H, H^g\rangle \leq N$ as $H \leq N \trianglelefteq G$. Thus $gu^{-1} \in N_G(H)$ and so $g \in N_G(H)N$.

(iv) This is obvious.

(v) By (iv), it suffices to show that if H is subnormal and pronormal in G then $H \trianglelefteq G$. In fact, we only need to prove the following: if $H \trianglelefteq K \trianglelefteq G$ and H is pronormal in G then $H \trianglelefteq G$. By applying (iii), we have $G = N_G(H)K$. However, as $H \trianglelefteq K$, $K \leq N_G(H)$ and so $G = N_G(H)$.

(vi) Since $H \leq L = \langle H^G \rangle \trianglelefteq G$, $G = N_G(H)L$ by (iii) and thus

$$L = \langle H^G \rangle = \langle H^{N_G(H)L} \rangle = \langle H^L \rangle \leq L.$$

Therefore $L = \langle H^L \rangle$. □

We next deduce some properties of extremely closed subgroups.

Lemma 2.2. *Let G be a finite group and let H be a p-subgroup of G for some prime p. Let $N \trianglelefteq G$ and assume that H is extremely closed in G. Let $\overline{G} = G/N$. Then the following hold.*

(i) *For every $g \in G$, we have $N_{\langle H, H^g \rangle}(H) = H$ and $H \in \mathrm{Syl}_p(\langle H, H^g \rangle)$.*

(ii) *H is pronormal in G.*

(iii) *If $H \leq L$, then H is extremely closed in L.*

(iv) *$N_{\overline{G}}(\overline{H}) = \overline{N_G(H)}$.*

(v) *If H is abelian, then $\langle H, H^g \rangle = HO_{p'}(\langle H, H^g \rangle)$, for every $g \in G$.*

(vi) *$\overline{H}$ is extremely closed in $\overline{G}$.*

(vii) *If $H \leq Q \leq G$, where Q is a p-group, then $N_G(Q) \leq N_G(H)$. In particular, if $H \leq P \in \mathrm{Syl}_p(G)$, then $P \leq N_G(H)$.*

Proof. (i) Let $g \in G$ and let $T = \langle H, H^g \rangle$. We have $H = T \cap N_G(H) = N_T(H)$ and so H is a Sylow p-subgroup of T by Sylow's theorem.

(ii) Let $g \in G$. As above, let $T = \langle H, H^g \rangle$. From part (i), H is a Sylow p-subgroup of T and since $|H^g| = |H|$ and $H^g \leq T$, H^g is also a Sylow p-subgroup of T. By Sylow's theorem, $H^g = H^u$ for some $u \in T$. Thus H is pronormal in G.

(iii) Let $g \in L$. Then $\langle H, H^g \rangle \cap N_L(H) = \langle H, H^g \rangle \cap N_G(H) \cap L = H \cap L = H$.

(iv) It suffices to show that $N_G(HN) \leq N_G(H)N$. Let $g \in N_G(HN)$. Then $H^g \leq HN$ and hence $T = \langle H, H^g \rangle \leq HN$. Since H is pronormal in G by (ii), we have $H^g = H^u$, for some $u \in T \leq HN$. Thus $gu^{-1} \in N_G(H)$ whence $g \in N_G(H)HN = N_G(H)N$.

(v) Assume that H is abelian. Let $g \in G$ and let $T = \langle H, H^g \rangle$. By (i), H is a self-normalizing abelian Sylow p-subgroup of T. The result now follows from Burnside's normal p-complement theorem ([22, Theorem 7.4.3]).

(vi) Applying (iv), we need to show that $\langle H, H^g \rangle N \cap N_G(H)N = HN$, for all $g \in G$. Let $T = \langle H, H^g \rangle$. By Dedekind's Modular law, we have

$$TN \cap N_G(H)N = N(TN \cap N_G(H)).$$

Hence, it suffices to show that $TN \cap N_G(H) \leq HN$. Let $y = xn \in TN \cap N_G(H)$, where $y \in N_G(H)$, $x \in T$ and $n \in N$. We have $H = H^y = H^{xn}$, it implies that $H^x = H^{n^{-1}}$. By (i), we have $N_T(H) = H$, and so by (ii) and Lemma 2.1(ii), H is abnormal in T. Thus $x \in \langle H, H^x \rangle = \langle H, H^{n^{-1}} \rangle \leq HN$. Therefore, $y = xn \in HN$.

(vii) Since H is pronormal and subnormal in $N_G(Q)$, by Lemma 2.1(v), H is normal in $N_G(Q)$. The remaining claim is obvious. □

Lemma 2.3. *Let G be a finite group, let $N \trianglelefteq G$ and let H be an extremely closed p-subgroup of G for some prime p. Let P be a Sylow p-subgroup of G containing H and let $Q = H \cap N$. Then the following hold.*

(i) *H is strongly closed in P with respect to G.*

(ii) *Q is strongly closed in $P \cap N$ with respect to N.*

(iii) *If $N \leq N_G(H)$, then $Q \trianglelefteq G$.*

Proof. Observe that $R := P \cap N \in \mathrm{Syl}_p(N)$ and by Lemma 2.2(vii), $P \leq N_G(H)$.

(i) For $g \in G$, we have $H^g \cap P \leq \langle H, H^g \rangle \cap N_G(H) = H$. So H is strongly closed in P with respect to G.

(ii) For $n \in N$, we have $Q^n \leq H^n$ and $R \leq P \leq N_G(H)$ and so

$$Q^n \cap R \leq \langle H, H^n \rangle \cap N_G(H) = H.$$

Furthermore, as $Q \leq N \trianglelefteq G$ and $R \leq N$, we have $Q^n \cap R \leq N$. Hence we obtain $Q^n \cap R \leq H \cap N = Q$.

(iii) Assume that $N \leq N_G(H)$. For each $g \in G$, we have

$$Q^g = Q^g \cap N \leq \langle H, H^g \rangle \cap N_G(H) \cap N = H \cap N = Q.$$

Hence $Q \trianglelefteq G$ as wanted. □

We next quote some results that we will need for the proofs of the main theorems.

Lemma 2.4. *Suppose that G is a finite group with $F(G) = 1$. Let L be a component of G. If $x \in G$ such that $x \notin N_G(L)$ and $x^2 \notin C_G(L)$ then there exists an element $g \in G$ such that $\langle x, x^g \rangle$ is not solvable.*

Proof. This is Lemma 1 in [25]. □

Lemma 2.5. *Let G be a finite almost simple group with socle L. Suppose that $x \in G$ is an element of order p, where p is an odd prime. Then one of the following holds*:

(i) *$\langle x, x^g \rangle$ is not solvable for some $g \in G$;*

(ii) *$p = 3$ and L is a finite simple group of Lie type defined over $\mathbb{F}_3$, a finite field with 3 elements, or $L \cong \mathrm{U}_n(2)$, $n \geq 4$. Moreover, the Lie rank of L is at least 2 unless $L \cong \mathrm{U}_3(3)$.*

Proof. This is Theorem A^* in [25]. □

Recall that a triple (G, M, H) with $H \trianglelefteq M \leq G$ is called a W-triple if $M \cap M^g \leq H$ for all $g \in G - M$.

Lemma 2.6. *Let G be a finite group and $H \trianglelefteq M \leq G$. Then (G, M, H) is a W-triple if and only if*

$$N_G(D) \leq M \text{ for all subgroups } D \leq M \text{ with } D \not\leq H.$$

Proof. This is Lemma 2.3 in [10]. □

Lemma 2.7 (Wielandt's theorem). *Let G be a finite group. If (G, M, H) is a W-triple, then G contains a normal subgroup K such that $G = MK$ and $M \cap K = H$. In particular, the triple (G, M, H) is special in G.*

Proof. The first claim is in [42] or [38, Exercise 1, p. 347] and the second is in [3, Lemma 9]. □

An automorphism θ of a finite group is Frobenius if each nontrivial power of θ is fixed point free.

Lemma 2.8. *Let G be a finite group and let D be a conjugacy class of G containing elements of order > 2. Assume that $G = \langle D \rangle$. Then some element of D induces a Frobenius automorphism on G' if and only if each pair of distinct elements in D generates a Frobenius group.*

Proof. This is [8, Satz I]. □

We also need the following results.

Lemma 2.9. *Let G be a finite group.*

(i) *Let π be a set of odd primes and suppose that the π-group P acts as a group of automorphisms on the solvable finite π'-group G. Then*

$$C_{[G,P]}(P) = \langle C_{[g,P]}(P) : g \in G \rangle.$$

(ii) *Let α be a coprime automorphism of odd order of G. Then*

$$C_{[G,\langle\alpha\rangle]}(\alpha) = \langle C_{[g,\langle\alpha\rangle]}(\alpha) : g \in [G, \langle\alpha\rangle] \rangle.$$

Proof. The first claim is [12, Theorem A] and the second is [14, Theorem 2]. □

We will use the next result repeatedly.

Lemma 2.10. *Let π be a nonempty set of primes. Let Q be a finite π-group which acts fixed point freely on a finite π'-group R, that is, $C_R(Q) = 1$, then R is solvable.*

Proof. For a proof, see Theorem 2.3 in [29]. □

We also need the following consequence of Burnside's normal p-complement theorem.

Lemma 2.11. *Let G be a finite group and let p be a prime. Let H be an abelian p-subgroup of G. Assume that $\langle H^G\rangle \cap N_G(H) = H$. Then $G = N_G(H)\,O_{p'}(G)$ and $\langle H^G\rangle$ is solvable. Moreover, $N_G(H)$ has a normal complement in G, which is $O_{p'}(\langle H^G\rangle)$.*

Proof. Let H be an abelian p-subgroup of G. Assume that $\langle H^G\rangle \cap N_G(H) = H$. Let $L = \langle H^G\rangle \trianglelefteq G$. Then $N_L(H) = L \cap N_G(H) = H$, so H is a self-normalizing abelian Sylow p-subgroup of L. In particular, $H \le Z(N_L(H))$; hence by Burnside's normal p-complement theorem [22, Theorem 7.4.3] $L = HO_{p'}(L)$. By Frattini's argument

$$G = N_G(H)\,L = N_G(H)\,O_{p'}(L) = N_G(H)\,O_{p'}(G).$$

The last equality holds since $O_{p'}(L) \le O_{p'}(G)$.

Let $L = \langle H^G\rangle$. Then $N_L(H) = H$, that is, H is a self-normalizing cyclic Sylow p-subgroup of L. By Burnside's normal p-complement theorem, L has a normal p-complement K , and hence $C_K(H) = 1$. By Lemma 2.10, K is solvable and thus $L = HK$ is solvable as well.

We now show that $O_{p'}(L)$ is a normal complement to $N_G(H)$ in G. To see this, observe that $L = HO_{p'}(L)$ and $H \cap O_{p'}(L) = 1$. Note that $O_{p'}(L) \trianglelefteq G$. Thus it suffices to show that $N_G(H) \cap O_{p'}(L) = 1$. Indeed, we have

$$N_G(H) \cap O_{p'}(L) = N_G(H) \cap L \cap O_{p'}(L) = H \cap O_{p'}(L) = 1. \qquad \square$$

Finally, we also need the following solvability result.

Lemma 2.12. *Let G be a finite group and let H be an abelian p-subgroup of G for some prime p. If H is extremely closed in G, then $\langle H, H^g\rangle$ is solvable for all $g \in G$.*

Proof. Let $g \in G$ and let $T = \langle H, H^g\rangle$. By Lemma 2.2(v), we have $T = HO_{p'}(T)$. Since $N_T(H) = H$, H acts fixed point freely and coprimely on $O_{p'}(T)$, the claim now follows from Lemma 2.10. $\square$

3. Extremely closed abelian p-subgroups

We are now ready to prove the main theorems. We first prove Theorem 1.4.

Proof of Theorem 1.4. Assume that M is a maximal subgroup of G, H is a maximal subgroup of M and that $\langle H, H^g\rangle \cap M = H$ for all $g \in G$. We will show that $\langle H^G\rangle \cap M = H$. The hypothesis implies that $H \trianglelefteq M$ and $|M : H|$ is a prime.

Clearly, if $H \trianglelefteq G$, then the conclusion holds. So, let $M = N_G(H)$. Suppose that there exists a subgroup $D \le M$ with $D \not\le H$ and $N_G(D) \not\le M$. Take $g \in N_G(D) - M$. Then $M = DH$ and $M^g = DH^g$. As $M \ne M^g$, $G = D\langle H, H^g\rangle$. We have

$$\langle H^G\rangle = \langle H^{D\langle H, H^g\rangle}\rangle \le \langle H, H^g\rangle \le \langle H^G\rangle$$

and so $\langle H^G\rangle = \langle H, H^g\rangle$. Thus $\langle H^G\rangle \cap M = \langle H, H^g\rangle \cap M = H$ and we are done.

So, we can assume that whenever $D \leq M$ with $D \not\leq H$ then $N_G(D) \leq M$. By Lemma 2.6, (G, M, H) is a W-triple and so the result follows from Lemma 2.7. □

Proof of Corollary 1.5. Let G be a finite nonabelian simple group. Let M be a maximal subgroup of G and let H be a normal subgroup of M such that $|M : H| = p$ is a prime. Suppose by contradiction that $G \neq \langle H, H^g \rangle$ for all $g \in G$. Since G is nonabelian simple, either $\langle H^G \rangle = G$ or $H = 1$. If $H = 1$, then M is a self-normalizing cyclic subgroup of G of prime order p and Burnside's normal p-complement theorem implies that G has a normal p-complement, a contradiction.

So $H \neq 1$ and $\langle H^G \rangle = G$. In particular, $M = N_G(H)$ and $\langle H^G \rangle \cap M = M > H$. Since H is maximal in M and M is maximal in G, Theorem 1.4 implies that there exists $g \in G$ such that $\langle H, H^g \rangle \cap M > H$. The maximality of H in M implies that $M \leq \langle H, H^g \rangle$. Hence $M = \langle H, H^g \rangle$ and $N_G(H^g) = M^g \neq M$ as $g \notin M$. Let $a \in M^g - M$, then $H^a H^g = M^a$ is a subgroup and $H^g \langle H, H^a \rangle = G$. Now $G = \langle H^G \rangle = H^{\langle H, H^a \rangle} \leq \langle H, H^a \rangle$, contradicting our assumption. Thus $G = \langle H, H^g \rangle$ for some $g \in G$. □

Proof of Theorem 1.6. Suppose we have proven that $\langle H^G \rangle \cap N_G(H) = H$. By Lemma 2.11, we have $G = N_G(H) O_{p'}(G)$ and $\langle H^G \rangle$ is solvable.

It remains to show that if H is extremely closed in G, then $\langle H^G \rangle \cap N_G(H) = H$. Let G be a counterexample to the claim with minimal order. Then H is a cyclic group of odd prime order p, $\langle H, H^g \rangle \cap N_G(H) = H$ for all $g \in G$ but $\langle H^G \rangle \cap N_G(H) \neq H$. Furthermore, since $|H| = p$ is prime, $H^g = H$ or $\langle H, H^g \rangle$ is a Frobenius group for all $g \in G$.

(1) We first claim that $\langle H^G \rangle = G$. Suppose by contradiction that $L := \langle H^G \rangle < G$. By Lemma 2.2(ii), H is pronormal in G and hence $L = \langle H^L \rangle$ by Lemma 2.1(vi). By Lemma 2.2(iii), H is extremely closed in L and so by the minimality of G, $\langle H^L \rangle \cap N_L(H) = H$. However, as $L = \langle H^L \rangle$, we have $N_L(H) = H$, and thus

$$\langle H^G \rangle \cap N_G(H) = L \cap N_G(H) = N_L(H) = H.$$

This contradiction proves the claim.

(2) Assume that $N_G(H) = C_G(H)$. Write $H = \langle x \rangle$. Then $\langle x, x^g \rangle$ is a Frobenius group for all $g \in G$. Hence x acts as a Frobenius automorphism on G' by Lemma 2.8 and so $G'H$ is a Frobenius group (as H has prime order). In particular, we have $G' \leq O_{p'}(G)$ and $G = N_G(H) O_{p'}(G)$ in this case. In addition, $\langle H^G \rangle = G'H$ and $\langle H^G \rangle \cap N_G(H) = H$, a contradiction.

(3) Assume that $N_G(H) > C_G(H)$. Let M be a maximal normal subgroup of G. Then G/M is a simple group. Since $G = \langle H^G \rangle$, we have $H \not\leq M$. Assume that $M > 1$. Then $|G/M| < |G|$. Lemma 2.2(vi) implies that HM/M is extremely closed in G/M. Hence, as G/M is simple, $G = HM$. Now $N_G(H) = N_M(H)H = C_G(H)$, which is a contradiction. Hence $M = 1$ and G is a nonabelian simple group.

(4) By Lemma 2.12, $\langle x, x^g\rangle$ is solvable for all $g \in G$ and so by Lemma 2.5, $p = 3$ and G is a finite simple group of Lie type defined over $\mathbb{F}_3$ or $G \cong \mathrm{U}_n(2)$ with $n \geq 4$. Furthermore, except for $\mathrm{U}_3(3)$, the Lie rank of G is strictly greater than 1. Now it is easy to see that a Sylow 3-subgroup of G is nonabelian. Let P be a Sylow 3-subgroup of G containing x. Then $H = \langle x\rangle$ is isolated in P with respect to G, that is, H does not conjugate in G to any subgroup in $P - H$. By [23, Theorem 4.250], we deduce that $G \cong \mathrm{U}_3(3)$. By [25, Theorem A*], x is a transvection. Now for any conjugate x^g of x different from x and x^{-1}, we have that $T := \langle x, x^g\rangle \cong \mathrm{SL}_2(3)$. However, $\mathrm{SL}_2(3)$ is not a Frobenius group. □

We will need the following result which is a consequence of Theorem A^* in [25] and Theorem 1.2 in [15].

Proposition 3.1. *Let G be a finite group and let p be an odd prime. Let H be a nontrivial abelian p-subgroup of G. Assume that G has a unique minimal normal subgroup N which is nonabelian such that $G = HN$. Then H is not extremely closed in G.*

Proof. Suppose by contradiction that H is an abelian extremely closed p-subgroup of G. By Lemma 2.12, we have $\langle H, H^g\rangle$ is solvable for all $g \in G$. In particular, if $x \in H$, then $\langle x, x^g\rangle$ is solvable for all $g \in G$.

By the uniqueness of N, we have $G = \langle H^G\rangle = HN$. We first show that G is an almost simple group with socle S. Let $W \trianglelefteq N$ be a component of G. Assume that $W \neq N$. As $W \trianglelefteq N$, $N \leq N_G(W)$ and so $N_G(W) = N_H(W)N$. Since $W < N$, $N_G(W) < G$ and so $N_H(W) < H$. Let $x \in H - N_H(W)$. Then $x \notin N_G(W)$ and $x^2 \notin C_G(W)$ since p is odd, whence x and W satisfy the hypothesis of Lemma 2.4 and hence $\langle x, x^g\rangle$ is not solvable for some $g \in G$, a contradiction. Thus $W \trianglelefteq G$ and so $G = HN$, where N is a nonabelian simple group which is also a minimal normal subgroup of G. Thus G is an almost simple group with socle S as wanted.

Let $x \in H$ be an element of order p. Then $\langle x, x^g\rangle$ is solvable for all $g \in G$. By Lemma 2.5, $p = 3$ and S is a finite simple group of Lie type defined over $\mathbb{F}_3$ or $S \cong \mathrm{U}_n(2)$, $n \geq 4$. If $|H| = 3$, then $\langle H^G\rangle$ is solvable by Theorem 1.6 and Lemma 2.11, which is impossible. Thus we may assume that $|H| \geq 9$. Since $G = HS$, we have $G/S \cong H/(H \cap S)$ is a 3-group and thus if $G \neq S$, then elements in $H - S$ induce outer automorphisms of S of 3-power order.

(i) Assume $S \cong \mathrm{U}_n(2)$, $n \geq 4$ or $G \neq S$. If $S \cong \mathrm{U}_n(2)$, then since $n \geq 4$, we have $|\mathrm{Out}(\mathrm{U}_n(2))| = 2(n, 3)$. If $G \neq S$ and $S \ncong \mathrm{U}_n(2)$, then $S \cong \mathrm{D}_4(3)$ or ${}^3\mathrm{D}_4(3)$ since S has no nontrivial field automorphism so $\mathrm{Out}(S)$ contains diagonal automorphisms and possibly graph automorphisms only. In all cases, the Sylow 3-subgroup of $\mathrm{Out}(S)$ has order at most 3. Hence $|G : S| = |H : S \cap H| \leq 3$.

Since $|H| \geq 9$, if G is not simple then $A = H \cap S > 1$. By Lemma 2.3(ii), A is strongly closed in S. If $G = S = \mathrm{U}_n(2)$, then $A = H$ is strongly closed in S. In

both cases, S contains a nontrivial abelian 3-subgroup A which is strongly closed in S. Since S is simple, $S = \langle A^S \rangle$. As a Sylow 3-subgroup of S is nonabelian, A cannot be a Sylow 3-subgroup of S; however this contradicts [15, Theorem 1.2(i)].

(ii) $G = S$ is a finite simple group of Lie type defined over $\mathbb{F}_3$. The possibilities for G can be read off from [25, Table 1] and note the correction in [37, Remark 5.2].

Let P be a Sylow p-subgroup of G containing H, and let M be a maximal parabolic subgroup of G containing the Borel subgroup $B := N_G(P)$. Let $R = O_p(M)$. Then $M = N_G(R)$. By induction, we have $\langle H^M \rangle \cap N_M(H) = H$, and hence $\langle H^M \rangle = HU$, where $U = O_{p'}(\langle H^M \rangle)$ and $C_U(H) = 1$. Therefore, U is a solvable normal p'-subgroup of M since $\langle H^M \rangle \trianglelefteq M$. As $H \leq \langle H^M \rangle \trianglelefteq M$ and H is pronormal in M, we have $M = N_M(H)U$. However, as M is a parabolic subgroup of a finite simple group of Lie type G, $F^*(M) = O_p(M)$ (Corollaries 3.1.4 and 3.1.5 in [24]), so $U = 1$. Hence $H \trianglelefteq M$ and $M = N_G(H)$. Thus M contains every maximal parabolic subgroup of G that contains the Borel subgroup B. However this happens only when the Lie rank of G is 1. Therefore $G \cong \mathrm{U}_3(3)$. (We can also use Theorem 1.2 in [15] to arrive at this conclusion.)

From the Atlas [7], we have $P \cong 3^{1+2}_+$ is an extraspecial group of order 27 and exponent 3. Thus $|H| = 9$. Since G has Lie rank 1, the Borel subgroup B of G is a maximal subgroup of G. Hence $B = N_G(H)$. Let $g \in G - B$ and $T = \langle H, H^g \rangle$. We know that $T = HO_{3'}(T)$ is solvable, so $T < G$ and thus T lies in some maximal subgroup of G whose order must be divisible by $|H| = 9$. Inspecting the list of maximal subgroups of G in the Atlas [7], the only maximal overgroups of H and T in G are the Borel subgroups B and its G-conjugates. Hence $\langle H, H^g \rangle \leq B^t$ for some $t \in G$. Note that $B = P : W$, where $W = \langle y \rangle$ is a cyclic group of order 8. Since B^t has a normal Sylow 3-subgroup, we must have that $H \leq P^t \trianglelefteq B^t$ and hence H is subnormal in B^t. Since H is pronormal in G and hence in B^t, we have $H \trianglelefteq B^t$ or $B^t = N_G(H) = B$. Therefore, $\langle H, H^g \rangle \leq B$ for all $g \in G$. However, this implies that $\langle H^G \rangle \leq B$, which is a contradiction. □

Proof of Theorem 1.7. Let G be a counterexample to Theorem 1.7 with minimal order. Then $\langle H, H^g \rangle \cap N_G(H) = H$ for all $g \in G$ but $G \neq N_G(H)\, O_{p'}(G)$, where H is an abelian p-subgroup of G.

(1) $\langle H^G \rangle = G$. Let $L = \langle H^G \rangle$. Assume $L \neq G$. By Lemmas 2.2(ii) and 2.1(iii), we have $G = N_G(H)\, L$. By Lemma 2.2(iii) and the minimality of G, we have $L = N_L(H)\, O_{p'}(L)$. As $L \trianglelefteq G$, $O_{p'}(L) \leq O_{p'}(G)$, and hence

$$G = N_G(H)\, L = N_G(H)\, O_{p'}(L) = N_G(H)\, O_{p'}(G).$$

This contradictions shows that $L = G$.

(2) $O_{p'}(G) = 1$. Suppose by contradiction that $N = O_{p'}(G) \neq 1$. It follows from Lemma 2.2(vi) that the hypothesis carries over to $\overline{G} = G/N$, and so by the minimality

of G, we obtain $\overline{G} = N_{\overline{G}}(\overline{H})\, O_{p'}(\overline{G})$. But $O_{p'}(\overline{G}) = O_{p'}(G/O_{p'}(G)) = 1$, and hence $\overline{G} = \overline{N_G(H)}$ by Lemma 2.2(iv). Therefore, $G = N_G(H)N = N_G(H)\, O_{p'}(G)$, which is a contradiction. This proves the claim.

(3) $O_p(G) = 1$. Assume $O_p(G) \neq 1$. Let $N \leq O_p(G)$ be a minimal normal subgroup of G and let $U = H \cap N$. Observe first that $O_p(G) \leq N_G(H)$ since $P \leq N_G(H)$ for any $H \leq P \in \mathrm{Syl}_p(G)$. It follows that $N \leq N_G(H)$. We will show that $N \leq Z(G)$. Suppose first that $U \neq 1$. Lemma 2.3(iii) implies that $U \trianglelefteq G$. By the minimality of N, we have $U = N$, and so $N \leq H$. As H is abelian, $H \leq C_G(N) \trianglelefteq G$. By (1), we have $C_G(N) = G$, and $N \leq Z(G)$. If $U = H \cap N = 1$, then as H and N are both normal in $N_G(H)$, we have $[H, N] \leq H \cap N = 1$ and thus $H \leq C_G(N) \trianglelefteq G$, so $N \leq Z(G)$.

By Lemma 2.2(vi), the hypothesis carries over to $\overline{G} = G/N$, and so by the minimality of G, we obtain $\overline{G} = N_{\overline{G}}(\overline{H})\, O_{p'}(\overline{G})$. Let $K \leq G$ be such that $N \leq K$ and $K/N = O_{p'}(G/N)$. Then $N \trianglelefteq K \trianglelefteq G$ and $|K : N|$ is odd. By the Schur–Zassenhaus theorem, $K = NT$ where T is Hall p'-subgroup of G. Moreover, since N is central in G, $T \trianglelefteq K$ and hence $T \trianglelefteq G$ as it is characteristic in $K \trianglelefteq G$. It follows that $T \leq O_{p'}(G) = 1$ and thus $G = N_G(H)K = N_G(H)N = N_G(H)$. This contradiction proves the claim.

(4) Let N be a minimal normal subgroup of G. We claim that $G = HN$ and N is the unique minimal normal subgroup of G. By (2) and (3), $N \cong S^k$ for some finite nonabelian simple group S with $p \mid |S|$ and some integer $k \geq 1$. Let $M = HN$. Suppose that $M < G$. By the minimality of G, we have $M = N_M(H)\, O_{p'}(M)$. Hence $O_{p'}(M) \leq N$, and so $O_{p'}(M) = 1$. We deduce that $H \trianglelefteq M$. As H is an abelian p-group and $N \cong S^k$, we have $H \cap N = 1$ and hence $[H, N] \leq H \cap N = 1$. Thus $H \leq C_G(N) \trianglelefteq G$. By (1), we have $C_G(N) = G$, and then $N \leq Z(G)$. This contradiction shows that $G = HN$. Since $G/N \cong H/H \cap N$ is abelian, N must be a unique minimal normal subgroup of G as wanted.

If p is odd, then Proposition 3.1 yields a contradiction. Thus for the remaining, we assume that $p = 2$.

(5) We next claim that G is finite nonabelian simple group. By (4), $F^*(G) = N$ and H is a strongly closed abelian 2-subgroup of G. Now by [20, Theorem A], we have $F^*(G) = G$ and so $N = G$. It follows that $G = S$ is simple. By [20, Theorem A] again, G is isomorphic to one of the following groups:

(i) $\mathrm{L}_2(2^n)$, $n \geq 3$; ${}^2\mathrm{B}_2(2^{2n+1})$, $n \geq 1$; or $\mathrm{U}_3(2^n)$, $n \geq 2$.

(ii) $\mathrm{L}_2(q)$, $q \equiv 3, 5 \pmod 8$.

(iii) ${}^2\mathrm{G}_2(3^{2n+1})$, $n \geq 1$; or J_1, the first Janko group.

By Glauberman's Z^*-theorem, we may assume that $|H| \geq 4$.

(6) The final contradiction. We now consider each case above separately.

(a) Assume G is isomorphic to one of the groups in (i). Let $H \leq P \in \mathrm{Syl}_2(G)$ and let $B = PT$ be the Borel subgroup of G containing P. By [20, (3.2)], $H = Z(P)$ is a noncyclic elementary abelian 2-group, P is a $T.I$ subgroup of G, and B is the unique maximal subgroup of G containing P. It follows that $B = N_G(H)$. Observe that for any $1 \neq x \in H$, $P \leq C_G(x) \leq B$, as P is uniquely contained in B. For $g \in G - B$, let $T = \langle H, H^g \rangle$. Then $N_T(H) = H$ and then by Burnside's normal p-complement theorem, we have $T = HU$, where $U = O_{2'}(T) \trianglelefteq T$. As H is noncyclic abelian, $U = \langle C_U(a) : 1 \neq a \in H \rangle$ [35, 8.3.4]. However, as $C_G(a) \leq B$ for all $1 \neq a \in H$, we have $U \leq B$ and so $T = \langle H, H^g \rangle \leq B$. From the hypothesis, we must have $T = T \cap B = H$, and hence $H^g = H$. This implies that $g \in B$, contradicting the choice of g.

(b) Assume that $G \cong \mathrm{L}_2(5) \cong \mathrm{A}_5$. By [20, (3.4)], $H \in \mathrm{Syl}_2(G)$, $N_G(H) \cong \mathrm{A}_4$ and $N_G(H)$ is the unique maximal subgroup of G containing $C_G(a)$, for all $1 \neq a \in H$. Take $g \in G - N_G(H)$, and let $T = \langle H, H^g \rangle$. Then $T = HU$, where $U = O_{2'}(T)$. As H is noncyclic, $U = \langle C_U(a) : 1 \neq a \in H \rangle \leq N_G(H)$. This leads to a contradiction as in the previous case.

(c) Assume G is isomorphic to one of the groups in (ii) with $q \geq 11$. By [20, (3.4)], we have $H \in \mathrm{Syl}_2(G)$ and $N_G(H) \cong \mathrm{A}_4$. Clearly, G contains a maximal subgroup M isomorphic to the dihedral group $D_{q\pm1}$ such that M does not contain H. Assume M is generated by two involutions a, b. We can choose a, b such that $b \in H$. Now $G = \langle M, H \rangle \leq \langle a, H \rangle$, and hence $G = \langle a, H \rangle$. By Sylow's theorem, there exists some $g \in G$ such that $a \in H^g$. Thus $G = \langle a, H \rangle \leq \langle H^g, H \rangle$, and then $G = \langle H, H^g \rangle$, which contradicts the hypothesis that $\langle H, H^g \rangle \cap N_G(H) = H$.

(d) Finally, assume that G is isomorphic to one of the groups in (iii). By [20, (3.4)], we have $H \in \mathrm{Syl}_2(G)$. Now H contains an involution t such that $C_G(t) = \langle t \rangle \times L$, where $L \cong \mathrm{L}_2(q)$, $q \equiv 3, 5 \pmod 8$ and $[G : C_G(t)]$ is odd [41]. As $H \leq C_G(t) < G$, by the minimality of G, we obtain $C_G(t) = (C_G(t) \cap N_G(H))\, O_{2'}(C_G(t))$. However, as $C_G(t) = \langle t \rangle \times L$, where L is nonabelian simple, it follows that $O_{2'}(C_G(t)) = 1$ whence $C_G(t) \leq N_G(H)$. Hence $C_G(t) = N_G(H)$ since $C_G(t)$ is maximal in G by [20, (3.4)]. It follows that $H \trianglelefteq C_G(t)$ and then $H \cap L \trianglelefteq L$, where $H \cap L \in \mathrm{Syl}_2(L)$, which contradicts the simplicity of L. □

Proof of Corollary 1.8. Let G be a counterexample to the corollary with minimal order. Then we have that H is an abelian p-subgroup for some odd prime p and $\langle H, H^g \rangle \cap N_G(H) = H$ for all $g \in G$ but $\langle H^G \rangle \cap N_G(H) \neq H$. By Theorem 1.7, we have $G = N_G(H)\, O_{p'}(G)$. Now we have that

$$\langle H^G \rangle = \langle H^{N_G(H) O_{p'}(G)} \rangle = \langle H^{O_{p'}(G)} \rangle \leq H O_{p'}(G).$$

Let $L = \langle H^G \rangle$. Then $L = H(L \cap O_{p'}(G)) = H O_{p'}(L)$ and so L has a normal p-complement. Let $L < G$. By the minimality of G, we have $\langle H^L \rangle \cap N_L(H) = H$.

However, $L = \langle H^L \rangle$ by Lemmas 2.2(ii) and 2.1(vi) and thus $N_L(H) = H$ or equivalently $\langle H^G \rangle \cap N_G(H) = H$, a contradiction. Therefore, we can assume that $G = \langle H^G \rangle = HO_{p'}(G)$.

Let $U = O_{p'}(G)$ and let $Q = [U, H]$. By [35, 8.2.7], $U = QC_U(H)$ and $Q = [Q, H]$. Furthermore, $N_G(H) = C_G(H) = HC_U(H)$ and thus $G = HC_U(H)Q$. As $G = \langle H^G \rangle = H^{HC_U(H)Q} \leq HQ$, we obtain $G = HQ$ whence $U = Q$.

Let $N \leq U$ be a minimal normal subgroup of G. Since HN/N is an abelian extremely closed p-subgroup of G/N by Lemma 2.2(vi) and $G/N = \langle (HN/N)^{G/N} \rangle$, we have $N_G(HN) = HN$ and hence by Lemma 2.2 (iv), $N_G(H)N = HN$. Moreover, $C_{U/N}(HN/N) = 1$. It follows that U/N is solvable by Lemma 2.10.

Assume that N is abelian. Then U is solvable. Let $1 \neq u \in U$ and let $T = \langle H, H^u \rangle$. Then $T = H[u, H]$, where $[u, H] = O_{p'}(T)$ and $C_{[u,H]}(H) = 1$ as H is self-normalizing in T. By Lemma 2.9(i), $C_{[U,H]}(H) = \langle C_{[u,H]}(H) : u \in U \rangle = 1$. Thus $C_U(H) = C_{[U,H]}(H) = 1$ and so $N_G(H) = C_G(H) = HC_U(H) = H$. Therefore, $\langle H^G \rangle \cap N_G(H) = H$, which is a contradiction.

Assume that $N \cong S^k$, where S is a nonabelian simple group and $k \geq 1$ is an integer. Assume that $K = HN < G$. By the minimality of G, we have $\langle H^K \rangle \cap N_K(H) = H$. Thus $\langle H^K \rangle = HQ$ and $C_Q(H) = 1$, where $Q = O_{p'}\langle H^K \rangle$. As Q is characteristic in $\langle H^K \rangle \trianglelefteq K$, it follows that $Q \trianglelefteq K$. Since $|K : N|$ is a power of p and Q is a p'-group, we must have $Q \leq N$ and hence $Q \trianglelefteq N$. By [35, 1.7.5], Q is isomorphic to a direct product of the nonabelian simple group S, so Q is not solvable or $Q = 1$. If the former case holds, then since $C_Q(H) = 1$ and Q is a p'-group, Lemma 2.10 implies that Q is solvable, which is a contradiction as it is a direct product of copies of S. Therefore, $Q = 1$ and hence $H \trianglelefteq K$. It follows that $R = H \cap N \trianglelefteq N$. However, as $N \cong S^k$ and $H \cap N$ is a normal p-subgroup of N, we must have $H \cap N = 1$. Hence $[H, N] \leq H \cap N = 1$ and so $H \leq C_G(N) \trianglelefteq G$. Since $G = \langle H^G \rangle$, we have $C_G(N) = G$ or $N \leq Z(G)$, a contradiction. Therefore $G = HN$ and since G/N is solvable, N is a unique minimal normal subgroup of G. Now Proposition 3.1 yields a contradiction. □

Proof of Corollary 1.9. Let H be an extremely closed abelian p-subgroup of G for some prime p. Assume first that $p = 2$. By Theorem 1.7, $G = N_G(H)\, O_{2'}(G)$ which implies that $HO_{2'}(G) \trianglelefteq G$ and clearly $HO_{2'}(G)$ is solvable and thus $H \subseteq R(G)$. Assume now that p is odd. By Corollary 1.8, we have $N_G(H) \cap \langle H^G \rangle = H$ and thus $\langle H^G \rangle$ is solvable by Lemma 2.11. □

4. A p-solvability criterion

Let p be a prime. A finite group G is said to be a minimal non-p-solvable group if G is not p-solvable but every proper subgroup of G is p-solvable. A minimal simple group is a nonabelian finite simple groups whose all proper subgroups are solvable.

Observe that minimal non-2-solvable simple groups are exactly the minimal simple groups and these groups are classified by Thompson in [39].

Lemma 4.1. *Every minimal simple group is isomorphic to one of the following simple groups*:

(1) $\mathrm{L}_2(2^r)$, *r is a prime.*

(2) $\mathrm{L}_2(3^r)$, *r is an odd prime.*

(3) $\mathrm{L}_2(r)$, *$r > 3$ is a prime such that $5 \mid r^2 + 1$.*

(4) ${}^2\mathrm{B}_2(2^r)$, *r is an odd prime.*

(5) $\mathrm{L}_3(3)$.

Proof. This is [39, Corollary 1]. □

The next result classifies minimal non-3-solvable simple groups.

Lemma 4.2. *Let G be a finite nonabelian simple group. Assume that every proper subgroup of G is 3-solvable. Then G is isomorphic to a minimal simple group or to the Suzuki group ${}^2\mathrm{B}_2(q)$ with $q = 2^{2m+1}$, $m \geq 1$.*

Proof. This is Lemma 5.3 in [28]. □

Finally, we need the classification of finite non-p-solvable simple groups for any primes $p \geq 5$.

Lemma 4.3. *Let G be a finite nonabelian simple group and let $p \geq 5$ be a prime dividing $|G|$. Assume that every proper subgroup of G is p-solvable. Then one of the following holds.*

(1) $G = \mathrm{L}_2(p)$.

(2) $G = \mathrm{A}_p$.

(3) $G = \mathrm{L}_2(q)$ *with* $p \mid q^2 - 1$.

(4) $G = \mathrm{L}_n(q)$, *$n \geq 3$ is odd, and p divides $q^n - 1$ but not $\prod_{i=1}^{n-1}(q^i - 1)$.*

(5) $G = \mathrm{U}_n(q)$, *$n \geq 3$ is odd, and p divides $q^n - (-1)^n$ but not $\prod_{i=1}^{n-1}(q^i - (-1)^i)$.*

(6) $G = {}^2\mathrm{B}_2(q)$ *with* $q = 2^{2m+1}$, $m \geq 1$.

(7) $G = {}^2\mathrm{G}_2(q)$ *and* $p \mid (q^2 - q + 1)$, *where* $q = 3^{2m+1}$, $m \geq 1$.

(8) $G = {}^2\mathrm{F}_4(q)$ *with* $q = 2^{2m+1}$, $m \geq 1$ *and* $p \mid (q^4 - q^2 + 1)$.

(9) $G = {}^3\mathrm{D}_4(q)$ *and* $p \mid (q^4 - q^2 + 1)$.

(10) $G = \mathrm{E}_8(q)$ *and p divides $(q^{30} - 1)$ but not $\prod_{i \in \{8,14,18,20,24\}}(q^i - 1)$.*

(11) (G, p) *is one of the following*: $(\mathrm{M}_{23}, 23)$, $(\mathrm{J}_1, 7$ *or* $19)$, $(\mathrm{Ly}, 37$ *or* $67)$, $(\mathrm{J}_4, 29$ *or* $43)$, $(\mathrm{Fi}'_{24}, 29)$, $(\mathrm{B}, 47)$ *or* $(\mathrm{M}, 41$ *or* 59 *or* $71)$.

Proof. This is Lemma 5.4 in [28]. □

Let q be a prime power and let $n \geq 2$ be an integer. A prime divisor p of $q^n - 1$ is called a primitive prime divisor or ppd of $q^n - 1$ if p does not divide $q^k - 1$ for all integers k with $1 \leq k < n$. Zsigmondy's theorem [43] states that such a ppd p exists unless $(n, q) = (6, 2)$ or $n = 2$ and q is a Mersenne prime. Now if $n > 1$ is an integer and p is a prime, then the p-part of n, denoted by n_p, is the largest power of p dividing n. We refer the reader to [5; 34] for the description of maximal subgroups of finite simple groups of Lie type.

Proposition 4.4. *Let G be a finite nonabelian simple group and p be a prime dividing $|G|$. Assume that every proper subgroup of G is p-solvable. Let P be a Sylow p-subgroup of G. Then $G = \langle P, P^g \rangle$ for some $g \in G$.*

Proof. Let G be a finite nonabelian simple group and let p be a prime dividing $|G|$. Let $P \in \mathrm{Syl}_p(G)$. Assume that every proper subgroup of G is p-solvable. Let $x \in P$ with $|x| = p$. Then $\langle x, x^g \rangle \leq \langle P, P^g \rangle$ for all $g \in G$ and thus $\langle x, x^g \rangle$ is p-solvable for all $g \in G$.

(a) If $p = 2$, then every finite nonabelian simple group G can be generated by two Sylow 2-subgroups by Theorem A in [26]. So, we may assume that $p > 2$.

(b) If G is a finite nonabelian simple group of Lie type in characteristic p, then G is generated by two Sylow p-subgroups by Proposition 2.5 in [6].

(c) Assume that $p = 3$. By Lemma 4.2, since 3 divides $|G|$, G is a minimal simple group. By part (b), we only need to consider the cases when G is isomorphic to $\mathrm{L}_2(2^r)$, r is a prime, or $\mathrm{L}_2(r)$, $r > 3$ is a prime and $5 \mid r^2 + 1$.

If $G \cong \mathrm{L}_2(4)$, then we can check by using GAP [17] that there exists $g \in G$ such that $G = \langle P, P^g \rangle$. Assume next that $G \cong \mathrm{L}_2(q)$, $q = 2^r$ or r, where r is an odd prime. By [25, Theorem A*], there exists an element $x \in G$ of order 3 such that $\langle x, x^g \rangle$ is nonsolvable for some $g \in G$. Since G is minimal simple, we must have $G = \langle x, x^g \rangle$ and thus $G = \langle P, P^g \rangle$.

(d) Assume that $p \geq 5$. By part (b), and Lemma 4.3, G is one of the groups listed in (2)–(11) in that lemma. We now consider each case in turn.

(1) Assume $G = \mathrm{A}_p$. Here $|P| = p$. Without loss of generality, take $P = \langle x \rangle$, where $x = (1, 2, \ldots, p)$ is a p-cycle in A_p. Let $y = (1, 2, p, p-1, p-2, \ldots, 3) \in \mathrm{A}_p$ be another p-cycle. Then $xy = (1, 3, 2)$ and clearly

$$\langle x, y \rangle = \langle (xy)^{-1}, x \rangle = \langle (1, 2, 3), (1, 2, \ldots, p) \rangle = \mathrm{A}_p$$

(see, e.g., [33, Theorem B]). Hence G is generated by two Sylow p-subgroups.

(2) Assume $G = \mathrm{L}_2(q)$ with $p \mid q^2 - 1$. If $q \leq 11$, we can check using GAP [17] that the result holds. Assume $q \geq 13$. Inspecting the argument in [25, Section 5.1.2], if x is any element of order p, then we can find $g \in G$ such that $\langle x, x^g \rangle \cong \mathrm{L}_2(q)$ and hence $\langle P, P^g \rangle = G$.

(3) Assume $G = \mathrm{L}_n(q)$, $n \geq 3$ is odd, and p divides $q^n - 1$ but not $\prod_{i=1}^{n-1}(q^i - 1)$. Write $q = s^f$, where s is a prime and $f \geq 1$ is an integer. In this case p is a ppd of $q^n - 1$. Hence $P \in \mathrm{Syl}_p(G)$ is cyclic of order $(q^n - 1)_p$. Since $n \equiv 1 \bmod p$ and $n \geq 3$ is odd, we have $p \geq 2n + 1$ and $p \nmid n$.

Assume that t is a prime divisor of n and write $n = tm$ for some integer $m \geq 1$. Assume that $m > 1$. Then G has a $\mathcal{C}_3$-subgroup H of type $\mathrm{GL}_m(q^t)$ (see [34, Table 3.5A]) which is maximal and contains a Sylow p-subgroup of G. Since $n \geq 3$ is odd, $m, t \geq 3$ and so $q^t \geq 2^t \geq 8$. Therefore, H is not p-solvable.

Thus we can assume that $n = t$ is an odd prime. If P lies in a unique maximal subgroup H of G, then H is of type $\mathrm{GL}_1(q^n)$ by [2, Table B]. We can choose $g \in G - H$ such that $P^g \not\leq H$ and hence $G = \langle P, P^g \rangle$. Assume that P lies in some other maximal subgroup M of G not of type $\mathrm{GL}_1(q^n)$. As in the proof of Case 3 of Proposition 6.2 in [2], $M \in \mathcal{C}_5$ is a subfield subgroup of type $\mathrm{GL}_n(q_0)$, where $q = q_0^k$, k is an odd prime and $(q_0^n - 1)_p = (q^n - 1)_p$ or $M \in \mathcal{S}$ is almost simple with socle $S \cong \mathrm{L}_2(p)$ and $n = \frac{1}{2}(p - 1)$. However, in both cases, M is not p-solvable.

(4) Assume $G = \mathrm{U}_n(q)$, $n \geq 3$ is odd, and p divides $q^n + 1$ but not $\prod_{i=1}^{n-1}(q^i - (-1)^i)$. Write $q = s^f$ where s is a prime and $f \geq 1$.

(a) Assume that $n = 3$. If $q = 3$, then $p = 7$. In this case, $|P| = 7$ and P lies in $\mathrm{L}_2(7)$ which is not 7-solvable. Similarly, if $q = 5$, then $p = 7$ and $|P| = 7$ and P lies in A_7.

First, let q be a prime. Let H be a maximal subgroup of G containing P. By the proof of [2, Proposition 6.3], either P lies in a unique maximal subgroup of G and we are done or P is contained in $\mathrm{L}_2(7)$ and $p = 7$; however, $\mathrm{L}_2(7)$ is not 7-solvable.

Assume $q = s^f$ with $f > 1$. In this case, if P is not contained in a unique maximal subgroup, then P can be contained in a subfield subgroup of type $\mathrm{GU}_3(q_0)$ with $q = q_0^k$, and k is an odd prime (see [5, Table 8.5]). However such a maximal subgroup is not p-solvable.

(b) Assume $n \geq 5$. Then p is a ppd of $q^{2n} - 1$. Hence $p \geq 2n + 1$.

Assume that $n = tm$, where t is a prime divisor of n and $m > 1$. Since $n \geq 5$ is odd, $t, m \geq 3$. Then G has a maximal subgroup of type $\mathrm{GU}_m(q^t)$ and contains a Sylow p-subgroup of G. Since $q^t \geq 2^t \geq 8$, such a maximal subgroup is not p-solvable.

Therefore, $n = t \geq 5$ is a prime. Argue as in case (3), if P lies in a unique maximal subgroup of G, then the conclusion holds. As in the proof of Proposition 6.4 in [2], P lies in a subfield subgroup H of type $\mathrm{GU}_n(q_0)$, where $q_0^k = q$ and $k \geq 3$ is a prime, or of type $\mathrm{O}_n(q)$ or H is an almost simple group with socle $\mathrm{L}_2(p)$ with $n = \frac{1}{2}(p - 1)$ and $7 \leq p \equiv 3 \bmod 4$. However, in all cases, these maximal subgroups are not p-solvable.

(5) $G = {}^2\mathrm{B}_2(q)$ with $q = 2^{2m+1}$, $m \geq 1$. Then $|G| = q^2(q-1)(q+s+1)(q-s+1)$, where $s = \sqrt{2q} = 2^{m+1}$. The maximal subgroups of G are listed in [5, Table 8.16]. Since p is not the characteristic of G, $p > 2$ and $p \mid q-1$ or $p \mid q \pm s + 1$.

Assume first that $p \mid q-1$. Then P lies in maximal subgroups of the form $[q^2] : (q-1)$ and $D_{2(q-1)}$. It follows that $\langle P, P^g \rangle$ is solvable for all $g \in G$. Let $x \in P$ with $|x| = p \geq 5$. Then $\langle x, x^g \rangle$ is solvable for all $g \in G$. However, this is impossible in view of Theorem A* in [25].

Assume that $p \mid q \pm s + 1$. In this case, P lies in a maximal subgroup of the form $(q \pm s + 1) : 4$ or a subfield subgroup of the form ${}^2\mathrm{B}_2(q_0)$, where $q_0^k = q$, $k \geq 3$ is a prime and $q_0 > 2$. Clearly, the subfield subgroup is not p-solvable (if it contains P). Hence P lies in a unique maximal subgroup of G and the result follows.

(6) $G = {}^2\mathrm{G}_2(q)$ with $q = 3^{2m+1}$, $m \geq 1$ and $p \mid q^2 - q + 1$. We can use the same argument as in the previous case using [5, Table 8.43].

(7) $G = {}^2\mathrm{F}_2(q)$ with $q = 2^{2m+1}$, $m \geq 1$ and $p \mid q^4 - q^2 + 1$. In this case, p is a ppd of $q^{12} - 1$. Using the argument in Proposition 7.2 in [2], either P lies in a unique maximal subgroup or it lies in a subfield subgroup ${}^2\mathrm{F}_2(q_0)$, which is not p-solvable.

(8) $G = {}^3\mathrm{D}_4(q)$ and $p \mid q^4 - q^2 + 1$. We can use the argument in Proposition 7.3 in [2] to obtain the conclusion as in the previous case.

(9) $G = \mathrm{E}_8(q)$ and p divides $(q^{30} - 1)$ but not $\prod_{i \in \{8,14,18,20,24\}}(q^i - 1)$. In this case, p is a ppd of $q^{30} - 1$. From Proposition 7.10 in [2], either P lies in a unique maximal subgroup and the result follows or P can lie in a maximal exotic local subgroup $2^{5+10} \cdot \mathrm{L}_5(2)$ when $|P| = p = 31$ or P lies in an almost simple group. In the last two possibilities, clearly, these maximal subgroups are not p-solvable.

(10) (G, p) is one of the following: $(\mathrm{M}_{23}, 23)$, $(\mathrm{J}_1, 7 \text{ or } 19)$, $(\mathrm{Ly}, 37 \text{ or } 67)$, $(\mathrm{J}_4, 29 \text{ or } 43)$, $(\mathrm{Fi}'_{24}, 29)$, $(\mathrm{B}, 47)$ or $(\mathrm{M}, 41 \text{ or } 59 \text{ or } 71)$.

By [2, Table D], P lies in the unique maximal subgroup of G and the result follows except for the case $(G, p) = (\mathrm{J}_1, 7)$. By the Atlas [7], the maximal subgroups of J_1 containing a Sylow 7-subgroup are isomorphic to either $2^3 : 7 : 3$ or $7 : 6$. Thus $\langle x, x^g \rangle$ is solvable for all $g \in \mathrm{J}_1$, where $x \in P$ with $|x| = 7$. However, this contradicts Theorem A* in [25]. □

Remark 4.5. It is conjectured in [6] that if G is a finite nonabelian simple group and if r and s are prime divisors of $|G|$, then G can be generated by a Sylow r-subgroup and a Sylow s-subgroup. The previous proposition is just a special case of this conjecture when $r = s = p$ and G is a minimal non-p-solvable simple group.

Proof of Theorem 1.10. Let G be a finite group and let p be a prime. Let P be a Sylow p-subgroup of G. If G is p-solvable, then every subgroup of G is p-solvable. Therefore, it suffices to show that if $\langle P, P^g \rangle$ is p-solvable for all $g \in G$, then G is

p-solvable. Suppose not and let G be a counterexample with minimal order. Then $\langle P, P^g\rangle$ is p-solvable for all $g \in G$ but G is not solvable.

We first claim that every proper subgroup of G is p-solvable and thus G is a minimal non-p-solvable group. Let H be a proper subgroup of G and let Q be a Sylow p-subgroup of H. Then $Q \le P^t$ for some $t \in G$. Now for every $h \in H$, we have

$$\langle Q, Q^h\rangle \le \langle P^t, (P^t)^h\rangle = \langle P, P^{tht^{-1}}\rangle^t.$$

Since $\langle P, P^{tht^{-1}}\rangle$ is p-solvable, $\langle Q, Q^h\rangle$ is p-solvable. Therefore, by the minimality of $|G|$, H is p-solvable.

By Proposition 4.4, we know that G is not a nonabelian simple group. Let N be a proper nontrivial normal subgroup of G. Now PN/N is a Sylow p-subgroup of G/N and it satisfies the hypothesis of the theorem. Since $|G/N| < |G|$, G/N is p-solvable. As in the previous claim, N is also p-solvable and thus G is p-solvable as well. This final contradiction proves the theorem. □

Acknowledgements

The author is grateful to the referee for numerous comments and suggestions that have significantly improved the exposition of the paper. The referee has simplified the proofs of both Theorem 1.4 and Corollary 1.5 as well as shortened the proof of Theorem 1.6 significantly. The author also thanks Chris Schroeder for careful reading of several versions of the paper.

References

[1] O. D. Artemovich, "Isolated elements of prime order in finite groups", *Ukrain. Mat. Zh.* **40**:3 (1988), 397–400, 408. MR Zbl

[2] B. Baumeister, T. C. Burness, R. M. Guralnick, and H. P. Tong-Viet, "On the maximal overgroups of Sylow subgroups of finite groups", *Adv. Math.* **444** (2024), art. id. 109632. MR Zbl

[3] Y. Berkovich, "Subgroups with the character restriction property and related topics", *Houston J. Math.* **24**:4 (1998), 631–638. MR Zbl

[4] M. Bianchi, A. G. B. Mauri, M. Herzog, and L. Verardi, "On finite solvable groups in which normality is a transitive relation", *J. Group Theory* **3**:2 (2000), 147–156. MR Zbl

[5] J. N. Bray, D. F. Holt, and C. M. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*, London Mathematical Society Lecture Note Series **407**, Cambridge University Press, Cambridge, 2013. MR Zbl

[6] T. Breuer and R. M. Guralnick, "Finite groups can be generated by a π-subgroup and a π'-subgroup", *Israel J. Math.* **257**:1 (2023), 55–69. MR Zbl

[7] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985. MR Zbl

[8] B. Fischer, "Frobeniusautomorphismen endlicher Gruppen", *Math. Ann.* **163** (1966), 273–298. MR Zbl

[9] P. Flavell, "Generating finite groups with conjugates of a subgroup", *Comm. Algebra* **23**:11 (1995), 3947–3959. MR Zbl

[10] P. Flavell, "Generating finite groups with maximal subgroups of maximal subgroups", *J. Algebra* **177**:2 (1995), 372–384. MR Zbl

[11] P. Flavell, "A characterisation of p-soluble groups", *Bull. London Math. Soc.* **29**:2 (1997), 177–183. MR Zbl

[12] P. Flavell, "The fixed points of coprime action", *Arch. Math.* (*Basel*) **75**:3 (2000), 173–177. MR Zbl

[13] P. Flavell, "Generating finite groups with conjugates of a subgroup, II", *J. Algebra* **232**:2 (2000), 578–616. MR Zbl

[14] P. Flavell and G. R. Robinson, "Fixed points of coprime automorphisms and generalizations of Glauberman's Z^*-theorem", *J. Algebra* **226**:2 (2000), 714–718. MR Zbl

[15] R. J. Flores and R. M. Foote, "Strongly closed subgroups of finite groups", *Adv. Math.* **222**:2 (2009), 453–484. MR Zbl

[16] R. Foote, "A characterization of finite groups containing a strongly closed 2-subgroup", *Comm. Algebra* **25**:2 (1997), 593–606. MR

[17] The GAP Group, *GAP (version 4.7.5)*, 2014, available at https://www.gap-system.org/. Zbl

[18] G. Glauberman, "Central elements in core-free groups", *J. Algebra* **4** (1966), 403–420. MR Zbl

[19] G. Glauberman, R. Guralnick, J. Lynd, and G. Navarro, "Centers of Sylow subgroups and automorphisms", *Israel J. Math.* **240**:1 (2020), 253–266. MR Zbl

[20] D. M. Goldschmidt, "2-fusion in finite groups", *Ann. of Math.* (2) **99** (1974), 70–117. MR Zbl

[21] N. Gordeev, F. Grunewald, B. Kunyavskiĭ, and E. Plotkin, "Baer–Suzuki theorem for the solvable radical of a finite group", *C. R. Math. Acad. Sci. Paris* **347**:5-6 (2009), 217–222. MR Zbl

[22] D. Gorenstein, *Finite groups*, 2nd ed., Chelsea Publishing Co., New York, 1980. MR Zbl

[23] D. Gorenstein, *Finite simple groups*: *an introduction to their classification*, Plenum Publishing Corp., New York, 1982. MR Zbl

[24] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups, I, Chapter A: Almost simple K-groups*, Mathematical Surveys and Monographs **40.3**, American Mathematical Society, Providence, RI, 1998. MR Zbl

[25] S. Guest, "A solvable version of the Baer–Suzuki theorem", *Trans. Amer. Math. Soc.* **362**:11 (2010), 5909–5946. MR Zbl

[26] R. M. Guralnick, "Generation of simple groups", *J. Algebra* **103**:1 (1986), 381–401. MR Zbl

[27] R. M. Guralnick and G. R. Robinson, "On extensions of the Baer–Suzuki theorem", *Israel J. Math.* **82**:1-3 (1993), 281–297. MR Zbl

[28] R. M. Guralnick and P. H. Tiep, "Lifting in Frattini covers and a characterization of finite solvable groups", *J. Reine Angew. Math.* **708** (2015), 49–72. MR Zbl

[29] R. M. Guralnick, G. Malle, and G. Navarro, "Self-normalizing Sylow subgroups", *Proc. Amer. Math. Soc.* **132**:4 (2004), 973–979. MR Zbl

[30] R. M. Guralnick, H. P. Tong-Viet, and G. Tracey, "Weakly subnormal subgroups and variations of the Baer–Suzuki theorem", preprint, 2024. arXiv 2402.00804

[31] T. Hawkes and J. Humphreys, "A character-theoretic criterion for the existence of normal complements to subgroups of finite groups", *J. Algebra* **94**:2 (1985), 382–387. MR Zbl

[32] I. M. Isaacs, "Subgroups with the character restriction property", *J. Algebra* **100**:2 (1986), 403–420. MR Zbl

[33] I. M. Isaacs and T. Zieschang, "Generating symmetric groups", *Amer. Math. Monthly* **102**:8 (1995), 734–739. MR Zbl

[34] P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, London Mathematical Society Lecture Note Series **129**, Cambridge University Press, Cambridge, 1990. MR Zbl

[35] H. Kurzweil and B. Stellmacher, *The theory of finite groups: an introduction*, Springer, New York, 2004. MR Zbl

[36] S. Li, "On minimal non-PE-groups", *J. Pure Appl. Algebra* **132**:2 (1998), 149–158. MR Zbl

[37] C. Parker and J. Saunders, "Squares of conjugacy classes and a variant on the Baer–Suzuki Theorem", preprint, 2022. arXiv 2210.06962

[38] M. Suzuki, *Group theory, II*, Grundl. Math. Wissen. **248**, Springer, New York, 1986. MR Zbl

[39] J. G. Thompson, "Nonsolvable finite groups all of whose local subgroups are solvable", *Bull. Amer. Math. Soc.* **74** (1968), 383–437. MR Zbl

[40] H. P. Tong-Viet, "Conjugacy classes of p-elements and normal p-complements", *Pacific J. Math.* **308**:1 (2020), 207–222. MR Zbl

[41] J. H. Walter, "Finite groups with abelian Sylow 2-subgroups of order 8", *Invent. Math.* **2** (1967), 332–376. MR Zbl

[42] H. Wielandt, "Über die Existenz von Normalteilern in endlichen Gruppen", *Math. Nachr.* **18** (1958), 274–280. MR Zbl

[43] K. Zsigmondy, "Zur Theorie der Potenzreste", *Monatsh. Math. Phys.* **3**:1 (1892), 265–284. MR Zbl

Received April 15, 2024. Revised May 10, 2024.

HUNG P. TONG-VIET
DEPARTMENT OF MATHEMATICS AND STATISTICS
BINGHAMTON UNIVERSITY
BINGHAMTON, NY 13902-6000
UNITED STATES
htongvie@binghamton.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 329, No. 2, 2024
https://doi.org/10.2140/pjm.2024.329.327

VISHIK EQUIVALENCE AND SIMILARITY OF QUASILINEAR p-FORMS AND TOTALLY SINGULAR QUADRATIC FORMS

KRISTÝNA ZEMKOVÁ

For quadratic forms over fields of characteristic different from two, there is a so-called Vishik criterion, giving a purely algebraic characterization of when two quadratic forms are motivically equivalent. In analogy to that, we define Vishik equivalence on quasilinear p-forms. We study the question whether Vishik equivalent p-forms must be similar. We prove that this is not true for quasilinear p-forms in general, but we find some families of totally singular quadratic forms (i.e., of quasilinear 2-forms) for which the question has a positive answer.

1. Introduction

Vishik [1997] defined an equivalence relation on quadratic forms over fields of characteristic other than 2 (see also [Karpenko 2000; Vishik 2004]): φ and ψ are equivalent if and only if

$$\text{(1-1)}\quad \dim\varphi = \dim\psi \quad \text{and} \quad i_W(\varphi_E) = i_W(\psi_E) \quad \text{for any field extension } E/F.$$

Vishik proved that this equivalence coincides with motivic equivalence. Hence, equation (1-1) is nowadays known as *Vishik's criterion* for motivic equivalence. Then the question was raised whether the equivalence defined by Vishik also coincides with similarity; in other words, the following question was asked:

Question A. Are quadratic forms satisfying Vishik's criterion (1-1) necessarily similar?

Izhboldin [1998; 2000] proved that the answer to Question A is positive for odd-dimensional quadratic forms, but negative for even-dimensional forms of dimension greater or equal to 8 (except possibly for the dimension 12). Hoffmann [2015] proves that, under some conditions on the base field, Question A has positive answer for even-dimensional quadratic forms as well.

This work was supported by DFG project HO 4784/2-1. The author further acknowledges support from the Pacific Institute for the Mathematical Sciences and a partial support from the National Science and Engineering Research Council of Canada. The research and findings may not reflect those of these institutions.

MSC2020: 11E04, 11E81.

Keywords: quasilinear p-forms, quadratic forms, finite characteristic, isotropy, equivalence relations.

© 2024 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

In the case of fields of characteristic 2, we have to distinguish between different types of quadratic forms — nonsingular and totally singular (the two extreme cases) and singular (the mixed type). Totally singular quadratic forms over fields of characteristic 2 have been generalized to *quasilinear p-forms* over fields of characteristic p in [Hoffmann 2004].

Let F be a field of characteristic p, and let φ and ψ be quasilinear p-forms over F. Inspired by Vishik's criterion, we define *Vishik equivalence* of φ and ψ as

$$\dim \varphi = \dim \psi \qquad \text{and} \qquad \mathrm{i_d}(\varphi_E) = \mathrm{i_d}(\psi_E) \quad \text{for any field extension } E/F,$$

where $\mathrm{i_d}(\tau)$ is the *defect* (sometimes also called the *quasilinear index*) of the quasilinear p-form τ. In analogy to Question A, we ask:

Question Q. Are Vishik equivalent quasilinear p-forms necessarily similar?

We show in Examples 3.14 and 3.15 that Question Q has a negative answer for p-forms if $p > 3$. Therefore, in Section 4, we will focus on totally singular quadratic forms, i.e., on the case when $p = 2$. In this case, we can give a positive answer to Question Q at least for some families of forms. For example, we prove:

Theorem 1.1 (see Theorem 4.6). *Let φ, ψ be totally singular quadratic forms over F such that φ_{an} is minimal over F. If φ and ψ are Vishik equivalent, then they are similar.*

See Theorem 4.17 and Corollary 4.18 for more families for which the answer to Question Q is positive. We would also like to point out Theorem 3.7 which shows that Vishik equivalence preserves similarity factors, even in the case of quasilinear p-forms.

This paper is based on the second and third chapters of the author's PhD thesis [Zemková 2022].

2. Preliminaries

All fields in this article are of characteristic $p > 0$. Whenever we talk about (totally singular) quadratic forms, we assume $p = 2$.

Quasilinear p-forms. Most of the necessary background on quasilinear p-forms can be found in [Hoffmann 2004]; we include it here for the readers' convenience.

Definition 2.1. Let F be a field and V a finite-dimensional F-vector space. A *quasilinear p-form* (or simply a *p-form*) *over* F is a map $\varphi : V \to F$ with the following properties:

(1) $\varphi(av) = a^p \varphi(v)$ for any $a \in F$ and $v \in V$, and

(2) $\varphi(v + w) = \varphi(v) + \varphi(w)$ for any $v, w \in V$.

The dimension of φ is defined as $\dim \varphi = \dim V$.

Any p-form φ on an F-vector space V can be associated with the polynomial $\sum_{i=1}^{n} a_i X_i^p \in F[X_1, \ldots, X_n]$, where $a_i = \varphi(v_i)$ with $\{v_1, \ldots, v_n\}$ a basis of V. In such case, we write φ as $\langle a_1, \ldots, a_n\rangle$.

We have $c\langle a_1, \ldots, a_n\rangle = \langle ca_1, \ldots, ca_n\rangle$ for any $c \in F^*$. If $\langle a_1, \ldots, a_n\rangle$ and $\langle b_1 \ldots, b_m\rangle$ are two p-forms over F, then we define

$$\langle a_1, \ldots, a_n\rangle \perp \langle b_1 \ldots, b_m\rangle = \langle a_1, \ldots, a_n, b_1 \ldots, b_m\rangle,$$
$$\langle a_1, \ldots, a_n\rangle \otimes \langle b_1 \ldots, b_m\rangle = a_1\langle b_1 \ldots, b_m\rangle \perp \ldots \perp a_n\langle b_1 \ldots, b_m\rangle.$$

Moreover, for a positive integer k, we write $k \times \varphi$ for the p-form $\varphi \perp \ldots \perp \varphi$ consisting of k copies of φ.

Two p-forms $\varphi : V \to F$ and $\psi : W \to F$ are called *isometric* (denoted $\varphi \simeq \psi$) if there exists a bijective homomorphism $f : V \to W$ of vector spaces such that $\varphi(v) = \psi(f(v))$ for any $v \in V$. If f is not bijective but injective, then φ is called a *subform* of ψ (denoted $\varphi \subseteq \psi$); then there exists a p-form σ over F such that $\psi \simeq \varphi \perp \sigma$. If $\varphi \simeq c\psi$ for some $c \in F^*$, then φ and ψ are called *similar* (denoted $\varphi \overset{\text{sim}}{\sim} \psi$).

A p-form $\varphi : V \to F$ is called *isotropic* if $\varphi(v) = 0$ for some $v \in V \setminus \{0\}$; otherwise, φ is called *anisotropic*. The p-form φ can be written as $\varphi \simeq \sigma \perp k \times \langle 0\rangle$ with σ an anisotropic p-form over F and k a nonnegative integer. Then σ is unique up to isometry, and it is called the *anisotropic part of* φ (denoted φ_{an}). The integer k is called the *defect* of φ (denoted $\mathrm{i_d}(\varphi)$).

Let φ be a p-form on an F-vector space V; we set

$$D_F(\varphi) = \{\varphi(v) \mid v \in V\} \quad \text{and} \quad D_F^*(\varphi) = D_F(\varphi) \setminus \{0\},$$
$$G_F^*(\varphi) = \{x \in F^* \mid x\varphi \simeq \varphi\} \quad \text{and} \quad G_F(\varphi) = G_F^*(\varphi) \cup \{0\}.$$

Note that $D_F(\varphi)$ is an F^p-vector space; in particular, if $\varphi \simeq \langle a_1, \ldots, a_n\rangle$, then $D_F(\varphi) = \mathrm{span}_{F^p}\{a_1, \ldots, a_n\}$.

Lemma 2.2 [Hoffmann 2004, Proposition 2.6]. *Let φ be a p-form over F.*

(i) *Let $\{c_1, \ldots, c_k\}$ be any F^p-basis of the vector space $D_F(\varphi)$. Then we have $\varphi_{\text{an}} \simeq \langle c_1, \ldots, c_k\rangle$.*

(ii) *If $a_1, \ldots, a_m \in D_F(\varphi)$, then $\langle a_1, \ldots, a_m\rangle_{\text{an}} \subseteq \varphi$.*

It follows that, for any $a, b \in F$ and $x \in F^*$, we have

$$\langle a, b\rangle \simeq \langle a + b, b\rangle \quad \text{and} \quad \langle a\rangle \simeq \langle ax^p\rangle.$$

For a p-form φ on an F-vector space V and a field extension E/F, we denote by φ_E the p-form on the E-vector space $V_E = E \otimes V$ defined by $\varphi_E(e \otimes v) = e^p\varphi(v)$ for any $e \in E$ and $v \in V$. It was proved in [Hoffmann 2004, Lemma 5.1] that if $\varphi \simeq \langle a_1, \ldots, a_n\rangle$, then $(\varphi_E)_{\text{an}} \simeq \langle a_{i_1}, \ldots, a_{i_k}\rangle$ for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.

Quasi-Pfister forms and quasi-Pfister neighbors. For $n > 0$, an *n-fold quasi-Pfister form* is a p-form $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle\langle a_1\rangle\rangle \otimes \cdots \otimes \langle\langle a_n\rangle\rangle$, where $\langle\langle a\rangle\rangle = \langle 1, a, \dots, a^{p-1}\rangle$. Moreover, $\langle 1\rangle$ is called the *0-fold quasi-Pfister form*.

A p-form φ over F is called a *quasi-Pfister neighbor* if $c\varphi \subseteq \pi$ for some $c \in F^*$ and a quasi-Pfister form π over F such that $\dim \varphi > \frac{1}{p} \dim \pi$. The value $\dim \pi - \dim \varphi$ is called the *codimension* of φ.

In the following lemma, we summarize some of the most important properties of quasi-Pfister forms and quasi-Pfister neighbors.

Lemma 2.3 [Hoffmann 2004, Propositions 4.6 and 4.14; Scully 2013, Lemma 2.6]. *Let $\pi \simeq \langle\langle a_1, \dots, a_n\rangle\rangle$ be a quasi-Pfister form over F, φ a quasi-Pfister neighbor of π, and E/F a field extension. Then*:

(i) $G_E(\pi) = D_E(\pi)$.

(ii) *There either exist $k > 0$ and a subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ such that $(\pi_E)_{\mathrm{an}} \simeq \langle\langle a_{i_1}, \dots, a_{i_k}\rangle\rangle_E$, or $(\pi_E)_{\mathrm{an}} \simeq \langle 1\rangle_E$.*

(iii) *φ_E is isotropic if and only if π_E is isotropic.*

p-bases, norm fields and norm forms. Let $a_1, \dots, a_n \in F$. We call the set $\{a_1, \dots, a_n\}$ *p-independent over F* if $[F^p(a_1, \dots, a_n) : F^p] = p^n$, and *$p$-dependent over F* otherwise. The set $\{a_1, \dots, a_n\}$ is p-independent over F if and only if the quasi-Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ is anisotropic over F [Zemková 2024, Corollary 2.8]. A (possibly infinite) set $\mathcal{S} \subseteq F$ is called *p-independent over F* if any finite subset of $\mathcal{S}$ is p-independent over F.

Let E be a field such that $F^p \subseteq E \subseteq F$; a set $\mathcal{B} \subseteq E$ is called a *p-basis of E over F* if $\mathcal{B}$ is p-independent over F and $E = F^p(\mathcal{B})$. Such a p-basis always exists, and any subset of E that is p-independent over F can be extended into a p-basis of E over F [Gille and Szamuely 2006, Corollary A.8.9]. A set $\mathcal{B} = \{b_i \mid i \in I\}$ is a p-basis of E over F if and only if

$$\widehat{\mathcal{B}} = \left\{ \prod_{i \in I} b_i^{\lambda(i)} \;\middle|\; \lambda : I \to \{0, \dots, p-1\}, \lambda(i) = 0 \text{ for almost all } i \in I \right\}$$

is an F^p-linear basis of E [Pickert 1949, page 27]; in such case, any $a \in E$ can be expressed *uniquely* as

$$a = \sum_{\lambda} x_\lambda^p \prod_{i \in I} b_i^{\lambda(i)}$$

for some $x_\lambda \in F$, almost all of them zero. By abuse of language, we say that *a can be expressed uniquely with respect to $\mathcal{B}$*.

Let φ be a p-form over F. We define the *norm field of φ over F* as the field

$$N_F(\varphi) = F^p\Big(\frac{a}{b} \;\Big|\; a, b \in D_F^*(\varphi)\Big).$$

By the definition, we have $N_F(\varphi) = N_F(\varphi_{\mathrm{an}})$ and also that $N_F(\varphi) = N_F(c\varphi)$ for any $c \in F^*$. Furthermore, if ψ is another p-form over F satisfying $\psi \simeq \varphi$, then $N_F(\psi) = N_F(\varphi)$. Finally, if $\tau_{\mathrm{an}} \subseteq c\varphi_{\mathrm{an}}$ for some p-form τ over F and $c \in F^*$, then $N_F(\tau) \subseteq N_F(\varphi)$.

Lemma 2.4 [Hoffmann 2004, Lemma 4.2 and Corollary 4.3]. *Let φ be a p-form over F.*

(i) *If $\varphi \simeq \langle a_0, \dots, a_n\rangle$ for some $n \geq 1$ and $a_i \in F$, $0 \leq i \leq n$, with $a_0 \neq 0$, then $N_F(\varphi) = F^p\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right)$.*

(ii) *Suppose $N_F(\varphi) = F^p(b_1, \dots, b_m)$ for some $b_i \in F^*$, $1 \leq i \leq m$, and let E/F be a field extension. Then $N_E(\varphi) = E^p(b_1, \dots, b_m)$.*

The following proposition provides an alternative possibility for a determination of the norm field.

Proposition 2.5. *Let $\varphi = \langle 1, a_1, \dots, a_n\rangle$ be a p-form over F, and let $b_1, \dots, b_m \in F$. Then*

$$(\varphi_{F(\sqrt[p]{b_1}, \dots, \sqrt[p]{b_m})})_{\mathrm{an}} \simeq \langle 1\rangle \quad \Longleftrightarrow \quad N_F(\varphi) \subseteq F^p(b_1, \dots, b_m).$$

If, moreover, $\{b_1, \dots, b_m\} \subseteq \{a_1, \dots, a_n\}$, then

$$(\varphi_{F(\sqrt[p]{b_1}, \dots, \sqrt[p]{b_m})})_{\mathrm{an}} \simeq \langle 1\rangle \quad \Longleftrightarrow \quad N_F(\varphi) = F^p(b_1, \dots, b_m).$$

In particular, if m is minimal with this property, then $\operatorname{ndeg}_F \varphi = p^m$.

Proof. We set $E = F(\sqrt[p]{b_1}, \dots, \sqrt[p]{b_m})$.

First, $(\varphi_E)_{\mathrm{an}} \simeq \langle 1\rangle$ is equivalent to $\operatorname{span}_{E^p}\{1\} = \operatorname{span}_{E^p}\{1, a_1, \dots, a_n\}$, which holds if and only if $a_i \in E^p$ for all $1 \leq i \leq n$. As $E^p = F^p(b_1, \dots, b_m)$, the latter condition is equivalent to $F^p(a_1, \dots, a_n) \subseteq F^p(b_1, \dots, b_m)$. But by Lemma 2.4 $F^p(a_1, \dots, a_n) = N_F(\varphi)$, so we are done.

If $\{b_1, \dots, b_m\} \subseteq \{a_1, \dots, a_n\}$, then $F^p(b_1, \dots, b_m) \subseteq N_F(\varphi)$, and the claim follows by the previous case. □

Note that $N_F(\varphi)$ is a finite field extension of F^p, and hence there always exists a finite p-basis $\{b_1, \dots, b_n\}$ of $N_F(\varphi)$ over F, i.e., $N_F(\varphi) = F^p(b_1, \dots, b_n)$ with $[F^p(b_1, \dots, b_n) : F^p] = p^n$. Then p^n is called the *norm degree of φ over F* and denoted by $\operatorname{ndeg}_F \varphi$. There is a relation between the norm degree and the dimension of a p-form.

Lemma 2.6 [Hoffmann 2004, Proposition 4.8]. *Let φ be a nonzero p-form with* $\operatorname{ndeg}_F \varphi = p^n$. *Then $n + 1 \leq \dim \varphi_{\mathrm{an}} \leq p^n$.*

By [Gille and Szamuely 2006, Corollary A.8.9], any p-generating set contains a p-basis. Therefore, part (i) of Lemma 2.4 implies that:

Lemma 2.7. *Assume that* $\varphi \simeq \langle a_0, \dots, a_n\rangle$ *for some* $n \geq 1$ *and* $a_0, \dots, a_n \in F$ *with* $a_0 \neq 0$. *Moreover, suppose that* $\operatorname{ndeg}_F \varphi = p^k$. *Then there exists a subset* $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ *such that* $\left\{\frac{a_{i_1}}{a_0}, \dots, \frac{a_{i_k}}{a_0}\right\}$ *is a* p*-basis of* $N_F(\varphi)$ *over* F.

In particular, if $1 \in D_F^*(\varphi)$, *then there exist* $b_1, \dots, b_n \in F$ *such that* $\varphi \simeq \langle 1, b_1, \dots, b_n\rangle$ *and* $N_F(\varphi) = F^p(b_1, \dots, b_k)$.

Let φ be a p-form over F with $\operatorname{ndeg}_F \varphi = p^n$ and $N_F(\varphi) = F^p(a_1, \dots, a_n)$ (so, in particular, $\{a_1, \dots, a_n\}$ is p-independent over F). Then we define the *norm form* of φ over F, denoted by $\hat{\nu}_F(\varphi)$, as the (necessarily anisotropic) quasi-Pfister form $\langle\!\langle a_1, \dots, a_n\rangle\!\rangle$. It follows that $\hat{\nu}_F(\varphi)$ is the smallest quasi-Pfister form that contains a scalar multiple of φ_{an} as its subform. In particular, if φ_{an} is a quasi-Pfister form itself, then we have $\hat{\nu}_F(\varphi) \simeq \varphi_{\text{an}}$.

Isotropy. Let φ be an anisotropic p-form over F, and let E/F be a field extension. If E/F is purely transcendental or separable, then φ_E remains anisotropic [Hoffmann 2004, Proposition 5.3].

Lemma 2.8 [Scully 2016, Lemma 2.27]. *Let* φ *be a* p*-form over* F *and let* $a \in F \setminus F^p$. *Then*:

(i) $D_{F(\sqrt[p]{a})}(\varphi) = D_F(\langle\!\langle a\rangle\!\rangle \otimes \varphi) = \sum_{i=0}^{p-1} a^i D_F(\varphi)$.

(ii) $\mathrm{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) = \frac{1}{p}\mathrm{i}_{\mathrm{d}}(\langle\!\langle a\rangle\!\rangle \otimes \varphi)$.

(iii) *If* φ *is anisotropic and* $\varphi_{F(\sqrt[p]{a})}$ *is isotropic, then* $a \in N_F(\varphi)$.

(iv) $\dim(\varphi_{F(\sqrt[p]{a})})_{\text{an}} \geq \frac{1}{p}\dim \varphi_{\text{an}}$.

(v) *Equality holds in (iv) if and only if there exists a* p*-form* γ *over* F *such that* $\varphi_{\text{an}} \simeq \langle\!\langle a\rangle\!\rangle \otimes \gamma$.

We want to point out that $N_F(\varphi)$ is the smallest field extension of F^p with property (iii) of Lemma 2.8. More precisely, we have:

Proposition 2.9. *Let* φ *be an anisotropic* p*-form over* F *and* E/F^p *be a field extension such that the following holds for any* $a \in F^*$:

$$(\divideontimes) \qquad \mathrm{i}_{\mathrm{d}}(\varphi_{F(\sqrt[p]{a})}) > 0 \Rightarrow a \in E.$$

Then $N_F(\varphi) \subseteq E$.

Proof. Let $\operatorname{ndeg}_F \varphi = p^k$. Since neither the isotropy nor the norm field depends on the choice of the representative of the similarity class, we can assume $1 \in D_F^*(\varphi)$. Invoking Lemma 2.7, we find $b_1, \dots, b_n \in F$ with $n \geq k$ such that $\varphi \simeq \langle 1, b_1, \dots, b_n\rangle$ and $\{b_1, \dots, b_k\}$ is a p-basis of $N_F(\varphi)$ over F. Since $\varphi_{F(\sqrt[p]{b_i})}$ is obviously isotropic for each $1 \leq i \leq k$, we get by the assumption $(\divideontimes)$ that $b_1, \dots, b_k \in E$. Since E is a field containing F^p, it follows that $N_F(\varphi) = F^p(b_1, \dots, b_k) \subseteq E$. □

Lemma 2.10. *Let φ be an anisotropic quasi-Pfister form or an anisotropic quasi-Pfister neighbor over F, and let $x \in F^*$. Then $\varphi \otimes \langle\langle x \rangle\rangle$ is isotropic if and only if $x \in N_F(\varphi)$.*

Proof. First, we prove the lemma in the case when $\varphi \simeq \pi$ is a quasi-Pfister form. Assume that $\pi \simeq \langle\langle a_1, \dots, a_n \rangle\rangle$, i.e., $N_F(\pi) = F^p(a_1, \dots, a_n)$. Note that π is anisotropic. Then the form $\pi \otimes \langle\langle x \rangle\rangle$ is isotropic if and only if $(\pi \otimes \langle\langle x \rangle\rangle)_{\text{an}} \simeq \pi$ if and only if $F^p(a_1, \dots, a_n) = F^p(a_1, \dots, a_n, x)$ if and only if $x \in N_F(\pi)$.

Now suppose that φ is a quasi-Pfister neighbor. Then $c\varphi \subseteq \pi$ for $\pi \simeq \hat{\nu}_F(\varphi)$ and some $c \in F^*$. It follows that $c\varphi \otimes \langle\langle x \rangle\rangle$ is a quasi-Pfister neighbor of $\pi \otimes \langle\langle x \rangle\rangle$, and hence $c\varphi \otimes \langle\langle x \rangle\rangle$ is isotropic if and only if $\pi \otimes \langle\langle x \rangle\rangle$ is isotropic (Lemma 2.3). By the first part of the proof, the latter is equivalent to $x \in N_F(\pi)$. Since we have $N_F(\pi) = N_F(c\varphi) = N_F(\varphi)$, the claim follows. □

Minimal forms. In this paper, we will work with minimal forms; they are, in a way, a counterpart to quasi-Pfister forms.

Definition 2.11. Let φ be an anisotropic p-form over F. We call φ *minimal over F* if $\operatorname{ndeg}_F \varphi = p^{\dim \varphi - 1}$.

Note that any one-dimensional anisotropic p-form is minimal. Minimal p-forms of dimension at least two are described by the following lemma:

Lemma 2.12 [Zemková 2024, Lemma 2.16]. *Let φ, ψ be p-forms over F of dimension at least two.*

(i) *The p-form $\langle 1, a_1, \dots, a_n \rangle$ is minimal over F if and only if the set $\{a_1, \dots, a_n\}$ is p-independent over F.*

(ii) *Minimality is not invariant under field extensions.*

(iii) *If φ is minimal over F and $\psi \overset{\text{sim}}{\sim} \varphi$, then ψ is minimal over F.*

(iv) *If φ is minimal over F and $\psi \subseteq c\varphi$ for some $c \in F^*$, then ψ is minimal over F.*

(v) *If $\operatorname{ndeg}_F \varphi = p^k$ with $k \geq 1$, then φ contains a minimal subform of dimension $k+1$.*

We can characterize elements represented by a given minimal p-form.

Lemma 2.13. *Let φ be a minimal p-form over F with $1 \in D_F(\varphi)$. Let $n \geq 2$ and $\{b_1, \dots, b_n\} \subseteq D_F(\varphi)$ be p-independent over F. Denote*

$$S = \big\{\lambda : \{1, \dots, n\} \to \{0, \dots, p-1\}\big\},$$

and consider

$$\beta = \sum_{\lambda \in S} x_\lambda^p \prod_{i=1}^{n} b_i^{\lambda(i)}$$

with $x_\lambda \in F$ for all $\lambda \in S$. Then the following are equivalent:

(i) $\beta \in D_F(\varphi)$.

(ii) $x_\lambda = 0$ *for all* $\lambda \in S$ *such that* $\sum_{i=1}^n \lambda(i) \geq 2$, *i.e.*, $\beta = y_0^p + \sum_{i=1}^n b_i y_i^p$ *for some* $y_i \in F$, $0 \leq i \leq n$.

Proof. Write $\tau \simeq \langle 1, b_1, \dots, b_n \rangle$. Then (ii) holds if and only if $\beta \in D_F(\tau)$. Since $\tau \subseteq \varphi$ by the assumptions, the implication (ii) $\Rightarrow$ (i) is obvious.

To prove (i) $\Rightarrow$ (ii), suppose that (ii) does not hold (e.g., $\beta = b_1 b_2$). Then we can see that $\beta \notin D_F(\tau)$. Hence, the p-form $\tau \perp \langle \beta \rangle$ is anisotropic. On the other hand, $\beta \in F^p(b_1, \dots, b_n)$, so $\tau \perp \langle \beta \rangle$ is not minimal over F by part (i) of Lemma 2.12. If $\beta \in D_F(\varphi)$, then $\tau \perp \langle \beta \rangle \subseteq \varphi$, in which case φ could not be minimal over F by part (iv) of the same lemma, which is a contradiction; therefore, $\beta \notin D_F(\varphi)$. □

Vishik equivalence. We introduce Vishik equivalence of p-forms. The name originates from *Vishik criterion* that, in characteristic different from 2, gives an algebraic characterization of the motivic equivalence (see [Hoffmann 2015]).

Definition 2.14. Let φ, ψ be p-forms over F. We say that they are *Vishik equivalent* and write $\varphi \overset{v}{\sim} \psi$ if $\dim \varphi = \dim \psi$ and the following holds:

$$(v) \qquad \mathrm{i_d}(\varphi_E) = \mathrm{i_d}(\psi_E) \quad \text{for all } E/F.$$

Remark 2.15. For quadratic forms φ and ψ over F (not necessarily totally singular) such that $\dim \varphi = \dim \psi$, we define Vishik equivalence as (see [Zemková 2022, Definition 1.57])

$$\varphi \overset{v}{\sim} \psi \quad \overset{\text{def}}{\iff} \quad \mathrm{i_W}(\varphi_E) = \mathrm{i_W}(\psi_E) \text{ and } \mathrm{i_d}(\varphi_E) = \mathrm{i_d}(\psi_E) \qquad \text{for all } E/F.$$

Note that this definition agree with Definition 2.14 in the special case of totally singular quadratic forms.

Lemma 2.16. *Let* φ, ψ *be* p*-forms over* F *such that* $\varphi \overset{v}{\sim} \psi$, *and let* E/F *be a field extension. Then* $(\varphi_E)_{\mathrm{an}} \overset{v}{\sim} (\psi_E)_{\mathrm{an}}$. *In particular,* $\varphi_{\mathrm{an}} \overset{v}{\sim} \psi_{\mathrm{an}}$.

Proof. Denote $\varphi' = (\varphi_E)_{\mathrm{an}}$ and $\psi' = (\psi_E)_{\mathrm{an}}$. Let K/E be another field extension. It holds that $\mathrm{i_d}(\varphi_K) = \mathrm{i_d}(\psi_K)$ and $\mathrm{i_d}(\varphi_E) = \mathrm{i_d}(\psi_E)$. Thus,

$$\mathrm{i_d}(\varphi'_K) = \mathrm{i_d}(\varphi_K) - \mathrm{i_d}(\varphi_E) = \mathrm{i_d}(\psi_K) - \mathrm{i_d}(\psi_E) = \mathrm{i_d}(\psi'_K).$$

Therefore, $\varphi' \overset{v}{\sim} \psi'$. □

Since similar p-forms must be of the same dimension and have the same defects over any field, we get:

Lemma 2.17. *If* $\varphi \overset{\mathrm{sim}}{\sim} \psi$, *then* $\varphi \overset{v}{\sim} \psi$.

Among others, Vishik equivalence implies that a p-form is isotropic over the function field of any Vishik equivalent p-form.

Lemma 2.18. *Let φ, ψ be p-forms over F of dimension at least two. If $\varphi \overset{v}{\sim} \psi$, then $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic.*

Proof. Since $\varphi_{F(\varphi)}$ is isotropic and we know that $\mathrm{i_d}(\psi_{F(\varphi)}) = \mathrm{i_d}(\varphi_{F(\varphi)})$, we get that $\psi_{F(\varphi)}$ is isotropic. Symmetrically, $\varphi_{F(\psi)}$ is isotropic. □

Lemma 2.19. *Let φ, ψ be anisotropic p-forms over F such that $\varphi \overset{v}{\sim} \psi$. Then $\hat{v}_F(\varphi) \simeq \hat{v}_F(\psi)$, $N_F(\varphi) = N_F(\psi)$ and $\mathrm{ndeg}_F(\varphi) = \mathrm{ndeg}_F(\psi)$.*

Proof. The claim is trivial if $\dim \varphi = \dim \psi = 1$. Assume that the dimension is at least two. By Lemma 2.18, we know that $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic. Thus, this is a consequence of [Hoffmann 2004, Lemma 7.12]. □

3. Vishik equivalence on p-forms

As an easy consequence of some know results, we get that Question Q has a positive answer for quasi-Pfister forms and quasi-Pfister neighbors of codimension one.

Proposition 3.1. *Let φ, ψ be anisotropic p-forms such that $\varphi \overset{v}{\sim} \psi$. If φ is a quasi-Pfister form or a quasi-Pfister neighbor of codimension one, then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. By Lemma 2.19, there exists a quasi-Pfister form π over F such that $\pi \simeq \hat{v}_F(\varphi) \simeq \hat{v}_F(\psi)$.

If φ is a quasi-Pfister form, then we can find $c, d \in F^*$ such that $c\varphi \simeq \pi \supseteq d\psi$; thus, $\varphi \overset{\mathrm{sim}}{\sim} \psi$ for dimension reasons.

Now assume that φ is a quasi-Pfister neighbor of π of codimension 1. Then $\dim \psi = \dim \varphi = \dim \pi - 1$. Moreover, $c\psi \subseteq \hat{v}_F(\psi) \simeq \pi$ for some $c \in F^*$. So, ψ is a quasi-Pfister neighbor of π of codimension 1. Using [Hoffmann 2004, Proposition 4.15], any two such p-forms are similar, i.e., $\varphi \overset{\mathrm{sim}}{\sim} \psi$ as claimed. □

Weak Vishik equivalence. Proving that two p-forms are Vishik equivalent might be difficult. Moreover, we do not always need the full strength of the Vishik equivalence; therefore, we define a weaker version.

Definition 3.2. Let φ, ψ be p-forms over F. We say that they are *weakly Vishik equivalent* and write $\varphi \overset{v_0}{\sim} \psi$ if $\dim \varphi = \dim \psi$ and the following holds:

$$(v_0) \qquad \mathrm{i_d}(\varphi_{F(\sqrt[p]{a})}) = \mathrm{i_d}(\psi_{F(\sqrt[p]{a})}) \quad \text{for all } a \in F.$$

The weak Vishik equivalence has three (rather obvious) properties.

Lemma 3.3. *Let φ, ψ be p-forms over F. Then*:

(i) *If $\varphi \overset{v_0}{\sim} \psi$, then $\mathrm{i_d}(\varphi) = \mathrm{i_d}(\psi)$.*

(ii) *$\varphi \overset{v_0}{\sim} \psi$ if and only if $\varphi_{\mathrm{an}} \overset{v_0}{\sim} \psi_{\mathrm{an}}$.*

(iii) *If $\varphi \overset{v}{\sim} \psi$, then $\varphi \overset{v_0}{\sim} \psi$.*

Proof. Part (i) follows directly from the definition, because we include the equality of the defects over the field $F(\sqrt[p]{1}) \simeq F$. Part (ii) is then an easy consequence of part (i). Finally, part (iii) is trivial. □

To prove that Vishik equivalent forms have the same norm field, we used the isotropy over the function fields of each other (see Lemma 2.19). Another possibility to determine the norm field is to use Proposition 2.5, which only needs one particular purely inseparable field extension of exponent one. But for weakly Vishik equivalent forms, neither of these is at our disposal; nevertheless, thanks to Proposition 2.9, we are still able to prove that they have the same norm field.

Proposition 3.4. *Let φ, ψ be p-forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Then $\hat{v}_F(\varphi) \simeq \hat{v}_F(\psi)$, $N_F(\varphi) = N_F(\psi)$ and $\operatorname{ndeg}_F(\varphi) = \operatorname{ndeg}_F(\psi)$.*

Proof. We prove only $N_F(\varphi) = N_F(\psi)$, then the rest follows. Invoking Lemma 3.3 and recalling that the norm field takes into account only the anisotropic part of a p-form, we can assume that φ and ψ are anisotropic. By the definition of the weak Vishik equivalence and by part (iii) of Lemma 2.8, we have for any $a \in F$:

$$\mathrm{i_d}(\varphi_{F(\sqrt[p]{a})}) > 0 \quad \Rightarrow \quad \mathrm{i_d}(\psi_{F(\sqrt[p]{a})}) > 0 \quad \Rightarrow \quad a \in N_F(\psi).$$

We can apply Proposition 2.9 on the p-form φ and the field $E = N_F(\psi)$; we obtain $N_F(\varphi) \subseteq N_F(\psi)$. By the symmetry of the argument, we get $N_F(\varphi) = N_F(\psi)$. □

Since (weakly) Vishik equivalent forms are always of the same dimension, we immediately get:

Corollary 3.5. *Let φ, ψ be p-forms over F such that $\varphi \overset{v_0}{\sim} \psi$.*

(i) *If φ is minimal over F, then ψ is also minimal over F.*

(ii) *If φ is a quasi-Pfister form or a quasi-Pfister neighbor of codimension one over F, then $\varphi \overset{\text{sim}}{\sim} \psi$.*

Proof. By Proposition 3.4, we have $\operatorname{ndeg}_F \varphi = \operatorname{ndeg}_F \psi$; since $\dim \varphi = \dim \psi$, part (i) follows immediately. To prove part (ii), note that $\hat{v}_F(\varphi) \simeq \hat{v}_F(\psi)$, and so we can proceed as in the proof of Proposition 3.1. □

Similarity factors. In this subsection, we will show that weak Vishik equivalence (and hence Vishik equivalence as well) preserves divisibility by quasi-Pfister forms. In particular, we will prove that if $\varphi \overset{v_0}{\sim} \psi$ for some p-forms φ, ψ with φ divisible by a quasi-Pfister form π, then ψ is divisible by π, too.

Let φ be a p-form defined over F. Recall that we write

$$G_F(\varphi) = \{x \in F^* \mid x\varphi \simeq \varphi\} \cup \{0\},$$

we call the nonzero elements of this set the *similarity factors* of φ. As observed in [Hoffmann 2004, Proposition 6.4], the set $G_F(\varphi)$ together with the usual operations

is a finite field extension of F^p inside $N_F(\varphi)$; in particular, there exists a p-independent set $\{a_1, \dots, a_m\} \subseteq F^*$ such that $G_F(\varphi) = F^p(a_1, \dots, a_m)$. We denote $\hat{\sigma}_F(\varphi) \simeq \langle\langle a_1, \dots, a_m \rangle\rangle$ and call it the *similarity form* of φ over F. Moreover, again by [Hoffmann 2004, Proposition 6.4], there exists a p-form γ over F such that $\varphi_{\mathrm{an}} \simeq \hat{\sigma}_F(\varphi) \otimes \gamma$. It holds that $D_F(\hat{\sigma}_F(\varphi)) = G_F(\varphi)$.

We will show that weak Vishik equivalent p-forms have the same similarity factors. But first, we need a simple lemma.

Lemma 3.6. *Let π_1, π_2 be anisotropic quasi-Pfister forms. Then $\pi_1 \subseteq \pi_2$ if and only if $\pi_2 \simeq \pi_1 \otimes \gamma$ for some p-form γ over F. In that case, γ can be chosen to be a quasi-Pfister form.*

Proof. Write $\pi_1 \simeq \langle\langle a_1, \dots, a_r \rangle\rangle$ and $\pi_2 \simeq \langle\langle b_1, \dots, b_s \rangle\rangle$, which means that we have $N_F(\pi_1) = F^p(a_1, \dots, a_r)$ and $N_F(\pi_2) = F^p(b_1, \dots, b_s)$. By [Hoffmann 2004, Proposition 4.6], there is a bijection between finite field extensions of F^p inside F and Pfister p-forms over F, which implies that $\pi_1 \subseteq \pi_2$ if and only if $N_F(\pi_1) \subseteq N_F(\pi_2)$.

If this holds, then we can extend $\{a_1, \dots, a_r\}$ to a p-basis of $F^p(b_1, \dots, b_s)$; thus, $F^p(b_1, \dots, b_s) = F^p(a_1, \dots, a_r, a_{r+1}, \dots, a_s)$ for some $a_{r+1}, \dots, a_s \in F^*$, and so $\langle\langle b_1, \dots, b_s \rangle\rangle \simeq \langle\langle a_1, \dots, a_r \rangle\rangle \otimes \langle\langle a_{r+1}, \dots, a_s \rangle\rangle$.

On the other hand, assume $\pi_2 \simeq \pi_1 \otimes \gamma$ for a p-form γ. If $1 \in D_F^*(\gamma)$, then we can write $\gamma \simeq \langle 1 \rangle \perp \gamma'$ for a suitable p-form γ'; in that case, $\pi_2 \simeq \pi_1 \perp \pi_1 \otimes \gamma'$ and we are done. More generally, if $c \in D_F^*(\gamma)$, then $c \in D_F^*(\pi_2) = G_F^*(\pi_2)$. Now $\pi_2 \simeq c\pi_2 \simeq \pi_1 \otimes c\gamma$ with $1 \in D_F^*(c\gamma)$, so we are done by the previous case. □

Theorem 3.7. *Let φ, ψ be anisotropic p-forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Then $G_F(\varphi) = G_F(\psi)$.*

Proof. The inclusion $F^p \subseteq G_F(\psi)$ is obvious. Hence, pick $a \in G_F(\varphi) \setminus F^p$. Since $G_F(\varphi) = D_F(\hat{\sigma}_F(\varphi))$ is a field, we get that $a^2, \dots, a^{p-1} \in D_F(\hat{\sigma}_F(\varphi))$ and $1 \in D_F(\hat{\sigma}_F(\varphi))$. As the p-form $\langle 1, a, \dots, a^{p-1} \rangle \simeq \langle\langle a \rangle\rangle$ is anisotropic, we get $\langle\langle a \rangle\rangle \subseteq \hat{\sigma}_F(\varphi)$ by Lemma 2.2; therefore, $\langle\langle a \rangle\rangle$ divides $\hat{\sigma}_F(\varphi)$ by Lemma 3.6. Since further $\hat{\sigma}_F(\varphi)$ divides φ, we can find a p-form φ' defined over F such that $\varphi \simeq \langle\langle a \rangle\rangle \otimes \varphi'$.

Set $E = F(\sqrt[p]{a})$; it holds that $(\varphi_E)_{\mathrm{an}} \simeq (\varphi'_E)_{\mathrm{an}}$. So, we have $\dim(\varphi'_E)_{\mathrm{an}} \leq \frac{1}{p} \dim \varphi$. But the reverse inequality is true by Lemma 2.8; hence, φ'_E is anisotropic and $\dim(\varphi_E)_{\mathrm{an}} = \frac{1}{p} \dim \varphi$.

Since $\varphi \overset{v_0}{\sim} \psi$, we have $\dim(\psi_E)_{\mathrm{an}} = \frac{1}{p} \dim \psi$. Let ψ' be a p-form over F such that $\psi'_E \simeq (\psi_E)_{\mathrm{an}}$. As it holds that

$$D_F(\psi) \subseteq D_E(\psi) = D_E(\psi') = D_F(\langle\langle a \rangle\rangle \otimes \psi'),$$

we get $\psi \subseteq \langle\langle a \rangle\rangle \otimes \psi'$ by Lemma 2.2. Comparing the dimensions implies that $\psi \simeq \langle\langle a \rangle\rangle \otimes \psi'$, and hence $a \in G_F(\psi)$.

All in all, we have proved $G_F(\varphi) \subseteq G_F(\psi)$. The other inclusion follows by the symmetry of the argument. □

Lemma 3.8. *Let τ be a p-form defined over F and K/F be a field extension such that $K^p \subseteq G_F(\tau)$. Let φ, ψ be p-forms defined over F, anisotropic over K and such that $\varphi_K \simeq \psi_K$. Then $\varphi \otimes \tau \simeq \psi \otimes \tau$ over F.*

Proof. Let $\varphi \simeq \langle b_1, \dots, b_m \rangle$ and $\psi \simeq \langle c_1, \dots, c_m \rangle$. It follows from the assumptions that $\{b_1, \dots, b_m\}$ and $\{c_1, \dots, c_m\}$ are two bases of the same K^p-vector space. Recall that we can get one basis from the other by a finite series of operations of the following type: an exchange of two basis elements, scalar multiplication of one basis element, and adding one basis element to another basis element. Thus, it is sufficient to show that the following hold for any $0 \leq i, j \leq m$, $i \neq j$, and $a \in K^p$:

(i) $\langle b_0, \dots, b_i, \dots, b_j, \dots, b_m \rangle \otimes \tau \simeq \langle b_0, \dots, b_j, \dots, b_i, \dots, b_m \rangle \otimes \tau$.

(ii) $\langle b_0, \dots, b_i, \dots, b_m \rangle \otimes \tau \simeq \langle b_0, \dots, ab_i, \dots, b_m \rangle \otimes \tau$.

(iii) $\langle b_0, \dots, b_i, \dots, b_j, \dots, b_m \rangle \otimes \tau \simeq \langle b_0, \dots, b_i + b_j, \dots, b_j, \dots, b_m \rangle \otimes \tau$.

Part (i) is obvious. Part (ii) follows from the fact that $b_i\tau \simeq ab_i\tau$ since $a \in G_F(\tau)$. To prove part (iii), note that $b_i\tau \perp b_j\tau \simeq (b_i + b_j)\,\tau \perp b_j\tau$. □

Proposition 3.9. *Let $\pi \simeq \langle\langle a_1, \dots, a_n \rangle\rangle$ be an anisotropic quasi-Pfister form over F and $K = F(\sqrt[p]{a_1}, \dots, \sqrt[p]{a_n})$. Assume that $\varphi \simeq \pi \otimes \varphi'$ and $\psi \simeq \pi \otimes \psi'$ are anisotropic p-forms over F such that $\varphi'_K \overset{\text{sim}}{\sim} \psi'_K$. Then $\varphi \overset{\text{sim}}{\sim} \psi$.*

Proof. Without loss of generality, assume that $1 \in D_F(\varphi') \cap D_F(\psi')$. Write $\varphi' \simeq \langle 1, b_1, \dots, b_m \rangle$, and let $c \in K^*$ be such that $c\psi'_K \simeq \varphi'_K$. Then

$$c \in D_K(\varphi') = \operatorname{span}_{K^p}\{1, b_1, \dots, b_m\} \subseteq F,$$

thus, $c\psi'$ is defined over F. By [Zemková 2024, Theorem 3.3], we have that $p^n \mathfrak{i}_{\mathrm{d}}(\varphi'_K) = \mathfrak{i}_{\mathrm{d}}(\varphi) = 0$, and hence φ'_K is anisotropic; analogously, we get that $c\psi'_K$ is anisotropic. Now Lemma 3.8 implies that $\pi \otimes \varphi' \simeq \pi \otimes c\psi'$. As $\pi \otimes c\psi' \simeq c(\pi \otimes \psi')$, the claim follows. □

Remark 3.10. Theorem 3.7 and Proposition 3.9 can be used for a simplification of the problem, whether Vishik equivalence implies the similarity. Namely, we can restrict ourselves to the case of forms with the similarity factors F^p: Let φ, ψ be two anisotropic p-forms over F such that $\varphi \overset{v}{\sim} \psi$. Since $G_F(\varphi) = G_F(\psi)$ by Theorem 3.7, we can write $\varphi \simeq \pi \otimes \varphi'$ and $\psi \simeq \pi \otimes \psi'$ with the quasi-Pfister form $\pi \simeq \hat{\sigma}_F(\varphi) \simeq \hat{\sigma}_F(\psi)$ and some anisotropic p-forms φ', ψ' defined over F. Let $\pi \simeq \langle\langle a_1, \dots, a_n \rangle\rangle$, and put $K = F(\sqrt[p]{a_1}, \dots, \sqrt[p]{a_n})$. Then φ'_K, ψ'_K are anisotropic by [Hoffmann 2004, Proposition 5.7]. Hence $(\varphi_K)_{\mathrm{an}} \simeq \varphi'_K$ and $(\psi_K)_{\mathrm{an}} \simeq \psi'_K$,

and we get $\varphi'_K \overset{v}{\sim} \psi'_K$ by Lemma 2.16 (note that to apply this lemma, we need the full strength of the Vishik equivalence, the weak Vishik equivalence does not have to be sufficient here). If we knew that $\varphi'_K \overset{\text{sim}}{\sim} \psi'_K$, it would follow by Proposition 3.9 that $\varphi \overset{\text{sim}}{\sim} \psi$.

Counterexamples. Recall that by Proposition 3.1, Question Q has a positive answer for quasi-Pfister forms and quasi-Pfister neighbors of codimension one (i.e., Vishik equivalence is sufficient for the p-forms to be similar). It would be nice if Question Q had a positive answer for all p-forms. Unfortunately, this is not true in general — as we will see, it is not even true for all quasi-Pfister neighbors. We will provide two examples of pairs of p-forms that are Vishik equivalent but not similar. However, both of the counterexamples have two things in common: First, the considered p-forms are subforms of $\langle\!\langle a \rangle\!\rangle$, and so they have norm degree one. Second, they do not work for $p = 2$ and $p = 3$. We will focus on the characteristic two case in Section 4, but the case $p = 3$ remains open as well as p-forms of higher norm degrees.

Before we can get to the counterexamples, we need some lemmas.

Lemma 3.11. *Let $1 \leq k, l \leq p-1$, $a \in F \setminus F^p$, and let E/F be a field extension. Then the following are equivalent*:

(i) $\langle 1, a^k \rangle$ *is isotropic over* E.

(ii) $\langle 1, a^l \rangle$ *is isotropic over* E.

(iii) $a^k \in E^p$.

(iv) $a^l \in E^p$.

Proof. Assume that $\langle 1, a^k \rangle$ is isotropic over E. This is equivalent to the existence of $x, y \in E$, at least one (and hence both) of them nonzero, such that $x^p + a^k y^p = 0$, which is equivalent to $a^k = \frac{(-x)^p}{y^p} \in E^p$. This proves both (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv).

Now let $1 \leq t \leq p-1$ be such that $kt \equiv l \pmod p$. If $a^k \in E^p$, then $a^{kt} \in E^p$, and hence also $a^l \in E^p$. This proves (iii) $\Rightarrow$ (iv) and, by the symmetry of the argument, also (iv) $\Rightarrow$ (iii). □

Lemma 3.12. *Let $a \in F \setminus F^p$ and $k, l \in \mathbb{Z}$ with $k, l \not\equiv 0 \pmod p$. Then we have $\langle 1, a^k \rangle \overset{\text{sim}}{\sim} \langle 1, a^l \rangle$ if and only if $k \equiv \pm l \pmod p$.*

Proof. If $k \equiv l \pmod p$, then $k = pm + l$ for some $m \in \mathbb{Z}$, and so we have $\langle 1, a^k \rangle \simeq \langle 1, a^{pm+l} \rangle \simeq \langle 1, a^l \rangle$. If $k \equiv -l \pmod p$, then we can write $k = pn - l$ for some $n \in \mathbb{Z}$, and then we have

$$\langle 1, a^k \rangle = \langle 1, a^{pn-l} \rangle \simeq \langle 1, a^{-l} \rangle \overset{\text{sim}}{\sim} a^l \langle 1, a^{-l} \rangle \simeq \langle a^l, 1 \rangle \simeq \langle 1, a^l \rangle.$$

To prove the opposite direction, assume that $\langle 1, a^k\rangle \overset{\text{sim}}{\sim} \langle 1, a^l\rangle$, i.e., there exists $c \in F^*$ such that $c\langle 1, a^k\rangle \simeq \langle 1, a^l\rangle$. Then necessarily $c \in D_F(\langle 1, a^l\rangle)$, so we can find $x, y \in F$ such that $c = x^p + a^l y^p$. On the other hand, $1 \in D_F(c\langle 1, a^k\rangle)$, and thus $1 = cu^p + ca^k v^p$ for some $u, v \in F$. Putting these together, we have

$$1 = (x^p + a^l y^p)\, u^p + (x^p + a^l y^p)\, a^k u^p = (xu)^p + a^l (yu)^p + a^k (xv)^p + a^{k+l} (yv)^p.$$

Suppose $k \not\equiv \pm l \pmod p$. Then $\tau = \langle 1, a^k, a^l, a^{k+l}\rangle$ is a subform of $\langle\langle a\rangle\rangle$, and hence anisotropic. Thus, τ represents every element of $D_F(\tau)$ uniquely, so we get $xu \neq 0$ and $yu = xv = yv = 0$. This implies $x, u \neq 0$ and $y = v = 0$; hence, $c = x^p$ and $\langle 1, a^k\rangle \simeq \langle 1, a^l\rangle$. Thus, $a^l \in D_F(\langle 1, a^k\rangle)$, which is impossible for $l \not\equiv 0, k \pmod p$. □

Lemma 3.13. *Let $a \in F \setminus F^p$ and φ, ψ be quasi-Pfister neighbors of $\langle\langle a\rangle\rangle$ of the same dimension. Then we have $\varphi \overset{v}{\sim} \psi$.*

Proof. Obviously, both φ and ψ are anisotropic. Let E/F be any field extension; it holds $(\varphi_E)_{\text{an}}, (\psi_E)_{\text{an}} \subseteq (\langle\langle a\rangle\rangle_E)_{\text{an}}$. Then either $\langle\langle a\rangle\rangle_E$ is anisotropic, and hence both φ_E and ψ_E are anisotropic, or we have $(\langle\langle a\rangle\rangle_E)_{\text{an}} \simeq \langle 1\rangle_E$ by Lemma 2.3, in which case $(\varphi_E)_{\text{an}} \simeq \langle 1\rangle_E \simeq (\psi_E)_{\text{an}}$. Consequently, $\varphi \overset{v}{\sim} \psi$. □

Example 3.14. Let $p > 3$, and let $\varphi \simeq \langle 1, a\rangle$, $\psi \simeq \langle 1, a^2\rangle$ be p-forms with $a \in F \setminus F^p$. Note that φ and ψ are quasi-Pfister neighbors of $\langle\langle a\rangle\rangle$; therefore, $\varphi \overset{v}{\sim} \psi$ by Lemma 3.13. On the other hand, Lemma 3.12 ensures that φ and ψ are not similar for any $p > 3$.

Note that the problems with characteristics two and three are different. If $p = 2$, then φ is anisotropic while ψ is isotropic. If $p = 3$, it holds that $a^2\varphi \simeq \psi$.

Example 3.15. Let $p = 5$ and $a \in F \setminus F^5$. Set $\pi \simeq \langle 1, a, a^2, a^3, a^4\rangle \simeq \langle\langle a\rangle\rangle$, $\varphi \simeq \langle 1, a, a^2\rangle$ and $\psi \simeq \langle 1, a, a^3\rangle$. It follows by Lemma 3.13 that $\varphi \overset{v}{\sim} \psi$.

Assume that $c \in F$ is such that $c\psi \simeq \varphi$. Note that

$$D_F(\varphi) = \{x_0^5 + ax_1^5 + a^2x_2^5 \mid x_i \in F,\ 0 \le i \le 2\},$$
$$D_F(\pi) = \{x_0^5 + ax_1^5 + a^2x_2^5 + a^3x_3^5 + a^4x_4^5 \mid x_i \in F,\ 0 \le i \le 4\},$$

and the expression of any element of $D_F(\varphi)$ (resp. $D_F(\pi)$) is unique thanks to the anisotropy of φ (resp. π). In particular, φ does not represent any term of the form a^3x^5 or a^4x^5 with $x \in F^*$.

As $c \in D_F(\varphi)$, we can write $c = x^5 + ay^5 + a^2z^5$ for some $x, y, z \in F$. Since $ca = ax^5 + a^2y^5 + a^3z^5 \in D_F(\varphi)$, it follows that $z = 0$. Furthermore, we have that $ca^3 = a^3x^5 + a^4y^5 + (az)^5 \in D_F(\varphi)$ implies that $x = y = 0$. But then $c = 0$, which is absurd. Therefore, φ and ψ are not similar.

4. Vishik equivalence on totally singular quadratic forms

In this section, we will answer Question Q for some families of totally singular quadratic forms. Note that totally singular quadratic forms are the special case of p-forms with $p = 2$, so we will use some results from the previous section. In particular, by Proposition 3.1, we already know that Question Q has a positive answer in case of quasi-Pfister forms and quasi-Pfister neighbors of codimension one.

There is one tool which is not available for $p > 2$ (at least not this strong version of it, see the note behind Question 4.7 in [Scully 2013]) that we will use repeatedly:

Proposition 4.1 [Laghribi 2006, Proposition 2.11]. *Let $a \in F \setminus F^2$ and let φ be an anisotropic totally singular quadratic form over F with $\dim \varphi \geq 2$. If φ becomes isotropic over $E = F(\sqrt{a})$, then there exists a totally singular quadratic form τ over F such that $\dim \tau = \mathrm{i}_{\mathrm{d}}(\varphi_E)$ and $\tau \otimes \langle 1, a\rangle \subseteq \varphi$.*

Minimal quadratic forms. Recall that a minimal quadratic form is an anisotropic totally singular quadratic form φ over F such that $\mathrm{ndeg}_F\, \varphi = 2^{\dim \varphi - 1}$.

We start with some preparatory lemmas on forms of dimensions two and three. Note that all anisotropic totally singular quadratic forms of dimensions two and three are minimal by Lemma 2.6.

Lemma 4.2. *Let $b, c \in F \setminus F^2$. Then $c \in D_F(\langle 1, b\rangle)$ if and only if $b \in D_F(\langle 1, c\rangle)$ if and only if $\langle 1, b\rangle \simeq \langle 1, c\rangle$.*

Proof. $c \in D_F(\langle 1, b\rangle)$ holds if we can find $x, y \in F$ such that $x^2 + by^2 = c$; as $c \notin F^2$, we have $y \neq 0$. It follows $\langle 1, b\rangle \simeq \langle 1, by^2\rangle \simeq \langle 1, x^2 + by^2\rangle \simeq \langle 1, c\rangle$, and thus also $b \in D_F(\langle 1, c\rangle)$. □

Lemma 4.3. *Let $a, b, x, y \in F$ and $y \neq 0$. Then*

$$\langle 1, a, bx^2 + aby^2\rangle \simeq \left(a + \left(\frac{x}{y}\right)^2\right)\langle 1, a, b\rangle.$$

Proof. We have

$$\begin{aligned}\langle 1, a, bx^2 + aby^2\rangle &\simeq \left\langle 1, a + \left(\frac{x}{y}\right)^2, b\left(a + \left(\frac{x}{y}\right)^2\right)\right\rangle \\ &\simeq \left(a + \left(\frac{x}{y}\right)^2\right)\left\langle a + \left(\frac{x}{y}\right)^2, 1, b\right\rangle \simeq \left(a + \left(\frac{x}{y}\right)^2\right)\langle 1, a, b\rangle. \quad \square\end{aligned}$$

Lemma 4.4. *Let $a, b, c \in F \setminus F^2$, and suppose that $a \in D_F(\langle 1, c, bc\rangle)$. Then there exists $s \in F$ such that $\langle 1, c, bc\rangle \simeq (a + s^2)\langle 1, a, b\rangle$.*

Proof. Let $x, y, z \in F$ be such that

$$a = x^2 + cy^2 + bcz^2. \tag{4-1}$$

Suppose first that $z = 0$; then $a \in D_F(\langle 1, c\rangle)$, and we have $\langle 1, c\rangle \simeq \langle 1, a\rangle$ by Lemma 4.2. Moreover, since $a \notin F^2$, it must be $y \neq 0$, and (4-1) can be rewritten as $c = \left(\frac{x}{y}\right)^2 + a\left(\frac{1}{y}\right)^2$. Putting these together, we obtain

$$\langle 1, c, bc\rangle \simeq \left\langle 1, a, b\left(\left(\frac{x}{y}\right)^2 + a\left(\frac{1}{y}\right)^2\right)\right\rangle \simeq \langle 1, a, b(x^2 + a)\rangle,$$

hence, $\langle 1, c, bc\rangle \simeq (a + x^2)\langle 1, a, b\rangle$ by Lemma 4.3.

Now let $z \neq 0$; then

$$\langle 1, c, bc\rangle \simeq \langle 1, c, x^2 + cy^2 + bcz^2\rangle \simeq \langle 1, a, c\rangle.$$

Note that $z \neq 0$ and $b \notin F^2$ imply $y^2 + bz^2 \neq 0$; hence, we can rewrite (4-1) as

$$c(y^2 + bz^2)^2 = (a + x^2)(y^2 + bz^2) = (xy)^2 + ay^2 + b(xz)^2 + abz^2.$$

Thus, we have

$$\langle 1, a, c\rangle \simeq \langle 1, a, b(xz)^2 + abz^2\rangle \simeq (a + x^2)\langle 1, a, b\rangle,$$

where the latter isometry follows from Lemma 4.3. Therefore, in this case we get $\langle 1, c, bc\rangle \simeq (a + x^2)\langle 1, a, b\rangle$, too. □

The following lemma mimics the situation we end up with after applying Proposition 4.1.

Lemma 4.5. *Let ψ be a totally singular quadratic form over F such that $1 \in D_F(\psi)$. Let $a, c \in F^*$ be such that $c\langle 1, a\rangle$ is anisotropic over F, and suppose $c\langle 1, a\rangle \subseteq \psi$. Then*

(i) *either $\langle 1, c, ac\rangle \subseteq \psi$ (this occurs if and only if $1 \notin D_F(c\langle 1, a\rangle)$),*

(ii) *or $\langle 1, a\rangle \subseteq \psi$ and $c \in D_F(\langle 1, a\rangle)$.*

Proof. If $1 \notin D_F(c\langle 1, a\rangle)$, then $\langle 1, a, ca\rangle$ is anisotropic, and so $\langle 1, c, ca\rangle \subseteq \psi$ according to Lemma 2.2. On the other hand, if $1 \in D_F(c\langle 1, a\rangle)$, then it follows that $c \in D_F(\langle 1, a\rangle) = G_F(\langle 1, a\rangle)$; thus, $\langle 1, a\rangle \simeq c\langle 1, a\rangle \subseteq \psi$. □

Now we prove that (weakly) Vishik equivalent minimal quadratic forms are always similar. The proof is based mainly on the 2-independence of the coefficients as explored in Lemma 2.13.

Theorem 4.6. *Let φ, ψ be totally singular quadratic forms over F such that φ_{an} is minimal over F. If $\varphi \overset{v_0}{\sim} \psi$, then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. Invoking Lemma 3.3 (and as obviously $\varphi \overset{\mathrm{sim}}{\sim} \psi$ if and only if $\varphi_{\mathrm{an}} \overset{\mathrm{sim}}{\sim} \psi_{\mathrm{an}}$), we can assume that the forms φ and ψ are anisotropic. By Lemma 2.12, we can

suppose that $\varphi \simeq \langle 1, a_1, \dots, a_n\rangle$ for some $\{a_1, \dots, a_n\} \subseteq F$ that is 2-independent over F; it follows that $N_F(\varphi) = F^2(a_1, \dots, a_n)$, and

$$\mathcal{B}_{(0)} = \{a_1, \dots, a_n\}$$

is a 2-basis of $N_F(\varphi)$ over F. Moreover, $N_F(\varphi) = N_F(\psi)$ by Proposition 3.4, and ψ is minimal over F by Corollary 3.5.

We start with an observation:

($\diamond$) For any $a \in D_F(\varphi) \setminus F^2$, the form $\varphi_{F(\sqrt{a})}$ is isotropic. As $\varphi \overset{v_0}{\sim} \psi$, ψ must be isotropic over $F(\sqrt{a})$ as well. By Proposition 4.1, we can find $c \in F^*$ such that $c\langle 1, a\rangle \subseteq \psi$.

The main idea is to look at such ca for some $a \in D_F(\varphi)$, and express it with respect to an appropriate 2-basis of $N_F(\psi)$ over F. Applying Lemma 2.13, we get that almost all coefficients must be zero. Via a combination of different values of a, we usually end up with the conclusion that c must be a square, which means that $a \in D_F(\psi)$. In particular, we will prove that there is a scalar multiple of ψ which represents all the values $1, a_1, \dots, a_n$.

We divide the proof into several steps and cases. To simplify the notation and omit multiple indices, we use the same letters repeatedly — the meaning of s_i, u_i, x_i, y_i, z_i changes in each subcase (although they are usually used in similar situations). On the other hand, the meaning of a_i, c_i, d_i, e_i is "global", i.e., does not change during the proof. Moreover, we consider only 2-independence and 2-bases over F, and so we omit to repeat "over F" each time. We also would like to recall the notation (see page 330): By a "unique expression with respect to a 2-basis" we actually mean the unique expression with respect to the corresponding F^2-linear basis.

(1) First of all, note that ($\diamond$) proves the claim completely if $\dim \psi = 2$. Therefore, suppose $\dim \psi \geq 3$.

As the first step, we will prove that ψ contains a subform similar to $\langle 1, a_1, a_2\rangle$: By ($\diamond$), we find $c_1 \in F^*$ such that $c_1\langle 1, a_1\rangle \subseteq \psi$. Since we are interested in ψ only up to similarity, we can assume without loss of generality that $c_1 = 1$. Now, $c_2\langle 1, a_2\rangle \subseteq \psi$ for some $c_2 \in F^*$. By Lemma 4.5, there are two possibilities: Either $\langle 1, a_2\rangle \subseteq \psi$ or $\langle 1, c_2, c_2a_2\rangle \subseteq \psi$.

(1A) If $\langle 1, a_2\rangle \subseteq \psi$, we get $\langle 1, a_1, a_2\rangle \subseteq \psi$ immediately (by Lemma 2.2).

(1B) Suppose that $\langle 1, c_2, c_2a_2\rangle \subseteq \psi$ holds. We have to further distinguish two cases, depending on whether a_1 is represented by the form $\langle 1, c_2, c_2a_2\rangle$ or not.

(1Bi) Assume that $a_1 \in D_F(\langle 1, c_2, c_2a_2\rangle)$. Then $\langle 1, c_2, c_2a_2\rangle \overset{\text{sim}}{\sim} \langle 1, a_1, a_2\rangle$ by Lemma 4.4.

(1Bii) Let $a_1 \notin D_F(\langle 1, c_2, c_2a_2\rangle)$; then $\langle 1, a_1, c_2, c_2a_2\rangle$ is an anisotropic subform of ψ, and hence $\psi \simeq \langle 1, a_1, c_2, c_2a_2, s_4, \dots, s_n\rangle$ for some suitable $s_4, \dots, s_n \in F^*$. Since ψ is minimal, the set

$$\mathcal{B}_{(1\text{Bii})} = \{a_1, c_2, c_2a_2, s_4, \dots, s_n\}$$

is 2-independent, and hence it is a 2-basis of $N_F(\psi)$. Since $a_2 \in N_F(\psi)$, the element a_2 has a unique expression with respect to $\mathcal{B}_{(1\text{Bii})}$:

$$a_2 = c_2 \cdot c_2a_2 \cdot \left(\frac{1}{c_2}\right)^2. \tag{4-2}$$

It follows by Lemma 2.13 that $a_2 \notin D_F(\psi)$.

Furthermore, as $a_1 + a_2 \in D_F(\varphi) \setminus F^2$, we use (✧) to find $d_2 \in F^*$ such that $d_2\langle 1, a_1 + a_2\rangle \subseteq \psi$. In particular, $d_2 \in D_F(\psi)$; hence,

$$d_2 = x_0^2 + a_1 \cdot x_1^2 + c_2 \cdot x_2^2 + c_2a_2 \cdot x_3^2 + \sum_{i=4}^{n} s_i \cdot x_i^2$$

for some suitable $x_0, \dots, x_n \in F$. Multiplying d_2 by $(a_1 + a_2)$ and using (4-2) (i.e., expressing $d_2(a_1 + a_2)$ with respect to the basis $\mathcal{B}_{(1\text{Bii})}$), we get

$$\begin{aligned} d_2(a_1 + a_2) = {} & (a_1x_1)^2 + a_1 \cdot x_0^2 + c_2 \cdot (a_2x_3)^2 + c_2a_2 \cdot x_2^2 \\ & + a_1 \cdot c_2 \cdot x_2^2 + a_1 \cdot c_2a_2 \cdot x_3^2 + \sum_{i=4}^{n} a_1 \cdot s_i \cdot x_i^2 + c_2 \cdot c_2a_2 \cdot \left(\frac{x_0}{c_2}\right)^2 \\ & + a_1 \cdot c_2 \cdot c_2a_2 \cdot \left(\frac{x_1}{c_2}\right)^2 + \sum_{i=4}^{n} c_2 \cdot c_2a_2 \cdot s_i \cdot \left(\frac{x_i}{c_2}\right)^2. \end{aligned}$$

We know that $d_2(a_1 + a_2)$ is represented by ψ; but that is impossible by Lemma 2.13 unless all the terms composed from at least two elements of $\mathcal{B}_{(1\text{Bii})}$ (i.e., all but the first four) are zero. It follows $x_i = 0$ for all $0 \le i \le n$; hence, $d_2 = 0$ which is absurd. Therefore, this case cannot happen at all.

We can conclude step (1): We have proved that ψ has a subform similar to $\langle 1, a_1, a_2\rangle$. If $\dim \psi = 3$, then there is nothing more to prove. From now on, we will assume that $\dim \psi \ge 4$ and $\langle 1, a_1, a_2\rangle \subseteq \psi$.

(2) Now let $k \in \{3, \dots, n\}$. By (✧), we find $c_k, d_k, e_k \in F^*$ such that $c_k\langle 1, a_k\rangle$, $d_k\langle 1, a_1 + a_k\rangle$ and $e_k\langle 1, a_2 + a_k\rangle$ are subforms of ψ. The case distinction is slightly different than in (1); here it depends on whether c_k is represented by $D_F(\langle 1, a_1, a_2\rangle)$ or not.

(2A) Assume $c_k \in D_F(\langle 1, a_1, a_2\rangle)$. Then

$$c_k = u_0^2 + a_1 \cdot u_1^2 + a_2 \cdot u_2^2 \tag{4-3}$$

for some $u_0, u_1, u_2 \in F$ (here we slightly abuse the notation; technically, u_0, u_1, u_2 depend on k), and so

$$c_k a_k = a_k \cdot u_0^2 + a_1 \cdot a_k \cdot u_1^2 + a_2 \cdot a_k \cdot u_2^2.$$

By the uniqueness of the expression of $c_k a_k$ with respect to $\mathcal{B}_{(0)}$, it follows by Lemma 2.13 that $c_k a_k \notin D_F(\langle 1, a_1, a_2\rangle)$. On the other hand, we know that $c_k a_k \in D_F(\psi)$; therefore, $\psi \simeq \langle 1, a_1, a_2, c_k a_k, s_4, \dots, s_n\rangle$ for some $s_4, \dots, s_n \in F^*$ (possibly different from the s_i's in case (1Bii) above), and

$$\mathcal{B}_{(2A)} = \{a_1, a_2, c_k a_k, s_4, \dots, s_n\}$$

is a 2-basis of $N_F(\psi)$ by the minimality of ψ. Obviously, $a_k = c_k \cdot c_k a_k \cdot c_k^{-2}$; rewriting c_k via (4-3), we get the unique expression of a_k with respect to $\mathcal{B}_{(2A)}$:

$$a_k = c_k a_k \cdot \left(\frac{u_0}{c_k}\right)^2 + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k}\right)^2 + a_2 \cdot c_k a_k \cdot \left(\frac{u_2}{c_k}\right)^2. \tag{4-4}$$

Furthermore, we have $d_k \in D_F(\psi)$, and hence

$$d_k = x_0^2 + a_1 \cdot x_1^2 + a_2 \cdot x_2^2 + c_k a_k \cdot x_3^2 + \sum_{i=4}^{n} s_i \cdot x_i^2$$

for suitable $x_0, \dots, x_n \in F$ (again, we omit to express the dependence on k). Multiplying d_k by $(a_1 + a_k)$ and using (4-4), we can express $d_k(a_1 + a_k)$ with respect to $\mathcal{B}_{(2A)}$ as

$$\begin{aligned}
d_k(a_1 + a_k) = {} & (a_1 x_1 + a_k u_0 x_3)^2 + a_1 \cdot (x_0 + a_k u_1 x_3)^2 + a_2 \cdot (a_k u_2 x_3)^2 \\
& + c_k a_k \cdot \left(\frac{u_0}{c_k} x_0 + \frac{a_1 u_1}{c_k} x_1 + \frac{a_2 u_2}{c_k} x_2\right)^2 + a_1 \cdot a_2 \cdot x_2^2 \\
& + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k} x_0 + \frac{u_0}{c_k} x_1 + x_3\right)^2 + \sum_{i=4}^{n} a_1 \cdot s_i \cdot x_i^2 \\
& + a_2 \cdot c_k a_k \cdot \left(\frac{u_2}{c_k} x_0 + \frac{u_0}{c_k} x_2\right)^2 + \sum_{i=4}^{n} c_k a_k \cdot s_i \cdot \left(\frac{u_0}{c_k} x_i\right)^2 \\
& + a_1 \cdot a_2 \cdot c_k a_k \cdot \left(\frac{u_2}{c_k} x_1 + \frac{u_1}{c_k} x_2\right)^2 + \sum_{i=4}^{n} a_1 \cdot c_k a_k \cdot s_i \cdot \left(\frac{u_1}{c_k} x_i\right)^2 \\
& + \sum_{i=4}^{n} a_2 \cdot c_k a_k \cdot s_i \cdot \left(\frac{u_2}{c_k} x_i\right)^2.
\end{aligned}$$

Similarly as before, since $d_k(a_1 + a_k) \in D_F(\psi)$ and ψ is minimal, it follows from Lemma 2.13 that all the "composed" terms, i.e., all the terms except for the first four, must be zero. In particular:

- The coefficients by $c_k a_k \cdot s_i$ and $a_1 \cdot c_k a_k \cdot s_i$ and $a_2 \cdot c_k a_k \cdot s_i$ are zero for each $4 \le i \le n$; since at least one of u_0, u_1 and u_2 is nonzero, we get that $x_i = 0$ for each $4 \le i \le n$.
- The coefficient by $a_1 \cdot a_2 \cdot$ equals zero; thus, $x_2 = 0$.
- The coefficient by $a_2 \cdot c_k a_k$ must be zero, i.e., $\frac{u_2}{c_k} x_0 + \frac{u_0}{c_k} x_2 = 0$. As $x_2 = 0$, we get $u_2 x_0 = 0$.
- The coefficient by $a_1 \cdot a_2 \cdot c_k a_k$ is zero, so $\frac{u_2}{c_k} x_1 + \frac{u_1}{c_k} x_2 = 0$. Again, as $x_2 = 0$, it must hold $u_2 x_1 = 0$.
- The coefficient by $a_1 \cdot c_k a_k$ must be zero, thus $x_3 + \frac{u_1}{c_k} x_0 + \frac{u_0}{c_k} x_1 = 0$.

If $u_2 \neq 0$, then $x_0 = x_1 = 0$, and in that case also $x_3 = 0$ (by the last bullet point); but then $d_k = 0$, which is absurd. Therefore, $u_2 = 0$.

We proceed analogously for e_k and $e_k(a_2 + a_k)$, only now we know $u_2 = 0$, which means that

$$c_k = u_0^2 + a_1 \cdot u_1^2 \quad \text{and} \quad a_k = c_k a_k \cdot \left(\frac{u_0}{c_k}\right)^2 + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k}\right)^2.$$

Since $e_k \in D_F(\psi)$, we have

$$e_k = y_0^2 + a_1 \cdot y_1^2 + a_2 \cdot y_2^2 + c_k a_k \cdot y_3^2 + \sum_{i=4}^{n} s_i \cdot y_i^2$$

for some $y_o, \ldots, y_n \in F$, and

$$\begin{aligned} e_k(a_2 + a_k) = {} & \left(a_2 y_2 + c_k a_k \frac{u_0}{c_k} y_3\right)^2 + a_1 \cdot \left(c_k a_k \frac{u_1}{c_k} y_3\right)^2 + a_2 \cdot y_0^2 \\ & + c_k a_k \cdot \left(\frac{u_0}{c_k} y_0 + a_1 \frac{u_1}{c_k} y_1\right)^2 + a_1 \cdot a_2 \cdot y_1^2 \\ & + a_1 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k} y_0 + \frac{u_0}{c_k} y_1\right)^2 + a_2 \cdot c_k a_k \cdot \left(y_3 + \frac{u_0}{c_k} y_2\right)^2 \\ & + \sum_{i=4}^{n} a_2 \cdot s_i \cdot y_i^2 + \sum_{i=4}^{n} c_k a_k \cdot s_i \cdot \left(\frac{u_0}{c_k} y_i\right)^2 \\ & + a_1 \cdot a_2 \cdot c_k a_k \cdot \left(\frac{u_1}{c_k} y_2\right)^2 + \sum_{i=4}^{n} a_1 \cdot c_k a_k \cdot s_i \cdot \left(\frac{u_1}{c_k} y_i\right)^2. \end{aligned}$$

Again, all the coefficients by the composed terms must be zero, so in particular:

- $y_i = 0$ for all $4 \le i \le n$ because of the coefficients by $a_2 \cdot s_i$.
- $y_1 = 0$ because of the coefficient by $a_1 \cdot a_2$.
- $u_1 y_0 = 0$ because of the coefficient by $a_1 \cdot c_k a_k$.

- $u_1 y_2 = 0$ because of the coefficient by $a_1 \cdot a_2 \cdot c_k a_k$.
- $y_3 + \frac{u_0}{c_k} y_2 = 0$ because of the coefficient by $a_2 \cdot c_k a_k$.

If $u_1 \neq 0$, then necessarily $y_0 = 0$ and $y_2 = 0$, which implies $y_3 = 0$; it would follow $e_k = 0$, which is absurd. Thus, we have $u_1 = 0$. Therefore, $c_k = u_0^2$, and we have $\langle 1, a_k \rangle \subseteq \psi$; in particular, $a_k \in D_F(\psi)$.

(2B) Suppose $c_k \notin D_F(\langle 1, a_1, a_2 \rangle)$; then $\langle 1, a_1, a_2, c_k \rangle \subseteq \psi$. Here we distinguish two subcases:

(2Bi) Let $c_k a_k \in D_F(\langle 1, a_1, a_2, c_k \rangle)$; then

$$c_k a_k = u_0^2 + a_1 \cdot u_1^2 + a_2 \cdot u_2^2 + c_k \cdot u_3^2$$

for some $u_0, \ldots, u_3 \in F$, and hence

$$a_k = u_3^2 + c_k \cdot \left(\frac{u_0}{c_k}\right)^2 + a_1 \cdot c_k \cdot \left(\frac{u_1}{c_k}\right)^2 + a_2 \cdot c_k \cdot \left(\frac{u_2}{c_k}\right)^2.$$

We have

$$\psi \simeq \langle 1, a_1, a_2, c_k, s_4, \ldots, s_n \rangle$$

for some $s_4, \ldots, s_n \in F$; then

$$\mathcal{B}_{(2\mathrm{Bi})} = \{a_1, a_2, c_k, s_4, \ldots, s_n\}$$

is a 2-basis of $N_F(\psi)$. As before, we consider the unique representations of d_k and e_k by ψ, and express $d_k(a_1 + a_k)$ and $e_k(a_2 + a_k)$ with respect to $\mathcal{B}_{(2\mathrm{Bi})}$. We obtain that $u_1 = u_2 = 0$; therefore, $a_k = u_3^2 + c_k\left(\frac{u_0}{c_k}\right)^2$, so in particular $a_k \in D_F(\psi)$.

(2Bii) If $c_k a_k \notin D_F(\langle 1, a_1, a_2, c_k \rangle)$, then we have $\dim \psi \geq 5$ and

$$\psi \simeq \langle 1, a_1, a_2, c_k, c_k a_k, s_5, \ldots, s_n \rangle$$

for some $s_5, \ldots, s_n \in F$, and the corresponding 2-basis of $N_F(\psi)$ is

$$\mathcal{B}_{(2\mathrm{Bii})} = \{a_1, a_2, c_k, c_k a_k, s_5, \ldots, s_n\}.$$

Again, we consider the unique representation of d_k by ψ. In this case, we express a_k with respect to $\mathcal{B}_{(2\mathrm{Bii})}$ as

$$a_k = c_k \cdot c_k a_k \cdot \left(\frac{1}{c_k}\right)^2.$$

This time, the consideration of the element $d_k(a_1 + a_k)$ with respect to $\mathcal{B}_{(2\mathrm{Bii})}$ already implies $d_k = 0$; that is absurd, and hence this case cannot happen.

(3) We have proved that, up to multiplying ψ by a constant from F^*, we have $\langle 1, a_1, a_2 \rangle \subseteq \psi$ (step (1)), and $a_k \in D_F(\psi)$ for all $3 \leq k \leq n$ (step (2)). Invoking Lemma 2.2, we get $\varphi \overset{\mathrm{sim}}{\sim} \psi$. □

Special quasi-Pfister neighbors. The definition of a special quasi-Pfister neighbor is motivated by its counterpart in characteristic different from 2, which appeared in [Ahmad and Ohm 1995].

Definition 4.7. We call a totally singular quadratic form φ over F a *special quasi-Pfister neighbor* if $\varphi \overset{\text{sim}}{\sim} \pi \perp b\sigma$ with π a quasi-Pfister form over F, $b \in F^*$ and $\sigma \subseteq \pi$. In such situation, we also say that φ is given by the triple (π, b, σ).

For a totally singular quadratic form φ over F, we define its *full splitting pattern* as

$$\mathrm{fSP}(\varphi) = \{\dim(\varphi_E)_{\mathrm{an}} \mid E/F \text{ a field extension}\}.$$

We will describe all special quasi-Pfister neighbors of dimension up to eleven and their full splitting pattern. In particular, we will see that all quasi-Pfister neighbors up to dimension eight are special. First, we recall a proposition from [Hoffmann and Laghribi 2004].

Lemma 4.8 [Hoffmann and Laghribi 2004, Proposition 8.12]. *Let φ be an anisotropic totally singular quadratic form over F with $\dim\varphi \leq 8$. Then φ is a quasi-Pfister neighbor if and only if*

(i) $\dim\varphi \leq 3$, *or*

(ii) $\dim\varphi = 2^n$ *for some $n \geq 1$ and φ is similar to an n-fold quasi-Pfister form, or*

(iii) *there exist $x, y, z \in F^*$ such that*

  (a) $\varphi \overset{\text{sim}}{\sim} \langle 1, x, y, xy, z\rangle$ *in the case of* $\dim\varphi = 5$,

  (b) $\varphi \overset{\text{sim}}{\sim} \langle 1, x, y, xy, z, xz\rangle$ *in the case of* $\dim\varphi = 6$,

  (c) $\varphi \overset{\text{sim}}{\sim} \langle x, y, z, xy, xz, yz, xyz\rangle$ *in the case of* $\dim\varphi = 7$.

With the previous lemma, it is easy to classify all special quasi-Pfister neighbors of dimensions up to 8: Any quasi-Pfister neighbor of dimension 2^n for some $n \geq 0$ is similar to an n-fold quasi-Pfister form, and hence special. For the other small dimensions, we have

- $\langle 1, x, y\rangle \simeq \langle\langle x\rangle\rangle \perp y\langle 1\rangle$,
- $\langle 1, x, y, xy, z\rangle \simeq \langle\langle x, y\rangle\rangle \perp z\langle 1\rangle$,
- $\langle 1, x, y, xy, z, xz\rangle \simeq \langle\langle x, y\rangle\rangle \perp z\langle 1, x\rangle$,
- $\langle x, y, z, xy, xz, yz, xyz\rangle \simeq xyz(\langle\langle x, y\rangle\rangle \perp z\langle 1, x, y\rangle)$;

therefore, we get the following corollary.

Corollary 4.9. *All anisotropic quasi-Pfister neighbors of dimensions up to* 8 *are special.*

Proposition 4.10. *Let φ be an anisotropic special quasi-Pfister neighbor.*

(i) *If* $\dim\varphi = 3$, *then* $\varphi \overset{\text{sim}}{\sim} \langle\langle a\rangle\rangle \perp d\langle 1\rangle$ *for some* $a, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3\}$.

(ii) *If* $\dim\varphi = 5$, *then we have* $\varphi \overset{\text{sim}}{\sim} \langle\langle a, b\rangle\rangle \perp d\langle 1\rangle$ *for some* $a, b, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 5\}$.

(iii) *If* $\dim\varphi = 6$, *then we have* $\varphi \overset{\text{sim}}{\sim} \langle\langle a, b\rangle\rangle \perp d\langle 1, a\rangle$ *for some* $a, b, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 6\}$.

(iv) *If* $\dim\varphi = 7$, *then we have* $\varphi \overset{\text{sim}}{\sim} \langle\langle a, b\rangle\rangle \perp d\langle 1, a, b\rangle$ *for some* $a, b, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 4, 7\}$.

(v) *If* $\dim\varphi = 9$, *then we have* $\varphi \overset{\text{sim}}{\sim} \langle\langle a, b, c\rangle\rangle \perp d\langle 1\rangle$ *for some* $a, b, c, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 5, 8, 9\}$.

(vi) *If* $\dim\varphi = 10$, *then* $\varphi \overset{\text{sim}}{\sim} \langle\langle a, b, c\rangle\rangle \perp d\langle 1, a\rangle$ *for some* $a, b, c, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 5, 6, 8, 10\}$.

(vii) *If* $\dim\varphi = 11$, *then* $\varphi \overset{\text{sim}}{\sim} \langle\langle a, b, c\rangle\rangle \perp d\langle 1, a, b\rangle$ *for some* $a, b, c, d \in F^*$ *and* $\mathrm{fSP}(\varphi) = \{1, 2, 3, 4, 6, 7, 8, 11\}$.

Proof. All special quasi-Pfister neighbors up to dimension 8 have been described by Lemma 4.8 and Corollary 4.9. If $\varphi \overset{\text{sim}}{\sim} \pi \perp d\sigma$ is a special quasi-Pfister neighbor and $9 \leq \dim\varphi \leq 11$, then necessarily $\dim\pi = 2^3$ and $1 \leq \dim\sigma \leq 3$. Recall that all totally singular quadratic forms of dimensions two or three are minimal; thus, without loss of generality, $\langle 1\rangle \subseteq \sigma \subseteq \langle 1, a, b\rangle$ for some 2-independent set $\{a, b\} \subseteq F$. Then we can find $c \in F^*$ such that $\{a, b, c\}$ is a 2-basis of $N_F(\pi)$ over F, and it follows that $\pi \simeq \langle\langle a, b, c\rangle\rangle$.

The full splitting pattern of φ follows directly from [Zemková 2024, Theorem 4.11]. □

In the following example, we show that not all quasi-Pfister neighbors are special.

Example 4.11. Let $\varphi \simeq \langle 1, a, b, c, d, ab, ac, ad, bc\rangle$ be a totally singular quadratic form such that $\{a, b, c, d\} \subseteq F$ is 2-independent over F. Then φ is a quasi-Pfister neighbor of $\langle\langle a, b, c, d\rangle\rangle$.

On the other hand, we have

$$\varphi_{F(\sqrt{b})} \simeq \langle 1, a, 1, c, d, a, ac, ad, c\rangle_{F(\sqrt{b})} \simeq \langle 1, a, c, d, ac, ad, 0, 0, 0\rangle_{F(\sqrt{b})},$$

since $\langle 1, a, c, d, ac, ad\rangle \subseteq \langle\langle a, c, d\rangle\rangle$ and $(\langle\langle a, c, d\rangle\rangle)_{F(\sqrt{b})} \simeq (\langle\langle a, b, c, d\rangle\rangle_{F(\sqrt{b})})_{\mathrm{an}}$ is anisotropic, we get

$$(\varphi_{F(\sqrt{b})})_{\mathrm{an}} \simeq \langle 1, a, c, d, ac, ad\rangle_{F(\sqrt{b})},$$

so we have in particular $6 \in \mathrm{fSP}(\varphi)$. As the full splitting pattern of any 9-dimensional special quasi-Pfister neighbor equals to $\{1, 2, 3, 4, 5, 8, 9\}$ by Proposition 4.10, it follows that φ cannot be any special quasi-Pfister neighbor.

Ideally, we would like to prove that if two totally singular forms are Vishik equivalent and at least one of them is a special quasi-Pfister neighbor, then they are similar. Unfortunately, we will need some additional assumptions. We start with a few lemmas.

Lemma 4.12 [Zemková 2024, Lemma 4.8]. *Let π be a quasi-Pfister form over F, $\sigma \subseteq \pi$ and $d \in F^*$ be such that the totally singular quadratic form $\varphi \simeq \pi \perp d\sigma$ is anisotropic. Let E/F be a field extension.*

(i) *If $d \in D_E(\pi)$, then $(\varphi_E)_{\mathrm{an}} \simeq (\pi_E)_{\mathrm{an}}$; in particular, $\mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\pi_E) + \dim \sigma$.*

(ii) *If $d \notin D_E(\pi)$, then $(\varphi_E)_{\mathrm{an}} \simeq (\pi_E)_{\mathrm{an}} \perp d(\sigma_E)_{\mathrm{an}}$; in particular, $\mathfrak{i}_{\mathrm{d}}(\varphi_E) = \mathfrak{i}_{\mathrm{d}}(\pi_E) + \mathfrak{i}_{\mathrm{d}}(\sigma_E)$.*

We prove that if a quasi-Pfister neighbor of norm degree 2^n contains a quasi-Pfister form of dimension 2^{n-1}, then it must be a special quasi-Pfister neighbor.

Lemma 4.13. *Let π be an anisotropic quasi-Pfister form over F. Moreover, let ψ be an anisotropic totally singular quadratic form over F such that $\pi \subseteq \psi$ and $\hat{\nu}_F(\psi) \simeq \pi \otimes \langle\langle b \rangle\rangle$ for some $b \in F^*$. Then there exists a totally singular form $\rho \subseteq \pi$ such that $\psi \simeq \pi \perp b\rho$.*

Proof. First, recall that a norm form is always anisotropic; hence, $\pi \otimes \langle\langle b \rangle\rangle$ is anisotropic, and we get by Lemma 2.10 that $b \notin N_F(\pi) = D_F(\pi)$.

Let ρ' be a totally singular quadratic form over F such that $\psi \simeq \pi \perp b\rho'$. Denote $s = \dim \rho'$ and $\rho' = \langle d'_1, \dots, d'_s \rangle$. For each $k \in \{1, \dots, s\}$, we proceed as follows: Since $bD_F(\rho') \subseteq D_F(\psi) \subseteq D_F(\pi \otimes \langle\langle b \rangle\rangle)$, we can write

$$bd'_k = \pi(\boldsymbol{\xi}_k) + b\pi(\boldsymbol{\zeta}_k)$$

for some appropriate vectors $\boldsymbol{\xi}_k, \boldsymbol{\zeta}_k$. If $\boldsymbol{\zeta}_k = 0$, then $bd'_k = \pi(\boldsymbol{\xi}_k) \in D_F(\pi)$, which contradicts the anisotropy of ψ; thus, the vector $\boldsymbol{\zeta}_k$ must be nonzero. It follows that $\pi(\boldsymbol{\zeta}_k) \neq 0$, and so we set $d_k = \pi(\boldsymbol{\zeta}_k)$; then

$$\pi \perp b\langle d'_k \rangle \simeq \pi \perp \langle \pi(\boldsymbol{\xi}_k) + bd_k \rangle \simeq \pi \perp b\langle d_k \rangle.$$

It follows that

$$\pi \perp b\langle d'_1, \dots, d'_k \rangle \perp b\langle d_{k+1}, \dots, d_s \rangle \simeq \pi \perp b\langle d'_1, \dots, d'_{k-1} \rangle \perp b\langle d_k, \dots, d_s \rangle$$

for any $1 \leq k \leq s$. Therefore,

$$\pi \perp b\langle d'_1, \dots, d'_s \rangle \simeq \pi \perp b\langle d_1, \dots, d_s \rangle,$$

where $\rho \simeq \langle d_1, \dots, d_s \rangle$ is a subform of π, because $d_k \in D_F(\pi)$ for each k. □

Proposition 4.14. *Let φ, ψ be anisotropic totally singular quadratic forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Assume that φ is a special quasi-Pfister neighbor given by a triple (π, b, σ). Moreover, suppose that $c\pi \subseteq \psi$ for some $c \in F^*$. Then ψ is a special*

quasi-Pfister neighbor given by a triple (π, b, ρ) *for some form* ρ *over* F *such that* $\rho \overset{v_0}{\sim} \sigma$.

Proof. Let $d \in D_F(\sigma)$. Then $d \in D_F(\pi) = G_F(\pi)$; hence, $d\varphi \simeq \pi \perp b(d\sigma)$ and $1 \in D_F(d\sigma)$. Therefore, we can assume that $1 \in D_F(\sigma)$. Then, since φ is anisotropic, we must have $b \notin D_F(\pi)$. It follows that $\hat{\nu}_F(\varphi) \simeq \pi \otimes \langle\langle b \rangle\rangle$. Since $\hat{\nu}_F(\psi) \simeq \hat{\nu}_F(\varphi)$ by Proposition 3.4, we also have $\hat{\nu}_F(\psi) \simeq \pi \otimes \langle\langle b \rangle\rangle$. Finally, we can suppose that $\pi \subseteq \psi$. Therefore, we can apply Lemma 4.13 to find a totally singular quadratic form $\rho \subseteq \pi$ such that $\psi \simeq \pi \perp b\rho$.

It remains to show that $\sigma \overset{v_0}{\sim} \rho$: The equality $\dim \sigma = \dim \rho$ follows directly from $\dim \varphi = \dim \psi$. So, consider the field $E = F(\sqrt{a})$ for some $a \in F^*$. If $a \notin N_F(\pi)$, then π_E is anisotropic by Lemma 2.8, and so are its subforms σ_E and ρ_E; in particular, $\mathrm{i_d}(\sigma_E) = \mathrm{i_d}(\rho_E)$. On the other hand, if $a \in N_F(\pi)$, then $(\pi \otimes \langle\langle a \rangle\rangle)_{\mathrm{an}} \simeq \pi$ by Lemma 2.10; together with Lemma 2.8, we obtain

$$D_E(\pi_E) = D_F(\pi \otimes \langle\langle a \rangle\rangle) = D_F(\pi).$$

It follows that $b \notin D_E(\pi_E)$, and so we get by Lemma 4.12 that

$$\mathrm{i_d}(\varphi_E) = \mathrm{i_d}(\pi_E) + \mathrm{i_d}(\sigma_E) \quad \text{and} \quad \mathrm{i_d}(\psi_E) = \mathrm{i_d}(\pi_E) + \mathrm{i_d}(\rho_E).$$

Since $\mathrm{i_d}(\varphi_E) = \mathrm{i_d}(\psi_E)$ by the assumption, we get $\mathrm{i_d}(\sigma_E) = \mathrm{i_d}(\rho_E)$. □

Remark 4.15. With the same notation as in Proposition 4.14, we can consider a stronger assumption $\varphi \overset{v}{\sim} \psi$ and ask whether it implies $\sigma \overset{v}{\sim} \rho$.

First, note that any field E with $b \in D_E(\pi)$ is problematic: In this case, we have

$$b\varphi_E \simeq b\pi_E \perp \sigma_E \simeq (\pi \perp \sigma)_E.$$

Hence, $(\varphi_E)_{\mathrm{an}} \simeq (\pi_E)_{\mathrm{an}}$, so we do not get any information about $\mathrm{i_d}(\sigma_E)$.

We can still give a more specific characterization of the problematic fields: As in the proof of Proposition 4.14, we can assume that $1 \in D_F(\sigma)$. First, let T and S be fields such that $F \subseteq T \subseteq S \subseteq E$, where T/F is purely transcendental, S/T is separable and E/S is purely inseparable.

If $b \in D_S(\pi)$, then $(\pi \perp \langle b \rangle)_S$ is isotropic, and hence $\pi \perp \langle b \rangle$ must be isotropic over F. But that is impossible because $\pi \perp \langle b \rangle \subseteq \pi \perp b\sigma$ and $\pi \perp b\sigma$ is anisotropic. Thus, $b \notin D_S(\pi)$.

Furthermore, isotropy is a finite problem; thus, we can construct a field L with $S \subseteq L \subseteq E$ such that L/S is finite, and $\mathrm{i_d}(\sigma_L) = \mathrm{i_d}(\sigma_E)$ and $\mathrm{i_d}(\rho_L) = \mathrm{i_d}(\rho_E)$. Then $L = S\left(\sqrt[2^{n_1}]{s_1}, \ldots, \sqrt[2^{n_k}]{s_k}\right)$ for some $k \geq 0$, $s_i \in S$ and $n_i \geq 1$. Set $K = S(\sqrt{s_1}, \ldots, \sqrt{s_k})$; then $\mathrm{i_d}(\sigma_K) = \mathrm{i_d}(\sigma_L)$ and $\mathrm{i_d}(\rho_K) = \mathrm{i_d}(\rho_L)$ by [Zemková 2024, Theorem 3.3].

If $b \notin D_K(\pi)$, then we can apply Lemma 4.12 to get $\mathrm{i_d}(\sigma_K) = \mathrm{i_d}(\rho_K)$; then $\mathrm{i_d}(\sigma_E) = \mathrm{i_d}(\rho_E)$ by the construction of K, and we are done. Therefore, assume $b \in D_K(\pi)$.

Now we can construct a field M such that $S \subseteq M \subseteq K$, it holds that $b \notin D_M(\pi)$, and for each $\varepsilon \in K \setminus M$, we have $b \in D_{M(\varepsilon)}(\pi)$. Then $(\varphi_M)_{\mathrm{an}} \simeq (\pi_M)_{\mathrm{an}} \perp b(\sigma_M)_{\mathrm{an}}$ and $(\psi_M)_{\mathrm{an}} \simeq (\pi_M)_{\mathrm{an}} \perp b(\rho_M)_{\mathrm{an}}$ and $\mathrm{i_d}(\sigma_M) = \mathrm{i_d}(\rho_M)$ by Lemma 4.12. Thus, let φ', ψ', π', σ', ρ' be forms over F such that $\varphi'_M \simeq (\varphi_M)_{\mathrm{an}}$, $\psi'_M \simeq (\psi_M)_{\mathrm{an}}$, etc. In particular, π'_M is a quasi-Pfister form by Lemma 2.3.

Let $\varepsilon \in K \setminus M$, and $e \in S$ be such that $\varepsilon^2 = e$. Then we know

$$b \in D_{M(\sqrt{e})}(\pi') \setminus D_M(\pi') = D_M(\pi' \otimes \langle\langle e \rangle\rangle) \setminus D_M(\pi'),$$

in particular, $D_M(\pi') \subsetneq D_M(\pi' \otimes \langle\langle e \rangle\rangle)$, which is only possible if $(\pi' \otimes \langle\langle e \rangle\rangle)_M$ is anisotropic. It follows by Lemma 2.8 that $\pi'_{M(\sqrt{e})}$ is anisotropic. In that case, its subforms $\sigma'_{M(\sqrt{e})}$, $\rho'_{M(\sqrt{e})}$ must be anisotropic, too. It follows that

$$\mathrm{i_d}(\sigma_{M(\sqrt{e})}) = \mathrm{i_d}(\sigma_M) = \mathrm{i_d}(\rho_M) = \mathrm{i_d}(\rho_{M(\sqrt{e})}).$$

However, the field extension K/M does not have to be simple, as we show in the following example: Let $\pi'_M \simeq \langle\langle a_1, \dots, a_n \rangle\rangle_M$ for some $a_1, \dots, a_n \in F^*$, and assume that $n \geq 2$. Set $K = M(\sqrt{b}, \sqrt{a_1 + a_2 b})$. It is easy to see that $\{b, a_1 + a_2 b\}$ is 2-independent over M, and hence $[K : M] = 4$. Since K does not contain any element of degree four over M, it follows that K/M is not simple. Now consider $\varepsilon \in K \setminus M$; then

$$\varepsilon = w + x\sqrt{b} + y\sqrt{b}\sqrt{a_1 + a_2 b} + z\sqrt{a_1 + a_2 b},$$

with $w, x, y, z \in M$, at least one of x, y, z nonzero. Then

$$D_{M(\varepsilon)}(\pi') = D_M(\pi' \otimes \langle\langle \varepsilon^2 \rangle\rangle) = M^2(a_1, \dots, a_n, \varepsilon^2).$$

Considering ε^2 as an element of $K^2/M^2(a_1, \dots, a_n)$, we get

$$\begin{aligned} \varepsilon^2 &= w^2 + bx^2 + a_1 by^2 + a_2(by)^2 + a_1 z^2 + a_2 bz^2 \\ &\equiv bx^2 + a_1 by^2 + a_2 bz^2 = b(x^2 + a_1 y^2 + a_2 z^2) \equiv b \mod M^2(a_1, \dots, a_n), \end{aligned}$$

where we used that $x^2 + a_1 y^2 + a_2 z^2 \neq 0$ (this holds because $\{a_1, a_2\}$ is 2-independent over M and at least one of x, y, z is nonzero by the assumption). Therefore, we have $D_{M(\varepsilon)}(\pi') = M^2(a_1, \dots, a_n, b)$. In particular, $b \in D_{M(\varepsilon)}(\pi')$ for any $\varepsilon \in K \setminus M$, and so we cannot find any field M' with $M \subsetneq M' \subseteq K$ such that $b \notin D_{M'}(\pi')$.

With the notation and assumptions of Proposition 4.14, we know that $\sigma \overset{v_0}{\sim} \rho$, and hence $N_F(\sigma) \simeq N_F(\rho)$ by Proposition 3.4. To conclude this section, we prove $N_F(\sigma) \simeq N_F(\rho)$ by a different approach.

Lemma 4.16. *Let $\varphi = \pi \perp b\sigma$, $\psi = \pi \perp b\rho$ be anisotropic totally singular quadratic forms over F with π a quasi-Pfister form, $b \in F^*$ and $\sigma, \rho \subseteq \pi$. Moreover, suppose that $1 \in D_F(\sigma) \cap D_F(\rho)$. If $\varphi \overset{v}{\sim} \psi$, then $N_F(\sigma) = N_F(\rho)$.*

Proof. First, note that the anisotropy of φ together with the assumption $1 \in D_F(\sigma)$ implies that $b \notin D_F(\pi)$.

By Lemma 2.7, we find $c_1, \dots, c_s \in F^*$ and $0 \le k \le s$ such that $\sigma = \langle 1, c_1, \dots, c_s\rangle$ and $\{c_1, \dots, c_k\}$ is a 2-basis of $N_F(\sigma)$ over F. Set $K = F(\sqrt{c_1}, \dots, \sqrt{c_k})$. Then $(\sigma_K)_{\mathrm{an}} \simeq \langle 1\rangle$ by Proposition 2.5. Write $\pi \simeq \langle\langle a_1, \dots, a_n\rangle\rangle$ and $\pi' \simeq (\pi_K)_{\mathrm{an}}$. Then

$$\begin{aligned} D_K(\pi') &= K^2(a_1, \dots, a_n) \\ &= F^2(c_1, \dots, c_k, a_1, \dots, a_n) \overset{N_F(\sigma)\subseteq N_F(\pi)}{=\!=} F^2(a_1, \dots, a_n) = D_F(\pi); \end{aligned}$$

hence, $b \notin D_K(\pi')$, and so $(\varphi_K)_{\mathrm{an}} \simeq \pi' \perp b\langle 1\rangle$. We will show that this is isometric to $(\psi_K)_{\mathrm{an}}$: Obviously, $\pi' \subseteq (\psi_K)_{\mathrm{an}}$. From the Vishik equivalence of φ and ψ we know $\dim(\psi_K)_{\mathrm{an}} = \dim \pi' + 1$. Set $\rho' \simeq (\rho_K)_{\mathrm{an}}$. Since $b \notin D_K(\pi')$, we get by Lemma 4.12 that $\pi' \perp b\rho'$ is anisotropic. It means that $\pi' \perp b\rho' \simeq (\psi_K)_{\mathrm{an}}$, and hence $\dim \rho' = 1$. As $1 \in D_F(\rho) \subseteq D_K(\rho')$, it follows that $\rho' \simeq \langle 1\rangle$. Consequently, again by Proposition 2.5, we get $N_F(\rho) \subseteq F^2(c_1, \dots, c_k) = N_F(\sigma)$. The other inclusion can be proved analogously. Therefore, $N_F(\sigma) = N_F(\rho)$. □

Conclusion. In this final part, we put together all our results on the Vishik equivalence of totally singular quadratic forms.

Before stating the main theorem, note that any anisotropic totally singular quadratic form of dimension less or equal to four is either minimal or it is similar to a quasi-Pfister form.

Theorem 4.17. *Let φ, ψ be totally singular quadratic forms over F such that $\varphi \overset{v_0}{\sim} \psi$. Assume that φ is a special quasi-Pfister neighbor given by the triple (π, b, σ), and $c\pi \subseteq \psi$ for some $c \in F^*$. Moreover, suppose that σ is either a quasi-Pfister form, a quasi-Pfister neighbor of codimension one, or a minimal form. Then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. By Proposition 4.14, there exists a totally singular quadratic form $\rho \subseteq \pi$ and $c' \in F^*$ such that $c'\psi \simeq \pi \perp b\rho$ and $\sigma \overset{v_0}{\sim} \rho$. Then, by Corollary 3.5, resp. by Theorem 4.6, we have $\sigma \overset{\mathrm{sim}}{\sim} \rho$.

Let $d \in F^*$ be such that $\rho \simeq d\sigma$, and let $a \in D_F^*(\sigma)$. Then

$$da \in D_F^*(d\sigma) = D_F^*(\rho) \subseteq D_F^*(\pi) = G_F^*(\pi).$$

As $a \in G_F^*(\pi)$ and $G_F^*(\pi)$ is a group, it follows that $d \in G_F^*(\pi)$. Hence,

$$d\varphi \simeq d\pi \perp b(d\sigma) \simeq \pi \perp b\rho \simeq c'\psi,$$

i.e., $\varphi \overset{\mathrm{sim}}{\sim} \psi$. □

Corollary 4.18. *Let φ, ψ be totally singular quadratic forms over F such that $\varphi \overset{v}{\sim} \psi$. Let K/F be a field extension such that $K^2 \simeq G_F(\varphi)$. Assume that $(\varphi_K)_{\mathrm{an}}$ is a special quasi-Pfister neighbor given by the triple (π, b, σ), and that $c\pi \subseteq \psi_K$ for some $c \in K^*$. Moreover, suppose that σ is either a quasi-Pfister form, a quasi-Pfister neighbor of codimension one, or a minimal form (over K). Then $\varphi \overset{\mathrm{sim}}{\sim} \psi$.*

Proof. Without loss of generality, assume that φ and ψ are anisotropic. Denote $\tau \simeq \hat{\sigma}_F(\varphi)$, and let $a_1, \dots, a_n \in F^*$ be such that $\tau \simeq \langle\langle a_1, \dots, a_n \rangle\rangle$. Then there exists a totally singular quadratic form φ' over F such that $\varphi \simeq \tau \otimes \varphi'$. Furthermore, as $G_F(\varphi) = G_F(\tau) = D_F(\tau) = F^2(a_1, \dots, a_n)$, we have $K = F(\sqrt{a_1}, \dots, \sqrt{a_n})$. By [Zemková 2024, Theorem 3.3], $\mathrm{i_d}(\varphi'_K) = \mathrm{i_d}(\tau \otimes \varphi') = 0$. Therefore, φ'_K is anisotropic, and we have $(\varphi_K)_{\mathrm{an}} \simeq \varphi'_K$. Since $G_F(\varphi) \simeq G_F(\psi)$ by Theorem 3.7, we can use analogous arguments to find a form ψ' over F such that $\psi \simeq \tau \otimes \psi'$ and $\psi'_K \simeq (\psi_K)_{\mathrm{an}}$.

Since $\varphi \overset{v}{\sim} \psi$, we get by Lemma 2.16 that $\varphi'_K \overset{v}{\sim} \psi'_K$. Since $c\pi \subseteq \psi_K$ and π is anisotropic over K, it follows that $c\pi \subseteq \psi'_K$. Note that $\varphi'_K \simeq (\varphi_K)_{\mathrm{an}}$ is a special quasi-Pfister neighbor given by the triple (π, b, σ). Hence, by Theorem 4.17, we have $\varphi'_K \overset{\mathrm{sim}}{\sim} \psi'_K$. Finally, Proposition 3.9 implies that $\varphi \overset{\mathrm{sim}}{\sim} \psi$. □

Remark 4.19. We need in the proof of Corollary 4.18 that $\varphi'_K \overset{v_0}{\sim} \psi'_K$. To be able to conclude that, we need the full strength of $\varphi \overset{v}{\sim} \psi$, it would not be sufficient to assume $\varphi \overset{v_0}{\sim} \psi$.

We would like to conclude the article with a determination of the smallest dimension in which the answer to Question Q is not fully known. To do that, we first need to characterize totally singular quadratic forms of low dimension.

Proposition 4.20. *Let φ be a totally singular quadratic form over F. In each of the following cases, there exists a 2-independent set $\{a, b, c, d, e\}$ over F such that*:

(i) *Let* $\dim \varphi = 1$ *and* $\mathrm{ndeg}_F \varphi = 1$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle 1 \rangle$.

(ii) *Let* $\dim \varphi = 2$ *and* $\mathrm{ndeg}_F \varphi = 2$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle 1, a \rangle$.

(iii) *Let* $\dim \varphi = 3$ *and* $\mathrm{ndeg}_F \varphi = 4$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle 1, a, b \rangle$.

(iv) *Let* $\dim \varphi = 4$;

- *if* $\mathrm{ndeg}_F \varphi = 4$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle\langle a, b \rangle\rangle$,
- *if* $\mathrm{ndeg}_F \varphi = 8$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle 1, a, b, c \rangle$.

(v) *Let* $\dim \varphi = 5$;

- *if* $\mathrm{ndeg}_F \varphi = 8$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle\langle a, b \rangle\rangle \perp c\langle 1 \rangle$,
- *if* $\mathrm{ndeg}_F \varphi = 16$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle 1, a, b, c, d \rangle$.

(vi) *Let* $\dim \varphi = 6$;

- *if* $\mathrm{ndeg}_F \varphi = 8$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle\langle a, b \rangle\rangle \perp c\langle 1, a \rangle$,
- *if* $\mathrm{ndeg}_F \varphi = 16$, *then there are two possibilities*: *either* $\varphi \overset{\mathrm{sim}}{\sim} \langle\langle a, b \rangle\rangle \perp \langle c, d \rangle$, *or* $\varphi \overset{\mathrm{sim}}{\sim} \langle 1, a, b, c, d, t + ad \rangle$ *for some* $t \in F^2(a, b, c)$,
- *if* $\mathrm{ndeg}_F \varphi = 32$, *then* $\varphi \overset{\mathrm{sim}}{\sim} \langle 1, a, b, c, d, e \rangle$.

Proof. Cases (i)–(iv) are obvious.

(v) Let $\dim\varphi = 5$. If $\operatorname{ndeg}_F\varphi = 8$, then φ is a quasi-Pfister neighbor, which is special by Corollary 4.9; the claim then follows from Proposition 4.10. If $\operatorname{ndeg}_F\varphi = 16 = 2^{\dim\varphi-1}$, then φ is minimal, and the claim follows from Lemma 2.12.

(vi) Assume that $\dim\varphi = 6$. If $\operatorname{ndeg}_F\varphi = 8$, then φ is a quasi-Pfister neighbor, special by Corollary 4.9, and we obtain the required form from Proposition 4.10.

If $\dim\varphi = 6$ and $\operatorname{ndeg}_F\varphi = 16$, then consider $\tau \subseteq \varphi$ with $\dim\tau = 5$, and let $x \in F$ be such that $\varphi \simeq \tau \perp \langle x\rangle$. By (v), we can assume that either $\tau \simeq \langle\langle a, b\rangle\rangle \perp \langle c\rangle$, or $\tau \simeq \langle 1, a, b, c, d\rangle$. In the former case, we must have $x \in F^2(a, b, c, d) \setminus F^2(a, b, c)$, as otherwise $\operatorname{ndeg}_F\varphi = 8$. Therefore, $N_F(\varphi) = F^2(a, b, c, d) = F^2(a, b, c, x)$, so we can exchange the d in the 2-basis of $N_F(\varphi)$ for x. Up to renaming, we get that $\varphi \simeq \langle\langle a, b\rangle\rangle \perp \langle c, d\rangle$.

Now, let $\tau \simeq \langle 1, a, b, c, d\rangle$, i.e., $\varphi \simeq \langle 1, a, b, c, d, x\rangle$ for some $x \in F^2(a, b, c, d)$. Write $x = y + dz$ with $y, z \in F^2(a, b, c)$. If $z \in F^2$, then $\varphi \simeq \langle 1, a, b, c, d, y\rangle$. Denote $\sigma \simeq \langle 1, a, b, c, y\rangle$; then $\dim\sigma = 5$ and $\operatorname{ndeg}_F\sigma = 8$, so (up to renaming) $\sigma \simeq \langle\langle a, b\rangle\rangle \perp \langle c\rangle$ by (v), and hence $\varphi \simeq \langle\langle a, b\rangle\rangle \perp \langle c, d\rangle$. Now, assume that $z \notin F^2$. Then we can write $z = z_1 + az_2$ with $z_1, z_2 \in F^2(b, c)$; we can assume that $z_2 \neq 0$ (otherwise exchange a with b or c). Hence, $F^2(a, b, c) = F^2(z, b, c)$; without loss of generality, we can assume $a = z$. We get $\varphi \simeq \langle 1, a, b, c, d, y + ad\rangle$ with $y \in F^2(a, b, c)$ as claimed.

In the remaining case, $\dim\varphi = 6$ and $\operatorname{ndeg}_F\varphi = 32$, we have $\operatorname{ndeg}_F\varphi = 2^{\dim\varphi-1}$. Hence, φ is minimal, and we conclude by applying Lemma 2.12. □

Corollary 4.21. *The answer to Question Q is positive for all totally singular quadratic forms of dimension* ≤ 5.

Proof. Let φ be a totally singular form of dimension ≤ 5. By Proposition 4.20, φ is minimal, or a quasi-Pfister form, or a special quasi-Pfister neighbor given by a triple (π, b, σ) with σ minimal. Hence, the claim follows by Theorem 4.6 (minimal forms), Proposition 3.1 (quasi-Pfister forms), and Theorem 4.17 (special quasi-Pfister neighbors). □

Remark 4.22. In dimension six, the only case that remains open is if the norm degree of the form is 16. This case will be covered in a forthcoming article.

References

[Ahmad and Ohm 1995] H. Ahmad and J. Ohm, "Function fields of Pfister neighbors", *J. Algebra* **178**:2 (1995), 653–664. MR Zbl

[Gille and Szamuely 2006] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101**, Cambridge University Press, Cambridge, 2006. MR Zbl

[Hoffmann 2004] D. W. Hoffmann, "Diagonal forms of degree p in characteristic p", pp. 135–183 in *Algebraic and arithmetic theory of quadratic forms*, Contemp. Math. **344**, Amer. Math. Soc., Providence, RI, 2004. MR Zbl

[Hoffmann 2015] D. W. Hoffmann, "Motivic equivalence and similarity of quadratic forms", *Doc. Math.* **Extra Vol.** (2015), 265–275. MR Zbl

[Hoffmann and Laghribi 2004] D. W. Hoffmann and A. Laghribi, "Quadratic forms and Pfister neighbors in characteristic 2", *Trans. Amer. Math. Soc.* **356**:10 (2004), 4019–4053. MR Zbl

[Izhboldin 1998] O. T. Izhboldin, "Motivic equivalence of quadratic forms", *Doc. Math.* **3** (1998), 341–351. MR Zbl

[Izhboldin 2000] O. T. Izhboldin, "Motivic equivalence of quadratic forms, II", *Manuscripta Math.* **102**:1 (2000), 41–52. MR Zbl

[Karpenko 2000] N. A. Karpenko, "Criteria of motivic equivalence for quadratic forms and central simple algebras", *Math. Ann.* **317**:3 (2000), 585–611. MR Zbl

[Laghribi 2006] A. Laghribi, "The norm theorem for totally singular quadratic forms", *Rocky Mountain J. Math.* **36**:2 (2006), 575–592. MR Zbl

[Pickert 1949] G. Pickert, "Inseparable Körpererweiterungen", *Math. Z.* **52** (1949), 81–136. MR Zbl

[Scully 2013] S. Scully, "Rational maps between quasilinear hypersurfaces", *Compos. Math.* **149**:3 (2013), 333–355. MR Zbl

[Scully 2016] S. Scully, "Hoffmann's conjecture for totally singular forms of prime degree", *Algebra Number Theory* **10**:5 (2016), 1091–1132. MR Zbl

[Vishik 1997] A. S. Vishik, *Integral motives of quadrics*, Ph.D. thesis, Harvard University, 1997, available at https://www.proquest.com/docview/304341627. MR Zbl

[Vishik 2004] A. Vishik, "Motives of quadrics with applications to the theory of quadratic forms", pp. 25–101 in *Geometric methods in the algebraic theory of quadratic forms*, Lecture Notes in Math. **1835**, Springer, Berlin, 2004. MR Zbl

[Zemková 2022] K. Zemková, *Equivalence relations of quadratic forms in characteristic* 2 *and quasilinear* p*-forms*, Ph.D. thesis, Technische Universität Dortmund, 2022.

[Zemková 2024] K. Zemková, "Isotropy and full splitting pattern of quasilinear p-forms", *J. Algebra* **657** (2024), 37–56. MR Zbl

Received February 12, 2024. Revised May 30, 2024.

KRISTÝNA ZEMKOVÁ
FAKULTÄT FÜR MATHEMATIK
TECHNISCHE UNIVERSITÄT DORTMUND
DORTMUND
GERMANY

and

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF VICTORIA
VICTORIA BC
CANADA

zemk.kr@gmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 329, No. 2, 2024
https://doi.org/10.2140/pjm.2024.329.357

RLL-REALIZATION OF TWO-PARAMETER QUANTUM AFFINE ALGEBRA IN TYPE $D_n^{(1)}$

RUSHU ZHUANG, NAIHONG HU AND XIAO XU

We obtain the basic R-matrix of the two-parameter quantum group $U = U_{r,s}(\mathfrak{so}_{2n})$ via its weight representation theory and determine its R-matrix with spectral parameters for the two-parameter quantum affine algebra $U = U_{r,s}(\widehat{\mathfrak{so}_{2n}})$. Using the Gauss decomposition of the R-matrix realization of $U = U_{r,s}(\mathfrak{so}_{2n})$, we study the commutation relations of the Gaussian generators and finally arrive at its RLL-formalism of the Drinfeld realization of two-parameter quantum affine algebra $U = U_{r,s}(\widehat{\mathfrak{so}_{2n}})$.

1. Introduction 357
2. Preliminaries 359
3. Basic R-matrix 362
4. Faddeev–Reshetikhin–Takhtajan realization of $U_{r,s}(\mathfrak{so}_{2n})$ 367
5. The algebra $\mathcal{U}(\widehat{R})$ and its Gauss decomposition 375
6. Drinfeld realization of $U_{r,s}(\widehat{\mathfrak{so}_{2n}})$ 379
Acknowledgements 394
References 394

1. Introduction

Quantum groups were independently discovered by Drinfeld [6; 8] and Jimbo [15], who showed that the universal enveloping algebra $U(\mathfrak{g})$ of any Kac–Moody algebra $\mathfrak{g}$ admits a certain q-deformation $U_q(\mathfrak{g})$ as a Hopf algebra. Their construction is given in terms of Chevalley generators and q-Serre relations. For the Yangian algebra $Y(\mathfrak{g})$ and the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ of complex simple Lie algebra $\mathfrak{g}$, Drinfeld [7] gave their new realizations, which are quantizations of the loop realizations of the classical loop and affine Lie algebras. Faddeev, Reshetikhin and Takhtajan [21]

Hu is the corresponding author. This work is supported by the NNSF of China (Grant 12171155), and in part by the Science and Technology Commission of Shanghai Municipality (Grant 22DZ2229014).
MSC2020: primary 16T25, 17B37, 81R50; secondary 17B67.
Keywords: basic R-matrix, Drinfeld realization, RLL formulation, quantum affine algebra, Gauss decomposition.

© 2024 MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

presented the RLL-realizations of $U_q(\mathfrak{g})$ [19] of the classical simple Lie algebras $\mathfrak{g}$ by means of solutions of the quantum Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{12} = R \otimes I$, etc., and $R \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$. This realization is a natural analog of the matrix realizations of the classical Lie algebras, which originated from the quantum inverse scattering method developed by the St. Petersburg school. Later on, the R-matrix realization of quantum loop algebra $U_q(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}])$ using a solution of the quantum Yang–Baxter equation with spectral parameters $z, w \in \mathbb{C}$,

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z),$$

where $R(z)$ is a rational function of z with values in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ was given by Faddeev, Reshetikhin and Takhtajan in [9].

As we know [10], the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ admits a natural realization as a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, when $\mathfrak{g}$ is a simple Lie algebra. In [20], Reshetikhin and Semenov-Tian-Shansky proved that the central extension can be viewed as the affine analog of the construction in [21]. In [5], by using the Gauss decomposition, Ding and Frenkel showed that the R-matrix realization and Drinfeld realization of quantum affine algebra $U_q(\widehat{\mathfrak{gl}_n})$ are isomorphic. Recently, Jing, Liu and Molev [17; 18] presented the isomorphisms between the R-matrix and Drinfeld presentations of one-parameter quantum affine algebras $U_q(\widehat{\mathfrak{g}})$ of affine types $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$.

On the other hand, in 2001, Benkart and Witherspoon [1], motivated by the two-parameters (r, s)-Serre relations satisfied by the up-down operators defined on posets, reobtained the two-parameter quantum enveloping algebras $U_{r,s}(\mathfrak{g})$ corresponding to the general linear Lie algebra $\mathfrak{gl}_n$ and the special linear Lie algebra $\mathfrak{sl}_n$, which were earlier defined by Takeuchi [22] in 1990. Bergeron, Gao, Hu in [2] found the defining structures of two-parameter quantum groups $U_{r,s}(\mathfrak{g})$ of orthogonal and symplectic Lie algebras, which are realized as the Drinfeld doubles and established their weight representation theory of category $\mathcal{O}^{r,s}$ [3]. Much research has been done for the other types including the affine types (see [4] and references therein). Hu, Rosso and Zhang [12; 13] defined and initiated the study of the vertex representations of Drinfeld realizations of two-parameter quantum affine algebras of untwisted types and constructed the quantum affine Lyndon bases. The RLL-realization of the two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{gl}_n})$ has been given by Jing and Liu [16] under the name of RTT-realization.

A natural open question is how to work out the RLL-realizations for the quantum affine algebras $U_{r,s}(\widehat{\mathfrak{g}})$ of types $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$. The difficulty lies in that there has been no information about two-parameter basic R-matrices in the corresponding cases for many years, let alone their Yang–Baxterizations. Recently we have

overcome such obstacles and solved these problems (also see the subsequent preprints [14; 23]).

The current paper is the first to give the RLL-realization of the two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{so}_{2n}})$ by the Reshetikhin and Semenov-Tian-Shansky method. We show that Drinfeld's construction can be naturally established in the Gaussian decomposition of a matrix composed of elements of the quantum affine algebra. The organization of the paper is as follows. In Section 2, we recall the basic results. In Section 3, we give the $\widehat{R}$-matrix of the two-parameter quantum group $U_{r,s}(\mathfrak{so}_{2n})$. In Section 4, we give the isomorphism between Faddeev–Reshetikhin–Takhtajan and Drinfeld–Jimbo definitions of $U_{r,s}(\mathfrak{so}_{2n})$, and further give the spectral parameter dependent R-matrix $\widehat{R}(z)$ as the Yang–Baxterization of the basic R-matrix we obtained. In Sections 5–6, we study the commutation relations between Gaussian generators and give the Drinfeld realization of $U_{r,s}(\widehat{\mathfrak{so}_{2n}})$ (modified version of [12]).

2. Preliminaries

In [2], let $\mathbb{K} = \mathbb{Q}(r, s)$ be a ground field of rational functions in r, s, where r, s are algebraically independent indeterminates. Assume Φ is a finite root system of type D_n with Π a base of simple roots. Regard Φ as a subset of a Euclidean space $E = \mathbb{R}^n$ with an inner product $(\ ,\ )$. Let $\epsilon_1, \ldots, \epsilon_n$ denote an orthonormal basis of E, and suppose $\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid 1 \le i < n\} \cup \{\alpha_n = \epsilon_{n-1} + \epsilon_n\}$ and $\Phi = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i \ne j \le n\}$. In this case, set $r_i = r^{(\alpha_i,\alpha_i)/2}$ and $s_i = s^{(\alpha_i,\alpha_i)/2}$, so that $r_1 = \cdots = r_n = r$ and $s_1 = \cdots = s_n = s$.

The Cartan matrix of D_n is

$$D_n = (a_{ij})_{n\times n} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix}. \tag{2-1}$$

The quantum structural constant matrix is of the form

$$(\langle w_i', w_j\rangle)_{n\times n} = \begin{pmatrix} rs^{-1} & r^{-1} & 1 & \cdots & 1 & 1 & 1 \\ s & rs^{-1} & r^{-1} & \cdots & 1 & 1 & 1 \\ 1 & s & rs^{-1} & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & rs^{-1} & r^{-1} & r^{-1} \\ 1 & 1 & 1 & \cdots & s & rs^{-1} & r^{-1}s^{-1} \\ 1 & 1 & 1 & \cdots & s & rs & rs^{-1} \end{pmatrix}. \tag{2-2}$$

In [2], let $U_{r,s}(\mathfrak{so}_{2n})$ be the unital associative algebra over $\mathbb{Q}(r,s)$ generated by e_i, f_i, $w_i^{\pm 1}$, $w_i'^{\pm 1}(1 \le i \le n)$, subject to the defining relations (D1)–(D6):

(D1) $w_i^{\pm 1}$, $w_i'^{\pm 1}$ all commute with one another and $w_i w_i^{-1} = w_i' w_i'^{-1} = 1$.

(D2) For $1 \le i \le n$, $1 \le j \le n$, and $1 \le k (\ne n-1) \le n$, we have

$$\begin{aligned}
&w_j e_i = r^{(\epsilon_j, \alpha_i)} s^{(\epsilon_{j+1}, \alpha_i)} e_i w_j, && w_j f_i = r^{-(\epsilon_j, \alpha_i)} s^{-(\epsilon_{j+1}, \alpha_i)} f_i w_j,\\
&w_j' e_i = s^{(\epsilon_j, \alpha_i)} r^{(\epsilon_{j+1}, \alpha_i)} e_i w_j', && w_j' f_i = s^{-(\epsilon_j, \alpha_i)} r^{-(\epsilon_{j+1}, \alpha_i)} f_i w_j',\\
&w_n e_k = r^{(\epsilon_{n-1}, \alpha_k)} s^{-(\epsilon_n, \alpha_k)} e_k w_n, && w_n f_k = r^{-(\epsilon_{n-1}, \alpha_k)} s^{(\epsilon_n, \alpha_k)} f_k w_n,\\
&w_n' e_k = s^{(\epsilon_{n-1}, \alpha_k)} r^{-(\epsilon_n, \alpha_k)} e_k w_n', && w_n' f_k = s^{-(\epsilon_{n-1}, \alpha_k)} r^{(\epsilon_n, \alpha_k)} f_k w_n'.
\end{aligned}$$

(D3)
$$\begin{aligned}
w_n e_{n-1} &= r^{(\epsilon_n, \alpha_{n-1})} s^{-(\epsilon_{n-1}, \alpha_{n-1})} e_{n-1} w_n,\\
w_n f_{n-1} &= r^{-(\epsilon_n, \alpha_{n-1})} s^{(\epsilon_{n-1}, \alpha_{n-1})} f_{n-1} w_n,\\
w_n' e_{n-1} &= s^{(\epsilon_n, \alpha_{n-1})} r^{-(\epsilon_{n-1}, \alpha_{n-1})} e_{n-1} w_n',\\
w_n' f_{n-1} &= s^{-(\epsilon_n, \alpha_{n-1})} r^{(\epsilon_{n-1}, \alpha_{n-1})} f_{n-1} w_n'.
\end{aligned}$$

(D4) For $1 \le i, j \le n$, we have

$$[e_i, f_j] = \delta_{ij} \frac{w_i - w_i'^{-1}}{r - s}.$$

(D5) For any $1 \le i \ne j \le n$ but $(i,j) \notin \{(n-1,n), (n,n-1)\}$ with $a_{ij} = 0$, we have

$$\begin{aligned}
&[e_i, e_j] = 0, \quad e_{n-1} e_n = rs e_n e_{n-1},\\
&[f_i, f_j] = 0, \quad f_n f_{n-1} = rs f_{n-1} f_n.
\end{aligned}$$

(D6) For $1 \le i < j \le n$ with $a_{ij} = -1$, we have (r,s)-Serre relations:

$$\begin{aligned}
e_i^2 e_j - (r+s) e_i e_j e_i + rs e_j e_i^2 &= 0,\\
f_j f_i^2 - (r+s) f_i f_j f_i + rs f_i^2 f_j &= 0,\\
e_j^2 e_i - (r^{-1} + s^{-1}) e_j e_i e_j + r^{-1} s^{-1} e_i e_j^2 &= 0,\\
f_i f_j^2 - (r^{-1} + s^{-1}) f_j f_i f_j + r^{-1} s^{-1} f_j^2 f_i &= 0.
\end{aligned}$$

Proposition 2.1. *The algebra $U_{r,s}(\mathfrak{so}_{2n})$ is a Hopf algebra with comultiplication Δ, counit ε and antipode S such that*

$$\begin{aligned}
&\Delta(e_i) = e_i \otimes 1 + w_i \otimes e_i, && \Delta(f_i) = 1 \otimes f_i + f_i \otimes w_i',\\
&\varepsilon(e_i) = 0, && \varepsilon(f_i) = 0,\\
&S(e_i) = -w_i^{-1} e_i, && S(f_i) = -f_i w_i'^{-1},
\end{aligned}$$

and w_i, w_i' are group-like elements for any $i \in I$.

Definition 2.2. We define the linear mapping $f : \Lambda \times \Lambda \to k^*$ as

$$f(\lambda, \mu) = \langle w'_\mu, w_\lambda \rangle^{-1},$$

which satisfies

$$f(\lambda + \nu, \mu) = f(\lambda, \mu) f(\nu, \mu), \quad f(\lambda, \mu + \nu) = f(\lambda, \mu) f(\lambda, \nu).$$

Definition 2.3. Let M, M' be finite-dimensional U-modules. Define an isomorphism of U-modules $\tilde{f} : M \otimes M' \to M \otimes M'$ as

$$\tilde{f}(m \otimes m') = f(\lambda, \mu) m \otimes m',$$

where $m \in M_\lambda$, $m' \in M'_\mu$ and $\lambda, \mu \in \Lambda$.

Corollary 2.4 [3]. *$U_{r,s}(\mathfrak{g}) \cong U_{r,s}(\mathfrak{n}^-) \otimes U^0 \otimes U_{r,s}(\mathfrak{n})$, as vector spaces. In particular, it induces $U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}^-) \otimes U_0 \otimes U_q(\mathfrak{n})$, as vector spaces.*

Let $Q = \mathbb{Z}\Phi$ denote the root lattice and set $Q^+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$. Then for any $\zeta = \sum_{i=1}^n \zeta_i \alpha_i \in Q$, we denote

$$\omega_\zeta = \omega_1^{\zeta_1} \cdots \omega_n^{\zeta_n}, \quad \omega'_\zeta = (\omega'_1)^{\zeta_1} \cdots (\omega'_n)^{\zeta_n}.$$

Then $U_{r,s}(\mathfrak{n}^\pm) = \bigoplus_{\eta \in Q^+} U_{r,s}^{\pm\eta}(\mathfrak{n}^\pm)$ $Q^\pm$-graded, where

$$U_{r,s}^{\eta}(\mathfrak{n}^\pm) = \left\{ a \in U_{r,s}(\mathfrak{n}^\pm) \mid \omega_\zeta \, a \, \omega_\zeta^{-1} = \langle \omega'_\eta, \omega_\zeta \rangle \, a, \ \omega'_\zeta a {\omega'_\zeta}^{-1} = \langle \omega'_\zeta, \omega_\eta \rangle^{-1} a \right\},$$

for $\eta \in Q^+ \cup Q^-$.

Lemma 2.5 [3]. *Set $d_\zeta = \dim_{\mathbb{K}} U_{r,s}^{+\zeta}(\mathfrak{n}^+)$. Consider the basis $\{u_k^\zeta\}_{k=1}^{d_\zeta}$ for $U_{r,s}^{+\zeta}(\mathfrak{n}^+)$, and $\{v_k^\zeta\}_{k=1}^{d_\zeta}$ is the dual basis for $U_{r,s}^{-\zeta}(\mathfrak{n}^-)$ with respect to the pairing. Now let*

$$\Theta_\zeta = \sum_{k=1}^{d_\zeta} v_k^\zeta \otimes u_k^\zeta \in U \otimes U. \tag{2-3}$$

All but finitely many terms in this sum will act as multiplication by 0 on any weight space M_λ of $M \in \mathcal{O}$. $\Theta = \sum_{\zeta \in Q^+} \Theta_\zeta$ is a well-defined operator on such $M \otimes M$.

Theorem 2.6 [3]. *For M, M' to be finite-dimensional modules of $U_{r,s}(\mathfrak{g})$, the map*

$$R_{M',M} = \Theta \circ \tilde{f} \circ P : M' \otimes M \to M \otimes M'$$

must be an isomorphism of $U_{r,s}(\mathfrak{g})$-module, where $P(m \otimes m') = m' \otimes m$, $m \in M$, $m' \in M'$.

Theorem 2.7 [3]. *For M, M', M'' to be finite-dimensional modules of $U_{r,s}(\mathfrak{g})$, the map must satisfy*

$$\Theta_{12}^f \circ \Theta_{13}^f \circ \Theta_{23}^f = \Theta_{23}^f \circ \Theta_{13}^f \circ \Theta_{12}^f.$$

Equivalently, we have

$$R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23}.$$

Remark 2.8. Denote $\Theta_{12}^f = (\Theta_{M,M'} \circ \tilde{f}) \otimes 1$, $\Theta_{23}^f = 1 \otimes (\Theta_{M',M''} \circ \tilde{f})$, $\Theta_{13}^f = (1 \otimes P) \circ (\Theta^f \otimes 1) \circ (1 \otimes P)$. Also, $\Theta \circ \tilde{f}$ is the solution of the quantum Yang–Baxter equation.

3. Basic *R*-matrix

In this section, we consider the R-matrix $R := R_{V,V}$ for the vector representation $T_1 = T_V$ of the Drinfeld–Jimbo algebra $U = U_{r,s}(\mathfrak{so}_{2n})$.

Definition 3.1. The vector representation T_1 of the Drinfeld–Jimbo algebra U is the irreducible type 1 representation with highest weight $\lambda = (1, 0, \ldots, 0)$ with respect to the ordered sequence $\alpha_1, \ldots, \alpha_n$ of simple roots.

Consider a $2n$-dimensional vector space V over $\mathbb{K}$ with basis $\{v_j \mid 1 \le j \le 2n\}$. We define an action of the generators of $U = U_{r,s}(\mathfrak{so}_{2n})$ by specifying their matrices relative to this basis:

Lemma 3.2. *Let E_{kl} be the $2n \times 2n$ matrix with 1 in the (k,l)-position and 0 elsewhere. The vector representation T_1 of $U_{r,s}(\mathfrak{so}_{2n})$ is described by the following list*:

$$\text{(B1)}\quad \begin{aligned} T_1(e_i) &= E_{i,i+1} - r^{-\frac{1}{2}} s^{-\frac{1}{2}} E_{(i+1)',i'}, \quad T_1(e_n) = E_{n,n+2} - r^{-\frac{1}{2}} s^{-\frac{1}{2}} E_{(n+2)',n'}, \\ T_1(f_i) &= E_{i+1,i} - r^{-\frac{1}{2}} s^{-\frac{1}{2}} E_{i',(i+1)'}, \quad T_1(f_n) = E_{n+2,n} - r^{-\frac{1}{2}} s^{-\frac{1}{2}} E_{n',(n+2)'}, \end{aligned}$$

$$\text{(B2)}\quad \begin{aligned} T_1(w_i) &= r E_{i,i} + s E_{i+1,i+1} + s^{-1} E_{(i+1)',(i+1)'} + r^{-1} E_{i',i'} + \sum_{j \neq \{i,i+1,i',(i+1)'\}} E_{j,j}, \\ T_1(w_i') &= s E_{i,i} + r E_{i+1,i+1} + r^{-1} E_{(i+1)',(i+1)'} + s^{-1} E_{i',i'} + \sum_{j \neq \{i,i+1,i',(i+1)'\}} E_{j,j}, \end{aligned}$$

$$\text{(B3)}\quad \begin{aligned} T_1(w_n) &= s^{-1} E_{n-1,n-1} + r E_{n,n} + r^{-1} E_{n',n'} + s E_{(n-1)',(n-1)'} + r^{-1} s^{-1} E_{1,1} \\ &\quad + r^{-1} s^{-1} \sum_{1 \le j \le n-2} E_{j,j} + rs \sum_{1 \le j \le n-2} E_{j',j'}, \\ T_1(w_n') &= r^{-1} E_{n-1,n-1} + s E_{n,n} + s^{-1} E_{n',n'} + r E_{(n-1)',(n-1)'} + r^{-1} s^{-1} E_{1,1} \\ &\quad + r^{-1} s^{-1} \sum_{1 \le j \le n-2} E_{j,j} + rs \sum_{1 \le j \le n-2} E_{j',j'}, \end{aligned}$$

where $1 \le i, j \le n-1$, and $i' = 2n + 1 - i$.

Proof. By straightforward calculations one checks that the preceding formulas define a weight representation T_1 of the algebra $U_{r,s}(\mathfrak{so}_{2n})$ on the vector space $\mathbb{C}^{2n}$. For the basis vector $\boldsymbol{e}_1 = (1, 0, \cdots, 0)$, we easily verify that $T_1(E_j)\boldsymbol{e}_1 = 0$, $T_1(w_1)\boldsymbol{e}_1 = r\boldsymbol{e}_1$, $T_1(w_i)\boldsymbol{e}_1 = \boldsymbol{e}_1$, $T_1(w_n)\boldsymbol{e}_1 = r^{-1}s^{-1}\boldsymbol{e}_1$, $T_1(w_1')\boldsymbol{e}_1 = s\boldsymbol{e}_1$, $T_1(w_i')\boldsymbol{e}_1 = \boldsymbol{e}_1$, and $T_1(w_n')\boldsymbol{e}_1 = rs\boldsymbol{e}_1$, for $1 \le j \le n$ and $2 \le i \le n-1$. Hence T_1 is the type 1 representation with highest weight $\lambda = \alpha_1 + \alpha_2 + \cdots + \frac{1}{2}(\alpha_{n-1} + \alpha_n) = \epsilon_1$. Thus T_1 is indeed the vector representation of $U_{r,s}(\mathfrak{so}_{2n})$. □

We illustrate through the following lemmas that the module $V \otimes V$ is decomposed into the direct sum of three simple submodules, $S^o(V \otimes V)$, $S'(V \otimes V)$ and $\Lambda(V \otimes V)$. These modules are defined and proved by the following three lemmas.

Lemma 3.3. *The module $S^o(V \otimes V)$ generated by $\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i$ is simple, and a_i satisfies*

$$a_i = \begin{cases} (rs^{-1})^{\frac{n-i}{2}} & \text{if } 1 \le i \le n, \\ 1 & \text{if } i = n+1, \\ (rs^{-1})^{\frac{n-i+1}{2}} & \text{if } n+2 \le i \le 2n. \end{cases}$$

Proof. The operators e_k, f_k, w_k, w_k' act on $\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i$, then the following computations show that $S^o(V \otimes V)$ is a simple module:

$$\begin{aligned} e_k . &\left(\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i \right) \\ &= (e_k \otimes 1 + w_k \otimes e_k) \cdot \left(\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i \right) \\ &= (a_{(k+1)'} - a_{k'} r^{\frac{1}{2}} s^{-\frac{1}{2}})\, v_k \otimes v_{(k+1)'} + (a_{k+1} s^{-1} - r^{-\frac{1}{2}} s^{-\frac{1}{2}} a_k)\, v_{(k+1)'} \otimes v_k \\ &= 0, \end{aligned}$$

$$\begin{aligned} e_n . &\left(\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i \right) \\ &= (e_n \otimes 1 + w_n \otimes e_n) \cdot \left(\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i \right) \\ &= (a_{n'} r^{\frac{1}{2}} s^{-\frac{1}{2}} - a_{(n+2)'})\, v_n \otimes v_{(n+2)'} + (r^{-\frac{1}{2}} s^{-\frac{1}{2}} a_n - a_{(n+2)'} s^{-1})\, v_{(n+2)'} \otimes v_n \\ &= 0. \end{aligned}$$

Moreover,

$$w_k . \left(\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i \right) = \sum_{i=1}^{2n} a_i v_{i'} \otimes v_i, \quad w_k' . \left(\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i \right) = \sum_{i=1}^{2n} a_i v_{i'} \otimes v_i,$$

and for $1 \le k \le n$. By similar calculations as above, we know that the operators f_k act trivially on $\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i$. So, it is easy to see that $S^o(V \otimes V)$ is simple. □

Lemma 3.4. *The simple module $S'(V \otimes V)$ is defined as follows*:

(i) $v_i \otimes v_i$, $1 \le i \le 2n$,

(ii) $v_i \otimes v_j + sv_j \otimes v_i$, $1 \le i \le n$ *and* $i+1 \le j \le n$ *or* $i'+1 \le j \le 2n$,

(iii) $v_i \otimes v_j + r^{-1}v_j \otimes v_i$, $1 \le i \le n-1$, $n+1 \le j \le 2n-i$ *or* $n+1 \le i \le 2n-1$, $i+1 \le j \le 2n$,

(iv) $v_i \otimes v_{i'} + r^{-1}sv_{i'} \otimes v_i - (r^{-1}s)^{\frac{1}{2}}(v_{(i+1)'} \otimes v_{i+1} + v_{i+1} \otimes v_{(i+1)'})$, $1 \le i \le n-1$,

where $v_1 \otimes v_1$ is the highest weight vector.

Proof. Operators e_k, f_k act on vectors in (i)–(iv). The following computations show that $S'(V \otimes V)$ is a simple module, because for case (i), we have

$$e_k.(v_i \otimes v_i) = \begin{cases} \delta_{i,k+1}(v_k \otimes v_{k+1} + sv_{k+1} \otimes v_k), \\ -\delta_{i,k'}(rs)^{-\frac{1}{2}}(v_{i-1} \otimes v_i + r^{-1}v_i \otimes v_{i-1}), \end{cases}$$

or

$$f_k.(v_i \otimes v_i) = \begin{cases} \delta_{i,k}(v_k \otimes v_{k+1} + sv_{k+1} \otimes v_k), \\ -\delta_{i,(k+1)'}(rs)^{-\frac{1}{2}}(v_i \otimes v_{i+1} + r^{-1}v_{i+1} \otimes v_i), \end{cases}$$

where $1 \le k \le n-1$, and for $k = n$, we have

$$e_n.(v_i \otimes v_i) = \begin{cases} \delta_{i,n+2}(v_n \otimes v_{n+2} + sv_{n+2} \otimes v_n), \\ -\delta_{i,n+1}(rs)^{-\frac{1}{2}}(v_{n-1} \otimes v_{n+1} + r^{-1}v_{n+1} \otimes v_{n-1}), \end{cases}$$

$$f_n.(v_i \otimes v_i) = \begin{cases} \delta_{i,n}(v_n \otimes v_{n+2} + sv_{n+2} \otimes v_n), \\ -\delta_{i,n-1}(rs)^{-\frac{1}{2}}(v_{n-1} \otimes v_{n+1} + r^{-1}v_{n+1} \otimes v_{n-1}). \end{cases}$$

For case (ii), we have

$$\begin{aligned} &e_k.(v_i \otimes v_j + sv_j \otimes v_i) \\ &\qquad = e_k(v_i) \otimes v_j + sw_k(v_j) \otimes e_k(v_i) + se_k(v_j) \otimes v_i + w_k(v_i) \otimes e_k(v_j), \end{aligned}$$

so we get

$$e_k.(v_i \otimes v_j + sv_j \otimes v_i) = \begin{cases} \delta_{i+j,2n+2}(v_i \otimes v_{i'} + r^{-1}sv_{i'} \otimes v_i \\ \quad - (r^{-1}s)^{\frac{1}{2}}v_{(i+1)'} \otimes v_{i+1} + v_{i+1} \otimes v_{(i+1)'}), \\ \delta_{i,k+1}(v_{i-1} \otimes v_j + sv_j \otimes v_{i-1}), \\ \delta_{j,k'}(v_i \otimes v_{j-1} + sv_{j-1} \otimes v_i), \\ \delta_{i,k}\delta_{k+1,j}(v_i \otimes v_i). \end{cases}$$

For operators f_k, we have

$$\begin{aligned} &f_k.(v_i \otimes v_j + sv_j \otimes v_i) \\ &\qquad = v_i \otimes f_k(v_j) + sv_j \otimes f_k(v_i) + f_k(v_i) \otimes w'_k(v_j) + sf_k(v_j) \otimes w'_k(v_i), \end{aligned}$$

so we have

$$f_k.(v_i\otimes v_j+sv_j\otimes v_i)=\begin{cases}\delta_{i+j,2n}(v_i\otimes v_{i'}+r^{-1}sv_{i'}\otimes v_i\\ -(r^{-1}s)^{\frac{1}{2}}v_{(i+1)'}\otimes v_{i+1}+v_{i+1}\otimes v_{(i+1)'}),\\ \delta_{j,(k+1)'}(v_i\otimes v_{j+1}+sv_{j+1}\otimes v_i),\\ \delta_{i,k}(v_{i+1}\otimes v_j+sv_j\otimes v_{i+1}),\\ \delta_{i,(k+1)'}\delta_{j,k'}(v_{i+1}\otimes v_{i+1}),\end{cases}$$

where $1\le k\le n-1$. For $k=n$, and $n'+2\le l\le 2n$, we have

$$f_n.(v_i\otimes v_j+sv_j\otimes v_i)=\begin{cases}v_{n-1}\otimes v_{(n-1)'}+r^{-1}sv_{(n-1)'}\otimes v_{n-1}\\ -(r^{-1}s)^{\frac{1}{2}}(v_n\otimes v_{n'}+v_{n'}\otimes v_n),\\ \delta_{i,n}(v_{n+2}\otimes v_l+r^{-1}v_l\otimes v_{n+2}),\\ \delta_{i,n-1}(v_{n+1}\otimes v_l+r^{-1}v_l\otimes v_{n+1}),\\ v_l\otimes v_l.\end{cases}$$

For case (iii), we have

$$\begin{aligned}&e_k\cdot(v_i\otimes v_j+r^{-1}v_j\otimes v_i)\\ &\quad=e_k(v_i)\otimes v_j+r^{-1}w_k(v_j)\otimes e_k(v_i)+r^{-1}e_k(v_j)\otimes v_i+w_k(v_i)\otimes e_k(v_j),\end{aligned}$$

so we get

$$e_k.(v_i\otimes v_j+r^{-1}v_j\otimes v_i)=\begin{cases}\delta_{i+j,2n+2}(v_i\otimes v_{i'}+r^{-1}s\,v_{i'}\otimes v_i\\ -(r^{-1}s)^{\frac{1}{2}}(v_{(i+1)'}\otimes v_{i+1}+v_{i+1}\otimes v_{(i+1)'})),\\ \delta_{i,k+1}(v_{i-1}\otimes v_j+r^{-1}v_j\otimes v_{i-1}),\\ \delta_{j,k'}(v_i\otimes v_{j-1}+r^{-1}v_{j-1}\otimes v_i),\\ \delta_{i,k}\delta_{k+1,j}(v_i\otimes v_i).\end{cases}$$

For operators f_k, we have

$$\begin{aligned}&f_k.(v_i\otimes v_j+r^{-1}v_j\otimes v_i)\\ &\quad=v_i\otimes f_k(v_j)+r^{-1}v_j\otimes f_k(v_i)+f_k(v_i)\otimes w_k'(v_j)+r^{-1}f_k(v_j)\otimes w_k'(v_i),\end{aligned}$$

so we have

$$f_k.(v_i\otimes v_j+r^{-1}v_j\otimes v_i)=\begin{cases}\delta_{i+j,2n}(v_i\otimes v_{i'}+r^{-1}sv_{i'}\otimes v_i\\ -(r^{-1}s)^{\frac{1}{2}}(v_{(i+1)'}\otimes v_{i+1}+v_{i+1}\otimes v_{(i+1)'})),\\ \delta_{j,(k+1)'}(v_i\otimes v_{j+1}+r^{-1}v_{j+1}\otimes v_i),\\ \delta_{i,k}(v_{i+1}\otimes v_j+r^{-1}v_j\otimes v_{i+1}),\\ \delta_{i,(k+1)'}\delta_{j,k'}(v_{i+1}\otimes v_{i+1}),\end{cases}$$

where $1 \le k \le n-1$. For $k = n$, and $n' + 2 \le l \le 2n$, we have

$$f_n.(v_i \otimes v_j + r^{-1} v_j \otimes v_i) = \begin{cases} v_{n-1} \otimes v_{(n-1)'} + r^{-1} s v_{(n-1)'} \otimes v_{n-1} \\ \quad -(r^{-1}s)^{\frac{1}{2}}(v_n \otimes v_{n'} + v_{n'} \otimes v_n), \\ \delta_{i,n}(v_{n+2} \otimes v_l + s v_l \otimes v_{n+2}), \\ \delta_{i,n-1}(v_{n+1} \otimes v_l + s v_l \otimes v_{n+1}), \\ v_l \otimes v_l. \end{cases}$$

Therefore, it is easy to see that $S'(V \otimes V)$ is simple. □

Lemma 3.5. *The simple module $\Lambda(V \otimes V)$ is defined as follows*:

(i) $v_i \otimes v_j - r v_j \otimes v_i$, $1 \le i \le n$ *and* $i+1 \le j \le n$ *or* $i' + 1 \le j \le 2n$,

(ii) $v_i \otimes v_j - s^{-1} v_j \otimes v_i$, $1 \le i \le n-1$, $n+1 \le j \le 2n-i$, *or* $n+1 \le i \le 2n-1$, $i+1 \le j \le 2n$,

(iii) $-(rs)^{-\frac{1}{2}} v_i \otimes v_{i'} - s^{-1} v_{(i+1)'} \otimes v_{i+1} + r^{-1} v_{i+1} \otimes v_{(i+1)'} + (rs)^{-\frac{1}{2}} v_{i'} \otimes v_i$, $1 \le i \le n-1$,

where $v_1 \otimes v_2 - r v_2 \otimes v_1$ is the highest weight vector.

Proof. We can check this lemma by repeating similar calculations to those in Lemma 3.4. □

Lemma 3.6. *The decomposition of $U_{r,s}(\mathfrak{so}_{2n})$-module $V \otimes V$ is*

$$V \otimes V = S^o(V \otimes V) \oplus S'(V \otimes V) \oplus \Lambda(V \otimes V).$$

Proof. In [11], Hu and Pei proved that as braided tensor categories, the categories $\mathcal{O}^{r,s}$ of finite-dimensional weight $U_{r,s}(\mathfrak{g})$-modules (of type 1) and $\mathcal{O}^q$ are monoidally equivalent. Referring to the book by Klimyk and Schmüdgen [19], $U_q(\mathfrak{so}_{2n})$-module $V \otimes V$ is completely reducible and can be decomposed into the direct sum of three simple modules. □

Proposition 3.7. *The minimum polynomial of $R = R_{V,V}$ on $V \otimes V$ is*

$$(t - r^{-\frac{1}{2}} s^{\frac{1}{2}}) \cdot (t + r^{\frac{1}{2}} s^{-\frac{1}{2}}) \cdot (t - r^{\frac{2n-1}{2}} s^{-\frac{2n-1}{2}}).$$

Proof. It follows from the definition of R that $R(v_1 \otimes v_1) = r^{-\frac{1}{2}} s^{\frac{1}{2}} v_1 \otimes v_1$ and $R(v_1 \otimes v_2 - r v_2 \otimes v_1) = -r^{\frac{1}{2}} s^{-\frac{1}{2}}(v_1 \otimes v_2 - r v_2 \otimes v_1)$. By the proceeding lemmas, $S'(V \otimes V)$ and $\Lambda(V \otimes V)$ are simple, and in fact, $v_1 \otimes v_1$ and $v_1 \otimes v_2 - r v_2 \otimes v_1$ are the highest weight vectors. In particular, each is a cyclic module generated by its highest weight vector, respectively, $R(a_1 v_{1'} \otimes v_1) = a_1 r^{\frac{1}{2}} s^{-\frac{1}{2}} v_1 \otimes v_{1'} + \cdots$, and $v_1 \otimes v_{1'}$ only occurs in $R(a_1 v_{1'} \otimes v_1)$. So we have the desired result

$$R\left(\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i\right) = r^{\frac{2n-1}{2}} s^{-\frac{2n-1}{2}} \left(\sum_{i=1}^{2n} a_i v_{i'} \otimes v_i\right).$$ □

Theorem 3.8. *The braiding R-matrix $R = R_{V,V}$ acts as*

$$
\begin{aligned}
R = {} & r^{-\frac{1}{2}} s^{\frac{1}{2}} \sum_{i=1}^{2n} E_{ii} \otimes E_{ii} + r^{\frac{1}{2}} s^{-\frac{1}{2}} \sum_{i=1}^{2n} E_{ii'} \otimes E_{i'i} \\
& + r^{-\frac{1}{2}} s^{-\frac{1}{2}} \Bigg\{ \sum_{\substack{1 \le i \le n-1 \\ i+1 \le j \le n}} E_{ij} \otimes E_{ji} + \sum_{\substack{1 \le i \le n-1 \\ i'+1 \le j \le 2n}} E_{ij} \otimes E_{ji} + \sum_{j=n+2}^{2n} E_{nj} \otimes E_{jn} \\
& \qquad + \sum_{\substack{n+1 \le i \le 2n-1 \\ i+1 \le j \le 2n}} E_{ji} \otimes E_{ij} + \sum_{\substack{1 \le i \le n-1 \\ n+1 \le j \le 2n-i}} E_{ji} \otimes E_{ij} \Bigg\} \\
& + r^{\frac{1}{2}} s^{\frac{1}{2}} \Bigg\{ \sum_{\substack{1 \le i \le n-1 \\ i+1 \le j \le n}} E_{ji} \otimes E_{ij} + \sum_{\substack{1 \le i \le n-1 \\ i'+1 \le j \le 2n}} E_{ji} \otimes E_{ij} + \sum_{j=n+2}^{2n} E_{jn} \otimes E_{nj} \\
& \qquad + \sum_{\substack{n+1 \le i \le 2n-1 \\ i+1 \le j \le 2n}} E_{ij} \otimes E_{ji} + \sum_{\substack{1 \le i \le n-1 \\ n+1 \le j \le 2n-i}} E_{ij} \otimes E_{ji} \Bigg\} \\
& + (r^{-\frac{1}{2}} s^{\frac{1}{2}} - r^{\frac{1}{2}} s^{-\frac{1}{2}}) \Bigg\{ \sum_{i<j} E_{jj} \otimes E_{ii} - \sum_{i>j} (r^{-\frac{1}{2}} s^{\frac{1}{2}})^{(\rho_i - \rho_j)} E_{ij'} \otimes E_{i'j} \Bigg\},
\end{aligned}
$$

where

$$
\rho_i = \begin{cases} n-i & \text{if } 1 \le i \le n \\ 0 & \text{if } n+1 \\ n-i+1 & \text{if } n+2 \le i \le 2n \end{cases}.
$$

Proof. We need to check that the braiding R-matrix R acts on $S^o(V \otimes V)$, $S'(V \otimes V)$ as multiplication by $r^{\frac{2n-1}{2}} s^{-\frac{2n-1}{2}}$, $r^{-\frac{1}{2}} s^{\frac{1}{2}}$, and on $\Lambda(V \otimes V)$ as multiplication by $-r^{\frac{1}{2}} s^{-\frac{1}{2}}$. By straightforward calculations, one checks that the expression formula of the basic R-matrix is correct. □

Remark 3.9. Consider the matrix $\widehat{R} = P \circ R$, where $P = \sum_{i,j} E_{ij} \otimes E_{ji}$, and R satisfying the braiding relations on the tensor power $V^{\otimes k}$:

$$
\begin{aligned}
R_i \circ R_{i+1} \circ R_i &= R_{i+1} \circ R_i \circ R_{i+1}, \\
R_i \circ R_j &= R_j \circ R_i,
\end{aligned}
$$

where $1 \le i < k$, $|i-j| \ge 2$, $R_i = id_V{}^{i-1} \otimes R \otimes id_V{}^{k-1}$.

4. Faddeev–Reshetikhin–Takhtajan realization of $U_{r,s}(\mathfrak{so}_{2n})$

In this section, we give an isomorphism between Faddeev–Reshetikhin–Takhtajan and Drinfeld–Jimbo definitions of $U_{r,s}(\mathfrak{so}_{2n})$, and the spectral parameter dependent $R(z)$. Let $\mathcal{B}$ (resp. $\mathcal{B}'$) denote the subalgebra of $U_{r,s}(\mathfrak{so}_{2n})$ generated by e_i, $w_i'^{\pm 1}$ (resp. f_i, $w_i^{\pm 1}$), $1 \le i \le n$.

Definition 4.1. $U(\widehat{R})$ is an associative algebra with unit. It has generators l_{ij}^{+}, l_{ji}^{-}, $1 \le i \le 2n$. Let $L^{\pm} = (l_{ij}^{\pm})$, $1 \le i, j \le 2n+1$, with $l_{ij}^{+} = l_{ji}^{-} = 0$, and $l_{ii}^{-} l_{ii}^{+} = l_{ii}^{+} l_{ii}^{-}$ for $1 \le j < i \le 2n+1$. The defining relations are given in matrix form as follows:

$$\widehat{R} L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} \widehat{R}, \quad \widehat{R} L_1^{+} L_2^{-} = L_2^{-} L_1^{+} \widehat{R}, \tag{4-1}$$

where $L_1^{\pm} = L^{\pm} \otimes 1$, $L_2^{\pm} = 1 \otimes L^{\pm}$.

Since $L^{\pm}$ are upper and lower triangular, respectively, and the diagonal elements of these matrix are invertible, $L^{\pm}$ have inverse $(L^{\pm})^{-1}$ as matrices with elements in $U(\widehat{R})$. The relations between $L_1^{\pm}$ and $L_2^{\pm}$ immediately imply the following theorem.

Theorem 4.2. *The mapping ϕ_n between $U(\widehat{R})$ and $U_{r,s}(\mathfrak{so}_{2n})$ is an algebraic homomorphism.*

Proof. We check the theorem for the case of $n = 4$. Let us consider $L^{\pm}$,

$$L^{+} = \begin{pmatrix} l_{11}^{+} & l_{12}^{+} & \cdots & l_{18}^{+} \\ 0 & l_{22}^{+} & \ddots & \vdots \\ \vdots & \ddots & \ddots & l_{78}^{+} \\ 0 & \cdots & 0 & l_{88}^{+} \end{pmatrix}_{8\times 8}, \quad L^{-} = \begin{pmatrix} l_{11}^{-} & 0 & \cdots & 0 \\ l_{21}^{-} & l_{22}^{-} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ l_{81}^{-} & \cdots & l_{87}^{+} & l_{88}^{-} \end{pmatrix}_{8\times 8}.$$

Then for the generators $L_1^{\pm}$, $L_2^{\pm}$, $\widehat{R}$, we have that

$$L_1^{+} = \begin{pmatrix} l_{11}^{+} I_8 & l_{12}^{+} I_8 & \cdots & l_{18}^{+} I_8 \\ 0 & l_{22}^{+} I_8 & \ddots & \vdots \\ \vdots & \ddots & \ddots & l_{78}^{+} I_8 \\ 0 & \cdots & 0 & l_{88}^{+} I_8 \end{pmatrix}_{64\times 64}, \quad L_1^{-} = \begin{pmatrix} l_{11}^{-} I_8 & 0 & \cdots & 0 \\ l_{21}^{-} I_8 & l_{22}^{-} I_8 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ l_{81}^{-} I_8 & \cdots & l_{87}^{+} I_8 & l_{88}^{-} I_8 \end{pmatrix}_{64\times 64},$$

$$L_2^{\pm} = \begin{pmatrix} L^{\pm} & 0 & \cdots & 0 \\ 0 & L^{\pm} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L^{\pm} \end{pmatrix}_{64\times 64}, \quad \widehat{R} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{18} \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{78} \\ 0 & \cdots & 0 & A_{88} \end{pmatrix}_{64\times 64},$$

$$A_{11} = \begin{pmatrix} A'_{11} & 0 \\ 0 & Q A'^{-1}_{11} Q \end{pmatrix}_{8\times 8}, \quad A'_{11} = \begin{pmatrix} r^{-\frac{1}{2}} s^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & r^{\frac{1}{2}} s^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & r^{\frac{1}{2}} s^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & r^{\frac{1}{2}} s^{\frac{1}{2}} \end{pmatrix}_{4\times 4},$$

$$A_{22} = \begin{pmatrix} A'_{22} & 0 \\ 0 & Q A'^{-1}_{22} Q \end{pmatrix}_{8\times 8}, \quad A'_{22} = \begin{pmatrix} r^{-\frac{1}{2}} s^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & r^{-\frac{1}{2}} s^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & r^{\frac{1}{2}} s^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & r^{\frac{1}{2}} s^{\frac{1}{2}} \end{pmatrix}_{4\times 4},$$

$$A_{33}=\begin{pmatrix} A'_{33} & 0 \\ 0 & QA'^{-1}_{33}Q \end{pmatrix}_{8\times 8}, \quad A'_{33}=\begin{pmatrix} r^{-\frac{1}{2}}s^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & r^{-\frac{1}{2}}s^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & r^{-\frac{1}{2}}s^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & r^{\frac{1}{2}}s^{\frac{1}{2}} \end{pmatrix}_{4\times 4},$$

$$A_{44}=\begin{pmatrix} A'_{44} & 0 \\ 0 & QA'^{-1}_{44}Q \end{pmatrix}_{8\times 8}, \quad A'_{44}=\begin{pmatrix} r^{-\frac{1}{2}}s^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & r^{-\frac{1}{2}}s^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & r^{-\frac{1}{2}}s^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & r^{-\frac{1}{2}}s^{\frac{1}{2}} \end{pmatrix}_{4\times 4},$$

where $Q=\sum_{i=1}^{4}E_{5-i,i}$, $A_{i'i'}=A_{ii}^{-1}$ and

$$A_{ij}=(r^{-\frac{1}{2}}s^{\frac{1}{2}}-r^{\frac{1}{2}}s^{-\frac{1}{2}})\{E_{ji}-(r^{-\frac{1}{2}}s^{\frac{1}{2}})^{(\rho_{i'}-\rho_{j'})}E_{i'j'}\},$$

$1\le i<j\le 8$, $E_{ij}\in M(8,\mathbb{K})$, where the multiplication between matrices $\widehat{R}$, $L_1^{\pm}$ and $L_2^{\pm}$ is matrix multiplication. From the equation $\widehat{R}L_1^+L_2^+=L_2^+L_1^+\widehat{R}$, we can derive the following calculations:

$$\widehat{R}L_1^+L_2^+(v_1\otimes v_j)=L_2^+L_1^+\widehat{R}(v_1\otimes v_j)\Rightarrow\begin{cases} l_{11}^+l_{12}^+=rl_{12}^+l_{11}^+, & l_{11}^+l_{23}^+=l_{23}^+l_{11}^+, \\ l_{11}^+l_{34}^+=l_{34}^+l_{11}^+, & l_{11}^+l_{35}^+=r^{-1}s^{-1}l_{35}^+l_{11}^+, \end{cases}$$

where $1\le j\le 7$;

$$\widehat{R}L_1^+L_2^+(v_2\otimes v_j)=L_2^+L_1^+\widehat{R}(v_2\otimes v_j)\Rightarrow\begin{cases} l_{22}^+l_{12}^+=sl_{12}^+l_{22}^+, & l_{22}^+l_{23}^+=rl_{23}^+l_{22}^+, \\ l_{22}^+l_{34}^+=l_{34}^+l_{22}^+, & l_{22}^+l_{35}^+=(rs)^{-1}l_{35}^+l_{22}^+, \\ l_{12}^+l_{34}^+=l_{34}^+l_{12}^+, & l_{12}^+l_{35}^+=(rs)^{-1}l_{35}^+l_{12}^+, \end{cases}$$

and we have

$$l_{12}^+l_{23}^+ + (r^{-1}-s^{-1})l_{22}^+l_{l3}^+ = l_{23}^+l_{12}^+, \tag{4-2}$$

$$l_{12}^+l_{13}^+ = rl_{13}^+l_{12}^+, \tag{4-3}$$

where $1\le j\le 8$ $(j\ne 7)$;

$$\widehat{R}L_1^+L_2^+(v_3\otimes v_j)=L_2^+L_1^+\widehat{R}(v_3\otimes v_j)\Rightarrow\begin{cases} l_{33}^+l_{12}^+=l_{12}^+l_{22}^+, & l_{33}^+l_{23}^+=sl_{23}^+l_{33}^+, \\ l_{33}^+l_{34}^+=rl_{34}^+l_{33}^+, & l_{33}^+l_{35}^+=s^{-1}l_{35}^+l_{33}^+, \end{cases}$$

and we have

$$l_{23}^+l_{34}^+ + (r^{-1}-s^{-1})l_{33}^+l_{24}^+ = l_{34}^+l_{23}^+, \tag{4-4}$$

$$l_{23}^+l_{35}^+ + (r^{-1}-s^{-1})l_{33}^+l_{25}^+ = (rs)^{-1}l_{35}^+l_{23}^+, \tag{4-5}$$

$$l_{23}^+l_{24}^+ = rl_{24}^+l_{23}^+, \tag{4-6}$$

$$l_{23}^+l_{25}^+ = s^{-1}l_{25}^+l_{23}^+, \tag{4-7}$$

$$l_{13}^+l_{23}^+ = s^{-1}l_{23}^+l_{13}^+, \tag{4-8}$$

where $1 \leq j \leq 8$ $(j \neq 6)$;

$$\widehat{R}L_1^+L_2^+(v_4 \otimes v_j) = L_2^+L_1^+\widehat{R}(v_4 \otimes v_j) \Rightarrow \begin{cases} l_{44}^+l_{12}^+ = l_{12}^+l_{44}^+, & l_{44}^+l_{23}^+ = l_{23}^+l_{44}^+, \\ l_{44}^+l_{34}^+ = sl_{34}^+l_{44}^+, & l_{44}^+l_{35}^+ = rl_{35}^+l_{44}^+, \end{cases}$$

and we have

$$l_{24}^+l_{34}^+ = s^{-1}l_{34}^+l_{24}^+, \tag{4-9}$$

$$l_{25}^+l_{35}^+ = s^{-1}l_{35}^+l_{25}^+, \tag{4-10}$$

where $1 \leq j \leq 8$ $(j \neq 5)$. In particular, we get

$$\widehat{R}L_1^+L_2^+(v_4 \otimes v_5) = L_2^+L_1^+\widehat{R}(v_4 \otimes v_5),$$
$$\widehat{R}L_1^+L_2^+(v_5 \otimes v_4) = L_2^+L_1^+\widehat{R}(v_5 \otimes v_4),$$

then we obtain

$$l_{34}^+l_{35}^+ + rs^{-1}(r^{-\frac{1}{2}}s^{\frac{1}{2}} - r^{\frac{1}{2}}s^{-\frac{1}{2}})l_{36}^+l_{33}^+ = rs^{-1}l_{35}^+l_{34}^+, \tag{4-11}$$

$$r^{-1}sl_{35}^+l_{34}^+ + (r^{-\frac{1}{2}}s^{\frac{1}{2}} - r^{\frac{1}{2}}s^{-\frac{1}{2}})l_{36}^+l_{33}^+ = l_{34}^+l_{35}^+. \tag{4-12}$$

By (4-2), (4-3) and (4-8), we get

$$l_{12}^{+\,2}l_{23}^+ + rs^{-1}l_{23}^+l_{12}^{+\,2} = (rs^{-1}+1)l_{12}^+l_{23}^+l_{12}^+,$$
$$l_{23}^{+\,2}l_{12}^+ + r^{-1}sl_{12}^+l_{23}^{+\,2} = (r^{-1}s+1)l_{23}^+l_{12}^+l_{23}^+.$$

By (4-4), (4-6) and (4-9), we get

$$l_{23}^{+\,2}l_{34}^+ + rs^{-1}l_{34}^+l_{23}^{+\,2} = (rs^{-1}+1)l_{23}^+l_{34}^+l_{23}^+,$$
$$l_{34}^{+\,2}l_{23}^+ + r^{-1}sl_{23}^+l_{34}^{+\,2} = (r^{-1}s+1)l_{34}^+l_{23}^+l_{34}^+.$$

By (4-5), (4-7) and (4-10), we get

$$s^2l_{23}^{+\,2}l_{35}^+ + (rs)^{-1}l_{35}^+l_{23}^{+\,2} = (r^{-1}s+1)l_{23}^+l_{35}^+l_{23}^+,$$
$$(rs)^{-1}l_{35}^{+\,2}l_{23}^+ + s^2l_{23}^+l_{35}^{+\,2} = (r^{-1}s+1)l_{35}^+l_{23}^+l_{35}^+.$$

By (4-11) and (4-12), we get

$$l_{34}^+l_{35}^+ = l_{35}^+l_{34}^+.$$

For the equation $\widehat{R}L_1^-L_2^- = L_2^-L_1^-\widehat{R}$, we can repeat the similar calculation process as above. Then we define a morphism $\phi_4 : U(\widehat{R}) \to U_{r,s}(\mathfrak{so}_8)$:

$$l_{11}^+ \mapsto (w_1'w_2'w_3'^{\frac{1}{2}}w_4'^{\frac{1}{2}})^{-1}, \qquad l_{12}^+ \mapsto (r-s)e_1l_{11}^+,$$
$$l_{22}^+ \mapsto (w_2'w_3'^{\frac{1}{2}}w_4'^{\frac{1}{2}})^{-1}, \qquad l_{23}^+ \mapsto (r-s)e_2l_{22}^+,$$

$$
\begin{aligned}
&l_{33}^{+} \mapsto (w_3' w_4')^{-\frac{1}{2}}, && l_{34}^{+} \mapsto (r-s)e_3 l_{33}^{+},\\
&l_{44}^{+} \mapsto (w_3'^{-1} w_4')^{-\frac{1}{2}}, && l_{35}^{+} \mapsto (r-s)e_4 l_{33}^{+},\\
&l_{11}^{-} \mapsto (w_1 w_2 (w_3 w_4)^{\frac{1}{2}})^{-1}, && l_{21}^{-} \mapsto -(r-s) l_{11}^{-} f_1,\\
&l_{22}^{-} \mapsto (w_2 (w_3 w_4)^{\frac{1}{2}})^{-1}, && l_{32}^{-} \mapsto -(r-s) l_{22}^{-} f_2,\\
&l_{33}^{-} \mapsto (w_3 w_4)^{-\frac{1}{2}}, && l_{43}^{-} \mapsto -(r-s) l_{33}^{-} f_3,\\
&l_{44}^{-} \mapsto (w_3^{-1} w_4)^{-\frac{1}{2}}, && l_{53}^{-} \mapsto -(r-s) l_{33}^{-} f_4,\\
&l_{i'i'}^{+} \mapsto (l_{ii}^{+})^{-1}, && l_{i'i'}^{-} \mapsto (l_{ii}^{-})^{-1},
\end{aligned}
$$

where $1 \le i \le 4$. It is obvious that ϕ_4 still preserves the algebra structure, the relations in $\mathcal{B}$ and $\mathcal{B}'$, respectively. Next, we need to ensure that ϕ_4 preserves the cross relations of $\mathcal{B}$ and $\mathcal{B}'$. Considering the equation $\widehat{R} L_1^{+} L_2^{-} = L_2^{-} L_1^{+} \widehat{R}$, we have:

$$
\widehat{R} L_1^{+} L_2^{-} (v_1 \otimes v_j) = L_2^{-} L_1^{+} \widehat{R} (v_1 \otimes v_j) \Rightarrow
\begin{cases}
l_{11}^{+} l_{21}^{-} = r^{-1} l_{21}^{-} l_{11}^{+}, & l_{11}^{+} l_{32}^{-} = l_{32}^{-} l_{11}^{+},\\
l_{11}^{+} l_{43}^{-} = l_{43}^{-} l_{11}^{+}, & l_{11}^{+} l_{53}^{-} = rs l_{53}^{-} l_{11}^{+},
\end{cases}
$$

where $1 \le j \le 7$;

$$
\widehat{R} L_1^{+} L_2^{-} (v_2 \otimes v_j) = L_2^{-} L_1^{+} \widehat{R} (v_2 \otimes v_j) \Rightarrow
\begin{cases}
l_{22}^{+} l_{21}^{-} = s^{-1} l_{21}^{-} l_{22}^{+}, & l_{22}^{+} l_{32}^{-} = r^{-1} l_{32}^{-} l_{22}^{+},\\
l_{22}^{+} l_{43}^{-} = l_{43}^{-} l_{22}^{+}, & l_{22}^{+} l_{53}^{-} = rs l_{53}^{-} l_{22}^{+},\\
rs l_{12}^{+} l_{21}^{-} - l_{21}^{-} l_{12}^{+} = (s-r)(l_{22}^{-} l_{11}^{+} - l_{22}^{+} l_{11}^{-}),\\
l_{32}^{-} l_{12}^{+} = r l_{12}^{+} l_{32}^{-}, & l_{43}^{-} l_{12}^{+} = l_{12}^{+} l_{43}^{-},\\
l_{12}^{+} l_{53}^{-} = rs l_{53}^{-} l_{12}^{+}, & l_{11}^{-} l_{12}^{+} = s l_{12}^{+} l_{11}^{-},\\
l_{22}^{-} l_{12}^{+} = r l_{12}^{+} l_{22}^{-}, & l_{33}^{-} l_{12}^{+} = l_{12}^{+} l_{33}^{-},\\
l_{44}^{-} l_{12}^{+} = l_{12}^{+} l_{44}^{-},
\end{cases}
$$

where $1 \le j \le 8$ $(j \ne 7)$;

$$
\widehat{R} L_1^{+} L_2^{-} (v_3 \otimes v_j) = L_2^{-} L_1^{+} \widehat{R} (v_3 \otimes v_j) \Rightarrow
\begin{cases}
l_{33}^{+} l_{21}^{-} = l_{21}^{-} l_{33}^{+}, & l_{33}^{+} l_{32}^{-} = s^{-1} l_{32}^{-} l_{33}^{+},\\
l_{33}^{+} l_{43}^{-} = r^{-1} l_{43}^{-} l_{33}^{+}, & l_{33}^{+} l_{53}^{-} = s l_{53}^{-} l_{33}^{+},\\
l_{23}^{+} l_{21}^{-} = s^{-1} l_{21}^{-} l_{23}^{+},\\
rs l_{23}^{+} l_{32}^{-} - l_{32}^{-} l_{23}^{+} = (s-r)(l_{33}^{-} l_{22}^{+} - l_{33}^{+} l_{22}^{-}),\\
l_{23}^{+} l_{43}^{-} = r^{-1} l_{43}^{-} l_{23}^{+}, & l_{23}^{+} l_{53}^{-} = s l_{53}^{-} l_{23}^{+},\\
l_{23}^{+} l_{11}^{-} = l_{11}^{-} l_{23}^{+}, & l_{23}^{+} l_{22}^{-} = s^{-1} l_{22}^{-} l_{23}^{+},\\
l_{23}^{+} l_{33}^{-} = r^{-1} l_{33}^{-} l_{23}^{+}, & l_{23}^{+} l_{44}^{-} = l_{44}^{-} l_{23}^{+},
\end{cases}
$$

where $1 \le j \le 8$ ($j \ne 6$);

$$\widehat{R}L_1^+L_2^-(v_4\otimes v_j)=L_2^-L_1^+\widehat{R}(v_4\otimes v_j)\Rightarrow\begin{cases} l_{34}^+l_{11}^-=l_{11}^-l_{34}^+, & l_{34}^+l_{22}^-=l_{22}^-l_{34}^+,\\ l_{34}^+l_{33}^-=s^{-1}l_{33}^-l_{34}^+, & l_{34}^+l_{44}^-=r^{-1}l_{44}^-l_{34}^+,\\ l_{34}^+l_{21}^-=l_{21}^-l_{34}^+, & l_{34}^+l_{32}^-=s^{-1}l_{32}^-l_{34}^+,\\ rsl_{34}^+l_{43}^--l_{43}^-l_{34}^+=(s-r)(l_{44}^-l_{33}^+-l_{44}^+l_{33}^-), & \\ l_{34}^+l_{53}^-=l_{53}^-l_{34}^+, & \\ l_{44}^+l_{21}^-=l_{21}^-l_{44}^+, & l_{44}^+l_{32}^-=l_{32}^-l_{44}^+,\\ l_{44}^+l_{43}^-=s^{-1}l_{43}^-l_{44}^+, & l_{44}^+l_{53}^-=sl_{53}^-l_{44}^+, \end{cases}$$

where $1 \le j \le 8$ ($j \ne 5$);

$$\widehat{R}L_1^+L_2^-(v_5\otimes v_j)=L_2^-L_1^+\widehat{R}(v_5\otimes v_j)\Rightarrow\begin{cases} l_{35}^+l_{11}^-=rsl_{11}^-l_{35}^+, & l_{35}^+l_{22}^-=rsl_{22}^-l_{35}^+,\\ l_{35}^+l_{33}^-=rl_{33}^-l_{35}^+, & l_{35}^+l_{44}^-=r^{-1}l_{44}^-l_{35}^+,\\ l_{35}^+l_{21}^-=rsl_{21}^-l_{35}^+, & l_{35}^+l_{32}^-=rl_{32}^-l_{35}^+,\\ l_{35}^+l_{43}^-=l_{43}^-l_{35}^+, & \\ rsl_{35}^+l_{53}^--l_{53}^-l_{35}^+=(s-r)(l_{55}^-l_{33}^+-l_{55}^+l_{33}^-), & \end{cases}$$

where $1 \le j \le 8$. Now we proceed to the case of general n, restricting the generating relations (4-1) to $E_{ij} \otimes E_{kl}$, $2 \le i, j, k, l \le 2n-1$, by induction, we get all commutation relations except those between $l_{11}^{\pm}, l_{12}^{+}, l_{21}^{-}$ and $l_{ii}^{\pm}, l_{ij}^{\pm}$. Repeating similar computations as above, we have the following relations:

(B1) The $l_{11}^{\pm 1}, l_{ii}^{\pm 1}$ all commute with one another and $l_{11}^{\pm 1}(l_{11}^{\pm 1})^{-1}=l_{ii}^{\pm 1}(l_{ii}^{\pm 1})^{-1}=1$.

(B2) For $3 \le i \le n$, we have

$$\begin{aligned} &l_{ii}^+l_{12}^+=l_{12}^+l_{ii}^+, && l_{ii}^-l_{12}^+=l_{12}^+l_{ii}^-,\\ &l_{ii}^+l_{21}^-=l_{21}^-l_{ii}^+, && l_{ii}^-l_{21}^-=l_{21}^-l_{ii}^-,\\ &l_{22}^+l_{12}^+=s^{-1}l_{12}^+l_{22}^+, && l_{22}^+l_{21}^-=sl_{21}^-l_{22}^+,\\ &l_{22}^-l_{12}^+=r^{-1}l_{12}^+l_{22}^-, && l_{22}^-l_{21}^-=rl_{21}^-l_{22}^-. \end{aligned}$$

(B3) For $1 \le i \le n$, we have

$$\begin{aligned} &l_{11}^+l_{i,i+1}^+=l_{i,i+1}^+l_{11}^+, && l_{11}^+l_{n-1,n+1}^+=(rs)^{-1}l_{n-1,n+1}^+l_{11}^+,\\ &l_{11}^-l_{i+1,i}^-=l_{i+1,i}^-l_{11}^-, && l_{11}^-l_{n+1,n-1}^-=(rs)^{-1}l_{n+1,n-1}^-l_{11}^-,\\ &l_{11}^-l_{i,i+1}^+=l_{i,i+1}^-l_{11}^+, && l_{11}^-l_{n-1,n+1}^+=rsl_{n-1,n+1}^+l_{11}^-,\\ &l_{11}^+l_{i+1,i}^-=l_{i+1,i}^-l_{11}^+, && l_{11}^+l_{n+1,n-1}^-=rsl_{n+1,n-1}^-l_{11}^+. \end{aligned}$$

(B4)
$$
\begin{aligned}
(l_{12}^+)^2 l_{23}^+ + rs^{-1} l_{23}^+ (l_{12}^+)^2 &= (rs^{-1}+1) l_{12}^+ l_{23}^+ l_{12}^+, \\
(l_{23}^+)^2 l_{12}^+ + r^{-1} s l_{12}^+ (l_{23}^+)^2 &= (r^{-1}s+1) l_{23}^+ l_{12}^+ l_{23}^+, \\
(l_{21}^-)^2 l_{32}^- + rs^{-1} l_{32}^- (l_{21}^-)^2 &= (rs^{-1}+1) l_{21}^- l_{32}^- l_{21}^-, \\
(l_{32}^-)^2 l_{21}^- + r^{-1} s l_{21}^- (l_{32}^-)^2 &= (r^{-1}s+1) l_{32}^- l_{21}^- l_{32}^-.
\end{aligned}
$$

(B5) For $3 \le i \le n-1$, we have

$$
\begin{aligned}
l_{12}^+ l_{i,i+1}^+ &= l_{i,i+1}^+ l_{12}^+, & l_{12}^+ l_{n-1,n+1}^+ &= (rs)^{-1} l_{n-1,n+1}^+ l_{12}^+, \\
l_{21}^- l_{i,i+1}^+ &= l_{i,i+1}^+ l_{21}^-, & l_{21}^- l_{n-1,n+1}^+ &= (rs)^{-1} l_{n-1,n+1}^+ l_{21}^-, \\
l_{12}^+ l_{i+1,i}^- &= l_{i+1,i}^- l_{12}^+, & l_{12}^+ l_{n+1,n-1}^- &= rs\, l_{n+1,n-1}^- l_{12}^+, \\
l_{21}^- l_{i+1,i}^- &= l_{i+1,i}^- l_{21}^-, & l_{21}^- l_{n+1,n-1}^- &= (rs)^{-1} l_{n+1,n-1}^- l_{21}^-.
\end{aligned}
$$

We give explicit expressions of the L-functionals $l_{ij}^{\pm}$ in terms of the generators of $U_{r,s}(\mathfrak{so}_{2n})$. Define $\phi_n : U(\widehat{R}) \to U_{r,s}(\mathfrak{so}_{2n})$ as follows:

$$
\begin{aligned}
l_{ii}^+ &\mapsto (w'_{\beta_i})^{-1}, & l_{i,i+1}^+ &\mapsto (r-s) e_i l_{ii}^+, \\
l_{ii}^- &\mapsto (w_{\beta_i})^{-1}, & l_{i+1,i}^- &\mapsto -(r-s) l_{ii}^- f_i, \\
l_{nn}^+ &\mapsto (w'_{\beta_n})^{-1}, & l_{n-1,n+1}^+ &\mapsto (r-s) e_n l_{n-1,n-1}^+, \\
l_{nn}^- &\mapsto (w_{\beta_n})^{-1}, & l_{n+1,n-1}^- &\mapsto -(r-s) l_{n-1,n-1}^- f_n, \\
l_{i'i'}^+ &\mapsto w'_{\beta_i}, & l_{i'i'}^- &\mapsto w_{\beta_i},
\end{aligned}
$$

where $\beta_i = \alpha_i + \cdots + \alpha_{n-2} + \frac{1}{2}(\alpha_{n-1} + \alpha_n)$, $\beta_n = \frac{1}{2}(\alpha_n - \alpha_{n-1})$, $1 \le i \le n-1$. By induction, we can prove that ϕ_n still preserves the structure of algebra $U_{r,s}(\mathfrak{so}_{2n})$. □

Theorem 4.3. *$\phi_n : U(\widehat{R}) \to U_{r,s}(\mathfrak{so}_{2n})$ is an algebraic isomorphism.*

Proof. It is easy to check that the image of ϕ_n contains all generators of $U_{r,s}(\mathfrak{so}_{2n})$. Therefore, ϕ_n is surjective.

It remains to show that ϕ_n is injective. To this end, we need to construct an algebra homomorphism $\psi_n : U_{r,s}(\mathfrak{so}_{2n}) \to U(\widehat{R})$,

$$
\begin{aligned}
e_i &\mapsto \frac{1}{r-s} l_{i,i+1}^+ (l_{ii}^+)^{-1}, & f_i &\mapsto \frac{1}{s-r} (l_{ii}^-)^{-1} l_{i+1,i}^-, \\
w'_i &\mapsto (l_{ii}^+)^{-1} l_{i+1,i+1}^+, & w_i &\mapsto (l_{ii}^-)^{-1} l_{i+1,i+1}^-, \\
e_n &\mapsto \frac{1}{r-s} l_{n-1,n+1}^+ (l_{n-1,n-1}^+)^{-1}, & f_n &\mapsto \frac{1}{s-r} (l_{n-1,n-1}^-)^{-1} l_{n+1,n-1}^-, \\
w'_n &\mapsto (l_{nn}^+)^{-1} l_{n-1,n-1}^+, & w_n &\mapsto (l_{nn}^-)^{-1} l_{n-1,n-1}^-,
\end{aligned}
$$

which satisfies $\psi_n \circ \phi_n = \mathrm{id}$.

To prove that ψ_n still preserves the algebra structure of $U(\widehat{R})$ is completely similar to that of Theorem 4.2. Hence, ϕ_n is injective. (For a similar proof in the one-parameter setting, one can refer to Section 8.5 of [17]). □

Proposition 4.4. *For the braiding R-matrix $R = R_{VV}$, the spectral parameter dependent $R(z)$ is given by*

$$\begin{aligned}
R(z) = &\sum_{i=1}^{2n} E_{ii}\otimes E_{ii} \\
&+\frac{rs(z-1)}{rz-s}\Bigg\{\sum_{\substack{1\le i\le n-1\\ i+1\le j\le n}} E_{ij}\otimes E_{ji}+\sum_{\substack{1\le i\le n-1\\ i'+1\le j\le 2n}} E_{ij}\otimes E_{ji}+\sum_{j=n+2}^{2n} E_{nj}\otimes E_{jn} \\
&\qquad\qquad +\sum_{\substack{n+1\le i\le 2n-1\\ i+1\le j\le 2n}} E_{ji}\otimes E_{ij}+\sum_{\substack{1\le i\le n-1\\ n+1\le j\le 2n-i}} E_{ji}\otimes E_{ij}\Bigg\} \\
&+\frac{z-1}{rz-s}\Bigg\{\sum_{\substack{1\le i\le n-1\\ i+1\le j\le n}} E_{ji}\otimes E_{ij}+\sum_{\substack{1\le i\le n-1\\ i'+1\le j\le 2n}} E_{ji}\otimes E_{ij}+\sum_{j=n+2}^{2n} E_{jn}\otimes E_{nj} \\
&\qquad\qquad +\sum_{\substack{n+1\le i\le 2n-1\\ i+1\le j\le 2n}} E_{ij}\otimes E_{ji}+\sum_{\substack{1\le i\le n-1\\ n+1\le j\le 2n-i}} E_{ij}\otimes E_{ji}\Bigg\} \\
&+\frac{r-s}{rz-s}\Bigg\{z\sum_{\substack{i<j\\ i'\neq j}} E_{jj}\otimes E_{ii}+\sum_{\substack{i>j\\ i\neq j'}} E_{jj}\otimes E_{ii}\Bigg\} \\
&+\frac{1}{(z-r^{1-n}s^{n-1})(rz-s)}\sum_{i,j=1}^{2n} d_{ij}(z)E_{ij'}\otimes E_{i'j},
\end{aligned}$$

where $d_{ij}(z)=\begin{cases}(s-r)z\{(r^{-\frac{1}{2}}s^{\frac{1}{2}})^{\rho_j-\rho_i}(z-1)-\delta_{ij'}[z-(rs^{-1})^{1-n}]\}, & \text{if } i>j,\\ (s-r)\{(r^{-\frac{1}{2}}s^{\frac{1}{2}})^{\rho_j-\rho_i+2n-2}(z-1)-\delta_{ij'}[z-(rs^{-1})^{1-n}]\}, & \text{if } i<j,\\ s[z-(rs^{-1})^{2-n}](z-1), & \text{if } i=j.\end{cases}$

Remark 4.5. Consider the $\widehat{R}$-matrix $\widehat{R}(z) = P\circ R(z)$, where P is defined as in Remark 3.9:

$$\begin{aligned}
\widehat{R}(z) = &\sum_{i=1}^{2n} E_{ii}\otimes E_{ii} \\
&+\frac{rs(z-1)}{rz-s}\Bigg\{\sum_{\substack{1\le i\le n-1\\ i+1\le j\le n}} E_{jj}\otimes E_{ii}+\sum_{\substack{1\le i\le n-1\\ i'+1\le j\le 2n}} E_{jj}\otimes E_{ii}+\sum_{j=n+2}^{2n} E_{jj}\otimes E_{nn} \\
&\qquad\qquad +\sum_{\substack{n+1\le i\le 2n-1\\ i+1\le j\le 2n}} E_{ii}\otimes E_{jj}+\sum_{\substack{1\le i\le n-1\\ n+1\le j\le 2n-i}} E_{ii}\otimes E_{jj}\Bigg\}
\end{aligned}$$

$$+\frac{z-1}{rz-s}\Bigg\{\sum_{\substack{1\le i\le n-1\\ i+1\le j\le n}} E_{ii}\otimes E_{jj}+\sum_{\substack{1\le i\le n-1\\ i'+1\le j\le 2n}} E_{ii}\otimes E_{jj}+\sum_{j=n+2}^{2n} E_{nn}\otimes E_{jj}$$
$$+\sum_{\substack{n+1\le i\le 2n-1\\ i+1\le j\le 2n}} E_{jj}\otimes E_{ii}+\sum_{\substack{1\le i\le n-1\\ n+1\le j\le 2n-i}} E_{jj}\otimes E_{ii}\Bigg\}$$
$$+\frac{r-s}{rz-s}\Bigg\{z\sum_{\substack{i<j\\ i'\neq j}} E_{ij}\otimes E_{ji}+\sum_{\substack{i>j\\ i\neq j'}} E_{ij}\otimes E_{ji}\Bigg\}+\sum_{i,j=1}^{2n} c_{ij}(z)E_{i'j'}\otimes E_{ij},$$

where

$$c_{ij}(z)=\frac{d_{ij}(z)}{(z-r^{1-n}s^{n-1})(rz-s)}.$$

It is easy to check that $\widehat{R}(z)$ satisfies the quantum Yang–Baxter equation

$$\widehat{R}_{12}(z)\widehat{R}_{13}(zw)\widehat{R}_{23}(w)=\widehat{R}_{23}(w)\widehat{R}_{13}(zw)\widehat{R}_{12}(z),$$

and the unitary condition

$$\widehat{R}_{21}(z)\widehat{R}(z^{-1})=\widehat{R}(z^{-1})\widehat{R}_{21}(z)=1. \tag{4-13}$$

5. The algebra $\mathcal{U}(\widehat{R})$ and its Gauss decomposition

Definition 5.1. The algebra $\mathcal{U}(\widehat{R})$ is an associative algebra with generators $l_{kl}^{\pm}[\mp m]$ ($m\in\mathbb{Z}_+\setminus\{0\}$), and $l_{kl}^{+}[0]$= $l_{lk}^{-}[0]$=0, $1\le l\le k\le n$ and the central element c via $r^{\frac{c}{2}}$ or $s^{\frac{c}{2}}$. Let $l_{ij}^{\pm}(z)=\sum_{m=0}^{\infty} l_{ij}^{\pm}[\mp m]z^{\pm m}$, and $L^{\pm}(z)=\sum_{i,j=1}^{n} E_{ij}\otimes l_{ij}^{\pm}(z)$. Then the relations are given by the following matrix equations on $\mathrm{End}(V^{\otimes 2})\otimes\mathcal{U}(\widehat{R})$:

(5-1) $l_{ii}^{+}[0],\ l_{ii}^{-}[0]$ are invertible and $l_{ii}^{+}[0]\,l_{ii}^{-}[0]=l_{ii}^{-}[0]\,l_{ii}^{+}[0]$,

$$\widehat{R}\Big(\frac{z}{w}\Big)L_1^{\pm}(z)L_2^{\pm}(w)=L_2^{\pm}(w)L_1^{\pm}(z)\widehat{R}\Big(\frac{z}{w}\Big), \tag{5-2}$$

$$\widehat{R}\Big(\frac{z_+}{w_-}\Big)L_1^{+}(z)L_2^{-}(w)=L_2^{-}(w)L_1^{+}(z)\widehat{R}\Big(\frac{z_-}{w_+}\Big), \tag{5-3}$$

where $z_+=zr^{\frac{c}{2}}$ and $z_-=zs^{\frac{c}{2}}$. Here (5-2) is expanded in the direction of either $\frac{z}{w}$ or $\frac{w}{z}$, and (5-3) is expanded in the direction of $\frac{z}{w}$.

Remark 5.2. From (5-3) and the unitary condition of $\widehat{R}$-matrix (4-13), we have

$$\widehat{R}\Big(\frac{z_\pm}{w_\mp}\Big)L_1^{\pm}(z)L_2^{\mp}(w)=L_2^{\mp}(w)L_1^{\pm}(z)\widehat{R}\Big(\frac{z_\mp}{w_\pm}\Big). \tag{5-4}$$

So the relations of generating series (5-2), (5-3) are equivalent to

$$L_1^{\pm}(z)^{-1}L_2^{\pm}(w)^{-1}\widehat{R}\Big(\frac{z}{w}\Big)=\widehat{R}\Big(\frac{z}{w}\Big)L_2^{\pm}(w)^{-1}L_1^{\pm}(z)^{-1}, \tag{5-5}$$

$$L_1^{\pm}(z)^{-1}L_2^{\mp}(w)^{-1}\widehat{R}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)=\widehat{R}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)L_2^{\mp}(w)^{-1}L_1^{\pm}(z)^{-1}. \tag{5-6}$$

They are also equivalent to

$$L_2^{\pm}(w)^{-1}\widehat{R}\Big(\frac{z}{w}\Big)L_1^{\pm}(z)=L_1^{\pm}(z)\widehat{R}\Big(\frac{z}{w}\Big)L_2^{\pm}(w)^{-1}, \tag{5-7}$$

$$L_2^{\mp}(w)^{-1}\widehat{R}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)L_1^{\pm}(z)=L_1^{\pm}(z)\widehat{R}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)L_2^{\mp}(w)^{-1}. \tag{5-8}$$

Remark 5.3. Here we present the specific matrix expression formulas for (5-2) and (5-3), and reveal the differences between type $D_n^{(1)}$ and type $A_n^{(1)}$. For $D_n^{(1)}$, write

$$L^{\pm}(z)=\begin{pmatrix} l_{11}^{\pm}(z) & l_{12}^{\pm}(z) & \cdots & l_{1,2n}^{\pm}(z) \\ l_{21}^{\pm}(z) & l_{22}^{\pm}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & l_{2n-1,2n}^{\pm}(z) \\ l_{2n,1}^{\pm}(z) & \cdots & l_{2n,2n-1}^{\pm}(z) & l_{2n,2n}^{\pm}(z) \end{pmatrix}_{2n\times 2n},$$

then for the generators $L_1^{\pm}(z)$, $L_2^{\pm}(z)$, $\widehat{R}(z)$, we have that

$$L_1^{\pm}(z)=\begin{pmatrix} l_{11}^{\pm}(z)I_{2n} & \cdots & l_{1,2n}^{\pm}(z)I_{2n} \\ \vdots & \ddots & \vdots \\ l_{2n,1}^{\pm}(z)I_{2n} & \cdots & l_{2n,2n}^{\pm}(z)I_{2n} \end{pmatrix}_{4n^2\times 4n^2},$$

$$L_2^{\pm}(z)=\begin{pmatrix} L^{\pm}(z) & 0 & \cdots & 0 \\ 0 & L^{\pm}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L^{\pm}(z) \end{pmatrix}_{4n^2\times 4n^2},$$

$$\widehat{R}(z)=\begin{pmatrix} B_{11}(z) & \cdots & B_{1,2n}(z) \\ \vdots & \ddots & \vdots \\ B_{2n,1}(z) & \cdots & B_{2n,2n}(z) \end{pmatrix}_{4n^2\times 4n^2},$$

$$B_{ll}(z)=\begin{pmatrix} a_{l1}(z) & 0 & \cdots & 0 \\ 0 & a_{l2}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{l,2n}(z) \end{pmatrix}_{2n\times 2n},$$

where $B_{ll}(z)$ is a diagonal matrix, and a_{lj} is the coefficient of element $E_{ll}\otimes E_{jj}$ in $\widehat{R}(z)$, and $B_{ij}(z)=b_{ij}(z)E_{ji}+c_{i'j'}(z)E_{i'j'}$, where $b_{ij}(z)$ is the coefficient of element $E_{ij}\otimes E_{ji}$ in $\widehat{R}(z)$, and $c_{ij}(z)$ is the coefficient of element $E_{i'j'}\otimes E_{ij}$ in $\widehat{R}(z)$.

The multiplication between matrices $\widehat{R}\big(\frac{z}{w}\big)$, $L_1^{\pm}(z)$, $L_2^{\pm}(w)$ is matrix multiplication. From

$$\widehat{R}\Big(\frac{z}{w}\Big)L_1^{\pm}(z)L_2^{\pm}(w) = L_2^{\pm}(w)L_1^{\pm}(z)\widehat{R}\Big(\frac{z}{w}\Big),$$

we can derive the following calculation:

$$\widehat{R}\Big(\frac{z}{w}\Big)L_1^{\pm}(z)L_2^{\pm}(w) = \begin{pmatrix} M_{11} & \cdots & M_{1,2n} \\ \vdots & \ddots & \vdots \\ M_{2n,1} & \cdots & M_{2n,2n} \end{pmatrix}_{4n^2\times 4n^2}, \quad M_{ij} \in M(2n,\mathbb{K}),$$

$$L_2^{\pm}(w)L_1^{\pm}(z)\widehat{R}\Big(\frac{z}{w}\Big) = \begin{pmatrix} M'_{11} & \cdots & M'_{1,2n} \\ \vdots & \ddots & \vdots \\ M'_{2n,1} & \cdots & M'_{2n,2n} \end{pmatrix}_{4n^2\times 4n^2}, \quad M'_{ij} \in M(2n,\mathbb{K}).$$

We only give two types of matrix expressions that will be used later. Taking $M_{ij} = M'_{ij}$, where $1 \le i, j \le n$, consider M_{ij}:

$$\text{(5-9)} \quad \begin{pmatrix} a_{i1}\big(\frac{z}{w}\big)l_{ij}^{\pm}(z) & & b_{i1}\big(\frac{z}{w}\big)l_{1j}^{\pm}(z) & & & \\ & \ddots & \vdots & & & \\ & & a_{ii}\big(\frac{z}{w}\big)l_{ij}^{\pm}(z) & & & \\ & & \vdots & \ddots & & \\ c_{i'1}\big(\frac{z}{w}\big)l_{1'j}^{\pm}(z) & \cdots & c_{i'i}\big(\frac{z}{w}\big)l_{i'j}^{\pm}(z) & \cdots & c_{i'i'}\big(\frac{z}{w}\big)l_{ij}^{\pm}(z) & \cdots & c_{i'1'}\big(\frac{z}{w}\big)l_{1j}^{\pm}(z) \\ & & \vdots & & & \ddots & \\ & & b_{i1'}\big(\frac{z}{w}\big)l_{1'j}^{\pm}(z) & & & & a_{i1'}\big(\frac{z}{w}\big)l_{ij}^{\pm}(z) \end{pmatrix} L^{\pm}(w),$$

and $1 \le i \le n$, $1+n \le j$, M_{ij}:

$$\text{(5-10)} \quad \begin{pmatrix} a_{i1}\big(\frac{z}{w}\big)l_{ij}^{\pm}(z) & & & & b_{i1}\big(\frac{z}{w}\big)l_{1j}^{\pm}(z) & & \\ & \ddots & & & \vdots & & \\ c_{i'1}\big(\frac{z}{w}\big)l_{1'j}^{\pm}(z) & \cdots & c_{i'i}\big(\frac{z}{w}\big)l_{ij}^{\pm}(z) & \cdots & c_{i'i'}\big(\frac{z}{w}\big)l_{i'j}^{\pm}(z) & \cdots & c_{i'1'}\big(\frac{z}{w}\big)l_{1j}^{\pm}(z) \\ & & & \ddots & \vdots & & \\ & & & & a_{ii}\big(\frac{z}{w}\big)l_{ij}^{\pm}(z) & & \\ & & & & \vdots & \ddots & \\ & & & & b_{i1'}\big(\frac{z}{w}\big)l_{1'j}^{\pm}(z) & & a_{i1'}\big(\frac{z}{w}\big)l_{ij}^{\pm}(z) \end{pmatrix} L^{\pm}(w),$$

where the elements in the i'-th row except for the element at position (i', i') are all zero for type $A_n^{(1)}$. Consider M'_{ij}, for $1 \le i, j \le n$:

$$L^{\pm}(w)\begin{pmatrix} a_{j1}\left(\frac{z}{w}\right)l_{ij}^{\pm}(z) & & & & c_{1j'}\left(\frac{z}{w}\right)l_{i1'}^{\pm}(z) & & \\ & \ddots & & & \vdots & & \\ b_{1j}\left(\frac{z}{w}\right)l_{i1}^{\pm}(z) & \cdots & a_{jj}\left(\frac{z}{w}\right)l_{ij}^{\pm}(z) & \cdots & c_{jj'}\left(\frac{z}{w}\right)l_{ij'}^{\pm}(z) & \cdots & b_{1'j}\left(\frac{z}{w}\right)l_{i1'}^{\pm}(z) \\ & & & \ddots & \vdots & & \\ & & & & c_{j'j'}\left(\frac{z}{w}\right)l_{ij}^{\pm}(z) & & \\ & & & & \vdots & \ddots & \\ & & & & c_{1'j'}\left(\frac{z}{w}\right)l_{i1}^{\pm}(z) & & a_{j1'}\left(\frac{z}{w}\right)l_{ij}^{\pm}(z) \end{pmatrix}, \tag{5-11}$$

moreover, $1 \le i \le n$, $1 + n \le j$:

$$L^{\pm}(w)\begin{pmatrix} a_{j1}\left(\frac{z}{w}\right)l_{ij}^{\pm}(z) & & c_{1j'}\left(\frac{z}{w}\right)l_{i1'}^{\pm}(z) & & & \\ & \ddots & \vdots & & & \\ & & c_{jj'}\left(\frac{z}{w}\right)l_{ij}^{\pm}(z) & & & \\ & & \vdots & \ddots & & \\ b_{1j}\left(\frac{z}{w}\right)l_{i1}^{\pm}(z) & \cdots & c_{j'j'}\left(\frac{z}{w}\right)l_{ij'}^{\pm}(z) & \cdots & a_{jj}\left(\frac{z}{w}\right)l_{ij}^{\pm}(z) & \cdots & b_{1'j}\left(\frac{z}{w}\right)l_{i1'}^{\pm}(z) \\ & & \vdots & & & \ddots & \\ & & c_{1'j'}\left(\frac{z}{w}\right)l_{i1}^{\pm}(z) & & & & a_{j1'}\left(\frac{z}{w}\right)l_{ij}^{\pm}(z) \end{pmatrix}, \tag{5-12}$$

where the elements in the j'-th column except for the element at position (j', j') are all zero for type $A_n^{(1)}$.

Definition 5.4. Let $X = (x_{ij})_{i,j=1}^{n}$ be a sequence matrix over a ring with identity. Denote by X^{ij} the submatrix obtained from X by deleting the i-th row and j-th column. Suppose that the matrix X^{ij} is invertible. The (i, j)-th quasideterminant $|X|_{ij}$ of X is defined by

$$|X|_{ij} = \begin{vmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{i1} & \cdots & \boxed{x_{ij}} & \cdots & x_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nn} \end{vmatrix} = x_{ij} - r_i^j (X^{ij})^{-1} c_j^i, \tag{5-13}$$

where r_i^j is the row matrix obtained from the i-th row of X by deleting the element x_{ij}, and c_j^i is the column matrix obtained from the j-th column of X by deleting the element x_{ij}.

Proposition 5.5. *$L^{\pm}(z)$ have the following unique decomposition*

$$L^{\pm}(z) = F^{\pm}(z)K^{\pm}(z)E^{\pm}(z), \tag{5-14}$$

by applying the Gauss decomposition to $L^{\pm}(z)$, *where we introduce matrices with* $N \times N$, *and* $N = 2n$,

$$F^{\pm}(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ f_{21}^{\pm}(z) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{N1}^{\pm}(z) & \cdots & f_{N,N-1}^{\pm}(z) & 1 \end{pmatrix}, \tag{5-15}$$

$$E^{\pm}(z) = \begin{pmatrix} 1 & e_{12}^{\pm}(z) & \cdots & e_{1N}^{\pm}(z) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & e_{N-1,N}^{\pm}(z) \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{5-16}$$

$$K^{\pm}(z) = \begin{pmatrix} k_1^{\pm}(z) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & k_N^{\pm}(z) \end{pmatrix}. \tag{5-17}$$

Their entries are found by the quasideterminant formulas

$$k_m^{\pm}(z) = \begin{vmatrix} l_{11}^{\pm}(z) & \cdots & l_{1,m-1}^{\pm}(z) & l_{1m}^{\pm}(z) \\ \vdots & \ddots & \vdots & \vdots \\ l_{m1}^{\pm}(z) & \cdots & l_{m,m-1}^{\pm}(z) & \boxed{l_{mm}^{\pm}(z)} \end{vmatrix} \tag{5-18}$$

for $1 \le m \le 2n$, $k_m^{\pm}(z) = \sum_{t \in \mathbb{Z}_+} k_m^{\pm}(\mp t) z^{\pm t}$;

$$e_{ij}^{\pm}(z) = k_i^{\pm}(z)^{-1} \begin{vmatrix} l_{11}^{\pm}(z) & \cdots & l_{1,i-1}^{\pm}(z) & l_{1j}^{\pm}(z) \\ \vdots & \ddots & \vdots & \vdots \\ l_{i1}^{\pm}(z) & \cdots & l_{i,i-1}^{\pm}(z) & \boxed{l_{ij}^{\pm}(z)} \end{vmatrix} \tag{5-19}$$

for $1 \le i < j \le 2n$; $e_{ij}^{\pm}(z) = \sum_{m \in \mathbb{Z}_+} e_{ij}^{\pm}(\mp m) z^{\pm m}$;

$$f_{ji}^{\pm}(z) = \begin{vmatrix} l_{11}^{\pm}(z) & \cdots & l_{1,i-1}^{\pm}(z) & l_{1i}^{\pm}(z) \\ \vdots & \ddots & \vdots & \vdots \\ l_{j1}^{\pm}(z) & \cdots & l_{j,i-1}^{\pm}(z) & \boxed{l_{ji}^{\pm}(z)} \end{vmatrix} k_i^{\pm}(z)^{-1} \tag{5-20}$$

for $1 \le i < j \le 2n$, $f_{ji}^{\pm}(z) = \sum_{m \in \mathbb{Z}_+} f_{ji}^{\pm}(\mp m) z^{\pm m}$.

6. Drinfeld realization of $U_{r,s}(\widehat{\mathfrak{so}_{2n}})$

In this section, we will study the commutation relations between the Gaussian generators and give the Drinfeld realization of $U_{r,s}(\widehat{\mathfrak{so}_{2n}})$.

Theorem 6.1. *In the algebra $\mathcal{U}(\widehat{R})$, we have*

$$X_i^+(z) = e_{i,i+1}^+(z_+) - e_{i,i+1}^-(z_-), \qquad X_n^+(z) = e_{n-1,n+1}^+(z_+) - e_{n-1,n+1}^-(z_-),$$
$$X_i^-(z) = f_{i+1,i}^+(z_-) - f_{i+1,i}^-(z_+), \qquad X_n^-(z) = f_{n+1,n-1}^+(z_-) - f_{n+1,n-1}^-(z_+).$$

For the generators $\{k_i^\pm(z), X_j^\pm(z), k_{n+1}^\pm(z), X_n^\pm(z) \mid 1 \le i \le n, 1 \le j \le n-1\}$, the relations between $k_i^\pm(z)$ and $X_j^\pm(z)$ are the same as those in $U_{r,s}(\widehat{\mathfrak{gl}_n})$. The other relations are those involving $k_i^\pm(z)$ and $k_{n+1}^\pm(w)$, that is,

$$k_i^\pm(z)k_{n+1}^\pm(w) = k_{n+1}^\pm(w)k_i^\pm(z),$$
$$k_{n+1}^+(z)k_{n+1}^-(w) = k_{n+1}^-(w)k_{n+1}^+(z),$$
$$k_i^\mp(z)k_{n+1}^\pm(w)\frac{z_\mp - w_\pm}{rz_\mp - sw_\pm} = \frac{z_\pm - w_\mp}{rz_\pm - sw_\mp}k_{n+1}^\pm(w)k_i^\mp(z).$$

The relations involving $k_t^\pm(w)$ $(1 \le t \le n+1)$ and $X_n^\pm(z)$ are

$$k_l^\pm(w)X_n^\pm(z) = rsX_n^\pm(z)k_l^\pm(w),$$
$$rsk_l^\pm(w)X_n^\mp(z) = X_n^\mp(z)k_l^\pm(w), \quad 1 \le l \le n-2,$$
$$(rw - sz_\pm)k_{n-1}^\pm(w)X_n^+(z) = rs(w - z_\pm)X_n^+(z)k_{n-1}^\pm(w),$$
$$rs(w - z_\mp)k_{n-1}^\pm(w)X_n^-(z) = (rw - sz_\mp)X_n^-(z)k_{n-1}^\pm(w),$$
$$k_n^\pm(w)X_n^+(z) = \frac{wr - sz_\pm}{w - z_\pm}X_n^+(z)k_n^\pm(w),$$
$$X_n^-(z)k_n^\pm(w) = \frac{wr - sz_\mp}{w - z_\mp}k_n^\pm(w)X_n^-(z),$$
$$k_{n+1}^\pm(w)X_n^+(z) = \frac{rs(w - z_\pm)}{sw - rz_\pm}X_n^+(z)k_{n+1}^\pm(w),$$
$$X_n^-(z)k_{n+1}^\pm(w) = \frac{rs(w - z_\mp)}{sw - rz_\mp}k_{n+1}^\pm(w)X_n^-(z).$$

The relations involving $k_{n+1}^\pm(z)$ and $X_t^\pm(w)$ $(1 \le t \le n-1)$ are

$$k_{n+1}^\pm(z)X_l^\mp(w) = X_l^\mp(w)k_{n+1}^\pm(z),$$
$$k_{n+1}^\pm(z)X_l^\pm(w) = X_l^\pm(w)k_{n+1}^\pm(z), \quad 1 \le l \le n-2,$$
$$k_{n+1}^\pm(w)X_{n-1}^+(z) = \frac{rs(z_\pm - w)}{z_\pm s - rw}X_{n-1}^+(z)k_{n+1}^\pm(w),$$
$$X_{n-1}^-(z)k_{n+1}^\pm(w) = \frac{rs(z_\mp - w)}{z_\mp s - rw}k_{n+1}^\pm(w)X_{n-1}^-(z).$$

For the relations involving $X_n^{\pm}(z)$ and $X_t^{\pm}(z)$ $(1 \le t \le n)$, we have

$$
\begin{aligned}
X_n^{\pm}(w)X_l^{\pm}(z) &= X_l^{\pm}(z)X_n^{\pm}(w),\\
X_n^{\pm}(w)X_l^{\mp}(z) &= X_l^{\mp}(z)X_n^{\pm}(w), \quad 1 \le l \le n-3,\\
X_n^{+}(w)X_{n-2}^{+}(z) &= \frac{rz-sw}{z-w}X_{n-2}^{+}(z)X_n^{+}(w),\\
X_n^{-}(w)X_{n-2}^{-}(z) &= \frac{z-w}{rz-sw}X_{n-2}^{-}(z)X_n^{-}(w),\\
X_n^{\pm}(w)X_{n-2}^{\mp}(z) &= X_{n-2}^{\mp}(z)X_n^{\pm}(w),\\
X_n^{\pm}(w)X_{n-1}^{\pm}(z) &= (rs)^{\pm 1}X_{n-1}^{\pm}(z)X_n^{\pm}(w),\\
X_{n-1}^{\pm}(w)X_n^{\mp}(z) &= X_n^{\mp}(z)X_{n-1}^{\pm}(w),\\
X_n^{+}(z)X_n^{+}(w) &= \frac{zr-ws}{zs-rw}X_n^{+}(w)X_n^{+}(z),\\
X_n^{-}(z)X_n^{-}(w) &= \frac{zs-wr}{zr-sw}X_n^{-}(w)X_n^{-}(z),\\
[\, X_n^{+}(z),\, X_t^{-}(w)\,] &= (s^{-1}-r^{-1})\delta_{nt}\Big\{\delta\Big(\frac{z_-}{w_+}\Big)k_{n+1}^{-}(w_+)k_n^{-}(w_+)^{-1}\\
&\qquad - \delta\Big(\frac{z_+}{w_-}\Big)k_{n+1}^{+}(z_+)k_n^{+}(z_+)^{-1}\Big\},
\end{aligned}
$$

and the following (r, s)-Serre relations hold in $\mathcal{U}(\widehat{R})$:

$$
\begin{aligned}
\text{(6-1)}\quad &\big\{X_{n-2}^{-}(z_1)X_{n-2}^{-}(z_2)X_n^{-}(w) - (r+s)X_{n-2}^{-}(z_1)X_n^{-}(w)X_{n-2}^{-}(z_2)\\
&\qquad + rsX_n^{-}(w)X_{n-2}^{-}(z_1)X_{n-2}^{-}(z_2)\big\} + \{z_1 \leftrightarrow z_2\} = 0,\\
\text{(6-2)}\quad &\big\{X_n^{+}(z_1)X_n^{+}(z_2)X_{n-2}^{+}(w) - (r+s)X_n^{+}(z_1)X_{n-2}^{+}(w)X_n^{+}(z_2)\\
&\qquad + rsX_{n-2}^{+}(w)X_n^{+}(z_1)X_n^{+}(z_2)\big\} + \{z_1 \leftrightarrow z_2\} = 0,\\
\text{(6-3)}\quad &\big\{rsX_{n-2}^{+}(z_1)X_{n-2}^{+}(z_2)X_n^{+}(w) - (r+s)X_{n-2}^{+}(z_1)X_n^{+}(w)X_{n-2}^{+}(z_2)\\
&\qquad + X_n^{+}(w)X_{n-2}^{+}(z_1)X_{n-2}^{+}(z_2)\big\} + \{z_1 \leftrightarrow z_2\} = 0,\\
\text{(6-4)}\quad &\big\{rsX_n^{-}(z_1)X_n^{-}(z_2)X_{n-2}^{-}(w) - (r+s)X_n^{-}(z_1)X_{n-2}^{-}(w)X_n^{-}(z_2)\\
&\qquad + X_{n-2}^{-}(w)X_n^{-}(z_1)X_n^{-}(z_2)\big\} + \{z_1 \leftrightarrow z_2\} = 0,
\end{aligned}
$$

where the formal delta function $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$.

Proof. The proof is based on the induction on n. We consider first the case $n = 4$,

$$
L^{\pm}(z) = \begin{pmatrix}
l_{11}^{\pm}(z) & l_{12}^{\pm}(z) & \cdots & l_{18}^{\pm}(z)\\
l_{21}^{\pm}(z) & l_{22}^{\pm}(z) & \ddots & \vdots\\
\vdots & \ddots & \ddots & l_{78}^{\pm}(z)\\
l_{81}^{\pm}(z) & \cdots & l_{87}^{\pm}(z) & l_{88}^{\pm}(z)
\end{pmatrix}_{8\times 8}.
$$

Observe (5-9) and (5-11) and restrict them to $E_{ij}\otimes E_{kl}$, $1\le i,j,k,l\le 4$, then

$$\widehat{R}_1\Big(\frac{z}{w}\Big)\widetilde{L}_1^{\pm}(z)\widetilde{L}_2^{\pm}(w)=\widetilde{L}_2^{\pm}(w)\widetilde{L}_1^{\pm}(z)\widehat{R}_1\Big(\frac{z}{w}\Big),$$
$$\widehat{R}_1\Big(\frac{z_+}{w_-}\Big)\widetilde{L}_1^{\pm}(z)\widetilde{L}_2^{\pm}(w)=\widetilde{L}_2^{\pm}(w)\widetilde{L}_1^{\pm}(z)\widehat{R}_1\Big(\frac{z_-}{w_+}\Big),$$
$$\widetilde{L}^{\pm}(z)=\begin{pmatrix} l_{11}^{\pm}(z) & l_{12}^{\pm}(z) & l_{13}^{\pm}(z) & l_{14}^{\pm}(z)\\ l_{21}^{\pm}(z) & l_{22}^{\pm}(z) & l_{23}^{\pm}(z) & l_{24}^{\pm}(z)\\ l_{31}^{\pm}(z) & l_{32}^{\pm}(z) & l_{33}^{\pm}(z) & l_{34}^{\pm}(z)\\ l_{41}^{\pm}(z) & l_{42}^{\pm}(z) & l_{43}^{\pm}(z) & l_{44}^{\pm}(z)\end{pmatrix}_{4\times 4},$$

and

$$\widehat{R}_1\Big(\frac{z}{w}\Big)=\sum_{i=1}^{4}E_{ii}\otimes E_{ii}+\frac{s(w-z)}{sw-zr}\Big(\sum_{i>j}E_{ii}\otimes E_{jj}+s^{-1}\sum_{i<j}E_{ii}\otimes E_{jj}\Big)$$
$$+\frac{(s-r)w}{sw-zr}\Big(\sum_{i>j}E_{ij}\otimes E_{ji}+\frac{z}{w}\sum_{i<j}E_{ij}\otimes E_{ji}\Big).$$

Jing and Liu [16] gave the following spectral parameter dependent $\widehat{R}_A\big(\frac{z}{w}\big)$ of $U_{r,s}(\widehat{\mathfrak{gl}}_n)$, in particular, setting $n=4$,

$$\widehat{R}_A\Big(\frac{z}{w}\Big)=\sum_{i=1}^{4}E_{ii}\otimes E_{ii}+\frac{w-z}{w-zrs^{-1}}\Big(\sum_{i>j}E_{ii}\otimes E_{jj}+s^{-1}\sum_{i<j}E_{ii}\otimes E_{jj}\Big)$$
$$+\frac{(1-rs^{-1})w}{w-zrs^{-1}}\Big(\sum_{i>j}E_{ij}\otimes E_{ji}+\frac{z}{w}\sum_{i<j}E_{ij}\otimes E_{ji}\Big),$$

we get $\widehat{R}_A\big(\frac{z}{w}\big)=\widehat{R}_1\big(\frac{z}{w}\big)$. Thereby, we can directly have the relations of generators $\{X_1^{\pm}(z),X_2^{\pm}(z),X_3^{\pm}(z),k_1^{\pm}(z),k_2^{\pm}(z),k_3^{\pm}(z),k_4^{\pm}(z)\}$.

Next, we need to obtain the relations between the remaining Gauss generators. First, using the Gauss decomposition, we write down $L^{\pm}(z)$ and $L^{\pm}(z)^{-1}$:

$$L^{\pm}(z)=\begin{pmatrix} k_1^{\pm}(z) & k_1^{\pm}(z)e_{12}^{\pm}(z) & \cdots\\ f_{21}^{\pm}(z)k_1^{\pm}(z) & \vdots & \vdots\\ \vdots & \vdots & \vdots\end{pmatrix},$$
$$L^{\pm}(z)^{-1}=\begin{pmatrix} \cdots & \cdots & \cdots\\ \cdots & \cdots & -e_{N-1,N}^{\pm}(z)k_N^{\pm}(z)^{-1}\\ \cdots & -k_N^{\pm}(z)^{-1}f_{N,N-1}^{\pm}(z) & k_N^{\pm}(z)^{-1}\end{pmatrix}.$$

Then using the generating series relations (5-2) and (5-4), we can complete our proof by using the following lemmas. □

Lemma 6.2. *The following equations hold in* $\mathcal{U}(\widehat{R})$:

$$k_5^{\pm}(w)X_i^{\pm}(z) = X_i^{\pm}(z)k_5^{\pm}(w), \tag{6-5}$$

$$k_5^{\pm}(w)X_i^{\mp}(z) = X_i^{\mp}(z)k_5^{\pm}(w), \tag{6-6}$$

$$k_j^{\pm}(z)k_5^{\pm}(w) = k_5^{\pm}(w)k_j^{\pm}(z), \tag{6-7}$$

$$k_j^{\pm}(z)k_5^{\mp}(w)\frac{z_{\pm}-w_{\mp}}{rz_{\pm}-sw_{\mp}} = \frac{z_{\mp}-w_{\pm}}{rz_{\mp}-sw_{\pm}}k_5^{\mp}(w)k_j^{\pm}(z), \tag{6-8}$$

where $1 \le i \le 2$ *and* $1 \le j \le 3$.

Proof. Due to the observation made in formulas (5-9) and (5-11), the relations between the Gaussian generators mentioned above follow from those for the quantum affine algebra $U_{r,s}(\widehat{\mathfrak{gl}_5})$; see [16] □

Lemma 6.3.

$$k_1^{\pm}(w)X_4^{+}(z) = rsX_4^{+}(z)k_1^{\pm}(w), \tag{6-9}$$

$$rsk_1^{\pm}(w)X_4^{-}(z) = X_4^{-}(z)k_1^{\pm}(w), \tag{6-10}$$

$$k_2^{\pm}(w)X_4^{+}(z) = rsX_4^{+}(z)k_2^{\pm}(w), \tag{6-11}$$

$$rsk_2^{\pm}(w)X_4^{-}(z) = X_4^{-}(z)k_2^{\pm}(w), \tag{6-12}$$

$$X_4^{\pm}(w)X_1^{\pm}(z) = X_1^{\pm}(z)X_4^{\pm}(w), \tag{6-13}$$

$$X_4^{\pm}(w)X_1^{\mp}(z) = X_1^{\mp}(z)X_4^{\pm}(w). \tag{6-14}$$

Proof. We only give details for one case of (6-9), (6-11), (6-13) and (6-14), the remaining relations are verified in a similar way. By (5-9) and (5-11), taking the equations $M_{11} = M'_{11}$, we get

$$a_{15}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)e_{3,5}^{\pm}(w)l_{11}^{\pm}(z) = a_{13}\Big(\frac{z}{w}\Big)l_{11}^{\pm}(z)k_3^{\pm}(w)e_{3,5}^{\pm}(w),$$

and the relations between $k_3^{\pm}(w)e_{3,5}^{\pm}(w)$ and $l_{11}^{\mp}(z)$ we have

$$a_{15}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)k_3^{\pm}(w)e_{3,5}^{\pm}(w)l_{11}^{\mp}(z) = a_{13}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)l_{11}^{\mp}(z)k_3^{\pm}(w)e_{3,5}^{\pm}(w),$$

so that $k_1^{+}(z)X_4^{+}(w) = rsX_4^{+}(w)k_1^{+}(z)$. Now apply $M_{12} = M'_{12}$ to obtain

$$a_{25}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)e_{3,5}^{\pm}(w)l_{12}^{\pm}(z) = a_{13}\Big(\frac{z}{w}\Big)l_{12}^{\pm}(z)k_3^{\pm}(w)e_{3,5}^{\pm}(w),$$

$$a_{25}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)k_3^{\pm}(w)e_{3,5}^{\pm}(w)l_{12}^{\mp}(z) = a_{13}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)l_{12}^{\mp}(z)k_3^{\pm}(w)e_{3,5}^{\pm}(w).$$

So we get $X_4^{+}(w)X_1^{+}(z) = X_1^{+}(z)X_4^{+}(w)$. Since $M_{21} = M'_{21}$, we get

$$a_{15}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)e_{3,5}^{\pm}(w)l_{21}^{\pm}(z) = a_{23}\Big(\frac{z}{w}\Big)l_{21}^{\pm}(z)k_3^{\pm}(w)e_{3,5}^{\pm}(w).$$

Furthermore,

$$a_{15}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)k_3^{\pm}(w)e_{3,5}^{\pm}(w)l_{21}^{\mp}(z)=a_{23}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)l_{21}^{\mp}(z)k_3^{\pm}(w)e_{3,5}^{\pm}(w),$$

thus we prove a case of relation in (6-14). Since $M_{22}=M'_{22}$, the formula is

$$a_{25}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)e_{3,5}^{\pm}(w)l_{22}^{\pm}(z)=a_{23}\Big(\frac{z}{w}\Big)l_{22}^{\pm}(z)k_3^{\pm}(w)e_{3,5}^{\pm}(w),$$

on the other hand, we have

$$a_{25}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)k_3^{\pm}(w)e_{3,5}^{\pm}(w)l_{22}^{\mp}(z)=a_{23}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)l_{22}^{\mp}(z)k_3^{\pm}(w)e_{3,5}^{\pm}(w).$$

As a final step, use the relations between $l_{21}^{\pm}(z)$ and $X_4^+(w)$ and those between $e_{12}^{\mp}(z)$ and $X_4^+(w)$, to come to the relation

$$k_2^+(z)X_4^+(w)=rsX_4^+(w)k_2^+(z),$$

as required. □

Lemma 6.4.

$$(rw-sz_{\pm})k_3^{\pm}(w)X_4^+(z)=rs(w-z_{\pm})X_4^+(z)k_3^{\pm}(w), \tag{6-15}$$

$$rs(w-z_{\mp})k_3^{\pm}(w)X_4^-(z)=(rw-sz_{\mp})X_4^-(z)k_3^{\pm}(w), \tag{6-16}$$

$$X_4^{\pm}(w)X_2^{\mp}(z)=X_2^{\mp}(z)X_4^{\pm}(w), \tag{6-17}$$

$$(z-w)X_4^+(w)X_2^+(z)=X_2^+(z)X_4^+(w)(rz-sw),$$

$$(rz-sw)X_4^-(w)X_2^-(z)=X_2^-(z)X_4^-(w)(z-w). \tag{6-18}$$

Proof. The arguments for both formulas are quite similar so we only give a proof of one case of (6-17) and (6-18), taking the equations $M_{13}=M'_{13}$, we get

$$a_{35}\Big(\frac{z}{w}\Big)l_{15}^{\pm}(w)l_{13}^{\pm}(z)+b_{53}\Big(\frac{z}{w}\Big)l_{13}^{\pm}(w)l_{15}^{\pm}(z)=l_{13}^{\pm}(z)l_{15}^{\pm}(w), \tag{6-19}$$

$$a_{35}\Big(\frac{z}{w}\Big)l_{25}^{\pm}(w)l_{13}^{\pm}(z)+b_{53}\Big(\frac{z}{w}\Big)l_{23}^{\pm}(w)l_{15}^{\pm}(z)=a_{12}\Big(\frac{z}{w}\Big)l_{13}^{\pm}(z)l_{25}^{\pm}(w)+b_{12}\Big(\frac{z}{w}\Big)l_{23}^{\pm}(z)l_{15}^{\pm}(w), \tag{6-20}$$

$$a_{35}\Big(\frac{z}{w}\Big)l_{35}^{\pm}(w)l_{13}^{\pm}(z)+b_{53}\Big(\frac{z}{w}\Big)l_{33}^{\pm}(w)l_{15}^{\pm}(z)=a_{13}\Big(\frac{z}{w}\Big)l_{13}^{\pm}(z)l_{35}^{\pm}(w)+b_{13}\Big(\frac{z}{w}\Big)l_{33}^{\pm}(z)l_{15}^{\pm}(w). \tag{6-21}$$

Using $f_{32}^{\pm}(w)f_{21}^{\pm}(w)\cdot$(6-19) $-$ $f_{32}^{\pm}(w)$(6-20) $-$ $f_{31}^{\pm}(w)\cdot$(6-19) $+$ (6-21), through a lot of calculations, we can obtain

$$\begin{aligned}(6\text{-}22)\quad & a_{35}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)e_{35}^{\pm}(w)l_{13}^{\pm}(z)+b_{53}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)l_{15}^{\pm}(z)\\ &\qquad +a_{13}\Big(\frac{z}{w}\Big)f_{32}^{\pm}(w)l_{13}^{\pm}(z)k_2^{\pm}(w)e_{25}^{\pm}(w)\\ &= a_{13}\Big(\frac{z}{w}\Big)l_{13}^{\pm}(z)f_{32}^{\pm}(w)k_2^{\pm}(w)e_{25}^{\pm}(w)+a_{13}\Big(\frac{z}{w}\Big)l_{13}^{\pm}(z)k_3^{\pm}(w)e_{35}^{\pm}(w)\\ &\qquad +b_{13}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)l_{11}^{\pm}(z)e_{15}^{\pm}(w).\end{aligned}$$

Taking the equations $M_{23}=M'_{23}$, we have

$$\begin{aligned}(6\text{-}23)\quad & a_{35}\Big(\frac{z}{w}\Big)l_{15}^{\pm}(w)l_{23}^{\pm}(z)+b_{53}\Big(\frac{z}{w}\Big)l_{13}^{\pm}(w)l_{25}^{\pm}(z)\\ &\qquad = a_{21}\Big(\frac{z}{w}\Big)l_{23}^{\pm}(z)l_{15}^{\pm}(w)+b_{21}\Big(\frac{z}{w}\Big)l_{13}^{\pm}(z)l_{25}^{\pm}(w),\\ (6\text{-}24)\quad & a_{35}\Big(\frac{z}{w}\Big)l_{25}^{\pm}(w)l_{23}^{\pm}(z)+b_{53}\Big(\frac{z}{w}\Big)l_{23}^{\pm}(w)l_{25}^{\pm}(z)\\ &\qquad = l_{23}^{\pm}(z)l_{25}^{\pm}(w),\\ (6\text{-}25)\quad & a_{35}\Big(\frac{z}{w}\Big)l_{35}^{\pm}(w)l_{23}^{\pm}(z)+b_{53}\Big(\frac{z}{w}\Big)l_{33}^{\pm}(w)l_{25}^{\pm}(z)\\ &\qquad = a_{23}\Big(\frac{z}{w}\Big)l_{23}^{\pm}(z)l_{35}^{\pm}(w)+b_{23}\Big(\frac{z}{w}\Big)l_{33}^{\pm}(z)l_{25}^{\pm}(w).\end{aligned}$$

A similar calculation, we come to the relation

$$\begin{aligned}(6\text{-}26)\quad & a_{35}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)e_{35}^{\pm}(w)l_{23}^{\pm}(z)+b_{53}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)l_{25}^{\pm}(z)\\ &=b_{21}\Big(\frac{z}{w}\Big)\{f_{32}^{\pm}(w)f_{21}^{\pm}(w)l_{13}^{\pm}(z)k_2^{\pm}(w)e_{25}^{\pm}(w)-f_{31}^{\pm}(w)l_{13}^{\pm}(z)k_2^{\pm}(w)e_{25}^{\pm}(w)\}\\ &\qquad +a_{23}\Big(\frac{z}{w}\Big)\{l_{23}^{\pm}(z)f_{32}^{\pm}(w)k_2^{\pm}(w)e_{25}^{\pm}(w)+l_{23}^{\pm}(z)k_3^{\pm}(w)e_{35}^{\pm}(w)\}\\ &+b_{23}\Big(\frac{z}{w}\Big)\{l_{33}^{\pm}(z)k_2^{\pm}(w)e_{25}^{\pm}(w)+l_{21}^{\pm}(z)k_3^{\pm}(w)e_{15}^{\pm}(w)\}\\ &\qquad -f_{32}^{\pm}(w)l_{23}^{\pm}(z)k_2^{\pm}(w)e_{25}^{\pm}(w).\end{aligned}$$

Furthermore, by using $-f_{21}^{\pm}(z)\cdot$(6-22) $+$ (6-26), we get

$$\begin{aligned}& a_{35}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)e_{35}^{\pm}(w)k_2^{\pm}(z)e_{23}^{\pm}(z)+b_{53}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)k_2^{\pm}(z)e_{25}^{\pm}(z)\\ &\qquad = b_{23}\Big(\frac{z}{w}\Big)k_2^{\pm}(z)k_3^{\pm}(w)e_{25}^{\pm}(w)+a_{23}\Big(\frac{z}{w}\Big)k_2^{\pm}(z)e_{23}^{\pm}(z)k_3^{\pm}(w)e_{35}^{\pm}(w),\end{aligned}$$

and taking into account the relations between $e^{\mp}_{3,5}(w)$ and $e^{\pm}_{23}(z)$, we have

$$a_{35}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)k^{\mp}_3(w)e^{\mp}_{35}(w)k^{\pm}_2(z)e^{\pm}_{23}(z)+b_{53}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)k^{\mp}_3(w)k^{\pm}_2(z)e^{\pm}_{25}(z)$$
$$=b_{23}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)k^{\pm}_2(z)k^{\mp}_3(w)e^{\mp}_{25}(w)+a_{23}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)k^{\pm}_2(z)e^{\pm}_{23}(z)k^{\mp}_3(w)e^{\mp}_{35}(w).$$

Therefore, we can arrive at $(z-w)X^+_4(w)X^+_2(z)=X^+_2(z)X^+_4(w)(rz-sw)$. Now turn to (6-24). Taking the equation $M_{31}=M'_{31}$, we get

$$\begin{aligned}(6\text{-}27)\quad & a_{15}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)k^{\mp}_3(w)e^{\mp}_{35}(w)l^{\pm}_{31}(z)\\ &=\Big\{l^{\pm}_{31}(z)-b_{32}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)f^{\mp}_{32}(w)l^{\pm}_{21}(z)\\ &\quad+b_{31}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)(f^{\mp}_{32}(w)f^{\mp}_{21}(w)-f^{\mp}_{31}(w))l^{\pm}_{11}(z)\Big\}k^{\mp}_3(w)e^{\mp}_{35}(w),\end{aligned}$$

$$\begin{aligned}(6\text{-}28)\quad & a_{15}\Big(\frac{z}{w}\Big)k^{\pm}_3(w)e^{\pm}_{35}(w)l^{\pm}_{31}(z)\\ &=\Big\{l^{\pm}_{31}(z)-b_{32}\Big(\frac{z}{w}\Big)f^{\pm}_{32}(w)l^{\pm}_{21}(z)\\ &\quad+b_{31}\Big(\frac{z}{w}\Big)(f^{\pm}_{32}(w)f^{\pm}_{21}(w)-f^{\pm}_{31}(w))l^{\pm}_{11}(z)\Big\}k^{\pm}_3(w)e^{\pm}_{35}(w).\end{aligned}$$

Using $M_{32}=M'_{32}$, consider the relations

$$\begin{aligned}(6\text{-}29)\quad & a_{15}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)k^{\mp}_3(w)e^{\mp}_{35}(w)l^{\pm}_{32}(z)\\ &=\Big\{l^{\pm}_{32}(z)-b_{32}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)f^{\mp}_{32}(w)l^{\pm}_{22}(z)\\ &\quad+b_{31}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)(f^{\mp}_{32}(w)f^{\mp}_{21}(w)-f^{\mp}_{31}(w))l^{\pm}_{12}(z)\Big\}k^{\mp}_3(w)e^{\mp}_{35}(w),\end{aligned}$$

$$\begin{aligned}(6\text{-}30)\quad & a_{15}\Big(\frac{z}{w}\Big)k^{\pm}_3(w)e^{\pm}_{35}(w)l^{\pm}_{32}(z)\\ &=\Big\{l^{\pm}_{32}(z)-b_{32}\Big(\frac{z}{w}\Big)f^{\pm}_{32}(w)l^{\pm}_{22}(z)\\ &\quad+b_{31}\Big(\frac{z}{w}\Big)(f^{\pm}_{32}(w)f^{\pm}_{21}(w)-f^{\pm}_{31}(w))l^{\pm}_{12}(z)\Big\}k^{\pm}_3(w)e^{\pm}_{35}(w).\end{aligned}$$

By $-(6\text{-}27)\cdot e^{\pm}_{12}(z)+(6\text{-}29)$ and $-(6\text{-}28)\cdot e^{\pm}_{12}(z)+(6\text{-}30)$, we can obtain

$$\begin{aligned}&a_{15}\Big(\frac{z}{w}\Big)k^{\pm}_3(w)e^{\pm}_{35}(w)f^{\pm}_{32}(z)k^{\pm}_2(z)\\ &\qquad=\Big\{f^{\pm}_{32}(z)k^{\pm}_2(z)-b_{32}\Big(\frac{z}{w}\Big)f^{\pm}_{32}(w)k^{\pm}_2(z)\Big\}k^{\pm}_3(w)e^{\pm}_{35}(w),\\ &a_{15}\Big(\frac{z_{\mp}}{w_{\pm}}\Big)k^{\mp}_3(w)e^{\mp}_{35}(w)f^{\pm}_{32}(z)k^{\pm}_2(z)\\ &\qquad=\Big\{f^{\pm}_{32}(z)k^{\pm}_2(z)-b_{32}\Big(\frac{z_{\pm}}{w_{\mp}}\Big)f^{\mp}_{32}(w)k^{\pm}_2(z)\Big\}k^{\mp}_3(w)e^{\mp}_{35}(w),\end{aligned}$$

and therefore $X^+_4(w)X^-_2(z)=X^-_2(z)X^+_4(w)$. □

Lemma 6.5. *In the algebra $\mathcal{U}(\widehat{R})$, we have*

$$e_{36}^{\pm}(z) = f_{63}^{\pm}(z) = 0, \quad e_{45}^{\pm}(z) = f_{54}^{\pm}(z) = 0.$$

Proof. We only verify a case of the first relation. By (5-9) and (5-11), we have $M_{13} = M'_{13}$, and so get the relations

$$\sum_{i=1}^{8} c_{i6}\Big(\frac{z}{w}\Big) l_{1i}^{\pm}(w) l_{1i'}^{\pm}(z) = l_{13}^{\pm}(z) l_{16}^{\pm}(w), \tag{6-31}$$

$$\sum_{i=1}^{8} c_{i6}\Big(\frac{z}{w}\Big) l_{2i}^{\pm}(w) l_{1i'}^{\pm}(z) = a_{12}\Big(\frac{z}{w}\Big) l_{13}^{\pm}(z) l_{26}^{\pm}(w) + b_{12}\Big(\frac{z}{w}\Big) l_{23}^{\pm}(z) l_{16}^{\pm}(w), \tag{6-32}$$

$$\sum_{i=1}^{8} c_{i6}\Big(\frac{z}{w}\Big) l_{3i}^{\pm}(w) l_{1i'}^{\pm}(z) = a_{13}\Big(\frac{z}{w}\Big) l_{13}^{\pm}(z) l_{36}^{\pm}(w) + b_{13}\Big(\frac{z}{w}\Big) l_{33}^{\pm}(z) l_{16}^{\pm}(w). \tag{6-33}$$

Let $(f_{32}^{\pm}(w) f_{21}^{\pm}(w) - f_{31}^{\pm}(w)) \cdot$ (6-31) $- f_{32}^{\pm}(w) \cdot$ (6-32) $+$ (6-33). Through a lot of calculations, we obtain

$$\begin{aligned} \sum_{i=3}^{8} c_{i6}\Big(\frac{z}{w}\Big) k_3^{\pm}(w) e_{3i}^{\pm}(w) l_{1i'}^{\pm}(z) \\ = a_{13}\Big(\frac{z}{w}\Big) l_{13}^{\pm}(z) k_3^{\pm}(w) e_{36}^{\pm}(w) + b_{13}\Big(\frac{z}{w}\Big) k_3^{\pm}(w) k_1^{\pm}(z) \\ \cdot \{(e_{12}^{\pm}(w) - e_{12}^{\pm}(z)) e_{26}^{\pm}(w) - e_{16}^{\pm}(w)\}. \end{aligned} \tag{6-34}$$

And from $M_{23} = M'_{23}$, we obtain

$$\sum_{i=1}^{8} c_{i6}\Big(\frac{z}{w}\Big) l_{1i}^{\pm}(w) l_{2i'}^{\pm}(z) = a_{21}\Big(\frac{z}{w}\Big) l_{23}^{\pm}(z) l_{16}^{\pm}(w) + b_{21}\Big(\frac{z}{w}\Big) l_{13}^{\pm}(z) l_{26}^{\pm}(w), \tag{6-35}$$

$$\sum_{i=1}^{8} c_{i6}\Big(\frac{z}{w}\Big) l_{2i}^{\pm}(w) l_{2i'}^{\pm}(z) = l_{23}^{\pm}(z) l_{26}^{\pm}(w), \tag{6-36}$$

$$\sum_{i=1}^{8} c_{i6}\Big(\frac{z}{w}\Big) l_{3i}^{\pm}(w) l_{2i'}^{\pm}(z) = a_{23}\Big(\frac{z}{w}\Big) l_{23}^{\pm}(z) l_{36}^{\pm}(w) + b_{23}\Big(\frac{z}{w}\Big) l_{33}^{\pm}(z) l_{26}^{\pm}(w). \tag{6-37}$$

Calculating $(f_{32}^{\pm}(w) f_{21}^{\pm}(w) - f_{31}^{\pm}(w)) \cdot$ (6-35) $- f_{32}^{\pm}(w) \cdot$ (6-36) $+$ (6-37) $- f_{21}^{\pm}(z) \cdot$ (6-34), we get

$$\begin{aligned} \sum_{i=3}^{8} c_{i6}\Big(\frac{z}{w}\Big) k_3^{\pm}(w) e_{3i}^{\pm}(w) k_2^{\pm}(z) e_{2i'}^{\pm}(z) \\ = a_{23}\Big(\frac{z}{w}\Big) k_2^{\pm}(z) e_{23}^{\pm}(z) k_3^{\pm}(w) e_{36}^{\pm}(w) \ + b_{23}\Big(\frac{z}{w}\Big) k_3^{\pm}(w) k_2^{\pm}(z) e_{26}^{\pm}(w). \end{aligned} \tag{6-38}$$

Using $M_{23} = M'_{23}$, we come by the relations

$$\sum_{i=1}^{8} c_{i6}\Big(\frac{z}{w}\Big) l_{1i}^{\pm}(w) l_{3i'}^{\pm}(z) = a_{31}\Big(\frac{z}{w}\Big) l_{33}^{\pm}(z) l_{16}^{\pm}(w) + b_{31}\Big(\frac{z}{w}\Big) l_{13}^{\pm}(z) l_{36}^{\pm}(w), \tag{6-39}$$

$$\sum_{i=1}^{8} c_{i6}\Big(\frac{z}{w}\Big) l_{2i}^{\pm}(w) l_{3i'}^{\pm}(z) = a_{32}\Big(\frac{z}{w}\Big) l_{33}^{\pm}(z) l_{26}^{\pm}(w) + b_{32}\Big(\frac{z}{w}\Big) l_{23}^{\pm}(z) l_{36}^{\pm}(w), \tag{6-40}$$

$$\sum_{i=1}^{8} c_{i6}\Big(\frac{z}{w}\Big) l_{3i}^{\pm}(w) l_{3i'}^{\pm}(z) = l_{33}^{\pm}(z) l_{36}^{\pm}(w). \tag{6-41}$$

Calculating

$$(f_{32}^{\pm}(w) f_{21}^{\pm}(w) - f_{31}^{\pm}(w)) \cdot (6\text{-}39) - f_{32}^{\pm}(w) \cdot (6\text{-}40) + (6\text{-}41) - f_{32}^{\pm}(z) \cdot (6\text{-}38) - f_{31}^{\pm}(z) \cdot (6\text{-}34),$$

we arrive at the relation

$$\sum_{i=3}^{8} c_{i6}\Big(\frac{z}{w}\Big) k_3^{\pm}(w) e_{3i}^{\pm}(w) l_{3i'}^{\pm}(z) = k_3^{\pm}(z) k_3^{\pm}(w) e_{36}^{\pm}(w). \tag{6-42}$$

Setting $z = w$, we get $e_{36}^{\pm}(z) = 0$. The remaining relations can be proved in a similar way. □

Lemma 6.6. *The following equations hold in* $\mathcal{U}(\widehat{R})$:

$$X_4^{\pm}(w) X_3^{\pm}(z) = (rs)^{\pm 1} X_3^{\pm}(z) X_4^{\pm}(w), \tag{6-43}$$

$$X_3^{\pm}(w) X_4^{\mp}(z) = X_4^{\mp}(z) X_3^{\pm}(w). \tag{6-44}$$

Proof. The arguments are similar for all relations so we only prove the relation $X_4^{+}(w) X_3^{+}(z) = rs X_3^{+}(z) X_4^{+}(w)$. By (5-9)–(5-12), we have $M_{34} = M'_{34}$, and then get the relation

$$\sum_{i=1}^{8} c_{i5}\Big(\frac{z}{w}\Big) l_{3i}^{\pm}(w) l_{3i'}^{\pm}(z) = l_{34}^{\pm}(z) l_{35}^{\pm}(w)$$

through a similar calculation process as in Lemma 6.5, which yields

$$\sum_{i=3}^{8} c_{i5}\Big(\frac{z}{w}\Big) k_3^{\pm}(w) e_{3i}^{\pm}(w) k_3^{\pm}(z) e_{3i'}^{\pm}(z) = k_3^{\pm}(z) e_{34}^{\pm}(z) k_3^{\pm}(w) e_{35}^{\pm}(w), \tag{6-45}$$

and from $M_{35} = M'_{35}$, we obtain

$$\sum_{i=3}^{8} c_{i4}\Big(\frac{z}{w}\Big) k_3^{\pm}(w) e_{3i}^{\pm}(w) k_3^{\pm}(z) e_{3i'}^{\pm}(z) = k_3^{\pm}(z) e_{35}^{\pm}(z) k_3^{\pm}(w) e_{34}^{\pm}(w). \tag{6-46}$$

Furthermore, $M_{33} = M'_{33}$ and $M_{36} = M'_{36}$ give that

$$\sum_{i=3}^{8} c_{i6}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)e_{3i}^{\pm}(w)k_3^{\pm}(z)e_{3i'}^{\pm}(z) = k_3^{\pm}(z)k_3^{\pm}(w)e_{36}^{\pm}(w), \tag{6-47}$$

$$\sum_{i=3}^{8} c_{i3}\Big(\frac{z}{w}\Big)k_3^{\pm}(w)e_{3i}^{\pm}(w)k_3^{\pm}(z)e_{3i'}^{\pm}(z) = k_3^{\pm}(z)e_{36}^{\pm}(z)k_3^{\pm}(w). \tag{6-48}$$

Combining (6-45) with (6-46), we get that

$$(rs)^{-1}X_4^+(w)X_3^+(z) - X_3^+(w)X_4^+(z) = X_3^+(z)X_4^+(w) - (rs)^{-1}X_4^+(z)X_3^+(w). \tag{6-49}$$

Taking (6-45) and (6-47), owing to Lemma 6.5 and the fact that $e_{36}^{\pm}(z) = 0$, we have

$$\begin{aligned}\sum_{i=4}^{5} k_3^{\pm}(w)e_{3i}^{\pm}(w)k_3^{\pm}(z)e_{3i'}^{\pm}(z)\Big(c_{i5}\Big(\frac{z}{w}\Big) - (r^{-1}s)^{\frac{1}{2}}c_{i6}\Big(\frac{z}{w}\Big)\Big)&\\ = k_3^{\pm}(z)e_{34}^{\pm}(z)k_3^{\pm}(w)e_{35}^{\pm}(w).&\end{aligned} \tag{6-50}$$

Using the relations (6-46) and (6-48), we get

$$\begin{aligned}\sum_{i=4}^{5} k_3^{\pm}(w)e_{3i}^{\pm}(w)k_3^{\pm}(z)e_{3i'}^{\pm}(z)\Big(c_{i4}\Big(\frac{z}{w}\Big) - (rs^{-1})^{\frac{1}{2}}c_{i3}\Big(\frac{z}{w}\Big)\Big)&\\ = k_3^{\pm}(z)e_{35}^{\pm}(z)k_3^{\pm}(w)e_{34}^{\pm}(w).&\end{aligned} \tag{6-51}$$

Exchanging z and w in the relation (6-51), and combining into (6-50), we obtain

$$\begin{aligned}z(rs)^{-1}X_4^+(w)X_3^+(z) - wX_3^+(w)X_4^+(z)&\\ = zX_3^+(z)X_4^+(w) - w(rs)^{-1}X_4^+(z)X_3^+(w)&\end{aligned} \tag{6-52}$$

By (6-49) and (6-52), we get $X_4^+(w)X_3^+(z) = rsX_3^+(z)X_4^+(w)$. □

Lemma 6.7. *The following equations hold in* $\mathcal{U}(\widehat{R})$:

$$\begin{aligned}[X_4^+(z), X_4^-(w)]&\\ = (s^{-1}-r^{-1})\Big\{\delta\Big(\frac{z_-}{w_+}\Big)k_5^-(w_+)k_3^-(w_+)^{-1} - \delta\Big(\frac{z_+}{w_-}\Big)k_5^+(z_+)k_3^+(z_+)^{-1}\Big\}.&\end{aligned} \tag{6-53}$$

Proof. By (5-10) and (5-12), we have $M_{35} = M'_{35}$ and then get the relations

$$a_{53}\Big(\frac{z}{w}\Big)l_{13}^{\pm}(w)l_{35}^{\pm}(z) + b_{35}\Big(\frac{z}{w}\Big)l_{15}^{\pm}(w)l_{33}^{\pm}(z) = b_{31}\Big(\frac{z}{w}\Big)l_{15}^{\pm}(z)l_{33}^{\pm}(w) + a_{31}\Big(\frac{z}{w}\Big)l_{35}^{\pm}(z)l_{13}^{\pm}(w),$$

$$a_{53}\Big(\frac{z}{w}\Big)l_{23}^{\pm}(w)l_{35}^{\pm}(z) + b_{35}\Big(\frac{z}{w}\Big)l_{25}^{\pm}(w)l_{33}^{\pm}(z) = b_{32}\Big(\frac{z}{w}\Big)l_{25}^{\pm}(z)l_{33}^{\pm}(w) + a_{32}\Big(\frac{z}{w}\Big)l_{35}^{\pm}(z)l_{23}^{\pm}(w),$$

$$a_{53}\Big(\frac{z}{w}\Big)l_{33}^{\pm}(w)l_{35}^{\pm}(z) + b_{35}\Big(\frac{z}{w}\Big)l_{35}^{\pm}(w)l_{33}^{\pm}(z) = l_{35}^{\pm}(z)l_{33}^{\pm}(w),$$

$$a_{53}\Big(\frac{z}{w}\Big)l_{53}^{\pm}(w)l_{35}^{\pm}(z) + b_{35}\Big(\frac{z}{w}\Big)l_{55}^{\pm}(w)l_{33}^{\pm}(z) = b_{35}\Big(\frac{z}{w}\Big)l_{55}^{\pm}(z)l_{33}^{\pm}(w) + a_{35}\Big(\frac{z}{w}\Big)l_{35}^{\pm}(z)l_{53}^{\pm}(w).$$

By straightforward calculations, one can check that

$$[X_4^+(z), X_4^-(w)] = (s^{-1}-r^{-1})\Big\{\delta\Big(\frac{z_-}{w_+}\Big)k_5^-(w_+)k_3^-(w_+)^{-1} - \delta\Big(\frac{z_+}{w_-}\Big)k_5^+(z_+)k_3^+(z_+)^{-1}\Big\}.$$

This completes the proof. □

Lemma 6.8. *The following equations hold in* $\mathcal{U}(\widehat{R})$:

$$k_5^\pm(w)X_3^+(z) = \frac{rs(z_\pm - w)}{z_\pm s - rw}X_3^+(z)k_5^\pm(w), \tag{6-54}$$

$$X_3^-(z)k_5^\pm(w) = \frac{rs(z_\mp - w)}{z_\mp s - rw}k_5^\pm(w)X_3^-(z), \tag{6-55}$$

$$k_4^\pm(w)X_4^+(z) = \frac{wr - sz_\pm}{w - z_\pm}X_4^+(z)k_4^\pm(w), \tag{6-56}$$

$$X_4^-(z)k_4^\pm(w) = \frac{wr - sz_\mp}{w - z_\mp}k_4^\pm(w)X_4^-(z). \tag{6-57}$$

Proof. We only give a proof of one case of (6-54). Similarly, we give the other identities. Using again $M_{35} = M'_{35}$ and $M_{34} = M'_{34}$, we get

$$\sum_{i=1}^{8} c_{i5}\Big(\frac{z}{w}\Big)l_{5i}^\pm(w)l_{3i'}^\pm(z) = b_{35}\Big(\frac{z}{w}\Big)l_{54}^\pm(z)l_{35}^\pm(w) + a_{35}\Big(\frac{z}{w}\Big)l_{34}^\pm(z)l_{55}^\pm(w)$$

through the same calculating process as in Lemma 6.5, which yields

$$\sum_{i=5}^{8} c_{i5}\Big(\frac{z}{w}\Big)k_5^\pm(w)e_{5i}^\pm(w)k_3^\pm(z)e_{3i'}^\pm(z) = a_{35}\Big(\frac{z}{w}\Big)k_3^\pm(z)e_{34}^\pm(z)k_5^\pm(w). \tag{6-58}$$

From $M_{35} = M'_{35}$, we obtain

$$\sum_{i=1}^{8} c_{i4}\Big(\frac{z}{w}\Big)l_{5i}^\pm(w)l_{3i'}^\pm(z) = b_{35}\Big(\frac{z}{w}\Big)l_{55}^\pm(z)l_{34}^\pm(w) + a_{35}\Big(\frac{z}{w}\Big)l_{35}^\pm(z)l_{54}^\pm(w),$$

we arrive at the relation

$$\sum_{i=5}^{8} c_{i4}\Big(\frac{z}{w}\Big)e_{5i}^\pm(w)k_3^\pm(z)e_{3i'}^\pm(z) = b_{35}\Big(\frac{z}{w}\Big)k_5^\pm(w)k_3^\pm(z)e_{34}^\pm(w). \tag{6-59}$$

Using (6-58) and (6-59), due to the relation $e_{45}^\pm(z) = f_{54}^\pm(z) = 0$, we obtain

$$k_5^\pm(w)k_3^\pm(z)e_{34}^\pm(z)\Big(c_{55}\Big(\frac{z}{w}\Big) - c_{54}\Big(\frac{z}{w}\Big)\Big) = a_{35}\Big(\frac{z}{w}\Big)k_3^\pm(z)e_{34}^\pm(z)k_5^\pm(w) - b_{35}\Big(\frac{z}{w}\Big)k_5^\pm(w)k_3^\pm(z)e_{34}^\pm(w).$$

Finally, we get

$$k_5^+(w)X_3^+(z) = \frac{rs(z_+ - w)}{z_+ s - rw} X_3^+(z)k_5^+(w).$$

This completes the proof. □

Lemma 6.9.

$$k_4^\pm(z)k_5^\pm(w) = k_5^\pm(w)k_4^\pm(z), \tag{6-60}$$

$$k_4^\pm(z)k_5^\mp(w)\frac{z_\pm - w_\mp}{rz_\pm - sw_\mp} = \frac{z_\mp - w_\pm}{rz_\mp - sw_\pm}k_5^\mp(w)k_4^\pm(z). \tag{6-61}$$

Proof. The proof is similar to that of Lemma 6.5. □

Lemma 6.10. *The following equations hold in* $\mathcal{U}(\widehat{R})$:

$$k_5^\pm(z)k_5^\pm(w) = k_5^\pm(w)k_5^\pm(z), \tag{6-62}$$

$$k_5^\pm(z)k_5^\mp(w) = k_5^\mp(w)k_5^\pm(z), \tag{6-63}$$

$$k_5^\pm(w)X_4^+(z) = \frac{rs(w - z_\pm)}{sw - rz_\pm} X_4^+(z)k_5^\pm(w), \tag{6-64}$$

$$X_4^-(z)k_5^\pm(w) = \frac{rs(w - z_\mp)}{sw - rz_\mp} k_5^\pm(w)X_4^-(z). \tag{6-65}$$

Proof. Here we only prove (6-64) as the other relations can be shown similarly. By (5-10) and (5-12), we have $M_{35} = M'_{35}$. Then we can get the relation

$$l_{55}^\pm(w)l_{35}^\pm(z) = b_{35}\left(\frac{z}{w}\right)l_{55}^\pm(z)l_{35}^\pm(w) + a_{35}\left(\frac{z}{w}\right)l_{35}^\pm(z)l_{55}^\pm(w).$$

By straightforward calculations, one checks that

$$k_5^+(w)X_4^+(z) = \frac{rs(z_+ - w)}{rz_+ - sw} X_4^+(z)k_5^+(w).$$

This completes the proof. □

Proposition 6.11. *The following equations hold in* $\mathcal{U}(\widehat{R})$

$$\begin{aligned}\big\{X_2^-(z_1)X_2^-(z_2)X_4^-(w) - (r+s)X_2^-(z_1)X_4^-(w)X_2^-(z_2)\\ + rsX_4^-(w)X_2^-(z_1)X_2^-(z_2)\big\} + \{z_1 \leftrightarrow z_2\} = 0,\end{aligned} \tag{6-66}$$

$$\begin{aligned}\big\{X_4^+(z_1)X_4^+(z_2)X_2^+(w) - (r+s)X_4^+(z_1)X_2^+(w)X_4^+(z_2)\\ + rsX_2^+(w)X_4^+(z_1)X_4^+(z_2)\big\} + \{z_1 \leftrightarrow z_2\} = 0,\end{aligned} \tag{6-67}$$

$$\begin{aligned}\big\{rsX_2^+(z_1)X_2^+(z_2)X_4^+(w) - (r+s)X_2^+(z_1)X_4^+(w)X_2^+(z_2)\\ + X_4^+(w)X_2^+(z_1)X_2^+(z_2)\big\} + \{z_1 \leftrightarrow z_2\} = 0,\end{aligned} \tag{6-68}$$

$$\begin{aligned}\big\{rsX_4^-(z_1)X_4^-(z_2)X_2^-(w) - (r+s)X_4^-(z_1)X_2^-(w)X_4^-(z_2)\\ + X_2^-(w)X_4^-(z_1)X_4^-(z_2)\big\} + \{z_1 \leftrightarrow z_2\} = 0.\end{aligned} \tag{6-69}$$

Proof. The equations can be proved similarly as in [16]. □

Now we proceed to the case of general n. We first restrict the relation to $E_{ij}\otimes E_{kl}$, $2\le i,j,k,l\le 2n-1$. By induction, we get all the commutation relations we need except those between $X_1^{\pm}(z)$, $k_1^{\pm}(z)$, and $X_n^{\pm}(z)$, $k_{n+1}^{\pm}(z)$.

Lemma 6.12. *The following equations hold in* $\mathcal{U}(\widehat{R})$:

$$k_1^{\pm}(z)X_n^{+}(w)=rsX_n^{+}(w)k_1^{\pm}(z), \tag{6-70}$$

$$rsk_1^{\pm}(z)X_n^{-}(w)=X_n^{-}(w)k_1^{\pm}(z), \tag{6-71}$$

$$X_n^{\pm}(w)X_1^{\pm}(z)=X_1^{\pm}(z)X_n^{\pm}(w), \tag{6-72}$$

$$X_n^{\pm}(w)X_1^{\mp}(z)=X_1^{\mp}(z)X_n^{\pm}(w), \tag{6-73}$$

$$k_1^{\pm}(z)k_{n+1}^{\pm}(w)=k_{n+1}^{\pm}(w)k_1^{\pm}(z), \tag{6-74}$$

$$k_{n+1}^{\pm}(z)k_{n+1}^{\pm}(w)=k_{n+1}^{\pm}(w)k_{n+1}^{\pm}(z), \tag{6-75}$$

$$k_{n+1}^{\pm}(z)k_{n+1}^{\mp}(w)=k_{n+1}^{\mp}(w)k_{n+1}^{\pm}(z), \tag{6-76}$$

$$k_{n+1}^{\pm}(z)X_n^{+}(w)=\frac{rs(w-z_{\pm})}{sw-rz_{\pm}}X_n^{+}(w)k_{n+1}^{\pm}(z), \tag{6-77}$$

$$X_n^{-}(w)k_{n+1}^{\pm}(z)=\frac{rs(w-z_{\mp})}{sw-rz_{\mp}}k_{n+1}^{\pm}(z)X_n^{-}(w), \tag{6-78}$$

$$\frac{w_{\pm}-z_{\mp}}{w_{\pm}r-z_{\mp}s}k_{n+1}^{\mp}(w)k_1^{\pm}(z)=\frac{w_{\mp}-z_{\pm}}{w_{\mp}r-z_{\pm}s}k_1^{\pm}(z)k_{n+1}^{\mp}(w), \tag{6-79}$$

$$\begin{aligned}[X_n^{+}(z),X_n^{-}(w)]=(s^{-1}-r^{-1})\Big\{&\delta\Big(\frac{z_-}{w_+}\Big)k_{n+1}^{-}(w_+)k_{n-1}^{-}(w_+)^{-1}\\&-\delta\Big(\frac{z_+}{w_-}\Big)k_{n+1}^{+}(z_+)k_{n-1}^{+}(z_+)^{-1}\Big\}.\end{aligned} \tag{6-80}$$

Proof. By straightforward calculations one checks that the preceding formulas are correct. □

Finally, we define the map $\tau: U_{r,s}(\widehat{\mathfrak{so}}_{2n})\to\mathcal{U}(\widehat{R})$ as follows:

$$\begin{aligned}
x_i^{\pm}(z)&\mapsto(r-s)^{-1}X_i^{\pm}(z(rs^{-1})^{\frac{i}{2}}),\\
x_n^{\pm}(z)&\mapsto(r-s)^{-1}X_n^{\pm}(z(rs^{-1})^{\frac{n-1}{2}}),\\
\varphi_i(z)&\mapsto k_{i+1}^{+}(z(rs^{-1})^{\frac{i}{2}})k_i^{+}(z(rs^{-1})^{\frac{i}{2}})^{-1},\\
\psi_i(z)&\mapsto k_{i+1}^{-}(z(rs^{-1})^{\frac{i}{2}})k_i^{-}(z(rs^{-1})^{\frac{i}{2}})^{-1},\\
\varphi_n(z)&\mapsto k_{n+1}^{+}(z(rs^{-1})^{\frac{n-1}{2}})k_{n-1}^{+}(z(rs^{-1})^{\frac{n-1}{2}})^{-1},\\
\psi_n(z)&\mapsto k_{n+1}^{-}(z(rs^{-1})^{\frac{n-1}{2}})k_{n-1}^{-}(z(rs^{-1})^{\frac{n-1}{2}})^{-1},
\end{aligned}$$

where $1\le i\le n-1$, and satisfy all the relations of Proposition 6.13.

Proposition 6.13. *In $U_{r,s}(\widehat{\mathfrak{so}_{2n}})$, the generating series $x_i^{\pm}(z)$, $\varphi_i(z)$, $\psi_i(z)$, $x_n^{\pm}(z)$, $\varphi_n(z)$, and $\psi_n(z)$, and the relations between $x_i^{\pm}(z)$, $\varphi_i(z)$, and $\psi_i(z)$ are the same as in $U_{r,s}(\widehat{\mathfrak{sl}_n})$; the other relations as follows*:

$$[\varphi_j(z), \varphi_n(w)] = 0, \quad [\psi_j(z), \psi_n(w)] = 0, \tag{6-81}$$

$$\varphi_j(z)\psi_n(w) = \frac{g_{jn}\left(\frac{z_-}{w_+}\right)}{g_{jn}\left(\frac{z_+}{w_-}\right)}\psi_n(w)\varphi_j(z), \quad 1 \le j \le n, \tag{6-82}$$

$$\varphi_n(z)x_{n-2}^{\pm}(w) = (rs)^{\pm 1}g_{n,n-2}\left(\frac{z}{w_{\pm}}\right)^{\pm 1}x_{n-2}^{\pm}(w)\varphi_n(z),$$

$$\psi_n(z)x_{n-2}^{\pm}(w) = (rs)^{\mp 1}g_{n,n-2}\left(\frac{w_{\mp}}{z}\right)^{\mp 1}x_{n-2}^{\pm}(w)\varphi_n(z), \tag{6-83}$$

$$\varphi_n(z)x_n^{\pm}(w) = g_{nn}\left(\frac{z}{w_{\pm}}\right)^{\pm 1}x_n^{\pm}(w)\varphi_n(z),$$

$$\psi_n(z)x_n^{\pm}(w) = g_{nn}\left(\frac{w_{\mp}}{z}\right)^{\mp 1}x_n^{\pm}(w)\varphi_n(z), \tag{6-84}$$

$$\varphi_n(z)x_{n-1}^{\pm}(w) = (rs)^{\pm 1}x_{n-1}^{\pm}(w)\varphi_n(z),$$

$$\psi_n(z)x_{n-1}^{\pm}(w) = (rs)^{\mp 1}x_{n-1}^{\pm}(w)\psi_n(z), \tag{6-85}$$

$$\varphi_n(z)x_l^{\pm}(w) = x_l^{\pm}(w)\varphi_n(z),$$

$$\psi_n(z)x_l^{\pm}(w) = x_l^{\pm}(w)\psi_n(z), \quad 1 \le l \le n-3, \tag{6-86}$$

$$\varphi_t(z)x_n^{\pm}(w) = x_n^{\pm}(w)\varphi_t(z),$$

$$\psi_t(z)x_n^{\pm}(w) = x_n^{\pm}(w)\varphi_t(z), \quad 1 \le t \le n-3,$$

$$\varphi_{n-2}(z)x_n^{\pm}(w) = g_{n,n-2}\left(\frac{z}{w_{\pm}}\right)^{\pm 1}x_n^{\pm}(w)\varphi_{n-2}(z),$$

$$\psi_{n-2}(z)x_n^{\pm}(w) = g_{n,n-2}\left(\frac{w_{\mp}}{z}\right)^{\mp 1}x_n^{\pm}(w)\varphi_{n-2}(z), \tag{6-87}$$

$$x_{n-2}^{\pm}(z)x_n^{\pm}(w) = g_{n,n-2}\left(\frac{z}{w}\right)^{\pm 1}x_n^{\pm}(w)x_{n-2}^{\pm}(z),$$

$$x_i^{\pm}(z)x_n^{\pm}(w) = \langle w_n', w_i\rangle^{\pm 1}x_n^{\pm}(w)x_i^{\pm}(z), \quad a_{in} = 0 \tag{6-88}$$

$$[x_n^+(z), x_j^-(w)] = (s^{-1}-r^{-1})\delta_{jn}\left\{\delta\left(\frac{z_-}{w_+}\right)\psi_n(w_+) - \delta\left(\frac{z_+}{w_-}\right)\varphi_n(z_+)\right\}, \quad j \le n, \tag{6-89}$$

$$\{x_{n-2}^{\pm}(z_1)x_{n-2}^{\pm}(z_2)x_n^{\pm}(w) - (r^{\pm 1}+s^{\pm 1})x_{n-2}^{\pm}(z_1)x_n^{\pm}(w)x_{n-2}^{\pm}(z_2) + (rs)^{\pm 1}x_n^{\pm}(w)x_{n-2}^{\pm}(z_1)x_{n-2}^{\pm}(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0, \tag{6-90}$$

$$\{x_n^{\pm}(z_1)x_n^{\pm}(z_2)x_{n-2}^{\pm}(w) - (r^{\mp 1}+s^{\mp 1})x_n^{\pm}(z_1)x_{n-2}^{\pm}(w)x_n^{\pm}(z_2) + (rs)^{\mp 1}x_{n-2}^{\pm}(w)x_n^{\pm}(z_1)x_n^{\pm}(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0, \tag{6-91}$$

where $z_+ = zr^{\frac{c}{2}}$ *and* $z_- = zs^{\frac{c}{2}}$. *We set* $g_{ij}^{\pm}(z) = \sum_{n\in\mathbb{Z}_+} c_{ijn}^{\pm} z^n$, *a formal power series in* z, *and it can be expressed as follows*:

$$g_{ij}^{\pm}(z) = \frac{\langle w_j', w_i\rangle^{\pm 1} z - (\langle w_j', w_i\rangle\langle w_i', w_j\rangle^{-1})^{\pm\frac{1}{2}}}{z - (\langle w_i', w_j\rangle\langle w_j', w_i\rangle)^{\pm\frac{1}{2}}}.$$

Acknowledgements

We thank Professors Zhengwei Liu, Jinsong Wu, and Yilong Wang for inviting Hu to report the current results in the international workshop on Advances in Quantum Algebras during Jan. 23–27, 2024 in BIMSA (Beijing Institute of Mathematical Sciences and Applications). We also thank ECNU for partial support of Xu's master thesis where the basic R-matrices were announced in March 2023.

References

[1] G. Benkart and S. Witherspoon, "Two-parameter quantum groups and Drinfel'd doubles", *Algebr. Represent. Theory* **7**:3 (2004), 261–286. MR Zbl

[2] N. Bergeron, Y. Gao, and N. Hu, "Drinfel'd doubles and Lusztig's symmetries of two-parameter quantum groups", *J. Algebra* **301**:1 (2006), 378–405. MR Zbl

[3] N. Bergeron, Y. Gao, and N. Hu, "Representations of two-parameter quantum orthogonal and symplectic groups", pp. 1–21 in *Proceedings of the International Conference on Complex Geometry and Related Fields*, edited by Z. Chen et al., AMS/IP Stud. Adv. Math. **39**, Amer. Math. Soc., Providence, RI, 2007. MR Zbl

[4] X. Y. Chen, N. H. Hu, and X. L. Wang, "Convex PBW-type Lyndon bases and restricted two-parameter quantum group of type F_4", *Acta Math. Sin.* (*Engl. Ser.*) **39**:6 (2023), 1053–1084. MR Zbl

[5] J. T. Ding and I. B. Frenkel, "Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}(n)})$", *Comm. Math. Phys.* **156**:2 (1993), 277–300. MR Zbl

[6] V. G. Drinfeld, "Hopf algebras and the quantum Yang-Baxter equation", *Dokl. Akad. Nauk SSSR* **283**:5 (1985), 1060–1064. In Russian; translated in *Sov. Math. Dokl.* **32** (1985), 245–258. MR Zbl

[7] V. G. Drinfeld, "A new realization of Yangians and of quantum affine algebras", *Dokl. Akad. Nauk SSSR* **296**:1 (1987), 13–17. In Russian; translated in *Sov. Math. Dokl.* **36** (1988), 212–216. MR Zbl

[8] V. G. Drinfeld, "Quantum groups", pp. 798–820 in *Proceedings of the International Congress of Mathematicians*, *Vol.* 1, 2 (Berkeley, CA, 1986), Amer. Math. Soc., Providence, RI, 1987. MR

[9] L. D. Faddeev, N. Yu. Reshetikhin, and L. A. Takhtajan, "Quantization of Lie groups and Lie algebras", pp. 299–309 in *Yang–Baxter equations in integrable systems*.

[10] H. Garland, "The arithmetic theory of loop groups", *Inst. Hautes Études Sci. Publ. Math.* 52 (1980), 5–136. MR Zbl

[11] N. Hu and Y. Pei, "Notes on two-parameter quantum groups, II", *Comm. Algebra* **40**:9 (2012), 3202–3220. MR Zbl

[12] N. Hu and H. Zhang, "Generating functions with τ-invariance and vertex representations of quantum affine algebras $U_{r,s}(\hat{\mathfrak{g}})$, I: simply-laced cases", 2014. arXiv 1401.4925

[13] N. Hu, M. Rosso, and H. Zhang, "Two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{sl}}_n)$, Drinfeld realization and quantum affine Lyndon basis", *Comm. Math. Phys.* **278**:2 (2008), 453–486. MR Zbl

[14] N. Hu, X. Xu, and R. Zhuang, "RLL-realization of two-parameter quantum affine algebra of type $B_n^{(1)}$", 2024. arXiv 2405.06587

[15] M. Jimbo, "A q-difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation", *Lett. Math. Phys.* **10**:1 (1985), 63–69. MR Zbl

[16] N. Jing and M. Liu, "R-matrix realization of two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{gl}}_n)$", *J. Algebra* **488** (2017), 1–28. MR Zbl

[17] N. Jing, M. Liu, and A. Molev, "Isomorphism between the R-matrix and Drinfeld presentations of quantum affine algebra: type C", *J. Math. Phys.* **61**:3 (2020), art. id. 031701. MR

[18] N. Jing, M. Liu, and A. Molev, "Isomorphism between the R-matrix and Drinfeld presentations of quantum affine algebra: types B and D", *SIGMA Symmetry Integrability Geom. Methods Appl.* **16** (2020), art. id. 043. MR Zbl

[19] A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, Springer, 1997. MR Zbl

[20] N. Y. Reshetikhin and M. A. Semenov-Tian-Shansky, "Central extensions of quantum current groups", *Lett. Math. Phys.* **19**:2 (1990), 133–142. MR Zbl

[21] N. Y. Reshetikhin, L. A. Takhtajan, and L. D. Faddeev, "Quantization of Lie groups and Lie algebras", *Algebra i Analiz* **1**:1 (1989), 178–206. In Russian; translated in *Leningrad Math. J.* **1** (1990), 193–225. MR Zbl

[22] M. Takeuchi, "A two-parameter quantization of GL(n) (summary)", *Proc. Japan Acad. Ser. A Math. Sci.* **66**:5 (1990), 112–114. MR Zbl

[23] X. Zhong, N. Hu, and N. Jing, "RLL-realization of two-parameter quantum affine algebra in type $C_n^{(1)}$", 2024. arXiv 2405.06597

Received February 8, 2024. Revised May 17, 2024.

RUSHU ZHUANG
52205500005@stu.ecnu.edu.cn

NAIHONG HU
nhhu@math.ecnu.edu.cn

XIAO XU
52275500004@stu.ecnu.edu.cn

(*all authors*)

SCHOOL OF MATHEMATICAL SCIENCES
MOE-KLMEA & SH-KLPMMP
EAST CHINA NORMAL UNIVERSITY
SHANGHAI
CHINA

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 329 No. 2 April 2024

Concentration inequalities for Paley–Wiener spaces 201
SYED HUSAIN and FRIEDRICH LITTMANN

Characterizing the Fourier transform by its properties 217
MATEUSZ KRUKOWSKI

Reduction types of CM curves 233
MENTZELOS MELISTAS

The local character expansion as branching rules: nilpotent cones and the case of SL(2) 259
MONICA NEVINS

Extremely closed subgroups and a variant on Glauberman's Z^*-theorem 303
HUNG P. TONG-VIET

Vishik equivalence and similarity of quasilinear p-forms and totally singular quadratic forms 327
KRISTÝNA ZEMKOVÁ

RLL-realization of two-parameter quantum affine algebra in type $D_n^{(1)}$ 357
RUSHU ZHUANG, NAIHONG HU and XIAO XU